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ON DISAGREEMENT PERCOLATION AND MAXIMALITY OF THE CRITICAL VALUE FOR IID PERCOLATION

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Abstract Two different problems are studied: (1) For an infinite locally finite connected graph G, let $p_c(G)$ be the critical value for the existence of an infinite cluster in iid bond percolation on G and let $P_c = \sup\{p_c(G) : G \text{ transitive}, p_c(G) < 1\}$. Is $P_c < 1$?

(2) Let G be transitive with $p_c(G) < 1$, let $p \in [0,1]$ and let X and Y be iid bond percolations on G with retention parameters $(1+p)/2$ and $(1-p)/2$ respectively. Is there a $q < 1$ such that $p > q$ implies that for any monotone coupling $(\hat{X}, \hat{Y})$ of X and Y the edges for which $\hat{X}$ and $\hat{Y}$ disagree form infinite connected component(s) with positive probability? Let $p_d(G)$ be the infimum of such q's (including $q = 1$) and let $P_d = \sup\{p_d(G) : G \text{ transitive}, p_c(G) < 1\}$. Is the stronger statement $P_d < 1$ true? On the other hand: Is it always true that $p_d(G) > p_c(G)$?

It is shown that if one restricts attention to biregular planar graphs G then these two problems can be treated in a similar way and all the above questions are positively answered. We also give examples to show that if one drops the assumption of transitivity, then the answer to the above two questions is no. Furthermore it is shown that for any bounded-degree bipartite graph G with $p_c(G) < 1$ one has $p_c(G) < p_d(G)$.

Problem (2) arises naturally from [6] where an example is given of a coupling of the distinct plus- and minus measures for the Ising model on a quasi-transitive graph at super-critical inverse temperature. We give an example of such a coupling on the r-regular tree, $\mathbb{T}_r$, for $r \geq 2$.

Keywords coupling, Ising model, random-cluster model, transitive graph, planar graph

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1 Introduction

Let $G = (V, E)$ be an infinite locally finite connected graph. (For the rest of this paper all graphs are assumed to have these properties if nothing else is explicitly stated.) Let $p_c(G)$ be the critical value for the existence of an infinite cluster for iid bond percolation on G. Then for any $p \in [0, 1]$ one can design G (e.g. by considering a properly chosen spherically symmetric tree, see e.g. [9]) in such a way that $p_c(G) = p$. In particular one can have $p_c(G) < 1$ but arbitrarily close to 1. An example of a sequence $\{G_n\}$ of graphs such that $p_c(G_n) < 1$ and $p_c(G_n) \uparrow 1$ is constructed by letting G_n be a binary tree with each edge replaced by an path of length n. However, if G is required to be *transitive*, i.e. if the automorphism group of G acts transitively on V, then this does not seem to be the case. We have

CONJECTURE 1.1 *Let $P_c = \sup\{p_c(G) : G$ transitive, $p_c(G) < 1\}$. Then $P_c < 1$.*

This was first stated by Olle Häggström (private communication). (One should note that there are of course transitive graphs G with $p_c(G) = 1$ even with arbitrarily high degree.)

We shall prove Conjecture 1.1 for a large class of *planar* graphs, i.e. graphs which can be embedded in the plane in such a way that no two edges cross. From now on we assume that a planar graph is embedded in such a way. For a planar graph G its *planar dual* is the planar (multi)graph $G^\dagger = (V^\dagger, E^\dagger)$ where $V^\dagger$ is the set of faces of G and two faces are joined by an edge in $E^\dagger$ precisely when they share an edge in E. Note that when the minimal degree in G is at least 3 then $G^\dagger$ is a graph. A graph G is said to be *regular* if all its vertices have the same degree and in this case we denote the common degree by d_G. If G is planar and regular and such that $G^\dagger$ is also regular then we say that G is *biregular*. In this case G is also transitive, see [8]. We will prove:

THEOREM 1.2 *Let $P_c^{plb} = \sup\{p_c(G) : G$ planar and biregular, $p_c(G) < 1\}$. Then $P_c^{plb} \leq \sqrt{5}/(1 + \sqrt{5})$.*

The proof is given in Section 3.

Let us now for a while turn to the second question of this paper. van den Berg [3] introduced the concept of *disagreement percolation* as a tool to prove Gibbsian uniqueness for Markov random fields: If X_1 and X_2 are picked independently according to two Gibbs measures μ_1 and μ_2 for the same specifications of a Markov random field on S^V, where S is some finite set, and $P(G$ contains an infinite connected component of vertices where X_1 and X_2 disagree$) = 0$, then $\mu_1 = \mu_2$. Later van den Berg and Maes [4] extended this to certain dependent couplings. One may perhaps be tempted to believe that the result in fact holds for *any* coupling of X_1 and X_2. However as is shown by Häggström [6], thereby confirming a conjecture of Steif, this is not true: One can find a G and a temperature such that the Ising model on G at that temperature has multiple Gibbs measures and a coupling $(\hat{X}_1, \hat{X}_2)$ of X_1 and X_2, where these are distributed according to the minus- and plus Gibbs states respectively, such that there is a.s. no infinite cluster of disagreements between $\hat{X}_1$ and $\hat{X}_2$. The graph G on which this example is constructed is not transitive, however; it is *quasi-transitive*, i.e. the action of the automorphism group on V partitions V into finitely many orbits. (In fact two orbits in this case.) Thus one may still ask if Steif's conjecture is false for transitive graphs. The answer to this also turns out to be no. We show:

