Electronic Journal of Probability

Vol. 8 (2003) Paper no. 21, pages 1–17.

Journal URL
http://www.math.washington.edu/~ejpecp/

Comparison Theorems for Small Deviations of Random Series

F. Gao[1], **J. Hannig** and **F. Torcaso**

Department of Mathematics, University of Idaho
Moscow, ID 83844-1103, USA
`fuchang@uidaho.edu`

Department of Statistics, Colorado State University
Ft. Collins, CO 80523-1877, USA
`hannig@stat.colostate.edu`

Department of Mathematical Sciences, Johns Hopkins University
Baltimore, MD 21218-2682, USA
`torcaso@mts.jhu.edu`

Abstract: Let $\{\xi_n\}$ be a sequence of i.i.d. positive random variables with common distribution function $F(x)$. Let $\{a_n\}$ and $\{b_n\}$ be two positive non-increasing summable sequences such that $\prod_{n=1}^{\infty}(a_n/b_n)$ converges. Under some mild assumptions on F, we prove the following comparison

$$\Pr\left(\sum_{n=1}^{\infty} a_n\xi_n \leq \varepsilon\right) \sim \left(\prod_{n=1}^{\infty} \frac{b_n}{a_n}\right)^{-\alpha} \Pr\left(\sum_{n=1}^{\infty} b_n\xi_n \leq \varepsilon\right),$$

where

$$\alpha = \lim_{x\to\infty} \frac{\log F(1/x)}{\log x} < 0$$

is the index of variation of $F(1/\cdot)$. When this result is applied to the case $\xi_n = |Z_n|^p$ and $\{Z_n\}$ is a sequence of i.i.d. standard Gaussian random variables, it affirms a conjecture of Li [9].

Keywords and phrases: small deviation, regular variation.

AMS 2000 subject classification: Primary 60G15

Submitted to EJP on August 8, 2003. Final version accepted on December 3, 2003.

[1]Research partially supported by NSF grant EPS-0132626, and a seed grant from the University of Idaho

1 Introduction

Let $\{\xi_n\}$ be a sequence of i.i.d. positive random variables with common distribution function $F(x)$ and $\{a_n\}$ a positive summable sequence. It is of great interest to know the small deviation probability of the sum

$$V = \sum_{n=1}^{\infty} a_n \xi_n.$$

For example, in the study of the natural rates of escape of infinite-dimensional Brownian motions, it is crucial to understand probabilities of this type (see, eg., Erickson [6] page 332, also see Cox[4]).

When $\xi_n = Z_n^2$ where $\{Z_n\}$ is a sequence of i.i.d. standard Gaussian random variables, by the Karhunen-Loève expansion, V is just the square of the L^2 norm of a centered Gaussian process on $[0,1]$ with $\{a_n\}$ being the sequence of eigenvalues of the corresponding covariance operator. In this case Sytaja [13] gave a complete description of the small deviation behavior in terms of the Laplace transform of V.

Lifshits [11] considerably extended this result to a large class of i.i.d. positive random variables that satisfy the following condition:

Condition 1. There exist constants $b \in (0,1)$, $c_1, c_2 > 1$ and $\varepsilon > 0$ such that for each $r \leq \varepsilon$ the inequality $c_1 F(br) \leq F(r) \leq c_2 F(br)$ holds.

Denote the Laplace transform of ξ_1 by

$$I(s) = \int_{[0,\infty)} e^{-sx} dF(x)$$

and the cumulant generating function of ξ_1 by

$$f(s) = \log I(s).$$

We can state the following result of Lifshits [11]:

Theorem 1. (Lifshits, 1997) *Under Condition 1, if the sequence $\{a_n\}$ is positive and summable, then as $\varepsilon \to 0^+$*

$$\Pr\left(V \leq \varepsilon\right) \sim \left(2\pi\gamma^2 h_a''(\gamma)\right)^{-1/2} \exp\left\{\gamma\varepsilon + h_a(\gamma)\right\}$$

where

$$h_a(\gamma) = \log E\left(\exp\{-\gamma V\}\right) = \sum_{n=1}^{\infty} f(a_n\gamma) \tag{1}$$

and $\gamma = \gamma(\varepsilon)$ satisfies

$$\lim_{\varepsilon \to 0} \frac{\gamma\varepsilon + \gamma h_a'(\gamma)}{\sqrt{\gamma^2 h_a''(\gamma)}} = 0.$$

Here and in what follows $x(\varepsilon) \sim y(\varepsilon)$ as $\varepsilon \to 0$ means $\lim_{\varepsilon\to 0} x(\varepsilon)/y(\varepsilon) = 1$.

Although Theorem 1 is extremely useful, it may be difficult to apply directly for a specific sequence $\{a_n\}$. This is because one would need to know to some extent the Laplace transform of V. The fact that γ is defined implicitly is just a matter of inconvenience. A useful method for obtaining closed form expressions for the Laplace transform $E\left(\exp\{-\gamma V\}\right)$, especially in the case where $\xi_n = Z_n^2$, is outlined in the recent paper Gao et al. [8]. Therefore, at least in situations where the method of [8] applies, the problem of obtaining the small deviation probability is simplified. However, closed form expressions for Laplace transforms are rarely obtained in general, and this greatly restricts the usefulness of Theorem 1.

By using Theorem 1, Dunker, Lifshits and Linde [5] obtain similar results when the random variables satisfy the following additional condition:

Condition 2. The function $sf'(s) = sI'(s)/I(s)$ is of bounded variation on $[0, \infty)$.

