Electronic Journal of Probability

Vol. 8 (2003) Paper no. 19, pages 1–18.

Journal URL
http://www.math.washington.edu/~ejpecp/

LONG-MEMORY STABLE ORNSTEIN-UHLENBECK PROCESSES

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Abstract The solution of the Langevin equation driven by a Lévy process noise has been well studied, under the name of Ornstein-Uhlenbeck type process. It is a stationary Markov process. When the noise is fractional Brownian motion, the covariance of the stationary solution process has been studied by the first author with different coauthors. In the present paper, we consider the Langevin equation driven by a linear fractional stable motion noise, which is a selfsimilar process with long-range dependence but does not have finite variance, and we investigate the dependence structure of the solution process.

Keywords and phrases: Langevin equation, linear fractional stable motion, long-range dependence, selfsimilar process.

AMS subject classification (2000): primary: 60H10; secondary: 60G10, 60G18

submitted to EJP on March 21, 2003. Final version accepted on October 2, 2003.

1. Introduction

In this paper, we consider a stochastic process $\{X(t), t \in \mathbb{R}\}$, which is a stationary solution of the Langevin equation

$$X(t) = X(0) - \lambda \int_0^t X(u)du + N(t) \tag{1.1}$$

with a stationary increment noise process $\{N(t), t \in \mathbb{R}\}$, where $\lambda > 0$. When such a stationary solution uniquely exists, we call $\{X(t)\}$ an Ornstein-Uhlenbeck process driven by a noise process $\{N(t)\}$. When $\{N(t)\}$ is Brownian motion, $\{X(t)\}$ is so-called (classical) Ornstein-Uhlenbeck process, which is a Gaussian Markov process.

We say that $\{N(t)\}$ is Lévy process if it has independent and stationary increments, is stochastically continuous and $N(0) = 0$ a.s. When $\{N(t)\}$ is a stable Lévy process, $\{X(t)\}$ is called the stable Ornstein-Uhlenbeck process, which was already studied in [5] and more deeply in [1]. It is also a Markov process. When $\{N(t)\}$ is a general Lévy process, the stationary solution $\{X(t)\}$ is called an Ornstein-Uhlenbeck type process, and its distributional properties were also well studied. (See [12] Section 17.)

Our concern here is what will happen if the noise process has dependent increments. In Gaussian case, the solution process $\{X(t)\}$ of (1.1) with fractional Brownian motion as $\{N(t)\}$ is studied in [3]. In this paper, we take a (non-Gaussian) linear fractional stable motion as a noise process, and investigate the dependence property of the stationary process $\{X(t)\}$.

2. Ornstein-Uhlenbeck processes driven by stationary increment noise processes

We first define Ornstein-Uhlenbeck processes driven by stationary increment noise processes. We follow an idea in [13].

Let $D[a,b]$ be a set of all functions $\phi : [a,b] \to \mathbb{R}$ such that they are left continuous and have right limits. Let $f \in \Re_m([a,b])$ be a set of all functions $f : [a,b] \to \mathbb{R}$ such that the Riemann integral $\int_a^b \phi(t)f(t)dt$ exists for any $\phi \in D[a,b]$.

Proposition 2.1. ([13]) *Let $\{N(t),\ t \in \mathbb{R}\}$ be a stationary increment measurable process with $N(0) = 0$ a.s. and $\lambda > 0$. If $\{N(t)\}$ satisfies conditions:*

(1) $\{N(t)\} \in \Re_m([0,1])$,

(2) $E\left[\ \log^+ \left|\int_0^1 e^t N(t)dt\right|\ \right] < \infty$

and

(3) $E\Big[\log^{+}|N(1)|\Big] < \infty,$

then

$$X(t) = N(t) - \lambda \int_{-\infty}^{t} e^{-\lambda(t-u)} N(u)du, \quad t \in \mathbb{R}, \tag{2.1}$$

is a stationary process well-defined in the sense of convergence in probability and the unique solution of the Langevin equation (1.1) *with a noise process* $\{N(t)\}$.

Remark 2.2. In their proof of Proposition 2.1 in [13], they show that $\lim_{n\to\infty} \int_{-n}^{t} e^{\lambda u} N(u)du$ exists a.s., but it is not clear that $\lim_{n\to\infty} \int_{-n-h}^{t} e^{\lambda u} N(u)du$ has the same value for all $h > 0$, which is needed for the stationarity of $\{X(t)\}$. However, we can show that

$$\lim_{n\to\infty} \int_{-n-h}^{-n} e^{\lambda u} N(u)du = 0 \quad \text{in probability}$$

under our assumptions as follows, and this is enough for our purpose. Actually we have

$$\begin{aligned}\int_{-n-h}^{n} e^{\lambda u} N(u)du &= e^{-\lambda n} \int_{-h}^{0} e^{\lambda v} N(v-n)dv \\ &= e^{-\lambda n} \int_{-h}^{0} e^{\lambda v}(N(v-n) - N(-n))dv + e^{-\lambda n} N(-n) \int_{-h}^{0} e^{\lambda v} dv \\ &=: I_1(n) + I_2(n).\end{aligned}$$

By the stationary increment property of $\{N(v)\}$,

$$I_1(n) \stackrel{\mathrm{d}}{=} e^{-\lambda n} \int_{-h}^{0} e^{\lambda v} N(v)dv,$$

which converges to 0 a.s. as $n \to \infty$, where $\stackrel{\mathrm{d}}{=}$ denotes equality in distribution, and thus $I_1(n) \to 0$ in probability. For $I_2(n)$, it is shown in [13] that $\sum_{n=1}^{\infty} e^{-n}|N(-n)| < \infty$ a.s. and thus $I_2(n) \to \infty$ a.s. as $n \to \infty$.

Remark 2.3. In their paper [13], they do not explicitly mention the uniqueness of the solution, but it is easily seen as follows. Suppose $X_1(t)$ and $X_2(t)$ are two solutions of (1.1) with $X_1(0) = X_2(0)$. Then

$$X_1(t) - X_2(t) = -\lambda \int_0^t (X_1(u) - X_2(u))du.$$

We thus have

$$X_1(t) - X_2(t) = \frac{(-\lambda)^n}{(n-1)!} \int_0^t (t-u)^{n-1}(X_1(u) - X_2(u))du$$

for $n = 1, 2, \ldots$. Since $((n-1)!)^{-1}(-\lambda)^n(t-u)^{n-1} \to 0$ uniformly in $u \in [0,t]$ as $n \to \infty$, we get $X_1(t) = X_2(t)$.

3. STABLE PROCESSES AND LINEAR FRACTIONAL STABLE MOTIONS

Throughout this paper, $\{Z_\alpha(x), x \in \mathbb{R}\}$ denotes a symmetric α-stable Lévy process with $E[e^{i\theta Z_\alpha(1)}] = e^{-|\theta|^\alpha}, \theta \in \mathbb{R}$. Linear fractional stable motions are defined as follows.