THEOREM 1.3 *Let $r \geq 2$, let $G = \mathbb{T}_r$, the $r+1$-regular tree and let β_c be the critical inverse temperature for the existence of multiple Gibbs mesures for the Ising model on G. Then there exists $\beta_d > \beta_c$ such that for all $\beta \in (\beta_c, \beta_d)$ there is a coupling $(\hat{X}_1, \hat{X}_2)$ of X_1 and X_2 (defined on a probability space with underlying probabilty measure P), where X_1 and X_2 are distributed according to the minus- and plus Gibbs measures respectively, such that P(there exists an infinite connected component of disagreements between $\hat{X}_1$ and $\hat{X}_2$) $= 0$.*

An example that proves Theorem 1.3 is given in Section 5.

In light of Theorem 1.3 many questions arise naturally. For example: Could $\mathbb{T}_r$ be replaced by any transitive graph G with $p_c(G) < 1$? Could β be arbitrarily large or must it be close to β_c? Is the set of β's for which such a coupling exists an interval? What if one considers Potts models with more than two different spins? Could one replace $\mathbb{T}_r$ by the integer lattice $\mathbb{Z}^d$? However, when considering questions of this sort there is no particular reason to restrict oneself to Ising/Potts models. Indeed one could ask these or similar questions about virtually any percolation process. However in order to make the problems precise one has to choose a model to work with. We shall instead of working with the Ising/Potts model make what seems the most natural choice and use the "cleanest" percolation process as our testing ground, namely iid (bond) percolation. We shall ask:

- Is there a graph G for which for any $p \in [0,1]$, iid (bond) percolation with retention parameter $(1+p)/2$ can be coupled to iid percolation with retention parameter $(1-p)/2$ with a.s. no infinite cluster of disagreement?

and the natural reverse question:

- If $(1+p)/2 - (1-p)/2 = p < p_c(G)$ then by coupling the two processes via independent edge-wise maximal couplings, there will a.s. be no disagreement percolation. Is there any G for which this is indeed the best one can do?

In the second case it is not hard to see that for many G one can do better than just edge-wise maximal couplings. For example when $G = \mathbb{T}_2$, the binary tree, one can by dividing the edges into properly chosen "families" construct a coupling without disagreement percolation even when p is slightly larger than $3/4$ (indeed, as it turns out the coupling in Section 5 is more or less a coupling of this kind, but for site percolation) and as we shall see below that this idea works in much larger generality.

We believe that the answers to the two questions are no in general, as long as we stick to "nice enough" graphs:

CONJECTURE 1.4 *If G is transitive and $p_c(G) < 1$ then there exists a $q < 1$ such that if $p > q$ then any coupling of iid-percolation(p) with iid-percolation(1-p) produces with positive probability an infinite cluster of disagreements. Moreover, with $p_d(G)$ being the infimum of such q's (including $q = 1$) and $P_d = \sup\{p_d(G) : G \text{ transitive}, p_c(G) < 1\}$, we have $P_d < 1$.*

On the other hand, if G is any bounded-degree graph with $p_c(G) < 1$, then $p_d(G) > p_c(G)$.

If we drop the assumption of transitivity, then the first part of the conjecture is false. Indeed in Section 4 we give a counterexample where G has $p_c(G) = 0$ and $p_d(G) = 1$. For the second part with the assumption of bounded degree dropped we do not know the answer. In Section 4 we also prove the following partial confirmation of Conjecture 1.4. Before stating it we need the definition of a *bipartite* graph: The graph $G = (V, E)$ is said to be bipartite if V can be partitioned into $V_o \cup V_e$, the sets of *odd* vertices and *even* vertices respectively, such that $u \in V_o, (u, v) \in E \Rightarrow v \in V_e$ and $u \in V_e, (u, v) \in E \Rightarrow v \in V_o$. Note that a graph is bipartite iff all its cycles are of even length. This applies to many of the commonly studied graphs, e.g. $\mathbb{Z}^d$, the hexagonal lattice and all trees (but not to the triangular lattice though).

THEOREM 1.5 *(a) Let G be planar and biregular with $p_c(G) < 1$. Then $p_d(G) < 1$. Moreover, with $P_d^{plb} = \sup\{p_d(G) : G$ planar and biregular, $p_c(G) < 1\}$, we have $P_d^{plb} < 1$.*

(b) Let G be a bounded-degree bipartite graph with $p_c(G) < 1$. Then $p_d(G) > p_c(G)$.

2 Preliminaries

The *(egde-)isoperimetric constant* of a graph $G = (V, E)$ is defined as

$$\kappa(G) = \inf\{\frac{|\partial_E W|}{|W|} : W \subset V, |W| < \infty\}$$

where $\partial_E W$ is the outer edge boundary of W, i.e. the set of edges with one endpoint in W and one endpoint in $V \setminus W$. If $\kappa(G) = 0$ then G is said to be *amenable* and if $\kappa(G) > 0$ then G is said to be *nonamenable.*

When G is transitive we have from [1] that $\kappa(G)$ coincides with

$$\kappa'(G) = \liminf_{N\to\infty}\{\frac{|\partial_E W|}{|W|} : W \text{ connected}, N < |W| < \infty\}.$$

In general we have $\kappa(G) \leq \kappa'(G)$. If G is nonamenable then $p_c(G) < 1$. Indeed, Benjamini and Schramm [2] show that:

LEMMA 2.1 $p_c(G) \leq 1/(1 + \kappa'(G))$.

