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Renewal convergence rates and correlation decay for homogeneous pinning models

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Abstract

A class of discrete renewal processes with exponentially decaying inter-arrival distributions coincides with the infinite volume limit of general homogeneous *pinning* models in their localized phase. Pinning models are statistical mechanics systems to which a lot of attention has been devoted both for their relevance for applications and because they are *solvable* models exhibiting a non-trivial phase transition. The spatial decay of correlations in these systems is directly mapped to the speed of convergence to equilibrium for the associated renewal processes. We show that close to criticality, under general assumptions, the correlation decay rate, or the renewal convergence rate, coincides with the inter-arrival decay rate. We also show that, in general, this is false away from criticality. Under a stronger assumption on the inter-arrival distribution we establish a local limit theorem, capturing thus the sharp asymptotic behavior of correlations.

Key words: Renewal Theory, Speed of Convergence to Equilibrium, Exponential Tails, Pinning Models, Decay of Correlations, Criticality.

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1 Introduction and main results

1.1 Renewals processes and the Renewal Theorem

Consider a discrete, non–delayed, persistent renewal process $\tau := \{\tau_j\}_{j\in\mathbb{N}\cup\{0\}}$, that is the sequence of random variables such that $\tau_0 = 0$, $\{\tau_j - \tau_{j-1}\}_{j\in\mathbb{N}}$ is IID and such that the law of τ_1, the *inter-arrival law*, takes values in $\mathbb{N} := \{1, 2, \ldots\}$. We introduce the notation $F(n) := \mathbf{P}(\tau_1 = n)$ and we observe that it is at times practical to look at τ as a random subset of $\mathbb{N}\cup\{0\}$, so in particular if we set $u(n) := \mathbf{P}(n \in \tau)$, then $\{u(n)\}_{n\in\mathbb{N}\cup\{0\}}$ is the *renewal sequence* of τ. Note that $u(0) = 1$ and, if $F(\cdot)$ is aperiodic (*i.e.* if $\gcd\{n : F(n) > 0\} = 1$), there exists $n_0 > 0$ such that $u(n) > 0$ for every $n \ge n_0$. The classical Renewal Theorem (see *e.g.* [2]) says that, if $F(\cdot)$ is aperiodic, we have

$$u(\infty) := \lim_{n\to\infty} u(n) = \frac{1}{\mathbf{E}[\tau_1]} \in [0, 1]. \tag{1.1}$$

Much effort has been put into refining such a result. Refinements are of course a very natural question when $\mathbf{E}[\tau_1] = +\infty$ (*e.g.* [11; 14]), but also when $\mathbf{E}[\tau_1] < +\infty$. In the latter case sharp estimates on $u(n) - u(\infty)$ have been obtained for sub-exponential tail decay of the inter-arrival distribution sequence, like for example in the case of $F(n) \overset{n\to\infty}{\sim} c_1/n^{2+c_2}$ (c_1 and $c_2 > 0$), and the tails of the two sequences are directly related (*e.g.* [18] and references therein). Throughout the text the notation $a_n \overset{n\to\infty}{\sim} b_n$ stands for $\lim_{n\to\infty} a_n/b_n = 1$.

When instead the inter-arrival distribution has exponential decay the situation is quite different. In fact, what can be proven in general is that, if there exists $c_1 > 0$ such that $\lim_{n\to\infty} \exp(c_1 n)F(n) = 0$, then there exists $c_2 > 0$ such that $\lim_{n\to\infty} \exp(c_2 n)|u(n)-u(\infty)| = 0$. However the precise decay, or even only the exponential asymptotic behavior (that is the supremum of the values of c_2 for which the previous equality holds), in general does not depend only on the tail behavior of the inter-arrival probability. This is definitely a very classical problem [20; 19], and a number of results have been proven in specific instances (see *e.g* [4; 5; 22]). We are now going to treat this point in some detail.

1.2 On exponentially decaying inter-arrival laws

From the very definition of renewal process one directly derives the equivalent expressions

$$u(n) = \mathbf{1}_{\{0\}}(n) + \sum_{j=0}^{n-1} u(j)F(n-j) \qquad \text{and} \qquad \widehat{u}(z) = \frac{1}{1-\widehat{F}(z)}, \tag{1.2}$$

with the notation $\widehat{f}(z) = \sum_{n=0}^{\infty} z^n f(n)$ ($\widehat{f}(\cdot)$ is the generating function, or *z-transform*, of $f(\cdot)$) and z is a complex number. Of course $\widehat{f}(\cdot)$ is a power series and $|z|$ a priori has to be chosen smaller than the radius of convergence, which, for the two series appearing in (1.2), is at least 1. As a matter of fact, we are interested (in particular) in the radius of convergence of

$$\Delta(z) := \sum_{n=0}^{\infty} (u(n) - u(\infty))z^n = \frac{1}{1-\widehat{F}(z)} - \frac{1}{\mathbf{E}[\tau_1](1-z)}. \tag{1.3}$$

If we assume that $\limsup_{n\to\infty} \exp(cn)F(n) < \infty$ for some $c > 0$, the radius of convergence of $\widehat{F}(\cdot)$ is at least $\exp(c)$, however it is not at all clear that the radius of convergence of $\Delta(\cdot)$ coincides with the radius of convergence of $\widehat{F}(\cdot)$. The problem does not come from the singularity at $z = 1$ since it is easily seen that it is removable ($\widehat{F}(z) = \mathbf{E}[\tau_1](1-z) + O((1-z)^2)$). And notice also that, when $F(\cdot)$ is aperiodic, $\widehat{F}(z) = 1$ on the unit circle only if $z = 1$, while of course $|\widehat{F}(z)| < 1$ for $|z| < 1$. What may happen is the existence of other solutions z to $\widehat{F}(z) = 1$ for z within the radius of convergence of $\widehat{F}(\cdot)$. And it may even happen that $\Delta(\cdot)$ can be analytically continued beyond the radius of convergence of $\widehat{F}(\cdot)$. Let us make this clear by giving two (classical) explicit examples:

- $F(1) = 1-p$, $F(2) = p$ and $F(n) = 0$ for $n = 3, 4, \ldots$ ($p \in (0,1)$). The radius of convergence of $\widehat{F}(\cdot)$ is ∞, but $\Delta(z) = p/((1+p)(1+pz))$ and therefore the radius of convergence of $\Delta(\cdot)$ is $1/p$, and in fact, by expanding $\Delta(z)$ around $z = 0$, we obtain $u(n) - u(\infty) = (-p)^n(p/(1+p))$ for $n = 1, 2, \ldots$.
- $F(n) = p^n(1-p)/p$, $p \in (0,1)$. In this case the radius of convergences of $\widehat{F}(\cdot)$ is $1/p$, but $\Delta(z) = p$ for every z, so the radius of convergence is ∞ and in fact $u(n) - u(\infty) = 0$ for every $n \geq 1$.