The advantage of the results in [5] over Theorem 1 is that the asymptotic behavior of the small deviation probability of V is expressed (implicitly) in terms of the Laplace transform of ξ_1 instead of the Laplace transform of V.

However, their results assume the existence of a function ϕ that is positive, logarithmically convex, twice differentiable and integrable on $[1, \infty)$ such that $a_n = \phi(n)$. In applications $\{a_n\}$ often oscillates along a sequence that satisfies these extra assumptions. For example, if V is the square of the L^2 norm of an m-times integrated Brownian motion, then $a_n = [\pi(n-1/2)+e_n]^{-(2m+2)}$, where the varying oscillation e_n decays exponentially.

The purpose of this paper is to provide a method that enables one to compute small deviations for a general sequence $\{a_n\}$ such as the one mentioned above. One of the first results in this direction is the following comparison theorem of Li [9]:

Theorem 2. (Li, 1992) *Let $\{a_n\}$ and $\{b_n\}$ be two positive non-increasing summable sequences such that $\sum_{n=1}^{\infty} |1 - a_n/b_n| < \infty$. Let $\{Z_n\}$ be a sequence of i.i.d. standard Gaussian random variables. Then as $\varepsilon \to 0$*

$$\Pr\left(\sum_{n=1}^{\infty} a_n Z_n^2 \le \varepsilon^2\right) \sim \left(\prod_{n=1}^{\infty} b_n/a_n\right)^{1/2} \Pr\left(\sum_{n=1}^{\infty} b_n Z_n^2 \le \varepsilon^2\right). \tag{2}$$

This comparison theorem is a very useful tool (see, e.g., [10]). Typically, $b_n = \phi(n)$ so that one can compute the small deviation asymptotics on the righthand side of (2) by using the results of [5]. $\{b_n\}$ can also be chosen so that one can find the exact expression of the Laplace transform of the corresponding series using the method of [8]. Recently Gao at al. [7] improved upon Theorem 2 by replacing the condition $\sum_{n=1}^{\infty} |1 - a_n/b_n| < \infty$ by the convergence of the infinite product $\prod_{n=1}^{\infty}(a_n/b_n)$.

We would like to extend Li's theorem to random variables that satisfy Conditions 1 and 2. Note that the Conditions 1 and 2 guarantee ([5], page 62) the existence of a finite negative constant

$$\alpha := \lim_{x\to\infty} \frac{\log F(1/x)}{\log x} = \lim_{s\to\infty} sf'(s) = -\lim_{s\to\infty} s^2 f''(s) = \frac{1}{2} \lim_{s\to\infty} s^3 f'''(s). \tag{3}$$

The constant α is called the index of variation of $F(1/\cdot)$ and arises in the theory of regular variation (see [2]).

In this paper, we prove

Theorem 3. *Let $\{\xi_n\}$ be a sequence of i.i.d. positive random variables having cumulative distribution function $F(x)$ satisfying conditions 1 and 2. Let $\{a_n\}$ and $\{b_n\}$ be positive, non-increasing, summable sequences that satisfy $\sum_{n=1}^{\infty} |1 - a_n/b_n| < \infty$. Then as $\varepsilon \to 0^+$*

$$\Pr\left(\sum_{n=1}^{\infty} a_n \xi_n \le \varepsilon\right) \sim \left(\prod_{n=1}^{\infty} \frac{b_n}{a_n}\right)^{-\alpha} \Pr\left(\sum_{n=1}^{\infty} b_n \xi_n \le \varepsilon\right),$$

where α is defined by (3).

Under a slightly stronger assumption on the distribution of ξ_1, we can replace the convergence of $\sum_{n=1}^{\infty} |1 - a_n/b_n|$ by the convergence of $\prod_{n=1}^{\infty}(a_n/b_n)$.

Theorem 4. *Let $\{\xi_n\}$ be a sequence of i.i.d. positive random variables having cumulative distribution function $F(x)$ satisfying condition 1 and 2. Let $\{a_n\}$ and $\{b_n\}$ be positive, non-increasing, summable sequences such that $\prod_{n=1}^{\infty}(a_n/b_n)$ converges. Further suppose $sI''(s)/I'(s)$ and $sI'''(s)/I''(s)$ are of bounded variation on $[0, \infty)$. Then as $\varepsilon \to 0^+$*

$$\Pr\left(\sum_{n=1}^{\infty} a_n \xi_n \le \varepsilon\right) \sim \left(\prod_{n=1}^{\infty} \frac{b_n}{a_n}\right)^{-\alpha} \Pr\left(\sum_{n=1}^{\infty} b_n \xi_n \le \varepsilon\right),$$

where α is defined by (3).

Remark 1. Though the conditions on bounded variation might seem difficult to check often it is not so. In fact there are many examples of random variables ξ_n that satisfy the conditions of this theorem (see Section 5). For a useful survey of results on the theory of bounded variation see also Section 5 of [5].

A particularly interesting example is when $\xi_n = |Z_n|^p$ where $\{Z_n\}$ is an i.i.d. sequence of standard Gaussian random variables. In this case $F(1/x) \sim x^{-1/p}\sqrt{2/\pi}$ and the conditions of Theorem 4 can be readily verified. The following corollary affirms a conjecture of Li [9].

