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Representation of continuous linear forms on the set of ladlag processes and the hedging of American claims under proportional costs

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Abstract

We discuss a d-dimensional version (for làdlàg optional processes) of a duality result by Meyer (1976) between bounded càdlàg adapted processes and random measures. We show that it allows to establish, in a very natural way, a dual representation for the set of initial endowments which allow to super-hedge a given American claim in a continuous time model with proportional transaction costs. It generalizes a previous result of Bouchard and Temam (2005) who considered a discrete time setting. It also completes the very recent work of Denis, De Vallière and Kabanov (2008) who studied càdlàg American claims and used a completely different approach.

Key words: Randomized stopping times, American options, transaction costs.

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1 Introduction

This paper is motivated by the study of d-dimensional markets with proportional transaction costs[1] in which each financial asset can possibly be exchanged directly against any other. This is typically the case on currency markets. The term *proportional transaction costs* refers to the fact that the buying and selling prices of the financial assets may differ but do not depend on the quantities that are exchanged.

More precisely, we study the set $\widehat{\mathscr{C}}^v$ of processes $\widehat{C}$ that can be *super-hedged* from an initial endowment v on $[0,T]$. This means that, by dynamically trading some d given underlying financial assets (stocks, bonds, currencies, etc.), it is possible to construct a portfolio $\widehat{V}$ such that $\widehat{V}_0 = v$ and $\widehat{V}$ is "larger" than $\widehat{C}$ at any time $t \in [0,T]$. Here and below, $\widehat{V}$ is a d-dimensional process corresponding to the different quantities $\widehat{V}^i$, $i \leq d$, of each financial asset i held in the portfolio. The superscript $\widehat{\ }$ is used to insist on the fact that we are dealing with quantities, as opposed to "amounts", and to be consistent with the established literature on the subject. Similarly, $\widehat{C}$ should be interpreted as a d-dimensional vector of quantities.

Obviously, in idealized financial markets where buying and selling each underlying asset i is done at a single price S^i in a given *numéraire*, such as Euros or Dollars, the value of the portfolio can be simply defined as the current value of the position holdings $V = SV := \sum_{i\leq d} S^i \widehat{V}^i$, $\widehat{C}$ can be represented as a real number $C = S\widehat{C}$, the value of $\widehat{C}$, and the term "larger" just means that $V_t \geq C_t$ at any time $t \in [0,T]$ $\mathbb{P}-$a.s., i.e. the net position $\widehat{V} - \widehat{C}$ of quantities has a non-negative value if evaluated at the price S. In this case, it can be typically shown that $\widehat{C} \in \widehat{\mathscr{C}}^v$ if and only if

$$\sup_{\tau\in\mathscr{T}} \sup_{\mathbb{Q}\in\mathscr{M}} \mathbb{E}^{\mathbb{Q}}\left[C_\tau\right] \leq S_0 v \iff \sup_{\tau\in\mathscr{T}} \sup_{\mathbb{Q}\in\mathscr{M}} \mathbb{E}^{\mathbb{Q}}\left[S_\tau(\widehat{C}_\tau - v)\right] \leq 0\,, \tag{1.1}$$

where $\mathscr{T}$ denotes the set of all $[0,T]$-valued stopping times and $\mathscr{M}$ is the set of equivalent probability measures that turn S into a martingale, see e.g. [16, Chapter 2 Theorem 5.3] or Section 3 for a more precise statement.

The so-called *dual formulation* (1.1) has important consequences. In the case where $\widehat{C}_t$ is interpreted as the payoff of an American option[2], in terms of quantities of the underlying assets to be delivered if the option is exercised at time t, it provides a way to compute the minimal value $p(\widehat{C})$ of $S_0 v$ such that $\widehat{C} \in \widehat{\mathscr{C}}^v$, or equivalently the corresponding "minimal" set of initial holdings $\{v \in \mathbb{R}^d \ : \ S_0 v = p(\widehat{C})\}$. The amount $p(\widehat{C})$ is the so-called *super-hedging price*. It is not only the minimal price at which the option can be sold without risk but also an upper-bound for *no-arbitrage* prices, i.e. the upper-bound of prices at which the option can be sold without creating an arbitrage opportunity. The dual formulation also plays a central role in discussing optimal management problems which are typically studied through the Fenchel duality approach, see e.g. [16, Section 6.5] for an introduction, [4] for models with transaction costs, the seminal paper [19] for frictionless markets, and [2] for wealth-path dependent problems where the notion of American options is involved. In this case, $\widehat{C}$ is related to the optimal variable in the associated dual problem. Proving existence in the original optimal management problem and the duality between the two corresponding value functions then essentially breaks down to proving that $\widehat{C} \in \widehat{\mathscr{C}}^v$, where v is the initial endowment.

[1] An excellent introduction to the concepts that will be described below can be found in [23, Section 1].

[2] See Section 2.5 of [16] for the financial definition.

This is typically obtained by using the optimality of $\widehat{C}$ together with some calculus of variations and a dual formulation for $\widehat{\mathscr{C}}^v$.

When transaction costs are taken into account, each financial asset i can no longer be described by a single value. It can only be described by its buying and selling values with respect to the other assets. These values are modeled as an adapted *càdlàg*[3] d-dimensional matrix valued process $\Pi = (\pi^{ij})_{1\leq i,j\leq d}$, on some complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F} := (\mathscr{F}_t)_{t\leq T}$ satisfying the usual assumptions. Each entry π_t^{ij} denotes the number of units of asset i which is required to obtain one unit of asset j at time t. They are assumed to satisfy the following natural conditions:

(i) $\pi_t^{ii} = 1$, $\pi_t^{ij} > 0$ for all $t \leq T$ and $1 \leq i, j \leq d$ $\mathbb{P}-$a.s.

(ii) $\pi_t^{ij} \leq \pi_t^{ik}\pi_t^{kj}$ for all $t \leq T$ and $1 \leq i, j, k \leq d$ $\mathbb{P}-$a.s.

The first condition has a clear interpretation: relative prices are positive. The second one means that it is always cheaper to directly exchange some units of i against units of j rather than first convert units of i into units of k and then exchange these units of k against units of j. One can actually always reduce to this case as explained in [23, Section 1].

In this framework, a position $\widehat{V}_t$ at time t is said to be *solvent* if an immediate exchange in the market allows to turn it into a vector with non-negative components. In mathematical terms, this means that it belongs to the closed convex cone $\widehat{K}_t(\omega)$ generated by the vectors e_i and $\pi_t^{ij}(\omega)e_i - e_j$, $1 \leq i, j \leq d$, with $(e_i)_{1\leq i\leq d}$ the canonical basis of $\mathbb{R}^d$, i.e., under the above conditions (i)-(ii):

$$\widehat{K}_t(\omega) := \left\{ \hat{v} \in \mathbb{R}^d \;:\; \exists\, \hat{a} \in \mathbb{M}_+^d \text{ s.t. } \hat{v}^i + \sum_{j=1}^d \left(\hat{a}^{ji} - \hat{a}^{ij}\pi_t^{ij}(\omega) \right) \geq 0 \;\; \forall\, i \leq d \right\}, \tag{1.2}$$

where $\mathbb{M}_+^d$ denotes the set of d-dimensional square matrices with non-negative entries. In the above equation $\hat{a}^{ji}$ should be interpreted as the number of units of the asset i which are obtained by exchanging $\hat{a}^{ji}\pi_t^{ji}$ units of the asset j.