Definition 3.1. Let $0 < \alpha \leq 2$, $0 < H < 1$, $a, b \in \mathbb{R}$, $|a| + |b| > 0$. $\{\Delta_{H,\alpha}(t), t \in \mathbb{R}\}$ is linear fractional stable motion, if

$$\Delta_{H,\alpha}(t) = \int_{-\infty}^{\infty} \Bigg(a\left[(t-x)_+^{H-1/\alpha} - (-x)_+^{H-1/\alpha}\right] + b\left[(t-x)_-^{H-1/\alpha} - (-x)_-^{H-1/\alpha}\right] \Bigg) dZ_\alpha(x),$$

where $u_+ = \max\{u, 0\}, u_- = \max\{-u, 0\}$ and $0^s = 0$ even for $s \leq 0$. When $\alpha = 2$, it is fractional Brownian motion.

The following are some known facts on linear fractional stable motions, (see [6] and [7]).

Proposition 3.2. *Linear fractional stable motions are symmetric α-stable selfsimilar processes with stationary increments.*

Proposition 3.3. *Let* $0 < \alpha < 2$.
(1) *If $H > 1/\alpha$, $\{\Delta_{H,\alpha}(t)\}$ has a continuous version.*
(2) *If $H = 1/\alpha$, $\{\Delta_{H,\alpha}(t)\}$ is an α-stable Lévy process. Hence, almost all sample paths are right continuous and have left limits.*
(3) *If $H < 1/\alpha$, all sample paths of $\{\Delta_{H,\alpha}(t)\}$ are nowhere bounded.*

Since $\{\Delta_{H,\alpha}(t)\}$ is an α-stable Lévy process when $H = 1/\alpha$, the problem was studied in [1]. When $H < 1/\alpha$, the dependence structure of $\{\Delta_{H,\alpha}(t)\}$ itself is studied in [2]. We can consider the Langevin equation (1.1) with a noise $\{\Delta_{H,\alpha}(t)\}$, $H < 1/\alpha$. However, we cannot expect the existence of the solution of (1.1) with a noise whose sample paths are nowhere bounded. For, if such a solution exists, then it should has an expression like (2.1), where the noise process should be integrable in some sense, but it cannot be expected. Therefore, in the rest of this paper, we always assume that $1/\alpha < H < 1, \alpha < 2$, which necessarily leads us to the case $1 < \alpha < 2$.

For later use, we mention here two propositions on stable integrals.

Proposition 3.4. ([11]) *Let $0 < \alpha < 2$, $S \subset \mathbb{R}$ and let $f(x)$ be a measurable function on S. If $\int_S |f(x)|^\alpha dx < \infty$, then the stable integral $\int_S f(x) dZ_\alpha(x)$ is well defined, and*

$$E\left[\exp\left\{i\theta\int_S f(x)dZ_\alpha(x)\right\}\right] = \exp\left\{-|\theta|^\alpha \int_S |f(x)|^\alpha dx\right\}, \quad \theta \in \mathbb{R}.$$

Combining Theorems 11.3.2 and 11.4.1 in [11], we get the following.

Proposition 3.5. *Let $0 < \alpha < 2$, $T, S \subset \mathbb{R}$ and let $f(u,x)$ be a measurable function on $T \times S$. If*

$$\int_T \left|\int_S f(u,x)dZ_\alpha(x)\right| du < \infty \quad a.s., \tag{3.1}$$

then

$$\int_T \left(\int_S f(u,x)dZ_\alpha(x)\right) du = \int_S \left(\int_T f(u,x)du\right) dZ_\alpha(x) \quad a.s.$$

A necessary and sufficient condition for the validity of (3.1) *when $1 < \alpha < 2$ is*

$$\int_T \left(\int_S |f(u,x)|^\alpha dx\right)^{1/\alpha} du < \infty. \tag{3.2}$$

When a stochastic process $\{X(t),\ t \in \mathbb{R}\}$ is stationary Gaussian, we can describe its dependence property by its covariance function $E[X(t)X(0)]$. However, in the non-Gaussian stable case, we cannot use the covariance function. Instead, we use the following. For a stationary process $\{X(t)\}$, let

$$\begin{aligned} r(t) &:= r(\theta_1, \theta_2; t) \\ &:= E\Big[\exp\{i(\theta_1 X(t) + \theta_2 X(0))\}\Big] \\ &\quad - E\Big[\exp\{i\theta_1 X(t)\}\Big] E\Big[\exp\{i\theta_2 X(0)\}\Big],\ \theta_1, \theta_2 \in \mathbb{R}, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} I(t) &:= I(\theta_1, \theta_2; t) \\ &:= -\log E\Big[\exp\{i(\theta_1 X(t) + \theta_2 X(0))\}\Big] \\ &\quad + \log E\Big[\exp\{i\theta_1 X(t)\}\Big] + \log E\Big[\exp\{i\theta_2 X(0)\}\Big],\ \theta_1, \theta_2 \in \mathbb{R}. \end{aligned}$$

The following relationship between $r(t)$ and $I(t)$ is valid:

$$r(\theta_1, \theta_2; t) = K(\theta_1, \theta_2; t)\left(e^{-I(\theta_1, \theta_2; t)} - 1\right),$$

where

$$
\begin{aligned}
K(\theta_1,\theta_2;t) &= E\Big[\exp\{i\theta_1 X(t)\}\Big]E\Big[\exp\{i\theta_2 X(0)\}\Big] \\
&= E\Big[\exp\{i\theta_1 X(0)\}\Big]E\Big[\exp\{i\theta_2 X(0)\}\Big] \\
&=: K(\theta_1,\theta_2) =: K,
\end{aligned}
$$

say. Further, if $I(t) \to 0$ as $t \to \infty$, then $r(t) \sim -KI(t)$ as $t \to \infty$, namely, $r(t)$ and $I(t)$ is asymptotically equal. If $\{X(t)\}$ is Gaussian, $-I(1,-1;t)$ coincides with the covariance function and thus $r(t)$ is comparable to it. (For the details about these notions, see [11].)

The quantity $r(t)$ was used in [2], where the authors studied the dependence structure of linear fractional stable motions. The quantity $r(t)$ is also found in [8], where a necessary and sufficient condition for the mixing property of stationary infinitely divisible processes was proved in terms of $r(t)$ implicitly. (See also [9].) Therefore, it is reasonable to use $r(t)$, or equivalently, $I(t)$ in measuring the dependence property of our stationary process $\{X(t)\}$.

We want to say that a symmetric α-stable stationary process $\{X(t)\}$ has *long-memory*, if $r(t)$ in (3.3) satisfies

$$
\sum_{n=0}^{\infty} |r(n)| = \infty. \tag{3.4}
$$

The concept of "long-memory" is still used in different ways. Historically, the long-memory is measured in terms of correlations when processes have finite variances. (See [4] and [11], Section 7.2.) In such a case, the condition (3.4) is well understood for long-memory, especially for the increments of fractional Brownian motion. However, Samorodnitsky ([10]) recently points out that correlations provide only very limited information about the process if the process is "not very close" to being Gaussian. In case when the variance is infinite, the situation is more chaotic. As we have mentioned above, a "correlation like" quantity $r(t)$ is taken as one candidate for measuring the dependence in [2] and [11], Section 7.10, and we are following their idea in the present paper. Once we adopt this quantity, it might be natural to call the process to have "long-memory" if it satisfies (3.4), even if the process has infinite variance.