Corollary 1. *Let $\{Z_n\}$ be a sequence of i.i.d. standard Gaussian random variables, and let $\{a_n\}$ and $\{b_n\}$ be two positive, non-increasing summable sequences such that $\prod_{n=1}^{\infty}(b_n/a_n) < \infty$. Then for any $p > 0$ as $\varepsilon \to 0^+$*

$$\Pr\left(\sum_{n=1}^{\infty} a_n |Z_n|^p \le \varepsilon^p\right) \sim \left(\prod_{n=1}^{\infty} \frac{b_n}{a_n}\right)^{1/p} \Pr\left(\sum_{n=1}^{\infty} b_n |Z_n|^p \le \varepsilon^p\right).$$

The following interesting fact proved in [7] shows that one can often calculate the product $\prod_{n=1}^{\infty}(b_n/a_n) < \infty$ explicitly.

Proposition 1. *Let $f(z)$ and $g(z)$ be entire functions with only positive real simple zeros. Denote the zeros of f by $\alpha_1 < \alpha_2 < \alpha_3 < \cdots$ and the zeros of g by $\beta_1 < \beta_2 < \beta_3 < \cdots$. If*

$$\lim_{k\to\infty} \max_{|z|=r_k} \left| \frac{f(z)}{g(z)} \right| = 1$$

where the sequence of radii r_k tending to ∞ are chosen so that $\beta_k < r_k < \beta_{k+1}$ for large k, then

$$\prod_{n=1}^{\infty} \frac{\alpha_n}{\beta_n} = \left| \frac{f(0)}{g(0)} \right|.$$

The rest of the paper is organized as follows. In Section 2 we show how such a comparison can be used to find exact small deviation rates. A short proof of Theorem 3 is supplied in Section 3. The proof of Theorem 4 is more involved and given in Section 4. The last section provides a proof of Corollary 1 and comments on how to verify the conditions of our theorems.

2 An example

Consider the random variable

$$\sum_{n=1}^{\infty} \pi_n |Z_n|^p, \tag{4}$$

where $\{Z_n\}$ is a sequence of i.i.d. standard Gaussian random variables,

$$\pi_n = \frac{\Gamma(n+d)}{\Gamma(d)\Gamma(n+1)}$$

and $-1/2 < d < 0$. This process is an example of a fractionally integrated ARMA process that has been used to model many so-called long memory time series such as annual minimal water levels of the Nile river and internet traffic data (see, for example, section 10.5 of [3]).

A simple calculation based on [1] (formula 6.1.47) shows that

$$\pi_n = \frac{1}{\Gamma(d)}(n + d/2 + e_n)^{-(1-d)},$$

where $e_n = O(1/n)$ and $-1/2 < d < 0$.

If we let

$$\rho_n = \frac{1}{\Gamma(d)}(n + d/2)^{-(1-d)},$$

then $\prod_{n=1}^{\infty} \pi_n/\rho_n$ converges, and Corollary 1 allows us to estimate the small deviation probability of (4).

Before continuing with this example, we first establish a small deviation result for $\sum_{n=1}^{\infty} b_n |Z_n|^p$, where $\{Z_n\}$ is a sequence of i.i.d. standard Gaussian random variables and $b_n = (n+c)^{-A}$ with $A > 1$.

For simplicity of notation we define

$$K = -\int_0^{\infty} t^{-1/A} I'(t)/I(t)\, dt, \quad \text{where} \quad I(t) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-tx^p - x^2/2}\, dx. \tag{5}$$

Notice $I(t)$ is the Laplace transform of the random variable $\xi = |Z_1|^p$ and K is a well defined constant depending only on A and p. However, although K can be calculated rather easily for $p = 2$, we are not aware of a closed form expression for K whenever $p \neq 2$.

The following lemma is a straightforward consequence of Corollary 3.2 of Dunker, Lifshits and Linde [5].

Lemma 1. *If $b_n = (n+c)^{-A}$, then as $\varepsilon \to 0^+$,*

$$\Pr\left(\sum_{n=1}^{\infty} b_n |Z_n|^p \leq \varepsilon^p\right) \sim C(c,A)\varepsilon^{\frac{p-A(1+2c)}{2(A-1)}} \exp\left(-(A-1)\left(\frac{K}{A}\right)^{\frac{A}{A-1}} \varepsilon^{-\frac{p}{A-1}}\right),$$

where

$$C(c,A) = \frac{2^{\frac{-3-2c+\frac{2A}{p}}{4}} A^{\frac{A+2Ac+p-2Ap}{2p-2Ap}} K^{\frac{A+2Ac-Ap}{-2p+2Ap}} \pi^{\frac{-1+2c+\frac{2A}{p}}{4}}}{\Gamma(1+\frac{1}{p})^{\frac{1}{2}+c} \Gamma(1+c)^{\frac{A}{p}} \sqrt{A-1}}.$$

When $p = 2$ this lemma has been proved recently by Nazarov and Nikitin [12]. For a calculation of the constant C_ϕ below see [12].

Proof. Using the notation of [5]:

$$\begin{aligned}
\phi(t) &= (t+c)^{-A}, \\
I_0(u) &= -(1+c)\log(\sqrt{2/\pi}\, I((1+c)^{-A}u) - Ku^{1/A} + \frac{A(1+c)}{p} + o(1), \\
I_1(u) &= -\frac{K}{A}u^{1/A} + \frac{1+c}{p} + o(1), \\
I_2(u) &= \frac{K(A-1)}{A^2}u^{1/A} + o(u^{1/A}), \\
C_\phi &= A\left\{\log\Gamma(1+c) - (\frac{1}{2}+c)\log(1+c) + (1+c) - \frac{1}{2}\log(2\pi)\right\}, \\
u &\sim \left(\frac{K}{A\varepsilon^p}\right)^{\frac{A}{A-1}}
\end{aligned}$$

as $u \to \infty$ (and, thus, as $\varepsilon \to 0^+$). The lemma now follows by plugging these quantities into Corollary 3.2 of Dunker, Lifshits and Linde [5]. $\square$