These examples show that the tail decay of $u(\cdot) - u(\infty)$ may have little to do with the tail decay of the $F(\cdot)$: in particular, changing fine details of $F(\cdot)$ may have a drastic effect on the decay of $u(\cdot) - u(\infty)$. For further examples of such a behavior see in particular [5], but also Section 4 below.

The main purpose of this note is, however, to point out that, in a suitable class of renewal processes motivated by statistical mechanics modeling, the tail decay of $u(\cdot) - u(\infty)$ is closely linked with the tail decay of $F(\cdot)$.

1.3 Our set–up

We introduce the class of renewals we are going to focus on without insisting on the physical motivations, that are postponed to § 1.5. We consider the aperiodic discrete probability density $K(\cdot)$ concentrated on $\mathbb{N}$ such that for some $\alpha > 0$ and some function $L(\cdot)$ which is slowly varying at infinity we have

$$\overline{K}(N) := \sum_{n>N} K(n) \overset{N\to\infty}{\sim} \frac{L(N)}{\alpha N^\alpha}. \tag{1.4}$$

We recall that a function $L(\cdot)$ defined on the positive semi-axis is slowly varying at infinity if it is positive, measurable and if $\lim_{t\to\infty} L(ct)/L(t) = 1$ for every $c > 0$. We refer to [6] for the full theory of slowly varying functions, recalling simply that both $L(t)$ and $1/L(t)$ are much smaller than t^δ (as $t \to \infty$), and this for any $\delta > 0$. It is customary to say that $\overline{K}(\cdot)$ varies regularly with exponent $-\alpha$. We point out that (1.4) and aperiodicity are implied by

$$K(n) \overset{n\to\infty}{\sim} \frac{L(n)}{n^{1+\alpha}}. \tag{1.5}$$

Starting from $K(\cdot)$, we introduce a family of discrete probability densities indexed by $b \geq 0$:

$$K_b(n) := c(b)K(n)\exp(-bn), \tag{1.6}$$

and $c(b) = 1/\sum_n K(n)\exp(-bn)$ (of course $c(0) = 1$). Our attention focuses on the renewal process $\tau(b) := \{\tau_j(b)\}_j$ with inter-arrival law $K_b(\cdot)$, that is the renewal process τ with $F(\cdot) = K_b(\cdot)$ with the notation in § 1.1. The renewal sequence this time is denoted by $\{u_b(n)\}_n$, that is $u_b(n) := \mathbf{P}(n \in \tau(b))$.

1.4 Main result

With the set-up of § 1.3 we have the following:

Theorem 1.1. *Given $K(\cdot)$ call $b_0(\in [0,\infty])$ the infimum of the values of $b > 0$ such that there exists z satisfying $1 < |z| \le \exp(b)$ and $\widehat{K_b}(z) = 1$.*

1. *For every choice of $K(\cdot)$ satisfying (1.4) we have $b_0 \in (0,\infty]$ and for every $b \in (0, b_0]$ we have*
$$\limsup_{n\to\infty} \frac{1}{n} \log |u_b(n) - u_b(\infty)| \;=\; -b, \tag{1.7}$$
while for $b > b_0$
$$\limsup_{n\to\infty} \frac{1}{n} \log |u_b(n) - u_b(\infty)| \;\ge\; -b. \tag{1.8}$$
2. *For every choice of $K(\cdot)$ satisfying (1.5) we have that for every $b \in (0, b_0)$*
$$u_b(n) - u_b(\infty) \overset{n\to\infty}{\sim} \frac{K_b(n)}{(c(b)-1)^2}, \tag{1.9}$$
which implies
$$\lim_{n\to\infty} \frac{1}{n} \log \left(u_b(n) - u_b(\infty)\right) \;=\; -b. \tag{1.10}$$

Remark 1.2. When there exists z_0, $1 < |z_0| < \exp(b)$, such that $\widehat{K_b}(z_0) = 1$ (therefore $b > b_0$) one can easily write down the sharp asymptotic behavior of $\{u_b(n) - u_b(\infty)\}_n$ in terms of the values of z_0 with minimal $|z_0|$ obtaining that the sequence changes sign infinitely often and that, while of course
$$\limsup_{n\to\infty} \frac{1}{n} \log |u_b(n) - u_b(\infty)| \;=\; -\log|z_0| \;>\; -b, \tag{1.11}$$
in general the superior limit cannot be replaced by a limit (see Section 4 for details). In Section 4 we also provide explicit examples showing that b_0 can be arbitrarily small by choosing $K(\cdot)$ suitably. In all the examples we have worked out the inequality in (1.8) is strict (for every $b > b_0$), but it is unclear to us whether or not this is a general phenomenon.

The proof of Theorem 1.1(1) can be found in Section 2 which is devoted to the study of
$$R_b \;:=\; \frac{1}{\limsup_n |u_b(n) - u_b(\infty)|^{1/n}}, \tag{1.12}$$
which of course is the radius of convergence of $\Delta_b(\cdot)$ (defined in analogy with (1.3)), and to establishing that b_0 is not zero, see Proposition 2.1. Theorem 1.1(2) follows instead by applying a well established technique [9]: this is detailed in Section 3.

In Section 2, Proposition 2.2 (see also Remark 3.1), we give a generalization of Theorem 1.1. This generalization deals with the case in which we do not assume (1.4), but we require $\sum_n nK(n) < \infty$. As a matter of fact, in proving Proposition 2.1, that yields Theorem 1.1(1), we use (1.4) only to prove $b_0 > 0$ and actually, as we shall see, the argument leading to $b_0 > 0$ goes through without assuming (1.4) if $\sum_n nK(n) < \infty$. We would like to point out that it is possible to show that $b_0 > 0$ also by coupling arguments, when $\sum_n nK(n) < \infty$. This is achieved by applying for example the results in [23], but we will not detail this here. The proof we give, when $\sum_n nK(n) < \infty$, is extremely short. Moreover, it does not seem to be easy to extract our results when $\sum_n nK(n) = \infty$ from coupling arguments: the results in the literature are either not as sharp or they are restricted to very particular cases (see the last part of § 1.5, notably Remark 1.3, for more details). Of course in the case $\sum_n nK(n) = \infty$ we do use (1.4) in order to establish $b_0 > 0$: this choice is driven by the applications we have in mind and we do not know to which extent one can relax it.