4. Long-memory stable Ornstein-Uhlenbeck processes

We are now going to investigate Ornstein-Uhlenbeck processes driven by non-Gaussian linear fractional stable motions as noise processes. Recall that we are restricting ourselves to the case $1/\alpha < H < 1, \alpha < 2$, and therefore $1 < \alpha < 2$.

Theorem 4.1.

$$X(t) = \Delta_{H,\alpha}(t) - \lambda \int_{-\infty}^{t} e^{-\lambda(t-u)} \Delta_{H,\alpha}(u) du \tag{4.1}$$

is the unique stationary solution of the Langevin equation,

$$X(t) = X(0) - \lambda \int_0^t X(u) du + \Delta_{H,\alpha}(t).$$

Proof. It is enough to show that when $H > 1/\alpha$, $\{\Delta_{H,\alpha}(t)\}$ satisfies Conditions (1), (2) and (3) in Proposition 2.1.

Condition (1): By Proposition 3.3, when $H > 1/\alpha$, $\Delta_{H,\alpha}(t)$ has a continuous version. For any sample path of such a version, $\phi(t)\Delta_{H,\alpha}(t)$ is integrable over $[0,1]$.

Condition (2): Let us write $\Delta_{H,\alpha}(t) = \int_{-\infty}^{\infty} f(t,x) dZ_\alpha(x)$. Then, we have

$$\begin{aligned} E\left[\log^+ \left| \int_0^1 e^t \Delta_{H,\alpha}(t) dt \right| \right] &\leq \sum_{n=0}^{\infty} P\left(\log \left| \int_0^1 e^t \Delta_{H,\alpha}(t) dt \right| \geq n \right) \\ &= \sum_{n=0}^{\infty} P\left(\left| \int_0^1 \left(e^t \int_{-\infty}^{\infty} f(t,x) dZ_\alpha(x) \right) dt \right| \geq e^n \right) \\ &\quad \text{(by Proposition 3.5)} \\ &= \sum_{n=0}^{\infty} P\left(\left| \int_{-\infty}^{\infty} \left(\int_0^1 e^t f(t,x) dt \right) dZ_\alpha(x) \right| \geq e^n \right). \end{aligned}$$

Since $\int_{-\infty}^{\infty} \left(\int_0^1 e^t f(t,x) dt \right) dZ_\alpha(x)$ is symmetric α-stable and the tail of its distribution has the power order of $-\alpha$, we have, with some $C_1 > 0$,

$$E\left[\log^+ \left| \int_0^1 e^t \Delta_{H,\alpha}(t) dt \right| \right] = C_1 \sum_{n=0}^{\infty} (e^n)^{-\alpha} < \infty \ .$$

Condition (3): We have

$$\begin{aligned} E\Big[\log^+ |\Delta_{H,\alpha}(1)| \Big] &\leq \sum_{n=0}^{\infty} P\Big(\log |\Delta_{H,\alpha}(1)| \geq n \Big) \\ &= \sum_{n=0}^{\infty} P\Big(|\Delta_{H,\alpha}(1)| \geq e^n \Big) \\ &\quad \text{(as above)} \end{aligned}$$

$$= C_2 \sum_{n=0}^{\infty} (e^n)^{-\alpha} < \infty \ ,$$

for some $C_2 > 0$. This completes the proof. $\square$

We want to investigate the behavior of $r(t) = r(\theta_1, \theta_2; t)$ as $t \to \infty$ in (3.3). However, since $\{X(t)\}$ is stationary,

$$P(X(t) \in A, X(0) \in B) = P(X(0) \in A, X(-t) \in B),$$

and thus $r(\theta_1, \theta_2; t) = r(\theta_2, \theta_1; -t)$. Therefore, the study of $r(\theta_1, \theta_2; t),\ t > 0$, as $t \to \infty$, is essentially the same as that of $r(\theta_1, \theta_2; t),\ t < 0$, as $t \to -\infty$. Since the calculation for $t < 0$ as $t \to -\infty$ is a bit easier in our setting, we will show a result on the behavior of $r(\theta_1, \theta_2; t),\ t < 0$, as $t \to -\infty$.

The following is our main theorem in this paper.

Theorem 4.2. *Let $1 < \alpha < 2, H > 1/\alpha$ and let $\{X(t)\}$ be the stochastic process in* (4.1). *Then as $t \to -\infty$,*

$$r(t) := r(\theta_1, \theta_2; t) \sim -KC|t|^{\alpha(H-1)}, \tag{4.2}$$

where

$$\begin{aligned} K = \exp\Bigg\{ &- (|\theta_1|^\alpha + |\theta_2|^\alpha) \\ &\times \Bigg(\int_{-\infty}^0 \bigg| \int_x^0 (H - \frac{1}{\alpha}) a e^{\lambda u} (u - x)^{H-\frac{1}{\alpha}-1} du \\ &\quad + \int_{-\infty}^x (H - \frac{1}{\alpha}) b e^{\lambda u} (x - u)^{H-\frac{1}{\alpha}-1} du \bigg|^\alpha dx \\ &\quad + \int_0^\infty \bigg| \int_{-\infty}^0 (H - \frac{1}{\alpha}) b e^{\lambda u} (x - u)^{H-\frac{1}{\alpha}-1} du \bigg|^\alpha dx \Bigg) \Bigg\} \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} C = {} & \lambda^{-\alpha} \left(H - \frac{1}{\alpha} \right)^\alpha \int_1^\infty \bigg(\Big| \theta_1 a (x-1)^{H-\frac{1}{\alpha}-1} + \theta_2 a x^{H-\frac{1}{\alpha}-1} \Big|^\alpha \\ & \quad - \Big| \theta_1 a (x-1)^{H-\frac{1}{\alpha}-1} \Big|^\alpha - \Big| \theta_2 a x^{H-\frac{1}{\alpha}-1} \Big|^\alpha \bigg) dx \\ & + \lambda^{-\alpha} \left(H - \frac{1}{\alpha} \right)^\alpha \int_0^1 \bigg(\Big| \theta_1 b (1-x)^{H-\frac{1}{\alpha}-1} + \theta_2 a x^{H-\frac{1}{\alpha}-1} \Big|^\alpha \\ & \quad - \Big| \theta_1 b (1-x)^{H-\frac{1}{\alpha}-1} \Big|^\alpha - \Big| \theta_2 a x^{H-\frac{1}{\alpha}-1} \Big|^\alpha \bigg) dx \\ & + \lambda^{-\alpha} \left(H - \frac{1}{\alpha} \right)^\alpha \int_{-\infty}^0 \bigg(\Big| \theta_1 b (1-x)^{H-\frac{1}{\alpha}-1} + \theta_2 b (-x)^{H-\frac{1}{\alpha}-1} \Big|^\alpha \end{aligned} \tag{4.4}$$

$$- \left|\theta_1 b(1-x)^{H-\frac{1}{\alpha}-1}\right|^\alpha - \left|\theta_2 b(-x)^{H-\frac{1}{\alpha}-1}\right|^\alpha \Big) dx.$$

Theorem 4.3. *Let* $1<\alpha<2$ *and* $H>1/\alpha$. *Then* $\{X(t)\}$ *in* (4.1) *has long-memory in the sense of* (3.4).