In this model, the term "larger" used above thus means $\widehat{V}_t - \widehat{C}_t \in \widehat{K}_t$ for all $t \leq T$ $\mathbb{P}-$a.s. (in short $\widehat{V} \succeq \widehat{C}$).

It remains to specify the dynamic of portfolio processes. This is done by noting that an immediate transaction on the market changes the portfolio by a vector of quantities of the form $\xi_t(\omega) \in -\partial\widehat{K}_t(\omega)$, the boundary of $-\widehat{K}_t(\omega)$. The terms $\hat{a}_t^{ij}(\omega)$ such that $\xi_t^i(\omega) = \sum_{j=1}^d \left(\hat{a}_t^{ji}(\omega) - \hat{a}_t^{ij}(\omega)\pi_t^{ij}(\omega) \right)$ for $i \leq d$ correspond to each transaction: one exchanges $\hat{a}_t^{ij}(\omega)\pi_t^{ij}(\omega)$ units of i against $\hat{a}_t^{ij}(\omega)$ units of j. It is thus natural to define *self-financing strategies* as vector processes $\widehat{V}$ such that $d\widehat{V}_t(\omega)$ belongs in some sense to $-\widehat{K}_t(\omega)$, the passage from $-\partial\widehat{K}_t(\omega)$ to $-\widehat{K}_t(\omega)$ reflecting the idea that one can always "throw away", or consume, some (non-negative) quantities of assets.

[3] The French acronym *càdlàg*, *continu à droite limité à gauche*, means "right continuous with left limits".

Such a modeling was introduced and studied at different levels of generality in [12], [13] and [5] among others, and it is now known from the work of [22] and [5] that a good definition of self-financing wealth processes is the following:

Definition 1.1. *We say that a* $\mathbb{R}^d$*-valued* làdlàg [4] *predictable process* $\widehat{V}$ *is a self-financing strategy if it has* $\mathbb{P}-$*a.s. finite total variation and:*

(i) $\dot{\widehat{V}}^c := d\widehat{V}^c/d\mathrm{Var}(\widehat{V}^c) \in -\widehat{K}\ d\mathrm{Var}(\widehat{V}^c) \otimes \mathbb{P}$*-a.e.* , *where* $\widehat{V}^c$ *denotes the continuous part of* $\widehat{V}$ *and* $\mathrm{Var}(\widehat{V}^c)$ *its total variation,*

(ii) $\Delta^+\widehat{V}_\tau := \widehat{V}_{\tau+} - \widehat{V}_\tau \in -\widehat{K}_\tau\ \mathbb{P}-$*a.s. for all stopping times* $\tau \le T$,

(iii) $\Delta\widehat{V}_\tau := \widehat{V}_\tau - \widehat{V}_{\tau-} \in -\widehat{K}_{\tau-}\ \mathbb{P}-$*a.s. for all predictable stopping times* $\tau \le T$.

Given $v \in \mathbb{R}^d$, *we denote by* $\widehat{\mathscr{V}}^v$ *the set of self-financing strategies* $\widehat{V}$ *such that* $\widehat{V}_0 = v$.

The set $\widehat{\mathscr{C}}^v$ is then naturally defined as the set of optional *làdlàg* processes $\widehat{C}$ such that $\widehat{V} \succeq \widehat{C}$.

A dual description of $\widehat{C}^v$ has already been obtained in discrete time models by [6] and [3], and extended to continuous time models in the very recent paper [9][5]. The argument used in [9] is based on a discrete time approximation of the super-hedging problem, completed by a passage to the limit. However, this technique requires some regularity and only allows to consider the case where $\widehat{C}$ is *càdlàg*. In particular, it does not apply to *self-financing strategies* which are, in general, only *làdlàg* , see Section 2.3 below for more comments.

In the present paper, we use a totally different approach which allows to consider optional *làdlàg* processes. It is based on a strong duality argument on the set $\mathscr{S}^1(\mathbb{Q})$ of optional *làdlàg* processes X such that $\|X\|_{\mathscr{S}^1(\mathbb{Q})} := \mathbb{E}^{\mathbb{Q}}\left[\sup_{t\le T}\|X_t\|\right] < \infty$, for some well chosen $\mathbb{P}$-equivalent probability measure $\mathbb{Q}$. Namely, we show that $\widehat{\mathscr{C}}^0 \cap \mathscr{S}^1(\mathbb{Q})$ is closed in $\mathscr{S}^1(\mathbb{Q})$ for some $\mathbb{Q} \sim \mathbb{P}$. We then use a Hahn-Banach type argument together with a version of the well-known result of Meyer [21], see Proposition 2.1 below, that provides a representation of continuous linear form on $\mathscr{S}^1(\mathbb{Q})$ in terms of random measures.

For technical reasons, see [5], we shall assume all over this paper that $\mathscr{F}_{T-} = \mathscr{F}_T$ and $\Pi_{T-} = \Pi_T$ $\mathbb{P}-$a.s. Note however, that we can always reduce to this case by considering a larger time horizon $T^* > T$ and by considering an auxiliary model where $\mathscr{F}_t = \mathscr{F}_{T^*}$ and $\Pi_t = \Pi_{T^*}$ $\mathbb{P}-$a.s. for $t \in [T, T^*]$.

We shall also need the following:

Standing assumption: There exists at least one *càdlàg* martingale Z such that

(i) $Z_t \in \widehat{K}^*_t$ for all $t \le T$, $\mathbb{P}-$a.s.

(ii) for every $[0,T] \cup \{\infty\}$-valued stopping times $Z_\tau \in \mathrm{Int}(\widehat{K}^*_\tau)$ $\mathbb{P}-$a.s. on $\{\tau < \infty\}$

(iii) for every predictable $[0,T] \cup \{\infty\}$-valued stopping times $Z_{\tau-} \in \mathrm{Int}(\widehat{K}^*_{\tau-})$ $\mathbb{P}-$a.s. on $\{\tau < \infty\}$.

[4] The French acronym *làdlàg* , *limité à droite limité à gauche*, means “with right and left limits”.

[5] We received this paper while preparing this manuscript. We are grateful to the authors for discussions we had on the subject at the Bachelier Workshop in Métabief, 2008.

Here, $\widehat{K}^*_t(\omega) := \{y \in \mathbb{R}^d \ : \ \sum_{i\le d} x^i y^i \ge 0 \ \ \forall\, x \in \widehat{K}_t(\omega)\}$ is the positive polar of $\widehat{K}_t(\omega)$. In the following, we shall denote by $\mathscr{Z}^s$ the set of processes satisfying the above conditions. We refer to [5] and [23] for a discussion on the link between the existence of these so-called *strictly consistent price processes* and the absence of arbitrage opportunities, see also Section 3.

The rest of this paper is organized as follows. In Section 2, we first state an abstract version of our main duality result in terms of a suitable set $\mathscr{D}$ of dual processes. We then provide a more precise description of the set $\mathscr{D}$ which allows us to state our duality result in a form which is more in the spirit of [9, Theorem 4.2]. In Section 3, we also discuss this result in the light of the literature on optimal stopping and American options pricing in frictionless markets. Section 4 presents the extension of Meyer's result. In Section 5, we prove the super-hedging theorem using the strong duality approach explained above.

