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# MUTUALLY CATALYTIC BRANCHING IN THE PLANE: INFINITE MEASURE STATES

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**Abstract** A two-type infinite-measure-valued population in $\mathsf{R}^2$ is constructed which undergoes diffusion and branching. The system is interactive in that the branching rate of each type is proportional to the local density of the other type. For a collision rate sufficiently small compared with the diffusion rate, the model is constructed as a pair of infinite-measure-valued processes which satisfy a martingale problem involving the collision local time of the solutions. The processes are shown to have densities at fixed times which live on disjoint sets and explode as they approach the interface of the two populations. In the long-term limit (in law), local extinction of one type is shown. Moreover the surviving population is uniform with random intensity. The process constructed is a rescaled limit of the corresponding $\mathsf{Z}^2$–lattice model studied by Dawson and Perkins (1998) and resolves the large scale mass-time-space behavior of that model under critical scaling. This part of a trilogy extends results from the finite-measure-valued case, whereas uniqueness questions are again deferred to the third part.

**Keywords** Catalyst, reactant, measure-valued branching, interactive branching, state-dependent branching, two-dimensional process, absolute continuity, self-similarity, collision measure, collision local time, martingale problem, moment equations, segregation of types, coexistence of types, self-duality, long-term behavior, scaling, Feynman integral

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# Contents

# 1 Introduction

## 1.1 Background and motivation

In [DF97a], a continuous super-Brownian reactant process $X^{\varrho}$ with a super-Brownian catalyst $\varrho$ was introduced. This pair $(\varrho, X^{\varrho})$ of processes serves as a model of a *chemical (or biological) reaction* of two substances, called *'catalyst'* and *'reactant'.* There the catalyst is modelled by an ordinary continuous super-Brownian motion $\varrho$ in $\mathsf{R}^d$, whereas the reactant is a continuous super-Brownian motion $X^{\varrho}$ whose branching rate, for 'particles' sitting at time $t$ in the space element $\mathrm{d}x$, is given by $\varrho_t(\mathrm{d}x)$ (random medium). This model has further been analyzed in [DF97b, EF98, FK99, DF01, FK00, FKX02]. Actually, the reactant process $X^{\varrho}$ makes non-trivial sense only in dimensions $d \leq 3$ since a "generic Brownian reactant particle" hits the super-Brownian catalyst only in these dimensions (otherwise $X^{\varrho}$ degenerates to the heat flow, [EP94, BP94]).

In a sense, $(\varrho, X^{\varrho})$ is a model with only a *'one-way interaction':* the catalyst $\varrho$ evolves autonomously, but it catalyzes the reactant $X^{\varrho}$. There is a natural desire to extend this model to the case in which *each* of the two substances catalyzes the other one, so that one has a *'true interaction'.* This, however, leads to substantial difficulties since the usual log-Laplace approach to superprocesses breaks down for such an interactive model. In particular, the analytic tool of diffusion-reaction equations is no longer available.

Dawson and Perkins [DP98, Theorem 1.7] succeeded in constructing such a continuum mutually catalytic model in the *one-dimensional* case, whereas in higher dimensions they obtained only a discrete version in which $\mathsf{R}^d$ is replaced by the lattice $\mathsf{Z}^d$, and Brownian motion is replaced by a random walk. More precisely, in the $\mathsf{R}$–case they showed that, for given (sufficiently nice) initial functions $\mathbf{X}_0 = \left(X_0^1, X_0^2\right)$, the following *system of stochastic partial differential equations* is uniquely solvable in a non-degenerate way:

$$\frac{\partial}{\partial t} X_t^i(x) \;=\; \frac{\sigma^2}{2} \Delta X_t^i(x) + \sqrt{\gamma\, X_t^1(x)\, X_t^2(x)}\, \dot{W}_t^i(x), \tag{1}$$

$(t,x) \in \mathsf{R}_+ \times \mathsf{R},\ i = 1,2.$ Here $\Delta$ is the one-dimensional Laplacian, $\sigma, \gamma$ are (strictly) positive constants (*migration and collision rate*, respectively), and $\dot{W}^1, \dot{W}^2$ are independent standard time-space white noises on $\mathsf{R}_+ \times \mathsf{R}$. The intuitive meaning of $X_t^i(x)$ is the *density* of mass of the $i^{\text{th}}$ substance at time $t$ at site $x$, which is dispersed in $\mathsf{R}$ according to a heat flow (Laplacian), but additionally branches with rate $\gamma X_t^j(x),\ j \neq i$ (and vice versa).

For the existence of a solution $\mathbf{X} = \left(X^1, X^2\right)$ to (1) they appealed to standard techniques as known, for instance, from [SS80], whereas uniqueness was made possible by Mytnik [Myt98] through a self-duality argument. For the existence part, their restriction to dimension one was substantial, and they pointed out that non-trivial existence of such a model (as measure-valued processes) in higher dimensional $\mathsf{R}^d$ remained open.

Major progress was made in Dawson et al. [DEF$^+$02] where it was shown that *also in* $\mathsf{R}^2$ such a *mutually catalytic branching process* $\mathbf{X}$ makes sense as a pair $\mathbf{X} = \left(X^1, X^2\right)$ of non-degenerate continuous finite-measure-valued Markov processes, provided that the collision rate $\gamma$ is not too large compared with the migration rate $\sigma$. In order to make this more precise, we write $\langle \mu, f \rangle$ or $\langle f, \mu \rangle$ to denote the integral of a function $f$ with respect to a measure $\mu$. Intuitively, $\mathbf{X} = \left(X^1, X^2\right)$ could be expected to satisfy the following system of stochastic partial differential equations

$$\begin{aligned} \left\langle X_t^i, \varphi^i \right\rangle \;=\; & \left\langle \mu^i, \varphi^i \right\rangle + \int_0^t \mathrm{d}s \left\langle X_s^i, \frac{\sigma^2}{2} \Delta \varphi^i \right\rangle \\ & + \int_{[0,t] \times \mathsf{R}^2} W^i\big(\mathrm{d}(s,x)\big)\ \varphi^i(x) \sqrt{\gamma\, X_s^1(x)\, X_s^2(x)}, \qquad t \geq 0, \end{aligned} \tag{2}$$

[compare with equation (1)]. Here the $\mu^i$ are sufficiently regular finite (initial) measures, the $\varphi^i$ are suitable test functions, $\Delta$ is the two–dimensional Laplacian, the $W^1\big(\mathrm{d}(s,x)\big), W^2\big(\mathrm{d}(s,x)\big)$ are independent standard time-space white noises on $\mathsf{R}_+ \times \mathsf{R}^2$, and $X_s^i(x)$ is the "generalized density" at $x$ of the measure $X_s^i(\mathrm{d}x)$.

More precisely, consider the following *martingale problem* $(\mathbf{MP})_\mu^{\sigma,\gamma}$ (for still more precise formulations, see Definition 3 below). For fixed constants $\sigma, \gamma > 0$, let $\mathbf{X} = (X^1, X^2)$ be a pair of continuous measure-valued processes such that

$$M_t^i(\varphi^i) \;:=\; \langle X_t^i, \varphi^i \rangle - \langle \mu^i, \varphi^i \rangle - \int_0^t \mathrm{d}s \, \Big\langle X_s^i, \frac{\sigma^2}{2} \Delta \varphi^i \Big\rangle, \tag{3}$$

$t \geq 0$, $i = 1, 2$, are orthogonal continuous square integrable (zero mean) martingales starting from $0$ at time $t = 0$ and with continuous square function

$$\langle\!\langle M^i(\varphi^i) \rangle\!\rangle_t \;=\; \gamma \int_{[0,t] \times \mathsf{R}^2} L_{\mathbf{X}}\big(\mathrm{d}(s,x)\big) \, (\varphi^i)^2(x). \tag{4}$$

Here $L_{\mathbf{X}}$ is the *collision local time* of $X^1$ and $X^2$, loosely described by

$$L_{\mathbf{X}}\big(\mathrm{d}(s,x)\big) \;=\; \mathrm{d}s \, X_s^1(\mathrm{d}x) \int_{\mathsf{R}^2} X_s^2(\mathrm{d}y) \, \delta_x(y) \tag{5}$$

(a precise description is given in Definition 1 below).

The main result of [DEF$^+$02] is that, provided the collision rate $\gamma$ is not too large compared with the migration rate $\sigma$, for initial states $\mu = (\mu^1, \mu^2)$ in the set $\mathcal{M}_{\mathrm{f,e}}$ of all pairs of finite measures on $\mathsf{R}^2$ satisfying the *energy condition*

$$\int_{\mathsf{R}^2} \mu^1(\mathrm{d}x^1) \int_{\mathsf{R}^2} \mu^2(\mathrm{d}x^2) \, \log^+ \frac{1}{|x^1 - x^2|} \;<\; \infty, \tag{6}$$

there is a (non-trivial) solution $\mathbf{X}$ to the martingale problem $(\mathbf{MP})_\mu^{\sigma,\gamma}$ with the property that $\mathbf{X}_t \in \mathcal{M}_{\mathrm{f,e}}$ for all $t > 0$ with probability 1.

## 1.2 Sketch of main results, and approach

The *main purpose* of this paper is to extend this existence result to certain *infinite* measures (see Theorem 4 below), where questions of long-term behavior can be properly studied. In contrast to the case of superprocesses, there does not seem to be a natural way to couple two versions of the process with different initial conditions and consequently we construct the process with infinite initial measures as a weak limit of a sequence of approximating processes. To this end, as in [DEF$^+$02], we start from the $\mathsf{Z}^2$–model ${}^1\mathbf{X}$ of [DP98], scale it to ${}^\varepsilon\mathbf{X}$ on $\varepsilon\mathsf{Z}^2$, and seek a limit as $\varepsilon \downarrow 0$. As in [DEF$^+$02], to prove tightness of the rescaled processes, we derive some uniform $4^{\text{th}}$ moment estimates in the case of a sufficiently small collision rate. But in contrast to [DEF$^+$02], we work with moment equations for ${}^\varepsilon\mathbf{X}$ instead of exploiting a moment dual process to ${}^\varepsilon\mathbf{X}$. We stress the fact that the construction of the infinite-measure-valued process is by no means a straightforward generalization of the finite-measure-valued case of [DEF$^+$02].

The proof of uniqueness of solutions to the martingale problem $(\mathbf{MP})_\mu^{\sigma,\gamma}$ is provided in the forthcoming paper [DFM$^+$02, Theorem 1.8] under an integrability condition (**IntC**) involving fourth moments. This integrability condition has been verified for a class of finite initial measures, and a simpler integrability condition (**SIntC**) (see Definition (45) below) that implies (**IntC**) has been verified for absolutely continuous measures with bounded densities. However it has not yet been verified for the class of infinite measures with sub-exponential growth at infinity which are treated in the present paper. Nevertheless we will be able to use the *self-duality* technique and convergence of the rescaled lattice models in the finite measures case to show that the lattice approximations for the case of infinite initial measures also converge weakly to a canonical solution of $(\mathbf{MP})_\mu^{\sigma,\gamma}$ (Theorem 6 below) and study this process.

We complement the existence result by showing that the process $\mathbf{X}$ which we construct has the following properties. (In the case of absolutely continuous measures, we often use the same symbol to denote both the measure and its density function.)

**(i)** For any fixed $t > 0$ and for each $i = 1, 2$, the state $X_t^i$ is *absolutely continuous,*

$$X_t^i(\mathrm{d}x) \;=\; X_t^i(x)\,\mathrm{d}x \quad \text{a.s.},$$

and for almost all $x \in \mathsf{R}^2$, the law of the vector $\mathbf{X}_t(x)$ of random densities at $x$ can explicitly be described in terms of the exit distribution of planar Brownian motion from the first quadrant. In particular the types are *separated:*

$$X_t^1(x)X_t^2(x) \;=\; 0 \quad \text{for Lebesgue almost all } x \in \mathsf{R}^2, \quad \text{a.s.},$$

and for both types the density blows up as one approaches the interface. See Theorem 11 below.

**(ii)** Starting $\mathbf{X}$ with multiples of Lebesgue measure $\ell$, that is $\mathbf{X}_0 = \mathbf{c}\ell = (c^1\ell, c^2\ell)$, $\mathbf{X}_t$ *converges in law* as $t \uparrow \infty$ to a limit $\mathbf{X}_\infty$ which can also explicitly be described:

$$\mathbf{X}_\infty \overset{\mathcal{L}}{=} \mathbf{X}_1(0)\,\ell \;=\; \bigl(X_1^1(0)\ell, X_1^2(0)\ell\bigr)$$

with $\mathbf{X}_1(0)$ the vector of random densities at time 1 at the origin 0 of $\mathsf{R}^2$ described in (i). In this case the law of $\mathbf{X}_1(0)$ is the exit distribution from the first quadrant of planar Brownian motion starting at $\mathbf{c}$. In particular, locally only one type survives in the limit (*non-coexistence of types*), and it is uniform as a result of the smearing out by the heat flow. See Theorem 13 below for the extension to more general initial states.

Clearly, the statements in (ii) are the continuum analogue of results in [DP98], and the interplay between $\mathbf{X}_\infty$ and $\mathbf{X}_1(0)$ is based on a self-similarity property of $\mathbf{X}$, starting with Lebesgue measures (see Proposition 16 (b) below).

We mention that the proofs of the aforementioned approximation theorem, of the separation of types, and of the long-term behavior require properties of the finite-measure-valued case which are based on uniqueness arguments provided in the forthcoming paper [DFM+02].

The problem of existence or non-existence of a mutually catalytic branching model in dimensions $d \geq 3$, as well as in $\mathsf{R}^2$ for large $\gamma$ remains unresolved.

# 2 Mutually catalytic branching X in $\mathsf{R}^2$ (results)

The purpose of this section is to rigorously introduce the infinite-measure-valued mutually catalytic branching process $\mathbf{X} = \bigl(X^1, X^2\bigr)$ in $\mathsf{R}^2$, and to state some of its properties.

## 2.1 Preliminaries: notation and some spaces

We use $c$ to denote a positive (finite) constant which may vary from place to place. A $c$ with some additional mark (as $\underline{c}$ or $c^1$) will, however, denote a specific constant. A constant of the form $c_{(\#)}$ or $c_{\#}$ means, this constant's first occurrence is related to formula line (#) or (for instance) to Lemma #, respectively.

Write $|\cdot|$ for the Euclidean norm in $\mathsf{R}^d$, $d \geq 1$. For $\mathbf{x} = \bigl(x^1, \ldots, x^n\bigr)$ in $(\mathsf{R}^d)^n$, $n \geq 1$, we set

$$\|\mathbf{x}\| \;:=\; |x^1| + \cdots + |x^n|. \tag{7}$$

For $\lambda \in \mathsf{R}$, introduce the *reference function* $\phi_\lambda = \phi_\lambda^d$ :

$$\phi_\lambda^d(x) \;:=\; \mathrm{e}^{-\lambda|x|}, \qquad x \in \mathsf{R}^d. \tag{8}$$

At some places we will need also a smoothed version $\tilde{\phi}_\lambda$ of $\phi_\lambda$. For this purpose, introduce the mollifier

$$\rho(x) \;:=\; c_{(9)}\, 1_{\{|x|<1\}} \exp\big[-1/(1-x^2)\big]\,, \qquad x \in \mathsf{R}, \tag{9}$$

with $c_{(9)}$ the normalizing constant such that $\int_{\mathsf{R}} \mathrm{d}x\, \rho(x) = 1$. For $\lambda \in \mathsf{R}$, set

$$\tilde{\phi}_\lambda^1(x) \;:=\; \int_{\mathsf{R}} \mathrm{d}y\; \phi_\lambda^1(y)\, \rho(y-x), \qquad x \in \mathsf{R}, \tag{10}$$

and introduce the *smoothed reference function*

$$\tilde{\phi}_\lambda(x) \;:=\; \tilde{\phi}_\lambda^1(x_1)\cdots\tilde{\phi}_\lambda^1(x_d), \qquad x = (x_1,\ldots,x_d) \in \mathsf{R}^d. \tag{11}$$

Note that to each $\lambda \in \mathsf{R}$ and $n \geq 0$ there are (positive) constants $\underline{c}_{\lambda,n}^1$ and $\overline{c}_{\lambda,n}^1$ such that

$$\underline{c}_{\lambda,n}^1\, \phi_\lambda(x) \;\leq\; \left|\frac{\mathrm{d}^n}{\mathrm{d}x^n}\tilde{\phi}_\lambda^1\,(x)\right| \;\leq\; \overline{c}_{\lambda,n}^1\, \phi_\lambda(x), \qquad x \in \mathsf{R}, \tag{12}$$

(cf. [Mit85, (2.1)]). Hence, for $\lambda \geq 0$ and $n \geq 0$,

$$\underline{c}_{\lambda,n}\, \phi_{\sqrt{d}\,\lambda}(x) \;\leq\; \left|\frac{\partial^n}{\partial x_i^n}\tilde{\phi}_\lambda\,(x)\right| \;\leq\; \overline{c}_{\lambda,n}\, \phi_\lambda(x), \tag{13}$$

$x = (x_1,\ldots,x_d) \in \mathsf{R}^d$, $1 \leq i \leq d$, for some constants $\underline{c}_{\lambda,n}$ and $\overline{c}_{\lambda,n}$. In particular, there exist constants $\underline{c}_\lambda$ and $\overline{c}_\lambda$ such that

$$\underline{c}_\lambda\, \phi_{\sqrt{d}\,\lambda}(x) \;\leq\; \left|\Delta\tilde{\phi}_\lambda\,(x)\right| \;\leq\; \overline{c}_\lambda\, \phi_\lambda(x), \qquad x \in \mathsf{R}^d. \tag{14}$$

For $f : \mathsf{R}^d \to \mathsf{R}$, put

$$|f|_\lambda \;:=\; \sup_{x\in\mathsf{R}^d} |f(x)|\,/\,\phi_\lambda(x), \qquad \lambda \in \mathsf{R}. \tag{15}$$

For $\lambda \in \mathsf{R}$, let $\mathcal{B}_\lambda = \mathcal{B}_\lambda(\mathsf{R}^d)$ denote the set of all measurable (real-valued) functions $f$ such that $|f|_\lambda$ is finite. Introduce the spaces

$$\mathcal{B}_{\mathrm{tem}} = \mathcal{B}_{\mathrm{tem}}(\mathsf{R}^d) := \bigcap_{\lambda>0} \mathcal{B}_{-\lambda}\,, \qquad \mathcal{B}_{\mathrm{exp}} = \mathcal{B}_{\mathrm{exp}}(\mathsf{R}^d) := \bigcup_{\lambda>0} \mathcal{B}_\lambda \tag{16}$$

of *tempered* and *exponentially decreasing* functions, respectively. (Roughly speaking, the functions in $\mathcal{B}_{\mathrm{tem}}$ are allowed to have a subexponential growth, whereas the ones in $\mathcal{B}_{\mathrm{exp}}$ have to decay at least exponentially.) Of course, $\mathcal{B}_{\mathrm{exp}} \subset \mathcal{B} = \mathcal{B}(\mathsf{R}^d)$, the set of all measurable functions on $\mathsf{R}^d$.

Let $\mathcal{C}_\lambda$ refer to the subsets of continuous functions $f$ in $\mathcal{B}_\lambda$ with the additional property that $f(x)/\phi_\lambda(x)$ has a finite limit as $|x| \uparrow \infty$. Define $\mathcal{C}_{\mathrm{tem}} = \mathcal{C}_{\mathrm{tem}}(\mathsf{R}^d)$ and $\mathcal{C}_{\mathrm{exp}} = \mathcal{C}_{\mathrm{exp}}(\mathsf{R}^d)$ analogously to (16), based on $\mathcal{C}_\lambda$. Write $\mathcal{C}_\lambda^{(m)} = \mathcal{C}_\lambda^{(m)}(\mathsf{R}^d)$ and $\mathcal{C}_{\mathrm{exp}}^{(m)} = \mathcal{C}_{\mathrm{exp}}^{(m)}(\mathsf{R}^d)$ if we additionally require that all partial derivatives up to the order $m \geq 1$ belong to $\mathcal{C}_\lambda$ and $\mathcal{C}_{\mathrm{exp}}$ , respectively.

For each $\lambda \geq 0$, the linear space $\mathcal{C}_\lambda$ equipped with the norm $|\cdot|_\lambda$ is a separable Banach space. The space $\mathcal{C}_{\mathrm{tem}}$ is topologized by the metric

$$\mathrm{d}_{\mathrm{tem}}(f,g) \;:=\; \sum_{n=1}^{\infty} 2^{-n}\left(|f-g|_{-1/n} \wedge 1\right), \qquad f,g \in \mathcal{C}_{\mathrm{tem}}\,, \tag{17}$$

making it a Polish space.

$\mathcal{C}_{\rm com} = \mathcal{C}_{\rm com}(\mathsf{R}^d)$ denotes the set of all $f$ in $\mathcal{C}_{\rm exp}$ with compact support, and we write $\mathcal{C}_{\rm com}^\infty = \mathcal{C}_{\rm com}^\infty(\mathsf{R}^d)$ if, in addition, they are infinitely differentiable.

If $E$ is a topological space, by 'measure on $E$' we mean a measure defined on the $\sigma$–field of all Borel subsets of $E$. If $\mu$ is a measure on a countable subset $E_0$ of a metric space $E$, then $\mu$ is also considered as a discrete measure on $E$. If $\mu$ is absolutely continuous with respect to some (fixed) measure $\nu$, then we often denote the density function (with respect to $\nu$) by the same symbol $\mu$, that is $\mu(\mathrm{d}x) = \mu(x)\,\nu(\mathrm{d}x)$, (and vice versa).

Let $\mathcal{M}_{\rm tem} = \mathcal{M}_{\rm tem}(\mathsf{R}^d)$ denote the set of all measures $\mu$ defined on $\mathsf{R}^d$ such that $\langle\mu,\phi_\lambda\rangle < \infty$, for all $\lambda > 0$. On the other hand, let $\mathcal{M}_{\rm exp} = \mathcal{M}_{\rm exp}(\mathsf{R}^d)$ be the space of all measures $\mu$ on $\mathsf{R}^d$ satisfying $\langle\mu,\phi_{-\lambda}\rangle < \infty$, for some $\lambda > 0$ (exponentially decreasing measures). Note that $\mathcal{M}_{\rm exp} \subset \mathcal{M}_{\rm f} = \mathcal{M}_{\rm f}(\mathsf{R}^d)$, the set of all finite measures on $\mathsf{R}^d$ equipped with the topology of weak convergence.

We topologize the set $\mathcal{M}_{\rm tem}$ of *tempered* measures by the metric

$$\mathrm{d}_{\rm tem}(\mu,\nu) \;:=\; \mathrm{d}_0(\mu,\nu) + \sum_{n=1}^\infty 2^{-n}\left(|\mu-\nu|_{1/n} \wedge 1\right), \qquad \mu,\nu \in \mathcal{M}_{\rm tem}\,. \tag{18}$$

Here $\mathrm{d}_0$ is a complete metric on the space of Radon measures on $\mathsf{R}^d$ inducing the vague topology, and $|\mu-\nu|_\lambda$ is an abbreviation for $\left|\langle\mu,\tilde\phi_\lambda\rangle - \langle\nu,\tilde\phi_\lambda\rangle\right|$. Note that $(\mathcal{M}_{\rm tem}\,,\mathrm{d}_{\rm tem})$ is a Polish space, and that $\mu_n \to \mu$ in $\mathcal{M}_{\rm tem}$ if and only if

$$\langle\mu_n\,,\varphi\rangle \;\underset{n\uparrow\infty}{\longrightarrow}\; \langle\mu,\varphi\rangle \quad \text{for all } \varphi\in\mathcal{C}_{\rm exp}\,. \tag{19}$$

For each $m \geq 1$, write $\mathbf{C} := \mathcal{C}\left(\mathsf{R}_+\,,\mathcal{M}_{\rm tem}^m\right)$ for the set of all continuous paths $t\mapsto\mu_t$ in $\mathcal{M}_{\rm tem}^m$, where $(\mathcal{M}_{\rm tem}^m, \mathrm{d}_{\rm tem}^m)$ is defined as the $m$–fold Cartesian product of $(\mathcal{M}_{\rm tem}, \mathrm{d}_{\rm tem})$. When equipped with the metric

$$\mathrm{d}_{\mathbf{C}}(\mu_\cdot\,,\mu'_\cdot) \;:=\; \sum_{n=1}^\infty 2^{-n}\Big(\sup_{0\leq t\leq n} \mathrm{d}_{\rm tem}^m\left(\mu_t\,,\mu'_t\right)\wedge 1\Big), \qquad \mu_\cdot\,,\mu'_\cdot\in\mathbf{C}, \tag{20}$$

$\mathbf{C}$ is a Polish space. Let $\mathbb{P}$ denote the set of all probability measures on $\mathbf{C}$. Endowed with the Prohorov metric $\mathrm{d}_{\mathbb{P}}$, $\mathbb{P}$ is a Polish space ([EK86, Theorem 3.1.7]).

Let $\mathrm{p}$ denote the heat kernel in $\mathsf{R}^d$ related to $\frac{\sigma^2}{2}\Delta$ :

$$\mathrm{p}_t(x) \;:=\; (2\pi\sigma^2 t)^{-d/2}\exp\Big[-\frac{|x|^2}{2\sigma^2 t}\Big], \qquad t>0, \quad x\in\mathsf{R}^d, \tag{21}$$

and $\{S_t : t\geq 0\}$ the corresponding heat flow semigroup. Write $\xi = (\xi,\Pi_x)$ for the related Brownian motion in $\mathsf{R}^d$, with $\Pi_x$ denoting the law of $\xi$ if $\xi_0 = x\in\mathsf{R}^d$.

Recall that $\ell$ refers to the (normalized) Lebesgue measure on $\mathsf{R}^d$. We use $\|\mu\|$ to denote the total mass of a measure $\mu$, whereas $|\mu|$ is the total variation measure of a signed measure $\mu$.

The upper or lower index $+$ on a set of real-valued functions will refer to the collection of all non-negative members of this set, similar to our notation $\mathsf{R}_+ = [0,\infty)$. The Kronecker symbol is denoted by $\delta_{k,\ell}$.

Random objects are always thought of as defined over a large enough stochastic basis $(\Omega,\mathcal{F},\mathcal{F}_\cdot,\mathcal{P})$ satisfying the usual hypotheses. If $Y = \{Y_t : t\geq 0\}$ is a random process starting at $Y_0 = y$, then as a rule the law of $Y$ is denoted by $P_y^Y$. If there is no ambiguity which process is meant, we also often simply write $P_y$ instead of $P_y^Y$. In particular, we usually use $P_y$ in situations in which $Y$ comes from the *unique* solution of a martingale problem. Also the corresponding $P$-letter (instead of $E$) is used in expectation expressions. We use $\mathcal{F}_t^Y$ to denote the completion of the $\sigma$–field $\bigcap_{\varepsilon>0}\sigma\left\{Y_s :\ s\leq t+\varepsilon\right\}$, $t\geq 0$.

As a rule, bold face letters refer to pairs as $\mathbf{X} = \left(X^1,X^2\right)$, $\mathbf{c}\ell = \left(c^1\ell,c^2\ell\right)$, etc.

Next we introduce a version of a definition from [DEF+02] which is used throughout this work.

**Definition 1 (Collision local time)** Let $\mathbf{X} = \left(X^1, X^2\right)$ be an $\mathcal{M}^2_{\text{tem}}$–valued continuous process. The *collision local time* of $\mathbf{X}$ (if it exists) is a continuous non-decreasing $\mathcal{M}_{\text{tem}}$–valued stochastic process $t \mapsto L_{\mathbf{X}}(t) = L_{\mathbf{X}}(t, \cdot\,)$ such that

$$\left\langle L^{*,\delta}_{\mathbf{X}}(t), \varphi\right\rangle \;\longrightarrow\; \left\langle L_{\mathbf{X}}(t), \varphi\right\rangle \quad \text{as} \quad \delta \downarrow 0 \quad \text{in probability,} \tag{22}$$

for all $t > 0$ and $\varphi \in \mathcal{C}_{\text{com}}(\mathsf{R}^d)$, where

$$L^{*,\delta}_{\mathbf{X}}(t, \mathrm{d}x) \;:=\; \frac{1}{\delta}\int_0^\delta \mathrm{d}r \int_0^t \mathrm{d}s\; X^1_s \ast \mathrm{p}_r(x)\, X^2_s \ast \mathrm{p}_r(x)\,\mathrm{d}x, \qquad t \geq 0, \quad \delta > 0.$$

The collision local time $L_{\mathbf{X}}$ will also be considered as a (locally finite) measure $L_{\mathbf{X}}\big(\mathrm{d}(s,x)\big)$ on $\mathsf{R}_+ \times \mathsf{R}^d$. $\diamond$

We now consider the scaled lattice $\varepsilon\mathsf{Z}^d$, for fixed $0 < \varepsilon \leq 1$. In much the same way as in the $\mathsf{R}^d$–case, we use the reference functions $\phi_\lambda$, $\lambda \in \mathsf{R}$, now restricted to $\varepsilon\mathsf{Z}^d$, to introduce $|f|_\lambda$, ${}^\varepsilon\mathcal{B}_\lambda = {}^\varepsilon\mathcal{B}_\lambda(\varepsilon\mathsf{Z}^d)$, ${}^\varepsilon\mathcal{B}_{\exp} = {}^\varepsilon\mathcal{B}_{\exp}(\varepsilon\mathsf{Z}^d)$, and ${}^\varepsilon\mathcal{B}_{\text{tem}} = {}^\varepsilon\mathcal{B}_{\text{tem}}(\varepsilon\mathsf{Z}^d)$. Let ${}^\varepsilon\Delta$ denote the discrete Laplacian:

$${}^\varepsilon\Delta f\,(x) \;:=\; \varepsilon^{-2} \sum_{\{y \in \varepsilon\mathsf{Z}^d:\ |y-x| = \varepsilon\}} \big[f(y) - f(x)\big], \qquad x \in \varepsilon\mathsf{Z}^d, \tag{23}$$

(acting on functions $f$ on $\varepsilon\mathsf{Z}^d$). Note that ${}^\varepsilon\Delta\phi_\lambda$ belongs to ${}^\varepsilon\mathcal{B}_\lambda$, for each positive $\lambda$. The spaces $({}^\varepsilon\mathcal{M}_{\text{tem}}\,,\, {}^\varepsilon\mathrm{d}_{\text{tem}})$ and $\mathcal{C}(\mathsf{R}_+\,,\, {}^\varepsilon\mathcal{M}^m_{\text{tem}})$ are also defined analogously to the $\mathsf{R}^d$–case.
Write

$${}^\varepsilon\ell \;:=\; \varepsilon^d \sum_{x \in \varepsilon\mathsf{Z}^d} \delta_x \tag{24}$$

for the Haar measure on $\varepsilon\mathsf{Z}^d$ (approximating the Lebesgue measure $\ell$ in $\mathcal{M}_{\text{tem}}(\mathsf{R}^d)$ as $\varepsilon \downarrow 0$). Let ${}^\varepsilon\mathrm{p}$ denote the transition density (with respect to ${}^\varepsilon\ell$) of the *simple* symmetric random walk $({}^\varepsilon\xi, {}^\varepsilon\Pi_a)$ on $\varepsilon\mathsf{Z}^d$ which jumps to a randomly chosen neighbor with rate $d\sigma^2/\varepsilon^2$, that is has generator $\frac{\sigma^2}{2}\,{}^\varepsilon\Delta$, with the related semigroup denoted by $\{{}^\varepsilon S_t : t \geq 0\}$. In other words, ${}^\varepsilon\mathrm{p}_t(x) := \varepsilon^{-d}\,{}^\varepsilon\Pi_0({}^\varepsilon\xi_t = x)$ and so

$${}^\varepsilon\mathrm{p}_t(x) \;=\; \varepsilon^{-d}\;{}^1\mathrm{p}_{\varepsilon^{-2}t}(\varepsilon^{-1}x), \qquad t \geq 0, \quad x \in \varepsilon\mathsf{Z}^d. \tag{25}$$

In the case $d = 2$ we will need some random walk kernel estimates that for convenience we now state as a lemma. For a proof, see, for instance, [DEF+02, Lemma 8].

**Lemma 2 (Random walk kernel estimates)**

**(a) (Local central limit theorem)** *For all* $t > 0$, *with the heat kernel* $\mathrm{p}$ *from* (21),

$$\lim_{\varepsilon \downarrow 0}\ \sup_{x \in \varepsilon\mathsf{Z}^2} \left|{}^\varepsilon\mathrm{p}_t(x) - \mathrm{p}_t(x)\right| \;=\; 0.$$

**(b) (Uniform bound)** *There exists an absolute constant* $c_{\text{rw}}$ *such that*

$$\sup_{t > 0,\ x \in \varepsilon\mathsf{Z}^2} \sigma^2\, t\; {}^\varepsilon\mathrm{p}_t(x) \;=\; c_{\text{rw}}, \qquad 0 < \varepsilon \leq 1, \quad \sigma > 0.$$

In fact $c_{\text{rw}} \in (.15, .55)$ (See Remark 9 in [DEF+02, Lemma 8].)
Often we will need the constant

$$c_2 \;:=\; c_2(\sigma) \;:=\; c_{\text{rw}}/\sigma^2 \tag{26}$$

instead of $c_{\text{rw}}$.

## 2.2 Existence of X on $\mathsf{R}^2$

First we want to introduce in detail the martingale problem $(\mathbf{MP})_\mu^{\sigma,\gamma}$ mentioned already in Subsection 1.1 (extended versions of the martingale problem will be formulated in Lemma 42 and Corollary 43 below). Let $d = 2$.

**Definition 3 (Martingale Problem $(\mathbf{MP})_\mu^{\sigma,\gamma}$)** Fix constants $\sigma,\gamma > 0$, and $\mu = (\mu^1,\mu^2) \in \mathcal{M}_{\mathrm{tem}}^2(\mathsf{R}^2)$. A continuous $\mathcal{F}_\cdot$–adapted and $\mathcal{M}_{\mathrm{tem}}^2(\mathsf{R}^2)$–valued process $\mathbf{X} = (X^1,X^2)$ [on a stochastic basis $(\Omega,\mathcal{F},\mathcal{F}_\cdot,\mathcal{P})$] is said to satisfy the martingale problem $(\mathbf{MP})_\mu^{\sigma,\gamma}$, if for all $\varphi^1,\varphi^2 \in C_{\exp}^{(2)}(\mathsf{R}^2)$,

$$M_t^i(\varphi^i) \;=\; \langle X_t^i,\varphi^i\rangle - \langle \mu^i,\varphi^i\rangle - \int_0^t \mathrm{d}s\,\Big\langle X_s^i,\frac{\sigma^2}{2}\Delta\varphi^i\Big\rangle, \quad t \ge 0,\;\; i = 1,2, \tag{27}$$

are orthogonal continuous (zero mean) square integrable $\mathcal{F}_\cdot^{\mathbf{X}}$–martingales such that $M_0^i(\varphi^i) = 0$ and

$$\langle\!\langle M^i(\varphi^i)\rangle\!\rangle_t \;=\; \gamma\,\big\langle L_{\mathbf{X}}(t),(\varphi^i)^2\big\rangle, \qquad t \ge 0, \quad i = 1,2, \tag{28}$$

(with $L_{\mathbf{X}}$ the collision local time of $\mathbf{X}$). ◇

The existence of the *infinite-measure-valued mutually catalytic branching process* $\mathbf{X} = \big(X^1,X^2\big)$ in $\mathsf{R}^2$ is established in the following theorem.

**Theorem 4 (Mutually catalytic branching in $\mathsf{R}^2$)** *Fix constants $\sigma,\gamma > 0$, and assume that*

$$\frac{\gamma}{\sigma^2} \;<\; \frac{1}{64\sqrt{6}\,\pi\,c_{\mathrm{rw}}}\,. \tag{29}$$

*Let $\mu = (\mu^1,\mu^2)$ be a pair of absolutely continuous measures on $\mathsf{R}^2$ with density functions in $\mathcal{B}_{\mathrm{tem}}(\mathsf{R}^2)$ (abbreviated to $\mu \in \mathcal{B}_{\mathrm{tem}}^2$).*

- **(a) (Existence)** *There exists a solution $\mathbf{X} = (X^1,X^2)$ to the martingale problem $(\mathbf{MP})_\mu^{\sigma,\gamma}$.*
- **(b) (Some moment formulae)** *For the process constructed in Theorem 6 below, $\mathbf{X} = (X^1,X^2)$, solving the martingale problem $(\mathbf{MP})_\mu^{\sigma,\gamma}$, the following moment formulae hold. The mean measures are given by*

$$P_\mu^{\mathbf{X}} X_t^i(\mathrm{d}x) \;=\; \mu^i * \mathrm{p}_t\,(x)\,\mathrm{d}x \;\in\; \mathcal{M}_{\mathrm{tem}}\,, \qquad i = 1,2, \quad t \ge 0,$$

*and $\mathbf{X}$ has covariance measures*

$$\begin{aligned}\mathrm{Cov}_\mu^{\mathbf{X}}\big(X_{t_1}^{i_1}\,,\,X_{t_2}^{i_2}\big)(\mathrm{d}\mathbf{z}) \;&=\; \mathrm{d}\mathbf{z}\;\delta_{i_1,i_2}\,\gamma\int_0^{t_1\wedge t_2}\mathrm{d}s\int_{\mathsf{R}^2}\mathrm{d}x\;\mu^1 * \mathrm{p}_s\,(x)\,\mu^2 * \mathrm{p}_s\,(x)\\ &\qquad\times\,\mathrm{p}_{t_1-s}(z^1-x)\,\mathrm{p}_{t_2-s}(z^2-x) \in \mathcal{M}_{\mathrm{tem}}^2\,,\end{aligned}$$

$t_1,t_2 > 0$, $i_1,i_2 \in \{1,2\}$, $\mathbf{z} = (z^1,z^2) \in (\mathsf{R}^2)^2$. *Moreover, for the expected collision local times we have*

$$P_\mu^{\mathbf{X}} L_{\mathbf{X}}(t)\,(\mathrm{d}x) \;=\; \mathrm{d}x\int_0^t \mathrm{d}s\;\mu^1{*}\mathrm{p}_s(x)\,\mu^2{*}\mathrm{p}_s(x) \;\in\; \mathcal{M}_{\mathrm{tem}}\,, \qquad t \ge 0.$$

Recall that

$$\mathrm{Cov}_\mu^{\mathbf{X}}\big(\big\langle X_{t_1}^{i_1},\varphi_1\big\rangle,\big\langle X_{t_1}^{i_1},\varphi_2\big\rangle\big) = \int_{\mathsf{R}^2\times\mathsf{R}^2}\mathrm{Cov}_\mu^{\mathbf{X}}\big(X_{t_1}^{i_1}\,,\,X_{t_2}^{i_2}\big)(\mathrm{d}\mathbf{z})\varphi_1(z_1)\varphi_2(z_2),$$

$\varphi_1,\varphi_2 \in C_{\exp}^{+}(\mathsf{R}^2)$.