Proof. Since $\alpha<2$ and $H>1/\alpha$, we have $\alpha(H-1)>1-\alpha>-1$. Thus by (4.2), $\sum_{n=0}^\infty |r(n)|=\infty$. □

5. Proof of the main theorem

In the following, for notational convenience, put $\beta = H-1/\alpha$. To prove our main theorem, Theorem 4.2, we start with stating a lemma, the proof of which will be given in the last section. Recall that $t<0$.

Lemma 5.1. *We have*

$$\int_{-\infty}^t \lambda e^{-\lambda(t-u)} \left(\int_{-\infty}^u \left|a(u-x)^\beta - a(-x)^\beta\right|^\alpha dx\right)^{1/\alpha} du < \infty, \tag{5.1}$$

$$\int_{-\infty}^t \lambda e^{-\lambda(t-u)} \left(\int_u^0 \left|b(x-u)^\beta - a(-x)^\beta\right|^\alpha dx\right)^{1/\alpha} du < \infty \tag{5.2}$$

and

$$\int_{-\infty}^t \lambda e^{-\lambda(t-u)} \left(\int_0^\infty \left|b(x-u)^\beta - bx^\beta\right|^\alpha dx\right)^{1/\alpha} du < \infty. \tag{5.3}$$

From (4.1) and the definition of $\{\Delta_{H,\alpha}(t)\}$, we have

$$\begin{aligned}X(t) = \Delta_{H,\alpha}(t) &- \int_{-\infty}^t \lambda e^{-\lambda(t-u)} \left(\int_{-\infty}^u \left(a(u-x)^\beta - a(-x)^\beta\right) dZ_\alpha(x)\right) du\\
&- \int_{-\infty}^t \lambda e^{-\lambda(t-u)} \left(\int_u^0 \left(b(x-u)^\beta - a(-x)^\beta\right) dZ_\alpha(x)\right) du\\
&- \int_{-\infty}^t \lambda e^{-\lambda(t-u)} \left(\int_0^\infty \left(b(x-u)^\beta - bx^\beta\right) dZ_\alpha(x)\right) du.\end{aligned}$$

Here we apply Proposition 3.5 to change the order of three integrals above. Condition (3.2) in Proposition 3.5 can be verified by Lemma 5.1, and note that the second integral becomes the sum of two integrals. Thus we have

$$X(t) = \int_{-\infty}^t \left(a(t-x)^\beta - a(-x)^\beta\right) dZ_\alpha(x)$$

$$
\begin{aligned}
&+\int_t^0 \left(b(x-t)^\beta - a(-x)^\beta\right) dZ_\alpha(x) + \int_0^\infty \left(b(x-t)^\beta - bx^\beta\right) dZ_\alpha(x)\\
&-\int_{-\infty}^t \left(\int_x^t a\lambda e^{-\lambda(t-u)}(u-x)^\beta du - a\left(1-e^{-\lambda(t-x)}\right)(-x)^\beta\right) dZ_\alpha(x)\\
&-\int_{-\infty}^t \left(\int_{-\infty}^x b\lambda e^{-\lambda(t-u)}(x-u)^\beta du - ae^{-\lambda(t-x)}(-x)^\beta\right) dZ_\alpha(x)\\
&-\int_t^0 \left(\int_{-\infty}^t b\lambda e^{-\lambda(t-u)}(x-u)^\beta du - a(-x)^\beta\right) dZ_\alpha(x)\\
&-\int_0^\infty \left(\int_{-\infty}^t b\lambda e^{-\lambda(t-u)}(x-u)^\beta du - bx^\beta\right) dZ_\alpha(x)\\
=&\int_{-\infty}^t \Bigg(a(t-x)^\beta - \int_x^t a\lambda e^{-\lambda(t-u)}(u-x)^\beta du\\
&\qquad - \int_{-\infty}^x b\lambda e^{-\lambda(t-u)}(x-u)^\beta du\Bigg) dZ_\alpha(x)\\
&+\int_t^\infty \left(b(x-t)^\beta - \int_{-\infty}^t b\lambda e^{-\lambda(t-u)}(x-u)^\beta du\right) dZ_\alpha(x),
\end{aligned}
$$

and hence

$$
\begin{aligned}
X(t) = &\int_{-\infty}^t \Bigg(\int_x^t \beta a e^{-\lambda(t-u)}(u-x)^{\beta-1} du\\
&\qquad + \int_{-\infty}^x \beta b e^{-\lambda(t-u)}(x-u)^{\beta-1} du\Bigg) dZ_\alpha(x)\\
&+\int_t^\infty \left(\int_{-\infty}^t \beta b e^{-\lambda(t-u)}(x-u)^{\beta-1} du\right) dZ_\alpha(x).
\end{aligned}
\tag{5.4}
$$

We first calculate K. By (5.4) and Proposition 3.4, we have

$$
\begin{aligned}
K =& E\Big[\exp\{i\theta_1 X(0)\}\Big] E\Big[\exp\{i\theta_2 X(0)\}\Big]\\
=& \exp\Bigg\{-\int_{-\infty}^0 \left|\int_x^0 \theta_1\beta a e^{\lambda u}(u-x)^{\beta-1} du + \int_{-\infty}^x \theta_1 \beta b e^{\lambda u}(x-u)^{\beta-1} du\right|^\alpha dx\\
&-\int_0^\infty \left|\int_{-\infty}^0 \theta_1\beta b e^{\lambda u}(x-u)^{\beta-1} du\right|^\alpha dx\\
&-\int_{-\infty}^0 \left|\int_x^0 \theta_2\beta a e^{\lambda u}(u-x)^{\beta-1} du + \int_{-\infty}^x \theta_2 \beta b e^{\lambda u}(x-u)^{\beta-1} du\right|^\alpha dx\\
&-\int_0^\infty \left|\int_{-\infty}^0 \theta_2\beta b e^{\lambda u}(x-u)^{\beta-1} du\right|^\alpha dx\Bigg\}\\
=& \exp\Bigg\{-\left(|\theta_1|^\alpha + |\theta_2|^\alpha\right)\Bigg(\int_{-\infty}^0 \left|\int_x^0 \beta a e^{\lambda u}(u-x)^{\beta-1} du + \int_{-\infty}^x \beta b e^{\lambda u}(x-u)^{\beta-1} du\right|^\alpha dx
\end{aligned}
$$

$$+\int_0^\infty \left| \int_{-\infty}^0 \beta b e^{\lambda u}(x-u)^{\beta-1} du \right|^\alpha dx \Big) \Big\}.$$

This is K in (4.3).