Combining Corollary 1 and this lemma one easily verifies

Corollary 2. *For π_n and Z_n defined above*

$$\Pr\left(\sum_{n=1}^{\infty} \pi_n |Z_n|^p \le \varepsilon^p\right) \sim C\varepsilon^{\frac{1-d^2-p}{2d}} \exp\left(d\left(\frac{K}{1-d}\right)^{\frac{d-1}{d}} \varepsilon^{\frac{p}{d}}\right),$$

where

$$C = C(d/2, 1-d) \prod_{n=1}^{\infty} \left(\frac{\Gamma(1+n)}{\Gamma(d+n)\,(n+d/2)^{1-d}}\right)^{1/p},$$

$C(\cdot,\cdot)$ was defined in Lemma 1 and K was defined in (5).

3 Proof of Theorem 3

We begin with some lemmas that will be used. From (1) we define

$$h_a(x) = \sum_{n=1}^{\infty} f(a_n x) \qquad \text{and} \qquad h_b(x) = \sum_{n=1}^{\infty} f(b_n x).$$

Lemma 2. *Suppose the cumulative distribution function F satisfies conditions 1 and 2, and let $y = y(x)$ be chosen to satisfy $h_a'(x) = h_b'(y)$. If $\sum_{n=1}^{\infty} |1 - a_n/b_n|$ converges, then as $x \to \infty$*

$$[xh_a'(x) - h_a(x)] - [yh_b'(y) - h_b(y)] \to -\alpha \log \prod_{n=1}^{\infty} \frac{a_n}{b_n}$$

where α is given by (3).

Proof. Without loss of generality, we assume $x \ge y$. By the mean value theorem, for any continuously differentiable function g on $(0, \infty)$,

$$g(x) - g(y) = \int_y^x tg'(t) \cdot \frac{1}{t}\, dt = \log\left(\frac{x}{y}\right) \theta g'(\theta), \tag{6}$$

for some θ between x and y. Applying (6) to the function $g(t) = tf'(t)$, we have

$$\begin{aligned} xh_a'(x) - xh_b'(x) &= \sum_{n=1}^{\infty} [a_n x f'(a_n x) - b_n x f'(b_n x)] \\ &= \sum_{n=1}^{\infty} \log\left(\frac{a_n}{b_n}\right) [\theta_n^2 f''(\theta_n) + \theta_n f'(\theta_n)], \end{aligned} \tag{7}$$

where θ_n is between $a_n x$ and $b_n x$.

Since $sf'(s)$ is of bounded variation, $s^2 f''(s) + sf'(s) \to 0$ as $s \to \infty$. Also, since we are assuming $\sum_{n=1}^{\infty} |1 - a_n/b_n| < \infty$, we have $\sum_{n=1}^{\infty} |\log(a_n/b_n)| < \infty$. Therefore, by the bounded convergence theorem (7) converges to 0 as $x \to \infty$.

On the other hand, by Mean Value Theorem, $h_b'(y) - h_b'(x) = (y - x)h_b''(\eta)$ for some η between x and y. Thus,

$$\begin{aligned} xh_a'(x) - xh_b'(x) &= x[h_b'(y) - h_b'(x)] \\ &= \sum_{n=1}^{\infty} b_n x(y - x) b_n f''(b_n \eta) \\ &= (y/x - 1)\frac{x^2}{\eta^2} \sum_{n=1}^{\infty} b_n^2 \eta^2 f''(b_n \eta). \end{aligned}$$

Because $s^2 f''(s) \to -\alpha > 0$ as $s \to \infty$,

$$\sum_{n=1}^{\infty} b_n^2 \eta^2 f''(b_n \eta) \to \infty.$$

On the other hand, we have shown that (7) converges to 0. This implies that $x \sim y$, which in turn implies that

$$\left(\frac{y}{x} - 1\right) \sum_{n=1}^{\infty} \eta_n^2 f''(\eta_n) \to 0 \tag{8}$$

for any η_n between $a_n x$ and $b_n y$ as $x \to \infty$.

Now, applying (6) to the function $sf'(s) - f(s)$, we obtain
$[xh_a'(x) - h_a(x)] - [yh_b'(y) - h_b(y)]$

$$\begin{aligned} &= \sum_{n=1}^{\infty} [a_n x f'(a_n x) - f(a_n x)] - [b_n y f'(b_n y) - f(b_n y)] \\ &= \sum_{n=1}^{\infty} \log\left(\frac{a_n x}{b_n y}\right) \cdot \zeta_n^2 f''(\zeta_n) \\ &= \sum_{n=1}^{\infty} \log\left(\frac{a_n}{b_n}\right) \cdot \zeta_n^2 f''(\zeta_n) + \log(x/y) \sum_{n=1}^{\infty} \zeta_n^2 f''(\zeta_n). \end{aligned}$$

From (8) the second series on the right hand side converges to 0. The lemma then follows by applying the bounded convergence theorem on the first series on the right. □

Lemma 3. *Under the assumptions of Lemma 2, we have $x^2 h_a''(x) \sim y^2 h_b''(y)$ as $x \to \infty$.*