Notations: From now on, we shall use the notation xy to denote the natural scalar product on $\mathbb{R}^d$. For a *làdlàg* optional process X, we define $\|X\|_* := \sup_{t\le T}\|X_t\|$. Given a process with bounded variations A, we write A^c and A^δ to denote its continuous and purely discontinuous parts, and by $\dot{A}$ its density with respect to the associated total variation process $\mathrm{Var}(A) := (\mathrm{Var}_t(A))_{t\le T}$. The integral with respect to A has to be understood as the sum of the integrals with respect to A^c and A^δ. Given a *làdlàg* measurable process X on $[0,T]$, we shall always use the conventions $X_{T+} = X_T$ and $X_{0-} = 0$.

2 Main results

2.1 Abstract formulation

Our dual formulation is based on the representation of continuous linear form on $\mathscr{S}^1(\mathbb{Q})$ in terms of elements of the set $\mathscr{R}$ of $\mathbb{R}^{3d}$-valued adapted *càdlàg* processes $A := (A^-, A^\circ, A^+)$ with $\mathbb{P}$-integrable total variation such that:

(i) A^- is predictable,

(ii) A^+ and A^- are pure jump processes,

(iii) $A^-_0 = 0$ and $A^+_T = A^+_{T-}$.

Letting $\mathscr{S}^\infty$ denote the collection of elements of $\mathscr{S}^1(\mathbb{Q})$ with essentially bounded supremum, we have:

Proposition 2.1. *Fix $\mathbb{Q} \sim \mathbb{P}$ and let μ be a continuous linear form on $\mathscr{S}^1(\mathbb{Q})$. Then, there exists $A := (A^-, A^\circ, A^+) \in \mathscr{R}$ such that:*

$$\mu(X) = (X|A] := \mathbb{E}\left[\int_0^T X_{t-}\, dA^-_t + \int_0^T X_t\, dA^\circ_t + \int_0^T X_{t+}\, dA^+_t\right] , \ \forall\, X \in \mathscr{S}^\infty .$$

Note that such a result is known from [1] or [21] in the case of *càdlàg* or *làdcàg* [6] processes, the one dimensional *làdlàg* case being mentioned in [10]. A complete proof will be provided in Section 4 below.

[6]The French acronym *làdcàg* , *limité à droite continu à gauche*, means “left continuous with right limits”.

To obtain the required dual formulation of $\widehat{\mathscr{C}}^0$, we then consider a particular subset $\mathscr{D} \subset \mathscr{R}$ of dual processes that takes into account the special structure of $\widehat{\mathscr{C}}^0$:

Definition 2.1. *Let $\mathscr{D}$ denote the set of elements $A \in \mathscr{R}$ such that*
(C1) $(\widehat{C}|A] \leq 0$*, for all $\widehat{C} \in \mathscr{S}^\infty$ satisfying $0 \succeq \widehat{C}$.*
(C2) $(\widehat{V}|A] \leq 0$*, for all $\widehat{V} \in \widehat{\mathscr{V}}^0$ with essentially bounded total variation.*

A more precise description of the set $\mathscr{D}$ will be given in Lemma 2.1 and Lemma 2.2 below. In particular, it will enable us to extend the linear form $(\cdot|A]$, with $A \in \mathscr{D}$, to elements of $\widehat{\mathscr{C}}_b^0 := \widehat{\mathscr{C}}^0 \cap \mathscr{S}_b$ where $\mathscr{S}_b$ denotes the set of *làdlàg* optional processes X satisfying $X \succeq a$ for some $a \in \mathbb{R}^d$.

This extension combined with a Hahn-Banach type argument, based on the key closure property of Proposition 5.1 below, leads to a natural polarity relation between $\mathscr{D}$ and $\widehat{\mathscr{C}}_b^0$. Here, given a subset E of $\mathscr{S}_b$, we define its polar as

$$E^\diamond := \{A \in \mathscr{R} \ : \ (X|A] \leq 0 \text{ for all } X \in E\} \ ,$$

and define similarly the polar of a subset F of $\mathscr{R}$ as

$$F^\diamond := \left\{X \in \mathscr{S}_b \ : \ (X|A] \leq 0 \text{ for all } A \in F\right\} \ ,$$

where we use the convention $(X|A] = \infty$ whenever $\int_0^T X_{t-} dA_t^- + \int_0^T X_t dA_t^\circ + \int_0^T X_{t+} dA_t^+$ is not $\mathbb{P}$-integrable.

Our main result reads as follows:

Theorem 2.1. $\mathscr{D}^\diamond = \widehat{\mathscr{C}}_b^0$ *and* $(\widehat{\mathscr{C}}_b^0)^\diamond = \mathscr{D}$.

The first statement provides a dual formulation for the set $\widehat{\mathscr{C}}_b^0$ of super-hedgeable American claims that are "bounded from below". The second statement shows that $\mathscr{D}$ is actually exactly the polar of $\widehat{\mathscr{C}}_b^0$ for the relation defined above.

Remark 2.1. Given $\widehat{C} \in \mathscr{S}_b$, let $\Gamma(\widehat{C})$ denote the set of initial portfolio holdings v such that $\widehat{C} \in \widehat{\mathscr{C}}^v$. It follows from the above theorem and the identity $\widehat{\mathscr{C}}^v = v + \widehat{\mathscr{C}}^0$ that

$$\Gamma(\widehat{C}) = \left\{v \in \mathbb{R}^d \ : \ (\widehat{C} - v|A] \leq 0 \text{ for all } A \in \mathscr{D}\right\} \ .$$

If the asset one is chosen as a *numéraire*, then the corresponding *super-hedging price* is given by

$$p(\widehat{C}) := \inf\left\{v^1 \in \mathbb{R} \ : \ (v^1, 0, \cdots, 0) \in \Gamma(\widehat{C})\right\} \ .$$

We shall continue this discussion in Remark 2.2 below.

2.2 Description of the set of dual processes $\mathscr{D}$

In this section, we provide a more precise description of the set of dual processes $\mathscr{D}$. The proofs of the above technical results are postponed to the Appendix.

Our first result concerns the property **(C1)**. It is the counterpart of the well-known one dimensional property: if μ admits the representation $\mu(X) = (X|A]$ and satisfies $\mu(X) \leq 0$ for all non-positive process X with essentially bounded supremum, then A has non-decreasing components. In our context, where the notion of non-positivity is replaced by $0 \succeq \widehat{C}$, it has to be expressed in terms of the positive polar sets process $\widehat{K}^*$ of $\widehat{K}$.

Lemma 2.1. *Fix $A := (A^-, A^\circ, A^+) \in \mathscr{R}$. Then* **(C1)** *holds if and only if*

(i) $\dot{A}^- \in \widehat{K}^*_-$ $d\mathrm{Var}(A^-) \otimes \mathbb{P}$-*a.e.,*

(ii) $\dot{A^{\circ c}} \in \widehat{K}^*$ $d\mathrm{Var}(A^{\circ c}) \otimes \mathbb{P}$-*a.e. and* $\dot{A^{\circ\delta}} \in \widehat{K}^*$ $d\mathrm{Var}(A^{\circ\delta}) \otimes \mathbb{P}$-*a.e.,*

(iii) $\dot{A^+} \in \widehat{K}^*$ $d\mathrm{Var}(A^+) \otimes \mathbb{P}$-*a.e.*

In the following, we shall denote by $\mathscr{R}_{\hat{K}}$ the subset of elements $A \in \mathscr{R}$ satisfying the above conditions (i)-(iii).