We next calculate $I(t)$. By (5.4) and Proposition 3.4 again, and by a standard calculation, we have

$$\begin{aligned}
I(t) = &-\log E\Big[\exp\{i(\theta_1 X(t)+\theta_2 X(0))\}\Big] \\
&\qquad +\log E\Big[\exp\{i\theta_1 X(t)\}\Big] + \log E\Big[\exp\{i\theta_2 X(0)\}\Big] \\
= &\int_{-\infty}^t \Big(\Big| \int_x^t \theta_1 \beta a e^{-\lambda(t-u)}(u-x)^{\beta-1} du + \int_{-\infty}^x \theta_1 \beta b e^{-\lambda(t-u)}(x-u)^{\beta-1} du \\
&\qquad + \int_x^0 \theta_2 \beta a e^{\lambda u}(u-x)^{\beta-1} du + \int_{-\infty}^x \theta_2 \beta b e^{\lambda u}(x-u)^{\beta-1} du \Big|^\alpha \\
&\qquad - \Big| \int_x^t \theta_1 \beta a e^{-\lambda(t-u)}(u-x)^{\beta-1} du + \int_{-\infty}^x \theta_1 \beta b e^{-\lambda(t-u)}(x-u)^{\beta-1} du \Big|^\alpha \\
&\qquad - \Big| \int_x^0 \theta_2 \beta a e^{\lambda u}(u-x)^{\beta-1} du + \int_{-\infty}^x \theta_2 \beta b e^{\lambda u}(x-u)^{\beta-1} du \Big|^\alpha \Big) dx \\
&+ \int_t^0 \Big(\Big| \int_{-\infty}^t \theta_1 \beta b e^{-\lambda(t-u)}(x-u)^{\beta-1} du \\
&\qquad + \int_x^0 \theta_2 \beta a e^{\lambda u}(u-x)^{\beta-1} du + \int_{-\infty}^x \theta_2 \beta b e^{\lambda u}(x-u)^{\beta-1} du \Big|^\alpha \\
&\qquad - \Big| \int_{-\infty}^t \theta_1 \beta b e^{-\lambda(t-u)}(x-u)^{\beta-1} du \Big|^\alpha \\
&\qquad - \Big| \int_x^0 \theta_2 \beta a e^{\lambda u}(u-x)^{\beta-1} du + \int_{-\infty}^x \theta_2 \beta b e^{\lambda u}(x-u)^{\beta-1} du \Big|^\alpha \Big) dx \\
&+ \int_0^\infty \Big(\Big| \int_{-\infty}^t \theta_1 \beta b e^{-\lambda(t-u)}(x-u)^{\beta-1} du + \int_{-\infty}^0 \theta_2 \beta b e^{\lambda u}(x-u)^{\beta-1} du \Big|^\alpha \\
&\qquad - \Big| \int_{-\infty}^t \theta_1 \beta b e^{-\lambda(t-u)}(x-u)^{\beta-1} du \Big|^\alpha - \Big| \int_{-\infty}^0 \theta_2 \beta b e^{\lambda u}(x-u)^{\beta-1} du \Big|^\alpha \Big) dx.
\end{aligned}$$

Here by change of variables $x = xt, u = ut$, we have

$$\begin{aligned}
I(t) = |t|^{\alpha H} \int_1^\infty \Big(\Big| &\int_1^x \theta_1 \beta a e^{\lambda t(u-1)}(x-u)^{\beta-1} du + \int_x^\infty \theta_1 \beta b e^{\lambda t(u-1)}(u-x)^{\beta-1} du \\
&+ \int_0^x \theta_2 \beta a e^{\lambda t u}(x-u)^{\beta-1} du + \int_x^\infty \theta_2 \beta b e^{\lambda t u}(u-x)^{\beta-1} du \Big|^\alpha \\
&- \Big| \int_1^x \theta_1 \beta a e^{\lambda t(u-1)}(x-u)^{\beta-1} du + \int_x^\infty \theta_1 \beta b e^{\lambda t(u-1)}(u-x)^{\beta-1} du \Big|^\alpha
\end{aligned}$$

$$
\begin{aligned}
&\qquad -\Big|\int_0^x \theta_2\beta a e^{\lambda tu}(x-u)^{\beta-1}du+\int_x^\infty \theta_2\beta b e^{\lambda tu}(u-x)^{\beta-1}du\Big|^\alpha\Big)dx\\
&+|t|^{\alpha H}\int_0^1\Big(\Big|\int_1^\infty \theta_1\beta b e^{\lambda t(u-1)}(u-x)^{\beta-1}du\\
&\qquad +\int_0^x \theta_2\beta a e^{\lambda tu}(x-u)^{\beta-1}du+\int_x^\infty \theta_2\beta b e^{\lambda tu}(u-x)^{\beta-1}du\Big|^\alpha\\
&\qquad -\Big|\int_1^\infty \theta_1\beta b e^{\lambda t(u-1)}(u-x)^{\beta-1}du\Big|^\alpha\\
&\qquad -\Big|\int_0^x \theta_2\beta a e^{\lambda tu}(x-u)^{\beta-1}du+\int_x^\infty \theta_2\beta b e^{\lambda tu}(u-x)^{\beta-1}du\Big|^\alpha\Big)dx\\
&+|t|^{\alpha H}\int_{-\infty}^0\Big(\Big|\int_1^\infty \theta_1\beta b e^{\lambda t(u-1)}(u-x)^{\beta-1}du+\int_0^\infty \theta_2\beta b e^{\lambda tu}(u-x)^{\beta-1}du\Big|^\alpha\\
&\qquad -\Big|\int_1^\infty \theta_1\beta b e^{\lambda t(u-1)}(u-x)^{\beta-1}du\Big|^\alpha-\Big|\int_0^\infty \theta_2\beta b e^{\lambda tu}(u-x)^{\beta-1}du\Big|^\alpha\Big)dx\\
&=: |t|^{\alpha H}\int_1^\infty(|f(t,x)+g(t,x)|^\alpha-|f(t,x)|^\alpha-|g(t,x)|^\alpha)dx\\
&+|t|^{\alpha H}\int_0^1(|h(t,x)+g(t,x)|^\alpha-|h(t,x)|^\alpha-|g(t,x)|^\alpha)dx\\
&+|t|^{\alpha H}\int_{-\infty}^0(|h(t,x)+k(t,x)|^\alpha-|h(t,x)|^\alpha-|k(t,x)|^\alpha)dx,
\end{aligned}
$$

where

$$
\begin{aligned}
f(t,x)&=\int_1^x \theta_1\beta a e^{\lambda t(u-1)}(x-u)^{\beta-1}du+\int_x^\infty \theta_1\beta b e^{\lambda t(u-1)}(u-x)^{\beta-1}du,\ x>1,\\
g(t,x)&=\int_0^x \theta_2\beta a e^{\lambda tu}(x-u)^{\beta-1}du+\int_x^\infty \theta_2\beta b e^{\lambda tu}(u-x)^{\beta-1}du,\ x>0,\\
h(t,x)&=\int_1^\infty \theta_1\beta b e^{\lambda t(u-1)}(u-x)^{\beta-1}du,\ x<1
\end{aligned}
$$

and

$$k(t,x)=\int_0^\infty \theta_2\beta b e^{\lambda tu}(u-x)^{\beta-1}du,\ x<0.$$

Thus

(5.5)

$$
\begin{aligned}
&\lim_{t\to-\infty}|t|^{\alpha(1-H)}I(t)\\
&\quad=\lim_{t\to-\infty}\int_1^\infty(|tf(t,x)+tg(t,x)|^\alpha-|tf(t,x)|^\alpha-|tg(t,x)|^\alpha)dx
\end{aligned}
$$

$$+\lim_{t\to-\infty}\int_0^1 (|th(t,x)+tg(t,x)|^\alpha - |th(t,x)|^\alpha - |tg(t,x)|^\alpha)dx$$
$$+\lim_{t\to-\infty}\int_{-\infty}^0 (|th(t,x)+tk(t,x)|^\alpha - |th(t,x)|^\alpha - |tk(t,x)|^\alpha)dx.$$