1.5 Homogeneous pinning models and decay of correlations

What motivated, and what even suggested the validity of the results in this note, is the behavior near criticality of homogeneous pinning models. As it as been pointed out in particular in [13], a large class of physical models boils down to a family of Gibbs measures that, in mathematical terms, are just obtained from discrete renewal processes *modified* by introducing an exponential weight, or Boltzmann factor, depending on $\mathcal{N}_N(\tau) := |\tau \cap (0, N]|$. More precisely if $\mathbb{P}$ is the law of τ and the latter is the renewal process with inter-arrival distribution $K(\cdot)$, we consider the family of probability measures $\{\mathbb{P}_{N,\beta}\}_{N\in\mathbb{N}}$ defined by

$$\frac{\mathrm{d}\mathbb{P}_{N,\beta}}{\mathrm{d}\mathbb{P}}(\tau) = \frac{1}{Z_{N,\beta}} \exp\left(\beta \mathcal{N}_N(\tau)\right), \tag{1.13}$$

with $Z_{N,\beta}$ the normalization constant. Then one can show ([8],[15, Ch. 2]) that the weak limit $\mathbb{P}_{\infty,\beta}$ of $\{\mathbb{P}_{N,\beta}\}_{N\in\mathbb{N}}$ exists for every $\beta \in \mathbb{R}$ (to be precise, this statement holds for every β assuming (1.5), but it holds also assuming only (1.4) if $\beta > 0$ and, as we shall see, this is the relevant regime for us). The parameter β actually plays a crucial role. In fact if $\beta < 0$ then τ, under $\mathbb{P}_{\infty,\beta}$, is a transient renewal and it contains therefore only a finite number of points (this is the so-called *delocalized phase*). If instead $\beta > 0$ then τ, again under $\mathbb{P}_{\infty,\beta}$, is a positive recurrent renewal with inter-arrival distribution given by $K_b(\cdot)$, with $b = b(\beta)$ unique real solution of $\sum_n K(n)\exp(-bn) = \exp(-\beta)$ (this is the *localized phase*). Note that if $\beta \searrow 0$, then $b \searrow 0$. We point also out that it is not difficult to see that b coincides with the limit as N tends to infinity of $(\log Z_{N,\beta})/N$ and it is hence the *free energy* of the system [15, Ch. 1]. In [13] and, more completely in [15, Ch. 2], one can find the analysis of $b(\beta)$ as $\beta \searrow 0$ (and (1.4) is the natural hypothesis to get a regular behavior of $b(\beta)$ as $\beta \searrow 0$).

As a consequence $\tau(b)$, for $b > 0$, does describe the localized regime of an infinite volume statistical mechanics system: if b is small, the system is close to *criticality*. The *correlation length* is a key quantity in statistical mechanics, see *e.g.* [13]. Moreover it is expected to scale *nicely* with β (or, which is equivalent, with b) approaching criticality, typically like β to some (negative) power, possibly times *logarithmic* corrections. The correlation length may be defined by introducing first the correlation function:

$$\mathbf{c}(n) := \lim_{m\to\infty} \frac{\mathbf{P}\left(m\in\tau(b),\, m+n\in\tau(b)\right)-\mathbf{P}\left(m\in\tau(b)\right)\mathbf{P}\left(m+n\in\tau(b)\right)}{\sqrt{\mathbf{P}\left(m\in\tau(b)\right)\left(1-\mathbf{P}\left(m\in\tau(b)\right)\right)\mathbf{P}\left(m+n\in\tau(b)\right)\left(1-\mathbf{P}\left(m+n\in\tau(b)\right)\right)}}$$
$$= \frac{\mathbf{E}\left[\tau_1(b)\right]}{\mathbf{E}\left[\tau_1(b)\right]-1}\left(\mathbf{P}\left(n\in\tau(b)\right)-\frac{1}{\mathbf{E}\left[\tau_1(b)\right]}\right), \quad (1.14)$$

where we have used the Renewal Theorem. Then the correlation length is just one over the decay rate $\xi(b)$ of $\mathbf{c}(\cdot)$: $\xi(b) := -1/\limsup_{n\to\infty} n^{-1}\log|\mathbf{c}(n)|$ and therefore

$$\xi(b) \;=\; -1/\limsup_{n\to\infty} n^{-1}\log|u_b(n)-u_b(\infty)|, \quad (1.15)$$

so that Theorem 1.1 (largely) guarantees that

$$\xi(b) \overset{b\searrow 0}{\sim} \frac{1}{b}, \quad (1.16)$$

which roughly can be rephrased by saying that the correlation length, close to criticality, scales like one over the free energy.

On physical grounds (1.16), or rather the weaker form $\log\xi(b) \sim -\log b$, is certainly expected [13], not only in the homogeneous set-up, but also in the *disordered* one. A disordered pinning model is defined by taking a typical realization of an IID sequence $\{\omega_1, \omega_2, \ldots\}$ of centered random variables and by replacing $\mathcal{N}_N(\tau)$ in (1.13) with $\mathcal{N}_N(\tau)+\varepsilon\sum_{n=1}^N \omega_n \mathbf{1}_{n\in\tau}$: if $\varepsilon\neq 0$ one can no longer solve exactly this model and, as a matter of fact, the disorder introduces some striking effects (see [1; 10; 15; 17] for the state of the art and further details). A proof of (1.16) has been given in [24] by coupling arguments for the case in which $K(\cdot)$ is given by the return times of a simple random walk and the proof is given also for disordered models. The result actually holds as an equality for every b (like the case presented in § 4.1 below: we point out that for $\alpha = 1/2$ the distribution $K(\cdot)$ treated in § 4.1 coincides with the distribution of the returns to zero of a simple random walk in the sense that $K(n)$ is the probability that the first return to zero of a simple random walk happens at time $2n$). In general, coupling arguments yield precise upper and lower bounds on the rate when suitable monotonicity properties are present (see in particular [21]): the returns of a simple random walk are in this class. In absence of monotonicity properties coupling arguments usually yield only upper bounds on the speed of convergence (and hence lower bounds on the rate, see [2] and references therein): in [25] a coupling argument is given for disordered pinning models and it yields in our homogeneous set-up that for every $\alpha > 0$

$$\limsup_{b\searrow 0} \frac{\log\xi(b)}{\log(b)} \;\leq\; -1, \quad (1.17)$$

under the stronger hypothesis (1.5) (compare (1.16) and (1.17)).