**Remark 5 (Non-deterministic limit)** The covariance formula in (b) shows that (for non-zero initial measures) the constructed process $\mathbf{X}$ is non-deterministic. Moreover, the variance densities explode along the diagonal, as can easily be checked in specific cases. For instance, if $\mu = \mathbf{c}\ell = \bigl(c^1\ell, c^2\ell\bigr)$ with $c^1, c^2 > 0$, the variance densities

$$\mathit{Var}^{\mathbf{X}}_{\mathbf{c}\ell} X^i_t(\mathbf{z}) \;=\; c^1\, c^2\, \gamma \int_0^t \mathrm{d}s\; \mathrm{p}_{2s}(z^1 - z^2), \qquad i = 1,2, \tag{30}$$

are *infinite* along the diagonal $\bigl\{z^1 = z^2\bigr\}$. $\diamond$

The existence claim in Theorem 4 (a) will be verified via a convergence theorem for $\varepsilon\mathsf{Z}^2$–approximations. Fix again $0 < \varepsilon \le 1$. Let ${}^{\varepsilon}\mu = ({}^{\varepsilon}\mu^1, {}^{\varepsilon}\mu^2) \in {}^{\varepsilon}\mathcal{M}^2_{\text{tem}}$ and let $({}^{\varepsilon}\mathbf{X}, P_{{}^{\varepsilon}\mu})$ denote the *mutually catalytic branching process* on $\varepsilon\mathsf{Z}^2$ based on the symmetric nearest neighbor random walk. This process was introduced in [DP98, Theorems 2.2 (a), (b)(iv) and 2.4 (a)] in the special case $\varepsilon = 1$, where it was constructed as the unique solution of the stochastic equation

$$\frac{\partial}{\partial t}\, {}^{1}\!X^i_t(x) \;=\; \frac{\sigma^2}{2}\; {}^{1}\!\Delta\, {}^{1}\!X^i_t\,(x) \;+\; \sqrt{\gamma\, {}^{1}\!X^1_t(x)\, {}^{1}\!X^2_t(x)}\; \dot{W}^i_t(x), \tag{31}$$

$(t,x) \in \mathsf{R}_+ \times \mathsf{Z}^2$, $i = 1,2$, where $\bigl\{W^i(x) : x \in \mathsf{Z}^2, i = 1,2\bigr\}$ is a family of independent standard Brownian motions in $\mathsf{R}$. Of course, (31) is the $\mathsf{Z}^2$–counterpart of the stochastic equation (1). We consider the process ${}^1X$ with the $\varepsilon$-dependent initial state ${}^1X_0(x) := {}^{\varepsilon}\!\mu(\varepsilon x)$, $x \in \mathsf{Z}^2$, and define ${}^{\varepsilon}X$ by rescaling

$${}^{\varepsilon}\!X^i_t(x) \;:=\; {}^{1}\!X^i_{\varepsilon^{-2}t}(\varepsilon^{-1}x), \qquad (t,x) \in \mathsf{R}_+ \times \varepsilon\mathsf{Z}^2, \quad i = 1,2. \tag{32}$$

We can interpret $\bigl\{{}^{\varepsilon}\!X^i_t(x) : x \in \varepsilon\mathsf{Z}^2\bigr\}$ as a density function with respect to ${}^{\varepsilon}\ell$ [defined in (24)] of the measure

$${}^{\varepsilon}\!X^i_t(B) \;:=\; \int_B {}^{\varepsilon}\ell(\mathrm{d}x)\, {}^{\varepsilon}\!X^i_t(x), \qquad B \subseteq \varepsilon\mathsf{Z}^2. \tag{33}$$

On the other hand, one can also define this process ${}^{\varepsilon}\mathbf{X}$ directly as the unique (in law) ${}^{\varepsilon}\mathcal{M}^2_{\text{tem}}$–valued continuous solution of the following system of equations:

$$\begin{aligned} \bigl\langle {}^{\varepsilon}\!X^i_t, \varphi^i \bigr\rangle \;=\; & \bigl\langle {}^{\varepsilon}\mu^i, \varphi^i \bigr\rangle + \int_0^t \mathrm{d}s\, \Bigl\langle {}^{\varepsilon}\!X^i_s,\, \frac{\sigma^2}{2}\, {}^{\varepsilon}\Delta\varphi^i \Bigr\rangle \\ & + \int_{\varepsilon\mathsf{Z}^2} {}^{\varepsilon}\ell(\mathrm{d}x) \int_0^t \mathrm{d}W^i_s(x)\, \varphi^i\,(x)\, \sqrt{\gamma\, {}^{\varepsilon}\!X^1_s(x)\, {}^{\varepsilon}\!X^2_s(x)}, \end{aligned} \tag{34}$$

$t \ge 0$, $i = 1,2$. Here $\bigl\{W^i(x) : x \in \varepsilon\mathsf{Z}^2,\ i = 1,2\bigr\}$ is again a family of independent standard Brownian motions in $\mathsf{R}$, the $\varphi^i$ are test functions in ${}^{\varepsilon}\mathcal{B}_{\exp}$ and the discrete Laplacian ${}^{\varepsilon}\Delta$ was defined in (23). Note that ${}^{\varepsilon}\mathbf{X} = \bigl({}^{\varepsilon}\!X^1, {}^{\varepsilon}\!X^2\bigr)$ satisfies the following *martingale problem* $(\mathbf{MP})^{\sigma,\gamma,\varepsilon}_{\mu}$ :

$$\left\{\begin{array}{l} {}^{\varepsilon}M^i_t(\varphi^i) \;:=\; \bigl\langle {}^{\varepsilon}\!X^i_t, \varphi^i \bigr\rangle - \bigl\langle {}^{\varepsilon}\mu^i, \varphi^i \bigr\rangle - \displaystyle\int_0^t \mathrm{d}s\, \Bigl\langle {}^{\varepsilon}\!X^i_s,\, \frac{\sigma^2}{2}\, {}^{\varepsilon}\Delta\varphi^i \Bigr\rangle, \qquad t \ge 0, \\[2ex] \varphi^i \in {}^{\varepsilon}\mathcal{B}_{\exp}\,,\ {}^{\varepsilon}\mu^i \in {}^{\varepsilon}\mathcal{M}_{\text{tem}}\,,\ i = 1,2, \text{ are continuous square integrable} \\[1ex] \text{(zero-mean) } \mathcal{F}^{{}^{\varepsilon}\mathbf{X}}_{\cdot}\text{–martingales with continuous square function} \\[1ex] \bigl\langle\!\bigl\langle {}^{\varepsilon}M^i(\varphi^i), {}^{\varepsilon}M^j(\varphi^j) \bigr\rangle\!\bigr\rangle_t \;=\; \delta_{i,j}\, \gamma\, \bigl\langle {}^{\varepsilon}L_{{}^{\varepsilon}\mathbf{X}}(t), \varphi^i\varphi^j \bigr\rangle, \quad \text{where} \\[1ex] \bigl\langle {}^{\varepsilon}L_{{}^{\varepsilon}\mathbf{X}}(t), \varphi \bigr\rangle := \displaystyle\int_0^t \mathrm{d}s \int_{\varepsilon\mathsf{Z}^2} {}^{\varepsilon}\ell(\mathrm{d}y)\, {}^{\varepsilon}\!X^1_s(y)\, {}^{\varepsilon}\!X^2_s(y)\, \varphi(y), \quad t \ge 0,\ \varphi \in {}^{\varepsilon}\mathcal{B}_{\exp}\,. \end{array}\right. \tag{35}$$

The continuous ${}^{\varepsilon}\mathcal{M}_{\text{tem}}$–valued random process ${}^{\varepsilon}L_{{}^{\varepsilon}\mathbf{X}}$ is the collision local time of ${}^{\varepsilon}\mathbf{X}$, in analogy to Definition 1.

The scaled process ${}^{\varepsilon}\mathbf{X} = ({}^{\varepsilon}\!X^1, {}^{\varepsilon}\!X^2)$ can be started with any pair ${}^{\varepsilon}\mathbf{X}_0 = {}^{\varepsilon}\mu$ of initial densities (with respect to ${}^{\varepsilon}\ell$) such that

$$\begin{cases} \text{for each } \lambda > 0 \text{ there is a constant } c_\lambda \text{ such that} \\ {}^{\varepsilon}\!\mu^i(x) \ \le\ c_\lambda \, \mathrm{e}^{\lambda|x|}, \qquad x \in \varepsilon\mathsf{Z}^2, \quad i = 1, 2. \end{cases} \tag{36}$$

It is also convenient for us to think of ${}^{\varepsilon}\mathbf{X}$ as continuous $\mathcal{M}^2_{\text{tem}}(\mathsf{R}^2)$–valued processes (recall our convention concerning measures on countable subsets). Now the existence Theorem 4 (a) will follow from the following convergence theorem.

**Theorem 6 (Lattice approximation)** *Let $\gamma, \sigma,$ and $\mu$ satisfy the conditions of Theorem 4. For each $\varepsilon \in (0,1]$, choose a pair ${}^{\varepsilon}\mathbf{X}_0 = {}^{\varepsilon}\mu = ({}^{\varepsilon}\!\mu^1, {}^{\varepsilon}\!\mu^2)$ of measures on $\varepsilon\mathsf{Z}^2$ with densities (with respect to ${}^{\varepsilon}\ell$) satisfying the domination condition (36) with the constants $c_\lambda$ independent of $\varepsilon$ and such that ${}^{\varepsilon}\mu \to \mu$ in $\mathcal{M}^2_{\text{tem}}(\mathsf{R}^2)$. Then the limit in law*

$$\lim_{\varepsilon \downarrow 0} {}^{\varepsilon}\mathbf{X} \ =:\ \mathbf{X} \ \textit{exists in}\ \mathcal{C}\big(\mathsf{R}_+, \mathcal{M}^2_{\text{tem}}(\mathsf{R}^2)\big), \tag{37}$$

*satisfies the martingale problem $(\mathbf{MP})^{\sigma,\gamma}_{\mu}$, and the law of $\mathbf{X}$ does not depend on the choice of the approximating family $\{{}^{\varepsilon}\mu : 0 < \varepsilon \le 1\}$ of $\mu$.*

For instance, the hypotheses on ${}^{\varepsilon}\mu$ will be satisfied if

$${}^{\varepsilon}\!\mu^i(x) \ :=\ \varepsilon^{-2}\mu^i\left(x + [0,\varepsilon)^2\right), \qquad x \in \varepsilon\mathsf{Z}^2, \quad i = 1, 2. \tag{38}$$

From now on, by *the mutually catalytic branching process* $\mathbf{X}$ on $\mathsf{R}^2$ with initial density $\mathbf{X}_0 = \mu \in \mathcal{B}^2_{\text{tem}}$ we mean the unique (in law) limiting process $\mathbf{X}$ from the previous theorem.

**Remark 7 (Uniqueness in $(\mathbf{MP})^{\sigma,\gamma}_{\mu}$ via self-duality)** Uniqueness of solutions to the martingale problem $(\mathbf{MP})^{\sigma,\gamma}_{\mu}$ under an additional integrability condition will be shown in [DFM+02]. This will be done via self-duality (see also Proposition 15 below) with the finite-measure-valued mutually catalytic branching process in $\mathsf{R}^2$ of *[DEF+ 02]*. However the integrability condition required for uniqueness will be established in [DFM+02, Theorem 1.11] for the solutions constructed in Theorem 6 only under the additional condition that the initial densities are uniformly bounded. ◇

**Remark 8 (Phase transition for higher moments)** In order to establish tightness of processes in Theorem 6, we will need to establish uniform bounds on the fourth moments of the increments of these processes (see Lemma 34 below). For $\gamma/\sigma^2$ large enough, it is not hard to see that these fourth moments (in fact even third moments) will explode as $\varepsilon$ approaches zero. The bound (29) is sufficient to ensure finiteness of these fourth moments for the limiting model; a somewhat more generous bound appeared in [DEF+02]. We believe Theorems 4 and 6 should be valid for all positive values of $\gamma$ and $\sigma$ as the existence of $2+\varepsilon$ moments should suffice for our tightness arguments, and for any given $\gamma$ and $\sigma$ these should be finite for sufficiently small $\varepsilon$. For this reason we have not tried very hard to find the critical value of $\gamma/\sigma^2$ for finiteness of fourth moments (but see the next remark). ◇

**Remark 9 (Bounded initial densities)** (i) If the initial densities are bounded, then Theorems 4 and 6 remain valid if

$$\gamma/\sigma^2 < \frac{1}{\sqrt{6}c_{\text{rw}}\pi}. \tag{39}$$

The proofs go through with minor changes, using Corollary 27 in place of Lemma 26.
(ii) An alternative construction of the process in Theorem 4(a) is also possible if the initial densities are bounded. This is briefly described in Remark 12(ii) of [DEF+02]. Here the process exists and the limiting 4th moments are finite if $\gamma/\sigma^2 < \sqrt{\frac{2}{3}} \sim 0.8$. These improved moment bounds are obtained using a modified version of the dual process introduced in [DEF+02]. Basically one then may replace $c_{\text{rw}}$ with its "limiting" value, namely $\frac{1}{2\pi}$ and this substitution in (39) gives the bound stated above. ◇

## 2.3 Properties of the states

To prepare for the next results, we need the following definition.

**Definition 10 (Brownian exit time $\tau$ from $(0,\infty)^2$)** For $\mathbf{a} \in \mathsf{R}_+^2$, let $\tau = \tau(\mathbf{a})$ denote the first time, Brownian motion $\xi$ in $\mathsf{R}^2$ starting from $\mathbf{a}$ hits the boundary of $\mathsf{R}_+^2$. $\diamond$

Here we state some properties of $\mathbf{X}$. Recall that we identify absolutely continuous measures with their density functions.

**Theorem 11 (State properties)** *Let $\mu = (\mu^1, \mu^2)$ denote a pair of absolutely continuous measures on $\mathsf{R}^2$ with density functions in $\mathcal{B}_{\text{tem}}^+(\mathsf{R}^2)$. Then the following statements hold. Fix any $t > 0$.*

**(a) (Absolutely continuous states)** *If $\mathbf{X}$ is any solution of the martingale problem $(\mathbf{MP})_\mu^{\sigma,\gamma}$, then, for $i = 1,2$, with probability one, $X_t^i$, is absolutely continuous:*

$$X_t^i(\mathrm{d}x) \;=\; X_t^i(x)\,\mathrm{d}x.$$

*Now let $\mathbf{X}$ be the mutually catalytic branching process from Theorem* 6.

**(b) (Law of the densities)** *For $\ell$–almost all $x \in \mathsf{R}^2$, the law of $\mathbf{X}_t(x)$ coincides with the law of the exit state $\xi_{\tau(\mathbf{a})}$ of planar Brownian motion starting from the point $\mathbf{a} := \big(\mu^1 * \mathrm{p}_t\,(x), \mu^2 * \mathrm{p}_t\,(x)\big)$. In particular,*

$$\mathrm{Var}_\mu^{\mathbf{X}}\, X_t^i(x) \;\equiv\; \infty, \qquad i = 1,2,$$

*provided that $\mu^j \neq 0$, $j = 1,2$.*

**(c) (Segregation of types)** *For $\ell$–almost all $x \in \mathsf{R}^2$,*

$$X_t^1(x)\, X_t^2(x) \;=\; 0, \ \ a.s.$$

**(d) (Blow-up at the interface)** *Define a canonical and jointly measurable density field $\underline{\mathbf{X}} = \big(\underline{X}^1, \underline{X}^2\big)$ of $\mathbf{X}$ on $\Omega \times \mathsf{R}_+ \times \mathsf{R}^2$ by*

$$\underline{X}_s^i(x) \;:=\; \begin{cases} \lim\limits_{n\uparrow\infty} X_s^i * \mathrm{p}_{2^{-n}}\,(x) & \text{if the limit exists,} \\ 0 & \text{otherwise,} \end{cases}$$

$s > 0$, $x \in \mathsf{R}^2$, $i = 1,2$. *Note that by* (a) *for all $t > 0$,*

$$\underline{\mathbf{X}}_t(x) \;=\; \mathbf{X}_t(x) \quad \text{for } \ell\text{–almost all } x, \ a.s.$$

*If $U$ is an open subset of $\mathsf{R}_+ \times \mathsf{R}^2$, write*

$$\|\,\underline{X}^i\|_U \;:=\; \operatorname*{ess\,sup}_{(s,x)\in U} \underline{X}_s^i(x), \quad i = 1,2,$$

*where the essential supremum is taken with respect to Lebesgue measure. Then*

$$L_{\mathbf{X}}(U) > 0 \quad \text{implies} \quad \|\,\underline{X}^1\|_U \;=\; \|\,\underline{X}^2\|_U \;=\; \infty, \quad a.s.$$

Consequently, at each *fixed* time point $t > 0$, our constructed mutually catalytic branching process $\mathbf{X}$ has absolutely continuous states with density functions which are *segregated:* at almost all space points there is only one type present (despite the spread by the heat flow), although the randomness of the process stems from the local branching interaction between types. On the other hand, if a density field $\underline{\mathbf{X}}$ is defined simultaneously for all times as in (d) (although the theorem leaves open whether non-absolutely continuous states might exist at some random times), then this field $\underline{\mathbf{X}}$ (related to the absolutely continuous parts of the measure states) *blows up* as one approaches the interface of the two types described by the support of the collision local time $L_{\mathbf{X}}$. This local unboundedness is reflected in simulations by "*hot spots*" at the interface of types.

At first sight, the separation of types looks paradoxical. But since the densities blow up as one approaches the interface of the two types, despite disjointness there might be a contribution to the collision local time which is defined only via a spatial smoothing procedure. In particular, the usual intuitive way of writing the collision local time as $L_{\mathbf{X}}\big(\mathrm{d}(s,x)\big) = \mathrm{d}s\, X_s^1(x) X_s^2(x)\,\mathrm{d}x$ gives the wrong picture in this case of locally unbounded densities.

**Remark 12 (State space for X)** Our construction of $\mathbf{X}$ (Theorem 6) was restricted to absolutely continuous initial states with *tempered densities.* The latter requirement is unnatural for this process because this regularity is not preserved by the dynamics of the process, which typically produces *locally unbounded densities* [recall Theorem 11 (d)].

It would be desirable to find a state space described by some energy condition in the spirit of (6). Our use of tempered initial densities is also an obstacle to establishing the Markov property for $\mathbf{X}$. Both problems are solved in the finite-measure case, see [DEF$^+$02, Theorem 11(b)] and [DFM$^+$02, Theorem 1.9(c)]. $\diamond$

## 2.4 Long-term behavior

Recall Definition 10 of the Brownian exit state $\xi_{\tau(\mathbf{a})}$. The long-term behavior of $\mathbf{X}$ is quite similar to that in the recurrent $\mathsf{Z}^d$ case (see [DP98]).

**Theorem 13 (Impossible longterm coexistence of types)** *Assume additionally that the initial state $\mathbf{X}_0 = \mu$ of our mutually catalytic branching process has bounded densities satisfying, for some $\mathbf{c} = (c^1, c^2) \in \mathsf{R}_+^2$,*

$$\mu^i * \mathrm{p}_t\,(x) \underset{t\uparrow\infty}{\longrightarrow} c^i, \qquad x \in \mathsf{R}^2, \quad i = 1,2. \tag{40}$$

*Then the following convergence in law holds:*

$$\mathbf{X}_t \underset{t\uparrow\infty}{\Longrightarrow} \xi_{\tau(\mathbf{c})}\boldsymbol{\ell}. \tag{41}$$

Consequently, if the initial densities are bounded and have an overall density in the sense of (40) [as trivially fulfilled in the case $X_0^i \equiv c^i\,\ell$], a long-term limit exists with full expectation (persistence), and the limit population is described in law by the state $\xi_{\tau(\mathbf{c})}$ of planar Brownian motion, starting from $\mathbf{c}$, at the time $\tau(\mathbf{c})$ of its exit from $(0,\infty)^2$. In particular, only one type survives locally in the limiting population (impossible coexistence of types) and it is uniform in space.

Of course, this does not necessarily mean that one type almost surely dies out as $t \uparrow \infty$. In fact, the method of [CK00] should show that as $t \uparrow \infty$, the predominant type in any compact set changes infinitely often, as they proved is the case for the lattice model. However, this would require the Markov property for our $\mathbf{X}$, and so we will not consider this question here.

**Remark 14 (Random initial states)** In Theorem 13 one may allow random initial states which satisfy

$$\sup_x P^{\mathbf{X}}(X_0^i(x)^2) < \infty$$

and

$$\lim_{t\to\infty} P^{\mathbf{X}}((X_0^i * p_t(x) - c^i)^2) = 0 \ \text{ for all } x,\, i = 1,2.$$

Note first that the law of $\mathbf{X}$ is a measurable function of the initial state by the self-duality in Proposition 15(b) below and so the process with random initial densities may be defined in the obvious manner. The derivation of (41) now proceeds with only minor changes in the proof below (see [CKP00] for the proof in the lattice case). $\diamond$

### 2.5 Self-duality, scaling, and self-similarity

Recall that we identify a non-negative $\varphi \in \mathcal{C}_{\exp}$ with the corresponding measure $\varphi(x)\,\mathrm{d}x$, also denoted by $\varphi$.

One of the crucial tools for investigating the mutually catalytic branching process is self-duality:

**Proposition 15 (Self-duality)** *Consider the mutually catalytic branching processes* $\mathbf{X} = (X^1, X^2)$ *and* $\widetilde{\mathbf{X}} = (\widetilde{X}^1, \widetilde{X}^2)$ *with initial densities* $\mathbf{X}_0 = \mu \in \mathcal{B}^2_{\text{tem}}(\mathsf{R}^2)$ *and* $\widetilde{\mathbf{X}}_0 = \varphi \in \mathcal{C}^2_{\exp}(\mathsf{R}^2)$, *respectively. Then the following two statements hold for each fixed* $t \geq 0$ :

**(a) (States in $\mathcal{M}^2_{\exp}$)** *With probability one,* $\widetilde{\mathbf{X}}_t \in \mathcal{M}^2_{\exp}(\mathsf{R}^2)$.

**(b) (Self-duality relation)** *The processes* $\mathbf{X}$ *and* $\widetilde{\mathbf{X}}$ *satisfy the self-duality relation*

$$\begin{aligned} & P^{\mathbf{X}}_{\mu} \exp\Big[-\big\langle X^1_t + X^2_t, \varphi^1 + \varphi^2\big\rangle + i\,\big\langle X^1_t - X^2_t, \varphi^1 - \varphi^2\big\rangle\Big] \\ &= P^{\widetilde{\mathbf{X}}}_{\varphi} \exp\Big[-\big\langle \mu^1 + \mu^2, \widetilde{X}^1_t + \widetilde{X}^2_t\big\rangle + i\,\big\langle \mu^1 - \mu^2, \widetilde{X}^1_t - \widetilde{X}^2_t\big\rangle\Big], \qquad t \geq 0, \end{aligned}$$

(*with* $i = \sqrt{-1}$).

Self-duality, for instance, makes it possible to derive the convergence Theorem 13, in the case of uniform initial states in a simple way from the total mass convergence of the finite-measure-valued mutually catalytic branching process of [DEF+02] (see Subsection 5.3 below).

Our class of mutually catalytic branching processes $\mathbf{X}$ on $\mathsf{R}^2$ is *invariant* under mass-time-space scaling, and spatial shift:

**Proposition 16** *Let* $\theta, \varepsilon > 0$ *and* $z \in \mathsf{R}^2$ *be fixed. Let* $\mathbf{X}$ *and* $\mathbf{X}^{(\varepsilon)}$ *denote the mutually catalytic branching processes with initial measures* $\mathbf{X}_0 = \mu$ *and* $\mathbf{X}^{(\varepsilon)}_0 = \mu^{(\varepsilon)} = \varepsilon^2\theta\mu\big(z + \varepsilon^{-1}\,\cdot\,\big)$, *respectively, with densities in* $\mathcal{B}^2_{\text{tem}}$. *Then, for* $t \geq 0$ *fixed, the following statements hold.*

**(a) (Scaling formula)** *The following pairs of random measures in* $\mathcal{M}_{\text{tem}}$ *coincide in law:*

$$\theta\varepsilon^2 \mathbf{X}_{\varepsilon^{-2}t}\big(z + \varepsilon^{-1}\,\cdot\,\big) \overset{\mathcal{L}}{=} \mathbf{X}^{(\varepsilon)}_t.$$

**(b) (Self-similarity)** *In the case of uniform initial states* $\mu = \mathbf{c}\ell$ $(\mathbf{c} \in \mathsf{R}^2_+)$,

$$\varepsilon^2 \mathbf{X}_{\varepsilon^{-2}t}(\varepsilon^{-1}\,\cdot) \overset{\mathcal{L}}{=} \mathbf{X}_t\,.$$

If $\mathbf{X}_0$ has bounded densities, the uniqueness of the solutions to $(\mathbf{MP})^{\sigma,\gamma}_{\mathbf{X}_0}$ established in [DFM+02, Theorem 1.11(b)] shows that the equivalence in (a) (and hence (b)) holds in the sense of processes in $t$.

**Remark 17 (Invariance of densities)** Together with spatial shift invariance, the self-similarity explains in particular why, in the case of uniform initial states, the law of the density at a point described in Theorem 11 (b) is *constant* in space and time. ◇

**Remark 18 (Growth of blocks of different types)** Recall that the types are segregated [Theorem 11 (c)], and in the long run compact sets are occupied by only one type (Theorem 13). So it is natural to ask about the growth of blocks of different types. To this end, for $\varepsilon, \beta > 0$, consider the scaled process $\mathbf{X}^{\varepsilon,\beta}$ defined by

$$X_t^{i,\varepsilon,\beta} \;:=\; \varepsilon^{2\beta} X^i_{\varepsilon^{-2}t}(\varepsilon^{-\beta}\,\cdot\,), \qquad t \geq 0, \quad i = 1, 2, \tag{42}$$

and start again with $\mathbf{X}_0 = \mathbf{c}\ell,\ \mathbf{c} \in \mathsf{R}^2_+$. Note that this scaling preserves the expectations: $P^{\mathbf{X}}_{\mathbf{c}\ell}\, \mathbf{X}_t^{\varepsilon,\beta} \equiv \mathbf{c}\ell$. If $\beta = 1$, we are in the self-similarity case of Proposition 16 (b), that is $\mathbf{X}^{\varepsilon,1} \equiv \mathbf{X}$. Consequently, essentially disjoint random blocks of linear size of order $\varepsilon^{-1}$ form at time $\varepsilon^{-2}t$. On the other hand, for any $\beta > 0$,

$$\mathbf{X}_t^{\varepsilon,\beta} \;=\; \varepsilon^{2(\beta-1)}\, \mathbf{X}_t^{\varepsilon,1}(\varepsilon^{1-\beta}\,\cdot\,) \;\overset{\mathcal{L}}{=}\; \varepsilon^{2(\beta-1)}\, \mathbf{X}_t(\varepsilon^{1-\beta}\,\cdot\,), \tag{43}$$

by self-similarity. If now $\beta > 1$, then by the $L^2$–ergodic theorem, using the covariance formula of Theorem 4 (b), from (43) it can easily be shown that in $L^2(P^{\mathbf{X}}_{\mathbf{c}\ell})$,

$$\mathbf{X}_t^{\varepsilon,\beta} \underset{\varepsilon\downarrow 0}{\longrightarrow} \mathbf{c}\ell \quad \text{in } \mathcal{M}^2_{\text{tem}}\,, \qquad t \geq 0. \tag{44}$$

That is, for $\beta > 1$, at length scales of $\varepsilon^{-\beta}$ one has instead a homogeneous mixing of types, so $\varepsilon^{-1}$ is the maximal order of pure type blocks. Finally, if $\beta < 1$, then from (43) by Theorem 11 (a),(b), we can derive the convergence in law

$$\langle X_t^{i,\varepsilon,\beta}, \varphi\rangle \underset{\varepsilon\downarrow 0}{\longrightarrow} X_t^i(0)\,\varphi(0) \;\overset{\mathcal{L}}{=}\; \xi_{\tau(\mathbf{c})}\,\varphi(0), \quad i = 1, 2,\ t \geq 0,\ \varphi \in \mathcal{C}_{\text{com}}(\mathsf{R}^2). \tag{45}$$

Consequently, in the $\beta < 1$ case, at blocks of order $\varepsilon^{-\beta}$ one sees essentially only one type.

This discussion also explains why in our construction of $\mathbf{X}$ starting from the lattice model ${}^{\varepsilon}\mathbf{X}$, we used the *critical* scaling, $\beta = 1$. Indeed, if instead we scaled with

$$\varepsilon^{2(\beta-1)}\, {}^{\varepsilon}\mathbf{X}_t(\varepsilon^{-\beta}\,\cdot\,)\, {}^{\varepsilon}\ell(\mathrm{d}x), \tag{46}$$

where $\beta \neq 1$, then we would have obtained a degenerate limit when $\varepsilon \to 0$, namely, for $\beta > 1$ a homogeneous mixing of types, whereas for $\beta < 1$ a pure type block behavior.

Moreover, from the point of view of the lattice model, our approximation Theorem 6 (under the critical scaling) together with the discussion above also leads to a description of the growth of blocks in the lattice model. In particular, at time $\varepsilon^{-2}t$ essentially disjoint blocks of linear size $\varepsilon^{-1}$ do form for solutions of (31) and by the above these are the largest pure blocks that form. (Recall, as in the $\varepsilon = 1$ case of [DP98] and as in Theorem 13, in ${}^{\varepsilon}\mathbf{X}_t$ locally only one type survives as $t \uparrow \infty$.) These considerations served as a motivation for us to start from the lattice model in constructing the two-dimensional continuum model $\mathbf{X}$.

Further elaboration on these ideas would involve the possibility of diffusive clustering phenomena, as, for instance, in the two-dimensional voter model [CG86] or for interacting diffusions on the hierarchical group in the recurrent but not strongly recurrent case [FG94, FG96]. In fact, the possibility of diffusive clustering phenomena of ${}^{\varepsilon}\mathbf{X}$ on $\varepsilon\mathsf{Z}^2$ is a topic of current study. ◇

## 2.6 Relation to the super-Brownian catalyst reactant pair

At the beginning of the paper we motivated the investigation of the mutually catalytic branching process $\mathbf{X}$ by the model of a super-Brownian reactant $X^{\varrho}$ with a super-Brownian catalyst $\varrho$ ([DF97a]). We now want to mention a few similarities in the models $(\varrho, X^{\varrho})$ and $\mathbf{X}$ in dimension two.

Both models can be described by a martingale problem, where the collision local time enters as an intrinsic part (see [DF01, Corollary 4]). Also for $X^{\varrho}$, one has to restrict the possible initial states for $\varrho$ (see [FK99, Proposition 5]). At each fixed time $t$, the measures $\varrho_t$ and $X_t^{\varrho}$ are separated, more precisely, the absolutely continuous reactant $X_t^{\varrho}$ lives outside the closed support of the catalyst $\varrho_t$ ([FK99, Theorem 1 (a)]), which however is singular. Moreover, under the annealed measure (the case in which the random law of $X^{\varrho}$ is replaced by its expectation obtained by integrating with respect to the law of $\varrho$), the variance of the random densities $X_t^{\varrho}(x)$ is infinite ([FK99, $4^{\text{th}}$ Remark after Theorem 1 ]), as in our mutually catalytic model.

In the case of uniform initial states, both models are self-similar ([DF97b, Proposition 13 (b)]), and in the long-term behavior of $X^{\varrho}$ one has persistent convergence in law to a non-degenerate random multiple of Lebesgue measure ([FK99, Corollary 2 (b)]), whereas $\varrho$ locally dies.

For a recent survey on catalytic super-Brownian motions, we refer to [DF02].

### 2.7 Outline

The remainder of the paper is organized as follows. In the next section, we start from the $\varepsilon\mathsf{Z}^2$–model ${}^{\varepsilon}\mathbf{X}$ of mutually catalytic branching and provide some fourth moment calculations that will lead to the uniform estimate of the second moment of the collision measure for sufficiently small parameters, see Corollary 30 below which is our key estimate for the existence proof. Via a tightness argument (Proposition 37), this then leads, in Section 4, to the proof of the approximation Theorem 6, hence to the construction of a solution $\mathbf{X}$ satisfying the martingale problem $(\mathbf{MP})_{\mu}^{\sigma,\gamma}$. In the last section, the claimed properties of $\mathbf{X}$ are verified. Finally, in the appendix, some auxiliary facts about random walks that we shall need are gathered together, a lengthy proof of a basic estimate related to our fourth moment calculations is provided, and a simple Feynman integral estimate is derived.

## 3 Mutually catalytic branching on lattice spaces

In this section we first recall the Green function representation of the $\varepsilon\mathsf{Z}^2$–version of the mutually catalytic branching process ${}^{\varepsilon}\mathbf{X}$. Then, in the case $\varepsilon = 1$ we will derive a $4^{\text{th}}$ moment formula, and in Subsection 3.5 a $4^{\text{th}}$ moment estimate. We will use this estimate in Proposition 29 to bound the second moment of the collision measure. After rescaling with $\varepsilon$, this finally gives a uniform estimate for second moments of collision measures (Corollary 30), the main result in the section.

### 3.1 Green function representation of ${}^{\varepsilon}\mathbf{X}$

An obvious adaptation of Theorem 2.2 (b) (ii) in [DP98] for the present simple random walk case (bearing in mind our Lemmas A1 and A2) gives that ${}^{\varepsilon}\mathbf{X}$ also satisfies the following *Green function representation* of the martingale problem $(\mathbf{MP})_{\mu}^{\sigma,\gamma,\varepsilon}$ :

For $\varphi^i \in {}^{\varepsilon}\mathcal{B}_{\text{exp}}$ and $\mu^i \in {}^{\varepsilon}\mathcal{M}_{\text{tem}}$,

$$\big\langle {}^{\varepsilon}X_t^i, \varphi^i \big\rangle - \big\langle \mu^i, {}^{\varepsilon}S_t \varphi^i \big\rangle \;=\; \int_{[0,t]\times\varepsilon\mathsf{Z}^2} {}^{\varepsilon}M^i\big(\mathrm{d}(s,x)\big)\, {}^{\varepsilon}S_{t-s}\varphi^i\,(x), \tag{47}$$

$t \geq 0,\ i = 1,2,$ where ${}^{\varepsilon}M^1, {}^{\varepsilon}M^2$ are (zero mean) $\mathcal{F}_{\cdot}^{{}^{\varepsilon}\mathbf{X}}$*–martingale measures with predictable square function*

$$\left\langle\!\!\left\langle \int_{[0,\,\cdot\,]\times\varepsilon\mathsf{Z}^2} {}^{\varepsilon}M^i\big(\mathrm{d}(s,x)\big)\, f^i(s,x),\; \int_{[0,\,\cdot\,]\times\varepsilon\mathsf{Z}^2} {}^{\varepsilon}M^j\big(\mathrm{d}(s,x)\big)\, f^j(s,x) \right\rangle\!\!\right\rangle_t$$

$$= \;\delta_{i,j}\,\gamma \int_{[0,t]\times\varepsilon\mathsf{Z}^2} {}^{\varepsilon}L_{{}^{\varepsilon}\mathbf{X}}\big(\mathrm{d}(s,x)\big)\, f^i(s,x)\, f^j(s,x), \tag{48}$$

$i,j\in\{1,2\}$ (see Chapter 2 of [Wal86] for information about martingale measures). Here $f^1,f^2$ belongs to the set of predictable functions $\psi$ defined on $\Omega\times\mathsf{R}_+\times\varepsilon\mathsf{Z}^2$ such that

$$P_\mu\int_{[0,t]\times\varepsilon\mathsf{Z}^2}{}^{\varepsilon}L_{{}^{\varepsilon}\mathbf{X}}\bigl(\mathrm{d}(s,x)\bigr)\,\psi^2(s,x)\;<\;\infty,\qquad t\geq 0. \tag{49}$$

Hence, the *expectation* of the Markov process ${}^{\varepsilon}\mathbf{X}=\bigl({}^{\varepsilon}X^1,\,{}^{\varepsilon}X^2\bigr)$ is given by

$$P_\mu\,{}^{\varepsilon}X^i_t(\mathrm{d}x)\;=\;\mu^i*{}^{\varepsilon}\mathrm{p}_t\,(x)\,{}^{\varepsilon}\ell(\mathrm{d}x). \tag{50}$$

In particular,

$$P_{\mathbf{c}\,{}^{\varepsilon}\ell}\,{}^{\varepsilon}X^i_t\,(x)\equiv c^i\quad\text{and}\quad P_\mu\,\bigl\|{}^{\varepsilon}X^i_t\bigr\|\equiv\bigl\|\mu^i\bigr\|\,. \tag{51}$$

On the other hand, by the Markov property, (50), and orthogonality (as in (48) the cross term cancels), the 'mixed' second moment measure equals

$$P_\mu\,{}^{\varepsilon}X^1_{t_1}(\mathrm{d}x^1)\,{}^{\varepsilon}X^2_{t_2}(\mathrm{d}x^2)\;=\;{}^{\varepsilon}\ell(\mathrm{d}x^1)\,{}^{\varepsilon}\ell(\mathrm{d}x^2)\,\mu^1*{}^{\varepsilon}\mathrm{p}_{t_1}(x^1)\,\mu^2*{}^{\varepsilon}\mathrm{p}_{t_2}(x^2) \tag{52}$$

(*bilinearity*). Thus, we get the following formula for the *expected collision local time:*

$$P_\mu\,{}^{\varepsilon}L_{{}^{\varepsilon}\mathbf{X}}\bigl(\mathrm{d}(s,x)\bigr)\;=\;\mathrm{d}s\,{}^{\varepsilon}\ell(\mathrm{d}x)\,\mu^1*{}^{\varepsilon}\mathrm{p}_s\,(x)\,\mu^2*{}^{\varepsilon}\mathrm{p}_s\,(x). \tag{53}$$

Moreover, again by the Markov property, (50), (47), (48), and (53), the second moment measure of ${}^{\varepsilon}X^i$ is given by

$$\begin{aligned}P_\mu\,{}^{\varepsilon}X^i_{t_1}(\mathrm{d}z^1)\,{}^{\varepsilon}X^i_{t_2}(\mathrm{d}z^2)\;&=\;{}^{\varepsilon}\ell(\mathrm{d}z^1)\,{}^{\varepsilon}\ell(\mathrm{d}z^2)\,\mu^i*{}^{\varepsilon}\mathrm{p}_{t_1}(z^1)\,\mu^i*{}^{\varepsilon}\mathrm{p}_{t_2}(z^2)\\ &+\;{}^{\varepsilon}\ell(\mathrm{d}z^1)\,{}^{\varepsilon}\ell(\mathrm{d}z^2)\,\gamma\int_0^{t_1\wedge t_2}\mathrm{d}s\int_{\varepsilon\mathsf{Z}^2}{}^{\varepsilon}\ell(\mathrm{d}x)\,\mu^1*{}^{\varepsilon}\mathrm{p}_s\,(x)\,\mu^2*{}^{\varepsilon}\mathrm{p}_s\,(x)\\ &\qquad\qquad{}^{\varepsilon}\mathrm{p}_{t_1-s}(z^1-x)\,{}^{\varepsilon}\mathrm{p}_{t_2-s}(z^2-x),\end{aligned} \tag{54}$$

$t_1,t_2>0$. Combined with (50) and (52), we get the following *covariance densities* with respect to ${}^{\varepsilon}\ell\times{}^{\varepsilon}\ell$ :

$$\begin{aligned}\mathrm{Cov}_\mu\bigl({}^{\varepsilon}X^{i_1}_{t_1},\,{}^{\varepsilon}X^{i_2}_{t_2}\bigr)(\mathbf{z})\;&=\;\delta_{i_1,i_2}\,\gamma\int_0^{t_1\wedge t_2}\mathrm{d}s\int_{\varepsilon\mathsf{Z}^2}{}^{\varepsilon}\ell(\mathrm{d}x)\\ &\mu^1*{}^{\varepsilon}\mathrm{p}_s\,(x)\,\mu^2*{}^{\varepsilon}\mathrm{p}_s\,(x)\,{}^{\varepsilon}\mathrm{p}_{t_1-s}(z^1-x)\,{}^{\varepsilon}\mathrm{p}_{t_2-s}(z^2-x),\end{aligned} \tag{55}$$

$i_1,i_2\in\{1,2\}$, $\mathbf{z}=(z^1,z^2)\in(\varepsilon\mathsf{Z}^2)^2$. In particular,

$$\mathrm{Cov}_{\mathbf{c}\,{}^{\varepsilon}\ell}\bigl({}^{\varepsilon}X^{i_1}_{t_1},\,{}^{\varepsilon}X^{i_2}_{t_2}\bigr)(\mathbf{z})\;=\;\delta_{i_1,i_2}\,c^1\,c^2\,\gamma\int_0^{t_1\wedge t_2}\mathrm{d}s\;{}^{\varepsilon}\mathrm{p}_{t_1+t_2-2s}(z^1-z^2) \tag{56}$$

and

$$\mathrm{Cov}_\mu\bigl(\|{}^{\varepsilon}X^{i_1}_{t_1}\|,\|{}^{\varepsilon}X^{i_2}_{t_2}\|\bigr)\;=\;\delta_{i_1,i_2}\,\gamma\int_{\varepsilon\mathsf{Z}^2}\mu^1(\mathrm{d}a^1)\int_{\varepsilon\mathsf{Z}^2}\mu^2(\mathrm{d}a^2)\int_0^{t_1\wedge t_2}\mathrm{d}s\;{}^{\varepsilon}\mathrm{p}_{2s}(a^1-a^2),$$

where by (53) the triple integral coincides with the expected collision local time $P_\mu\,{}^{\varepsilon}L_{{}^{\varepsilon}\mathbf{X}}\bigl(t_1\wedge t_2\,,\varepsilon\mathsf{Z}^2\bigr)$.