In order to apply the dominated convergence theorem, we need the following lemmas.

Lemma 5.2. *For $1<\alpha<2$ and for any $r,s\in\mathbb{R}$, it holds that*

$$|\,|r+s|^\alpha - |r|^\alpha - |s|^\alpha\,| \le \alpha|r||s|^{\alpha-1} + (\alpha+1)|r|^\alpha.$$

Lemma 5.3. *There exist constants K_1, K_2, K_3 and K_4 such that for any $t<0$,*

$$|tf(t,x)| \le K_1(x-1)^{\beta-1},\ \textit{for any } x>1, \tag{5.6}$$
$$|tg(t,x)| \le K_2x^{\beta-1},\ \textit{for any } x>0,$$
$$|th(t,x)| \le K_3(1-x)^{\beta-1},\ \textit{for any } x<1 \tag{5.7}$$

and

$$|tk(t,x)| \le K_4(-x)^{\beta-1},\ \textit{for any } x<0.$$

Lemma 5.4.

$$\lim_{t\to-\infty} tf(t,x) = \frac{1}{\lambda}\theta_1\beta a(x-1)^{\beta-1}, \tag{5.8}$$
$$\lim_{t\to-\infty} tg(t,x) = \frac{1}{\lambda}\theta_2\beta a x^{\beta-1},$$
$$\lim_{t\to-\infty} th(t,x) = \frac{1}{\lambda}\theta_1\beta b(1-x)^{\beta-1} \tag{5.9}$$

and

$$\lim_{t\to-\infty} tk(t,x) = \frac{1}{\lambda}\theta_2\beta b(-x)^{\beta-1}.$$

The proofs of these lemmas will also be given in the last section.

By Lemma 5.2 with $r=tg(t,x), s=tf(t,x)$ and Lemma 5.3, we have

$$\begin{aligned}
&|tf(t,x)+tg(t,x)|^\alpha - |tf(t,x)|^\alpha - |tg(t,x)|^\alpha\\
&\quad\le \alpha|tg(t,x)||tf(t,x)|^{\alpha-1} + (\alpha+1)|tg(t,x)|^\alpha\\
&\quad\le \alpha K_2x^{\beta-1}(K_1(x-1)^{\beta-1})^{\alpha-1} + (\alpha+1)(K_2x^{\beta-1})^\alpha\\
&\quad= \alpha K_1^{(\alpha-1)(\beta-1)}K_2x^{\beta-1}(x-1)^{(\alpha-1)(\beta-1)} + (\alpha+1)K_2^\alpha x^{\alpha(\beta-1)}.
\end{aligned}$$

Note that $(\alpha-1)(\beta-1)>-1$ and $\alpha(\beta-1)<-1$, which we are going to use below frequently. Thus the above function of x belongs to $L^1(1,\infty)$.

When $0 \le x \le 1/2$, by Lemma 5.2 with $r = th(t,x), s = tg(t,x)$ and Lemma 5.3, we have

$$\begin{aligned}
&|th(t,x) + tg(t,x)|^\alpha - |th(t,x)|^\alpha - |tg(t,x)|^\alpha \\
&\quad \le \alpha |th(t,x)||tg(t,x)|^{\alpha-1} + (\alpha+1)|th(t,x)|^\alpha \\
&\quad \le \alpha K_3 (1-x)^{\beta-1} (K_2 x^{\beta-1})^{\alpha-1} + (\alpha+1)(K_3(1-x)^{\beta-1})^\alpha \\
&\quad = \alpha K_2^{(\alpha-1)(\beta-1)} K_3 (1-x)^{\beta-1} x^{(\alpha-1)(\beta-1)} + (\alpha+1) K_3^\alpha (1-x)^{\alpha(\beta-1)},
\end{aligned}$$

which belongs to $L^1(0,1/2)$.

When $1/2 < x \le 1$, by Lemma 5.2 with $r = tg(t,x), s = th(t,x)$ and Lemma 5.3, we have

$$\begin{aligned}
&|th(t,x) + tg(t,x)|^\alpha - |th(t,x)|^\alpha - |tg(t,x)|^\alpha \\
&\quad \le \alpha |tg(t,x)||th(t,x)|^{\alpha-1} + (\alpha+1)|tg(t,x)|^\alpha \\
&\quad \le \alpha K_2 x^{\beta-1} (K_3(1-x)^{\beta-1})^{\alpha-1} + (\alpha+1)(K_2 x^{\beta-1})^\alpha \\
&\quad = \alpha K_2 K_3^{(\alpha-1)(\beta-1)} x^{\beta-1} (1-x)^{(\alpha-1)(\beta-1)} + (\alpha+1) K_2^\alpha x^{\alpha(\beta-1)},
\end{aligned}$$

which belongs to $L^1(1/2,1)$.

By Lemma 5.2 with $r = th(t,x), s = tk(t,x)$ and Lemma 5.3, we have

$$\begin{aligned}
&|th(t,x) + tk(t,x)|^\alpha - |th(t,x)|^\alpha - |tk(t,x)|^\alpha \\
&\quad \le \alpha |th(t,x)||tk(t,x)|^{\alpha-1} + (\alpha+1)|th(t,x)|^\alpha \\
&\quad \le \alpha K_3 (1-x)^{\beta-1} (K_4(-x)^{\beta-1})^{\alpha-1} + (\alpha+1)(K_3(1-x)^{\beta-1})^\alpha \\
&\quad = \alpha K_4^{(\alpha-1)(\beta-1)} K_3 (1-x)^{\beta-1} (-x)^{(\alpha-1)(\beta-1)} + (\alpha+1) K_3^\alpha (1-x)^{\alpha(\beta-1)}.
\end{aligned}$$

This belongs to $L^1(-\infty,0)$.