Remark 1.3. The result (1.17) is obtained [25] by a coupling argument that is substantially more complex than the *over-jumps* coupling technique in [23]. The argument, which tries to mimic the proof for the simple random walk, involves suitably chosen Bessel processes and it is tuned to the regular variation character of $K(\cdot)$, *i.e.* to hypothesis (1.5). It yields however stronger results in the direction of having more quantitative bounds, namely results that hold for every n. It must be said that coupling results typically do yield quantitative estimates, but also the generating function techniques can be pushed beyond asymptotic results like the one we have presented (see *e.g.* [5], but also [4]): we have not pursued this direction. On the other hand, it is less obvious how to apply generating function techniques when disorder is present.

We conclude this introduction by recalling that the class of pinning models we have considered is sometimes presented as the class of $(1+d)$–*dimensional* pinning models. The name comes from the *directed* viewpoint on Markov chains: if one considers a Markov chain S with state space $\mathbb{Z}^d$, the state space of the *directed process* $\{(n, S_n)\}_n$ is $\mathbb{Z}^{1+d}$. The renewal structure in this case is simply given by the successive returns to $0 \in \mathbb{Z}^d$ by S or, equivalently, by the intersections of the directed process with the line $\{(n,0) \in \mathbb{Z}^{1+d} : n = 0, 1, 2, \ldots\}$. This viewpoint is important in order to understand the spectrum of applications of pinning models, that includes interfaces in two dimensional space. We are not going to discuss this further here, and we refer to [15; 26], but we do point out that precise estimates catching the order of magnitude of the correlation length in a class of interface pinning models in d-dimensional space (Gaussian effective interfaces pinned at an (hyper-)plane) have been obtained in [7].

2 The radius of convergence of $\Delta_b(\cdot)$

In this section we work in the most general set-up, *i.e.* we assume (1.4). Recall the definition of b_0 from the statement of Theorem 1.1 and recall (1.12).

Proposition 2.1. *$R_b \leq \exp(b)$ and, for every choice of $K(\cdot)$, $b_0 > 0$ and therefore $R_b = \exp(b)$ for $b \in (0, b_0]$.*

Note that this result implies (1.7) and (1.8).

Proof. We are going to show that $R_b \leq \exp(b)$ by making use only of $\widehat{K_b}(\exp(b)) < \infty$ and of the fact that the radius of convergence of $\widehat{K_b}(\cdot)$ is $\exp(b)$.

Of course we may assume that $\Delta_b(\cdot)$ is analytic in the centered ball of radius $\exp(b)$, since otherwise there is nothing to prove. Let us suppose that $\Delta_b(\cdot)$ has an analytic extension to the open ball of radius $R > \exp(b)$. From (1.3) we immediately derive an expression for $\widehat{K_b}(z)$ in terms of $\Delta_b(z)$, for $|z| < \exp(b)$, and this gives the meromorphic extension of $\widehat{K_b}(\cdot)$ to the centered ball of radius R. However we know that the radius of convergence of $\widehat{K_b}(\cdot)$ is $\exp(b)$ and that $|\widehat{K_b}(z)| \leq c(b) \sum_n K(n) < \infty$ if $|z| = \exp(b)$. So the singularity of $\widehat{K_b}(\cdot)$ cannot be a pole and therefore $\widehat{K_b}(\cdot)$ does not have a meromorphic extension. This implies that $\Delta_b(\cdot)$ cannot be analytically continued beyond the centered ball of radius $\exp(b)$.

The question that we have to address in order to complete the proof of Proposition 2.1, that is proving $b_0 > 0$, can be rephrased as: do there exist two sequences $\{b_j\}_j$, $b_j \searrow 0$ and $\{z_j\}_j$, $1 < |z_j| \leq \exp(b_j)$ such that $\widehat{K_{b_j}}(z_j) = 1$ for every j? Of course, if this is not the case, $\widehat{K_b}(z) \neq 1$ if $\log|z| (> 0)$ is sufficiently small.

We make some preliminary observations: first, we may assume $\Im(z_j) \geq 0$, since if $\widehat{K_b}(z) = 1$, we have $\widehat{K_b}(\overline{z}) = 1$ too. Then let us remark that, by writing $z_j = r_j \exp(i\theta_j)$, we can pass to the limit in the equation $\widehat{K_{b_j}}(z_j) = 1$: by the Lebesgue Dominated Convergence Theorem we have that every limit point $(1, \theta)$ of $\{(r_j, \theta_j)\}_j$ satisfies

$$\sum_n K(n) \exp(in\theta) = 1, \tag{2.1}$$

which gives $\theta = 0$ by aperiodicity. This tells us that, for b small, singularities have necessarily positive real part and small imaginary part (in short, they are close to 1). Moreover, by monotonicity, we see that the imaginary part cannot be zero (and therefore we assume that it is positive, since solutions come in conjugate pairs).

Let us now assume by contradiction that there exists a triplet of sequences

$$\left(\{b_j\}_j,\, \{\delta_j\}_j,\, \{\theta_j\}_j\right), \tag{2.2}$$

tending to zero, with the requirements that $0 \leq \delta_j < b_j$, $\theta_j > 0$ for every j and such that $\widehat{K_{b_j}}(\exp(b_j - \delta_j)\exp(i\theta_j)) = 1$ for every j. Of course the triplet corresponds to the poles of the associated $\Delta_{b_j}(\cdot)$ function at $z_j = \exp\left((b_j - \delta_j) + i\theta_j\right)$. We are going to show that such a triplet does not exist since we are able to extract subsequences such that

$$\widehat{K_{b_j}}(\exp(b_j - \delta_j)\exp(i\theta_j)) \neq 1, \tag{2.3}$$

for every j in the subsequence.

Let us consider the auxiliary sequence of non-negative numbers $\{\delta_j/\theta_j\}_j$. By choosing a subsequence we may assume that this sequence converges to a limit point $\gamma \in [0, \infty]$.

We consider first the case of $\alpha \in (0,1)$. We distinguish the two cases $\gamma < \infty$ and $\gamma = \infty$.