### 3.2 Finite higher moments on $\mathsf{Z}^2$

As announced, we need some higher moment bounds, uniformly in $\varepsilon$. But first we proceed with $\varepsilon=1$, and in Subsection 3.8 we will go back to general $\varepsilon$ by scaling.

Using differential notation, we can rewrite (31) as

$$\mathrm{d}\,{}^{1}X^i_t(x)\;=\;\frac{\sigma^2}{2}\;{}^{1}\Delta\,{}^{1}X^i_t\,(x)\,\mathrm{d}t\;+\;\sqrt{\gamma\,{}^{1}X^1_t(x)\,{}^{1}X^2_t(x)}\,\mathrm{d}W^i_t(x), \tag{57}$$

$(t,x) \in \mathsf{R}_+ \times \mathsf{Z}^2,\ i=1,2.$

Fix for now $\mu \in {}^1\mathcal{M}^2_{\text{tem}}$ . For $n \geq 1$, $\mathbf{i} = (i_1, \ldots, i_n) \in \{1,2\}^n$, $\mathbf{x} = \left(x^1, \ldots, x^n\right) \in (\mathsf{Z}^2)^n$, and $t \geq 0$, we introduce the following *higher moment density* notation:

$$ {}^1m^{\mathbf{i}}_t(\mathbf{x}) \ :=\ P_\mu \prod_{j=1}^n {}^1X^{i_j}_t(x^j). \tag{58} $$

Note that these moment density expressions are invariant with respect to *simultaneous* reordering of $\mathbf{i}$ and $\mathbf{x}$. For instance,

$$ P_\mu\, {}^1X^1_t(x^1)\, {}^1X^2_t(x^2) \ =\ P_\mu\, {}^1X^2_t(x^2)\, {}^1X^1_t(x^1). \tag{59} $$

First we check that the fourth moments are finite:

**Lemma 19 (Finite fourth moments)** *Let* $\mu \in {}^1\mathcal{M}^2_{\text{tem}}(\mathsf{Z}^2)$ *and* $\lambda > 0$. *Then*

$$ \sup_{0\leq t\leq T} P_\mu \sum_{i=1,2} \sum_{x\in\mathsf{Z}^2} \left({}^1X^i_t(x)\right)^4 \mathrm{e}^{-\lambda|x|} \ <\ \infty, \qquad T>0. \tag{60} $$

**Proof** Itô's formula gives for $t \geq 0$,

$$ \begin{aligned} \sum_{i=1,2} \sum_{x\in\mathsf{Z}^2} \left({}^1X^i_t(x)\right)^4 \mathrm{e}^{-\lambda|x|} \ =&\ \sum_{i=1,2} \sum_{x\in\mathsf{Z}^2} \left(\mu^i(x)\right)^4 \mathrm{e}^{-\lambda|x|} \\ &+ 2\sigma^2 \sum_{i=1,2} \sum_{x\in\mathsf{Z}^2} \int_0^t \mathrm{d}s\ \left({}^1X^i_s(x)\right)^3\ {}^1\Delta\, {}^1X^i_s(x)\, \mathrm{e}^{-\lambda|x|} \\ &+ 4 \sum_{i=1,2} \sum_{x\in\mathsf{Z}^2} \int_0^t \mathrm{d}W^i_s(x) \sqrt{\gamma\, {}^1X^1_s(x)\, {}^1X^2_s(x)}\ \left({}^1X^i_s(x)\right)^3 \mathrm{e}^{-\lambda|x|} \\ &+ 6\gamma \sum_{i=1,2} \sum_{x\in\mathsf{Z}^2} \int_0^t \mathrm{d}s\ {}^1X^1_s(x)\, {}^1X^2_s(x)\ \left({}^1X^i_s(x)\right)^2 \mathrm{e}^{-\lambda|x|}. \end{aligned} \tag{61} $$

Note that the convergence of each of the series and continuity in $t$ follows from the fact that the ${}^1X^i$ are ${}^1\mathcal{M}_{\text{tem}}$–valued processes (use the convergence of the predictable square function to handle the local martingale term). The continuity allows us to introduce a sequence of stopping times $T_n \uparrow \infty$ as $n \uparrow \infty$, in such a way that each term in (61) is bounded if $t$ is replaced by $t \wedge T_n$. Then, by Hölder's inequality,

$$ \begin{aligned} &P_\mu \sum_{i=1,2} \left\langle ({}^1X^i_{t\wedge T_n})^4, \phi_\lambda \right\rangle \\ &\leq\ \sum_{i=1,2} \left\langle (\mu^i)^4, \phi_\lambda \right\rangle \ +\ c_{\gamma,\sigma}\, P_\mu \int_0^{t\wedge T_n} \mathrm{d}s \sum_{i=1,2} \left\langle ({}^1X^i_s)^4, \phi_\lambda \right\rangle \end{aligned} \tag{62} $$

for some constant $c_{\gamma,\sigma}$. But the latter expectation expression can further be bounded from above by

$$ \int_0^t \mathrm{d}s\, P_\mu \sum_{i=1,2} \left\langle ({}^1X^i_{s\wedge T_n})^4, \phi_\lambda \right\rangle. \tag{63} $$

A simple application of the Gronwall and Fatou Lemmas now gives the claim. □

**Remark 20 (Refinement)** By a refinement of the previous proof, the supremum could be moved under the expectation sign. Clearly, also the fourth moment could be replaced by a moment of any higher order, but fourth moments are enough for our purpose. ◇

### 3.3 Moment equations

From (57), by Itô's formula,

$$\begin{aligned} \mathrm{d} \prod_{j=1}^{4} {}^1X_t^{i_j}(x^j) \;&=\; \frac{\sigma^2}{2} \sum_{k=1}^{4} {}^1\Delta_{x^k} \prod_{j=1}^{4} {}^1X_t^{i_j}(x^j)\, \mathrm{d}t \;+\; \mathrm{d}\,(\text{martingale}) \\ &\quad + \gamma \sum_{1 \le j < k \le 4} \delta_{(i_j,x^j),(i_k,x^k)}\, {}^1X_t^{i^\wedge}(x^\wedge)\, {}^1X_t^{i^\vee}(x^\vee)\, {}^1X_t^1(x^j)\, {}^1X_t^2(x^k)\, \mathrm{d}t, \end{aligned}$$

where ${}^1\Delta_{x^k}$ indicates that ${}^1\Delta$ is applied to the variable $x^k \in \mathsf{Z}^2$, and the local martingale term is a martingale by Lemma 19. Moreover, the index $\wedge$ stands for the number $\min(\{1,\ldots,4\} \setminus \{j,k\})$ whereas $\vee$ refers to $\max(\{1,\ldots,4\} \setminus \{j,k\})$. Taking expectations and using Lemma 19 to see that ${}^1m_t^{\mathbf{i}}(x) < \infty$ for $\mathbf{i} \in \{1,2\}^4$, we immediately get the following result:

**Lemma 21 ($4^{\text{th}}$ moment equations)** *Let $\mu \in \mathcal{M}^2_{\text{tem}}(\mathsf{Z}^2)$ and $\lambda > 0$. Then the $4^{\text{th}}$ moment density functions are finite and satisfy the following closed linear system of equations:*

$$\begin{aligned} \frac{\partial}{\partial t}\, {}^1m_t^{\mathbf{i}}(\mathbf{x}) \;&=\; \frac{\sigma^2}{2} \sum_{k=1}^{4} {}^1\Delta_{x^k}\; {}^1m_t^{\mathbf{i}}(\mathbf{x}) \\ &\quad + \gamma \sum_{1 \le j < k \le 4} \delta_{(i_j,x^j),(i_k,x^k)}\, {}^1m_t^{(i^\wedge, i^\vee, 1, 2)}\left(x^\wedge, x^\vee, x^j, x^k\right), \end{aligned} \tag{64}$$

$\mathbf{i} = (i_1, \ldots, i_4) \in \{1,2\}^4,\;\; \mathbf{x} = \left(x^1, \ldots, x^4\right) \in (\mathsf{Z}^2)^4,$ *and* $t > 0.$

Let $\mathbf{i}^{\text{c}}$ arise from $\mathbf{i}$ by *interchanging* the types 1 and 2. Pass in (64) from $\mathbf{i}$ to $\mathbf{i}^{\text{c}}$. Note that concerning the new Kronecker symbol expression, $i_j^{\text{c}} = i_k^{\text{c}}$ holds if and only if $i_j = i_k$ is true. Thus we can add up the new system with the original one, and we get a system in terms of functions which are *invariant* according to the transition $\mathbf{i} \mapsto \mathbf{i}^{\text{c}}$. This justifies the following convention.

**Convention 22 (Type symmetrization)** For our later purpose of establishing upper moment estimates, by an abuse of notation we assume that the moment density functions ${}^1m^{\mathbf{i}}$, $\mathbf{i} \in \{1,2\}^4$, are invariant with respect to the type interchange $\mathbf{i} \mapsto \mathbf{i}^{\text{c}}$. In short, we will now be writing ${}^1m^{\mathbf{i}}$ for ${}^1m^{\mathbf{i}} + {}^1m^{\mathbf{i}^{\text{c}}}$ without changing our notation. Also, for simplification of notation, in calculations we often drop the upper index 1 in front of $m$, p, and $S$, and we delete some commas in writing $m^{1122}$ instead of $m^{1,1,2,2}$, for instance. ◇

Our next goal is to derive a formula for $m_{s_0}^{1122}(x_0^1, x_0^2, x_0^1, x_0^3)$, with $s_0 > 0$ and $\mathbf{x}_0 = (x_0^1, x_0^2, x_0^3) \in (\mathsf{Z}^2)^3$. For this purpose, set

$$\begin{aligned} &{}^1f_{s_0}(\mathbf{x}_0) \;:=\; {}^1\mathbf{S}_{s_0} m_0^{1122}\,(x_0^1, x_0^2, x_0^1, x_0^3) \;+\; \gamma \int_0^{s_0} \mathrm{d}s_1 \sum_{\mathbf{x}_1 \in (\mathsf{Z}^2)^3} \\ &\left[{}^1\mathrm{p}_{s_0-s_1}(x_0^2 - x_1^3)\, {}^1\mathrm{p}_{s_0-s_1}(x_0^3 - x_1^1) + {}^1\mathrm{p}_{s_0-s_1}(x_0^3 - x_1^3)\, {}^1\mathrm{p}_{s_0-s_1}(x_0^2 - x_1^1)\right] \\ &{}^1\mathrm{p}_{s_0-s_1}(x_0^1 - x_1^2)\, {}^1\mathrm{p}_{s_0-s_1}(x_0^1 - x_1^3)\, {}^1\mathbf{S}_{s_1} m_0^{1112}\,(x_1^1, x_1^2, x_1^3, x_1^3), \end{aligned} \tag{65}$$

where ${}^1\mathbf{S}$ denotes the semigroup of four *independent* random walks each with generator $\frac{\sigma^2}{2}\, {}^1\Delta$. Moreover, for $s_0 > \cdots > s_{2n} > 0$, and $\mathbf{x}_\ell \in (\mathsf{Z}^2)^3$, $1 \le \ell \le 2n$, write $\Pi_n(\mathbf{s}_{2n}; \mathbf{x}_0, \ldots, \mathbf{x}_{2n})$ for the $n$–fold product

$$\prod_{j=1}^{n} \Bigg\{ \Big[ {}^1\mathrm{p}_{s_{2j-2}-s_{2j-1}}(x^2_{2j-2}-x^3_{2j-1})\, {}^1\mathrm{p}_{s_{2j-2}-s_{2j-1}}(x^3_{2j-2}-x^1_{2j-1})$$
$$+\ {}^1\mathrm{p}_{s_{2j-2}-s_{2j-1}}(x^3_{2j-2}-x^3_{2j-1})\, {}^1\mathrm{p}_{s_{2j-2}-s_{2j-1}}(x^2_{2j-2}-x^1_{2j-1}) \Big]$$
$${}^1\mathrm{p}_{s_{2j-2}-s_{2j-1}}(x^1_{2j-2}-x^2_{2j-1})\, {}^1\mathrm{p}_{s_{2j-2}-s_{2j-1}}(x^1_{2j-2}-x^3_{2j-1})\, {}^1\mathrm{p}_{s_{2j-1}-s_{2j}}(x^3_{2j-1}-x^3_{2j})$$
$$\Big( {}^1\mathrm{p}_{s_{2j-1}-s_{2j}}(x^1_{2j-1}-x^1_{2j})\, {}^1\mathrm{p}_{s_{2j-1}-s_{2j}}(x^2_{2j-1}-x^1_{2j})\, {}^1\mathrm{p}_{s_{2j-1}-s_{2j}}(x^3_{2j-1}-x^2_{2j})$$
$$+\ {}^1\mathrm{p}_{s_{2j-1}-s_{2j}}(x^1_{2j-1}-x^1_{2j})\, {}^1\mathrm{p}_{s_{2j-1}-s_{2j}}(x^2_{2j-1}-x^2_{2j})\, {}^1\mathrm{p}_{s_{2j-1}-s_{2j}}(x^3_{2j-1}-x^1_{2j})$$
$$+\ {}^1\mathrm{p}_{s_{2j-1}-s_{2j}}(x^1_{2j-1}-x^2_{2j})\, {}^1\mathrm{p}_{s_{2j-1}-s_{2j}}(x^2_{2j-1}-x^1_{2j})\, {}^1\mathrm{p}_{s_{2j-1}-s_{2j}}(x^3_{2j-1}-x^1_{2j}) \Big) \Bigg\}.$$

### 3.4 A $4^{\text{th}}$ moment density formula on $\mathsf{Z}^2$

Here now is the desired formula which we derive from (64) by iteration and using our Convention 22:

**Lemma 23 (A fourth moment density formula)** *Under Convention* 22, *for* $s_0 > 0$ *and* $\mathbf{x}_0 = (x^1_0, x^2_0, x^3_0)$ *in* $(\mathsf{Z}^2)^3$,

$$ {}^1m^{1122}_{s_0}(x^1_0, x^2_0, x^1_0, x^3_0) \;=\; {}^1f_{s_0}(\mathbf{x}_0) \;+\; \sum_{n=1}^{\infty} \gamma^{2n} \int_0^{s_0} \mathrm{d}s_1 \cdots \int_0^{s_{2n-1}} \mathrm{d}s_{2n}$$
$$\sum_{\mathbf{x}_\ell \in (\mathsf{Z}^2)^3 \text{ for } 1 \le \ell \le 2n} {}^1f_{s_{2n}}(\mathbf{x}_{2n})\, \Pi_n(\mathbf{s}_{2n}; \mathbf{x}_0, \ldots, \mathbf{x}_{2n}).$$

**Proof** Take $i_1 = i_2 = 1$ and $i_3 = i_4 = 2$ in (64) and using simultaneous (in both $i_j$ and $x_j$) reordering as well as our Convention 22, we obtain for $t > 0$ and $\mathbf{x}_0 \in (\mathsf{Z}^2)^4$,

$$\frac{\partial}{\partial t} m^{1122}_t(\mathbf{x}_0) \;=\; \frac{\sigma^2}{2} \sum_{k=1}^{4} {}^1\Delta_{x^k_0}\, m^{1122}_t(\mathbf{x}_0) + \gamma\, \delta_{x^1_0 x^2_0}\, m^{1112}_t(\overline{\mathbf{x}_0}) + \gamma\, \delta_{x^3_0 x^4_0}\, m^{1112}_t(\mathbf{x}_0),$$

where $\overline{\mathbf{x}_0} := (x^4_0, \ldots, x^1_0)$. By integration,

$$m^{1122}_t(\mathbf{x}_0) \;=\; {}^1\mathbf{S}_t m^{1122}_0\,(\mathbf{x}_0) + \gamma \int_0^t \mathrm{d}s \sum_{\mathbf{x}_1 \in (\mathsf{Z}^2)^4} \prod_{i=1}^{4} \mathrm{p}_{t-s}(x^i_0 - x^i_1) \Big( \delta_{x^1_1 x^2_1}\, m^{1112}_s(\overline{\mathbf{x}_1}) + \delta_{x^3_1 x^4_1}\, m^{1112}_s(\mathbf{x}_1) \Big). \tag{66}$$

Specializing the $\mathbf{x}_0$–vector as well as using simultaneous reordering and renaming of the summation variables, we get, for $\mathbf{x}_0 = (x^1_0, x^2_0, x^1_0, x^3_0) \in (\mathsf{Z}^2)^4$ and $s_0 > 0$,

$$m^{1122}_{s_0}(x^1_0, x^2_0, x^1_0, x^3_0) \;=\; {}^1\mathbf{S}_{s_0} m^{1122}_0\,(x^1_0, x^2_0, x^1_0, x^3_0) \;+\; \gamma \int_0^{s_0} \mathrm{d}s_1 \sum_{\mathbf{x}_1 \in (\mathsf{Z}^2)^3} \Big[ \mathrm{p}_{s_0-s_1}(x^2_0 - x^3_1)\, \mathrm{p}_{s_0-s_1}(x^3_0 - x^1_1) + \mathrm{p}_{s_0-s_1}(x^3_0 - x^3_1)\, \mathrm{p}_{s_0-s_1}(x^2_0 - x^1_1) \Big] \mathrm{p}_{s_0-s_1}(x^1_0 - x^2_1)\, \mathrm{p}_{s_0-s_1}(x^1_0 - x^3_1)\, m^{1112}_{s_1}(x^1_1, x^2_1, x^3_1, x^3_1). \tag{67}$$

On the other hand, from (64) combined with our Convention 22, we have for $\mathbf{x}_1 \in (\mathsf{Z}^2)^4$,

$$\frac{\partial}{\partial t} m_t^{1112}(\mathbf{x}_1) \;=\; \frac{\sigma^2}{2} \sum_{k=1}^{4} {}^1\Delta_{x^k}\, m_t^{1112}(\mathbf{x}_1) + \gamma\, \delta_{x^1 x^2}\, m_t^{1122}(x_1^1, x_1^3, x_1^2, x_1^4) \qquad (68)$$
$$+\; \gamma\, \delta_{x_1^1 x_1^3}\, m_t^{1122}(\mathbf{x}_1) + \gamma\, \delta_{x_1^2 x_1^3}\, m_t^{1122}(\mathbf{x}_1).$$

A similar derivation to that of equation (67) above yields, for $\mathbf{x}_1 \in (\mathsf{Z}^2)^3$,

$$m_{s_1}^{1112}(x_1^1, x_1^2, x_1^3, x_1^3) \;=\; {}^1\mathbf{S}_{s_1} m_0^{1112}\,(x_1^1, x_1^2, x_1^3, x_1^3) \qquad (69)$$
$$+\; \gamma \int_0^{s_1} \mathrm{d}s_2 \sum_{\mathbf{x}_2 \in (\mathsf{Z}^2)^3} \Big( \mathrm{p}_{s_1-s_2}(x_1^1 - x_2^1)\, \mathrm{p}_{s_1-s_2}(x_1^2 - x_2^1)\, \mathrm{p}_{s_1-s_2}(x_1^3 - x_2^2)$$
$$+\; \mathrm{p}_{s_1-s_2}(x_1^1 - x_2^1)\, \mathrm{p}_{s_1-s_2}(x_1^2 - x_2^2)\, \mathrm{p}_{s_1-s_2}(x_1^3 - x_2^1)$$
$$+\; \mathrm{p}_{s_1-s_2}(x_1^1 - x_2^2)\, \mathrm{p}_{s_1-s_2}(x_1^2 - x_2^1)\, \mathrm{p}_{s_1-s_2}(x_1^3 - x_2^1)\Big)$$
$$\mathrm{p}_{s_1-s_2}(x_1^3 - x_2^3)\, m_{s_2}^{1122}(x_2^1, x_2^2, x_2^1, x_2^3).$$

Substituting (69) into (67) gives the following *"closed" equation* for the moment density $m_{s_0}^{1122}(x_0^1, x_0^2, x_0^1, x_0^3)$ :

$$m_{s_0}^{1122}(x_0^1, x_0^2, x_0^1, x_0^3) \;=\; {}^1\!f_{s_0}(\mathbf{x}_0) + \gamma^2 \int_0^{s_0} \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \sum_{\mathbf{x}_1, \mathbf{x}_2 \in (\mathsf{Z}^2)^3} \qquad (70)$$
$$\Big[\mathrm{p}_{s_0-s_1}(x_0^2 - x_1^3)\, \mathrm{p}_{s_0-s_1}(x_0^3 - x_1^1) + \mathrm{p}_{s_0-s_1}(x_0^3 - x_1^3)\, \mathrm{p}_{s_0-s_1}(x_0^2 - x_1^1)\Big]$$
$$\mathrm{p}_{s_0-s_1}(x_0^1 - x_1^2)\, \mathrm{p}_{s_0-s_1}(x_0^1 - x_1^3)\, \mathrm{p}_{s_1-s_2}(x_1^3 - x_2^3)$$
$$\Big(\mathrm{p}_{s_1-s_2}(x_1^1 - x_2^1)\, \mathrm{p}_{s_1-s_2}(x_1^2 - x_2^1)\, \mathrm{p}_{s_1-s_2}(x_1^3 - x_2^2)$$
$$+\; \mathrm{p}_{s_1-s_2}(x_1^1 - x_2^1)\, \mathrm{p}_{s_1-s_2}(x_1^2 - x_2^2)\, \mathrm{p}_{s_1-s_2}(x_1^3 - x_2^1)$$
$$+\; \mathrm{p}_{s_1-s_2}(x_1^1 - x_2^2)\, \mathrm{p}_{s_1-s_2}(x_1^2 - x_2^1)\, \mathrm{p}_{s_1-s_2}(x_1^3 - x_2^1)\Big) m_{s_2}^{1122}(x_2^1, x_2^2, x_2^1, x_2^3),$$

where ${}^1\!f_{s_0}(\mathbf{x}_0)$ was defined in (65).

Denote by $S_\infty$ the right-hand side of the claimed identity in Lemma 23 (series expansion). Recall the notation $\Pi_n(\mathbf{s}_{2n}; \mathbf{x}_0, \ldots, \mathbf{x}_{2n})$ introduced immediately before the lemma, and set

$$T_n(\mathbf{s}_{2n}; \mathbf{x}_0, \ldots, \mathbf{x}_{2n}) \;:=\; \sum_{\mathbf{x}_\ell \in (\mathsf{Z}^2)^3 \text{ for } 1 \le \ell \le 2n-1} \Pi_n(\mathbf{s}_{2n}; \mathbf{x}_0, \ldots, \mathbf{x}_{2n}). \qquad (71)$$

Iteration of the closed equation (70) implies that

$$m_{s_0}^{1122}(x_0^1, x_0^2, x_0^1, x_0^3) \;=\; S_\infty + \lim_{n\uparrow\infty} \gamma^{2n} \int_0^{s_0} \mathrm{d}s_1 \cdots \int_0^{s_{2n-1}} \mathrm{d}s_{2n}$$
$$\sum_{\mathbf{x}_{2n} \in (\mathsf{Z}^2)^3} T_n(\mathbf{s}_{2n}; \mathbf{x}_0, \ldots, \mathbf{x}_{2n})\, m_{s_{2n}}^{1122}(\mathbf{x}_{2n}), \qquad (72)$$

where the series $S_\infty$ and the latter limit must converge by the monotonicity of the partial sums and the finiteness of the left-hand side (by Lemma 19). To finish the proof, we have to show that the limit expression in (72) will disappear.

If $\lambda > 0$, then Lemma 19 implies that

$$m_{s_{2n}}^{1122}(\mathbf{x}_{2n}) \;\le\; c_{(73)} \exp\big[\lambda \|\mathbf{x}_{2n}\|\big] \qquad (73)$$

(recall the definition (7) of the norm $\|\cdot\|$ and that $s_{2n}\le s_0$) for some constant $c_{(73)}=c_{(73)}(\lambda,s_0)$. In Lemma 24 (see Remark 25) we will show that

$$\sum_{\mathbf{x}_{2n}\in(\mathsf{Z}^2)^3} T_n(\mathbf{s}_{2n};\mathbf{x}_0,\ldots,\mathbf{x}_{2n})\exp\bigl[\lambda\,\|\mathbf{x}_{2n}\|\bigr] \;\le\; c_{(74)}\exp\bigl[2\lambda\,\|\mathbf{x}_0\|\bigr]\,6^n \tag{74}$$

for some constant $c_{(74)}=c_{(74)}(s_0,\lambda,\sigma)$ [note that the left hand side of (74) is $L_n(\mathbf{1})$]. Use (73) and (74) to see that the limit in (72) is bounded by

$$\overline{\lim_{n\uparrow\infty}}\, c_{(73)}c_{(74)}\,(6\gamma^2)^n\,\frac{s_0^{2n}}{(2n)!}\,\exp\bigl[2\lambda\|\mathbf{x}_0\|\bigr] \;=\; 0. \tag{75}$$

Thus, the limit expression in (72) vanishes, and the proof is finished. □

### 3.5 A $4^{\text{th}}$ moment density estimate on $\mathsf{Z}^2$

Now we temporarily fix a $\lambda\ge 0$, and assume that the initial state ${}^1\mathbf{X}_0={}^1\mu\in{}^1\mathcal{M}^2_{\text{tem}}$ is deterministic with density function (also denoted by ${}^1\mathbf{X}_0={}^1\mu$) satisfying

$${}^1\mu^i(x)\;\le\; c_\lambda\,\mathrm{e}^{\lambda|x|},\qquad x\in\mathsf{Z}^2,\quad i=1,2, \tag{76}$$

for some constant $c_\lambda$. (In other words, ${}^1\mu^i\in\mathcal{B}_{-\lambda}$.) For ${}^1f_{s_0}(\mathbf{x}_0)$, defined in (65), with $0<s_0\le T$ and $\mathbf{x}_0=(x_0^1,x_0^2,x_0^3)\in(\mathsf{Z}^2)^3$, by Lemma A2 in the appendix we obtain

$$\begin{aligned}{}^1f_{s_0}(\mathbf{x}_0)\;\le\;& c_\lambda^4\,c_{\mathrm{A2}}^4\exp\bigl[2\lambda|x_0^1|+\lambda|x_0^2|+\lambda|x_0^3|\bigr]\;+\;\gamma\int_0^{s_0}\mathrm{d}s_1\sum_{\mathbf{x}_1\in(\mathsf{Z}^2)^3}\\ &\Bigl[\mathrm{p}_{s_0-s_1}(x_0^2-x_1^3)\,\mathrm{p}_{s_0-s_1}(x_0^3-x_1^1)+\mathrm{p}_{s_0-s_1}(x_0^3-x_1^3)\,\mathrm{p}_{s_0-s_1}(x_0^2-x_1^1)\Bigr]\\ &\mathrm{p}_{s_0-s_1}(x_0^1-x_1^2)\,\mathrm{p}_{s_0-s_1}(x_0^1-x_1^3)\,c_\lambda^4\,c_{\mathrm{A2}}^4\,\exp\bigl[\lambda\left(|x_1^1|+|x_1^2|\right)\bigr]\exp\bigl[2\lambda|x_1^3|\bigr],\end{aligned}$$

with $c_{\mathrm{A2}}=c_{\mathrm{A2}}(T,\lambda,\sigma)\ge 1$ (defined in that lemma). For the integral term on the right hand side, we again use Lemma A2 (to eliminate the summation variables $x_1^1$ and $x_1^2$,) to obtain the upper estimate

$$\begin{aligned}\gamma\,c_\lambda^4\,c_{\mathrm{A2}}^6\int_0^{s_0}\mathrm{d}s_1\sum_{x_1^3\in\mathsf{Z}^2}&\Bigl[\mathrm{p}_{s_0-s_1}(x_0^2-x_1^3)\exp\bigl[\lambda\left(|x_0^1|+|x_0^3|\right)\bigr]\\ &+\;\mathrm{p}_{s_0-s_1}(x_0^3-x_1^3)\exp\bigl[\lambda\left(|x_0^1|+|x_0^2|\right)\bigr]\Bigr]\mathrm{p}_{s_0-s_1}(x_0^1-x_1^3)\exp\bigl[2\lambda|x_1^3|\bigr].\end{aligned}$$

Then, by Lemma A6, altogether we obtain

$$\begin{aligned}{}^1f_{s_0}(\mathbf{x}_0)\;\le\;& c_\lambda^4\,c_{\mathrm{A2}}^4\,\exp\bigl[2\lambda|x_0^1|+\lambda|x_0^2|+\lambda|x_0^3|\bigr]\\ &+\;\gamma\,c_\lambda^4\,c_{\mathrm{A2}}^6\,c_{\mathrm{A6}}\exp\bigl[2\lambda|x_0^1|+\lambda|x_0^2|+\lambda|x_0^3|\bigr]\int_0^{s_0}\mathrm{d}s_1\sum_{k=2}^{3}\mathrm{p}_{2\mathrm{e}^{20\lambda^2}(s_0-s_1)}(x_0^1-x_0^k)\end{aligned}$$

with $c_{\mathrm{A6}}=c_{\mathrm{A6}}(T,2\lambda,\sigma)\ge 1$.

To apply this estimate to ${}^1f_{s_{2n}}(\mathbf{x}_{2n})$ occurring in the $4^{\text{th}}$ moment density formula of Lemma 23, it is convenient to introduce two quantities $L_n(\mathbf{a})$ and $M_n^k(\mathbf{a},b)$. To describe them, we set $\mathbf{a}:=\bigl(a^1,a^2,a^3\bigr)$ with numbers $a^i\in[0,2]$ satisfying $a^1+a^2+a^3=4$, and write $\mathbf{A}$ for the set of all these $\mathbf{a}$. Moreover, with a slight abuse of notation (7), we set

$$\|\mathbf{a}\mathbf{x}\|:=a^1|x^1|+a^2|x^2|+a^3|x^3|\quad\text{if}\quad\mathbf{x}=\bigl(x^1,x^2,x^3\bigr)\in(\mathsf{Z}^2)^3. \tag{77}$$

Here now is the definition of $L_n(\mathbf{a}),\ n \geq 1$ :

$$L_n(\mathbf{a}) \;=\; L_n(\mathbf{a}, \lambda, \mathbf{s}_{2n}; \mathbf{x}_0) \tag{78}$$
$$:= \sum_{\mathbf{x}_\ell \in (\mathsf{Z}^2)^3 \text{ for } 1 \leq \ell \leq 2n} \exp\bigl[\lambda \,\|\mathbf{a}\mathbf{x}_{2n}\|\bigr]\, \Pi_n(\mathbf{s}_{2n}; \mathbf{x}_0, \ldots, \mathbf{x}_{2n}),$$

with $\Pi_n(\mathbf{s}_{2n}; \mathbf{x}_0, \ldots, \mathbf{x}_{2n})$ as introduced in the end of Subsection 3.3. On the other hand, $M_n^k(\mathbf{a}, b) = M_n^k(\mathbf{a}, b, \lambda, \mathbf{s}_{2n+1}; \mathbf{x}_0)$, with $k = 2, 3$ and $b \geq 1$, is defined as $L_n(\mathbf{a})$ but with the additional factor $\mathrm{p}_{2b\,(s_{2n}-s_{2n+1})}(x_{2n}^1 - x_{2n}^k)$ under the summation symbol. With these definitions, the moment density function of Lemma 23 becomes

$$m_{s_0}^{1122}(x_0^1, x_0^2, x_0^1, x_0^3) \;\leq\; c_\lambda^4\, c_{\mathrm{A}2}^4 \Bigl\{ \exp\bigl[\lambda \,\|\mathbf{a}\mathbf{x}_0\|\bigr] \tag{79}$$
$$+\, \gamma\, c_{\mathrm{A}2}^2\, c_{\mathrm{A}6} \exp\bigl[\lambda \,\|\mathbf{a}\mathbf{x}_0\|\bigr] \int_0^{s_0} \mathrm{d}s_1 \sum_{k=2}^{3} \mathrm{p}_{2b\,(s_0-s_1)}(x_0^1 - x_0^k)$$
$$+ \sum_{n=1}^{\infty} \gamma^{2n} \int_0^{s_0} \mathrm{d}s_1 \cdots \int_0^{s_{2n-1}} \mathrm{d}s_{2n}\; L_n(\mathbf{a}, \lambda, \mathbf{s}_{2n}; \mathbf{x}_0)$$
$$+\, c_{\mathrm{A}2}^2 c_{\mathrm{A}6} \sum_{n=1}^{\infty} \gamma^{2n+1} \int_0^{s_0} \mathrm{d}s_1 \cdots \int_0^{s_{2n}} \mathrm{d}s_{2n+1} \sum_{k=2}^{3} M_n^k(\mathbf{a}, b, \lambda, \mathbf{s}_{2n+1}; \mathbf{x}_0) \Bigr\}$$

where $\mathbf{a} := (2,1,1)$, $b = \mathrm{e}^{20\lambda^2}$, $c_{\mathrm{A}2} = c_{\mathrm{A}2}(T, \lambda, \sigma)$ and $c_{\mathrm{A}6} = c_{\mathrm{A}6}(T, 2\lambda, \sigma)$.

Now we need estimates for $L_n(\mathbf{a})$ and $M_n^k(\mathbf{a}, b)$. Recall the definition (7) of the norm $\|\cdot\|$.

**Lemma 24 (Basic estimates)** *For* $\lambda \geq 0$, $n \geq 1$, $T \geq s_0 > \cdots > s_{n+1} > 0$, $\mathbf{x}_0 \in (\mathsf{Z}^2)^3$, $\mathbf{a} \in \mathbf{A}$, $b \geq 1$, *and* $k = 2, 3$,

$$L_n(\mathbf{a}) \;\leq\; \frac{c_{\mathrm{A}2}^2\, c_{24}^{2n-1}}{\prod_{j=2}^{2n} (s_{j-2} - s_j)}\, \mathrm{e}^{2\lambda \|\mathbf{x}_0\|} \sum_{\substack{1 \leq i \leq 6^n/2 \\ k=2,3}} {}^{1}\mathrm{p}_{2b_{i,k}\,(s_0-s_1)}(x_0^1 - x_0^k), \tag{80}$$

$$M_n^k(\mathbf{a}, b) \;\leq\; \frac{c_{24}^{2n}}{\prod_{j=2}^{2n+1} (s_{j-2} - s_j)}\, \mathrm{e}^{2\lambda \|\mathbf{x}_0\|} \sum_{\substack{1 \leq \bar{\imath} \leq 6^n/2 \\ \bar{k}=2,3}} {}^{1}\mathrm{p}_{2b_{\bar{\imath},\bar{k}}\,(s_0-s_1)}(x_0^1 - x_0^{\bar{k}}), \tag{81}$$

*where the* $b_{i,k} \geq 1$ *might depend on* $\mathbf{a}, \lambda, \mathbf{s}_{2n-1}$, *and the* $b_{\bar{\imath},\bar{k}} \geq 1$ *even on* $\mathbf{a}, b, k, \lambda, \mathbf{s}_{2n+1}$. *Moreover,* $c_{A2} = c_{A2}(T, 2\lambda, \sigma) \geq 1$ *and*

$$c_{24} \;=\; c_{24}(T, \lambda, \sigma) \;:=\; \tilde{c}_{24}\, \sigma^{-2} \exp\Bigl[6\, \sigma^2 T \bigl(\mathrm{e}^{80\lambda^2} - 1\bigr)\Bigr], \tag{82}$$

*with the absolute constant* $\tilde{c}_{24} := 64\, \tilde{c}_2$.

The proof of this lemma will be postponed to the appendix (Subsection A.2).

**Remark 25 (Simplified bound)** The proof of (80) will also show that

$$L_n(\mathbf{a}) \;\leq\; c_{\mathrm{A}2}^2\, 6^n\, \mathrm{e}^{2\lambda \|\mathbf{x}_0\|}. \tag{83}$$

To see this, instead of using Lemma 2 (b) to bound $\mathrm{p}_s(x)$, use the trivial bound of 1 throughout the proof and the factors of $\frac{c_{24}}{s_{2j-2}-s_{2j}}$ effectively disappear. This bound was already exploited in (74) but will not be of further use because it does not scale properly. ◇

Inserting these bounds into (79) gives the following result.

**Lemma 26 (4th moment density estimate)** *Suppose the initial state* ${}^1\mathbf{X}_0 = {}^1\mu$ *has density functions satisfying* (76) *for some* $\lambda \geq 0$. *Then, for* $0 < s_0 \leq T$ *and* $\mathbf{x} \in (\mathsf{Z}^2)^3$,

$$\begin{aligned}
&{}^1m^{1122}_{s_0}(x^1, x^2, x^1, x^3) \\
&\leq c_\lambda^4\, c_{\mathrm{A2}}^4\, \mathrm{e}^{2\lambda\|\mathbf{x}\|} \Bigg\{ 1 + \gamma\, c_{\mathrm{A2}}^2\, c_{\mathrm{A6}} \int_0^{s_0} \mathrm{d}s_1 \sum_{k=2}^{3} {}^1\mathrm{p}_{2b\,(s_0 - s_1)}(x^1 - x^k) \\
&\qquad + c_{\mathrm{A2}}^2 \sum_{n=1}^{\infty} \gamma^{2n} c_{24}^{2n-1} \int_0^{s_0} \mathrm{d}s_1 \cdots \int_0^{s_{2n-1}} \mathrm{d}s_{2n} \\
&\qquad\qquad \frac{1}{\prod_{j=2}^{2n} (s_{j-2} - s_j)} \sum_{\substack{1 \leq i \leq 6^n/2 \\ k=2,3}} {}^1\mathrm{p}_{2b_{i,k}\,(s_0 - s_1)}(x^1 - x^k) \\
&\qquad + c_{\mathrm{A2}}^2 c_{\mathrm{A6}} \sum_{n=1}^{\infty} \gamma^{2n+1} c_{24}^{2n} \int_0^{s_0} \mathrm{d}s_1 \cdots \int_0^{s_{2n}} \mathrm{d}s_{2n+1} \\
&\qquad\qquad \frac{1}{\prod_{j=2}^{2n+1} (s_{j-2} - s_j)} \sum_{\substack{1 \leq \bar{\imath} \leq 6^n \\ \bar{k}=2,3}} {}^1\mathrm{p}_{2b_{\bar{\imath},\bar{k}}\,(s_0 - s_1)}(x^1 - x^{\bar{k}}) \Bigg\}
\end{aligned}$$

*where* $b = \mathrm{e}^{20\lambda^2}$, *whereas* $b_{i,k} \geq 1$ *and* $b_{\bar{\imath},\bar{k}} \geq 1$ *might depend on* $\lambda, \mathbf{s}_{2n-1}$ *and* $b, \lambda, \mathbf{s}_{2n+1}$, *respectively. Moreover,* $c_{\mathrm{A2}} = c_{\mathrm{A2}}(T, 2\lambda, \sigma) \geq 1$, $c_{\mathrm{A6}} = c_{\mathrm{A6}}(T, 2\lambda, \sigma) \geq 1$, *and* $c_{24} = c_{24}(T, \lambda, \sigma)$.