We now discuss the implications of the constraint **(C2)**. From now on, given $A := (A^-, A^\circ, A^+) \in \mathscr{R}$, we shall denote by $\bar{A}^-$ (resp. $\bar{A}^+$) the predictable projection (resp. optional) of $(\delta A^-_t)_{t\le T}$ (resp. $(\delta A^+_t)_{t\le T}$), where $\delta A^-_t := A^-_T - A^-_t + A^\circ_T - A^\circ_{t-} + A^+_T - A^+_{t-}$ and $\delta A^+_t := A^-_T - A^-_t + A^\circ_T - A^\circ_t + A^+_T - A^+_{t-}$.

Lemma 2.2. *Fix $A := (A^-, A^\circ, A^+) \in \mathscr{R}$. Then* **(C2)** *holds if and only if*

(i) $\bar{A}^-_\tau \in \widehat{K}^*_{\tau-}$ $\mathbb{P}-$*a.s. for all predictable stopping times $\tau \le T$,*

(ii) $\bar{A}^+_\tau \in \widehat{K}^*_\tau$ $\mathbb{P}-$*a.s. for all stopping times $\tau \le T$.*

In the following, we shall denote by $\mathscr{R}_{\Delta\hat{K}}$ the subset of elements $A \in \mathscr{R}$ satisfying the above conditions (i)-(ii).

Note that combining the above Lemmas leads to the following precise description of $\mathscr{D}$:

Corollary 2.1. $\mathscr{D} = \mathscr{R}_{\hat{K}} \cap \mathscr{R}_{\Delta\hat{K}}$.

Remark 2.2. Since $\widehat{K} \supset [0,\infty)^d$, recall (1.2), it follows that $\widehat{K}^* \subset [0,\infty)^d$. The fact that $\pi^{ij}_t e_i - e_j \in \widehat{K}_t$ and $\pi^{ij}_t > 0$ for all $i, j \le d$ thus implies that $y^1 = 0 \Rightarrow y = 0$ for all $y \in \widehat{K}^*_t(\omega)$. It then follows from Lemma 2.1 that for $A \in \mathscr{D}$, $(e_1|A] \ge 0$ and $(e_1|A] = 0 \Rightarrow (X|A] = 0$ for all $X \in \mathscr{S}_b$. In view of Remark 2.1, this shows that

$$p(\widehat{C}) = \sup_{B \in \mathscr{D}_1} (\widehat{C}|B] \quad \text{for all } \widehat{C} \in \mathscr{S}_b\,,$$

where $\mathscr{D}_1 := \{B = A/(e_1, A],\ A \in \mathscr{D} \text{ s.t. } (e_1, A] > 0\} \cup \{0\}$.

2.3 Alternative formulation

The dual formulation of Theorem 2.1 is very close to the one obtained in [3, Theorem 2.1], for discrete time models, and more recently in [9, Theorem 4.2], for *càdlàg* processes in continuous time models. Their formulation is of the form: if $\widehat{C} \succeq a$ for some $a \in \mathbb{R}^d$, then

$$\widehat{C} \in \widehat{\mathscr{C}}^\nu \iff \sup_{A^\circ \in \tilde{\mathscr{D}}} \mathbb{E}\left[\int_0^T (\widehat{C}_t - \nu)\, dA^\circ_t\right] \le 0\,, \tag{2.1}$$

where $\tilde{\mathscr{D}}$ is a family of *càdlàg* adapted processes A° with integrable total variation such that

1. $A^\circ_{0-} = 0$

2. There is a deterministic finite non-negative measure ν° on $[0,T]$ and an adapted process Z° such that $Z^\circ \in \widehat{K}^*$ $\mathbb{P} \otimes \nu^\circ$-a.e., $A^\circ = \int_0^\cdot Z^\circ_t \nu^\circ(dt)$ and $\nu^\circ([0,T]) = 1$.

3. The optional projection $\bar{A}^\circ$ of $(A^\circ_T - A^\circ_t)_{t\le T}$ satisfies $\bar{A}^\circ_t \in \widehat{K}^*_t$ for all $t \le T$ $\mathbb{P}-$a.s.

In this section, we show that a similar representation holds in our framework. Namely, let $\mathcal{N}$ denote the set of triplets of non-negative random measures $\nu := (\nu^-, \nu^\circ, \nu^+)$ such that ν^- is predictable, ν° and ν^+ are optional and $(\nu^- + \nu^\circ + \nu^+)([0,T]) = 1$ $\mathbb{P}-$a.s.

Note that ν is usually called a randomized quasi-stopping time, and a randomized stopping time if $\nu^+ = \nu^- = 0$.

Given $\nu \in \mathcal{N}$, we next define $\tilde{Z}(\nu)$ as the set of $\mathbb{R}^{3d}$-valued processes $Z := (Z^-, Z^\circ, Z^+)$ such that:

(i) Z^i is $\nu^i(dt,\omega)d\mathbb{P}(\omega)$ integrable for $i \in \{-,\circ,+\}$, Z^- is predictable and Z°, Z^+ are optional.
(ii) $A = (A^-, A^\circ, A^+)$ defined by $A^i_\cdot = \int_0^\cdot Z^i_t\,\nu^i(dt)$ for $i \in \{-,\circ,+\}$ belongs to $\mathcal{D}$.

Corollary 2.2. *Let $\widehat{C}$ be an element of $\mathcal{S}_b$. Then, $\widehat{C} \in \widehat{\mathcal{C}}^0$ if and only if*

$$\mathbb{E}\left[\int_0^T \widehat{C}_{t-} Z^-_t \nu^-(dt) + \int_0^T \widehat{C}_t Z^\circ_t \nu^\circ(dt) + \int_0^T \widehat{C}_{t+} Z^+_t \nu^+(dt)\right] \le 0 \tag{2.2}$$

for all $\nu \in \mathcal{N}$ and $Z \in \tilde{Z}(\nu)$.

Remark 2.3. It follows from Remark 2.2 and Corollary 2.2 that, for $\widehat{C} \in \mathcal{S}_b$,

$$p(\widehat{C}) = \sup_{(\nu,Z)\in\mathcal{N}\times\tilde{Z}(\nu)_1} \mathbb{E}\left[\int_0^T \widehat{C}_{t-} Z^-_t \nu^-(dt) + \int_0^T \widehat{C}_t Z^\circ_t \nu^\circ(dt) + \int_0^T \widehat{C}_{t+} Z^+_t \nu^+(dt)\right] ,$$

where $\tilde{Z}(\nu)_1$ is defined as

$$\left\{ Z \in \tilde{Z}(\nu) \;:\; \mathbb{E}\left[\int_0^T Z^{-,1}_t \nu^-(dt) + \int_0^T Z^{\circ,1}_t \nu^\circ(dt) + \int_0^T Z^{+,1}_t \nu^+(dt)\right] = 1 \right\} \cup \{0\} ,$$

and $Z^{-,1}$, $Z^{\circ,1}$, $Z^{+,1}$ are the first components of Z^-, Z°, Z^+ appearing in the decomposition of Z.