If $\gamma < \infty$ we have the asymptotic relation

$$\sum_n K(n) \exp(-\delta_j n) \sin(\theta_j n) \overset{j\to\infty}{\sim} \theta_j^\alpha L(1/\theta_j) \int_0^\infty \frac{\exp(-\gamma s)\sin(s)}{s^{1+\alpha}}\, \mathrm{d}s\,, \tag{2.4}$$

that follows from a Riemann sum approximation and the uniform convergence property of slowly varying functions [6, § 1.5] if the sum is restricted to $\theta_j n \in (\varepsilon, 1/\varepsilon)$. The rest is then controlled for small n's ($n \leq \varepsilon/\theta_j$) by replacing $\sin(x)$ with x and using summation by parts which tells us that $\sum_{n=1}^N nK(n)$ is equal to $\sum_{n=0}^{N-1} \overline{K}(n) - N\overline{K}(N)$ and the latter behaves for large values of N as $N^{1-\alpha}L(N)/(1-\alpha)$ [6, § 1.5]. For large n's the rest is controlled by using $|\exp(-\delta_j n)\sin(\theta_j n)| \leq 1$. Overall the absolute value of the rest is bounded by $c\theta_j^\alpha L(1/\theta_j)(\varepsilon^{1-\alpha} + \varepsilon^\alpha)$ for some $c > 0$, with c not depending on ε, for j sufficiently large (for example, $\theta_j < \varepsilon$) and (2.4) follows.

Observe that the left-hand side of (2.4) is asymptotically equivalent to the imaginary part of $\widehat{K_b}(\exp(b_j - \delta_j)\exp(i\theta_j))$, apart for the multiplicative constant $c(b_j) = 1 + o(1) \in \mathbb{R}$. The integral can be explicitly computed and it is equal to

$$\left(1 + \gamma^2\right)^{\alpha/2} \Gamma(1-\alpha) \sin\left(\alpha \arctan(1/\gamma)\right), \tag{2.5}$$

which is positive for every $\gamma \in [0, \infty)$, therefore for j sufficiently large (2.3) holds (the definition of $\Gamma(\cdot)$ is recalled in Section 4).

If $\gamma = \infty$ instead we write

$$\sum_n K(n)\exp(-\delta_j n)\sin(\theta_j n) \;=\; R_j^< + R_j^> \,, \tag{2.6}$$

with $R_j^<$ the sum for $n \le \varepsilon/\theta_j$ and $R_j^>$ is the rest ($0 < \varepsilon \le \pi/2$ is a fixed positive constant). Setting $s_\varepsilon := \sin(\varepsilon)/\varepsilon$ we have

$$R_j^< \;\ge\; s_\varepsilon\, \theta_j \sum_{n \le \varepsilon/\theta_j} nK(n)\exp(-\delta_j n) \overset{j\to\infty}{\sim} s_\varepsilon \Gamma(1-\alpha) L(1/\delta_j) \left(\frac{\theta_j}{\delta_j}\right) \delta_j^\alpha \,. \tag{2.7}$$

To obtain (2.7) we have used summation by parts, namely the identity:

$$\sum_{n=1}^{\infty} nK(n)\exp(-\delta_j n) \;=\;$$

$$\sum_{n=0}^{\infty} \overline{K}(n)\exp(-\delta_j(n+1)) \;-\; (1-\exp(-\delta_j)) \sum_{n=1}^{\infty} n\overline{K}(n)\exp(-\delta_j n). \tag{2.8}$$

On the other hand

$$\left| R_j^> \right| \;\le\; \exp\left(-(\delta_j/\theta_j)\varepsilon\right) \sum_{n > \varepsilon/\theta_j} K(n) \overset{j\to\infty}{\sim} \exp\left(-(\delta_j/\theta_j)\varepsilon\right) \frac{L(1/\theta_j)}{\alpha} (\theta_j/\varepsilon)^\alpha \,, \tag{2.9}$$

therefore

$$\left| \frac{R_j^>}{R_j^<} \right| \;\le\; c\, \exp\left(-(\delta_j/\theta_j)\varepsilon\right) \frac{L(1/\theta_j)}{L(1/\delta_j)} \left(\frac{\theta_j}{\delta_j}\right)^{\alpha-1} \;\le\; c'\, \exp\left(-(\delta_j/\theta_j)\varepsilon\right) \left(\frac{\theta_j}{\delta_j}\right)^{\alpha-2} \,, \tag{2.10}$$

where c, c' are positive constants (we have explicitly used the fact that, for every $c_1 > 1$ and every $c_2 > 0$ there exists $c_3 > 0$ such that $L(x)/L(y) \le c_1 (x/y)^{c_2}$ whenever $x/y \ge c_3$ [6, Th. 1.5.6]). Therefore $|R_j^>/R_j^<| \to 0$ as $j \to \infty$ and for j sufficiently large we have

$$\sum_n K(n)\exp(-\delta_j n)\sin(\theta_j n) \;\ge\; \frac{1}{2} s_\varepsilon \Gamma(1-\alpha) L(1/\delta_j) \frac{\theta_j}{\delta_j} \delta_j^\alpha \,, \tag{2.11}$$

and then also in this regime (2.3) holds.

The marginal case of $\alpha = 1$ and $\sum_n nK(n) = +\infty$ is treated as follows.

If $\gamma \in [0, \infty)$ for the step analogous to (2.4) we split the sum according to whether $\theta_j n \le \varepsilon$ or $\theta_j n > \varepsilon$. Summing by parts we obtain

$$\sum_{n=1}^{N} nK(n) \;=\; \sum_{n=0}^{N-1} \overline{K}(n) \;-\; N\overline{K}(N) \overset{N\to\infty}{\sim} \sum_{n=1}^{N} \frac{L(n)}{n} \;=:\; \widehat{L}(N), \tag{2.12}$$

where in the asymptotic limit we have used [6, Prop. 1.5.9a] that guarantees that $\widehat{L}(\cdot)$ is slowly varying and that $\lim_{n\to\infty} \widehat{L}(n)/L(n) = +\infty$. From this we directly obtain that the first term in the splitting, *i.e.* the sum over $\theta_j n \le \varepsilon$, is bounded below by a positive constant, depending on ε

and γ (this constant can be chosen bounded away from zero for γ in any compact subset of $[0,\infty)$) times $\theta_j \widehat{L}(1/\delta_j)$. The rest instead is bounded, in absolute value, by a constant (independent of γ) times $\theta_j L(1/\theta_j)$, for j sufficiently large (just use $|\sin(\theta_j n)\exp(-\delta_j n)| \le 1$). Using once again $\widehat{L}(n) \gg L(n)$ for large n, we obtain that $\sum_n K(n)\exp(-\gamma_j n)\sin(\theta_j n) > 0$ for j sufficiently large. If instead $\gamma = +\infty$ we restart from (2.6) and, by proceeding like in (2.7) and (2.9), we obtain that for j sufficiently large

$$\sum_n K(n)\exp(-\delta_j n)\sin(\theta_j n) \ \ge\ \frac{1}{2} s_\varepsilon \widehat{L}(1/\delta_j)\left(\frac{\theta_j}{\delta_j}\right)\delta_j \ -\ \frac{2}{\varepsilon}\exp\left(-(\delta_j/\theta_j)\varepsilon\right)\, L(1/\theta_j)\theta_j, \quad (2.13)$$

which is positive for j sufficiently large and the case $\alpha = 1$ and $\sum_n nK(n) = \infty$ is under control.