We will use this along with the following formula from [8] (Theorem 4.1):

LEMMA 2.2 *Let G be planar and biregular with dual $G^\dagger$ and assume that $d_{G^\dagger} < \infty$. Then*

$$\kappa(G) = (d_G - 2)\sqrt{1 - \frac{4}{(d_G - 2)(d_{G^\dagger} - 2)}}.$$

In case $G^\dagger$ is not regular the formula of Lemma 2.2 has no meaning. We believe that if $d_{G^\dagger}$ is replaced with the harmonic average of the numbers of edges of the d_G different faces of G that any vertex of G is incident to, then the formula still holds or at least is not far from the truth. Unfortunately we do not know how to prove such a result and this is the reason for restricting ourselves to planar biregular graphs instead of all planar transitive graphs in Theorems 1.2 and 1.5(a). If our belief holds then these theorems would extend in a straightforward way to the class of all planar transitive graphs.

We will also need the following result of Kesten [11]:

Lemma 2.3 *Let G be a regular graph and for $n = 1, 2, \dots$ and $v \in V$, let $C_n(v)$ be the number of connected subsets of n vertices that contain v. Then*

$$C_n(v) \leq (ed_G)^n.$$

Finally before moving on to the main sections we need to introduce the concept of stochastic domination. Let $(S, \mathcal{S})$ be some measurable partially ordered space. An event $A \in \mathcal{S}$ is said to be increasing if $x \in A, x \leq y \Rightarrow y \in A$. If μ and ν are two probability measures on $\mathcal{S}$ then we say that μ is stochastically dominated by ν if $\mu(A) \leq \nu(A)$ for every increasing event A and in this case we write $\mu \leq_d \nu$. If X and Y are S-valued random variables distributed to μ and ν respectively, then we say that X is stochastically dominated by Y and write $X \leq_d Y$. By Strassen's Theorem $\mu \leq_d \nu$ is equivalent to the existence of a coupling $(\hat{X}, \hat{Y})$ of X and Y such that $X \leq Y$ a.s.

3 Maximality of $p_c(G)$

Proof of Theorem 1.2. In order to prove Theorem 1.2 we first take care of the cases when $d_{G^\dagger} = \infty$, i.e. when G is a regular tree. Then if $d_G = 2$ we have $p_c(G) = 1$ and if $d_G \geq 3$ then $\kappa(G) = d_G - 2$ and $p_c(G) = 1/(d_G - 1)$ as is well known.

Assume second that $7 \leq d_{G^\dagger} < \infty$. Then by Lemma 2.2 we have

$$\kappa(G) = (d_G - 2)\sqrt{1 - \frac{4}{(d_G - 2)(d_{G^\dagger} - 2)}} \geq 1/\sqrt{5}$$

and so by Lemma 2.1, $p_c(G) \leq \sqrt{5}/(1 + \sqrt{5})$.

If $d_{G^\dagger} \in \{5, 6\}$ and $d_G \geq 4$ then again by Lemma 2.2, $\kappa(G) \geq 2/\sqrt{3}$ so that $p_c(G) \leq \sqrt{3}/(2+\sqrt{3})$. For $d_{G^\dagger} = 4$ and $d_G \geq 5$ we get similarly that $\kappa(G) \geq \sqrt{3}$ and $p_c(G) \leq 1/(1 + \sqrt{3})$ and for $d_{G^\dagger} = 3$ and $d_G \geq 7$ we get $\kappa(G) \geq \sqrt{5}$ and $p_c(G) \leq 1/(1 + \sqrt{5})$.

The only remaining cases are now the square, triangular and hexagonal lattices for which $p_c(G)$ is $1/2$, $2\sin(\pi/18)$ and $1 - 2\sin(\pi/18)$ respectively. (Note that if $d_G = 3$ and $d_{G^\dagger} \leq 5$ or vice versa then G is finite.)

Summarizing we have seen that $P_c^{plb} \leq \sqrt{5}/(1 + \sqrt{5}) < 0.7$, proving Theorem 1.2. □

Remark. By considering all "standard" examples of nonamenable transitive graphs it is easy to believe that if one puts $K = \inf\{\kappa(G) : G \text{ transitive and nonamenable}\}$, then $K > 0$. In light of Lemma 2.1 this would then have implied an analog to Theorem 1.2 for the class of all nonamenable transitive graphs. However it turns out that $K = 0$. This was recently established by Grigorchuk and de la Harpe [5] who construct a sequence $\{G_n\}$ of nonamenable Cayley graphs such that $\kappa(G_n) \downarrow 0$. The sequence $\{G_n\}$ is of course a candidate for a counterexample to Conjecture 1.1. However we believe, but do not know how to prove, that $\inf_n p_c(G_n) > 0$.