## 3.6 A 4th moment estimate on $\mathsf{Z}^2$ under bounded initial densities

For the forthcoming paper [DFM+02] we will need the following more handy version of the previous estimate concerning the special case $\lambda = 0$.

**Corollary 27 (Bounded initial densities)** *Let* $0 < p < 1$. *Assume*

$$\frac{\gamma}{\sigma^2} < \frac{\sin\big[\pi\,(1-p)\big]}{\sqrt{6}\, c_{\mathrm{rw}}\, \pi}, \tag{84}$$

*and that the initial state* ${}^1\mathbf{X}_0 = {}^1\mu$ *has bounded density functions,* $\big\|{}^1X_0^i\big\|_\infty \leq a$, *say,* $i = 1, 2$. *Then for* $s_0 > 0$, *and* $\mathbf{x} = (x^1, x^2, x^3, x^4) \in (\mathsf{Z}^2)^4$,

$${}^1m^{1122}_{s_0}(\mathbf{x}) \leq a^4 \bigg(1 + c_{27}\, s_0^p \int_0^{s_0} \mathrm{d}s_1\; s_1^{-p} \left[ {}^1\mathrm{p}_{2s_1}(x^1 - x^2) + {}^1\mathrm{p}_{2s_1}(x^3 - x^4) \right] \bigg)$$

*and*

$${}^1m^{1112}_{s_0}(x^1, x^2, x^3, x^3) \leq a^4 \bigg(1 + c_{27}\, s_0^p \int_0^{s_0} \mathrm{d}s_1\; s_1^{-p} \sum_{1 \leq j < k \leq 3} {}^1\mathrm{p}_{2s_1}(x^j - x^k) \bigg)$$

*for some constant* $c_{27} = c_{27}(p, \gamma, \sigma)$.

**Proof** *Step* 1°. First we restrict our attention to $\mathbf{x} = (x^1, x^2, x^3)$ in $(\mathsf{Z}^2)^3$. According to Remark A7, in the $\lambda = 0$ case, $c_{\mathrm{A2}} = 1 = c_{\mathrm{A6}}$, hence, by (A40), we can chose $c_{24} = c_2 = c_{\mathrm{rw}}\, \sigma^{-2}$. Moreover, under

$\lambda = 0$, the $b$ in (79) equals one, therefore all the $b$'s in Lemma 24 and its proof are one. Putting these simplifications in the inequality in Lemma 26 yields (with $a$ instead of $c_\lambda$)

$$m_{s_0}^{1122}(x^1,x^2,x^1,x^3) \;\le\; a^4 \left\{ 1 + \gamma \int_0^{s_0} \mathrm{d}s_1 \; g_{s_0-s_1}(\mathbf{x}) \right.$$
$$+ \sum_{n=1}^{\infty} \gamma^{2n}\, c_2^{2n-1} \int_0^{s_0} \mathrm{d}s_1 \cdots \int_0^{s_{2n-1}} \mathrm{d}s_{2n} \, \frac{1}{\prod_{j=2}^{2n} (s_{j-2}-s_j)} \, \frac{6^n}{2}\, g_{s_0-s_1}(\mathbf{x})$$
$$\left. + \sum_{n=1}^{\infty} \gamma^{2n+1}\, c_2^{2n} \int_0^{s_0} \mathrm{d}s_1 \cdots \int_0^{s_{2n}} \mathrm{d}s_{2n+1} \, \frac{1}{\prod_{j=2}^{2n+1} (s_{j-2}-s_j)} \, 6^n \, g_{s_0-s_1}(\mathbf{x}) \right\},$$

where for $\mathbf{x} = (x^1,x^2,x^3) \in (\mathsf{Z}^2)^3$ we put

$$g_s(\mathbf{x}) \;:=\; \sum_{k=2}^{3} \mathrm{p}_{2s}(x^1-x^k), \qquad s>0. \tag{85}$$

Applying the Feynman integral estimate of Lemma A8 with $n$ replaced by $2n$ and $2n+1$, respectively, we obtain

$$m_{s_0}^{1122}(x^1,x^2,x^1,x^3) \;\le\; a^4 \left\{ 1 + \gamma \int_0^{s_0} \mathrm{d}s_1 \; g_{s_0-s_1}(\mathbf{x}) \right. \tag{86}$$
$$+ \frac{1}{p} \sum_{n=1}^{\infty} \frac{6^n}{2}\, \gamma^{2n}\, c_2^{2n-1} \int_0^{s_0} \mathrm{d}s_1 \; c_{\mathrm{A8}}^{2n-2} \left( \frac{s_0}{s_0-s_1} \right)^{p} g_{s_0-s_1}(\mathbf{x})$$
$$\left. + \frac{1}{p} \sum_{n=1}^{\infty} 6^n\, \gamma^{2n+1} c_2^{2n} \int_0^{s_0} \mathrm{d}s_1 \; c_{\mathrm{A8}}^{2n-1} \left( \frac{s_0}{s_0-s_1} \right)^{p} g_{s_0-s_1}(\mathbf{x}) \right\}.$$

Changing variable in the integration (to interchange $s_0 - s_1$ and $s_1$), and recalling that with (84) we assumed that $\sqrt{6}\,\gamma\, c_2\, c_{\mathrm{A8}} < 1$, we may sum the series (adding the initial term in the second case) to obtain the estimate

$$m_{s_0}^{1122}(x^1,x^2,x^1,x^3) \;\le\; a^4 \left\{ 1 + \gamma \int_0^{s_0} \mathrm{d}s_1 \; g_{s_1}(\mathbf{x}) \right. \tag{87}$$
$$\left. + \frac{1}{p}\, \gamma \,[3\,\gamma\, c_2 + c_{\mathrm{A8}}^{-1}] \, \frac{1}{1 - 6\,\gamma^2\, c_2^2\, c_{\mathrm{A8}}^2} \int_0^{s_0} \mathrm{d}s_1 \; (s_0/s_1)^p \, g_{s_1}(\mathbf{x}) \right\}.$$

Hence,

$$m_{s_0}^{1122}(x^1,x^2,x^1,x^3) \;\le\; a^4 \left\{ 1 + c_{(88)} \int_0^{s_0} \mathrm{d}s_1 \; (s_0/s_1)^p \, g_{s_1}(\mathbf{x}) \right\} \tag{88}$$

for some constant $c_{(88)} = c_{(88)}(p,\gamma,\sigma)$.

*Step* $2^\circ$. Next we want to substitute this estimate into (69) to derive the second of the claimed inequalities. For this purpose, for $\mathbf{x} = (x^1,x^2,x^3)$ and $\mathbf{y} = (y^1,y^2,y^3)$ in $(\mathsf{Z}^2)^3$ and $r>0$, set

$$\left\{ \begin{aligned} I_r(\mathbf{x},\mathbf{y}) \;:=\; \Big[ & \mathrm{p}_r(x^1-y^1)\, \mathrm{p}_r(x^2-y^1)\, \mathrm{p}_r(x^3-y^2) \\ & + \mathrm{p}_r(x^1-y^1)\, \mathrm{p}_r(x^2-y^2)\, \mathrm{p}_r(x^3-y^1) \\ & + \mathrm{p}_r(x^1-y^2)\, \mathrm{p}_r(x^2-y^1)\, \mathrm{p}_r(x^3-y^1) \Big]\, \mathrm{p}_r(x^3-y^3), \end{aligned} \right. \tag{89}$$

to obtain from (69) and (88),

$$m_{s_0}^{1112}(x^1,x^2,x^3,x^3) \;\le\; a^4 + \gamma\int_0^{s_0} \mathrm{d}s_1 \sum_{\mathbf{y}\in(\mathsf{Z}^2)^3} I_{s_0-s_1}(\mathbf{x},\mathbf{y}) \\ a^4\Big\{1 + c_{(88)}\int_0^{s_1}\mathrm{d}s_2\;(s_1/s_2)^p\, g_{s_2}(\mathbf{y})\Big\}. \tag{90}$$

First we calculate two sums over $\mathbf{y}$. Trivially,

$$\sum_{\mathbf{y}\in(\mathsf{Z}^2)^3} I_r(\mathbf{x},\mathbf{y}) \;=\; \sum_{1\le j<k\le 3} \mathrm{p}_{2r}(x^j-x^k) \;=:\; h_r(\mathbf{x}), \tag{91}$$

whereas

$$\begin{aligned} &\sum_{\mathbf{y}\in(\mathsf{Z}^2)^3} I_r(\mathbf{x},\mathbf{y})\,\mathrm{p}_{2s_2}(y^1-y^2) && (92)\\ &= \sum_{y^1\in\mathsf{Z}^2}\Big[\mathrm{p}_r(x^1-y^1)\,\mathrm{p}_r(x^2-y^1)\,\mathrm{p}_{r+2s_2}(x^3-y^1)\\ &\qquad +\;\mathrm{p}_r(x^1-y^1)\,\mathrm{p}_{r+2s_2}(x^2-y^1)\,\mathrm{p}_r(x^3-y^1)\\ &\qquad +\;\mathrm{p}_{r+2s_2}(x^1-y^1)\,\mathrm{p}_r(x^2-y^1)\,\mathrm{p}_r(x^3-y^1)\Big]\\ &\le\; c_2\,(r+2s_2)^{-1}\,h_r(\mathbf{x}), \end{aligned}$$

and a similar calculation gives

$$\sum_{\mathbf{y}\in(\mathsf{Z}^2)^3} I_r(\mathbf{x},\mathbf{y})\,\mathrm{p}_{2s_2}(y^1-y^3) \;\le\; c_2\,\frac{h_r(\mathbf{x})}{r+2s_2}\,. \tag{93}$$

Recalling the definition (85) of $g_{s_2}(\mathbf{y})$, put these three bounds into (90) to conclude

$$m_{s_0}^{1112}(x^1,x^2,x^3,x^3) \;\le\; a^4 + a^4\,\gamma\int_0^{s_0}\mathrm{d}s_1\;h_{s_0-s_1}(\mathbf{x}) \\ +\,a^4\,c_{(94)}\int_0^{s_0}\mathrm{d}s_1\int_0^{s_1}\mathrm{d}s_2\;(s_1/s_2)^p\,\frac{h_{s_0-s_1}(\mathbf{x})}{s_0-s_1+2s_2} \tag{94}$$

for some constant $c_{(94)}=c_{(94)}(p,\gamma,\sigma)$. The substitution $r := \Big(\frac{2s_2}{s_0-s_1}\Big)^{1-p}$ gives

$$\begin{aligned} \int_0^{s_1}\mathrm{d}s_2\;\frac{s_1^p}{s_2^p\,(s_0-s_1+2s_2)} \;&\le\; \frac{1}{2(1-p)}\Big(\frac{2s_0}{s_0-s_1}\Big)^p\int_0^\infty \mathrm{d}r\;\frac{1}{1+r^{\frac{1}{1-p}}} && (95)\\ &=\; c_{(95)}\,s_0^p\,(s_0-s_1)^{-p} \end{aligned}$$

with a constant $c_{(95)}=c_{(95)}(p)$. Consequently,

$$\begin{aligned} &m_{s_0}^{1112}(x^1,x^2,x^3,x^3)\\ &\le\; a^4\;+\;a^4\,\gamma\int_0^{s_0}\mathrm{d}s_1\;h_{s_1}(\mathbf{x})\;+\;a^4\,c_{(94)}\,s_0^p\int_0^{s_0}\mathrm{d}s_1\;c_{(95)}\,(s_0-s_1)^{-p}\,h_{s_0-s_1}(\mathbf{x})\\ &\le\; a^4\Big(1+c_{(96)}\,s_0^p\int_0^{s_0}\mathrm{d}s_1\;s_1^{-p}\,h_{s_1}(\mathbf{x})\Big) && (96) \end{aligned}$$

with a constant $c_{(96)}=c_{(96)}(p,\gamma,\sigma)$. This gives the second estimate claimed in the corollary [recall the definition (91) of $h_{s_1}(\mathbf{x})$].

*Step* 3°. It remains to prove the first estimate claimed in Corollary 27. According to (66), for $\mathbf{x} = (x^1, x^2, x^3, x^4) \in (\mathsf{Z}^2)^4$,

$$\begin{aligned} m_{s_0}^{1122}(\mathbf{x}) \;&=\; \mathbf{S}_{s_0} m_0^{1122}(\mathbf{x}) \,+\, \gamma \int_0^{s_0} \mathrm{d}s_1 \sum_{\mathbf{y} \in (\mathsf{Z}^2)^3} \\ &\Big[\mathrm{p}_{s_0-s_1}(x^1-y^3)\,\mathrm{p}_{s_0-s_1}(x^2-y^3)\,\mathrm{p}_{s_0-s_1}(x^3-y^2)\,\mathrm{p}_{s_0-s_1}(x^4-y^1) \\ &\quad+ \mathrm{p}_{s_0-s_1}(x^1-y^1)\,\mathrm{p}_{s_0-s_1}(x^2-y^2)\,\mathrm{p}_{s_0-s_1}(x^3-y^3)\,\mathrm{p}_{s_0-s_1}(x^4-y^3)\Big] \\ &\qquad m_{s_1}^{1112}(y^1, y^2, y^3, y^3). \end{aligned} \tag{97}$$

Substituting (96) and using the definition (91) of $h_{s_2}(\mathbf{y})$ gives

$$\begin{aligned} m_{s_0}^{1122}(\mathbf{x}) \;&\le\; a^4 + \gamma \int_0^{s_0} \mathrm{d}s_1 \sum_{\mathbf{y} \in (\mathsf{Z}^2)^3} \\ &\Big[\mathrm{p}_{s_0-s_1}(x^1-y^3)\,\mathrm{p}_{s_0-s_1}(x^2-y^3)\,\mathrm{p}_{s_0-s_1}(x^3-y^2)\,\mathrm{p}_{s_0-s_1}(x^4-y^1) \\ &\quad+ \mathrm{p}_{s_0-s_1}(x^1-y^1)\,\mathrm{p}_{s_0-s_1}(x^2-y^2)\,\mathrm{p}_{s_0-s_1}(x^3-y^3)\,\mathrm{p}_{s_0-s_1}(x^4-y^3)\Big] \\ &a^4 \bigg(1 + c_{(96)}\, s_1^p \int_0^{s_1} \mathrm{d}s_2\; s_2^{-p} \\ &\qquad \big[\mathrm{p}_{2s_2}(y^1-y^2) + \mathrm{p}_{2s_2}(y^1-y^3) + \mathrm{p}_{2s_2}(y^2-y^3)\big]\bigg). \end{aligned} \tag{98}$$

By Chapman-Kolmogorov we have

$$\begin{aligned} &\sum_{\mathbf{y} \in (\mathsf{Z}^2)^3} \mathrm{p}_r(x^1-y^3)\,\mathrm{p}_r(x^2-y^3)\,\mathrm{p}_r(x^3-y^2)\,\mathrm{p}_r(x^4-y^1) \\ &\qquad \big[\mathrm{p}_{2s_2}(y^1-y^2) + \mathrm{p}_{2s_2}(y^1-y^3) + \mathrm{p}_{2s_2}(y^2-y^3)\big] \\ &= \sum_{y^3 \in \mathsf{Z}^2} \mathrm{p}_r(x^1-y^3)\,\mathrm{p}_r(x^2-y^3) \\ &\qquad \big[\mathrm{p}_{2r+2s_2}(x^3-x^4) + \mathrm{p}_{r+2s_2}(x^4-y^3) + \mathrm{p}_{r+2s_2}(x^3-y^3)\big]. \end{aligned} \tag{99}$$

According to Lemma 2 (b),

$$\mathrm{p}_{2r+2s_2}(x^3-x^4) \;\le\; c_2\, \frac{1}{2r+2s_2}\,, \tag{100}$$

whereas for the second and third term in the final bracket of (99) one gets twice this estimate. Thus, again by Chapman-Kolmogorov, (99) can be bounded by

$$5\, c_2\, \frac{1}{2r+2s_2}\, \mathrm{p}_{2r}(x^1-x^2). \tag{101}$$

Use this estimate, a symmetrical counterpart, and the fact that (99) without the square bracket expressions equals $\mathrm{p}_{2r}(x^1 - x^2)$, to conclude from (98) that

$$m^{1122}_{s_0}(\mathbf{x}) \le a^4 + \gamma \int_0^{s_0} \mathrm{d}s_1 \; \big[\mathrm{p}_{2(s_0-s_1)}(x^1-x^2) + \mathrm{p}_{2(s_0-s_1)}(x^3-x^4)\big] \\ a^4 \left(1 + c_{(96)}\, 5\, c_2\, s_1^p \int_0^{s_1} \mathrm{d}s_2 \; \frac{1}{s_2^p\, (2(s_0-s_1)+2s_2)}\right)$$

$$\le a^4 + \gamma \int_0^{s_0} \mathrm{d}s_1 \; \big[\mathrm{p}_{2(s_0-s_1)}(x^1-x^2) + \mathrm{p}_{2(s_0-s_1)}(x^3-x^4)\big] \\ a^4 \left(1 + c_{(96)}\, 5\, c_2\, s_1^p\, c_{(95)} \; \big(2(s_0-s_1)\big)^{-p}\right),$$

where in the last step we used (95). Consequently,

$$m^{1122}_{s_0}(\mathbf{x}) \; \le \; a^4 \Big(1 + c_{(102)}\, s_0^p \int_0^{s_0} \mathrm{d}s_1 \; s_1^{-p} \, \big[\mathrm{p}_{2s_1}(x^1-x^2) + \mathrm{p}_{2s_1}(x^3-x^4)\big]\Big), \tag{102}$$

finishing the proof. □

### 3.7 An estimate for the 2nd moment of the collision measure on $\mathsf{Z}^2$

For the desired tightness properties, we will restrict our consideration to a finite time interval $[0,T]$. So let us fix now a $T>0$.

Later we will need estimates for certain moments in the case of tempered initial density functions and we will provide them for $\gamma/\sigma^2$ not too large. More precisely, we will impose the following hypothesis.

**Hypothesis 28 (Small collision rate)** Assume that

$$0 \; < \; \gamma \; < \; \frac{\sigma^2}{\sqrt{6}\,\pi\,\tilde{c}_{24}} \quad =: \; \gamma_\sigma\,, \tag{103}$$

with the absolute constant $\tilde{c}_{24} = 64\, c_{\mathrm{rw}}$ from Lemma 24. ◇

Later we will consider initial density functions ${}^1X_0^i = {}^1\mu^i$ belonging to ${}^1\mathcal{B}_{\mathrm{tem}} \subset {}^1\mathcal{B}_{-\lambda}$, $\lambda > 0$. Actually, under Hypothesis 28, we will restrict ourselves to those $\lambda \in [0,1]$ satisfying

$$\gamma \, \exp\!\Big[6\,\sigma^2 T \big(\mathrm{e}^{80\lambda^2} - 1\big)\Big] \; < \; \gamma_\sigma\,. \tag{104}$$

We will use Lemma 26 to derive the following statement. Recall that we fixed $T>0$.

**Proposition 29 (2nd moment of collision measure)** *Assume that both $\gamma > 0$ and $\lambda \in [0,1]$ are small as in Hypothesis 28 and condition (104), respectively. Suppose that ${}^1\mathbf{X}_0 = {}^1\mu$ has density functions satisfying (76) (for the present $\lambda$). Then, for $0 < t \le T$ and non-negative test functions $\varphi$,*

$$P_{{}^1\mu}\Big[\sum_{x\in\mathsf{Z}^2} {}^1X_t^1(x)\, {}^1X_t^2(x)\, \varphi(x)\Big]^2 \; \le \; c^0_{29} \left[\Big(\sum_{x\in\mathsf{Z}^2} \varphi(x)\, \mathrm{e}^{4\lambda|x|}\Big)^2 + \; c_{29}\, t \sum_{x\in\mathsf{Z}^2} \varphi^2(x)\, \mathrm{e}^{8\lambda|x|}\right]$$

*where* $c^0_{29} \; = \; c^0_{29}(T,\lambda,\sigma) \; := \; 2\, c_\lambda^4\, c_{\mathrm{A}2}^6(T,2\lambda,\sigma)\, c_{\mathrm{A}6}(T,4\lambda,\sigma)$ *and*

$$0 \; < \; c_{29} \; = \; c_{29}(T,\lambda,\sigma,\gamma) \; := \; \gamma + 24\,\gamma^2\, c_{24}\, \frac{(1 + 2\,\pi\,\gamma\, c_{24})}{1 - 6\,\pi^2\,\gamma^2\, c_{24}^2} \; < \; \infty \tag{105}$$

*with* $c_{24} = c_{24}(T,\lambda,\sigma)$ *from* (82).

**Proof** The left hand moment expression in the claim (with $t = s_0$) equals

$$\sum_{x^1, x^2 \in \mathsf{Z}^2} \varphi(x^1)\,\varphi(x^2)\, m_{s_0}^{1122}(x^1, x^2, x^1, x^2). \tag{106}$$

By Lemma 26 with $x^2 = x^3$, we bound the latter sum by

$$\begin{aligned} c_{29}^0 \sum_{\mathbf{x}=(x^1,x^2)\in(\mathsf{Z}^2)^2} \varphi(x^1)\,\varphi(x^2)\,\mathrm{e}^{4\lambda\|\mathbf{x}\|} \Bigg\{ 1 \,+\, \gamma \int_0^{s_0} \mathrm{d}s_1\; \mathrm{p}_{2b\,(s_0-s_1)}(x^1-x^2) \\ +\sum_{n=1}^{\infty} \gamma^{2n} c_{24}^{2n-1} \int_0^{s_0} \mathrm{d}s_1 \cdots \int_0^{s_{2n-1}} \mathrm{d}s_{2n} \\ \frac{1}{\prod_{j=2}^{2n} (s_{j-2}-s_j)} \sum_{\substack{1\le i\le 6^n/2 \\ k=2,3}} {}^1\mathrm{p}_{2b_{i,k}\,(s_0-s_1)}(x^1-x^2) \\ +\sum_{n=1}^{\infty} \gamma^{2n+1} c_{24}^{2n} \int_0^{s_0} \mathrm{d}s_1 \cdots \int_0^{s_{2n}} \mathrm{d}s_{2n+1} \\ \frac{1}{\prod_{j=2}^{2n+1} (s_{j-2}-s_j)} \sum_{\substack{1\le \bar{\imath}\le 6^n \\ \bar{k}=2,3}} {}^1\mathrm{p}_{2b_{\bar{\imath},\bar{k}}\,(s_0-s_1)}(x^1-x^2) \Bigg\} \end{aligned}$$

(note that we passed from $\mathbf{x} \in (\mathsf{Z}^2)^3$ to $\mathbf{x} \in (\mathsf{Z}^2)^2$). We write the right hand side as $c_{29}^0\,(\mathsf{S}_1 + \cdots + \mathsf{S}_4)$ in the obvious correspondence. Trivially,

$$\mathsf{S}_1 \;=\; \Bigg(\sum_{x\in\mathsf{Z}^2} \varphi(x)\,\mathrm{e}^{4\lambda|x|}\Bigg)^2 \tag{107}$$

(recall (7) which now reads as $\|\mathbf{x}\| = |x^1| + |x^2|$) giving the first term in the claim. By Chapman-Kolmogorov and a change of variable,

$$\begin{aligned} \mathsf{S}_2 \;&=\; \gamma \int_0^{s_0} \mathrm{d}s_1 \sum_{y\in\mathsf{Z}^2} \sum_{x^1,x^2\in\mathsf{Z}^2} \varphi(x^1)\,\varphi(x^2)\,\mathrm{e}^{4\lambda\|\mathbf{x}\|}\,\mathrm{p}_{bs_1}(x^1-y)\,\mathrm{p}_{bs_1}(x^2-y) \\ &=\; \gamma \int_0^{s_0} \mathrm{d}s_1 \sum_{y\in\mathsf{Z}^2} \Bigg(\sum_{x\in\mathsf{Z}^2} \varphi(x)\,\mathrm{e}^{4\lambda|x|}\,\mathrm{p}_{bs_1}(x-y)\Bigg)^2. \end{aligned}$$

By Jensen and $L^1$–invariance, we may bound the latter expression by

$$\gamma \int_0^{s_0} \mathrm{d}s_1 \sum_{y\in\mathsf{Z}^2} \sum_{x\in\mathsf{Z}^2} \varphi^2(x)\,\mathrm{e}^{8\lambda|x|}\,\mathrm{p}_{bs_1}(x-y) \;=\; \gamma\, s_0 \sum_{x\in\mathsf{Z}^2} \varphi^2(x)\,\mathrm{e}^{8\lambda|x|} \tag{108}$$

which gives rise to the second term in the required upper bound. Treating $\mathsf{S}_3$ this way, but without performing the integral in $s_1$, we get

$$\mathsf{S}_3 \;\le\; \sum_{n=1}^{\infty} \gamma^{2n} c_{24}^{2n-1} \int_0^{s_0} \mathrm{d}s_1 \cdots \int_0^{s_{2n-1}} \mathrm{d}s_{2n}\; \frac{1}{\prod_{j=2}^{2n} (s_{j-2}-s_j)}\; 6^n \sum_{x\in\mathsf{Z}^2} \varphi^2(x)\,\mathrm{e}^{8\lambda|x|}.$$

By the Feynman integral Lemma A8 with $2n$ instead of $n$ and with $p = 1/2$, this in turn is

$$\le\; 2 \sum_{x\in\mathsf{Z}^2} \varphi^2(x)\,\mathrm{e}^{8\lambda|x|} \sum_{n=1}^{\infty} \gamma^{2n} c_{24}^{2n-1} 6^n\, \pi^{2n-2} \int_0^{s_0} \mathrm{d}s_1\, \sqrt{\frac{s_0}{s_0-s_1}}\,. \tag{109}$$

But $\int_0^t \mathrm{d}s\,\sqrt{t/s} \;=\; 2t,$ and

$$\sum_{n=1}^{\infty} \gamma^{2n} c_{24}^{2n-1} 6^n\, \pi^{2n-2} \;=\; \frac{6\,\gamma^2\, c_{24}}{1-6\,\pi^2\,\gamma^2\, c_{24}^2} \tag{110}$$

since $6\,\pi^2\,\gamma^2\, c_{24}^2 < 1$ by Hypothesis 28 and assumption (104) on $\gamma$ and $\lambda$, respectively. So

$$\mathsf{S}_3 \;\le\; \frac{24\, s_0\,\gamma^2\, c_{24}}{1-6\,\pi^2\,\gamma^2\, c_{24}^2} \sum_{x\in\mathsf{Z}^2} \varphi^2(x)\,\mathrm{e}^{8\lambda|x|}. \tag{111}$$

Finally,

$$\mathsf{S}_4 \;\le\; \sum_{n=1}^{\infty} \gamma^{2n+1} c_{24}^{2n} \int_0^{s_0} \mathrm{d}s_1 \cdots \int_0^{s_{2n}} \mathrm{d}s_{2n+1}\, \frac{1}{\prod_{j=2}^{2n+1}(s_{j-2}-s_j)}\; 2{\cdot}6^n \sum_{x\in\mathsf{Z}^2} \varphi^2(x)\,\mathrm{e}^{8\lambda|x|}.$$

Lemma A8 applied to $K_{2n+1}(s_0,s_1)$ and $p=1/2$ gives that this is

$$\begin{aligned}
&\le\; 4 \sum_{x\in\mathsf{Z}^2} \varphi^2(x)\,\mathrm{e}^{8\lambda|x|} \sum_{n=1}^{\infty} \gamma^{2n+1}\, c_{24}^{2n}\, 6^n\, \pi^{2n-1} \int_0^{s_0} \mathrm{d}s_1 \sqrt{\frac{s_0}{s_0-s_1}} \\
&\le\; \frac{48\, s_0\, \pi\,\gamma^3\, c_{24}^2}{1-6\,\pi^2\,\gamma^2\, c_{24}^2} \sum_{x\in\mathsf{Z}^2} \varphi^2(x)\,\mathrm{e}^{8\lambda|x|}.
\end{aligned} \tag{112}$$

Combining the estimates (111) and (112) for $\mathsf{S}_3$ and $\mathsf{S}_4$, respectively, gives rise to the second term of $c_{29}$, and we are done. □

### 3.8 Uniform bound for second moment of collision measure on $\varepsilon\mathsf{Z}^2$

Recall that the mutually catalytic branching processes ${}^{\varepsilon}\mathbf{X} = \bigl({}^{\varepsilon}\!X^1,\, {}^{\varepsilon}\!X^2\bigr)$ in $\varepsilon\mathsf{Z}^2$, $0<\varepsilon\le 1$, introduced before Theorem 6, can be defined through ${}^{1}\mathbf{X}$ via their densities with respect to ${}^{\varepsilon}\ell$ [defined in (24)]:

$${}^{\varepsilon}\!X_t^i(x) \;=\; {}^{1}\!X_{\varepsilon^{-2}t}^i(\varepsilon^{-1}x), \qquad t\ge 0, \quad x\in\varepsilon\mathsf{Z}^2, \quad i=1,2. \tag{113}$$

That is, the ${}^{\varepsilon}\mathcal{M}_{\mathrm{tem}}^2$–valued process ${}^{\varepsilon}\mathbf{X}$ satisfies the martingale problem $(\mathbf{MP})_{{}^{\varepsilon}\mu}^{\sigma,\gamma,\varepsilon}$ in (35) if and only if the ${}^{1}\mathcal{M}_{\mathrm{tem}}^2$–valued ${}^{1}\mathbf{X}$ satisfies $(\mathbf{MP})_{{}^{1}\mu}^{\sigma,\gamma,1}$, where (113) also determines the relationship between ${}^{\varepsilon}\mu$ and ${}^{1}\mu$. Let

$${}^{\varepsilon}m_t^{\mathbf{i}}(\mathbf{x}) := P_{{}^{\varepsilon}\mu}\left[\prod_{j=1}^{n} {}^{\varepsilon}\!X_t^{i_j}(x_j)\right]$$

be the corresponding moment densities.

Recall that we fixed $T>0$. Instead of imposing (104) we will consider now $\lambda\in[0,1]$ satisfying

$$\gamma\,\exp\bigl[480\,\sigma^2\,T\,\lambda^2\,\mathrm{e}^{80}\bigr] \;<\; \gamma_\sigma \tag{114}$$

(with $\gamma_\sigma$ from Hypothesis 28). The following statement is crucial for our development.

**Corollary 30 (Scaled 2nd moment of collision measure)** *Assume that $\gamma>0$ and $\lambda\in[0,1]$ are small as in Hypothesis 28 and assumption (114), respectively. Suppose that the (deterministic) initial densities ${}^{\varepsilon}\mathbf{X}_0 \;=\; {}^{\varepsilon}\mu$ satisfy*

$$\left\{\begin{array}{l} {}^{\varepsilon}\mu^i(x) \;\le\; c_\lambda\,\mathrm{e}^{\lambda|x|}, \qquad x\in\varepsilon\mathsf{Z}^2, \quad i=1,2, \quad \varepsilon\in(0,1], \\ \textit{for some constant } c_\lambda \ (\textit{independent of } \varepsilon). \end{array}\right. \tag{115}$$

*Then there is a constant* $c_{30} = c_{30}(T,\lambda,\sigma,\gamma)$ *independent of* $\varepsilon$, *such that for* $0<t\le T$ *and non-negative test functions* $\varphi$ *on* $\varepsilon\mathsf{Z}^2$,

$$P_{^\varepsilon\mu}\Big[\int_{\varepsilon\mathsf{Z}^2} {}^\varepsilon\ell(\mathrm{d}y)\,{}^\varepsilon X_t^1(y)\,{}^\varepsilon X_t^2(y)\,\varphi(y)\Big]^2 \;\le\; c_{30}\int_{\varepsilon\mathsf{Z}^2} {}^\varepsilon\ell(\mathrm{d}y)\,\varphi^2(y)\,\mathrm{e}^{10\lambda|y|}. \tag{116}$$

**Proof** By definition, the left hand side of (116) can be written as

$$P_{^1\mu}\Big[\sum_{x\in\mathsf{Z}^2} {}^1X^1_{\varepsilon^{-2}t}(x)\,{}^1X^2_{\varepsilon^{-2}t}(x)\,\varepsilon^2\varphi(\varepsilon x)\Big]^2, \tag{117}$$

where ${}^1\mu^i(x) = {}^\varepsilon\mu^i(\varepsilon x) \le c_\lambda\,\mathrm{e}^{\varepsilon\lambda|x|}$, by (115). Now we want to apply Proposition 29 with $T$, $\lambda$, and $\varphi$ replaced by $\varepsilon^{-2}T$, $\varepsilon\lambda$ and $\varepsilon^2\varphi(\varepsilon\,\cdot)$, respectively. This is actually possible, since

$$\mathrm{e}^{c\varepsilon}-1 \;\le\; c\,\varepsilon\,\mathrm{e}^c, \qquad 0<\varepsilon\le 1, \quad c>0, \tag{118}$$

hence, by (114) and since $\lambda\le 1$,

$$\gamma\,\exp\Big[6\,\sigma^2\,\varepsilon^{-2}T\big(\mathrm{e}^{80\,\varepsilon^2\lambda^2}-1\big)\Big] \;\le\; \gamma\,\exp\big[480\,\sigma^2\,T\,\lambda^2\mathrm{e}^{80}\big] \;<\; \gamma_\sigma\,. \tag{119}$$

Thus, Proposition 29 gives the following upper bound for (117):

$$c_{29}^0\bigg[\Big(\sum_{x\in\mathsf{Z}^2}\varepsilon^2\,\varphi(\varepsilon x)\,\mathrm{e}^{4\varepsilon\lambda|x|}\Big)^2 + c_{29}\,\varepsilon^{-2}\,T\sum_{x\in\mathsf{Z}^2}\varepsilon^4\,\varphi^2(\varepsilon x)\,\mathrm{e}^{8\varepsilon\lambda|x|}\bigg], \tag{120}$$

with $c_{29}^0 = c_{29}^0\left(\varepsilon^{-2}T,\varepsilon\lambda,\sigma\right)$ and $c_{29} = c_{29}\left(\varepsilon^{-2}T,\varepsilon\lambda,\sigma,\gamma\right)$. Concerning their $\varepsilon$–dependence, these constants depend only on terms of the form

$$c^1\,\exp\Big[c^2\,\varepsilon^{-2}T\big(\mathrm{e}^{c^3\varepsilon^2\lambda^2}-1\big)\Big] \tag{121}$$

with constants $c^1,c^2,c^3$ independent of $\varepsilon$. Using again the trivial estimate (118), the latter expression can be bounded from above by

$$c^1\,\exp\Big[c^2\,T\,c^3\,\lambda^2\,\mathrm{e}^{c^3\lambda^2}\Big] \tag{122}$$

which is independent of $\varepsilon$. Moreover, the second term in (120) is of the form of the integral on the right hand side of (116) [except the enlargement of the constant 8 to 10]. Finally, using Cauchy-Schwarz, the squared sum in (120) can be bounded from above by

$$\sum_{x\in\mathsf{Z}^2}\varepsilon^4\,\varphi^2(\varepsilon x)\,\mathrm{e}^{10\,\varepsilon\lambda|x|}\sum_{x\in\mathsf{Z}^2}\mathrm{e}^{-2\,\varepsilon\lambda|x|}, \tag{123}$$

where the second sum equals $c\,(\varepsilon\lambda)^{-2}$. Combining the arguments above gives (116), completing the proof. □

The following bounds on the scaled fourth moment densities will be used in [DFM+02] and follow directly from Corollary 27.

**Corollary 31 (Scaled moment density bounds)** *Let* $0<p<1$. *Assume*

$$\frac{\gamma}{\sigma^2} \;<\; \frac{\sin\big[\pi\,(1-p)\big]}{\sqrt{6}\,c_{\mathrm{rw}}\,\pi}\,, \tag{124}$$

*and that the initial state* ${}^{\varepsilon}\mathbf{X}_0 = {}^{\varepsilon}\mu$ *has bounded density functions,* $\left\| {}^{\varepsilon}X_0^i \right\|_\infty \le a$, *say,* $i = 1, 2$. *Then for* $\varepsilon > 0$, $s_0 \ge 0$, *and* $\mathbf{x} = (x^1, x^2, x^3, x^4) \in (\varepsilon \mathsf{Z}^2)^4$,

$$ {}^{\varepsilon}m_{s_0}^{1122}(\mathbf{x}) \;\le\; a^4 \left( 1 + c_{27}\, s_0^p \int_0^{s_0} \mathrm{d}s_1 \; s_1^{-p} \left[ {}^{\varepsilon}\mathrm{p}_{2s_1}(x^1 - x^2) + {}^{\varepsilon}\mathrm{p}_{2s_1}(x^3 - x^4) \right] \right) $$

*and*

$$ {}^{\varepsilon}m_{s_0}^{1112}(x^1, x^2, x^3, x^3) \;\le\; a^4 \left( 1 + c_{27}\, s_0^p \int_0^{s_0} \mathrm{d}s_1 \; s_1^{-p} \sum_{1 \le j < k \le 3} {}^{\varepsilon}\mathrm{p}_{2s_1}(x^j - x^k) \right) $$

*for the constant* $c_{27} = c_{27}(p, \gamma, \sigma)$.

# 4 Construction of X

In this section, the approximation Theorem 6, hence Theorem 4 (a) will be proved which states the existence of a mutually catalytic branching process $\mathbf{X}$ on $\mathsf{R}^2$, satisfying the martingale problem $(\mathbf{MP})_\mu^{\sigma,\gamma}$.

## 4.1 Tightness on path space

The purpose of this subsection is to derive some uniform moment estimates, which imply the tightness on path space (Proposition 37 below).

It is convenient to introduce the following hypothesis.

**Hypothesis 32 (Uniformly tempered initial densities)** Assume that the initial densities ${}^{\varepsilon}\mathbf{X}_0 = {}^{\varepsilon}\mu$ satisfy the uniform domination condition (115) for all $\lambda > 0$. $\diamond$

Recall that measures on $\varepsilon\mathsf{Z}^2$ will also be considered as (discrete) measures on $\mathsf{R}^2$.