The proof of the above Corollary is an immediate consequence of Theorem 2.1 and the following representation result.

Proposition 2.2. *Let $A = (A^-, A^\circ, A^+)$ be a $\mathbb{R}^{3d}$-valued process with integrable total variation. Then, $A \in \mathcal{D}$ if and only if there exists $\nu := (\nu^-, \nu^\circ, \nu^+) \in \mathcal{N}$ and $Z := (Z^-, Z^\circ, Z^+) \in \tilde{Z}(\nu)$ such that*

$$A^i_\cdot = \int_0^\cdot Z^i_t\,\nu^i(dt) \;,\; i \in \{-,\circ,+\} . \tag{2.3}$$

Proof. It is clear that given $(\nu^-, \nu^\circ, \nu^+) \in \mathcal{N}$ and $(Z^-, Z^\circ, Z^+) \in \tilde{Z}(\nu)$, the process defined in (2.3) belongs to $\mathcal{D}$. We now prove the converse assertion.

1. We first observe that, given $A = (A^-, A^\circ, A^+) \in \mathcal{R}$, we can find a $\mathbb{R}^{3d}$-adapted process $Z := (Z^-, Z^\circ, Z^+)$ and a triplet of real positive random measures $\nu := (\nu^-, \nu^\circ, \nu^+)$ on $[0,T]$ such that Z^- and ν^- are predictable, (Z°, Z^+) and (ν°, ν^+) are optional, and $A^i = \int_0^\cdot Z^i_t \nu^i(dt)$ for $i \in \{-,\circ,+\}$.

2. We can then always assume that $\bar{\nu} := \nu^- + \nu^\circ + \nu^+$ satisfies $\bar{\nu}([0,T]) \leq 1$ $\mathbb{P}-$a.s. Indeed, let f be some strictly increasing function mapping $[0,\infty)$ into $[0,1/3)$. Then, for $i \in \{-,\circ,+\}$, ν^i is absolutely continuous with respect to $\tilde{\nu}^i := f(\nu^i)$ and thus admits a density. Replacing ν^i by $\tilde{\nu}^i$ and multiplying Z^i by the optional (resp. predictable) projection of the associated density leads to the required representation for $i \in \{\circ,+\}$ (resp. $i = -$).

3. Finally, we can reduce to the case where $\bar{\nu}([0,T]) = 1$ $\mathbb{P}-$a.s. Indeed, since ν^- is only supported by graphs of $[0,T]$-valued random variables (recall that A^- is a pure jump process), we know that it has no continuous part at $\{T\}$. We can thus replace ν^- by $\tilde{\nu}^- := \nu^- + \delta_{\{T\}}(1 - \bar{\nu}([0,T]))$ where $\delta_{\{T\}}$ denotes the Dirac mass at T. We then also replace Z^- by

$$\tilde{Z}^- := Z^-[\mathbf{1}_{\{t<T\}} + \mathbf{1}_{\{t=T\}}\mathbf{1}_{\{\bar{\nu}([0,T])<1\}}\nu^-(\{T\})(\nu^-(\{T\}) + 1 - \bar{\nu}([0,T]))^{-1}]$$

so that $A^- = \int_0^\cdot \tilde{Z}_t^- \tilde{\nu}^-(dt)$. Observe that the assumption $\mathscr{F}_{T-} = \mathscr{F}_T$ ensures that $\tilde{\nu}^-$ and $\tilde{Z}^-$ are still predictable. □

Remark 2.4. Note that only the measure ν° appears in the formulation (2.1) and that it is deterministic. In this sense our result is less tractable than the one obtained in [9] for continuous time models. However, as already pointed out in the introduction, the latter applies only to *càdlàg* processes.

The reason for this it that their approach relies on a discrete time approximation of the super-hedging problem. Namely, they first prove that the result holds if we only impose $\widehat{V}_t - \widehat{C}_t \in \widehat{K}_t$ on a finite number of times $t \leq T$, and then pass to the limit. Not surprisingly, this argument requires some regularity.

At first glance, this restriction may not seem important, but, it actually does not apply to admissible self-financing portfolios of the set $\widehat{\mathscr{V}}^\nu$, since they are only assumed to be *làdlàg* (except when Π is continuous in which case the portfolios can be taken to be continuous, see the final discussion in [9]).

3 Comparison with frictionless markets

Let us first recall that the frictionless market case corresponds to the situation where selling and buying is done at the same price, i.e. $\pi^{ij} = 1/\pi^{ji}$ for all $i, j \leq d$. In this case, the price process (say in terms of the first asset) is $S^i := \pi^{1i}$ and is a *càdlàg* semimartingale, see [7]. In order to avoid technicalities, it is usually assumed to be locally bounded. The no-arbitrage condition, more precisely no free lunch with vanishing risk, implies that the set $\mathscr{M}$ of equivalent measures $\mathbb{Q}$ under which $S = (S^i)_{i \leq d}$ is a local martingale is non-empty. Such measures should be compared to the strictly consistent price processes Z of $\mathscr{Z}^s$. Indeed, if H denotes the density process associated to $\mathbb{Q}$, then HS is “essentially” an element of $\mathscr{Z}^s$, and conversely, up to an obvious normalization. The term “essentially” is used here because in this case the interior of $\widehat{K}^*$ is empty and the notion of interior has to be replaced by that of relative interior. See the comments in [23, Section 1].

As already explained in the introduction, in such models, the wealth process can be simply represented by its value $V = S\widehat{V}$. The main difference is that the set of admissible strategies is no more described by $\widehat{\mathscr{V}}^0$ but in terms of stochastic integrals with respect to S.

In the case where $\mathscr{M} = \{\mathbb{Q}\}$, the so-called complete market case, the super-hedging price of an American claim $\widehat{C}$, such that $C := S\widehat{C}$ is bounded from below, coincides with the value at time 0 of the Snell envelope of C computed under $\mathbb{Q}$, see e.g. [18] and the references therein. Equivalently, the American claim $\widehat{C}$ can be super-hedged from a zero initial endowment if and only if the $\mathbb{Q}$-Snell envelope of C at time 0 is non-positive.