4 Disagreement percolation

4.1 Proof of Theorem 1.5(a)

Let $G = (V, E)$ be a graph and let X and Y be bond percolations on G with retention parameters $(1+p)/2$ and $(1-p)/2$ respectively. We saw in the proof of Theorem 1.2 above that if G is any planar biregular graph other than the square, triangular or hexagonal lattice, then G is nonamenable with $\kappa(G) \geq 1/\sqrt{5}$. Indeed Lemma 2.2 tells us that for all these graphs we have $\kappa(G) \geq \kappa_0 d_G$ where $\kappa_0 = (3\sqrt{5})^{-1}$. Now let $(\hat{X}, \hat{Y})$ be an arbitrary coupling of X and Y with underlying probability measure P. Fix a positive integer m and a connected set $K \subset V$ with $|K| = m$. In order for no vertex $v \in K$ to be in an infinite cluster of disagreements between $\hat{X}$ and $\hat{Y}$ there must be some positive integer n and some connected $W \subset V$ such that $|W| = n$, $W \supseteq K$ and such that $\hat{X}$ and $\hat{Y}$ agree on every edge of $\partial_E W$. However

$$
\begin{aligned}
P(\hat{X}(\partial_E W) \equiv \hat{Y}(\partial_E W)) &\leq P(|\hat{X}(\partial_E W)| \leq n/2) + P(|\hat{Y}(\partial_E W)| \geq n/2) \\
&= 2P(|\hat{X}(\partial_E W)| \leq n/2).
\end{aligned}
$$

Since $|\hat{X}(\partial_E W)|$ has a binomial distribution with parameters $|\partial_E W|$ and p, a standard tail estimate in the binomial distribution yields for some constant C (depending on p)

$$
\begin{aligned}
P(\hat{X}(\partial_E W) \equiv \hat{Y}(\partial_E W)) &\leq 2C\binom{n}{\lfloor n/2 \rfloor} 2^{-n}(1+p)^{\lceil n/2 \rceil}(1-p)^{\lfloor n/2 \rfloor} \\
&\leq 2C(1-p^2)^{n/2}.
\end{aligned}
$$

Summing over n and all W's with $|W| = n$, using Lemma 2.3 applied with v being any vertex of K, we get

$$
\begin{aligned}
&P(\text{No } v \in K \text{ is in an infinite cluster of disagreements between } \hat{X} \text{ and } \hat{Y}) \\
&\qquad \leq 2C \sum_{n=m}^{\infty} (ed_G)^n((1-p^2)^{\kappa_0 d_G/2})^n
\end{aligned}
$$

which is less than 1 for large enough m as soon as

$$ed_G(1-p^2)^{\kappa_0 d_G/2} < 1$$

i.e. as soon as

$$p^2 \geq 1 - \frac{1}{(ed_G)^{2/(\kappa_0 d_G)}} = 1 - \frac{1}{(ed_G)^{6\sqrt{5}/d_G}}.$$

The "worst case" is when $d_G = 3$ in which case we get that $p_d(G)$ is bounded above by

$$\sqrt{1 - \frac{1}{(3e)^{2\sqrt{5}}}}$$

which is less than 0.9999581.

(Note also that as $d_G \to \infty$ we get that $p_d(G) = O(\sqrt{\log d_G / d_G})$ which can be compared to the bound given on $p_c(G)$ by Lemma 2.1 and Lemma 2.2: $p_c(G) = O(1/d_G)$.)

Now assume that G is one of the square lattice, the triangular lattice and the hexagonal lattice. In each of these case we have $d_{G^\dagger} \leq 6$, so for each integer n the number of cutsets (i.e. edge boundaries of finite connected subsets of V containing K) of size n is bounded by $n5^n$. By using the same arguments as above we get that

$$P(\text{No } v \in K \text{ is in an infinite cluster of disagreements between } \hat{X} \text{ and } \hat{Y})$$

$$\leq 2 \sum_{n=m}^{\infty} n5^n((1-p^2)^{1/2})^n$$

which is less than 1 for large enough m as soon as

$$p \geq \frac{\sqrt{24}}{5}.$$

Thus $p_d(G) \leq \sqrt{24}/5 < 0.9798$. Theorem 1.5(a) follows with $P_d^{plb} < 0.9999581$. □

4.2 A non-transitive counterexample

Now we shall drop the assumption of transitivity and see that we can construct a G with $p_c(G) = 0$ such that X and Y can for any p be coupled without an infinite cluster of disagreements.