Proof. Because $x^2 h_a''(x) = \sum_{n=1}^{\infty} a_n^2 x^2 f''(a_n x)$ is bounded away from 0, we just need to show that $x^2 h_a''(x) - y^2 h_b''(y) \to 0$ as $x \to \infty$. Applying (6) to the function $g(t) = t^2 f''(t)$, we have

$$
\begin{aligned}
x^2 h_a''(x) - y^2 h_b''(y) &= \sum_{n=1}^{\infty} [a_n^2 x^2 f''(a_n x) - b_n^2 y^2 f''(b_n y)] \\
&= \left(1 - \frac{y}{x}\right) \sum_{n=1}^{\infty} a_n^2 x^2 f''(a_n x) \cdot \frac{\log(x/y)}{1 - \frac{y}{x}} \cdot \frac{a_n^2 x^2 f''(a_n x) - b_n^2 y^2 f''(b_n y)}{a_n^2 x^2 f''(a_n x)(\log x - \log y)} \\
&= \left(1 - \frac{y}{x}\right) \sum_{n=1}^{\infty} a_n^2 x^2 f''(a_n x) \cdot \frac{\log(x/y)}{1 - \frac{y}{x}} \cdot \frac{t_n^3 f'''(t_n) + 2t_n^2 f''(t_n)}{a_n^2 x^2 f''(a_n x)},
\end{aligned}
$$

where t_n lies between $a_n x$ and $b_n y$. The second and third factors in the summation on the right hand side are bounded. In fact, the third factor goes to 0 as $x \to \infty$ (see (3)). What remains is exactly the sum in (8) and converges to 0 as $x \to \infty$. Therefore, the series above converges to 0 as $x \to \infty$. □

Proof of Theorem 3. ¿From Theorem 1 we have

$$
\frac{P\left(\sum_{n=1}^{\infty} a_n \xi_n \le \varepsilon\right)}{P\left(\sum_{n=1}^{\infty} b_n \xi_n \le \varepsilon\right)} = \left(\frac{x^2 h_a''(x)}{y^2 h_b''(y)}\right)^{-1/2} \exp\left\{-\left([x h_a'(x) - h_a(x)] - [y h_b'(y) - h_b(y)]\right)\right\}.
$$

The proof now follows easily from Lemmas 2 and 3. □

4 Proof of Theorem 4

The proof of Theorem 4 is a little more involved. It is more convenient to use the function $I(s)$ instead of $f(s)$.

Lemma 4. *For all $s > 0$ and $|\varepsilon| < 1$, we have*

$$
\frac{I^{(j)}((1+\varepsilon)s)}{I^{(j)}(s)} \ge 1 + \frac{s I^{(j+1)}(s)}{I^{(j)}(s)} \varepsilon \qquad \textit{for } j = 0, 1, 2.
$$

Proof. Note that $I^{(j)}(s) = (-1)^j \int_0^\infty t^j e^{-st} dF(t)$. Taylor's theorem gives

$$
I^{(j)}(s + \varepsilon s) \quad = \quad I^{(j)}(s) + I^{(j+1)}(x)\varepsilon s + \frac{(\varepsilon s)^2}{2} I^{(j+2)}(\xi)
$$

Thus,

$$
\frac{I^{(j)}(s + \varepsilon s)}{I^{(j)}(s)} \ge 1 + \frac{I^{(j+1)}(s)}{I^{(j)}(s)} \varepsilon s.
$$

□

Lemma 5. *Suppose $\sum_{n=1}^{\infty} c_n$ converges, and suppose g has total variation D on $[0,\infty)$. Then, for any monotonic non-negative sequence $\{d_n\}$,*

$$\left|\sum_{n>N} c_n g(d_n)\right| \leq D \sup_{k>N} \left|\sum_{n=k}^{\infty} c_n\right|.$$

This lemma is a consequence of Abel's summation formula. The proof is elementary and we omit the details.

Lemma 6. *Suppose $\{a_n\}$ and $\{b_n\}$ are two positive, non-increasing summable sequences such that $\prod_{n=1}^{\infty}(a_n/b_n)$ converges, and suppose $sI^{(j+1)}(s)/I^{(j)}(s)$ are of bounded variation on $[0,\infty)$, $j = 0, 1$ and 2. Then, $\displaystyle\prod_{n=N}^{\infty} \frac{I^{(j)}(a_n x)}{I^{(j)}(b_n x)}$ converges to 1 uniformly as $N \to \infty$.*

Proof. Set $C_j(s) = sI^{(j+1)}(s)/I^{(j)}(s)$ and let $a_n = (1+\varepsilon_n)b_n$. Then $\sum_{n=1}^{\infty} \varepsilon_n$ converges. Since $C_j(s)$ is bounded, there exists $M_j > 0$ and $N \in \mathbb{N}$, such that for $n > N$ and $s > 0$, $|C_j(s) \cdot \varepsilon_n| \leq M_j |\varepsilon_n| < 1$. By Lemma 4, we have

$$\prod_{n>N} \frac{I^{(j)}(a_n x)}{I^{(j)}(b_n x)} \geq \prod_{n>N} (1 + C_j(b_n x)\varepsilon_n).$$

Applying Lemma 5 with $c_n = \varepsilon_n$ and $d_n = b_n x$ and $g(x) = C_j(x)$, we obtain

$$\left|\sum_{n>N} C_j(b_n x)\varepsilon_n\right| \leq D \sup_{k>N} \left|\sum_{n=k}^{\infty} \varepsilon_n\right|.$$

Thus, $\sum_{n>N} C_j(b_n x)\varepsilon_n$ converges to 0 uniformly (in x) as $N \to \infty$, which implies that $\prod_{n>N}(1 - C_j(b_n x)\varepsilon_n)$ converges to 1 uniformly as $N \to \infty$. Thus,

$$\lim_{N\to\infty} \inf_{x\in(0,\infty)} \prod_{n=N}^{\infty} \frac{I^{(j)}(a_n x)}{I^{(j)}(b_n x)} \geq 1.$$