Let us now consider the case $\sum_n nK(n) < \infty$. In this case for every $\gamma \in [0,\infty]$ we observe that $\lim_{j\to\infty} \exp(-\delta_j n)\sin(\theta_j n)/\theta_j = n$ and that $|\exp(-\delta_j n)\sin(\theta_j n)/\theta_j| \le n$, so that by Dominated Convergence we have

$$\sum_n K(n)\exp(-\delta_j n)\frac{\sin(\theta_j n)}{\theta_j} \stackrel{j\to\infty}{\longrightarrow} \sum_n nK(n), \quad (2.14)$$

and therefore the left-hand side is positive for j sufficiently large. This concludes the proof of Proposition 2.1. $\square$

The very last part of the previous proof (formula (2.14)), that is when $\sum_n nK(n) < \infty$, clearly requires *no regular variation* assumption. More precisely we have proven the following generalization of Proposition 2.1:

Proposition 2.2. *Assume that inter-arrival laws are of the form* (1.6)*, with* $K(\cdot)$ *an aperiodic discrete probability density such that* $\sum_n nK(n) < \infty$*. If the radius of convergence of* $\widehat{K_b}(z)$ *is* $\exp(b)$*, then* $b_0 > 0$ *and* $R_b = \exp(b)$ *for* $b \in (0, b_0]$.

This of course immediately generalizes Theorem 1.1(1) (for what concerns Theorem 1.1(2), see Remark 3.1).

3 Sharp estimates

Throughout this section $K(\cdot)$ satisfies (1.5), we assume $b > 0$ and we set $\nabla u_b(n) := u_b(n) - u_b(n-1)$ for $n = 0, 1, \ldots$ ($u_b(-1) := 0$). We also introduce the discrete probability density μ_b on $\mathbb{N} \cup \{0\}$ defined by

$$\mu_b(n) \ :=\ \overline{K_b}(n)/m_b, \quad (3.1)$$

with $m_b := \sum_n nK_b(n)$ and $\overline{K_b}(n) := \sum_{j>n} K_b(j)$. Let us observe that

$$m_b\mu_b(n) \ =\ K_b(n)\sum_{j=1}^{\infty}\frac{K(n+j)}{K(n)}\exp(-bj) \stackrel{n\to\infty}{\sim} \frac{1}{\exp(b)-1}K_b(n), \quad (3.2)$$

and that this directly implies the properties

$$\frac{\sum_{j=0}^{n} \mu_b(j)\mu_b(n-j)}{\mu_b(n)} \overset{n\to\infty}{\sim} 2\widehat{\mu}_b(\exp(b)) \quad \text{and} \quad \frac{\mu_b(n+1)}{\mu_b(n)} \overset{n\to\infty}{\sim} \exp(-b). \tag{3.3}$$

We point out also that from (1.2) we get

$$\widehat{\nabla u_n}(z) \,=\, \phi_b\left(\widehat{\mu}_b(z)\right), \quad \text{with} \quad \phi_b(z) \,:=\, \frac{1}{m_b z}, \tag{3.4}$$

at least for $|z| < 1$, like for (1.3). Of course the domain of analyticity of $\phi_b(\cdot)$ is $\mathbb{C} \setminus \{0\}$ and if we observe that, by direct computation, we have

$$\widehat{\mu}_b(z) \,=\, \frac{1-\widehat{K_b}(z)}{m_b(1-z)}, \tag{3.5}$$

one can then extend the validity of (3.4) to all values of z satisfying $|z| \le \exp(b)$ and $|z| < \inf\{|\zeta| > 1 : \widehat{K_b}(\zeta) = 1\}$.

Proof of Theorem 1.1(2). Let us choose $b < b_0$. We observe that the two properties in (3.3) are the hypotheses (α) and (β) of [9, Theorem 1]. Hypothesis (γ) of the same theorem, that is that $\widehat{\mu}_b(z)$ converges at its radius of convergence $(\exp(b))$, is verified too. Since $b < b_0$, $\{\widehat{\mu}_b(z) : |z| \le \exp(b)\} \subset \mathbb{C} \setminus \{0\}$, *i.e.* the range of the power series $\widehat{\mu}_b(\cdot)$ is a subset of the analyticity domain of $\phi_b(\cdot)$. Therefore [9, Theorem 1] yields

$$\nabla u_b(n) \overset{n\to\infty}{\sim} \phi_b'\left(\widehat{\mu}_b(\exp(b))\right) \mu_b(n) \,=\, -\frac{\mu_b(n)}{\left(\widehat{\mu}_b(\exp(b))\right)^2 m_b}, \tag{3.6}$$

and by (3.2) we have

$$\nabla u_b(n) \overset{n\to\infty}{\sim} -\frac{c(b)(\exp(b)-1)}{(c(b)-1)^2} K(n)\exp(-bn). \tag{3.7}$$

We conclude by observing that this yields

$$u_b(n) \,=\, -\sum_{j>n} \nabla u_b(j) \overset{n\to\infty}{\sim} \frac{c(b)}{(c(b)-1)^2} K(n)\exp(-bn) \,=\, \frac{K_b(n)}{(c(b)-1)^2}, \tag{3.8}$$

and the proof is complete. □

Remark 3.1. The validity of the results in [9] go beyond the assumption (1.5), that, in fact, has been used to verify (3.3). Since, as pointed out in Proposition 2.2, we do not make use of the regularly varying character of $K(\cdot)$ in establishing $b_0 > 0$ when $\sum_n nK(n) < \infty$, the results in this section (and therefore Theorem 1.1(2)) can be generalized to the set-up of Proposition 2.2, assuming in addition (3.3). The hypotheses (3.3) characterize, in a rather implicit way, a class of distribution that goes under the name of discrete sub-exponential [6, App. 4]. Just to make an example, Theorem 1.1 holds also for $K(n) = L(n)n^q \exp(-n^\gamma)$, with $q \in \mathbb{R}$ and $\gamma \in (0,1)$. Whether sub-exponentiality could replace in general our hypotheses seems to be a delicate point and in the literature there are some incorrect statements (for example we point out that [6, Th. A.4], cited from [12], is not correct, as it is proven by the examples we work out in the next section).