**Lemma 33 (Uniform first absolute moments)** *Under Hypothesis* 32, *for each* $T > 0$ *and* $\varphi \in \mathcal{C}_{\exp}(\mathsf{R}^2)$,

$$ \sup_{0<\varepsilon\le 1} P_{{}^{\varepsilon}\mu} \sup_{0\le t\le T} \left| \langle {}^{\varepsilon}X_t^i, \varphi \rangle \right| \;<\; \infty, \qquad i = 1, 2. \tag{125} $$

**Proof** Fix $T > 0$ and $i = 1, 2$. We may assume that $\varphi \in \mathcal{C}_{\sqrt{2}\lambda}(\mathsf{R}^2)$, $\lambda > 0$. Since $|\varphi| \le |\varphi|_{\sqrt{2}\lambda}\, \phi_{\sqrt{2}\lambda}$ [recall notation (15)], and using the first inequality in (13) in the case $n = 0$, it suffices to verify the claim (125) with $\varphi$ replaced by $\tilde{\phi}_\lambda$. By the martingale problem $(\mathbf{MP})_\mu^{\sigma,\gamma,\varepsilon}$ in (35),

$$ \begin{aligned} P_{{}^{\varepsilon}\mu} \sup_{0\le t\le T} \langle {}^{\varepsilon}X_t^i, \tilde{\phi}_\lambda \rangle \;&\le\; c\, P_{{}^{\varepsilon}\mu} \sup_{0\le t\le T} \left| {}^{\varepsilon}M_t^i(\tilde{\phi}_\lambda) \right| \;+\; c\, \langle {}^{\varepsilon}\mu^i, \tilde{\phi}_\lambda \rangle \\ &\quad + \; c \int_0^T \mathrm{d}s \; \Big\langle {}^{\varepsilon}S_s\, {}^{\varepsilon}\mu^i, \; \frac{\sigma^2}{2} \left| {}^{\varepsilon}\Delta \tilde{\phi}_\lambda \right| \Big\rangle, \end{aligned} \tag{126} $$

where in the last term we have used the expectation formula (50). Write $c\,(\mathsf{S}_1 + \mathsf{S}_2 + \mathsf{S}_3)$ for the right hand side (in the obvious correspondence). For $\mathsf{S}_2$ we use (115) with $\lambda$ replaced by a $\lambda' \in (0, \lambda)$, and the upper estimate of (13) in the case $n = 0$ to get a finite bound, independent of $\varepsilon$.

Next, using the mean value theorem (twice), and then the second part of (13) in the case $n = 2$, there is a constant $\overline{c}_\lambda$ independent of $\varepsilon$ such that

$$ \left| {}^{\varepsilon}\Delta \tilde{\phi}_\lambda(x) \right| \;\le\; \overline{c}_\lambda\, \mathrm{e}^{-\lambda |x|}, \qquad x \in \varepsilon\mathsf{Z}^2. \tag{127} $$

On the other hand, due to (115) with $\lambda$ replaced by $\lambda' \in (0,\lambda)$,

$$ {}^{\varepsilon}\mu \;\le\; c_{\lambda'}\,\phi_{-\lambda'}\,, \tag{128} $$

whereas by Corollary A3

$$ {}^{\varepsilon}S_s\phi_{-\lambda'} \;\le\; c_{\mathrm{A3}}\,\phi_{-\lambda'} \tag{129} $$

with $c_{\mathrm{A3}} = c_{\mathrm{A3}}(T,\lambda',\sigma)$. Together these give

$$ {}^{\varepsilon}S_s\,{}^{\varepsilon}\mu^i \;\le\; c\,\phi_{-\lambda'}\,, \qquad 0<\lambda'<\lambda, \tag{130} $$

with a constant $c$ depending on $\lambda'$. Combining these estimates, $\mathsf{S}_3$ also behaves nicely. Finally, to $\mathsf{S}_1$ we apply Burkholder's inequality to get the upper bound

$$ c\,P_{{}^{\varepsilon}\mu}\sqrt{\gamma\left\langle {}^{\varepsilon}L_{{}^{\varepsilon}\mathbf{X}}(T),\tilde{\phi}^2_\lambda\right\rangle} \;\le\; c\sqrt{P_{{}^{\varepsilon}\mu}\,\gamma\left\langle {}^{\varepsilon}L_{{}^{\varepsilon}\mathbf{X}}(T),\tilde{\phi}^2_\lambda\right\rangle} \tag{131} $$

where we have also used Jensen's inequality. By the expectation formula (53) and the second part of (13) in the case $n=0$, for the expectation under the root we get the upper bound

$$ c\,\gamma\int_0^T \mathrm{d}s\int_{\varepsilon\mathsf{Z}^2} {}^{\varepsilon}\ell(\mathrm{d}x)\;{}^{\varepsilon}\mu^1 * {}^{\varepsilon}\mathrm{p}_s\,(x)\;{}^{\varepsilon}\mu^2 * {}^{\varepsilon}\mathrm{p}_s\,(x)\,\mathrm{e}^{-2\lambda|x|}. \tag{132} $$

Applying (130) twice, we are done. $\square$

For the rest of this subsection we *assume* that the collision rate $\gamma>0$ is small as in Hypothesis 28, and that the initial densities ${}^{\varepsilon}\mathbf{X}_0 = {}^{\varepsilon}\mu$ are uniformly tempered as in Hypothesis 32.

**Lemma 34 (Uniform 4$^{\text{th}}$ moments of increments)** *Fix a $\varphi\ge 0$ belonging to $\mathcal{C}^{(2)}_{\exp}(\mathsf{R}^2)$. Then there is a constant $c_{34}=c_{34}(T,\gamma,\sigma,\varphi)$ such that*

$$ \sup_{0<\varepsilon\le 1} P_{{}^{\varepsilon}\mu}\left\langle {}^{\varepsilon}X^i_{t'} - {}^{\varepsilon}X^i_t,\,\varphi\right\rangle^4 \;\le\; c_{34}\,|t'-t|^2, \qquad 0\le t<t'\le T, \quad i=1,2. $$

**Proof** Fix $T,\gamma,\sigma,i$ as in the lemma, and take $\varphi\in\mathcal{C}^{(2)}_\lambda(\mathsf{R}^2)$, $\lambda>0$. By the Green function representation of the martingale problem $(\mathbf{MP})^{\sigma,\gamma,\varepsilon}_{{}^{\varepsilon}\mu}$ in Subsection 3.1,

$$ \begin{aligned} \left\langle {}^{\varepsilon}X^i_{t'} - {}^{\varepsilon}X^i_t,\varphi\right\rangle^4 \;\le\; & c\,\left\langle {}^{\varepsilon}\mu^i,\,{}^{\varepsilon}S_{t'}\varphi - {}^{\varepsilon}S_t\varphi\right\rangle^4 \\ & +\; c\left|\int_{[0,t]\times\varepsilon\mathsf{Z}^2} {}^{\varepsilon}M^i\big(\mathrm{d}(s,x)\big)\left[{}^{\varepsilon}S_{t'-s}\varphi(x) - {}^{\varepsilon}S_{t-s}\varphi(x)\right]\right|^4 \\ & +\; c\left|\int_{[t,t']\times\varepsilon\mathsf{Z}^2} {}^{\varepsilon}M^i\big(\mathrm{d}(s,x)\big)\,{}^{\varepsilon}S_{t'-s}\varphi(x)\right|^4. \end{aligned} \tag{133} $$

Write the right hand side as $c\,(\mathsf{S}_1+\mathsf{S}_2+\mathsf{S}_3)$ (in the obvious correspondence). We will use the fact that

$$ {}^{\varepsilon}S_{t'}\varphi - {}^{\varepsilon}S_t\varphi \;=\; \int_t^{t'}\mathrm{d}s\;\frac{\sigma^2}{2}\,{}^{\varepsilon}S_s\,{}^{\varepsilon}\Delta\varphi. \tag{134} $$

By the mean value theorem, and since by assumption $\Delta\varphi$ belongs to $\mathcal{C}_\lambda(\mathsf{R}^2)$, we conclude that

$$ \left|{}^{\varepsilon}\Delta\varphi\right|(x) \;\le\; c\,\mathrm{e}^{-\lambda|x|}, \qquad x\in\varepsilon\mathsf{Z}^2. \tag{135} $$

Then by (130), the term $\mathsf{S}_1$ has the required property.

By Burkholder's inequality, (48), and the definition of ${}^{\varepsilon}L_{{}^{\varepsilon}\mathbf{X}}$,

$$P_{{}^{\varepsilon}\mu}\,\mathsf{S}_3 \;\le\; c\,P_{{}^{\varepsilon}\mu}\Big(\gamma\int_t^{t'}\mathrm{d}s\int_{\varepsilon\mathsf{Z}^2}{}^{\varepsilon}\ell(\mathrm{d}x)\,{}^{\varepsilon}X_s^1(x)\,{}^{\varepsilon}X_s^2(x)\,\big({}^{\varepsilon}S_{t'-s}\varphi(x)\big)^2\Big)^2 \tag{136}$$
$$\le\; c\gamma^2\,|t'-t|\int_t^{t'}\mathrm{d}s\;P_{{}^{\varepsilon}\mu}\Big(\int_{\varepsilon\mathsf{Z}^2}{}^{\varepsilon}\ell(\mathrm{d}x)\,{}^{\varepsilon}X_s^1(x)\,{}^{\varepsilon}X_s^2(x)\,{}^{\varepsilon}S_{t'-s}\varphi^2\,(x)\Big)^2,$$

where we have also used the Cauchy-Schwarz and Jensen's inequalities. By Corollary 30 with $\varphi$ replaced by ${}^{\varepsilon}S_{t'-s}\varphi^2$, and $\lambda$ by a $\lambda'$ satisfying additionally $\lambda'\in(0,2\lambda/5)$, the latter second moment expression can be bounded from above by

$$c_{30}\int_{\varepsilon\mathsf{Z}^2}{}^{\varepsilon}\ell(\mathrm{d}y)\;\big({}^{\varepsilon}S_{t'-s}\varphi^2\big)^2\,(y)\,\mathrm{e}^{10\,\lambda'|y|}. \tag{137}$$

But by Corollary A3 (a),

$${}^{\varepsilon}S_{t'-s}\varphi^2 \;\le\; c\,\phi_{2\lambda} \tag{138}$$

with a constant $c$ depending on $T$ and $\lambda$. Hence, by our assumption on $\lambda'$, the integral in formula line (137) is bounded by a constant, uniformly in $\varepsilon, s, t'$. Altogether, $\mathsf{S}_3$ behaves as we want it to.

Similarly, $\mathsf{S}_2$ can be handled by using (134), finishing the proof. □

Since each $\varphi\in\mathcal{C}_\lambda(\mathsf{R}^2)$, $\lambda>0$, satisfies $|\varphi|\le|\varphi|_\lambda\,\phi_\lambda\le|\varphi|_\lambda\,\tilde{\phi}_{\lambda/\sqrt{2}}$, and $\tilde{\phi}_{\lambda/\sqrt{2}}$ belongs to $\mathcal{C}^{(2)}_{\lambda/\sqrt{2}}(\mathsf{R}^2)$, the previous lemma immediately implies the following result.

**Corollary 35 (Uniform fourth moments)** *Let* $\varphi\in\mathcal{C}_{\exp}(\mathsf{R}^2)$. *Then*

$$\sup_{0<\varepsilon\le 1,\;0\le t\le T} P_{{}^{\varepsilon}\mu}\big\langle{}^{\varepsilon}X_t^i,\varphi\big\rangle^4 \;<\;\infty,\qquad i=1,2. \tag{139}$$

We also need the following lemma.

**Lemma 36 (2nd moment of collision local time increments)** *Fix a* $\varphi\ge 0$ *in* $\mathcal{C}_{\exp}(\mathsf{R}^2)$. *Then there is a constant* $c_{36}=c_{36}(T,\gamma,\sigma,\varphi)$ *such that*

$$\sup_{0<\varepsilon\le 1} P_{{}^{\varepsilon}\mu}\Big\langle{}^{\varepsilon}L_{{}^{\varepsilon}\mathbf{X}}(t)-{}^{\varepsilon}L_{{}^{\varepsilon}\mathbf{X}}(t'),\varphi\Big\rangle^2 \;\le\; c_{36}\,|t'-t|^2,\qquad 0\le t<t'\le T. \tag{140}$$

**Proof** The proof requires us to estimate

$$P_{{}^{\varepsilon}\mu}\Big(\gamma\int_t^{t'}\mathrm{d}s\int_{\varepsilon\mathsf{Z}^2}{}^{\varepsilon}\ell(\mathrm{d}x)\,{}^{\varepsilon}X_s^1(x)\,{}^{\varepsilon}X_s^2(x)\,\varphi(x)\Big)^2, \tag{141}$$

and this can be done in the same way as in the proof of Lemma 34 [recall (136)]. □

Here is the essential result of this subsection.

**Proposition 37 (Tightness)** *Under Hypotheses* 28 *and* 32, *the family of random processes* $\big\{({}^{\varepsilon}\mathbf{X},{}^{\varepsilon}L_{{}^{\varepsilon}\mathbf{X}}):\varepsilon\in(0,1]\big\}$ *is tight (in law) in* $\mathcal{C}\big(\mathsf{R}_+,\mathcal{M}^3_{\mathrm{tem}}(\mathsf{R}^2)\big)$.

**Proof** Fix a $T > 0$. We want to exploit [EK86, Theorem 3.9.1]. For this purpose, we use the relatively compact subsets

$$K = K\left((c_n)_{n\geq 1}\right) := \left\{\nu \in \mathcal{M}_{\text{tem}}:\ \langle \nu, \tilde{\phi}_{1/n}\rangle \leq c_n\,,\ n \geq 1\right\} \tag{142}$$

of $\mathcal{M}_{\text{tem}}$, where $(c_n)_{n\geq 1}$ is a sequence of positive numbers. For $0 < \varepsilon \leq 1$, using Lemma 33, we can find a sequence $(c_n)_{n\geq 1}$ such that for $i = 1, 2$,

$$P_{^\varepsilon\mu}\Big(\sup_{0\leq t\leq T}\left|\left\langle {}^\varepsilon X_t^i,\, \tilde{\phi}_{1/n}\right\rangle\right| \geq c_n\Big) \leq \varepsilon/2^n. \tag{143}$$

Then

$$P_{^\varepsilon\mu}\left({}^\varepsilon X_t^i \in K\left((c_n)_{n\geq 1}\right) \text{ for all } t \in [0,T]\right) \geq 1 - \varepsilon. \tag{144}$$

By the Lemmas 34 and 36 we obtain that, for every non-negative $\varphi \in \mathcal{C}_{\exp}^{(2)}$, the families

$$\left\{\langle {}^\varepsilon X^i, \varphi\rangle:\ 0 < \varepsilon \leq 1\right\},\quad i = 1,2,\quad \text{and}\quad \left\{\langle {}^\varepsilon L_{^\varepsilon\mathbf{X}}, \varphi\rangle:\ 0 < \varepsilon \leq 1\right\} \tag{145}$$

of random processes, restricted to $[0,T]$, are tight (in law) in $\mathcal{C}\big([0,T], \mathsf{R}\big)$. Then by [EK86, Theorem 3.9.1] the claim follows. (In fact, since our processes are all continuous, tightness in the Skorohod space then yields the tightness in our $\mathcal{C}$–space.) $\square$

## 4.2 Limiting martingale problem (proof of Theorem 4)

As the main task, here we want to verify the following proposition which implies Theorem 4 (a).

**Proposition 38 (Limiting martingale problem)** *Fix $\gamma, \sigma, \mu$ as in Theorem 4, hence as in Theorem 6, and, for $0 < \varepsilon \leq 1$, choose ${}^\varepsilon\mathbf{X}_0 = {}^\varepsilon\mu \in {}^\varepsilon\mathcal{B}^2_{\text{tem}}(\varepsilon\mathsf{Z}^2)$ as in Theorem 6, that is, satisfying Hypothesis 32 and converging in $\mathcal{M}^2_{\text{tem}}(\mathsf{R}^2)$ to $\mu$ as $\varepsilon \downarrow 0$. Then for each limit point $(\mathbf{X}, \Lambda)$ of $\left\{({}^\varepsilon\mathbf{X},\, {}^\varepsilon L_{^\varepsilon\mathbf{X}}):\ \varepsilon \in (0,1]\right\}$ in $\mathcal{C}\big(\mathsf{R}_+, \mathcal{M}^3_{\text{tem}}(\mathsf{R}^2)\big)$ we have $\Lambda = L_{\mathbf{X}}$, and $\mathbf{X}$ satisfies the martingale problem $(\mathbf{MP})_\mu^{\sigma,\gamma}$.*

The proof will be divided into a series of lemmas. For this purpose, in this subsection we fix $\gamma$, $\sigma$, and ${}^\varepsilon\mu \to \mu$ as $\varepsilon \downarrow 0$, as well as $(\mathbf{X}, \Lambda)$ as in the proposition. Note that then the Hypotheses 28 and 32 hold. Take a sequence $({}^{\varepsilon_n}\mathbf{X},\, {}^{\varepsilon_n}L)$ with $0 < \varepsilon_n \downarrow 0$ as $n \uparrow \infty$ such that

$$({}^{\varepsilon_n}\mathbf{X},\, {}^{\varepsilon_n}L) \underset{n\uparrow\infty}{\longrightarrow} (\mathbf{X}, \Lambda) \quad \text{in } \mathcal{C}\big(\mathsf{R}_+, \mathcal{M}^3_{\text{tem}}(\mathsf{R}^2)\big) \tag{146}$$

in law. By Skorohod's theorem, we may (and shall) assume that this convergence is almost sure on the stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_\cdot, \mathcal{P})$.

Since each ${}^{\varepsilon_n}\mathbf{X}$ is a time-homogeneous Markov process, from the expected collision local time formula (53) we immediately get the following statement: for fixed $\varepsilon_n$, $0 \leq s \leq t$, and $\varphi \in \mathcal{C}_{\exp}(\mathsf{R}^2)$,

$$\begin{aligned} &\mathcal{P}\left\{\left\langle {}^{\varepsilon_n}L_{^{\varepsilon_n}\mathbf{X}}(t) - {}^{\varepsilon_n}L_{^{\varepsilon_n}\mathbf{X}}(s), \varphi\right\rangle \,\middle|\, \mathcal{F}_s\right\} \\ &= \int_s^t \mathrm{d}r \int_{^{\varepsilon_n}\mathsf{Z}^2} {}^{\varepsilon_n}\ell(\mathrm{d}x)\ {}^{\varepsilon_n}X_s^1 * {}^{\varepsilon_n}\mathrm{p}_{r-s}\,(x)\, {}^{\varepsilon_n}X_s^2 * {}^{\varepsilon_n}\mathrm{p}_{r-s}\,(x)\, \varphi(x) \end{aligned} \tag{147}$$

$\mathcal{P}$–a.s. (conditional expected approximated collision local time).

**Lemma 39 (Uniform integrability)** *For any fixed $t > 0$ and $\varphi \in \mathcal{C}_{\exp}(\mathsf{R}^2)$, the random variables*

$$\left\langle {}^{\varepsilon_n}X_t^1, \varphi\right\rangle^2,\ \left\langle {}^{\varepsilon_n}X_t^2, \varphi\right\rangle^2,\ \left\langle {}^{\varepsilon_n}L_{^\varepsilon\mathbf{X}}(t), \varphi\right\rangle,\quad n \geq 1, \tag{148}$$

*are uniformly integrable with respect to $\mathcal{P}$.*

**Proof** Fix $t$ and $\varphi$ as in the lemma. By Corollaries 35 and 30 the fourth moment of $\langle {}^{\varepsilon_n}X^i_t, \varphi\rangle$ and the second moment of the collision measure, respectively, are bounded, uniformly in $n$. The conclusion of the lemma is then immediate. □

From the previous lemma it easily follows that the limit point $(\mathbf{X}, \Lambda)$ *satisfies the martingale problem* $(\mathbf{MP})^{\sigma,\gamma}_{\mu}$ of Definition 3, but with $L_{\mathbf{X}}$ replaced by $\Lambda$. In order to complete the proof of Proposition 38, the only point which remains to be checked is that $\Lambda$ is in fact the collision local time $L_{\mathbf{X}}$. This we will achieve by some $L^1$–arguments based on the additional smoothing imposed in Definition 1 on the collision local time. The first technical result in this direction is the following lemma.

**Lemma 40 (Convergence of expected collision local times)** *For every* $0 \le s < t$ *and* $\varphi \in \mathcal{C}^+_{\exp}(\mathsf{R}^2)$,

$$\begin{aligned}\int_s^t \mathrm{d}r\Big[\int_{\varepsilon_n\mathsf{Z}^2} {}^{\varepsilon_n}\ell(\mathrm{d}x)\, {}^{\varepsilon_n}X^1_s * {}^{\varepsilon_n}\mathrm{p}_{r-s}(x)\, {}^{\varepsilon_n}X^2_s * {}^{\varepsilon_n}\mathrm{p}_{r-s}(x)\,\varphi(x)\Big] \\ \underset{n\uparrow\infty}{\longrightarrow} \int_s^t \mathrm{d}r \int_{\mathsf{R}^2} \ell(\mathrm{d}x)\; X^1_s * \mathrm{p}_{r-s}(x)\, X^2_s * \mathrm{p}_{r-s}(x)\,\varphi(x)\end{aligned} \tag{149}$$

*in* $L^1(\mathcal{P})$.

**Proof** Consider $s, t, \varphi$ as in the lemma. By the expectation formula (50), the expectation of the integrand in square brackets in (149) equals

$$\int_{\varepsilon_n\mathsf{Z}^2} {}^{\varepsilon_n}\ell(\mathrm{d}x)\, {}^{\varepsilon_n}\mu^1 * {}^{\varepsilon_n}\mathrm{p}_r(x)\, {}^{\varepsilon_n}\mu^2 * {}^{\varepsilon_n}\mathrm{p}_r(x)\,\varphi(x). \tag{150}$$

Since the ${}^{\varepsilon_n}\mu$ satisfy (115) for all $\lambda > 0$, and $t$ is fixed, by Corollary A3 (a),

$${}^{\varepsilon_n}\mu^i * {}^{\varepsilon_n}\mathrm{p}_r \;\le\; c_\lambda\, c_{\mathrm{A3}}\, \phi_{-\lambda}, \qquad n \ge 1, \quad i = 1,2, \quad r \le t. \tag{151}$$

On the other hand, $\varphi \le c_{\lambda'}\,\phi_{\lambda'}$ and choosing $\lambda' > 2\lambda$, the integral in (150) is bounded from above by $c\,\langle {}^{\varepsilon_n}\ell, \phi_{\lambda'-2\lambda}\rangle \le c$, uniformly in $r$ and $\varepsilon_n$. Similarly, the expectation of the corresponding integrand on the right hand side of (149) is uniformly bounded. Thus, by bounded convergence, it is enough to show that for fixed $r > s \ge 0$ and $\varphi$,

$$\begin{aligned}\mathcal{P}\Big|\int_{\varepsilon_n\mathsf{Z}^2} {}^{\varepsilon_n}\ell(\mathrm{d}x)\, {}^{\varepsilon_n}X^1_s * {}^{\varepsilon_n}\mathrm{p}_{r-s}(x)\, {}^{\varepsilon_n}X^2_s * {}^{\varepsilon_n}\mathrm{p}_{r-s}(x)\,\varphi(x) \\ - \int_{\mathsf{R}^2} \ell(\mathrm{d}x)\; X^1_s * \mathrm{p}_{r-s}(x)\, X^2_s * \mathrm{p}_{r-s}(x)\,\varphi(x)\Big| \underset{n\uparrow\infty}{\longrightarrow} 0.\end{aligned} \tag{152}$$

Next we bring in the additional terms

$$\mp\int_{\varepsilon_n\mathsf{Z}^2} {}^{\varepsilon_n}\ell(\mathrm{d}x)\, {}^{\varepsilon_n}X^1_s * \mathrm{p}_{r-s}(x)\, {}^{\varepsilon_n}X^2_s * \mathrm{p}_{r-s}(x)\,\varphi(x). \tag{153}$$

This time we want to apply dominated convergence. Besides domination estimates, the key is the following variance estimate: by the covariance formula (55), for $i = 1, 2$,

$$\begin{aligned}&\mathrm{Var}\, {}^{\varepsilon_n}X^i_s * \big[{}^{\varepsilon_n}\mathrm{p}_{r-s}(x) - \mathrm{p}_{r-s}(x)\big] \\ &= \gamma\int_0^s \mathrm{d}u \int_{\varepsilon_n\mathsf{Z}^2} {}^{\varepsilon_n}\ell(\mathrm{d}y)\, {}^{\varepsilon_n}\mu^1 * {}^{\varepsilon_n}\mathrm{p}_u(y)\, {}^{\varepsilon_n}\mu^2 * {}^{\varepsilon_n}\mathrm{p}_u(y) \\ &\qquad \Big[\int_{\varepsilon_n\mathsf{Z}^2} {}^{\varepsilon_n}\ell(\mathrm{d}z)\, \big[{}^{\varepsilon_n}\mathrm{p}_{r-s}(z-x) - \mathrm{p}_{r-s}(z-x)\big]\, {}^{\varepsilon_n}\mathrm{p}_{s-u}(z-y)\Big]^2.\end{aligned} \tag{154}$$

For fixed $\lambda > 0$, by (151), and using Jensen's inequality, we may bound this expression from above by

$$\leq c \int_0^s \mathrm{d}u \int_{\varepsilon_n \mathsf{Z}^2} {}^{\varepsilon_n}\ell(\mathrm{d}y)\, \phi_{-2\lambda}(y) \tag{155}$$
$$\int_{\varepsilon_n \mathsf{Z}^2} {}^{\varepsilon_n}\ell(\mathrm{d}z)\, \left[{}^{\varepsilon_n}\mathrm{p}_{r-s}(z-x) - \mathrm{p}_{r-s}(z-x)\right]^2 {}^{\varepsilon_n}\mathrm{p}_{s-u}(z-y).$$

Interchanging the order of integration, and exploiting Corollary A3 (a), we get the bound

$$\leq c \int_{\varepsilon_n \mathsf{Z}^2} {}^{\varepsilon_n}\ell(\mathrm{d}z)\, \left[{}^{\varepsilon_n}\mathrm{p}_{r-s}(z-x) - \mathrm{p}_{r-s}(z-x)\right]^2 \phi_{-2\lambda}(z). \tag{156}$$

But $\phi_{-2\lambda}(z) \leq \phi_{-2\lambda}(x)\, \phi_{-2\lambda}(z-x)$, and, since $r-s>0$ is fixed, by Lemma 2 (a), given $\delta > 0$ we may choose $N = N(\delta)$ such that for all $n > N$,

$$\left|{}^{\varepsilon_n}\mathrm{p}_{r-s}(z-x) - \mathrm{p}_{r-s}(z-x)\right| \;\leq\; \delta. \tag{157}$$

Therefore, we may bound the expression (156) by

$$\leq c\,\delta\,\phi_{-2\lambda}(x) \int_{\varepsilon_n \mathsf{Z}^2} {}^{\varepsilon_n}\ell(\mathrm{d}z)\, {}^{\varepsilon_n}\mathrm{p}_{r-s}(z)\, \phi_{-2\lambda}(z) \tag{158}$$
$$+\, c\,\delta\,\phi_{-2\lambda}(x) \int_{\varepsilon_n \mathsf{Z}^2} {}^{\varepsilon_n}\ell(\mathrm{d}z)\, \mathrm{p}_{r-s}(z)\, \phi_{-2\lambda}(z), \qquad n > N.$$

By Corollary A3, the integrals are bounded in $\varepsilon_n$. Therefore, the variance expressions in (154) tend to 0 as $n \uparrow \infty$. It is easy to derive bounds in the $x$ variable which allow us to apply Dominated Convergence by using the fact that $\varphi \leq c_{\lambda'}\, \phi_{\lambda'}$ and choosing $\lambda' > 2\lambda$.

Summarizing, it is enough to show that for our fixed $r > s \geq 0$ and $\varphi$,

$$\mathcal{P}\biggl|\int_{\varepsilon_n \mathsf{Z}^2} {}^{\varepsilon_n}\ell(\mathrm{d}x)\, {}^{\varepsilon_n}X_s^1 * \mathrm{p}_{r-s}(x)\, {}^{\varepsilon_n}X_s^2 * \mathrm{p}_{r-s}(x)\, \varphi(x)$$
$$- \int_{\mathsf{R}^2} \ell(\mathrm{d}x)\, X_s^1 * \mathrm{p}_{r-s}(x)\, X_s^2 * \mathrm{p}_{r-s}(x)\, \varphi(x)\biggr| \underset{n\uparrow\infty}{\longrightarrow} 0. \tag{159}$$

But this follows from the assumed a.s. convergence ${}^{\varepsilon_n}\mathbf{X} \to \mathbf{X}$ in $\mathcal{C}\bigl(\mathsf{R}_+, \mathcal{M}_{\mathrm{tem}}(\mathsf{R}^2)\bigr)$ by domination arguments using the uniform finiteness of fourth moments of Corollary 35. □

By the assumed a.s. convergence in (146), Lemma 39, the identity (147) and Lemma 40, we have, for each $\varphi \in \mathcal{C}_{\exp}(\mathsf{R}^2)$, the following convergence in $L^1(\mathcal{P})$:

$$\left\{\begin{aligned} &\mathcal{P}\left\{\bigl\langle \Lambda(t) - \Lambda(s), \varphi\bigr\rangle \,\Big|\, \mathcal{F}_s\right\} \;=\; \lim_{n\uparrow\infty} \mathcal{P}\left\{\Bigl\langle {}^{\varepsilon_n}L_{{}^{\varepsilon_n}\mathbf{X}}(t) - {}^{\varepsilon_n}L_{{}^{\varepsilon_n}\mathbf{X}}(s), \varphi\Bigr\rangle \,\Big|\, \mathcal{F}_s\right\} \\ &= \lim_{n\uparrow\infty} \int_s^t \mathrm{d}r \int_{\varepsilon_n \mathsf{Z}^2} {}^{\varepsilon_n}\ell(\mathrm{d}x)\, {}^{\varepsilon_n}X_s^1 * {}^{\varepsilon_n}\mathrm{p}_{r-s}(x)\, {}^{\varepsilon_n}X_s^2 * {}^{\varepsilon_n}\mathrm{p}_{r-s}(x)\, \varphi(x) \\ &= \int_s^t \mathrm{d}r \int_{\mathsf{R}^2} \ell(\mathrm{d}x)\, X_s^1 * \mathrm{p}_{r-s}(x)\, X_s^2 * \mathrm{p}_{r-s}(x)\, \varphi(x). \end{aligned}\right. \tag{160}$$

Recalling Definition 1 of collision local time, we now prove the following result.

**Lemma 41 (Identifying the collision local time)** *For all* $\varphi \in \mathcal{C}_{\exp}(\mathsf{R}^2)$ *and* $t \geq 0$, *we have the following convergence in* $L^1(\mathcal{P})$:

$$\bigl\langle L_{\mathbf{X}}^{*,\delta}(t), \varphi\bigr\rangle \;\longrightarrow\; \bigl\langle L_{\mathbf{X}}(t), \varphi\bigr\rangle \quad \text{as} \quad \delta \downarrow 0, \tag{161}$$

*and* $\Lambda = L_{\mathbf{X}}$.

**Proof** For $\varphi \in \mathcal{C}_{\exp}(\mathsf{R}^2)$, by (160) we have

$$\left\langle L_{\mathbf{X}}^{*,\delta}(t), \varphi \right\rangle \;=\; \int_0^t \mathrm{d}s \;\frac{1}{\delta}\,\mathcal{P}\left\{ \left\langle \Lambda(s+\delta) - \Lambda(s), \varphi \right\rangle \;\Big|\; \mathcal{F}_s \right\}, \quad \mathcal{P}\text{–a.s.} \tag{162}$$

Theorem 37 of [Mey66, p.126] and the continuity of $t \mapsto \Lambda(t)$ in $\mathcal{M}_{\mathrm{tem}}$ yield that the latter integral term converges to $\left\langle \Lambda(t), \varphi \right\rangle$ in $L^1(\mathcal{P})$ as $\delta \downarrow 0$, for each $t \geq 0$ and $\varphi \in \mathcal{C}_{\exp}(\mathsf{R}^2)$. Since $\Lambda$ is a continuous non-decreasing $\mathcal{M}_{\mathrm{tem}}$–valued process, the identity (162) and Definition 1 tell us that the collision local time $L_{\mathbf{X}}$ exists, coincides with $\Lambda$, and we have the convergence claimed in the lemma. This finishes the proof. □

Note that *we have now proved Proposition* 38 and hence *Theorem* 4 (a).

*Proof of Theorem* 4 (b). The claimed moment formula for the collision local time easily follows from the corresponding formula (53) for the approximating processes ${}^{\varepsilon}\mathbf{X}$, the limiting martingale Proposition 38, and Lemmas 40 (deterministic case $s = 0$) and 39. Argue similarly for the remaining two moment formulae. □

### 4.3 Extended martingale problem and Green function representation

In this subsection we present two immediate consequences of the martingale problem $(\mathbf{MP})_\mu^{\sigma,\gamma}$ of Definition 3.

For $T, \lambda > 0$, denote by $\mathcal{C}_{T,\lambda}^{(1,2)}$ the set of all real-valued functions $\psi$ defined on $[0,T] \times \mathsf{R}^2$ such that $t \mapsto \psi(t,\,\cdot\,)$, $t \mapsto \frac{\partial}{\partial t}\psi(t,\,\cdot\,)$, and $t \mapsto \Delta\psi(t,\,\cdot\,)$ are continuous $\mathcal{C}_\lambda$–valued functions. Set $\mathcal{C}_{T,\exp}^{(1,2)} := \bigcup_{\lambda>0} \mathcal{C}_{T,\lambda}^{(1,2)}$.

**Lemma 42 (Extension of the martingale problem $(\mathbf{MP})_\mu^{\sigma,\gamma}$)** *Let* $\mathbf{X}$ *be any solution of the martingale problem* $(\mathbf{MP})_\mu^{\sigma,\gamma}$ *of Definition* 3. *Then, for* $\psi^1, \psi^2$ *in* $\mathcal{C}_{T,\exp}^{(1,2)}$,

$$\begin{aligned} \left\langle X_t^i, \psi^i(t) \right\rangle \;&=\; \left\langle \mu^i, \psi^i(0) \right\rangle \;+\; \int_0^t \mathrm{d}s \left\langle X_s^i\,,\, \frac{\sigma^2}{2}\,\Delta\psi^i(s) + \frac{\partial}{\partial s}\psi^i\,(s) \right\rangle \\ &\quad + \int_{[0,t]\times\mathsf{R}^2} M^i\big(\mathrm{d}(s,x)\big)\,\psi^i(s,x), \qquad 0 \leq t \leq T,\;\; i = 1,2, \end{aligned} \tag{163}$$

*where* $M^i\big(\mathrm{d}(s,x)\big)$ *are the (zero-mean) martingale measures such that*

$$\begin{aligned} &\left\langle\!\!\left\langle \int_{[0,\,\cdot\,]\times\mathsf{R}^2} M^i\big(\mathrm{d}(s,x)\big)\, f^i(s,x),\; \int_{[0,\,\cdot\,]\times\mathsf{R}^2} M^j\big(\mathrm{d}(s,x)\big)\, f^j(s,x) \right\rangle\!\!\right\rangle_t \\ &= \;\delta_{i,j}\,\gamma \int_{[0,t]\times\mathsf{R}^2} L_{\mathbf{X}}\big(\mathrm{d}(s,x)\big)\, f^i(s,x)\, f^j(s,x), \qquad 0 \leq t \leq T, \quad i,j = 1,2. \end{aligned} \tag{164}$$

*Here* $f^1, f^2$ *belong to the set of predictable functions* $f$ *defined on* $\Omega \times \mathsf{R}_+ \times \mathsf{R}^2$ *such that*

$$P_\mu^{\mathbf{X}} \int_{[0,t]\times\mathsf{R}^2} L_{\mathbf{X}}\big(\mathrm{d}(s,x)\big)\, f^2(s,x) \;<\; \infty, \qquad t \geq 0. \tag{165}$$

**Proof** We will only outline the proof which is standard. We may fix a $\lambda > 0$ and note that $S$ is a strongly continuous semigroup acting on the separable Banach space $\mathcal{C}_\lambda$, and that each $S_t$ maps $\mathcal{C}_\lambda^{(2)}$ into itself. We then use Proposition 1.3.3 of [EK86] to bootstrap up to the domain of the generator of time-space Brownian motion on $C_0\big([0,T]\times\mathsf{R}^2\big)$ (the space of continuous functions $[0,T]\times\mathsf{R}^2$ vanishing at infinity), and this domain contains $\mathcal{C}_{T,\exp}^{(1,2)}$. Approximate $\psi \in \mathcal{C}_{T,\exp}^{(1,2)}$ by an appropriate sequence of step functions in the time variable, and then proceed as in the proof of Proposition II.5.7 of [Per00]. $\square$

**Corollary 43 (Green function representation of $(\mathbf{MP})_\mu^{\sigma,\gamma}$)** *Let* $\mathbf{X}$ *be as in Lemma* 42 *above. Then, for* $\varphi$ *in* $\mathcal{C}^2_{\exp}$, $i=1,2,$ *and* $t\ge 0$,

$$\big\langle X_t^i,\varphi^i\big\rangle \;=\; \big\langle \mu^i,S_t\varphi^i\big\rangle + \int_{[0,t]\times\mathsf{R}^2} M^i\big(\mathrm{d}(s,x)\big)\,S_{t-s}\varphi^i(x) \tag{166}$$

*with the martingale measures* $M^i$ *satisfying* (164)*. Further, if in addition* $\mu\in\mathcal{C}^2_{\exp}$, *then equation* (166) *holds for* $\varphi\in\mathcal{C}^2_{\mathrm{tem}}$.