Similarly,

$$\lim_{N\to\infty} \inf_{x\in(0,\infty)} \prod_{n=N}^{\infty} \frac{I^{(j)}(b_n x)}{I^{(j)}(a_n x)} \geq 1.$$

The lemma follows. □

Lemma 7. *Under the assumptions of Lemma 6, as $x \to \infty$ we have*

1. $h_a(x) - h_b(x) \to \alpha \log \displaystyle\prod_{n=1}^{\infty} \frac{a_n}{b_n}$,
2. $xh_a'(x) - xh_b'(x) \to 0$, *and*

3. $x^2 h_a''(x) - x^2 h_b''(x) \to 0$.

Proof. First,

$$h_a(x) - h_b(x) = \log \prod_{n=1}^{\infty} \frac{I(a_n x)}{I(b_n x)}.$$

The fact that $I(s) \sim \Gamma(1-\alpha)F(1/s)$ implies for any fixed N

$$\prod_{n=1}^{N-1} \frac{I(a_n x)}{I(b_n x)} \to \left(\prod_{n=1}^{N-1} \frac{a_n}{b_n} \right)^{\alpha}$$

as $x \to \infty$. The first statement of the lemma now follows from Lemma 6.

Second,

$$\begin{aligned} x h_a'(x) - x h_b'(x) &= \sum_{n=1}^{\infty} \left(\frac{a_n x I'(a_n x)}{I(a_n x)} - \frac{b_n x I'(b_n x)}{I(b_n x)} \right) \\ &= \sum_{n=1}^{N} \left(\frac{a_n x I'(a_n x)}{I(a_n x)} - \frac{b_n x I'(b_n x)}{I(b_n x)} \right) \\ &\quad + \sum_{n>N} \frac{b_n x I'(b_n x)}{I(b_n x)} \left(\frac{a_n}{b_n} \cdot \frac{I'(a_n x)}{I'(b_n x)} \cdot \frac{I(b_n x)}{I(a_n x)} - 1 \right) \end{aligned}$$

In light of (3) it is easy to see the finite sum vanishes as $x \to \infty$. To see that the tail sum vanishes let us denote

$$\bar{c}_n = \frac{a_n}{b_n} \cdot \frac{I'(a_n x)}{I'(b_n x)} \cdot \frac{I(b_n x)}{I(a_n x)}.$$

First, by Lemma 6, the infinite product $\prod_{n>N} \bar{c}_n$ converges to 1 uniformly as $N \to \infty$. Thus, $\sum_{n>N} (\bar{c}_n - 1)$ converges to 0 uniformly as $N \to \infty$. Second, note that by assumption the quantity

$$\frac{b_n x I'(b_n x)}{I(b_n x)} = C_0(b_n x)$$

is of bounded variation on $[0, \infty)$. Applying Lemma 5 with $d_n = b_n x$ and $c_n = \bar{c}_n - 1$ we obtain

$$\sum_{n>N} \frac{b_n x I'(b_n x)}{I(b_n x)} \left(\frac{a_n}{b_n} \cdot \frac{I'(a_n x)}{I'(b_n x)} \cdot \frac{I(b_n x)}{I(a_n x)} - 1 \right) \to 0$$

uniformly as $N \to \infty$.

The proof of $x^2 h_a''(x) - x^2 h_b''(x) \to 0$ is similar. $\square$

Lemma 8. *If $y = y(x)$ is chosen to satisfy $h_a'(x) = h_b'(y)$ then as $x \to \infty$ we have*

1. *$x \sim y$;*

2. $[xh_a'(x) - h_a(x)] - [yh_b'(y) - h_b(y)] \to -\alpha \log \prod_{n=1}^{\infty} \frac{a_n}{b_n}$*;*

3. $h_a''(x) \sim h_b''(y)$ *as* $x \to \infty$

where α *is given by (3).*

Proof. To see that $x \sim y$ we notice from the last lemma

$$xh_b'(y) - xh_b'(x) = xh_a'(x) - xh_b'(x) \to 0 \tag{9}$$

as $x \to \infty$. Moreover, by the Mean Value Theorem $h_b'(x) - h_b'(y) = (x-y)h_b''(\xi)$, we have

$$\begin{aligned} xh_b'(x) - xh_b'(y) &= \sum_{n=1}^{\infty} \left(\frac{b_n x I'(b_n x)}{I(b_n x)} - \frac{b_n x I'(b_n y)}{I(b_n y)} \right) \\ &= \sum_{n=1}^{\infty} b_n x \cdot b_n (x-y) \left(\frac{I'(s)}{I(s)} \right)' |_{s=b_n\xi} \\ &= \left(1 - \frac{y}{x}\right) \sum_{n=1}^{\infty} (b_n x)^2 \cdot \frac{I''(b_n\xi) I(b_n\xi) - [I'(b_n\xi)]^2}{I^2(b_n\xi)}. \end{aligned} \tag{10}$$

Since

$$\frac{I''(s)I(s) - [I'(s)]^2}{I^2(s)} = f''(s) > 0 \text{ for all } s > 0,$$

$$|xh_b'(y) - xh_b'(x)| \geq \left|1 - \frac{y}{x}\right| \cdot (b_1 x)^2 f''(b_1 \xi).$$

Combining this with (9) and the fact that $s^2 f''(s) \to -\alpha > 0$ as $s \to \infty$, we have $y \sim x$ as $x \to \infty$.