A graph G is said to be a *spherically symmetric tree* if it is a tree and if for some vertex $v \in V$ it is the case that for every positive integer n all vertices at graphical distance n from v have the same degree. The vertex v is called a root of G. Now let G be a spherically symmetric tree specified in the following way: Let a vertex o, which is to be the root, have degree 3. Let d_n be the degree of vertices at distance n from o and let

$$\{d_1, d_2, d_3, \ldots\}$$

$$= \{3,2,4,4,2,2,5,5,5,2,2,2,6,6,6,6,2,2,2,2,7,7,7,7,7,2,2,2,2,2,\ldots\}.$$

Then a simple branching process argument shows that $p_c(G) = 0$. However, all infinite connected paths in G contain arbitrarily long induced paths, i.e. paths containing only vertices of degree 2. For a given p let N be the smallest integer such that $((1+p)/2)^N \leq 1/2$ and assume for simplicity that $((1+p)/2)^N = 1/2$. Now for an edge e not in an induced path containing at least N edges, let $(\hat{X}(e), \hat{Y}(e))$ be independent of $(\hat{X}(E \setminus \{e\}), \hat{Y}(E \setminus \{e\}))$ and such that $\hat{X}(e) \geq \hat{Y}(e)$. For an induced path W containing at least N edges, let $E(W)$ be its set of edges and let $E(W) = E_1 \cup E_2$ where E_1 consists of the N edges that are nearest to o and $E_2 = E(W) \setminus E_1$. For $e \in E_2$, couple $\hat{X}(e)$ and $\hat{Y}(e)$ independently of all other edges like for the edges not in long induced paths above. For E_1, let $\hat{X}(E_1) \equiv 1$ iff $\hat{Y}(E_1) \not\equiv 0$. This is possible since $((1+p)/2)^N = 1/2$. Couple all induced paths containing at least N edges independently of each other in this way. Now since each such induced path contains at least one edge for which $\hat{X}(e) = \hat{Y}(e)$ it follows that no infinite cluster of disagreements can exist.

Finally if $((1+p)/2)^N < 1/2$, then minor modifications which are left to the reader yields a monotone coupling where $\hat{X}(E_1) \equiv 1 \Rightarrow \hat{Y}(E_1) \not\equiv 0$.

Remark. The above example was constructed so that $p_c(G) = 0$ in order to be as spectacular as possible. This of course requires unbounded degree. If one instead wants a bounded-degree example then an obvious modification of the above will do fine, but with $0 < p_c(G) < 1$.

4.3 Proof of Theorem 1.5(b)

Let Q be the probability measure that underlies the coupling (X', Y') of X and Y via independent edge-wise maximal couplings, i.e. for which $Q(X'(e) = Y'(e) = 1) = Q(X'(e) = Y'(e) = 0) = (1-p)/2$ and $Q(X'(e) = 1, Y'(e) = 0) = p$, independently for different edges. By letting $Z'(e) = I_{\{X'(e) \neq Y'(e)\}}$, the indicator of disagreement at e, $\{Z'(e)\}_{e \in E}$ gets exactly the distribution of iid bond percolation(p).

Since G is assumed to be bipartite one can partition E into *stars*, i.e. write $V = V_o \cup V_e$, the sets of odd vertices and even vertices respectively, and let for each $v \in V_o$ the star S_v be the subset of E consisting of the d_v edges that are incident to v. Then since G is bipartite the S_v's are disjoint and $\bigcup_{v \in V_o} S_v = E$. The crucial observation is now that for determining whether or not there exists an infinite cluster for a bond percolation process on G it is not interesting to know the states of the individual edges of a star as long as one knows which of their even end-vertices that are connected to each other through the star. (This is not an original observation of this paper. It was originally used by Wierman as one ingredient for determining the exact critical value for bond percolation on the triangular and hexagonal lattices, see [12].)

Now fix some star, i.e. some $v \in V_o$, and label the edges incident to v in some order $e_1, e_2, \ldots, e_{d_v}$ and label their even end-vertices $v_1, v_2, \ldots, v_{d_v}$ accordingly. For any $\mathbf{x}, \mathbf{y} \in \{0,1\}^{S_v}$ such that $\mathbf{x} \geq \mathbf{y}$ we have

$$Q(X'(S_v) = \mathbf{x}, Y'(S_v) = \mathbf{y})$$
$$= \prod_{k=1}^{d_v} (\frac{1+p}{2})^{x_k} (\frac{1-p}{2})^{1-x_k} (\frac{1-p}{1+p})^{y_k} (2p)^{1-y_k} > 0.$$

Let $\delta_v = \min_{\mathbf{x}, \mathbf{y} \in \{0,1\}^{S_v} : \mathbf{x} \geq \mathbf{y}} Q(X'(S_v) = \mathbf{x}, Y'(S_v) = \mathbf{y})$. Now we make the following coupling $(\hat{X}, \hat{Y})$ of X and Y (with underlying probability measure P) by making a "cyclic adjustment" of (X', Y'): Let

$$P(\hat{X}(S_v) = (1,1,\ldots,1), \hat{Y}(S_v) = (0,\ldots,0)) =$$
$$Q(X'(S_v) = (1,1,\ldots,1), Y'(S_v) = (0,\ldots,0)) - \delta_v,$$

$$P(\hat{X}(S_v) = (1,1,\ldots,1), \hat{Y}(S_v) = (0,1,\ldots,1)) =$$
$$Q(X'(S_v) = (1,1,\ldots,1), Y'(S_v) = (0,1,\ldots,1)) + \delta_v,$$

$$P(\hat{X}(S_v) = (0,1,\ldots,1), \hat{Y}(S_v) = (0,1,\ldots,1)) =$$
$$Q(X'(S_v) = (0,1,\ldots,1), Y'(S_v) = (0,1,\ldots,1)) - \delta_v,$$