In the case where C is *làdlàg* and of class (D), the $\mathbb{Q}$-Snell envelope $J^{\mathbb{Q}}$ of C satisfies, see [10, p. 135] and [11, Proposition 1],

$$J_0^{\mathbb{Q}} \;=\; \sup_{\tau\in\mathscr{T}} \mathbb{E}^{\mathbb{Q}}\left[C_\tau\right] = \sup_{(\tau^-,\tau^\circ,\tau^+)\in\tilde{\mathscr{T}}} \mathbb{E}^{\mathbb{Q}}\left[C_{\tau^- -} + C_{\tau^\circ} + C_{\tau^+ +}\right] \tag{3.1}$$

where $\mathscr{T}$ is the set of all $[0,T]$-valued stopping times, $\tilde{\mathscr{T}}$ is the set quasi-stopping times, i.e. the set of triplets of $[0,T]\cup\{\infty\}$-valued stopping times $(\tau^-,\tau^\circ,\tau^+)$ such that τ^- is predictable and, a.s., only one of them is finite. Here, we use the convention $C_{\infty-}=C_\infty=C_{\infty+}=0$. The first formulation is simple but does not allow to provide an existence result, while the second does. Indeed, [11, Proposition 1],

$$J_0^{\mathbb{Q}} \;=\; \mathbb{E}^{\mathbb{Q}}\left[C_{\hat\tau-}\mathbb{1}_{B^-} + C_{\hat\tau}\mathbb{1}_{B^\circ} + C_{\hat\tau+}\mathbb{1}_{B^+}\right]$$

where

$$\hat\tau := \inf\{t\in[0,T]\;:\; J^{\mathbb{Q}}_{t-}=C_{t-} \text{ or } J^{\mathbb{Q}}_t = C_t \text{ or } J^{\mathbb{Q}}_{t+}=C_{t+}\}$$

and

$$B^- := \{J^{\mathbb{Q}}_{t-}=C_{t-}\}\;,\; B^\circ := \{J^{\mathbb{Q}}_t = C_t\}\cap(B^-)^c\;,\; B^+ := (B^-\cup B^\circ)^c\;.$$

It thus suffices to set $\hat\tau^i := \hat\tau\mathbb{1}_{B^i} + \infty\mathbb{1}_{(B^i)^c}$ for $i\in\{-,\circ,+\}$ to obtain

$$J_0^{\mathbb{Q}} = \mathbb{E}^{\mathbb{Q}}\left[C_{\hat\tau^- -} + C_{\hat\tau^\circ} + C_{\hat\tau^+ +}\right]\;.$$

This shows that, in general, one needs to consider quasi stopping times instead of stopping times if one wants to establish an existence result, see also [1, Proposition 1.2] for the case of *càdlàg* processes.

In the case of incomplete markets, the super-hedging price is given by the supremum over all $\mathbb{Q}\in\mathscr{M}$ of $J_0^{\mathbb{Q}}$, [18, Theorem 3.3] . See also [14] for the case of portfolio constraints.

In our framework, the measure $\nu\in\mathscr{N}$ that appears in (2.2) can be interpreted as a randomized version of the quasi-stopping times while the result of [9], of the form (2.1), should be interpreted as a formulation in terms of randomized stopping times, recall the definitions given in Section 2.3 after the introduction of $\mathscr{N}$ as well as Remark 2.3. Both are consistent with the results of [3] and [6] that show that the duality does not work in discrete time models if we restrict to (non-randomized) stopping times. In both cases the process $Z\in\tilde{Z}(\nu)$ plays the role of $H^{\mathbb{Q}}S$ where $H^{\mathbb{Q}}$ is the density process associated to the equivalent martingale measures $\mathbb{Q}$ mentioned above. These two formulations thus correspond to the two representations of the Snell envelope in (3.1). As in frictionless markets, the formulation of [9] is simpler while ours should allow to find the optimal randomized quasi-stopping time, at least when Z is fixed. We leave this point for further research.

4 On continuous linear forms for *làdlàg* processes

We first provide an extension of Theorem 27 in Chapter VI of [21] to the case of *làdlàg* processes. It is obtained by following almost line by line Meyer's proof. We then provide the proof of Proposition 2.1, which is inspired from the arguments used in [1, Proposition 1.3].

4.1 Extension of Meyer's result

We first state a version of Theorem 27 in Chapter VI in [21] for the set $\tilde{\mathscr{S}}^\infty$ of *làdlàg* $\mathscr{B}([0,T])\otimes\mathscr{F}$-measurable $\mathbb{P}$-essentially bounded processes.

Theorem 4.1. *Let $\tilde{\mu}$ be a linear form on $\tilde{\mathscr{S}}^\infty$ such that:*

(**A1**) $\tilde{\mu}(X^n)\to 0$ *for all sequence $(X^n)_{n\geq 0}$ of positive elements of $\tilde{\mathscr{S}}^\infty$ such that* $\sup_n \|X^n\|_{\tilde{\mathscr{S}}^\infty}\leq M$ *for some $M>0$ and satisfying $\|X^n\|_*\to 0$ $\mathbb{P}$-a.s.*

Then, there exists three measures α_-, $\alpha_\circ$ and α_+ on $[0,T]\times\Omega$ such that

1. α_- is carried by $(0,T]\times\Omega$ and by a countable union of graphs of $[0,T]$-valued $\mathscr{F}$-measurable random variables.

2. α_+ is carried by $[0,T)\times\Omega$ and by a countable union of graphs of $[0,T]$-valued $\mathscr{F}$-measurable random variables.

3. $\alpha_\circ=\alpha_\circ^\delta+\alpha_\circ^c$ where $\alpha_\circ^\delta$ is carried by $[0,T]\times\Omega$ and by a countable union of graphs of $[0,T]$-valued $\mathscr{F}$-measurable random variables, $\alpha_\circ^c$ is carried by $[0,T]\times\Omega$ and does not charge any graph of $[0,T]$-valued $\mathscr{F}$-measurable random variable.

4. For all $X\in\tilde{\mathscr{S}}^\infty$, we have

$$\tilde{\mu}(X)=\int_\Omega\int_0^T X_{t-}(\omega)\alpha_-(dt,d\omega)+\int_\Omega\int_0^T X_t(\omega)\alpha_\circ(dt,d\omega)+\int_\Omega\int_0^T X_{t+}(\omega)\alpha_+(dt,d\omega)\,.$$

This decomposition is unique among the set of measures satisfying the above conditions 1., 2. and 3.

The proof can be decomposed in four main steps:
Step 1. To a process X in $\tilde{\mathscr{S}}^\infty$, we associate

$$\bar{X}(t,\omega,-):=X_{t-}(\omega)\,,\ \bar{X}(t,\omega,\circ):=X_t\quad\text{and}\quad\bar{X}(t,\omega,+):=X_{t+}(\omega)\,,$$

so as to keep track of the right and left limits and isolate the point-value. Note that $\bar{X}$ is a measurable map on

$$W:=((0,T]\times\Omega\times\{-\})\cup([0,T]\times\Omega\times\{\circ\})\cup([0,T)\times\Omega\times\{+\})$$

endowed with the sigma-algebra $\mathscr{W}:=\sigma(\bar{X},\,\bar{X}\in\bar{\mathscr{S}}^\infty)$, where $\bar{\mathscr{S}}^\infty:=\{\bar{X}\mid X\in\tilde{\mathscr{S}}^\infty\}$.
Step 2. Since $\bar{\mathscr{S}}^\infty$ is a lattice and $X\mapsto\bar{X}$ is a bijection, we next observe that a linear form $\tilde{\mu}$ on $\tilde{\mathscr{S}}^\infty$ can always be associated to a linear form $\bar{\mu}$ on $\bar{\mathscr{S}}^\infty$ by $\bar{\mu}(\bar{X}):=\tilde{\mu}(X)$.
Step 3. We then deduce from the above condition (**A1**) that Daniell's condition holds for $\bar{\mu}$, see e.g. [17]. This allows to construct a signed bounded measure $\bar{\nu}$ on $(W,\mathscr{W})$ such that $\tilde{\mu}(X)=\bar{\mu}(\bar{X})=\bar{\nu}(\bar{X})$.
Step 4. The rest of the proof consists in identifying the triplet $(\alpha_-,\alpha_\circ,\alpha_+)$ of Theorem 4.1 in terms of $\bar{\nu}$ defined on $(W,\mathscr{W})$.