¿From Lemma 7 and (9) it is enough to prove

$$\Delta := [xh_b'(x) - h_b(x)] - [yh_b'(y) - h_b(y)] \to 0 \quad \text{as } x \to \infty. \tag{11}$$

By the Mean Value Theorem,

$$\begin{aligned} |\Delta| &= |(x-y)\,\theta\, h_b''(\theta)| \\ &= |x-y|\,\theta \sum_{n=1}^{\infty} b_n^2 \frac{I''(b_n\theta) I(b_n\theta) - [I'(b_n\theta)]^2}{I^2(b_n\theta)}. \end{aligned}$$

Lemma 4 and the assumption of bounded variation (Condition 2) guarantee the existence of a constant $M > 0$ such that by the triangle inequality

$$\frac{I(b_n\xi)}{I(b_n\theta)} \leq \left(1 + M \left|\frac{\xi}{\theta} - 1\right|\right). \tag{12}$$

Assume for now that there exists $0 < \varepsilon < 1$, such that for all $|\xi/\theta - 1| < \varepsilon$

$$\frac{I''(b_n\theta)I(b_n\theta) - [I'(b_n\theta)]^2}{I''(b_n\xi)I(b_n\xi) - [I'(b_n\xi)]^2} \le \left(1 + K\left|\frac{\xi}{\theta} - 1\right|\right) \tag{13}$$

for some constant K. Then, since $\xi \sim \theta$, we eventually obtain

$$\begin{aligned} |\Delta| &\le \left|1 - \frac{y}{x}\right| \sum_{n=1}^{\infty} b_n^2 x^2 \frac{I''(b_n\xi)I(b_n\xi) - [I'(b_n\xi)]^2}{I^2(b_n\xi)} \cdot \frac{\theta}{x}\left[1 + K\left|\frac{\xi}{\theta} - 1\right|\right]\left[1 + M\left|\frac{\xi}{\theta} - 1\right|\right]^2 \\ &= |xh_b'(y) - xh_b'(x)| \cdot \frac{\theta}{x}\left[1 + K\left|\frac{\xi}{\theta} - 1\right|\right]\left[1 + M\left|\frac{\xi}{\theta} - 1\right|\right]^2 \to 0, \end{aligned}$$

where in the last equality we used (10). Thus, we have shown (11).

To see (13) observe

$$\phi(z) := I''(z)I(z) - [I'(z)]^2 = \int_0^\infty \int_0^\infty (t-s)^2 e^{-z(t+s)} dF(t)dF(s).$$

Clearly

$$\phi^{(j)}(z) = (-1)^j \int_0^\infty \int_0^\infty (t+s)^j (t-s)^2 e^{-z(t+s)} dF(t)dF(s)$$

and a similar argument as in Lemma 4 gives us

$$\frac{\phi(b_n\xi)}{\phi(b_n\theta)} \ge 1 + \frac{b_n\theta\phi'(b_n\theta)}{\phi(b_n\theta)}\left(\frac{\xi}{\theta} - 1\right),$$

for all $|\xi/\theta - 1| < 1$. Thus, in order to prove (13) it is enough to see that $b_n\theta\phi'(b_n\theta)/\phi(b_n\theta)$ is bounded. We will do more and prove that the function $z\phi'(z)/\phi(z)$ is of bounded variation.

Notice that

$$\frac{z\phi'(z)}{\phi(z)} = \frac{z\frac{I'''(z)}{I''(z)} - z\frac{I'(z)}{I(z)}}{1 - \frac{I'(z)}{I''(z)} \cdot \frac{I'(z)}{I(z)}}. \tag{14}$$

By assumption both the numerator and denominator of the right-hand-side of (14) are of bounded variation. To prove that $z\phi'(z)/\phi(z)$ is of bounded variation it is enough to show that the denominator above is bounded away from 0. However, for all $z > 0$ we have

$$0 < \frac{I'(z)}{I''(z)} \cdot \frac{I'(z)}{I(z)} < 1.$$

Moreover,

$$\lim_{z\to\infty} \frac{I'(z)}{I''(z)} \cdot \frac{I'(z)}{I(z)} = \frac{\alpha}{\alpha - 1} < 1 \quad \text{and} \quad \lim_{z\to 0+} \frac{I'(z)}{I''(z)} \cdot \frac{I'(z)}{I(z)} = \frac{(EX)^2}{E(X^2)},$$

where X is a random variable having distribution F. If both of the expectations are ∞ the limit can be shown to be 0. Thus, the denominator of the right-hand-side of (14) is bounded away from 0, and $z\phi'(z)/\phi(z)$ is of bounded variation.

To see $h_a''(x) \sim h_b''(y)$ as $x \to \infty$ recall $|x^2 h_a''(x)| \to \infty$ as $x \to \infty$ and, Lemma 7 implies $h_a''(x) \sim h_b''(x)$. Therefore, it is enough to show $h_b''(x) \sim h_b''(y)$. However, by replacing θ and ξ with x and y, respectively, in both (12) and (13) we obtain

$$\frac{f''(b_n x)}{f''(b_n y)} \leq \left(1 + K\left|\frac{y}{x} - 1\right|\right)\left(1 + M\left|\frac{y}{x} - 1\right|\right)^2$$

and consequently

$$\frac{h_b''(x)}{h_b''(y)} = \frac{\sum_{n=1}^{\infty} b_n^2 f''(b_n x)}{\sum_{n=1}^{\infty} b_n^2 f''(b_n y)} \leq \left(1 + K\left|\frac{y}{x} - 1\right|\right)\left(1 + M\left|\frac{y}{x} - 1\right|\right)^2.$$

Thus, $\limsup_{x,y\to 0} h_b''(x)/h_b''(y) \leq 1$ and $h_b''(x) \sim h_b''(y)$ follows by reversing the roles of x and y. □

Proof of Theorem 4. This is just a consequence of Theorem 1 and the last two lemmas. □

5 Proof of Corollary 1

Proof of Corollary 1. We will prove that if $\xi_i = |Z_i|^p$, $p > 0$, where Z_i are independent standard Gaussian variables, then $sI^{(j+1)}(s)/I^{(j)}(s)$, $j = 0, 1, 2$, is bounded and decreasing, and thus of bounded variation on $[0, \infty)$.