$$P(\hat{X}(S_v) = (0,1,\ldots,1), \hat{Y}(S_v) = (0,0,1,\ldots,1)) =$$
$$Q(X'(S_v) = (0,1,\ldots,1), Y'(S_v) = (0,0,1,\ldots,1)) + \delta_v,$$

$$P(\hat{X}(S_v) = (0,0,1,\ldots,1, \hat{Y}(S_v) = (0,0,1,\ldots,1)) =$$
$$Q(X'(S_v) = (0,0,1,\ldots,1, Y'(S_v) = (0,0,1,\ldots,1)) - \delta_v,$$

$$P(\hat{X}(S_v) = (0,0,1,\ldots,1), \hat{Y}(S_v) = (0,0,0,1,\ldots,1)) =$$
$$Q(X'(S_v) = (0,0,1,\ldots,1), Y'(S_v) = (0,0,0,1,\ldots,1)) + \delta_v,$$

$$\vdots$$

$$P(\hat{X}(S_v) = (0,\ldots,0,1), \hat{Y}(S_v) = (0,\ldots,0,1)) =$$
$$Q(X'(S_v) = (0,\ldots,0,1), Y'(S_v) = (0,\ldots,0,1)) - \delta,$$

$$P(\hat{X}(S_v) = (0,\ldots,0,1), \hat{Y}(S_v) = (0,\ldots,0)) =$$
$$Q(X'(S_v) = (0,\ldots,0,1), Y'(S_v) = (0,\ldots,0)) + \delta.$$

Do this independently for all stars S_v, $v \in V_o$.

In analogy with how Z' was defined, set $\hat{Z}(e) = I_{\{\hat{X}(e) \neq \hat{Y}(e)\}}$. Again fix $v \in V_o$. For all pairs $\{v_i, v_j\}$ of vertices of $\{v_1, \ldots, v_{d_v}\}$, set $U'(v_i, v_j)$ to be the indicator of the event that v_i and v_j are connected by edges $e \in S_v$ for which $Z'(e) = 1$ and define $\hat{U}(v_i, v_j)$ correspondingly. Then by the nature of the couplings we have that the Q-probability and the P-probability for any particular possible outcome of the $U'(v_i, v_j)$'s and the corresponding outcome of the $\hat{U}(v_i, v_j)$'s respectively are the same with the exceptions that $Q(U'(v_i, v_j) = 1$ for every i and $j) = P(\hat{U}(v_i, v_j) = 1$ for every i and $j) + \delta_v$ and $Q(U'(v_i, v_j) = 0$ for every i and $j) = P(\hat{U}(v_i, v_j) = 0$ for every i and $j) - \delta_v$. Therefore $\hat{U}$ is stochastically dominated by U' in such a way that for any increasing event $A \in \{0,1\}^{S_v^{(2)}}$, (where $S_v^{(2)}$ is the set of subsets of order 2 of $\{v_1, \ldots, v_{d_v}\}$) which is not the whole set $\{0,1\}^{S_v^{(2)}}$, we have $P(\hat{U} \in A) \leq Q(U' \in A) - \delta_v$.

Now observe that the probabilities for any of the outcomes discussed above are continuous as functions of p and that the δ_v's only depend on v through the degree of v so that $\min_{v \in V_o} \delta_v > 0$. It thus follows that the $\hat{U}$ corresponding to $(\hat{X}, \hat{Y})$ for p' larger than but close enough to p is also stochastically dominated by the U' corresonding to (X', Y') for p and so Theorem 1.5(b) follows from Strassen's Theorem. $\square$

5 A coupling for the Ising model

5.1 Definitions and basic facts on the Ising and random-cluster models

DEFINITION 5.1 *Let $G = (V, E)$ be a graph and let $\beta \geq 0$. If μ_G^β is a probability measure on $\{-1,1\}^V$ such that if X is a random variable with distribution μ_G^β then for every finite $W \subset V$, every $\omega \in \{-1,1\}^W$ and a.e. $\omega' \in \{-1,1\}^{V \setminus W}$*

$$P(X(W) = \omega | X(V \setminus W) = \omega') = \frac{1}{Z} e^{-2\beta D(\omega, \omega')} \tag{5.1}$$

where

$$D(\omega, \omega') = \sum_{u,v \in W : (u,v) \in E} I_{\{\omega(u) \neq \omega(v)\}} + \sum_{u \in W, v \in V \setminus W : (u,v) \in E} I_{\{\omega(u) \neq \omega'(v)\}}$$

and Z is a normalizing constant, then μ_G^β is called a Gibbs measure for the Ising model on G at inverse temperature β.

Gibbs measures satisfying (5.1) can be constructed by letting W_n, $n = 1, 2, \ldots$, be finite subsets of V such that $W_n \uparrow V$ and *defining* measures $\mu^{\beta}_{+,n}$ and $\mu^{\beta}_{-,n}$ according to (5.1) with $W = W_n$ and $\omega' \equiv 1$ and $\omega' \equiv -1$ respectively. It is well known that there exist weak limits $\mu^{\beta}_{+} = \lim_{n\to\infty} \mu^{\beta}_{+,n}$ and $\mu^{\beta}_{-} = \lim_{n\to\infty} \mu^{\beta}_{-,n}$, that these limit measures satisfy (5.1) and that for every other Gibbs measure μ satisfying (5.1) on gets $\mu^{\beta}_{-} \leq_d \mu \leq_d \mu^{\beta}_{+}$. Hence there are multiple Gibbs measures iff $\mu^{\beta}_{+} \neq \mu^{\beta}_{-}$. It is also known that there exists a critical inverse temperature $\beta_c \in [0, \infty]$ such that for smaller inverse temperatures there is a unique Gibbs measure and for larger inverse temperatures there are multiple Gibbs measures.