It is clear that we can always reduce to the one dimensional case since $\tilde{\mu}$ is linear. From now on, we shall therefore only consider the case $d=1$. We decompose the proof in different Lemmata.

We first show that Daniell's condition holds for $\bar{\mu}$, whenever (**A1**) holds.

Lemma 4.1. *Assume that* (**A1**) *holds. Then, there exists a signed bounded measure $\bar{\nu}$ on $(W,\mathscr{W})$ such that $\tilde{\mu}(X)=\bar{\mu}(\bar{X})=\bar{\nu}(\bar{X})$ and $|\tilde{\mu}|(X)=|\bar{\mu}|(\bar{X})=|\bar{\nu}|(\bar{X})$ for all $X\in\tilde{\mathscr{S}}^{\infty}$.*

Proof. We first assume that the linear form $\tilde{\mu}$ is non-negative. We only have to prove that $\bar{\mu}$ satisfies the Daniell's condition:

(**A2**) If $(\bar{X}^n)_{n\geq 0}$ decreases to zero then $\bar{\mu}(\bar{X}^n)\to 0$.

Let $(\bar{X}^n)_{n\geq 0}$ be a sequence of non-negative elements of $\bar{\mathscr{S}}^{\infty}$ that decreases to 0. For $\epsilon>0$, we introduce the sets

$$\begin{aligned}
A_n(\omega) &:= \{t\in[0,T] \mid X^n_{t+}(\omega)\geq\epsilon \text{ or } X^n_{t-}(\omega)\geq\epsilon\}\,,\\
B_n(\omega) &:= \{t\in[0,T] \mid X^n_{t}(\omega)\geq\epsilon\}\,,\\
K_n(\omega) &:= A_n(\omega)\cup B_n(\omega)\,.
\end{aligned}\tag{4.1}$$

Obviously, $K_{n+1}(\omega)\subset K_n(\omega)$, $\bigcap_{n\geq 0}K_n(\omega)=\emptyset$ and $A_n(\omega)$ is closed. Let $(t_k)_{k\geq 1}$ be a sequence of $K_n(\omega)$ converging to $s\in[0,T]$. If there is a subsequence $(t_{\phi(k)})_{k\geq 1}$ such that $X_{t_{\phi(k)}}\in A_n(\omega)$ for all $k\geq 0$, then $s\in K_n(\omega)$, since $A_n(\omega)$ is closed. If not, we can suppose than t_k belongs to $B_n(\omega)$ for all $k\geq 1$, after possibly passing to a subsequence. Since $X(\omega)$ is *làdlàg* and bounded, we can extract a subsequence $(t_{\phi(k)})_{k\geq 1}$ such that $\lim X_{t_{\phi(k)}}(\omega)\in\{X_{s-}(\omega),X_s(\omega),X_{s+}(\omega)\}$. Since $X_{t_{\phi(k)}(\omega)}\geq\epsilon$, we deduce that $s\in K_n(\omega)$. This proves that $K_n(\omega)$ is closed. Using the compactness of $[0,T]$, we then obtain that there exists some $N_\epsilon>0$ for which $\cup_{n\geq N_\epsilon}K_n(\omega)=\emptyset$. Thus, $\|X^n(\omega)\|_*<\epsilon$ for $n\geq N_\epsilon$. Since $\tilde{\mu}$ satisfies (**A1**), this implies that $\bar{\mu}$ satisfies Daniell's condition (**A2**).

To cover the case where $\tilde{\mu}$ is not non-negative and prove the last assertion of the Theorem, we can follow exactly the same arguments as in [21, Chapter VI]. We first use the standard decomposition argument $\tilde{\mu}=\tilde{\mu}^+-\tilde{\mu}^-$ where $\tilde{\mu}^+$ and $\tilde{\mu}^-$ are non-negative and satisfy (**A1**). This allows to construct two signed measures $\bar{\nu}^+$ and $\bar{\nu}^-$ on $(W,\mathscr{W})$ such that $\bar{\mu}^+=\bar{\nu}^+$, $\bar{\mu}^-=\bar{\nu}^-$ and therefore $\bar{\mu}=\bar{\nu}:=\bar{\nu}^+-\bar{\nu}^-$. Finally, we observe that, for non-negative X, $|\tilde{\mu}|(X)=\sup\{\tilde{\mu}(Y),\ Y\in\tilde{S}^{\infty},|Y|\leq X\}=|\bar{\mu}|(\bar{X})=\sup\{\bar{\mu}(\bar{Y}),\ \bar{Y}\in\bar{S}^{\infty},|\bar{Y}|\leq\bar{X}\}=|\bar{\nu}|(\bar{X})=\sup\{\bar{\nu}(\bar{Y}),\ \bar{Y}\in\bar{S}^{\infty},|\bar{Y}|\leq\bar{X}\}$, and recall that $\mathscr{W}$ is generated by $\bar{S}^{\infty}$. □

To conclude the proof, it remains to identify the triplet $(\alpha_-,\alpha_\circ,\alpha_+)$ of Theorem 4.1 in terms of $\bar{\nu}$ defined on $(W,\mathscr{W})$. This is based on the two following Lemmas.

From now on, to a function c on $[0,T]\times\Omega$ we associate the three functions c_-, $c_\circ$ and c_+ defined on W by

$$\begin{aligned}
c_-(t,w,+)=c_-(t,\omega,\circ)=0 \ \text{ and } \ c_-(t,\omega,-)=c(t,\omega)\,,\\
c_\circ(t,w,+)=c_\circ(t,\omega,-)=0 \ \text{ and } \ c_\circ(t,\omega,\circ)=c(t,\omega)\,,\\
c_+(t,w,-)=c_+(t,\omega,\circ)=0 \ \text{ and } \ c_+(t,\omega,+)=c(t,\omega)\,.
\end{aligned}$$

Lemma 4.2. *If S is a $\mathscr{F}$-measurable $[0,T]$-valued random variable, then $[[S]]_+,[[S]]_\circ$ and $[[S]]_-$ belongs to $\mathscr{W}$.*

Proof. For $\epsilon>0$, we set $X^\epsilon:=\mathbf{1}_{]]S,(S+\epsilon)\wedge T[[}$ which belongs to $\tilde{\mathscr{S}}^{\infty}$. The associated process $\bar{X}^\epsilon$ is the indicator function of the set $I^\epsilon:=]]S,(S+\epsilon)\wedge T]]_-\cup]]S,(S+\epsilon)\wedge T[[_\circ\cup[[S,(S+\epsilon)\wedge T[[_+$ which belongs to $\mathscr{W}$. Taking $\epsilon_n:=1/n$ with $n\geq 1$, we thus obtain $\cap_{n\geq 1}I^{\epsilon_n}=[[S]]_+\in\mathscr{W}$. Using the same arguments with $X^\epsilon:=\mathbf{1}_{]]0\vee(S-\epsilon),S[[}$, we get that $[[S]]_-\in\mathscr{W}$. Finally working with $X^\epsilon:=$

$\mathbf{1}_{[[S,(S+\epsilon)\wedge T[[}$, we also obtain that $[[S]]_+ \cup [[S]]_\circ \in \mathscr{W}$. Since $[[S]]_\circ = ([[S]]_+ \cup [[S]]_\circ) \cap ([[S]]_+)^c$, this shows that $[[S]]_\circ \in \mathscr{W}$. □