Let $g_j(s) = s^{-p/2} I^{(j+1)}(s^{-p/2})/I^{(j)}(s^{-p/2})$, $0 \leq s < \infty$. Since $s^{-p/2}$ is a monotone function, $sI^{(j+1)}(s)/I^{(j)}(s)$ is of bounded variation if $g_j(s)$ is of bounded variation. Thus it suffices to prove that $g_j(s)$ is a bounded increasing function. We show $g_j'(s) > 0$. By making the change of variable $u = s^{-1/2}t$, we have

$$\begin{aligned} g_j(s) &= -\frac{\int_0^\infty [s^{-1/2}t]^{jp+p} e^{-s^{-p/2}t^p} e^{-t^2/2} dt}{\int_0^\infty [s^{-1/2}t]^{jp} e^{-s^{-p/2}t^p} e^{-t^2/2} dt} \\ &= -\frac{\int_0^\infty u^{jp+p} e^{-u^p} e^{-su^2/2} du}{\int_0^\infty u^{jp} e^{-u^p} e^{-su^2/2} du}. \end{aligned} \tag{15}$$

A direct calculation gives

$$g_j'(s) = \frac{\int_0^\infty \int_0^\infty (u^{jp+p} - t^{jp+p})(u^{jp+2} - t^{jp+2}) e^{-u^p - su^2/2} e^{-t^p - st^2/2} du dt}{4\left[\int_0^\infty u^{jp} e^{-u^p - su^2/2} du\right]^2} > 0.$$

To prove that g_j is bounded, use (15) and the bounded convergence theorem to obtain

$$\lim_{s\to 0+} g_j(s) = -\frac{\int_0^\infty u^{jp+p} e^{-u^p} du}{\int_0^\infty u^{jp} e^{-u^p} du} = -\frac{\Gamma(j + 1 + 1/p)}{\Gamma(j + 1/p)} = -(j + 1/p).$$

Together with monotonicity, we obtain $|g_j(s)| \leq j + 1/p$.

Finally, $\alpha = -1/p$ follows from the observation that

$$\alpha = \lim_{s\to\infty} sf'(s) = \lim_{s\to\infty} g_0(s^{-2/p}) = -1/p.$$

Therefore, Corollary 1 follows from Theorem 4 with $\alpha = -1/p$. $\square$

Remark 2. The same proof will work for any $\xi_i = X_i^p$, where $\{X_i\}$ is a sequence of independent non-negative random variables having a density $F'(x)$ that is continuously differentiable, non-increasing, and $-\infty < F''(0) < 0$.

Remark 3. If we assume the cumulative distribution function F satisfies both conditions 1 and 2, then $F(1/x) = x^\alpha l(x)$ where $\alpha < 0$ is the index of variation and $l(x)$ is a slowly varying function. On the other hand, if $F(1/x) = x^\alpha l(x)$ and $l(x) = c + dx^\rho + o(x^\rho)$ as $x \to \infty$, where $c > 0$, $d \neq 0$ and $\rho < 0$, then conditions 1 and 2 hold (cf. [5] page 73).

Now, if we suppose $F(1/x) = x^\alpha(c + dx^\rho + o(x^\rho))$ as $x \to \infty$ then we can actually verify that $sI^{(j+1)}(s)/I^{(j)}(s)$ is of bounded variation on $[0,\infty)$ for any j such that $E\xi^{j+1} = \int_0^\infty x^{j+1} dF(x) < \infty$. Indeed,

$$I^{(j)}(s) = (-1)^j \int_0^\infty x^j e^{-tx} dF(x).$$

Thus $I^{(j+1)}(s)/I^{(j)}(s) = \bar{I}'(s)/\bar{I}(s)$ where $\bar{I}(s)$ is the Laplace transform of a random variable $\bar{\xi}$, where $\bar{\xi}$ has the distribution function

$$\bar{F}(x) = \frac{\int_0^x s^j \, dF(s)}{E\xi^j}.$$

An integration by parts gives

$$\bar{F}(1/x) = \frac{1}{E\xi^j}\left(\frac{F(1/x)}{x^j} - \int_0^{1/x} js^{j-1}F(s)\, ds\right) = \frac{x^{\alpha-j}}{E\xi^j}\left(\frac{-\alpha c}{j-\alpha} + \frac{-(\alpha+\rho)d}{j-\alpha-\rho}x^\rho + o(x^\rho)\right)$$

so that $s\bar{I}'(s)/\bar{I}(s)$ is of bounded variation, and therefore $sI^{(j+1)}(s)/I^{(j)}(s)$ is also.

6 Acknowledgment

The authors are profoundly grateful to Professor Wenbo V. Li for posing the question about the l_p comparison to us, which was the inspiration for this paper. We also thank Professor P. J. Brockwell for useful conversations regarding fractionally integrated ARMA processes.

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