There is a close connection between the Ising model and the *random-cluster model* that we will need.

Definition 5.2 *Let $p \in [0, 1]$ and $q > 0$ and let $\gamma^{p,q}$ be a probability measure on $\{0, 1\}^E$ such that if Y is a random variable with distribution $\gamma^{p,q}$ then for every finite $S \subset E$, every $\eta \in \{0, 1\}^S$ and a.e. $\eta' \in \{0, 1\}^{E \setminus S}$*

$$P(Y(S) = \eta | Y(E \setminus S) = \eta') = \frac{1}{Z}\big(\prod_{e \in S} p^{\eta(e)}(1 - p)^{1-\eta(e)}\big)q^{k(\eta,\eta')} \tag{5.2}$$

where $k(\eta, \eta')$ is the number of **finite** *clusters in the open subgraph given by η and η' that intersect the set of vertices incident to at least one edge in S, and Z is a normalizing constant. Then $\gamma^{p,q}$ is called a (DLR-wired) random-cluster measure on G with parameters p and q.*

Just as for the Ising model, there may be multiple random-cluster measures, see [10]. Here however we will only be concerned with one random-cluster measure, namely $\gamma^{p,q}_w$ which is the weak limit of the sequence $\{\gamma^{p,q}_n\}$ where these are defined by letting $S_n \subset E$ be finite and such that $S_n \uparrow E$ and defining $\gamma^{p,q}_n$ according to (5.2) with $S = S_n$ and $\eta' \equiv 1$. (The subscript w stands for "wired".) Then $\gamma^{p,q}_w$ satisfies (5.2), see [10] again.

The following connection between the random-cluster model and the Ising model is well known:

Lemma 5.3 *For $\beta \geq 0$, let $p = 1 - e^{-2\beta}$. Let $i \in \{-1, 1\}$. Pick $X \in \{-1, 1\}^V$ by first picking $Y \in \{0, 1\}^E$ according to $\gamma^{p,2}_w$ and then for each finite cluster in the open subgraph of Y picking a spin in $\{-1, 1\}$ uniformly at random and assigning this spin to every vertex of that cluster. Do this independently for all finite clusters. Finally assign the spin i to all vertices of infinite clusters. Then X has distribution μ^{β}_{i}. (We identify $\{-1, 1\}$ with $\{-, +\}$.)*

An immediate consequence of Lemma 5.3 is that $\mu^{\beta}_{+} \neq \mu^{\beta}_{-}$ iff $\mu^{p,2}_w$ gives rise to infinite cluster(s). In the special case when $G = \mathbb{T}_r$, it is known (see [7]) that when $p/(2 - p) \leq 1/r$, i.e. when $p \leq 2/(1 + r)$, then $\gamma^{p,2}_w$ is just iid bond percolation with retention parameter $p/(2 - p)$. One consequence is that $p = 2/(1 + r)$ is the critical value for percolation for $\gamma^{p,2}_w$ and another is that as $p \downarrow 2/(1 + r)$ the probability for a given edge to be open in a $\gamma^{p,2}_w$-configuration approaches $1/r$ and the probability for a given vertex to be in an infinite open cluster approaches 0. In other words the phase transition in the corresponding Ising model is second order.

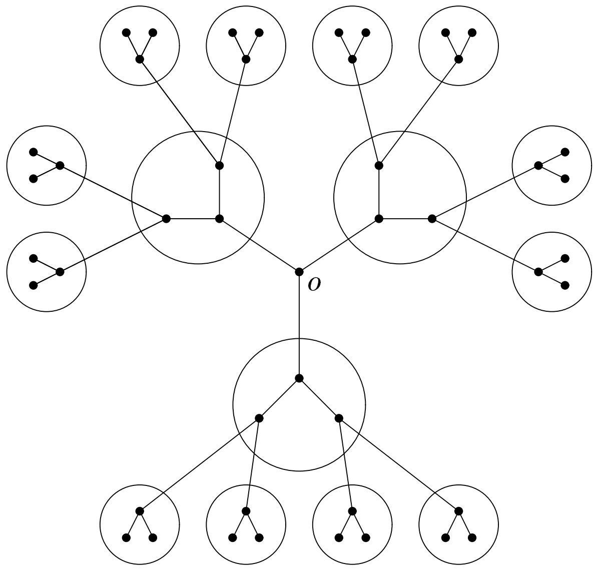

Figure 1: Partitioning the vertices of $\mathbb{T}_2$ into families.