**Proof** The first part is standard. Now additionally assume $\mu\in\mathcal{C}^2_{\exp}$, hence $\mu\in\mathcal{C}^2_\lambda$ for some $\lambda>0$, and consider non-negative $\varphi\in\mathcal{C}^2_{\mathrm{tem}}$. Take non-negative $\varphi_n\in\mathcal{C}^2_{\exp}$ with $\varphi_n\uparrow\varphi\in\mathcal{C}^2_{\mathrm{tem}}$ as $n\uparrow\infty$. Note that

$$\big\langle X_t^i,\varphi_n^i\big\rangle \;=\; \big\langle \mu^i,S_t\varphi_n^i\big\rangle + \int_{[0,t]\times\mathsf{R}^2} M^i\big(\mathrm{d}(s,x)\big)\,S_{t-s}\varphi_n^i(x), \tag{167}$$

by (166). By monotone convergence,

$$\big\langle X_t^i,\varphi_n^i\big\rangle\uparrow\big\langle X_t^i,\varphi^i\big\rangle \quad\text{and}\quad \big\langle \mu^i,S_t\varphi_n^i\big\rangle\uparrow\big\langle \mu^i,S_t\varphi^i\big\rangle \quad\text{as } n\uparrow\infty. \tag{168}$$

On the other hand,

$$\begin{aligned} &P_\mu^{\mathbf{X}}\Big[\int_{[0,t]\times\mathsf{R}^2} M^i\big(\mathrm{d}(s,x)\big)\,\big(S_{t-s}\varphi_n^i(x)-S_{t-s}\varphi^i(x)\big)\Big]^2\\ &\quad=\int_0^t\mathrm{d}s\int_{\mathsf{R}^2}\mathrm{d}x\;\big(S_{t-s}\left(\varphi_n^i-\varphi^i\right)(x)\big)^2\;S_s\mu^1(x)\,S_s\mu^2(x). \end{aligned} \tag{169}$$

But the integrand in (169) is bounded by

$$c_{\lambda,\lambda'}\,\left(S_{t-s}\phi_{-\lambda}\right)^2(x)\;\left(S_s\phi_{\lambda'}\right)^2(x)\;\le\;c\,\phi_{2(\lambda'-\lambda)}\,(x) \tag{170}$$

for any $\lambda'>0$. Take $\lambda'>\lambda$, and dominated convergence implies that (169) tends to 0 as $n\uparrow\infty$. Therefore (166) is satisfied also for $\mu\in\mathcal{C}^2_{\exp}$ and $\varphi\in\mathcal{C}^2_{\mathrm{tem}}$. $\square$

## 4.4 Convergence of dual processes

The main purpose of this subsection is to define a process $\widetilde{\mathbf{X}}$ which later will be shown to be dual to $\mathbf{X}$. For convenience, we introduce now the following notation. For

$$(\nu,\widetilde{\nu}) \;=\; \big((\nu^1,\nu^2),(\tilde\nu^1,\tilde\nu^2)\big)\;\in\;\mathcal{M}^2_{\mathrm{tem}}(\mathsf{R}^2)\times\mathcal{B}^2_+(\mathsf{R}^2)\ \text{ or }\ \mathcal{B}^2_+(\mathsf{R}^2)\times\mathcal{M}^2_{\mathrm{f}}(\mathsf{R}^2)$$

set

$$\mathfrak{E}(\nu,\widetilde{\nu}) \;:=\; \exp\Big[-\big\langle \nu^1+\nu^2,\,\tilde\nu^1+\tilde\nu^2\big\rangle + i\,\big\langle \nu^1-\nu^2,\,\tilde\nu^1-\tilde\nu^2\big\rangle\Big], \tag{171}$$

where the right hand side of (171) is defined to be 0 if $\langle \nu^1+\nu^2,\, \tilde{\nu}^1+\tilde{\nu}^2\rangle=\infty$, and where here $i=\sqrt{-1}$. We may apply this definition of $\mathfrak{E}(\nu,\widetilde{\nu})$ also if $\mathsf{R}^2$ is replaced by $\varepsilon\mathsf{Z}^2$, $0<\varepsilon\le 1$, everywhere. In particular, we may apply it in the situation of the following lemma.

Theorem 2.4 (b) in [DP98], rescaling as in (32) and (33), and using our identification convention for density functions and corresponding measures gives the following self-duality relation for the discrete space processes as introduced at the beginning of Subsection 3.1. [DP98] deals with a smaller space of initial measures than $\mathcal{B}_{\exp}$ (called $\mathcal{M}_{\mathrm{rap}}$ there) but the proof carries over without significant change.

**Lemma 44 (Self-duality: lattice case)** *Fix* $0<\varepsilon\le 1$. *Let* ${}^{\varepsilon}\mathbf{X}=({}^{\varepsilon}X^1,{}^{\varepsilon}X^2)$ *and* ${}^{\varepsilon}\widetilde{\mathbf{X}}=({}^{\varepsilon}\widetilde{X}^1,{}^{\varepsilon}\widetilde{X}^2)$ *denote independent mutually catalytic branching processes in* $\varepsilon\mathsf{Z}^2$ *with initial states* ${}^{\varepsilon}\mathbf{X}_0={}^{\varepsilon}\mu=({}^{\varepsilon}\mu^1,{}^{\varepsilon}\mu^2)\in{}^{\varepsilon}\mathcal{M}^2_{\mathrm{tem}}$ *and* ${}^{\varepsilon}\widetilde{\mathbf{X}}_0={}^{\varepsilon}\varphi=({}^{\varepsilon}\varphi^1,{}^{\varepsilon}\varphi^2)\in{}^{\varepsilon}\mathcal{B}^2_{\exp}$, *respectively. Then with probability one,* ${}^{\varepsilon}\widetilde{\mathbf{X}}_t\in{}^{\varepsilon}\mathcal{B}^2_{\exp}$ *for all* $t\ge 0$, *and the following duality relation holds:*

$$P_{{}^{\varepsilon}\mu}\mathfrak{E}({}^{\varepsilon}\mathbf{X}_t\,,{}^{\varepsilon}\varphi)\;=\;P_{{}^{\varepsilon}\varphi}\mathfrak{E}({}^{\varepsilon}\mu,{}^{\varepsilon}\widetilde{\mathbf{X}}_t),\qquad t\ge 0. \tag{172}$$

Fix again $\gamma$, $\sigma$, and

$${}^{\varepsilon}\mu\underset{\varepsilon\downarrow 0}{\longrightarrow}\mu,\ \text{ as well as }\ ({}^{\varepsilon_n}\mathbf{X},\,{}^{\varepsilon_n}L_{{}^{\varepsilon_n}\mathbf{X}})\underset{n\uparrow\infty}{\longrightarrow}(\mathbf{X},L_{\mathbf{X}})\ \text{ almost surely} \tag{173}$$

as in Proposition 38, respectively in its proof. The ${}^{\varepsilon}\mu$ continue to satisfy the uniform domination Hypothesis 32. Fix also

$$0\;\le\;\varphi\;=\;(\varphi^1,\varphi^2)\;\in\;\mathcal{C}^2_{\exp}(\mathsf{R}^2). \tag{174}$$

For each $0<\varepsilon\le 1$, let

$${}^{\varepsilon}\varphi\;=\;({}^{\varepsilon}\varphi^1,{}^{\varepsilon}\varphi^2)\;\in\;{}^{\varepsilon}\mathcal{B}^2_{\exp}(\varepsilon\mathsf{Z}^2)\ \text{ denote the restriction of }\varphi\text{ to }\varepsilon\mathsf{Z}^2, \tag{175}$$

and consider the mutually catalytic branching process

$${}^{\varepsilon}\widetilde{\mathbf{X}}\ \text{ in }\varepsilon\mathsf{Z}^2\ \text{ starting from }\ {}^{\varepsilon}\widetilde{\mathbf{X}}_0\;=\;{}^{\varepsilon}\varphi. \tag{176}$$

Then for each $n\ge 1$, we may apply the duality relation (172) of Lemma 44 to $({}^{\varepsilon_n}\mathbf{X},{}^{\varepsilon_n}\widetilde{\mathbf{X}})$ [with $\varepsilon_n$ from (173)]. Later we want to pass to the limit as $n\uparrow\infty$ in the duality relation (172). For this we need, in particular, the convergence of ${}^{\varepsilon_n}\widetilde{\mathbf{X}}$ to some limit process. To make this more precise, we introduce the following definition.

**Definition 45 (Strong integrability condition (SIntC))** For $\delta>0$, set

$$H_\delta(\nu)\;:=\;\int_{\mathsf{R}^2}\mathrm{d}x\int_{\mathsf{R}^2}\mathrm{d}y\,\Big[1+\frac{1}{|x-y|}\Big]S_\delta\nu^1(x)\,S_\delta\nu^2(x)\,S_\delta\nu^1(y)\,S_\delta\nu^2(y), \tag{177}$$

where $\nu=(\nu^1,\nu^2)$ is a pair of measures in $\mathcal{M}_{\mathrm{f}}(\mathsf{R}^2)$. A continuous $\mathcal{M}^2_{\mathrm{f}}$–valued process $\mathbf{Y}$ is said to satisfy the *strong integrability condition* (**SIntC**), if

$$\overline{\lim_{\delta\downarrow 0}}\,\mathcal{P}\int_0^T\mathrm{d}s\;H_\delta(\mathbf{Y}_s)\;<\;\infty, \tag{178}$$

for all $T>0$. $\diamond$

**Proposition 46 (Exponentially decreasing initial densities)** *Fix* $\varphi\ge 0$ *in* $\mathcal{C}^2_{\exp}(\mathsf{R}^2)$.

**(a) (Uniqueness)** *There exists a unique solution* $\widetilde{\mathbf{X}}$ *of the martingale problem* $(\mathbf{MP})^{\sigma,\gamma}_{\varphi}$ *of Definition* 3 *which satisfies the strong integrability condition* (**SIntC**) *of Definition* 45.

**(b) (Convergence)** *For* $\{{}^{\varepsilon}\widetilde{\mathbf{X}}: 0<\varepsilon\leq 1\}$ *as in* (175)–(176) *and* $\widetilde{\mathbf{X}}$ *of* (a), *the convergence in law*

$$\lim_{\varepsilon\downarrow 0} {}^{\varepsilon}\widetilde{\mathbf{X}} \;=\; \widetilde{\mathbf{X}} \text{ holds in } \mathcal{C}\big(\mathsf{R}_+, \mathcal{M}_{\mathrm{f}}^2(\mathsf{R}^2)\big).$$

**(c) (Exponentially decreasing states)** *For fixed* $t\geq 0$,

$$\widetilde{\mathbf{X}}_t \in \mathcal{M}^2_{\exp}(\mathsf{R}^2), \text{ almost surely.}$$

**Proof** Fix $\varphi$ as in (174). In order to apply a result stated in [DEF+02], we first recall the notation $\mathcal{M}_{\mathrm{f,se}}$ from there. $\mathcal{M}_{\mathrm{f,se}}$ is the set of all pairs $\nu=(\nu^1,\nu^2)$ in $\mathcal{M}_{\mathrm{f}}^2$ satisfying the following *strong energy condition:* for any $p\in(0,1)$, there is a constant $c_{(179)}=c_{(179)}(\nu,p)$ such that

$$\max_{1\leq i,j\leq 2}\int_{\mathsf{R}^2}\nu^i(\mathrm{d}x)\int_{\mathsf{R}^2}\nu^j(\mathrm{d}y)\,\mathrm{p}_r(x-y) \;\leq\; c_{(179)}\,r^{-p}, \qquad 0<r<1. \tag{179}$$

(a) Clearly, $\varphi\in\mathcal{M}_{\mathrm{f,se}}$ and so by [DEF+02, Theorem 11 (a,b)] there is a unique solution $\widetilde{\mathbf{X}}$ of the martingale problem $(\mathbf{MP})_{\varphi}^{\sigma,\gamma}$ there, satisfying (**SIntC**). Certainly, this $\widetilde{\mathbf{X}}$ solves also our martingale problem $(\mathbf{MP})_{\varphi}^{\sigma,\gamma}$ of Definition 3 since the $\varphi$ in $\mathcal{C}^2_{\exp}(\mathsf{R}^2)$ own the needed boundedness properties.

Let ${}'\widetilde{\mathbf{X}}$ be another solution to our martingale problem $(\mathbf{MP})_{\varphi}^{\sigma,\gamma}$ and $\psi=(\psi^1,\psi^2)$ be a pair of non-negative test functions as in the martingale problem $(\mathbf{MP})_{\varphi}^{\sigma,\gamma}$ of [DEF+02] (that is, twice continuously differentiable with bounded derivatives). Choose non-negative $\psi_n\in\mathcal{C}^2_{\exp}(\mathsf{R}^2)$ such that $\psi_n\uparrow\psi$ as $n\uparrow\infty$. By monotone convergence,

$$\big\langle {}'\widetilde{X}^i_t,\psi^i_n\big\rangle \underset{n\uparrow\infty}{\nearrow} \big\langle {}'\widetilde{X}^i_t,\psi^i\big\rangle, \qquad i=1,2, \quad t\geq 0. \tag{180}$$

Hence, by simple moment calculations, ${}'\widetilde{\mathbf{X}}$ satisfies the martingale problem $(\mathbf{MP})_{\varphi}^{\sigma,\gamma}$ of [DEF+02]. But by the uniqueness there, ${}'\widetilde{\mathbf{X}}=\widetilde{\mathbf{X}}$, and the proof of (a) is complete.

(b) Statement (b) is a variant of [DEF+02, Theorem 11 (c)]. In fact, by [DEF+02, Remark 12 (i)] we need only check that the Lemmas 35 and 45 (a) there are satisfied by our sequence of initial measures ${}^{\varepsilon}\varphi$, and this is trivial to verify. This gives the convergence statement in (b).

(c) We may assume that $\varphi$ belongs to $\mathcal{C}^2_\lambda$ for some $\lambda>0$. From the expectation formula in Theorem 4 (b) [or from the Green function representation of $(\mathbf{MP})_{\varphi}^{\sigma,\gamma}$ in Corollary 43],

$$P_{\varphi}\big\langle \widetilde{X}^j_t,\phi_{-\lambda'}\big\rangle \;=\; \big\langle\varphi^j, S_t\phi_{-\lambda'}\big\rangle \;<\;\infty, \qquad j=1,2, \quad 0<\lambda'<\lambda. \tag{181}$$

Claim (c) follows, finishing the proof. □

## 4.5 A regularization procedure for dual processes

We also need the following two regularization lemmas.

**Lemma 47 (Regularization for** ${}^{\varepsilon}\widetilde{\mathbf{X}}$**)** *Fix* $r\geq 0$, $t>0$, $0\leq\varphi\in\mathcal{C}^2_{\exp}(\mathsf{R}^2)$, *and initial densities* ${}^{\varepsilon}\mu$, $0<\varepsilon\leq 1$, *satisfying the uniform domination Hypothesis 32. For each* $\varepsilon\in(0,1]$, *consider the independent mutually catalytic branching processes* ${}^{\varepsilon}\mathbf{X}$ *and* ${}^{\varepsilon}\widetilde{\mathbf{X}}$ *on* $\varepsilon\mathsf{Z}^2$ *with initial states* ${}^{\varepsilon}\mathbf{X}_0={}^{\varepsilon}\mu$ *and* ${}^{\varepsilon}\widetilde{\mathbf{X}}_0={}^{\varepsilon}\varphi$, *the restriction of* $\varphi$ *to* $\varepsilon\mathsf{Z}^2$, *respectively. Then there is a constant* $c_{47}$ *such that for all bounded measurable complex-valued functions* $f$ *on* $\mathcal{C}\big(\mathsf{R}_+,\mathcal{M}^2_{\mathrm{tem}}(\mathsf{R}^2)\big)$, *and all* $\delta\in(0,t)$,

$$\sup_{0<\varepsilon\leq 1}\Big|P_{{}^{\varepsilon}\mu}\,f({}^{\varepsilon}\mathbf{X})\,P_{{}^{\varepsilon}\varphi}\Big[\mathfrak{E}\big({}^{\varepsilon}\mathbf{X}_r,{}^{\varepsilon}\widetilde{\mathbf{X}}_t\big) \;-\; \mathfrak{E}\big({}^{\varepsilon}\mathbf{X}_r,{}^{\varepsilon}S_\delta{}^{\varepsilon}\widetilde{\mathbf{X}}_{t-\delta}\big)\Big]\Big| \;\leq\; c_{47}\,\delta\,\|f\|_\infty. \tag{182}$$

**Proof** For the moment, fix $\varepsilon$. From the Green function representation of the martingale problem $(\mathbf{MP})^{\sigma,\gamma,\varepsilon}_{{}^{\varepsilon}\varphi}$ [see (47) – (49)], conditioning on $\mathcal{F}^{{}^{\varepsilon}\widetilde{\mathbf{X}}}_{t-\delta}$, and Itô's formula (applied to the process ${}^{\varepsilon}\widetilde{\mathbf{X}}$), the expectation expression on the left hand side of estimate (182) equals

$$4\gamma P_{{}^{\varepsilon}\mu} f({}^{\varepsilon}\mathbf{X}) P_{{}^{\varepsilon}\varphi} \int_{[t-\delta,t]\times\varepsilon\mathsf{Z}^2} {}^{\varepsilon}L_{{}^{\varepsilon}\widetilde{\mathbf{X}}}\big(\mathrm{d}(s,x)\big)\, \mathfrak{E}\big({}^{\varepsilon}S_{t-s}{}^{\varepsilon}\mathbf{X}_r, {}^{\varepsilon}\widetilde{\mathbf{X}}_s\big)\, {}^{\varepsilon}S_{t-s}{}^{\varepsilon}X^1_r(x)\, {}^{\varepsilon}S_{t-s}{}^{\varepsilon}X^2_r(x).$$

Hence, the absolute value expression in (182) can be bounded from above by

$$\begin{aligned} &4\gamma\|f\|_\infty\, P_{{}^{\varepsilon}\mu}\, P_{{}^{\varepsilon}\varphi} \int_{[t-\delta,t]\times\varepsilon\mathsf{Z}^2} {}^{\varepsilon}L_{{}^{\varepsilon}\widetilde{\mathbf{X}}}\big(\mathrm{d}(s,x)\big)\ {}^{\varepsilon}S_{t-s}{}^{\varepsilon}X^1_r(x)\, {}^{\varepsilon}S_{t-s}{}^{\varepsilon}X^2_r(x) \\ &= 4\gamma\|f\|_\infty P_{{}^{\varepsilon}\mu} P_{{}^{\varepsilon}\varphi} \int_{t-\delta}^{t} \mathrm{d}s \int_{\varepsilon\mathsf{Z}^2} {}^{\varepsilon}\ell(\mathrm{d}x)\, {}^{\varepsilon}S_{s-r}{}^{\varepsilon}\widetilde{X}^1_r(x)\, {}^{\varepsilon}S_{t-s}{}^{\varepsilon}\widetilde{X}^2_r(x)\, {}^{\varepsilon}S_{t-s}{}^{\varepsilon}X^1_r(x)\, {}^{\varepsilon}S_{t-s}{}^{\varepsilon}X^2_r(x) \\ &= 4\gamma\|f\|_\infty \int_{t-\delta}^{t} \mathrm{d}s \int_{\varepsilon\mathsf{Z}^2} {}^{\varepsilon}\ell(\mathrm{d}x)\, {}^{\varepsilon}S_s{}^{\varepsilon}\varphi^1(x)\, {}^{\varepsilon}S_s{}^{\varepsilon}\varphi^2(x)\, {}^{\varepsilon}S_{t-s+r}{}^{\varepsilon}\mu^1(x)\, {}^{\varepsilon}S_{t-s-r}{}^{\varepsilon}\mu^2(x), \end{aligned}$$

where we first used the expectation formula (53) for the collision local time, and then the mixed second moment formula (52). Now take $0<\lambda<\tilde{\lambda}$ and exploit the fact that, by assumption, ${}^{\varepsilon}\mu^j \le c_\lambda\,\phi_{-\lambda}$ and ${}^{\varepsilon}\varphi^j \le c_{\tilde{\lambda}}\,\phi_{\tilde{\lambda}}$ for some constants $c_\lambda$ and $c_{\tilde{\lambda}}$, $j=1,2$. Then the claim follows from Corollary A3 (a). □

Here is a continuum analog of the previous lemma:

**Lemma 48 (Regularization for $\widetilde{\mathbf{X}}$)** *Fix $r\ge 0$, $t>0$, $0\le\varphi\in\mathcal{C}^2_{\mathrm{exp}}(\mathsf{R}^2)$, and $\mu\in\mathcal{B}^2_{\mathrm{tem}}(\mathsf{R}^2)$. Let $\mathbf{X}$ be a solution of the martingale problem $(\mathbf{MP})^{\sigma,\gamma}_{\mu}$ occurring in Proposition 38 and $\widetilde{\mathbf{X}}$, independent of $\mathbf{X}$, be the unique solution of $(\mathbf{MP})^{\sigma,\gamma}_{\varphi}$ as in Proposition 46. Then there is a constant $c_{48}$ such that for all bounded measurable complex-valued functions $f$ on $\mathcal{C}\big(\mathsf{R}_+,\mathcal{M}^2_{\mathrm{tem}}(\mathsf{R}^2)\big)$, and all $\delta\in(0,t)$,*

$$\varlimsup_{\eta\downarrow 0}\Big| P^{\mathbf{X}}_{\mu}\, f(\mathbf{X})\, P^{\widetilde{\mathbf{X}}}_{\varphi}\Big[\mathfrak{E}\big(S_\eta\mathbf{X}_r, \widetilde{\mathbf{X}}_t\big) \;-\; \mathfrak{E}\big(\mathbf{X}_r, S_\delta\widetilde{\mathbf{X}}_{t-\delta}\big)\Big]\Big| \;\le\; c_{48}\,\delta\,\|f\|_\infty. \tag{183}$$

Note that according to our general convention, in case of $\mathbf{X}$ we must write $P^{\mathbf{X}}$ since we do not have uniqueness in the martingale problem at this point.

**Remark 49 (Case $r=0$)** Note that in the case $r=0$, one can immediately pass to the limit as $\eta\downarrow 0$ on the left hand side of (183), that is, the additional smoothing with $S_\eta$ can be dropped. ◇

*Proof of Lemma* 48. We need only slightly modify the proof of Lemma 47. From the Green function representation of the martingale problem $(\mathbf{MP})^{\sigma,\gamma}_{\varphi}$ of Corollary 43, conditioning on $\mathcal{F}^{\widetilde{\mathbf{X}}}_{t-\delta}$, and Itô's formula, the expectation on the left hand side of (183) equals

$$\begin{aligned} 4\gamma P^{\mathbf{X}}_{\mu} f(\mathbf{X}) P^{\widetilde{\mathbf{X}}}_{\varphi} &\int_{[t-\delta,t]\times\mathsf{R}^2} L_{\widetilde{\mathbf{X}}}\big(\mathrm{d}(s,x)\big)\, \mathfrak{E}\big(S_{\eta+t-s}\mathbf{X}_r, \widetilde{\mathbf{X}}_s\big)\, S_{\eta+t-s}X^1_r(x)\, S_{\eta+t-s}X^2_r(x) \\ &+\; P^{\mathbf{X}}_{\mu}\, f(\mathbf{X})\, P^{\widetilde{\mathbf{X}}}_{\varphi}\mathfrak{E}\big(S_{\eta+\delta}\mathbf{X}_r, \widetilde{\mathbf{X}}_{t-\delta}\big) \;-\; P^{\mathbf{X}}_{\mu}\, f(\mathbf{X})\, P^{\widetilde{\mathbf{X}}}_{\varphi}\mathfrak{E}\big(\mathbf{X}_r, S_\delta\widetilde{\mathbf{X}}_{t-\delta}\big). \end{aligned}$$

Hence, using the moment formulae in Theorem 4, the absolute value expression in formula line (183) can be bounded from above by

$$4\gamma\|f\|_\infty \int_{t-\delta}^{t}\mathrm{d}s \int_{\mathsf{R}^2}\mathrm{d}x\ S_s\varphi^1(x)\, S_s\varphi^2(x)\, S_{\eta+t-s+r}\mu^1(x)\, S_{\eta+t-s+r}\mu^2(x) \tag{184}$$

$$+\;\|f\|_\infty\, P^{\mathbf{X}}_{\mu}\Big| P^{\widetilde{\mathbf{X}}}_{\varphi}\Big[\mathfrak{E}\big(S_{\eta+\delta}\mathbf{X}_r, \widetilde{\mathbf{X}}_{t-\delta}\big) \;-\; \mathfrak{E}\big(S_\delta\mathbf{X}_r, \widetilde{\mathbf{X}}_{t-\delta}\big)\Big]\Big|. \tag{185}$$

Now from Corollary A3 (b), the term in (184) gives the desired bound in (183), uniformly in $\eta\in(0,1]$. Letting $\eta\downarrow 0$, the expression in (185) will disappear by bounded convergence, finishing the proof. □

## 4.6 Convergence of one-dimensional distributions

As a first step to the approximation Theorem 6 we show convergence of one-dimensional (in time) marginals. For this we need a technical lemma.

**Lemma 50 (Continuous convergence)** *For* $0 < \varepsilon \leq 1$, *let* ${}^{\varepsilon}\nu \in \mathcal{M}^2_{\text{tem}}(\varepsilon\mathsf{Z}^2)$. *Suppose* ${}^{\varepsilon}\nu \to \nu$ *in* $\mathcal{M}^2_{\text{tem}}(\mathsf{R}^2)$. *Moreover, let* $\varphi \in \mathcal{B}^2_{\exp}(\mathsf{R}^2)$, *and* ${}^{\varepsilon}\varphi$ *the restriction of* $\varphi$ *to* $\varepsilon\mathsf{Z}^2$, $0 < \varepsilon \leq 1$. *Consider the related processes* ${}^{\varepsilon}\widetilde{\mathbf{X}} \Rightarrow \widetilde{\mathbf{X}}$ *(as* $\varepsilon \downarrow 0$*) as in Proposition* 46. *Then, for fixed* $j,k = 1,2$ *and* $s,t > 0$,

$$\big\langle {}^{\varepsilon}\nu^j,\, {}^{\varepsilon}S_s\, {}^{\varepsilon}\widetilde{X}^k_t \big\rangle \underset{\varepsilon\downarrow 0}{\Longrightarrow} \big\langle \nu^j,\, S_s \widetilde{X}^k_t \big\rangle. \tag{186}$$

**Proof** We may assume that even ${}^{\varepsilon}\widetilde{\mathbf{X}} \to \widetilde{\mathbf{X}}$ a.s. as $\varepsilon \downarrow 0$. For $R \geq 1$, choose a continuous function $f_R : \mathsf{R}^2 \to \mathsf{R}_+$ such that $1_{B(R)} \leq f_R \leq 1_{B(R+1)}$, where $B(R)$ is the centered open ball in $\mathsf{R}^2$ with radius $R$. Then

$$\begin{aligned} \Big|\big\langle {}^{\varepsilon}\nu^j,\, {}^{\varepsilon}S_s\, {}^{\varepsilon}\widetilde{X}^k_t \big\rangle - \big\langle \nu^j,\, S_s \widetilde{X}^k_t \big\rangle\Big| \;&\leq\; \Big|\big\langle {}^{\varepsilon}\nu^j,\, {}^{\varepsilon}S_s\, {}^{\varepsilon}\widetilde{X}^k_t \big\rangle - \big\langle {}^{\varepsilon}\nu^j,\, f_R\, {}^{\varepsilon}S_s\, {}^{\varepsilon}\widetilde{X}^k_t \big\rangle\Big| \\ + \;\Big|\big\langle {}^{\varepsilon}\nu^j,\, f_R\, {}^{\varepsilon}S_s\, {}^{\varepsilon}\widetilde{X}^k_t \big\rangle - \big\langle \nu^j,\, f_R S_s \widetilde{X}^k_t \big\rangle\Big| &+ \Big|\big\langle \nu^j,\, f_R S_s \widetilde{X}^k_t \big\rangle - \big\langle \nu^j,\, S_s \widetilde{X}^k_t \big\rangle\Big|. \end{aligned} \tag{187}$$

Since $\varphi$ belongs to $\mathcal{C}^2_\lambda$ for some $\lambda > 0$, by (50) the expectation of the first term on the right hand side of (187) equals

$$\big\langle {}^{\varepsilon}\nu^j,\, (1-f_R)\, {}^{\varepsilon}S_{s+t}\, {}^{\varepsilon}\varphi^k \big\rangle \;\leq\; \big\langle {}^{\varepsilon}\nu^j,\, (1-f_R)\, c_\lambda\, {}^{\varepsilon}S_{s+t}\phi_\lambda \big\rangle \;\leq\; c\,\big\langle {}^{\varepsilon}\nu^j,\, (1-f_R)\, \phi_\lambda \big\rangle,$$

where for the second estimate we used Corollary A3. Now take any $\delta > 0$. Then the latter expression can be made smaller than $\delta$, uniformly in $\varepsilon$ by choosing $R$ sufficiently large. Similarly, enlarging $R$ if necessary, the expectation of the last term on the right hand side of (187) is smaller than $\delta$.

Concerning the remaining middle term on the right hand side of (187), first of all we have, as $\varepsilon \downarrow 0$, the (a.s.) convergence of finite measures ${}^{\varepsilon}\widetilde{X}^k_t \to \widetilde{X}^k_t$ and the continuous convergence ${}^{\varepsilon}\mathrm{p}_s({}^{\varepsilon}y) \to \mathrm{p}_s(y)$ whenever ${}^{\varepsilon}y \to y$ [by Lemma 2 (a)]. This certainly implies ${}^{\varepsilon}S_s\, {}^{\varepsilon}\widetilde{X}^k_t({}^{\varepsilon}x) \to S_s \widetilde{X}^k_t(x)$ whenever ${}^{\varepsilon}x \to x$. But we have also the following convergence of finite measures: ${}^{\varepsilon}\nu^j(\mathrm{d}x)\, f_R(x) \to \nu^j(\mathrm{d}x)\, f_R(x)$. Therefore, the remaining term tends to 0 a.s. as $\varepsilon \downarrow 0$. Altogether, we have proved convergence in probability instead of (186), and the claim follows. □

Restricting our attention to a fixed time $t \geq 0$, we know so far only that the random measures ${}^{\varepsilon}\mathbf{X}_t$ are tight in law as $\varepsilon \downarrow 0$. Now we will basically show their convergence in law.

**Lemma 51 (Convergence of one-dimensional distributions)** *Assume* ${}^{\varepsilon}\mu \in {}^{\varepsilon}\mathcal{B}^2_{\text{tem}}(\varepsilon\mathsf{Z}^2)$ *are as in Hypothesis* 32 and converge to $\mu$ in $\mathcal{M}^2_{\text{tem}}(\mathsf{R}^2)$. Then, for each non-negative $\varphi \in \mathcal{C}^2_{\exp}(\mathsf{R}^2)$ and the related unique process $\widetilde{\mathbf{X}}$ from Proposition *46,*

$$\Big| P_{{}^{\varepsilon}\mu}\mathfrak{E}\big({}^{\varepsilon}\mathbf{X}_t\,, \varphi\big) \;-\; P_{\varphi}\mathfrak{E}\big(\mu, \widetilde{\mathbf{X}}_t\big)\Big| \underset{\varepsilon\downarrow 0}{\longrightarrow} 0, \qquad t \geq 0. \tag{188}$$

**Proof** We may assume that $t > 0$. Let ${}^{\varepsilon}\varphi$ denote the restriction of $\varphi$ to $\varepsilon\mathsf{Z}^2$, $0 < \varepsilon \leq 1$. By the self-duality relation (172), the absolute value expression in (188) equals

$$\Big| P_{{}^{\varepsilon}\varphi}\mathfrak{E}\big({}^{\varepsilon}\mu,\, {}^{\varepsilon}\widetilde{\mathbf{X}}_t\big) \;-\; P_{\varphi}\mathfrak{E}\big(\mu, \widetilde{\mathbf{X}}_t\,\big)\Big|. \tag{189}$$

Take $0<\delta<t,$ then (189) can be bounded from above by

$$\Big|P_{^\varepsilon\varphi}\mathfrak{E}\big({}^\varepsilon\mu,{}^\varepsilon\widetilde{\mathbf{X}}_t\big)-P_{^\varepsilon\varphi}\mathfrak{E}\big({}^\varepsilon\mu,{}^\varepsilon S_\delta\,{}^\varepsilon\widetilde{\mathbf{X}}_{t-\delta}\big)\Big|+\Big|P_{^\varepsilon\varphi}\mathfrak{E}\big({}^\varepsilon\mu,{}^\varepsilon S_\delta\,{}^\varepsilon\widetilde{\mathbf{X}}_{t-\delta}\big)-P_\varphi\mathfrak{E}\big(\mu,S_\delta\widetilde{\mathbf{X}}_{t-\delta}\big)\Big|$$
$$+\;\Big|P_\varphi\mathfrak{E}\big(\mu,\,S_\delta\widetilde{\mathbf{X}}_{t-\delta}\big)-P_\varphi\mathfrak{E}\big(\mu,\,\widetilde{\mathbf{X}}_t\big)\Big|.$$

By the Lemmas 47 and 48 with $r=0$ and $f=1,$ and Remark 49, the first and last terms are bounded from above by $c_{47}\,\delta$ and $c_{48}\,\delta,$ uniformly in $\varepsilon.$ Since $\delta$ can be made arbitrarily small, it remains to show that

$$\Big|P_{^\varepsilon\varphi}\mathfrak{E}\big({}^\varepsilon\mu,\,{}^\varepsilon S_\delta\,{}^\varepsilon\widetilde{\mathbf{X}}_{t-\delta}\big)-P_\varphi\mathfrak{E}\big(\mu,\,S_\delta\widetilde{\mathbf{X}}_{t-\delta}\big)\Big|\;\underset{\varepsilon\downarrow 0}{\longrightarrow}\;0, \tag{190}$$

for fixed $\delta.$ But this follows from the continuous convergence Lemma 50 applied to ${}^\varepsilon\nu\equiv{}^\varepsilon\mu.$ ☐

## 4.7 Convergence of finite-dimensional distributions

The purpose of this subsection is to complete the proof of the approximation Theorem 6. This will be achieved by the following lemma.

**Lemma 52 (Convergence of finite-dimensional distributions)** *Let* $\mathbf{X}$ *denote any limit point of* $\{{}^\varepsilon\mathbf{X}:{}^\varepsilon\mathbf{X}_0={}^\varepsilon\mu,\ 0<\varepsilon\le 1\}$ *occurring in Proposition* 38 *above* (*where* ${}^\varepsilon\mu\to\mu$). *Moreover, let* $\big\{{}^\varepsilon\hat\mu\in{}^\varepsilon\mathcal{B}^2_{\mathrm{tem}}(\varepsilon\mathsf{Z}^2):0<\varepsilon\le 1\big\}$ *be any family* (*possibly different from* $\{{}^\varepsilon\mu:0<\varepsilon\le 1\}$) *also satisfying the domination condition* (36) *and converging in* $\mathcal{M}^2_{\mathrm{tem}}(\mathsf{R}^2)$ *to the same* $\mu$ *as* $\varepsilon\downarrow 0.$ *Finally, for each* $\varepsilon\in(0,1],$ *let* ${}^\varepsilon\hat{\mathbf{X}}$ *be the solution to the martingale problem* $(\mathbf{MP})^{\sigma,\gamma,\varepsilon}_{^\varepsilon\hat\mu}$ (*introduced in Subsection* 3.1). *Then, for each finite sequence* $0\le t_1\le\cdots\le t_m,$ *the following convergence in law holds:*

$$\big({}^\varepsilon\hat{\mathbf{X}}_{t_1},\ldots,{}^\varepsilon\hat{\mathbf{X}}_{t_m}\big)\;\underset{\varepsilon\downarrow 0}{\Longrightarrow}\;\big(\mathbf{X}_{t_1},\ldots,\mathbf{X}_{t_m}\big). \tag{191}$$

Note that (191) yields the desired uniqueness of limit points as well as the independence of the choice of the approaching ${}^\varepsilon\mu,$ thus *completing the proof of the lattice approximation Theorem* 6.

*Proof of Lemma* 52. We will proceed by induction. First assume that $m=1.$ We will apply Lemma 51 with $\{{}^\varepsilon\mu:0<\varepsilon\le 1\}$ replaced by $\{{}^\varepsilon\hat\mu:0<\varepsilon\le 1\}.$ Since there the pair $0\le\varphi\in\mathcal{C}^2_{\exp}(\mathsf{R}^2)$ of test functions is arbitrary, this lemma implies that ${}^\varepsilon\hat{\mathbf{X}}_{t_1}$ has a limit in law as $\varepsilon\downarrow 0,$ which is independent of the choice of the family $\{{}^\varepsilon\hat\mu:0<\varepsilon\le 1\}$ and so must coincide in law with $\mathbf{X}_{t_1}.$ This implies (191) in the present $m=1$ case.

Suppose now that (191) holds for some $m\ge 1,$ and we want to check it for $m+1.$ For this we may assume that $t_m<t_{m+1},$ and that for this $m$ we have almost sure convergence in (191). Take $0\le\varphi_j\in\mathcal{C}^2_{\exp}(\mathsf{R}^2),$ $1\le j\le m+1.$ We only need to show that

$$P_{^\varepsilon\hat\mu}\prod_{j=1}^{m+1}\mathfrak{E}\big({}^\varepsilon\hat{\mathbf{X}}_{t_j},\varphi_j\big) \tag{192}$$

has a limit as $\varepsilon\downarrow 0,$ which is independent of the choice of $\{{}^\varepsilon\hat\mu:0<\varepsilon\le 1\}.$ Trivially, the expectation expression (192) equals

$$P_{^\varepsilon\hat\mu}\prod_{j=1}^{m}\mathfrak{E}\big({}^\varepsilon\hat{\mathbf{X}}_{t_j},\varphi_j\big)\,P_{^\varepsilon\hat\mu}\Big\{\mathfrak{E}\big({}^\varepsilon\hat{\mathbf{X}}_{t_{m+1}},\varphi_{m+1}\big)\;\Big|\;\mathcal{F}^{^\varepsilon\hat{\mathbf{X}}}_{t_m}\Big\}. \tag{193}$$

By time-homogeneity, with probability one the latter conditional expectation can be written as

$$P_{^\varepsilon\hat{\mathbf{X}}_{t_m}}\mathfrak{E}\big({}^\varepsilon\hat{\mathbf{X}}_{t_{m+1}-t_m},\varphi_{m+1}\big)\;=\;P^{^\varepsilon\widetilde{\mathbf{X}}}_{^\varepsilon\varphi_{m+1}}\mathfrak{E}\big({}^\varepsilon\hat{\mathbf{X}}_{t_m},{}^\varepsilon\widetilde{\mathbf{X}}_{t_{m+1}-t_m}\big), \tag{194}$$

where ${}^{\varepsilon}\varphi_{m+1}$ is the restriction of $\varphi_{m+1}$ to $\varepsilon\mathsf{Z}^2$, $0<\varepsilon\le 1$, and in the last step we exploited the self-duality relation (172). Now we want to proceed in a way similar to the proof of the convergence statement (188). It suffices to show that

$$P_{{}^{\varepsilon}\hat{\mu}} \prod_{j=1}^{m+1} \mathfrak{E}\big({}^{\varepsilon}\hat{\mathbf{X}}_{t_j}, \varphi_j\big) \;-\; \overline{\lim_{\eta\downarrow 0}}\, P_{\mu}^{\mathbf{X}} \prod_{j=1}^{m} \mathfrak{E}\big(\mathbf{X}_{t_j}, \varphi_j\big)\, P_{\varphi_{m+1}}^{\widetilde{\mathbf{X}}} \mathfrak{E}\big(S_\eta \mathbf{X}_{t_m}, \widetilde{\mathbf{X}}_{t_{m+1}-t_m}\big) \tag{195}$$

converges to 0 as $\varepsilon\downarrow 0$. In fact, using (192)–(194), and taking $0<\delta<t_{m+1}-t_m$, the absolute value of the expressions in (195) can be bounded from above by

$$\begin{aligned}
&\Big| P_{{}^{\varepsilon}\hat{\mu}}^{{}^{\varepsilon}\hat{\mathbf{X}}} \prod_{j=1}^{m} \mathfrak{E}\big({}^{\varepsilon}\hat{\mathbf{X}}_{t_j}, \varphi_j\big)\, P_{{}^{\varepsilon}\varphi_{m+1}}^{{}^{\varepsilon}\widetilde{\mathbf{X}}} \Big[\mathfrak{E}\big({}^{\varepsilon}\hat{\mathbf{X}}_{t_m}, {}^{\varepsilon}\widetilde{\mathbf{X}}_{t_{m+1}-t_m}\big) - \mathfrak{E}\big({}^{\varepsilon}\hat{\mathbf{X}}_{t_m}, {}^{\varepsilon}S_\delta\, {}^{\varepsilon}\widetilde{\mathbf{X}}_{t_{m+1}-t_m-\delta}\big)\Big]\Big| \\
&+ \Big| P_{{}^{\varepsilon}\hat{\mu}}^{{}^{\varepsilon}\hat{\mathbf{X}}} \prod_{j=1}^{m} \mathfrak{E}\big({}^{\varepsilon}\hat{\mathbf{X}}_{t_j}, \varphi_j\big)\, P_{{}^{\varepsilon}\varphi_{m+1}}^{{}^{\varepsilon}\widetilde{\mathbf{X}}} \mathfrak{E}\big({}^{\varepsilon}\hat{\mathbf{X}}_{t_m}, {}^{\varepsilon}S_\delta\, {}^{\varepsilon}\widetilde{\mathbf{X}}_{t_{m+1}-t_m-\delta}\big) \\
&\qquad\qquad - P_{\mu}^{\mathbf{X}} \prod_{j=1}^{m} \mathfrak{E}\big(\mathbf{X}_{t_j}, \varphi_j\big)\, P_{\varphi_{m+1}}^{\widetilde{\mathbf{X}}} \mathfrak{E}\big(\mathbf{X}_{t_m}, S_\delta \widetilde{\mathbf{X}}_{t_{m+1}-t_m-\delta}\big)\Big| \\
&+ \overline{\lim_{\eta\downarrow 0}} \Big| P_{\mu}^{\mathbf{X}} \prod_{j=1}^{m} \mathfrak{E}\big(\mathbf{X}_{t_j}, \varphi_j\big) \\
&\qquad\qquad P_{\varphi_{m+1}}^{\widetilde{\mathbf{X}}} \Big[\mathfrak{E}\big(\mathbf{X}_{t_m}, S_\delta \widetilde{\mathbf{X}}_{t_{m+1}-t_m-\delta}\big) - \mathfrak{E}\big(S_\eta \mathbf{X}_{t_m}, \widetilde{\mathbf{X}}_{t_{m+1}-t_m}\big)\Big]\Big|.
\end{aligned}$$

By the induction hypothesis and Skorohod's representation we may assume that the convergence statement (191) holds in $\mathcal{M}_{\text{tem}}^m$ almost surely. Lemma 47 with ${}^{\varepsilon}\mathbf{X}$ replaced by ${}^{\varepsilon}\hat{\mathbf{X}}$, $r=t_m$ and $t=t_{m+1}-t_m$ shows that the first absolute value in the above display is bounded by $c_{47}\,\delta$, uniformly in $\varepsilon$. Similarly, by Lemma 48, the $\overline{\lim}$–term is bounded by $c_{48}\,\delta$. Finally, by the continuous convergence of Lemma 50 applied to ${}^{\varepsilon}\nu \equiv {}^{\varepsilon}\hat{\mathbf{X}}_{t_m}$, our induction hypothesis and bounded convergence, the middle term converges to 0 as $\varepsilon\downarrow 0$. Thus, (195) converges to 0, finishing the proof.