4 Some examples and further considerations

Recall that $\Gamma(z) := \int_0^\infty t^{z-1}\exp(-t)\,\mathrm{d}t$ for $\Re(z) > 0$, that $\Gamma(\cdot)$ can be extended as a meromorphic function to $\mathbb{C}$ and that $\Gamma(z+1) = z\Gamma(z)$ for $z \notin \{0, -1, -2, \dots\}$ (therefore $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$). Much of the content of this section is based on the fact that for $\beta \in \mathbb{R} \setminus \{0, -1, -2, \dots\}$ and $|x| < 1$ we have

$$\sum_{n=0}^{\infty} \frac{\Gamma(\beta+n)}{n!} x^n \;=\; \Gamma(\beta)(1-x)^{-\beta}. \tag{4.1}$$

This is just a matter of realizing that for $n \geq 1$

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(1-x)^{-\beta} \;=\; \beta(\beta+1)\dots(\beta+n-1)(1-x)^{-\beta-n}, \tag{4.2}$$

and the formula is the Taylor expansion in $x = 0$.

Since $\mathrm{sign}(\Gamma(\beta)) = (-1)^{\lceil |\beta| \rceil}$ for $\beta < 0$ ($|\beta| \notin \mathbb{N}$) the first terms of the series in (4.1) have alternating signs, but for n sufficiently large the sign stabilizes and, by Stirling's formula

$$\Gamma(x) \overset{x\to\infty}{\sim} \exp(-x)x^{x-(1/2)}\sqrt{2\pi}, \tag{4.3}$$

one readily sees that $\Gamma(n-\alpha)/n! \overset{n\to\infty}{\sim} 1/n^{1+\alpha}$. Therefore, with the help of (4.1) we can build probability inter-arrival distributions with the type of decay we are interested in and for which the generating function is explicit.

Remark 4.1. It is not difficult to see that one can differentiate, say j times, the expression in (4.1) generating thus sequences which decay like $(\log n)^j/n^{1+\alpha}$ and that, for sufficiently large n, do not change sign. This provides examples involving slowly varying functions.

Since we are just developing examples and that generalizations are straightforward, we specialize to the case of $-\beta = \alpha \in (0,1)$.

4.1 The basic example

In this section we study the case of

$$K(n) \;:=\; \frac{\Gamma(n-\alpha)}{-\Gamma(-\alpha)\,n!} \overset{n\to\infty}{\sim} \frac{n^{-1-\alpha}}{-\Gamma(-\alpha)}. \tag{4.4}$$

Note that $\sum_{n=1}^\infty K(n) = 1$ follows from (4.1), with $\beta = -\alpha$, as well as, with reference to (1.6), $c(b) = 1/(1-(1-\exp(-b))^\alpha)$ and

$$\widehat{K_b}(z) \;=\; \frac{1-(1-z\exp(-b))^\alpha}{1-(1-\exp(-b))^\alpha}. \tag{4.5}$$

In defining z^α for α non integer, we choose the cut line $\{z \in \mathbb{R} : z < 0\}$. With this choice $(1-z\exp(-b))^\alpha$, and therefore $\widehat{K_b}(\cdot)$, has a discontinuity on the line $\{z \in \mathbb{R} : z > \exp(b)\}$.

We observe that, for every $b > 0$, $\widehat{K_b}(z) = 1$ for $|z| \leq \exp(b)$ only if $z = 1$, therefore Theorem 1.1 holds with $b_0 = \infty$.

Remark 4.2. In the special case under consideration, but also in all the other cases considered in this section, one can obtain and go beyond Theorem 1.1 by direct computations. In fact if we set $q(z) := (1 - z\exp(-b))^\alpha$ we have for $|q(z)| < |q(1)|$

$$\frac{1}{1-\widehat{K_b}(z)} \;=\; \frac{1-q(1)}{q(z)-q(1)} \;=\; -\frac{1-q(1)}{q(1)} \sum_{j=0}^{\infty} \left(\frac{q(z)}{q(1)}\right)^j . \tag{4.6}$$

Now we set

$$R_m(z) \;:=\; \Delta_b(z) \;+\; \frac{1-q(1)}{q(1)} \sum_{j=1}^{m} \left(\frac{q(z)}{q(1)}\right)^j , \tag{4.7}$$

and we note that $(q(z))^j = (1 - z\exp(-b))^{j\alpha}$ and therefore once again (4.1) provides the expansion for $(q(z))^j$ if $j\alpha \notin \mathbb{N}$ and the n-th term in the power series (of $(q(z))^j$) behaves, as $n \to \infty$, like $c\exp(-nb)n^{-1-j\alpha}$, $c \neq 0$. Note that if $j\alpha \in \mathbb{N}$ the arising expression is just a polynomial and hence does not contribute to the asymptotic behavior of the series expansion.

Finally, the series expansion $\sum_n r^{(m)}(n)z^n$ of $R_m(\cdot)$ can be controlled by observing that this function is analytic in the centered ball of radius $\exp(b)$ and by using the formula

$$r^{(m)}(n) \;=\; \frac{1}{2\pi i}\oint \frac{R_m(z)}{z^{n+1}}\,\mathrm{d}z \;=\; \frac{\exp(-bn)}{2\pi}\int_0^{2\pi} R_m\left(\exp(b+i\theta)\right)\exp\left(-in\theta\right)\,\mathrm{d}\theta, \tag{4.8}$$

where the contour in the middle term is (say) $|z| = r$, for $r \in (0, \exp(\beta))$, and the last term is obtained by letting $r \nearrow \exp(b)$, using the fact that $R_m(\exp(b+i\theta))$ is bounded. In fact, from the explicit expression and by construction, one readily sees that $R_m\left(\exp(b+i\theta)\right)$ is smooth except at $\theta = 2\pi k$, $k \in \mathbb{Z}$, where it is $C^{\lfloor (m+1)\alpha \rfloor}$. By using the fact that n-th Fourier coefficient of a C^k function is $o(n^{-k})$, we see that $r^{(m)}(n) \;=\; \exp(-bn)o(1/n^{\lfloor (m+1)\alpha \rfloor})$.

The chain of considerations we have just made leads to an explicit expansion to all orders for $\exp(bn)(u_b(n) - u_b(\infty))$ as a sum of terms of the form $c_{j_1,j_2}n^{-j_1-\alpha j_2}$, for suitable (explicit) real coefficients c_{j_1,j_2} (j_1, $j_2 \in \mathbb{N}$).