5.2 Proof of Theorem 1.3

For simplicity we will do the promised coupling on $G = \mathbb{T}_2$; the general case is analogous. Thus $1/r$ and $2/(1+r)$ are now $1/2$ and $2/3$ respectively. Fix p "slightly" larger than $2/3$, (where the "slightly" will be specified below) and let $\beta = -\log(1-p)/2$ be the corresponding inverse temperature for the Ising model and note that by Lemma 5.3 we have that $\mu_+^\beta \neq \mu_-^\beta$. Fix a vertex $o \in V$. Couple $X_1, X_2 \in \{-1,1\}^V$, where X_i is distributed according to μ_i^β, in the following way: First pick $(\hat{X}_1(o), \hat{X}_2(o))$ with the correct marginals in such a way that $\hat{X}_1(o) \leq \hat{X}_2(o)$ and independently of all other vertices. Next partition $V \setminus \{o\}$ into *families*, where each family is formed by one vertex at odd distance from o (called the mother) together with the two neighbors of that vertex that are one step further away from o (called the daughters). See Figure 1 for an illustration.

Note how this in a natural way gives rise to a "super-tree" where every "vertex" (i.e. family) but o has degree 5. Now order the families and define the coupling recursively: At each step pick the first family in the ordering for which the coupling has not yet been specified and which is connected to a family for which the coupling has been specified. Denote the vertices in this family m, d_1 and d_2; mother, daughter 1 and daughter 2 respectively. Let m_0 be the "grand-mother", i.e. the neighbor of m which is not d_1 or d_2. If $\hat{X}_1(m_0) = \hat{X}_2(m_0)$ then no disagreement percolation is possible through this family, so define the coupling in any way you like only so that $\hat{X}_1(m, d_1, d_2) \leq \hat{X}_2(m, d_1, d_2)$ and independently of all other families. Assume now that $\hat{X}_1(m_0) = -1$ and $\hat{X}_2(m_0) = 1$. Now if p had been exactly $2/3$ then since the random-cluster would have been iid bond percolation with retention parameter $1/2$, the conditional distributions for $\hat{X}_1(m, d_1, d_2)$ and $\hat{X}_2(m, d_1, d_2)$ would then by Lemma 5.3 have been:

Outcome	$\hat{X}_1(m,d_1,d_2)$	$\hat{X}_2(m,d_1,d_2)$
$(-1,-1,-1)$	$27/64$	$9/64$
$(-1,+1,-1)$	$9/64$	$3/64$
$(-1,-1,+1)$	$9/64$	$3/64$
$(-1,+1,+1)$	$3/64$	$1/64$
$(+1,+1,+1)$	$9/64$	$27/64$
$(+1,-1,+1)$	$3/64$	$9/64$
$(+1,+1,-1)$	$3/64$	$9/64$
$(+1,-1,-1)$	$1/64$	$3/64$

We would then have been able to couple $\hat{X}_1(m,d_1,d_2)$ and $\hat{X}_2(m,d_1,d_2)$ by letting:

$$P((\hat{X}_1,\hat{X}_2)(m,d_1,d_2)=(\mathbf{x},\mathbf{y}))=\begin{cases} 9/64, & \mathbf{x}=\mathbf{y}=(1,1,1)\\ 3/64, & \mathbf{x}=(1,-1,1),\mathbf{y}=(1,1,1)\\ 3/64, & \mathbf{x}=(1,1,-1),\mathbf{y}=(1,1,1)\\ 3/64, & \mathbf{x}=(1,-1,-1),\mathbf{y}=(1,1,1)\\ 1/64, & \mathbf{x}=((-1,1,1),\mathbf{y}=(1,1,1)\\ 8/64, & \mathbf{x}=(-1,-1,-1),\mathbf{y}=(1,1,1)\\ 9/64, & \mathbf{x}=(-1,1,-1),\mathbf{y}=(1,1,-1)\\ 9/64, & \mathbf{x}=(-1,-1,1),\mathbf{y}=(1,-1,1)\\ 3/64, & \mathbf{x}=(-1,-1,-1),\mathbf{y}=(-1,1,-1)\\ 3/64, & \mathbf{x}=(-1,-1,-1),\mathbf{y}=(-1,-1,1)\\ 3/64, & \mathbf{x}=(-1,-1,-1),\mathbf{y}=(1,-1,-1)\\ 1/64, & \mathbf{x}=(-1,-1,-1),\mathbf{y}=(-1,1,1)\\ 9/64, & \mathbf{x}=\mathbf{y}=(-1,-1,-1) \end{cases}$$

The only case for which there is a path of disagreements through the family is when $\mathbf{y}=-\mathbf{x}=(1,1,1)$ and this case gives rise to four new families in the "super-tree" through which disagreement percolation can continue. Thus the expected number of such families is $4\cdot 8/64=1/2<1$ so that basic branching process theory tells us that the process of disagreement percolation will a.s. die out.

However, this is what would have been had p been exactly $2/3$. On the other hand, by facts noted above, we can by letting p be larger than but close enough to $2/3$ arrange things so that none of the conditional probabilities above changes by more than, say, $1/48$ no matter what earlier couplings of other families have revealed so far. Then the above coupling does not have to be changed by more than so that the outcome $(\hat{X}_1(m,d_1,d_2),\hat{X}_2(m,d_1,d_2))=((-1,-1,-1),(1,1,1))$ gets an extra probability mass of $6/48$. Thus the expected number of new families through which disagreement percolation can continue is bounded by $4(8/64+6/48)=3/4<1$ and we are still safe. This completes the desired coupling. $\square$

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