Altogether, by applying the dominated convergence theorem to (5.5) and by Lemma 5.4, we have

$$\begin{aligned}
&\lim_{t\to-\infty} |t|^{\alpha-\alpha H} I(t) \\
&= \lambda^{-\alpha}\beta^\alpha \int_1^\infty \left(|\theta_1 a(x-1)^{\beta-1} + \theta_2 a x^{\beta-1}|^\alpha - |\theta_1 a(x-1)^{\beta-1}|^\alpha - |\theta_2 a x^{\beta-1}|^\alpha \right) dx \\
&\quad + \lambda^{-\alpha}\beta^\alpha \int_0^1 \left(|\theta_1 b(1-x)^{\beta-1} + \theta_2 a x^{\beta-1}|^\alpha - |\theta_1 b(1-x)^{\beta-1}|^\alpha - |\theta_2 a x^{\beta-1}|^\alpha \right) dx \\
&\quad + \lambda^{-\alpha}\beta^\alpha \int_{-\infty}^0 \left(|\theta_1 b(1-x)^{\beta-1} + \theta_2 b(-x)^{\beta-1}|^\alpha - |\theta_1 b(1-x)^{\beta-1}|^\alpha - |\theta_2 b(-x)^{\beta-1}|^\alpha \right) dx,
\end{aligned}$$

which is C in (4.4). Thus $I(t) \to 0$ as $t \to \infty$, and hence

$$r(t) \sim -KI(t) \sim -KC|t|^{\alpha(H-1)}$$

as $t \to \infty$. This completes the proof of Theorem 4.2.

6. Proofs of Lemmas

Proof Lemma 5.1. (5.1) can be proved as

$$\begin{aligned}
&\int_{-\infty}^{t} \lambda e^{-\lambda(t-u)} \left(\int_{-\infty}^{u} \left| a(u-x)^{\beta} - a(-x)^{\beta} \right|^{\alpha} dx \right)^{1/\alpha} du \\
&= \int_{-\infty}^{t} \lambda e^{-\lambda(t-u)} \left(\int_{-\infty}^{-1} \left| a(u+uy)^{\beta} - a(uy)^{\beta} \right|^{\alpha} (-u) dy \right)^{1/\alpha} du \\
&= \int_{-\infty}^{t} \lambda e^{-\lambda(t-u)} |u|^{H} du \left(\int_{1}^{\infty} \left| a(y-1)^{\beta} - a y^{\beta} \right|^{\alpha} dy \right)^{1/\alpha} < \infty,
\end{aligned}$$

and (5.2) and (5.3) can also be shown similarly. The proof is thus completed. □

Proof of Lemma 5.2. We consider two cases when $r, s > 0$ and when $r > 0, s < 0$.
(1) When $r, s > 0$. Since $(r+s)^{\alpha} - r^{\alpha} - s^{\alpha} > 0$, it is enough to show that $f(r) := -(r+s)^{\alpha} + \alpha r s^{\alpha-1} + (\alpha+2) r^{\alpha} + s^{\alpha} \geq 0$. Using an inequality $r^{\alpha-1} + s^{\alpha-1} - (r+s)^{\alpha-1} > 0$, we have

$$f(0) = 0, \ f'(r) = -\alpha(r+s)^{\alpha-1} + \alpha s^{\alpha-1} + \alpha(\alpha+2) r^{\alpha-1} \geq 0.$$

(2) When $r > 0, s < 0$. Since $|r+s|^{\alpha} - r^{\alpha} - |s|^{\alpha} < 0$, it is enough to show that $f(r) := |r+s|^{\alpha} + \alpha r |s|^{\alpha-1} + \alpha r^{\alpha} - |s|^{\alpha} \geq 0$. When $r + s > 0$,

$$f(0) = 0, \ f'(r) = \alpha(r+s)^{\alpha-1} + \alpha(-s)^{\alpha-1} + \alpha^2 r^{\alpha-1} \geq 0,$$

and when $r + s < 0$

$$f(0) = 0, \ f'(r) = -\alpha(-r-s)^{\alpha-1} + \alpha(-s)^{\alpha-1} + \alpha^2 r^{\alpha-1} \geq 0.$$

This completes the proof. □

Proof of Lemma 5.3. We first note that ${\theta_1}^{-1} f(t,x) = {\theta_2}^{-1} g(t, x-1)$ and ${\theta_1}^{-1} h(t,x) = {\theta_2}^{-1} k(t, x-1)$. Thus it is enough to prove (5.6) and (5.7).

Proof of (5.6): We have

$$\begin{aligned}
|tf(t,x)| &= \left| \int_{1}^{x} \theta_1 \beta a t e^{\lambda t(u-1)} (x-u)^{\beta-1} du + \int_{x}^{\infty} \theta_1 \beta b t e^{\lambda t(u-1)} (u-x)^{\beta-1} du \right| \\
&\leq \left| \int_{1}^{(x+1)/2} \theta_1 \beta a t e^{\lambda t(u-1)} (x-u)^{\beta-1} du \right| + \left| \int_{(x+1)/2}^{x} \theta_1 \beta a t e^{\lambda t(u-1)} (x-u)^{\beta-1} du \right|
\end{aligned}$$

$$
\begin{aligned}
&+\left|\int_x^{2x-1} \theta_1\beta bte^{\lambda t(u-1)}(u-x)^{\beta-1}du\right| + \left|\int_{2x-1}^{\infty} \theta_1\beta bte^{\lambda t(u-1)}(u-x)^{\beta-1}du\right| \\
&=: F_1+F_2+F_3+F_4,
\end{aligned}
$$

say. We have

$$F_1 \le |\theta_1\beta at| \left(\frac{x-1}{2}\right)^{\beta-1} \int_1^{(x+1)/2} e^{\lambda t(u-1)}du \le \frac{1}{\lambda}|a\theta_1|\beta\left(\frac{x-1}{2}\right)^{\beta-1}.$$

We have

$$
\begin{aligned}
F_2 &\le |\theta_1\beta at|\, e^{\lambda t(x-1)/2}\int_{(x+1)/2}^{x}(x-u)^{\beta-1}du \\
&= |a\theta_1|\left(\frac{x-1}{2}\right)^{\beta}|t|e^{\lambda t(x-1)/2} \le \frac{1}{\lambda e}|a\theta_1|\left(\frac{x-1}{2}\right)^{\beta-1},
\end{aligned}
$$

since $z(t) := |t|e^{\lambda t(x-1)/2}$ takes its maximum value $2/(\lambda(x-1)e)$ at $t=-2/(\lambda(x-1))$. Similarly, we have

$$F_3 \le |\theta_1\beta bt|\, e^{\lambda t(x-1)}\int_x^{2x-1}(u-x)^{\beta-1}du \le \frac{1}{\lambda e}|b\theta_1|(x-1)^{\beta-1}.$$

We finally have

$$F_4 \le |\theta_1\beta bt|\,(x-1)^{\beta-1}\int_{2x-1}^{\infty}e^{\lambda t(u-1)}du \le \frac{1}{\lambda}|b\theta_1|\beta(x-1)^{\beta-1}.$$

Altogether, we have

$$|tf(t,x)| = F_1+F_2+F_3+F_4 \le K_1(x-1)^{\beta-1}.$$

Proof of (5.7): We have

$$|th(t,x)| \le |\theta_1\beta bt|\,(1-x)^{\beta-1}\int_1^{\infty}e^{\lambda t(u-1)}du = \frac{1}{\lambda}|b\theta_1|\beta(1-x)^{\beta-1},$$

and thus (5.7). The proof of Lemma 5.3 is now completed. □

Proof of Lemma 5.4. By the same reasoning mentioned in the beginning of the proof of Lemma 5.3, we need to prove only (5.8) and (5.9).