□

# 5 Properties of X

Here we will verify the claimed properties of our mutually catalytic branching process $\mathbf{X}$ in $\mathsf{R}^2$.

## 5.1 Self-duality, scaling and self-similarity

*Completion of the proof of the self-duality Proposition* 15. By (188), the left hand side in the duality relation (172) converges to the right hand side of the self-duality claim in Proposition 15 (b) (recall (176)). But trivially, by (191), it converges also to $P_{\mu}^{\mathbf{X}}\mathfrak{E}(\mathbf{X}_t, \varphi)$, that is, part (b) is proved. But from Proposition 46 (c) we also get claim (a), completing the proof. □

*Proof of the scaling Proposition* 16. We only have to prove (a), since (b) is a special case of (a) in which $\theta=1$ and $z=0$. Fix $\theta,\varepsilon,t,z,\mathbf{X},\mathbf{X}^{(\varepsilon)}$ as in the proposition. Set $\hat{\mathbf{X}}_t^{(\varepsilon)} := \theta\varepsilon^2\mathbf{X}_{\varepsilon^{-2}t}\big(z+\varepsilon^{-1}(\,\cdot\,)\big)\in\mathcal{M}_{\text{tem}}^2$. By the self-duality of Proposition 15, applied to $\mathbf{X}^{(\varepsilon)}$ instead of $\mathbf{X}$, for the process $\widetilde{\mathbf{X}}$ with initial density $\widetilde{\mathbf{X}}_0=\varphi\in\mathcal{C}_{\text{exp}}^2$ we have

$$P_{\mu^{(\varepsilon)}}^{\mathbf{X}^{(\varepsilon)}} \mathfrak{E}\big(\mathbf{X}_t^{(\varepsilon)}, \varphi\big) \;=\; P_{\varphi}^{\widetilde{\mathbf{X}}} \mathfrak{E}\big(\mu^{(\varepsilon)}, \widetilde{\mathbf{X}}_t\big) \;=\; P_{\varphi}^{\widetilde{\mathbf{X}}} \mathfrak{E}\Big(\mu,\, \varepsilon^2\theta\widetilde{\mathbf{X}}_t\big(-\varepsilon z+\varepsilon(\,\cdot\,)\big)\Big). \tag{196}$$

But by scaling of the finite-measure-valued mutually catalytic branching process $\widetilde{\mathbf{X}}$ (see [DEF+02, Theorem 11 (d)]), the chain (196) of equations can be continued with

$$= P^{\widetilde{\mathbf{X}}}_{\theta\varepsilon^2\varphi(-\varepsilon z+\varepsilon(\,\cdot\,))}\,\mathfrak{E}\big(\mu,\widetilde{\mathbf{X}}_{\varepsilon^{-2}t}\big) \;=\; P^{\mathbf{X}}_{\mu}\,\mathfrak{E}\Big(\mathbf{X}_{\varepsilon^{-2}t}\,,\,\theta\varepsilon^2\varphi\big(-\varepsilon z+\varepsilon(\,\cdot\,)\big)\Big) \\ = P^{\mathbf{X}}_{\mu}\,\mathfrak{E}\big(\hat{\mathbf{X}}^{(\varepsilon)}_t,\varphi\big), \tag{197}$$

where we have once again used the self-duality of Proposition 15. Since $\varphi$ is arbitrary, the claim follows.□

## 5.2 Absolute continuity, law of densities, segregation, and blow-up

Now we are ready to show that our $\mathbf{X}$ has absolutely continuous states and to determine the law of densities at a point.

*Proof of Theorem* 11 (a) *(absolute continuity).* This is a direct application of a more general absolute continuity result in [DEF+02]. We now check the hypotheses required to apply this result. Let $\mu,\mathbf{X},t$ as in Theorem 11 (a) and $\varphi\in\mathcal{C}^{\infty}_{\mathrm{com}}(\mathsf{R}^2)$. Use the fact that the function $\psi:=\{S_{T-t}\varphi:\,0\le t\le T\}$ belongs to each $\mathcal{C}^{(1,2)}_{T,\lambda}$, $\lambda>0$, (introduced at the beginning of Subsection 4.3). Thus, we can apply the extended martingale problem of Lemma 42 to $\psi$ to see that the hypotheses of the general absolute continuity Theorem 57 of [DEF+02] are satisfied with $d=2$, $Q=S$, and $\Lambda=L_{\mathbf{X}}$. The result then follows from the fact that Brownian motion has absolutely continuous laws for positive times. □

*Proof of Theorem* 11 (b) *and* (c) *(law of densities and segregation).* Actually this requires only some minor modifications to the proofs in the finite measure case of [DEF+02]. In fact, in some respects the proof is even easier since the key ingredient is the self-duality of Proposition 15, whereas in the general finite measure case only a limiting duality was available.

Take $\mathbf{X}$ with $\mathbf{X}_0=\mu\in\mathcal{B}^2_{\mathrm{tem}}(\mathsf{R}^2)$ as in the theorem, and $\widetilde{\mathbf{X}}$ with $\widetilde{\mathbf{X}}_0=\varphi$ in $\mathcal{C}^2_{\exp}(\mathsf{R}^2)$ as in the self-duality Proposition 15. Set

$$U\;:=\;X^1+X^2,\qquad V\;:=\;X^1-X^2. \tag{198}$$

Moreover, for $a^1,a^2\ge 0$, put $a:=a^1+a^2$ and $b:=a^1-a^2$. Recall that for $t>0$ fixed, $\mathbf{X}_t$ is a pair of absolutely continuous measures, by Theorem 11 (a). Writing $\mathrm{p}_{\delta,x}:=\mathrm{p}_\delta(\,\cdot\,-x)$, by standard differentiation theory, for fixed $t>0$ and Lebesgue-almost all $x\in\mathsf{R}^2$,

$$P^{\mathbf{X}}_{\mu}\exp\big[-aU_t(x)+ibV_t(x)\big] \;=\; \lim_{\delta\downarrow 0}\,P^{\mathbf{X}}_{\mu}\exp\big[-a\,\langle U_t\,,\mathrm{p}_{\delta,x}\rangle+ib\,\langle V_t\,,\mathrm{p}_{\delta,x}\rangle\big].$$

By the self-duality of Proposition 15, we conclude that for Lebesgue-almost all $x\in\mathsf{R}^2$,

$$P^{\mathbf{X}}_{\mu}\exp\big[-aU_t(x)+ibV_t(x)\big] \;=\; \lim_{\delta\downarrow 0}P^{\widetilde{\mathbf{X}}^{\delta,x}}_{\varphi}\exp\Big[-\big\langle U_0\,,\widetilde{U}^{\delta,x}_t\big\rangle+\big\langle iV_0\,,\widetilde{V}^{\delta,x}_t\big\rangle\Big] \tag{199}$$

with $\varphi^i:=a^i\mathrm{p}_{\delta,x}$, $i=1,2$, $\widetilde{\mathbf{X}}^{\delta,x}=(\widetilde{X}^{1,\delta,x},\widetilde{X}^{2,\delta,x})$ and

$$\widetilde{U}^{\delta,x}\;:=\;\widetilde{X}^{1,\delta,x}+\widetilde{X}^{2,\delta,x},\qquad \widetilde{V}^{\delta,x}\;:=\;\widetilde{X}^{1,\delta,x}-\widetilde{X}^{2,\delta,x}. \tag{200}$$

Fix $x$ such that (199) holds, and take $\delta\in(0,1]$. By the formula

$$\big\langle\widetilde{X}^i_s,f\big\rangle\;:=\;\Big\langle\widetilde{X}^{i,\delta,x}_{\delta s},f\big((\,\cdot\,-x)/\sqrt{\delta}\big)\Big\rangle,\quad s\ge 0,\;\;i=1,2,\;\;f\in\mathcal{C}^{+}_{\mathrm{com}}(\mathsf{R}^2), \tag{201}$$

we introduce a process $\widetilde{\mathbf{X}} = (\widetilde{X}^1, \widetilde{X}^2)$. According to the scaling Proposition 16, $\widetilde{\mathbf{X}}$ is our mutually catalytic branching process in $\mathsf{R}^2$ starting from $\widetilde{\mathbf{X}}_0^{\delta,x} = \mathbf{a}\nu$, where $\mathbf{a} = (a^1, a^2)$ and $\nu$ is the normal law on $\mathsf{R}^2$ with density function $\mathrm{p}_1$. Now the definition (201) of $\widetilde{\mathbf{X}}$ turns (199) into

$$\begin{aligned} &P_\mu^{\mathbf{X}} \exp\bigl[-aU_t(x) + ibV_t(x)\bigr] \\ &= \lim_{\delta\downarrow 0} P_{\mathbf{a}\nu}^{\widetilde{\mathbf{X}}} \exp\Bigl[-\bigl\langle \widetilde{U}_{t/\delta}, U_0(\,\cdot\,\sqrt{\delta}+x)\bigr\rangle + i\bigl\langle \widetilde{V}_{t/\delta}, V_0(\,\cdot\,\sqrt{\delta}+x)\bigr\rangle\Bigr], \end{aligned} \tag{202}$$

where $\widetilde{U}$ and $\widetilde{V}$ are defined in an analogous way to (200). Applying the Green function representation in Corollary 43 with $\mathbf{X}, t$ replaced by $\widetilde{\mathbf{X}}$, $t/\delta$, respectively, and with

$$\varphi \;:=\; \mu(\,\cdot\,\sqrt{\delta}+x) - S_t\mu\,(x), \tag{203}$$

but the indices interchanged, we get

$$\langle \widetilde{X}^j_{t/\delta}, \varphi^i\rangle \;=\; a_j\,\langle \nu, S_{t/\delta}\,\varphi^i\rangle + \int_{[0,\,t/\delta\,]\times\mathsf{R}^2} M^j\bigl(\mathrm{d}(r,y)\bigr)\, S_{t/\delta-r}\varphi^i\,(y) \;\text{ a.s.} \tag{204}$$

For $r < t/\delta$, by scaling of the heat kernel, we have

$$S_{t/\delta-r}\varphi^i\,(y) \;=\; \int_{\mathsf{R}^2}\mathrm{d}z\;\mu^i(z)\,\Bigl[\mathrm{p}_1\Bigl(\frac{z-x-y\sqrt{\delta}}{\sqrt{t-\delta/r}}\Bigr) - \mathrm{p}_1\Bigl(\frac{z-x}{\sqrt{t}}\Bigr)\Bigr]. \tag{205}$$

Clearly, the integrand converges to 0 as $\delta\downarrow 0$. Also,

$$\begin{aligned} \mathrm{p}_1\Bigl(\frac{z-x-y\sqrt{\delta}}{\sqrt{t-\delta/r}}\Bigr) \;&\le\; \mathrm{p}_1\Bigl(\frac{z-x-y\sqrt{\delta}}{\sqrt{t}}\Bigr) \;\le\; \mathrm{p}_1\Bigl(\frac{z-x}{\sqrt{t}}\Bigr)\,\exp\Bigl[\frac{|z-x|\,|y|\sqrt{\delta}}{\sigma^2 t}\Bigr] \\ &\le\; \mathrm{p}_1\Bigl(\frac{z-x}{\sqrt{t}}\Bigr)\,\exp\Bigl[\frac{|z-x|\,|y|}{\sigma^2 t}\Bigr] \;=:\; g_{x,y,t}(z), \end{aligned}$$

since $0<\delta\le 1$. But for fixed $x,y,t$, the dominating function $g_{x,y,t}$ is integrable with respect to $\mu^i(z)\,\mathrm{d}z$. Thus it follows from dominated convergence that

$$S_{t/\delta-r}\varphi^i\,(y) \to 0 \;\text{ as }\; \delta\downarrow 0. \tag{206}$$

But, again by scaling,

$$S_{t/\delta-r}\varphi^i(y) \;=\; S_{t-\delta r}\varphi^i(\,\cdot\,/\sqrt{\delta})\,(y\sqrt{\delta}), \qquad y\in\mathsf{R}^2, \tag{207}$$

and, for fixed $\lambda>0$,

$$\mu^i \;\le\; c_\lambda\,\phi_{-\lambda}\,. \tag{208}$$

This gives

$$|\varphi^i| \;\le\; c_{\lambda,x,t}\,\phi_{-\lambda\sqrt{\delta}}\,. \tag{209}$$

Thus, by Lemma A2, there is a constant $c_{(210)} = c_{(210)}(t,\lambda,\sigma,x)$ such that for $r<t/\delta$,

$$\bigl|S_{t/\delta-r}\varphi^i(y)\bigr| \;\le\; c_{(210)}\,\phi_{-\lambda\sqrt{\delta}}\,, \tag{210}$$

which is $\nu$–integrable. Then from (206), the first term on the right hand side of equation (204) approaches 0 as $\delta\downarrow 0$, by dominated convergence.

Writing

$$N_s^\delta \;:=\; \int_{[0,s]\times\mathsf{R}^2} M^j\bigl(\mathrm{d}(r,y)\bigr)\; S_{t/\delta-r}\varphi^i\,(y), \qquad s\le t/\delta, \tag{211}$$

then from (164)

$$\bigl\langle\!\bigl\langle N^\delta\bigr\rangle\!\bigr\rangle_{t/\delta} \;=\; \gamma\int_{\mathsf{R}_+\times\mathsf{R}^2} L_{\widetilde{\mathbf{X}}}\bigl(\mathrm{d}(r,y)\bigr)\; 1_{[0,t/\delta]}(r)\,\bigl[\,S_{t/\delta-r}\varphi^i\,(y)\bigr]^2. \tag{212}$$

For $K>0,$ write $I^K$ for this integral, if the integrand is additionally restricted to $|y|\le K/\sqrt{\delta},$ and $J^K$ in the opposite case. The integrand of $I^K$ approaches 0 as $\delta\downarrow 0$ by (206), and is bounded by $c\,\mathrm{e}^{2\lambda K}$ by (210). But $L_{\widetilde{\mathbf{X}}}$ is finite $P_{\mathbf{a}\nu}^{\widetilde{\mathbf{X}}}$–a.s. This shows that

$$\lim_{\delta\downarrow 0} I^K \;=\; 0,\;\; P_{\mathbf{a}\nu}^{\widetilde{\mathbf{X}}}\text{–a.s. for each } K>0. \tag{213}$$

Now use (210) and then the expectation formula for the collision local time [Theorem 4 (b)] to see that

$$P_{\mathbf{a}\nu}^{\widetilde{\mathbf{X}}} J^K \;\le\; c\,a_1a_2\int_0^{t/\delta}\mathrm{d}r\int_{|y|\,>\,K/\sqrt{\delta}}\mathrm{d}y\;\phi_{-2\lambda\sqrt{\delta}}(y)\,\mathrm{p}_{1+r}^2(y). \tag{214}$$

By scaling, the right hand side equals

$$c\int_0^t\mathrm{d}r\int_{|y|\,>\,K}\mathrm{d}y\;\phi_{-2\lambda}(y)\,\mathrm{p}_{\delta+r}^2(y)\;\le\; c\int_0^t\mathrm{d}r\int_{|y|\,>\,K}\mathrm{d}y\;\frac{1}{\delta+r}\,\mathrm{p}_{\delta+r}(y) \tag{215}$$

where we have used the trivial estimate

$$\phi_{-2\lambda}(y)\,\mathrm{p}_{\delta+r}(y)\;\le\; c\,\frac{1}{\delta+r}\,,\qquad r\le t. \tag{216}$$

Therefore, (214) and (215) give

$$P_{\mathbf{a}\nu}^{\widetilde{\mathbf{X}}} J^K \;\le\; c\int_0^{t+1}\mathrm{d}r\int_{|y|\,>\,K}\mathrm{d}y\;\frac{1}{r}\,\mathrm{p}_r(y)\;\underset{K\uparrow\infty}{\longrightarrow}\;0. \tag{217}$$

The statements (213) and (217) easily show that $\big\langle\!\big\langle N^\delta\big\rangle\!\big\rangle_{t/\delta}\to 0$ in $P_{\mathbf{a}\nu}^{\widetilde{\mathbf{X}}}$–probability as $\delta\downarrow 0$. By a standard martingale inequality, the second term on the right hand side of (204) (that is $N_{t/\delta}^{\delta}$) also converges to 0 in $P_{\mathbf{a}\nu}^{\widetilde{\mathbf{X}}}$–probability as $\delta\downarrow 0$.
Summarizing, we have proved

$$\langle \widetilde{X}_{t/\delta}^{j},\varphi^i\rangle\;\underset{\delta\downarrow 0}{\longrightarrow}\;0\text{ in }P_{\mathbf{a}\nu}^{\widetilde{\mathbf{X}}}\text{–probability}, \tag{218}$$

and so (202) now gives

$$\begin{aligned} &P_\mu^{\mathbf{X}}\exp\big[-aU_t(x)+ibV_t(x)\big]\\ &=\lim_{\delta\downarrow 0}P_{\mathbf{a}\nu}^{\widetilde{\mathbf{X}}}\exp\Big[-\big\langle\widetilde{U}_{t/\delta}\,,1\big\rangle S_tU_0(x)+i\big\langle\widetilde{V}_{t/\delta}\,,1\big\rangle S_tV_0(x)\Big]. \end{aligned} \tag{219}$$

According to the convergence Theorem 21 in [DEF+02], the total masses $\big\langle\widetilde{X}_T^j,1\big\rangle,\ j=1,2,$ of the pair $\widetilde{\mathbf{X}}_T$ of finite measures has a limit in law as $T\uparrow\infty$ which can be described by the exit state $\xi_\tau$ of planar Brownian motion started at $\mathbf{a}$ (recall Definition 10). Therefore, the limit in (219) can be computed and equals

$$\begin{aligned} &\Pi_{\mathbf{a}}\exp\Big[-S_tU_0(x)(\xi_\tau^1+\xi_\tau^2)+iS_tV_0(x)(\xi_\tau^1-\xi_\tau^2)\Big]\\ &=\Pi_{\left(S_t\mu^1(x),\,S_t\mu^2(x)\right)}\exp\big[-a(\xi_\tau^1+\xi_\tau^2)+ib(\xi_\tau^1-\xi_\tau^2)\big] \end{aligned} \tag{220}$$

(recall that $\Pi$ refers to the law of Brownian motion). In fact, the last equality is an easy exercise in harmonic analysis which may be found in the proof of [DP98, Theorem 1.5]. An easy application of the Stone-Weierstrass Theorem, as in the proof of [DP98, Lemma 2.3(b)], shows that the latter joint Laplace-Fourier transform for $\mathbf{a}\in\mathsf{R}_+^2$ uniquely determines the law of $\mathbf{X}_t(x)$ to be that claimed in Theorem 11 (b).

Both, the variance formula and the segregation follow from simple properties of planar Brownian motion, completing the proof of Theorem 11 (b) and (c).

*Proof of Theorem* 11(d) *(blow-up at the interface).* The detailed proof of the blow-up property, is omitted, since it is similar to the one of Corollary 19 in [DEF+02], which gives the result in the context of finite measures. In fact, one must simply replace $X_s^i(\mathsf{R}^2)$ with $\langle X_s^i, \phi_\lambda\rangle$ for $\lambda > 0$ in some places (notably in the inequality prior to the estimate (207) in [DEF+02]) to accommodate our $\mathcal{M}_{\text{tem}}$–setting. □

### 5.3 Long-term behavior (proof of Theorem 13)

First we additionally assume that $\mu = \mathbf{c}\ell = (c^1\ell, c^2\ell)$, with $\mathbf{c} = (c^1, c^2) \in \mathsf{R}_+^2$. Take a non-negative $\varphi \in \mathcal{C}_{\text{com}}^2$, and consider the mutually catalytic branching process $\widetilde{\mathbf{X}}$ but starting from $\varphi$. By the self-duality Proposition 15 [recall the notation $\mathfrak{E}$ from (171)],

$$P_{\mathbf{c}\ell}^{\mathbf{X}}\, \mathfrak{E}\big(\mathbf{X}_t, \varphi\big) \;=\; P_\varphi^{\widetilde{\mathbf{X}}}\, \mathfrak{E}\big(\mathbf{c}\ell, \widetilde{\mathbf{X}}_t\big), \qquad t \ge 0. \tag{221}$$

But again according to Theorem 20 of [DEF+02], the right hand side of (221) converges to

$$\begin{aligned} &\Pi_{(\langle\varphi^1,1\rangle,\langle\varphi^2,1\rangle)} \exp\Big[-(c^1+c^2)(\xi_\tau^1+\xi_\tau^2) + i\,(c^1-c^2)(\xi_\tau^1-\xi_\tau^2)\Big] \\ &= \Pi_{\mathbf{c}} \exp\Big[-\big(\xi_\tau^1+\xi_\tau^2\big)\left\langle \ell, \varphi^1+\varphi^2\right\rangle + i\,\big(\xi_\tau^1-\xi_\tau^2\big)\left\langle \ell, \varphi^1-\varphi^2\right\rangle\Big] \end{aligned} \tag{222}$$

as $t \uparrow \infty$, where the last identity is again a simple exercise in harmonic analysis (see [DP98, proof of Theorem 1.5]). This gives the required convergence for a determining class of functionals in $\mathcal{M}_{\text{tem}}^2$ (see [DP98, Lemma 6.7]). Moreover, the required tightness follows from

$$P_{\mathbf{c}\ell}^{\mathbf{X}}\left\langle X_t^1 + X_t^2, \phi_\lambda\right\rangle \;=\; (c^1+c^2)\left\langle \ell, S_t\phi_\lambda\right\rangle \;\equiv\; (c^1+c^2)\left\langle \ell, \phi_\lambda\right\rangle \;<\; \infty \tag{223}$$

[by the expectation formula in Theorem 4 (b)]. More precisely, [DP98, Lemma 6.7] (trivially extended to $\mathsf{R}^2$) gives the required result in the case $\mu = \mathbf{c}\ell$.

Using the method of [CKP00], we remove now the additional assumption $\mu = \mathbf{c}\ell$. In fact, let the initial densities $\mathbf{X}_0 = \mu$ be bounded and satisfy (40). Consider $\widetilde{\mathbf{X}}$ with $\widetilde{\mathbf{X}}_0 = \varphi \in \mathcal{C}_{\exp}^2$ from the self-duality Proposition 15. Then this proposition gives, for $t \ge 0$,

$$\Big|P_{\mathbf{c}\ell}^{\mathbf{X}}\, \mathfrak{E}\big(\mathbf{X}_t, \varphi\big) - P_\mu^{\mathbf{X}}\, \mathfrak{E}\big(\mathbf{X}_t, \varphi\big)\Big| \;\le\; P_\varphi^{\widetilde{\mathbf{X}}}\Big|\mathfrak{E}\big(\mathbf{c}\ell, \widetilde{\mathbf{X}}_t\big) - \mathfrak{E}\big(\mu, \widetilde{\mathbf{X}}_t\big)\Big|. \tag{224}$$

To show this approaches 0 as $t \uparrow \infty$, it suffices to show that

$$\left\langle \mu^j - c^j, \widetilde{X}_t^k\right\rangle \underset{t\uparrow\infty}{\longrightarrow} 0 \quad \text{in probability}, \quad j,k = 1,2. \tag{225}$$

It suffices to show this for $j = k = 1$. Put $\psi := \mu^1 - c^1 \in \mathcal{C}_{\text{tem}}$. Then by the martingale problem in the Green function representation of Corollary 43,

$$\left\langle \widetilde{X}_t^1, \psi\right\rangle \;=\; \left\langle \widetilde{X}_0^1, S_t\psi\right\rangle + \int_{[0,t]\times\mathsf{R}^2} \widetilde{M}^i\big(\mathrm{d}(s,x)\big)\, S_{t-s}\psi(x). \tag{226}$$

Now

$$\int_{[0,t]\times\mathsf{R}^2} L_{\widetilde{\mathbf{X}}}\big(\mathrm{d}(s,x)\big)\,\big(S_{t-s}\psi(x)\big)^2 \underset{t\uparrow\infty}{\longrightarrow} 0 \quad \text{a.s.} \tag{227}$$

by dominated convergence, the assumption (40), and since $L_{\widetilde{\mathbf{X}}}(\mathsf{R}_+ \times \mathsf{R}^2)$ is finite a.s. Moreover, $\lim_{t\uparrow\infty}\left\langle \widetilde{X}_0^1, S_t\psi\right\rangle = 0$ by the same reasoning. Consequently, $\left\langle \widetilde{X}_t^1, \psi\right\rangle \to 0$ in probability and we have reduced the general case to the special case already proved. □

# Appendix: auxiliary facts and remaining proofs

## A.1 Some random walk estimates

Recall that ${}^1S$ denotes the semigroup of the simple symmetric random walk in $\mathsf{Z}^d$ with jump rate $\frac{\sigma^2}{2}$, and that $\phi_{-\lambda}(x) = \mathrm{e}^{\lambda|x|}$.

**Lemma A1 (Preservation of tempered functions)** *For* $\lambda \geq 0$,

$$\left|{}^1\Delta\phi_{-\lambda}\right| \;\leq\; c_{\mathrm{A1}}\,\phi_{-\lambda}\,, \tag{A1}$$

*where* $c_{\mathrm{A1}} = c_{\mathrm{A1}}(\lambda) := \frac{d}{2}\left(\mathrm{e}^{\lambda} - 1\right)$.

**Proof** Take the definition (23) of ${}^1\Delta$ (case $\varepsilon = 1$) and use $\mathrm{e}^{\lambda|y|} \leq \mathrm{e}^{\lambda|x|}\,\mathrm{e}^{\lambda|y-x|}$. □

**Lemma A2 (Preservation of tempered functions)** *For* $t > 0$ *and* $\lambda \in \mathsf{R}$,

$${}^1S_t\phi_\lambda \;\leq\; c_{\mathrm{A2}}\,\phi_\lambda \tag{A2}$$

*with* $c_{\mathrm{A2}} = c_{\mathrm{A2}}(t,\lambda,\sigma) := 2^d \exp\!\left[d\sigma^2 t\,(\mathrm{e}^{\lambda^2} - 1)\right]$.

**Proof** First we assume that $d = 1$. Let $\{\zeta_n : n \geq 0\}$ denote the discrete time simple symmetric random walk in $\mathsf{Z}$ starting from $0$. Then, for $\lambda \in \mathsf{R}$,

$$\sum_{k\in\mathsf{Z}} {}^1\mathrm{p}_t(k)\,\mathrm{e}^{\lambda|k|} \;=\; \mathrm{e}^{-\sigma^2 t}\sum_{n=0}^{\infty}\frac{(\sigma^2 t)^n}{n!}\,\mathcal{P}\mathrm{e}^{\lambda|\zeta_n|}. \tag{A3}$$

But

$$\mathrm{e}^{\lambda|a|} \;\leq\; \mathrm{e}^{\lambda a} + \mathrm{e}^{-\lambda a}, \qquad a \in \mathsf{R}, \tag{A4}$$

and by symmetry we get

$$\mathcal{P}\mathrm{e}^{\lambda|\zeta_n|} \;\leq\; 2\,\mathcal{P}\mathrm{e}^{\lambda\zeta_n} \;=\; 2\left(\mathcal{P}\mathrm{e}^{\lambda\zeta_1}\right)^n, \tag{A5}$$

where we additionally used that $\zeta_n$ has i.i.d. increments. But

$$\mathcal{P}\mathrm{e}^{\lambda\zeta_1} \;=\; \frac{1}{2}\left(\mathrm{e}^{\lambda} + \mathrm{e}^{-\lambda}\right) \;\leq\; \mathrm{e}^{\lambda^2}. \tag{A6}$$

(To see the latter inequality, multiply by $\mathrm{e}^{\lambda}$, differentiate, multiply by $\mathrm{e}^{-2\lambda}$, and differentiate again.) Inserting (A6) into (A5) and (A3) gives

$$\sum_{k\in\mathsf{Z}} {}^1\mathrm{p}_t(k)\,\mathrm{e}^{\lambda|k|} \;\leq\; 2\,\mathrm{e}^{-\sigma^2 t}\sum_{n=0}^{\infty}\frac{\left(\sigma^2 t\,\mathrm{e}^{\lambda^2}\right)^n}{n!} \;=\; 2\exp\!\left[\sigma^2 t\,(\mathrm{e}^{\lambda^2} - 1)\right]. \tag{A7}$$

Turning back to $d \geq 1$ dimensions, we note first that the $d$–dimensional continuous time simple symmetric random walk can be considered as $d$ *independent* one-dimensional random walks each with generator $\frac{\sigma^2}{2d}\,{}^1\Delta$. Hence, using the elementary inequality

$$|k^1| \;\leq\; |k| \;\leq\; |k^1| + \cdots + |k^d|, \qquad k = \left(k^1,\ldots,k^d\right) \in \mathsf{Z}^d, \tag{A8}$$

from (A7) we get

$$\sum_{k\in\mathsf{Z}^d} {}^1\mathrm{p}_t(k)\,\mathrm{e}^{\lambda|k|} \;\leq\; 2^d\exp\!\left[d\sigma^2 t\,(\mathrm{e}^{\lambda^2} - 1)\right] \;=:\; c_{\mathrm{A2}}\,. \tag{A9}$$

Thus, for $x \in \mathsf{Z}^d$,

$$\sum_{y\in\mathsf{Z}^d} {}^1\mathrm{p}_t(x-y)\,\mathrm{e}^{\lambda|y|} \;=\; \sum_{y\in\mathsf{Z}^d} {}^1\mathrm{p}_t(y)\,\mathrm{e}^{\lambda|x+y|} \;\le\; c_{\mathrm{A2}}\,\mathrm{e}^{\lambda|x|}, \tag{A10}$$

since $|x|+|y| \ge |x+y| \ge |x|-|y|$, giving the required estimate. $\square$

Combining Lemma A2 with the scaling formula (25) and the trivial estimate (118) we get (a) of the following result. Part (b) is standard (see, for example, Lemma 6.2(ii) of [Shi94]). Recall the definition (8) of $\phi_\lambda$.

**Corollary A3 (Uniform preservation of tempered functions)**
**(a)** *For* $0<\varepsilon\le 1$ *and* $t>0$, *as well as* $\lambda\in\mathsf{R}$,

$${}^{\varepsilon}S_t\phi_\lambda \;\le\; c_{\mathrm{A3}}\,\phi_\lambda$$

*with* $c_{\mathrm{A3}}=c_{\mathrm{A3}}(t,\lambda,\sigma):=2^d\exp\bigl[d\sigma^2t\lambda^2\,\mathrm{e}^{\lambda^2}\bigr]$ *independent of* $\varepsilon$.
**(b)** *For each* $T>0$ *and* $\lambda\in\mathsf{R}$ *there is a* $\tilde c_{\mathrm{A3}}=\tilde c_{\mathrm{A3}}(T,\lambda,\sigma)$ *such that*

$$\sup_{t\le T} S_t\phi_{-\lambda} \;\le\; \tilde c_{\mathrm{A3}}\,\phi_{-\lambda}\,.$$

Next we need the following estimate.

**Lemma A4 (Binomial estimate)** *For* $N\ge 0$ *and* $\lambda\ge 0$,

$$\sum_{m=0}^{N}\binom{N}{m}p^m(1-p)^{N-m}\,\mathrm{e}^{\lambda|m-Np|} \;\le\; 2\,\mathrm{e}^{\lambda^2N}, \qquad 0\le p\le 1. \tag{A11}$$

**Proof** Let $\xi_N$ be distributed according to the binomial distribution $B(N,p)$, and set $\eta_N := N-\xi_N$, which has the law $B(N,1-p)$. Then the left hand side of the claim (A11) equals $\mathcal{P}\exp\bigl[\lambda|\xi_N-Np|\bigr]$. Using the elementary inequality (A4), we see that the left hand side of (A11) is

$$\le\; \mathrm{e}^{-\lambda Np}\mathcal{P}\mathrm{e}^{\lambda\xi_N} \;+\; \mathrm{e}^{-\lambda N(1-p)}\mathcal{P}\mathrm{e}^{\lambda\eta_N}. \tag{A12}$$

But

$$\mathcal{P}\mathrm{e}^{\lambda\xi_N} \;=\; \bigl(\mathcal{P}\mathrm{e}^{\lambda\xi_1}\bigr)^N \tag{A13}$$

and

$$\mathcal{P}\mathrm{e}^{\lambda\xi_1} \;=\; p\mathrm{e}^{\lambda} \;+\; (1-p), \tag{A14}$$

hence

$$\mathrm{e}^{-\lambda p}\mathcal{P}\mathrm{e}^{\lambda\xi_1} \;=\; p\mathrm{e}^{\lambda(1-p)} \;+\; (1-p)\mathrm{e}^{-\lambda p} \;\le\; \mathrm{e}^{\lambda^2}, \qquad 0\le p\le 1. \tag{A15}$$

(To see the latter inequality, multiply by $\mathrm{e}^{\lambda p}$, differentiate with respect to $\lambda$, multiply by $\mathrm{e}^{-\lambda}$, and differentiate again.) Putting together (A15) and (A13) gives

$$\mathrm{e}^{-\lambda Np}\mathcal{P}\mathrm{e}^{\lambda\xi_N} \;\le\; \mathrm{e}^{\lambda^2N}. \tag{A16}$$

Replacing $p$ by $(1-p)$, the second term in (A12) has the same bound. This completes the proof. $\square$

**Lemma A5 (Hypergeometric estimate)** *For* $0 \le m, \ell \le N$, *let* $\xi_\ell$ *be distributed according to the hypergeometric distribution* $HG(m, N-m, \ell)$, *that is*

$$\mathcal{P}\left(\xi_\ell = k\right) \;=\; \frac{\binom{Np}{k}\binom{N(1-p)}{\ell-k}}{\binom{N}{\ell}}, \qquad 0 \le k \le \ell, \tag{A17}$$

*where* $p := \frac{m}{N}$ *(taken to be* $1$ *in the case* $N = m = 0$*). Then, for all* $\lambda \ge 0$,

$$\mathcal{P}\exp\bigl[\lambda|\xi_\ell - \ell p|\bigr] \;\le\; 2\,\mathrm{e}^{\lambda^2\ell}. \tag{A18}$$

**Proof** Set $\eta_\ell = \ell - \xi_\ell$. Note that $\eta_\ell$ has the law $HG(N-m, m, \ell)$. As in the previous proof,

$$\mathcal{P}\exp\bigl[\lambda|\xi_\ell - \ell p|\bigr] \;\le\; \mathrm{e}^{-\lambda\ell p}\,\mathcal{P}\mathrm{e}^{\lambda\xi_\ell} \;+\; \mathrm{e}^{-\lambda\ell(1-p)}\,\mathcal{P}\mathrm{e}^{\lambda\eta_\ell}. \tag{A19}$$

By "symmetry", it suffices to show that

$$\mathrm{e}^{-\lambda\ell p}\,\mathcal{P}\mathrm{e}^{\lambda\xi_\ell} \;\le\; \mathrm{e}^{\lambda^2\ell}, \qquad \lambda \ge 0. \tag{A20}$$

This trivially holds for $\ell = 0$. Assume that (A20) is true for some $0 \le \ell \le N-1$. Then

$$\mathcal{P}\mathrm{e}^{\lambda\xi_{\ell+1}} \;=\; \mathcal{P}\mathrm{e}^{\lambda\xi_\ell}\left[q\mathrm{e}^{\lambda} + (1-q)\right] \;=\; \mathcal{P}\mathrm{e}^{\lambda\xi_\ell}\left[q\mathrm{e}^{\lambda(1-q)} + (1-q)\mathrm{e}^{-\lambda q}\right]\mathrm{e}^{\lambda q}, \tag{A21}$$

where $q := \frac{m-\xi_\ell}{N-\ell}$. By (A15), this is

$$\le\; \mathcal{P}\mathrm{e}^{\lambda\xi_\ell}\,\mathrm{e}^{\lambda^2}\,\mathrm{e}^{\lambda q}. \tag{A22}$$

Hence

$$\mathrm{e}^{-\lambda(\ell+1)p}\,\mathcal{P}\mathrm{e}^{\lambda\xi_{\ell+1}} \;\le\; \mathrm{e}^{\lambda^2}\mathcal{P}\mathrm{e}^{\lambda\xi_\ell}\mathrm{e}^{-\lambda(\ell+1)p+\lambda q} \;=\; \mathrm{e}^{\lambda^2}\mathrm{e}^{-\widetilde{\lambda}\ell p}\,\mathcal{P}\mathrm{e}^{\widetilde{\lambda}\xi_\ell} \tag{A23}$$

with $0 \le \widetilde{\lambda} := \lambda\,\frac{N-\ell-1}{N-\ell} \le \lambda$. By the induction hypothesis, for this we get the bound

$$\mathrm{e}^{\lambda^2}\,\mathrm{e}^{\widetilde{\lambda}^2\ell} \;\le\; \mathrm{e}^{\lambda^2(\ell+1)}, \tag{A24}$$

and we are done. □

**Lemma A6 (A collision estimate)** *For* $\lambda \in \mathsf{R}$, $0 < s, t \le T$ *and* $x, y$ *in* $\mathsf{Z}^d$,

$$\sum_{z\in\mathsf{Z}^d} {}^1\mathrm{p}_s(x-z)\,{}^1\mathrm{p}_t(y-z)\,\mathrm{e}^{\lambda|z|} \;\le\; c_{\mathrm{A6}}\;{}^1\mathrm{p}_{\mathrm{e}^{5\lambda^2}(s+t)}(x-y)\exp\left[|\lambda|\frac{|tx+sy|}{s+t}\right],$$