4.2 Singularities and slower decay of correlations

From the basic example one can actually build a large number of *exactly solvable* cases that display the more general phenomenology hinted by Theorem 1.1: in particular that, in general, $b_0 < \infty$.

For example, fix $m \in \mathbb{N}$ and define

$$K(n) \;:=\; \begin{cases} \Gamma(n-m-\alpha)/\left(-\Gamma(-\alpha)\,(n-m)!\right) & \text{for } n = m+1, m+2, \ldots \\ 0 & \text{for } n = 1, 2, \ldots, m. \end{cases} \tag{4.9}$$

Note that this is nothing but the previous choice of $K(\cdot)$ *translated* to the right by m steps. Therefore

$$\widehat{K_b}(z) \;=\; z^m\,\frac{(1-(1-z\exp(-b))^\alpha)}{(1-(1-\exp(-b))^\alpha)}. \tag{4.10}$$

Once again the radius of convergence is $\exp(b)$, but this time, in general, it is no longer true that one cannot find a solution z_0 to $\widehat{K_b}(z_0) = 1$ in the annulus $1 < |z_0| < \exp(b)$.

Let us choose $\alpha = 1/2$ and let us first look at the case of $m = 1$. One can directly verify that

$$z_0 \;=\; -\frac{1}{2}\left(1+\sqrt{8\exp(b)\left(1-\sqrt{1-\exp(-b)}\right)-3}\right) \;<\; -1, \tag{4.11}$$

solves $\widehat{K_b}(z_0) = 1$, that it is the unique solution (except the trivial solution $z_0 = 1$), and $|z_0| < \exp(b)$ for $b > b_0$ with

$$b_0 \;:=\; \log\left(3/2+\sqrt{2}-\sqrt{\sqrt{2}+5/4}\right) \;=\; 0.248399... \tag{4.12}$$

So, if $b > b_0$, since z_0 is a (simple) pole singularity of $\Delta_b(\cdot)$ we can write

$$\Delta_b(z) \;=\; \frac{1}{z_0 K_b'(z_0)\,(1-(z/z_0))} \;+\; f(z), \tag{4.13}$$

with $f(\cdot)$ a function which is analytic on the centered ball of radius $\exp(b)$. Therefore

$$u_b(n) - u_b(\infty) \;=\; \frac{1}{z_0 K_b'(z_0)} z_0^{-n} + \varepsilon(n), \tag{4.14}$$

and $\limsup_{n\to\infty}(1/n)\log|\varepsilon(n)| = -b$.

Remark 4.3. Note that $z_0 = -1 - \exp(-b)/4 + O(\exp(-2b))$ for b large, so that the rate of convergence of $u_b(n) - u_\infty(n)$ becomes smaller and smaller as b becomes large. This is not a general phenomenon, for example if one chooses $\delta \in (0,1)$ and defines an inter-arrival distribution taking value δ for $n = 1$ and value $(1-\delta)K(n)$ for $n \geq 2$, $K(\cdot)$ as in (4.9) with $m = 1$ and $\alpha = 1/2$, then for $\delta \in (0, \sqrt{2}-1)$ there exists z_0, simple pole singularity of the corresponding $\Delta_b(\cdot)$ function, for b sufficiently large. But we have $z_0 \overset{b\to\infty}{\sim} -\delta(2+\delta)\exp(b)$.

Going back to (4.9), for m larger than 3 one can no longer explicitly find all the solutions z to $\widehat{K_b}(z) = 1$. However we have the following:

Proposition 4.4. *For every $b > 0$ and $\alpha \in (0,1)$ one can find $m \in \mathbb{N}$ such that if $K(\cdot)$ is given by* (4.9) *then there exists a solution z_0 to $\widehat{K_b}(z_0) = 1$ with $1 < |z_0| < \exp(b)$.*

Remark 4.5. In general, once the solutions to $\widehat{K_b}(\cdot) = 1$ of minimal absolute value (in the annulus $\{z : 1 < |z| < \exp(b)\}$) are known, it is straightforward to write the sharp asymptotic behavior of $u_b(n) - u_b(\infty)$. For example if z_0 is a complex solution, then also its conjugate is a solution. If these have minimal absolute value among the solutions and if they are simple solutions, for a suitable (and computable) real constants c_1 and c_2 ($|c_1| + |c_2| > 0$) we have

$$u_b(n) - u_b(\infty) \overset{n\to\infty}{\sim} |z_0|^{-n}\left(c_1\cos\left(n\arg\left(z_0\right)\right) + c_2\sin\left(n\arg\left(z_0\right)\right)\right). \tag{4.15}$$

An analogous formula is easily written in the general case.

Proof of Proposition 4.4. In reality, we are going to do something rather cheap, but we are actually proving more than what is stated: we are going to show that for every $b > 0$ and every $r \in (0, \exp(b))$ we can find an m such that there are m zeros of $\widehat{K_b}(\cdot) - 1$ in the annulus $\{z : 1 < |z| < r\}$.

Given $b > 0$, since the only solution z to $1 - (1 - z\exp(-b))^\alpha) = 0$ is $z = 0$, then for every $r \in (1, \exp(b))$ we have

$$x_r := \inf_\theta \left| \frac{1 - (1 - r\exp(-b + i\theta))^\alpha)}{1 - (1 - \exp(-b))^\alpha)} \right| > 0. \tag{4.16}$$

Therefore (recall (4.10)) $|\widehat{K_b}(z)| \ge r^m x_r$, if $|z| = r$. Therefore for m sufficiently large we have $|\widehat{K_b}(z)| > 1$ for $|z| = r$: let us fix such a couple (m, r). Rouché's Theorem (e.g. [3, p. 153]) guarantees that if f and g are analytic in a simply connected domain containing the simple closed curve γ and if $|f(z) - g(z)| < |f(z)|$ for $z \in \gamma$, then f and g have the same number of zeros enclosed by γ. Let us apply Rouché's Theorem with $f(z) := \widehat{K_b}(z)$ and $g(z) := 1 - \widehat{K_b}(z)$ and $\gamma := \{z : |z| = r\}$, so that $|f(z) - g(z)| = 1 < |f(z)|$ for $z \in \gamma$, by the choice of m. But $\widehat{K_b}(\cdot)$ has precisely $m+1$ zeros (they are all in 0) and therefore also $1 - \widehat{K_b}(\cdot)$ has $m+1$ zeros enclosed by γ. Of course $1 - \widehat{K_b}(\cdot)$ has a zero in 1 and all the other zeros have absolute value in $(1, r)$. $\square$

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