Proof of (5.8): We have

$$
\begin{aligned}
\lim_{t\to-\infty} tf(t,x) &= \lim_{t\to-\infty}\int_1^{(x+1)/2}\theta_1\beta ate^{\lambda t(u-1)}(x-u)^{\beta-1}du \\
&\quad + \lim_{t\to-\infty}\int_{(x+1)/2}^{x}\theta_1\beta ate^{\lambda t(u-1)}(x-u)^{\beta-1}du \\
&\quad + \lim_{t\to-\infty}\int_x^{\infty}\theta_1\beta bte^{\lambda t(u-1)}(u-x)^{\beta-1}du \\
&=: f_1+f_2+f_3, \qquad (6.1)
\end{aligned}
$$

say. We have

$$f_1 = -\lim_{t\to-\infty}\int_{t(x-1)/2}^{0} \theta_1\beta a e^{\lambda v}\left(x-\frac{v}{t}-1\right)^{\beta-1}dv$$
$$= -\lim_{t\to-\infty}\int_{-\infty}^{0} \mathbf{1}_{\{t(x-1)/2\le v\le 0\}}\theta_1\beta a e^{\lambda v}\left(x-\frac{v}{t}-1\right)^{\beta-1}dv,$$

where

$$\mathbf{1}_{\{t(x-1)/2\le v\le 0\}}e^{\lambda v}\left(x-\frac{v}{t}-1\right)^{\beta-1}\le e^{\lambda v}\left(\frac{x-1}{2}\right)^{\beta-1},$$

which is integrable with respect to v over $(-\infty,0)$. Hence, by the dominated convergence theorem,

$$f_1 = \int_{-\infty}^{0}\theta_1\beta a e^{\lambda v}(x-1)^{\beta-1}dv \;=\; \frac{1}{\lambda}\theta_1\beta a(x-1)^{\beta-1}. \tag{6.2}$$

As to f_2,

$$|f_2| \le \lim_{t\to-\infty}|\theta_1\beta a t|e^{\lambda t(x-1)/2}\int_{(x+1)/2}^{x}(x-u)^{\beta-1}du = 0. \tag{6.3}$$

We finally have, by a standard argument,

$$f_3 = \lim_{t\to-\infty}\theta_1\beta b\int_{-\infty}^{t(x-1)} e^{\lambda v}\left(\frac{v}{t}+1-x\right)^{\beta-1}dv = 0. \tag{6.4}$$

Thus, it follows from (6.1)–(6.4) that

$$\lim_{t\to-\infty} tf(t,x) = \lambda^{-1}\theta_1\beta a(x-1)^{\beta-1},$$

which is (5.8).

Proof of (5.9): We have

$$\lim_{t\to-\infty} th(t,x) \;=\; \lim_{t\to-\infty}\int_{1}^{2}\theta_1\beta b t e^{\lambda t(u-1)}(u-x)^{\beta-1}du$$
$$+\lim_{t\to-\infty}\int_{2}^{\infty}\theta_1\beta b t e^{\lambda t(u-1)}(u-x)^{\beta-1}du$$
$$=: h_1+h_2, \tag{6.5}$$

say. We have

$$h_1 = \lim_{t\to-\infty}\int_{t}^{0}\theta_1\beta b e^{\lambda v}\left(\frac{v}{t}+1-x\right)^{\beta-1}dv$$
$$= \lim_{t\to-\infty}\int_{-\infty}^{0}\mathbf{1}_{\{t\le v\le 0\}}\theta_1\beta b e^{\lambda v}\left(\frac{v}{t}+1-x\right)^{\beta-1}dv,$$

where

$$\mathbf{1}_{\{t\le v\le 0\}}e^{\lambda v}\left(\frac{v}{t}+1-x\right)^{\beta-1}\le e^{\lambda v}(1-x)^{\beta-1},$$

which is integrable with respect to v over $(-\infty, 0)$, and thus

$$h_1 = \int_{-\infty}^{0} \theta_1 \beta b e^{\lambda v}(1-x)^{\beta-1} dv = \frac{1}{\lambda}\theta_1 \beta b(1-x)^{\beta-1}.$$

$h_2 = 0$ is shown similarly as (6.4), and we thus have

$$\lim_{t\to-\infty} th(t,x) = \lambda^{-1}\theta_1\beta b(1-x)^{\beta-1},$$

which is (5.9). This completes the proof of Lemma 5.4. □

Acknowledgement

The authors would like to express their thanks to the referee and an associate editor for their many essential comments, which improved the first version of this paper.

References

[1] Adler, R.J., Cambanis, S. and Samorodnitsky, G. (1990) : On stable Markov processes, *Stoch. Proc. Appl.* **34**, 1-17.

[2] Astrauskas, A., Levy, J.B. and Taqqu, M.S. (1991) : The asymptotic dependence structure of the linear fractional Lévy motion, *Lietuvos Matematikos Rinkinys* (*Lithuanian Mathematical Journal*) **31**(1), 1–28.

[3] Cheridito, P., Kawaguchi, H. and Maejima, M. (2003) : Fractional Ornstein-Uhlenbeck processes, *Electron. J. Probab.* **8**, paper no. 3, 1–14.

[4] Cox, D.R. (1984) : Long-range dependence : A review, In *Statistics : An Appraisal*, The Iowa University Press, 55–74.

[5] Doob, J.L. (1942) : The Brownian movement and stochastic equations, *Ann. Math.* **43**, 351–369.

[6] Maejima, M. (1983) : On a class of self-similar processes, *Z. Wahrsheinlichkeitstheorie verw. Gebiete* **62**, 235–245.

[7] Maejima, M. (1983) : A self-similar process with nowhere bounded sample paths, *Z. Wahrsheinlichkeitstheorie verw. Gebiete* **65**, 115–119.

[8] Maruyama, G. (1970) : Infinitely divisible processes, *Theory Probab. Appl.* **15**, 1–22.

[9] Rosiński, J. and Żak, T. (1995) : Simple conditions for mixing of infinitely divisible processes, *Stoch. Proc. Appl.* **61**, 277–288.

[10] Samorodnitsky, G. (2002) : Long Range Dependence, Heavy Tails and Rare Events, Lecture Notes Series, No. 12, MaPhySto, University of Aarhus.

[11] Samorodnitsky, G. and Taqqu, M.S. (1994) : *Stable Non-Gaussian Random Processes*, Chapman & Hall.

[12] Sato, K. (1999) : *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press.

[13] Surgailis, D., Rosiński, J., Mandrekar, V. and Cambanis, S. (2001) : On the mixing structure of stationary increment and self-similer $S\alpha S$ processes, Preprint.