*where*

$$c_{\mathrm{A6}} \;=\; c_{\mathrm{A6}}(T,\lambda,\sigma) \;:=\; 4^d\,\exp\Bigl[2d\sigma^2T\bigl(\mathrm{e}^{5\lambda^2}-1\bigr)\Bigr]. \tag{A25}$$

**Remark A7 (Case** $\lambda = 0$**)** In the $\lambda = 0$ case, the constants in Lemmas A2 and A6 can be improved to $c_{\mathrm{A2}} = 1 = c_{\mathrm{A6}}$, that is the inequalities are not sharp. This is trivial for Lemma A6 and for $c_{\mathrm{A2}}$ is immediate from the proof of Lemma A2. ◇

*Proof of Lemma* A6. The left hand side of the claimed inequality can be bounded from above by

$$\exp\left[\lambda\frac{|tx+sy|}{s+t}\right]\sum_{z\in\mathsf{Z}^d} {}^1\mathrm{p}_s(x-z)\,{}^1\mathrm{p}_t(y-z)\exp\left[|\lambda|\left|z-\frac{tx+sy}{s+t}\right|\right]. \tag{A26}$$

Switching from $x-z$ to $z$, for the series this gives

$$\sum_{z\in\mathsf{Z}^d} {}^1\mathrm{p}_s(z)\,{}^1\mathrm{p}_t(x-y-z)\exp\Big[|\lambda|\Big|z-\frac{s(x-y)}{s+t}\Big|\Big]. \tag{A27}$$

Assume for the moment that $d=1$. Setting $p:=s/(s+t)$ and $a:=x-y$, the latter formula line can be written as

$$\mathrm{e}^{-\sigma^2(s+t)}\sum_{m,n=0}^{\infty}\frac{(\sigma^2 s)^m}{m!}\,\frac{(\sigma^2 t)^n}{n!} \tag{A28}$$
$$\sum_{z\in\mathsf{Z}}\mathcal{P}\Big(\zeta_m=z,\ \zeta'_n=a-z\Big)\exp\Big[|\lambda|\,|z-pa|\Big],$$

where $\zeta$ and $\zeta'$ are independent discrete time simple symmetric random walks in $\mathsf{Z}$, starting from 0. The latter series coincides with the following restricted expectation:

$$\mathcal{P}\Big\{\exp\Big[|\lambda|\,|\zeta_m-pa|\Big]\,;\ \zeta_{m+n}=a\Big\}. \tag{A29}$$

Substituting $m+n=:N$, we rewrite (A28) as

$$\mathrm{e}^{-\sigma^2(s+t)}\sum_{N=0}^{\infty}\frac{\big(\sigma^2(s+t)\big)^N}{N!}\,\mathcal{P}\left(\zeta_N=a\right) \tag{A30}$$
$$\sum_{m=0}^{N}\binom{N}{m}p^m(1-p)^{N-m}\,\mathcal{P}\Big\{\exp\Big[|\lambda|\,|\zeta_m-pa|\Big]\ \Big|\ \zeta_N=a\Big\}.$$

Setting $\widetilde{\zeta}_n=(\zeta_n+n)/2$, which has the binomial law $B(n,\frac12)$, $n\ge 0$, the latter conditional expectation can be written as

$$\mathcal{P}\Big\{\exp\Big[|\lambda|\,\Big|\widetilde{\zeta}_m-m-pa\Big|\Big]\ \Big|\ \widetilde{\zeta}_N=\frac{a+N}{2}\Big\}. \tag{A31}$$

Now, $\widetilde{\zeta}_m$ conditioned on $\widetilde{\zeta}_N=\frac{a+N}{2}=:\ell$ is hypergeometric $HG(m,N-m,\ell)$, denoted by $\xi_\ell$. Thus, (A31) coincides with

$$\mathcal{P}\exp\Big[|\lambda|\,|2\xi_\ell-m-pa|\Big]\ \le\ \exp\Big[|\lambda|\Big|\frac{2\ell-N}{N}\,(m-Np)\Big|\Big]\,\mathcal{P}\exp\Big[2|\lambda|\Big|\xi_\ell-\frac{m\ell}{N}\Big|\Big].$$

By Lemma A5, this is

$$\le\ \exp\Big[|\lambda|\Big|\frac{2\ell-N}{N}\,(m-Np)\Big|\Big]\,2\,\mathrm{e}^{4\lambda^2N}\ \le\ 2\,\mathrm{e}^{|\lambda|\,|m-Np|}\,\mathrm{e}^{4\lambda^2N}. \tag{A32}$$

Thus, for (A31) we obtain the upper bound

$$2\,\mathrm{e}^{|\lambda|\,|m-Np|}\,\mathrm{e}^{4\lambda^2N}. \tag{A33}$$

Hence, for (A30) we get the upper estimate

$$2\,\mathrm{e}^{-\sigma^2(s+t)}\sum_{N=0}^{\infty}\frac{\big(\sigma^2(s+t)\big)^N}{N!}\,\mathrm{e}^{4\lambda^2N}\,\mathcal{P}\left(\zeta_N=a\right)$$
$$\sum_{m=0}^{N}\binom{N}{m}p^m(1-p)^{N-m}\,\mathrm{e}^{2|\lambda|\,|m-Np|}\,. \tag{A34}$$

Apply Lemma A4 to bound this by

$$
\begin{aligned}
&\leq\ 4\,\mathrm{e}^{-\sigma^2(s+t)} \sum_{N=0}^{\infty} \frac{\big(\sigma^2(s+t)\big)^N}{N!}\, \mathrm{e}^{5\lambda^2 N}\, \mathcal{P}\left(\zeta_N = a\right) \\
&\leq\ 4\, \exp\Big[2\sigma^2 T\big(\mathrm{e}^{5\lambda^2}-1\big)\Big]\ {}^1\mathrm{p}_{\mathrm{e}^{5\lambda^2}(s+t)}(a). \qquad \text{(A35)}
\end{aligned}
$$

Turning back to $d$ dimensions, we need only note that the series (A27) can be bounded from above by a $d$–fold product of corresponding one-dimensional expressions. This finishes the proof. $\square$

## A.2 Proof of Lemma 24 (basic estimates)

Our strategy is as follows. We will first bound $L_n(\mathbf{a})$ in terms of some $M^k_{n-1}$ [see (A37) below]. After this we will exploit some of the used techniques to derive an iteration inequality for the $M^k_n$ [see (A41) below]. Then the claim will follow.

In the definition (78) of $L_n(\mathbf{a})$, consider the summands for $\ell = 2n-1$ and $\ell = 2n$, as well as the factor for $j = n$ within the product abbreviated by $\Pi_n(\mathbf{s}_{2n}; \mathbf{x}_0, \ldots, \mathbf{x}_{2n})$ [introduced in the end of Subsection 3.3]:

$$
\begin{aligned}
&\sum_{\mathbf{x}_{2n-1},\mathbf{x}_{2n}\,\in\,(\mathsf{Z}^2)^3} \exp\big[\lambda\,\|\mathbf{a}\mathbf{x}_{2n}\|\big] \\
&\Big[\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^2_{2n-2}-x^3_{2n-1})\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^3_{2n-2}-x^1_{2n-1}) \\
&\qquad\qquad +\ \mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^3_{2n-2}-x^3_{2n-1})\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^2_{2n-2}-x^1_{2n-1})\Big] \\
&\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^1_{2n-2}-x^2_{2n-1})\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^1_{2n-2}-x^3_{2n-1})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^3_{2n}) \\
&\Big(\mathrm{p}_{s_{2n-1}-s_{2n}}(x^1_{2n-1}-x^1_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^2_{2n-1}-x^1_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^2_{2n}) \\
&\quad +\ \mathrm{p}_{s_{2n-1}-s_{2n}}(x^1_{2n-1}-x^1_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^2_{2n-1}-x^2_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^1_{2n}) \\
&\quad +\ \mathrm{p}_{s_{2n-1}-s_{2n}}(x^1_{2n-1}-x^2_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^2_{2n-1}-x^1_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^1_{2n})\Big)
\end{aligned}
$$

[which is the "abundance" of $L_n(\mathbf{a})$ over $L_{n-1}(\mathbf{a})$]. By Chapman-Kolmogorov, summing over $x^2_{2n-1}$ and $x^1_{2n-1}$ gives

$$
\begin{aligned}
&\sum_{x^3_{2n-1}\in\mathsf{Z}^2,\ \mathbf{x}_{2n}\in(\mathsf{Z}^2)^3} \exp\big[\lambda\,\|\mathbf{a}\mathbf{x}_{2n}\|\big] \\
&\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^2_{2n-2}-x^3_{2n-1})\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^1_{2n-2}-x^3_{2n-1})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^3_{2n}) \\
&\Big(\mathrm{p}_{s_{2n-2}-s_{2n}}(x^3_{2n-2}-x^1_{2n})\mathrm{p}_{s_{2n-2}-s_{2n}}(x^1_{2n-2}-x^1_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^2_{2n}) \\
&\quad +\ \mathrm{p}_{s_{2n-2}-s_{2n}}(x^3_{2n-2}-x^1_{2n})\mathrm{p}_{s_{2n-2}-s_{2n}}(x^1_{2n-2}-x^2_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^1_{2n}) \\
&\quad +\ \mathrm{p}_{s_{2n-2}-s_{2n}}(x^3_{2n-2}-x^2_{2n})\mathrm{p}_{s_{2n-2}-s_{2n}}(x^1_{2n-2}-x^1_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^1_{2n})\Big)
\end{aligned}
$$

$$+ \sum_{x^3_{2n-1}\in\mathsf{Z}^2,\ \mathbf{x}_{2n}\in(\mathsf{Z}^2)^3} \exp\big[\lambda\,\|\mathbf{a}\mathbf{x}_{2n}\|\big]$$

$$\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^3_{2n-2}-x^3_{2n-1})\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^1_{2n-2}-x^3_{2n-1})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^3_{2n})$$

$$\Big(\mathrm{p}_{s_{2n-2}-s_{2n}}(x^2_{2n-2}-x^1_{2n})\mathrm{p}_{s_{2n-2}-s_{2n}}(x^1_{2n-2}-x^1_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^2_{2n})$$

$$+\ \mathrm{p}_{s_{2n-2}-s_{2n}}(x^2_{2n-2}-x^1_{2n})\mathrm{p}_{s_{2n-2}-s_{2n}}(x^1_{2n-2}-x^2_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^1_{2n})$$

$$+\ \mathrm{p}_{s_{2n-2}-s_{2n}}(x^2_{2n-2}-x^2_{2n})\mathrm{p}_{s_{2n-2}-s_{2n}}(x^1_{2n-2}-x^1_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^1_{2n})\Big).$$

Using Lemma 2 (b) six times we get the bound

$$\frac{c_2}{s_{2n-2}-s_{2n}} \sum_{x^3_{2n-1}\in\mathsf{Z}^2,\ \mathbf{x}_{2n}\in(\mathsf{Z}^2)^3} \exp\big[\lambda\,\|\mathbf{a}\mathbf{x}_{2n}\|\big]$$

$$\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^2_{2n-2}-x^3_{2n-1})\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^1_{2n-2}-x^3_{2n-1})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^3_{2n})$$

$$\Big(\mathrm{p}_{s_{2n-2}-s_{2n}}(x^1_{2n-2}-x^1_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^2_{2n})$$

$$+\ \mathrm{p}_{s_{2n-2}-s_{2n}}(x^1_{2n-2}-x^2_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^1_{2n})$$

$$+\ \mathrm{p}_{s_{2n-2}-s_{2n}}(x^3_{2n-2}-x^2_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^1_{2n})\Big)$$

$$+\ \frac{c_2}{s_{2n-2}-s_{2n}} \sum_{x^3_{2n-1}\in\mathsf{Z}^2,\ \mathbf{x}_{2n}\in(\mathsf{Z}^2)^3} \exp\big[\lambda\,\|\mathbf{a}\mathbf{x}_{2n}\|\big]$$

$$\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^3_{2n-2}-x^3_{2n-1})\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^1_{2n-2}-x^3_{2n-1})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^3_{2n})$$

$$\Big(\mathrm{p}_{s_{2n-2}-s_{2n}}(x^1_{2n-2}-x^1_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^2_{2n})$$

$$+\ \mathrm{p}_{s_{2n-2}-s_{2n}}(x^1_{2n-2}-x^2_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^1_{2n})$$

$$+\ \mathrm{p}_{s_{2n-2}-s_{2n}}(x^2_{2n-2}-x^2_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^1_{2n})\Big).$$

Exploit now Lemma A2 in the summation over $\mathbf{x}_{2n}$ to obtain

$$\frac{c^3_{\mathrm{A2}}c_2}{s_{2n-2}-s_{2n}} \sum_{x^3_{2n-1}\in\mathsf{Z}^2} \mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^2_{2n-2}-x^3_{2n-1})\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^1_{2n-2}-x^3_{2n-1})$$

$$\Big(\exp\Big[\lambda a^1|x^1_{2n-2}| + \lambda(a^2+a^3)|x^3_{2n-1}|\Big] + \exp\Big[\lambda a^2|x^1_{2n-2}| + \lambda(a^1+a^3)|x^3_{2n-1}|\Big]$$

$$+\ \exp\Big[\lambda a^2|x^3_{2n-2}| + \lambda(a^1+a^3)|x^3_{2n-1}|\Big]\Big)$$

$$+\ \frac{c^3_{\mathrm{A2}}c_2}{s_{2n-2}-s_{2n}} \sum_{x^3_{2n-1}\in\mathsf{Z}^2} \mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^3_{2n-2}-x^3_{2n-1})\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^1_{2n-2}-x^3_{2n-1})$$

$$\Big(\exp\Big[\lambda a^1|x^1_{2n-2}| + \lambda(a^2+a^3)|x^3_{2n-1}|\Big] + \exp\Big[\lambda a^2|x^1_{2n-2}| + \lambda(a^1+a^3)|x^3_{2n-1}|\Big]$$

$$+\ \exp\Big[\lambda a^2|x^2_{2n-2}| + \lambda(a^1+a^3)|x^3_{2n-1}|\Big]\Big),$$

where $c_{\mathrm{A2}} = c_{\mathrm{A2}}(T,2\lambda,\sigma)$. Next we apply Lemma A6 to $x^3_{2n-1}$ to arrive at

$$
\begin{aligned}
&\frac{c_{\mathrm{A2}}^3\, c_{\mathrm{A6}}\, c_2}{s_{2n-2}-s_{2n}} \Bigg( \exp\Big[\lambda a^1 |x^1_{2n-2}| + \frac{\lambda}{2}(a^2+a^3)|x^1_{2n-2}| + \frac{\lambda}{2}(a^2+a^3)|x^2_{2n-2}|\Big] \\
&\qquad\qquad \mathrm{p}_{2\mathrm{e}^{5\lambda^2(a^2+a^3)^2}(s_{2n-2}-s_{2n-1})}(x^1_{2n-2}-x^2_{2n-2}) \\
&\quad + \exp\Big[\lambda a^2 |x^1_{2n-2}| + \frac{\lambda}{2}(a^1+a^3)|x^1_{2n-2}| + \frac{\lambda}{2}(a^1+a^3)|x^2_{2n-2}|\Big] \\
&\qquad\qquad \mathrm{p}_{2\mathrm{e}^{5\lambda^2(a^1+a^3)^2}(s_{2n-2}-s_{2n-1})}(x^1_{2n-2}-x^2_{2n-2}) \\
&\quad + \exp\Big[\lambda a^2 |x^3_{2n-2}| + \frac{\lambda}{2}(a^1+a^3)|x^1_{2n-2}| + \frac{\lambda}{2}(a^1+a^3)|x^2_{2n-2}|\Big] \\
&\qquad\qquad \mathrm{p}_{2\mathrm{e}^{5\lambda^2(a^1+a^3)^2}(s_{2n-2}-s_{2n-1})}(x^1_{2n-2}-x^2_{2n-2})\Bigg) \\
&+ \frac{c_{\mathrm{A2}}^3\, c_{\mathrm{A6}}\, c_2}{s_{2n-2}-s_{2n}} \Bigg( \exp\Big[\lambda a^1 |x^1_{2n-2}| + \frac{\lambda}{2}(a^2+a^3)|x^1_{2n-2}| + \frac{\lambda}{2}(a^2+a^3)|x^2_{2n-2}|\Big] \\
&\qquad\qquad \mathrm{p}_{2\mathrm{e}^{5\lambda^2(a^2+a^3)^2}(s_{2n-2}-s_{2n-1})}(x^1_{2n-2}-x^3_{2n-2}) \\
&\quad + \exp\Big[\lambda a^2 |x^1_{2n-2}| + \frac{\lambda}{2}(a^1+a^3)|x^1_{2n-2}| + \frac{\lambda}{2}(a^1+a^3)|x^2_{2n-2}|\Big] \\
&\qquad\qquad \mathrm{p}_{2\mathrm{e}^{5\lambda^2(a^1+a^3)^2}(s_{2n-2}-s_{2n-1})}(x^1_{2n-2}-x^3_{2n-2}) \\
&\quad + \exp\Big[\lambda a^2 |x^2_{2n-2}| + \frac{\lambda}{2}(a^1+a^3)|x^1_{2n-2}| + \frac{\lambda}{2}(a^1+a^3)|x^2_{2n-2}|\Big] \\
&\qquad\qquad \mathrm{p}_{2\mathrm{e}^{5\lambda^2(a^1+a^3)^2}(s_{2n-2}-s_{2n-1})}(x^1_{2n-2}-x^3_{2n-2})\Bigg)
\end{aligned}
$$

with $c_{\mathrm{A6}} = c_{A6}(T, 4\lambda, \sigma)$. This is our estimate for that part of $L_n(\mathbf{a})$ [abundance over $L_{n-1}(\mathbf{a})$]. It can be written as

$$
\frac{c_{\mathrm{A2}}^3\, c_{\mathrm{A6}}\, c_2}{s_{2n-2}-s_{2n}} \sum_{i=1}^{3} \exp\left[\lambda \,\|\mathbf{a}_i \mathbf{x}_{2n-2}\|\right] \sum_{k=2}^{3} \mathrm{p}_{2b_i\,(s_{2n-2}-s_{2n-1})}(x^1_{2n-2}-x^k_{2n-2}), \tag{A36}
$$

with some $\mathbf{a}_i \in \mathbf{A}$ and $b_i \geq 1$, where the $\mathbf{a}_i$ depend on $\mathbf{a}$, however the $b_i$ on $\mathbf{a}$ and $\lambda$. But by our definition of $M_n^k(\mathbf{a}, b)$ [introduced after (77)], this means

$$
L_n(\mathbf{a}) \;\leq\; \frac{c_{\mathrm{A2}}^3\, c_{\mathrm{A6}}\, c_2}{s_{2n-2}-s_{2n}} \sum_{i=1}^{3}\sum_{k=2}^{3} M^k_{n-1}(\mathbf{a}_i, b_i), \qquad n \geq 2. \tag{A37}
$$

In the definition of $M_n^k(\mathbf{a}, b)$, we restrict our attention to the summands for $\ell = 2n-1$ and $\ell = 2n$, and again to the factor concerning $j = n$ (also some type of abundance):

$$
\begin{aligned}
&\sum_{\mathbf{x}_{2n-1}, \mathbf{x}_{2n} \in (\mathsf{Z}^2)^3} \exp\left[\lambda \,\|\mathbf{a}\mathbf{x}_{2n}\|\right] \mathrm{p}_{2b\,(s_{2n}-s_{2n+1})}(x^1_{2n} - x^k_{2n}) \\
&\Big[\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^2_{2n-2}-x^3_{2n-1})\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^3_{2n-2}-x^1_{2n-1}) \\
&\qquad + \mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^3_{2n-2}-x^3_{2n-1})\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^2_{2n-2}-x^1_{2n-1})\Big] \\
&\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^1_{2n-2}-x^2_{2n-1})\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^1_{2n-2}-x^3_{2n-1})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^3_{2n}) \\
&\Big(\mathrm{p}_{s_{2n-1}-s_{2n}}(x^1_{2n-1}-x^1_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^2_{2n-1}-x^1_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^2_{2n}) \\
&\;+ \mathrm{p}_{s_{2n-1}-s_{2n}}(x^1_{2n-1}-x^1_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^2_{2n-1}-x^2_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^1_{2n}) \\
&\;+ \mathrm{p}_{s_{2n-1}-s_{2n}}(x^1_{2n-1}-x^2_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^2_{2n-1}-x^1_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^1_{2n})\Big).
\end{aligned}
$$

As in the first two estimation steps at the beginning of this subsection, by Chapman-Kolmogorov, we sum over $x^2_{2n-1}$ and $x^1_{2n-1}$, and use Lemma 2 (b) six times to obtain the upper bound

$$\frac{c_2}{s_{2n-2}-s_{2n}} \sum_{x^3_{2n-1}\in \mathsf{Z}^2,\ \mathbf{x}_{2n}\in(\mathsf{Z}^2)^3} \exp\big[\lambda\,\|\mathbf{a}\mathbf{x}_{2n}\|\big]\,\mathrm{p}_{2b\,(s_{2n}-s_{2n+1})}(x^1_{2n}-x^k_{2n})$$
$$\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^2_{2n-2}-x^3_{2n-1})\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^1_{2n-2}-x^3_{2n-1})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^3_{2n})$$
$$\Big(\mathrm{p}_{s_{2n-2}-s_{2n}}(x^1_{2n-2}-x^1_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^2_{2n})$$
$$+\ \mathrm{p}_{s_{2n-2}-s_{2n}}(x^1_{2n-2}-x^2_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^1_{2n})$$
$$+\ \mathrm{p}_{s_{2n-2}-s_{2n}}(x^3_{2n-2}-x^2_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^1_{2n})\Big)$$

$$+\ \frac{c_2}{s_{2n-2}-s_{2n}} \sum_{x^3_{2n-1}\in \mathsf{Z}^2,\ \mathbf{x}_{2n}\in(\mathsf{Z}^2)^3} \exp\big[\lambda\,\|\mathbf{a}\mathbf{x}_{2n}\|\big]\,\mathrm{p}_{2b\,(s_{2n}-s_{2n+1})}(x^1_{2n}-x^k_{2n})$$
$$\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^3_{2n-2}-x^3_{2n-1})\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^1_{2n-2}-x^3_{2n-1})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^3_{2n})$$
$$\Big(\mathrm{p}_{s_{2n-2}-s_{2n}}(x^1_{2n-2}-x^1_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^2_{2n})$$
$$+\ \mathrm{p}_{s_{2n-2}-s_{2n}}(x^1_{2n-2}-x^2_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^1_{2n})$$
$$+\ \mathrm{p}_{s_{2n-2}-s_{2n}}(x^2_{2n-2}-x^2_{2n})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^1_{2n})\Big).$$

Lemma A6, applied to the sum over $x^1_{2n}$, leads to factors $c_{\mathrm{A}6}=c_{\mathrm{A}6}(T,2\lambda,\sigma)$, to several new

$$\mathbf{a}_i\in\mathbf{A}\ \text{ depending on }\ \mathbf{a},b,k,\lambda,\ \text{ and }\ s_{2n-2},\ldots,s_{2n+1}, \tag{A38}$$

and replacements of $\mathbf{x}_{2n}$ as $(x^1_{2n-2},x^2_{2n},x^3_{2n})$ in the exponential expressions, and certain p–terms. For the p–terms we use Lemma 2 (b), estimating additionally their time expression as follows:

$$\mathrm{e}^{5\lambda^2a_i^2}\,[2b\,(s_{2n}-s_{2n+1})+s_{2n-2}-s_{2n}]\ \geq\ s_{2n-1}-s_{2n+1}\,. \tag{A39}$$

This way we get the bound

$$c_{\mathrm{A}6}\,\frac{c_2}{s_{2n-2}-s_{2n}}\,\frac{c_2}{s_{2n-1}-s_{2n+1}} \sum_{x^3_{2n-1},x^2_{2n},x^3_{2n}\in\mathsf{Z}^2}$$
$$\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^2_{2n-2}-x^3_{2n-1})\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^1_{2n-2}-x^3_{2n-1})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^3_{2n})$$
$$\Big(\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^2_{2n})\exp\Big[\lambda\,\big\|\mathbf{a}_1(x^1_{2n-2},x^2_{2n},x^3_{2n})\big\|\Big]$$
$$+\ \mathrm{p}_{s_{2n-2}-s_{2n}}(x^1_{2n-2}-x^2_{2n})\exp\Big[\lambda\,\big\|\mathbf{a}_2(x^3_{2n-1},x^2_{2n},x^3_{2n})\big\|\Big]$$
$$+\ \mathrm{p}_{s_{2n-2}-s_{2n}}(x^3_{2n-2}-x^2_{2n})\exp\Big[\lambda\,\big\|\mathbf{a}_3(x^3_{2n-1},x^2_{2n},x^3_{2n})\big\|\Big]\Big)$$
$$+\ c_{\mathrm{A}6}\,\frac{c_2}{s_{2n-2}-s_{2n}}\,\frac{c_2}{s_{2n-1}-s_{2n+1}} \sum_{x^3_{2n-1},x^2_{2n},x^3_{2n}\in\mathsf{Z}^2}$$
$$\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^3_{2n-2}-x^3_{2n-1})\mathrm{p}_{s_{2n-2}-s_{2n-1}}(x^1_{2n-2}-x^3_{2n-1})\mathrm{p}_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^3_{2n})$$

$$\Big( p_{s_{2n-1}-s_{2n}}(x^3_{2n-1}-x^2_{2n}) \exp\Big[\lambda \,\|\mathbf{a}_1(x^1_{2n-2},x^2_{2n},x^3_{2n})\|\Big]$$
$$+\ p_{s_{2n-2}-s_{2n}}(x^1_{2n-2}-x^2_{2n}) \exp\Big[\lambda \,\|\mathbf{a}_2(x^3_{2n-1},x^2_{2n},x^3_{2n})\|\Big]$$
$$+\ p_{s_{2n-2}-s_{2n}}(x^2_{2n-2}-x^2_{2n}) \exp\Big[\lambda \,\|\mathbf{a}_3(x^3_{2n-1},x^2_{2n},x^3_{2n})\|\Big]\Big).$$

By Lemma A2, the sum over $x^3_{2n}$ and $x^2_{2n}$ gives the estimate

$$c_{\mathrm{A}6}\,\frac{c_{\mathrm{A}2}\,c_2}{s_{2n-2}-s_{2n}}\,\frac{c_{\mathrm{A}2}\,c_2}{s_{2n-1}-s_{2n+1}} \sum_{x^3_{2n-1}\in \mathsf{Z}^2}$$
$$p_{s_{2n-2}-s_{2n-1}}(x^2_{2n-2}-x^3_{2n-1})p_{s_{2n-2}-s_{2n-1}}(x^1_{2n-2}-x^3_{2n-1})$$
$$\Big(\exp\Big[\lambda a^1_1|x^1_{2n-2}|+\lambda(a^2_1+a^3_1)|x^3_{2n-1}|\Big]+\exp\Big[\lambda a^2_2|x^1_{2n-2}|+\lambda(a^1_2+a^3_2)|x^3_{2n-1}|\Big]$$
$$+\ \exp\Big[\lambda a^2_3|x^3_{2n-2}|+(a^1_3+a^3_3)|x^3_{2n-1}|\Big]\Big)$$

$$+\ c_{\mathrm{A}6}\,\frac{c_{\mathrm{A}2}\,c_2}{s_{2n-2}-s_{2n}}\,\frac{c_{\mathrm{A}2}\,c_2}{s_{2n-1}-s_{2n+1}} \sum_{x^3_{2n-1}\in \mathsf{Z}^2}$$
$$p_{s_{2n-2}-s_{2n-1}}(x^3_{2n-2}-x^3_{2n-1})p_{s_{2n-2}-s_{2n-1}}(x^1_{2n-2}-x^3_{2n-1})$$
$$\Big(\exp\Big[\lambda a^1_1|x^1_{2n-2}|+\lambda(a^2_1+a^3_1)|x^3_{2n-1}|\Big]+\exp\Big[\lambda a^2_2|x^1_{2n-2}|+\lambda(a^1_2+a^3_2)|x^3_{2n-1}|\Big]$$
$$+\ \exp\Big[\lambda a^2_3|x^2_{2n-2}|+(a^1_3+a^3_3)|x^3_{2n-1}|\Big]\Big).$$

Finally, by Lemma A6, the sum over $x^3_{2n-1}$ amounts to

$$\frac{c_{\mathrm{A}2}\,c_{\mathrm{A}6}\,c_2}{s_{2n-2}-s_{2n}}\,\frac{c_{\mathrm{A}2}\,c_{\mathrm{A}6}\,c_2}{s_{2n-1}-s_{2n+1}}$$
$$\sum_{\bar\imath=1}^{3}\sum_{\bar k=2}^{3} p_{2b_{\bar\imath}\,(s_{2n-2}-s_{2n-1})}(x^1_{2n-2}-x^{\bar k}_{2n-2})\ \exp\big[\lambda\,\|\mathbf{a}_{\bar\imath,\bar k}\mathbf{x}_{2n-2}\|\big]$$

with $c_{\mathrm{A}6}=c_{\mathrm{A}6}(T,4\lambda,\sigma)$ and some $\mathbf{a}_{\bar\imath,\bar k}\in\mathbf{A}$ and $b_{\bar\imath}\geq 1$, where the $\mathbf{a}_{\bar\imath,\bar k}$ and $b_{\bar\imath}$ depend on $\mathbf{a},b,k,\lambda,$ and $s_{2n-2},\ldots,s_{2n+1}$ [via $\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3$ – recall (A38) – which enter into the $\mathrm{e}^{5\lambda^2}$–factor in Lemma A6]. This is our estimate for that abundance part of $M^k_n(\mathbf{a},b)$. Since

$$c_{\mathrm{A}2}(T,2\lambda,\sigma)\,c_{\mathrm{A}6}(T,4\lambda,\sigma)\,c_2(\sigma)\ \leq\ c_{24}(T,\lambda,\sigma) \tag{A40}$$

as defined in the lemma, this means that

$$M^k_n(\mathbf{a},b)\ \leq\ \frac{c_{24}}{s_{2n-2}-s_{2n}}\,\frac{c_{24}}{s_{2n-1}-s_{2n+1}}\sum_{\bar\imath=1}^{3}\sum_{\bar k=2}^{3} M^{\bar k}_{n-1}(\mathbf{a}_{\bar\imath,\bar k},b_{\bar\imath}),\qquad n\geq 2, \tag{A41}$$

and

$$M^k_1(\mathbf{a},b)\ \leq\ \frac{c_{24}}{s_0-s_2}\,\frac{c_{24}}{s_1-s_3}\,\mathrm{e}^{2\lambda\|\mathbf{x}_0\|}\sum_{\bar\imath=1}^{3}\sum_{\bar k=2}^{3} p_{2b_{\bar\imath}\,(s_0-s_1)}(x^1_0-x^{\bar k}_0). \tag{A42}$$

Iteration gives (81), and inserting (81) into (A37) amounts to (80), finishing the proof of Lemma 24. □

## A.3 A Feynman integral estimate

We need also the following simple estimate.

**Lemma A8 (Feynman integral estimate)** *For* $n \geq 2$ *and* $s_0 > s_1 > 0$, *set*

$$K_n(s_0, s_1) := \int_0^{s_1} \mathrm{d}s_2 \cdots \int_0^{s_{n-1}} \mathrm{d}s_n \, \frac{1}{\prod_{j=2}^{n} (s_{j-2} - s_j)} . \tag{A43}$$

*Then, for each* $p \in (0, 1)$,

$$K_n(s_0, s_1) \;\leq\; \frac{1}{p} \, c_{\mathrm{A8}}^{n-2} \Big( \frac{s_0}{s_0 - s_1} \Big)^p , \tag{A44}$$

*where*

$$c_{\mathrm{A8}} \;=\; c_{\mathrm{A8}}(p) \;:=\; \pi / \sin\big[\pi\,(1-p)\big] .$$

**Proof** We proceed by induction. If $n = 2$, then the left hand side of (A44) equals

$$\int_0^{s_1} \frac{\mathrm{d}s_2}{s_0 - s_2} \;=\; \log \frac{s_0}{s_0 - s_1} \;\leq\; \frac{1}{p} \Big( \frac{s_0}{s_0 - s_1} \Big)^p , \tag{A45}$$

where we used the elementary inequality

$$\log r \;\leq\; p^{-1} \, r^p , \qquad r \geq 1 . \tag{A46}$$

Hence, (A44) holds in the case $n = 2$. Suppose now that it is true for $n \geq 2$. Then,

$$\begin{aligned} K_{n+1}(s_0, s_1) \;&=\; \int_0^{s_1} \frac{\mathrm{d}s_2}{(s_0 - s_1) + (s_1 - s_2)} \, K_n(s_1, s_2) \\ &\leq\; \frac{1}{p} \, c_{\mathrm{A8}}^{n-2} \, s_1^p \int_0^{s_1} \frac{\mathrm{d}s_2}{(s_0 - s_1) + s_2} \, \frac{1}{s_2^p} . \end{aligned} \tag{A47}$$

Substituting $r := s_2/(s_0 - s_1)$ the right hand side is

$$\begin{aligned} &=\; \frac{1}{p} \, c_{\mathrm{A8}}^{n-2} \Big( \frac{s_1}{s_0 - s_1} \Big)^p \int_0^{s_1/(s_0 - s_1)} \frac{\mathrm{d}r}{(1 + r)\, r^p} \\ &\leq\; \frac{1}{p} \, c_{\mathrm{A8}}^{n-2} \Big( \frac{s_0}{s_0 - s_1} \Big)^p \int_0^{\infty} \frac{\mathrm{d}r}{(1 + r)\, r^p} \;=\; \frac{1}{p} \, c_{\mathrm{A8}}^{n-1} \Big( \frac{s_0}{s_0 - s_1} \Big)^p \end{aligned} \tag{A48}$$

by a standard residue calculation. The result follows for $n + 1$. □

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# References


[BP94] M.T. Barlow and E.A. Perkins. On the filtration of historical Brownian motion. *Ann. Probab.*, 22:1273–1294, 1994.

[CG86] J.T. Cox and D. Griffeath. Diffusive clustering in the two dimensional voter model. *Ann. Probab.*, 14:347–370, 1986.

[CK00] J.T. Cox and A. Klenke. Recurrence and ergodicity of interacting particle systems. *Probab. Theory Related Fields*, 116(2):239–255, 2000.

[CKP00] J.T. Cox, A. Klenke, and E.A. Perkins. Convergence to equilibrium and linear systems duality. In Luis G. Gorostiza and B. Gail Ivanoff, editors, *Stochastic Models*, volume 26 of *CMS Conference Proceedings*, pages 41–66. Amer. Math. Soc., Providence, 2000.

[DEF+02] D.A. Dawson, A.M. Etheridge, K. Fleischmann, L. Mytnik, E.A. Perkins, and J. Xiong. Mutually catalytic branching in the plane: Finite measure states. WIAS Berlin, Preprint No. 615, 2000, Ann. Probab., to appear 2002.

[DF97a] D.A. Dawson and K. Fleischmann. A continuous super-Brownian motion in a super-Brownian medium. *Journ. Theoret. Probab.*, 10(1):213–276, 1997.

[DF97b] D.A. Dawson and K. Fleischmann. Longtime behavior of a branching process controlled by branching catalysts. *Stoch. Process. Appl.*, 71(2):241–257, 1997.

[DF02] D.A. Dawson and K. Fleischmann. Catalytic and mutually catalytic super-Brownian motions. In *Ascona 1999 Conference*, volume 52 of *Progress in Probability*, pages 89-110, Birkhäuser Verlag, 2002.

[DFM+02] D.A. Dawson, K. Fleischmann, L. Mytnik, E.A. Perkins, and J. Xiong. Mutually catalytic branching in the plane: Uniqueness. WIAS Berlin, Preprint No. 641, *Ann. Inst. Henri Poincaré Probab. Statist.* (in print), 2002.

[DP98] D.A. Dawson and E.A. Perkins. Long-time behavior and coexistence in a mutually catalytic branching model. *Ann. Probab.*, 26(3):1088–1138, 1998.

[DF01] J.-F. Delmas and K. Fleischmann. On the hot spots of a catalytic super-Brownian motion. *Probab. Theory Relat. Fields*, 121(3):389-421, 2001.

[EF98] A.M. Etheridge and K. Fleischmann. Persistence of a two-dimensional super-Brownian motion in a catalytic medium. *Probab. Theory Relat. Fields*, 110(1):1–12, 1998.

[EK86] S.N. Ethier and T.G. Kurtz. *Markov Processes: Characterization and Convergence.* Wiley, New York, 1986.

[EP94] S.N. Evans and E.A. Perkins. Measure-valued branching diffusions with singular interactions. *Canad. J. Math.*, 46(1):120–168, 1994.

[FG94] K. Fleischmann and A. Greven. Diffusive clustering in an infinite system of hierarchically interacting diffusions. *Probab. Theory Relat. Fields*, 98:517–566, 1994.

[FG96] K. Fleischmann and A. Greven. Time-space analysis of the cluster-formation in interacting diffusions. *Electronic J. Probab.*, 1(6):1–46, 1996.

[FK99] K. Fleischmann and A. Klenke. Smooth density field of catalytic super-Brownian motion. *Ann. Appl. Probab.*, 9(2):298–318, 1999.

[FK00] K. Fleischmann and A. Klenke. The biodiversity of catalytic super-Brownian motion. *Ann. Appl. Probab.*, 10(4):1121-1136, 2000.

[FKX02] K. Fleischmann, A. Klenke and J. Xiong. Pathwise convergence of a rescaled super-Brownian catalyst reactant process. Preprint, University Koeln, Math. Instit., in preparation, `www.aklenke.de/preprints`, 2002.

[Mey66] P.-A. Meyer. *Probability and Potentials.* Blaisdell Publishing Company, Toronto, 1966.

[Mit85] I. Mitoma. An $\infty$–dimensional inhomogeneous Langevin equation. *J. Functional Analysis*, 61:342–359, 1985.

[Myt98] L. Mytnik. Uniqueness for a mutually catalytic branching model. *Probab. Theory Related Fields*, 112(2):245–253, 1998.

[Per00] E.A. Perkins. Dawson-Watanabe superprocesses and measure-valued diffusions. In *École d'été de probabilités de Saint Flour XXIX–1999*, Lecture Notes in Mathematics. Springer-Verlag, Berlin, to appear 2000.

[Shi94] T. Shiga. Two contrasting properties of solutions for one-dimensional stochastic partial differential equations. *Can. J. Math.*, 46:415–437, 1994.

[SS80] T. Shiga and A. Shimizu. Infinite-dimensional stochastic differential equations and their applications. *J. Mat. Kyoto Univ.*, 20:395–416, 1980.

[Wal86] J.B. Walsh. An introduction to stochastic partial differential equations. volume 1180 of *Lecture Notes Math.*, pages 266–439. École d'Été de Probabilités de Saint-Flour XIV – 1984, Springer-Verlag Berlin, 1986.