Similarly, given a subset C of $[0,T] \times \Omega$, we set

$$C_- = \{(t,\omega,-) \in W \mid (t,\omega) \in C\,,\ t > 0\}$$
$$C_\circ = \{(t,\omega,\circ) \in W \mid (t,\omega) \in C\}$$
$$C_+ = \{(t,\omega,+) \in W \mid (t,\omega) \in C\,,\ t < T\}\,.$$

Lemma 4.3. *If C is a measurable set of $[0,T] \times \Omega$, then $C_+ \cup C_\circ \cup C_- \in \mathscr{W}$.*

Proof. Since $\mathscr{B}([0,T]) \otimes \mathscr{F}$ is generated by continuous measurable processes, it suffices to check that $X_- + X + X_+$ is $\mathscr{W}$-measurable whenever X is continuous and measurable. This is obvious since $\bar{X} = X_- + X_\circ + X_+$ in this case. □

We can now conclude the proof of Theorem 4.1.

Proof of Theorem 4.1. We first define $\mathscr{H}$ as the collection of sets of the form $A = \bigcup_{n\geq 0} [[S_n]]_+$ for a given sequence $(S_n)_{n\geq 0}$ of $[0,T]$-valued $\mathscr{F}$-measurable random variables. This set is closed under countable union. The quantity $\sup_{A\in\mathscr{H}} |\bar{\nu}|(A) =: M$ is well defined since $\bar{\nu}$ is bounded. Let $(A_n)_{n\geq 1}$ be a sequence such that $\lim |\bar{\nu}|(A_n) = M$ and set $G_+ := \bigcup_{n\geq 0} A_n$, so that $|\bar{\nu}|(G_+) = M$. Observe that we can easily reduce to the case where the G_+ is the union of disjoint graphs. We then define the measure $\bar{\nu}_+ := \bar{\nu}(\cdot \cap G_+)$ and, recall Lemma 4.3,

$$\alpha_+(C) := \bar{\nu}_+(C_+ \cup C_- \cup C_\circ) = \bar{\nu}_+(C_+)$$

for $C \in \mathscr{B}([0,T]) \otimes \mathscr{F}$. The measure α_+ is carried by graphs of $[0,T]$-valued $\mathscr{F}$-measurable random variable. Moreover, for all $[0,T]$-valued $\mathscr{F}$-measurable random variable S, we have

$$\alpha_+([[S]]) = \bar{\nu}([[S]]_+)\,.$$

Indeed, $\bar{\nu}([[S]]_+) > \bar{\nu}([[S]]_+ \cap G_+)$ implies $\bar{\nu}([[S]]_+ \cup G_+) > \bar{\nu}(G_+)$, which contradicts the maximality of G_+.

We construct G_-, $G_\circ$ and the measures α_- and $\bar{\nu}_-$ similarly. The measure $\bar{\nu}_\circ^\delta$ is defined by $\bar{\nu}_\circ^\delta := \bar{\nu}(.\cap G_\circ)$ and the measure $\alpha_\circ^\delta$ by $\alpha_\circ^\delta(C) := \bar{\nu}_\circ^\delta(C_+ \cup C_- \cup C_\circ)$, for $C \in \mathscr{B}([0,T]) \otimes \mathscr{F}$.

We then set $\bar{\nu}_\circ^c := \bar{\nu} - \bar{\nu}_+ - \bar{\nu}_- - \bar{\nu}_\circ^\delta$ and define $\alpha_\circ^c$ by $\alpha_\circ^c(C) := \bar{\nu}_\circ^c(C_+ \cup C_\circ \cup C_-)$ for $C \in \mathscr{B}([0,T]) \times \mathscr{F}$, recall Lemma 4.3 again. Observe that $\bar{\nu}_\circ^\delta$, $\bar{\nu}_\circ^c$ and $\bar{\nu}_-$ do not charge any element of the form $[[S]]_+$ with S a $[0,T]$-valued $\mathscr{F}$-measurable random variable. This follows from the maximal property of G_+. Similarly, $\bar{\nu}_\circ^c$, $\bar{\nu}_\circ^\delta$ and $\bar{\nu}_+$ do not charge any element of the form $[[S]]_-$ and $\bar{\nu}_\circ^c$, $\bar{\nu}_-$ and $\bar{\nu}_+$ do not charge any element of the form $[[S]]_\circ$.

We now fix $X \in \tilde{\mathscr{S}}^\infty$ and set $u : (t,\omega) \mapsto X_{t-}(\omega)$, $v : (t,\omega) \mapsto X_t(\omega)$ and $w : (t,\omega) \mapsto X_{t+}(\omega)$. Then, $\bar{X} = u_- + v_\circ + w_+$ and, by Lemma 4.1,

$$\tilde{\mu}(X) = \bar{\nu}(\bar{X}) = (\bar{\nu}_- + \bar{\nu}_\circ^\delta + \bar{\nu}_\circ^c + \bar{\nu}_+)(u_- + v_\circ + w_+)\,.$$

Since $\bar{\nu}_-$ is carried by G_-, $\bar{\nu}_+$ by G_+, $\bar{\nu}_\circ^\delta$ by $G_\circ$ and $\bar{\nu}_\circ^c$ does not charge any graph of $[0,T]$-valued $\mathscr{F}$-measurable random variable, we deduce that

$$\bar{\nu}_+(u_- + v_\circ + w_+) = \bar{\nu}_+(w_+) = \alpha_+(w),$$

where the last equality comes from the definition of α_+ and w. Similarly, we have

$$\bar{v}_-(u_- + v_\circ + w_+) = \bar{v}_-(u_-) = \alpha_-(u),$$
$$\bar{v}_\circ^\delta(u_- + v_\circ + w_+) = \bar{v}_\circ^\delta(v_\circ) = \alpha_\circ^\delta(v).$$

Since u, v and w differ only on a countable union of graphs, it also follows that $\bar{v}_\circ^c(u_- + v_\circ + w_+) = \bar{v}_\circ^c(v_\circ) = \alpha_\circ^c(v)$. Hence

$$\mu(X) = \alpha_-(u) + \alpha_\circ^c(v) + \alpha_\circ^\delta(v) + \alpha_+(w)$$

which is assertion 3. of the Theorem with $\alpha_\circ := \alpha_\circ^c + \alpha_\circ^\delta$.

To prove the uniqueness of the decomposition, it suffices to show that for a measure $\tilde{\mu}$ satisfying 1, 2 and 3 of Theorem 4.1 then $\tilde{\mu} = 0$ implies $\alpha_- = \alpha_+ = \alpha_\circ = 0$. This follows from similar arguments as the one used in the proof of Lemma 2.1 above. One first proves that $\alpha_- = \alpha_+ = 0$, using the fact that they are carried by a countable union of graphs of $[0, T]$-valued $\mathscr{F}$-measurable random variables. For α_+, the result is obtained by considering processes $X^{\xi,S,\epsilon}$ of the form $X^{\xi,S,\epsilon} := \xi \mathbf{1}_{(S,(S+\epsilon)\wedge T)}$, where S is a $[0, T]$-valued $\mathscr{F}$-measurable random variable, ξ a real $\mathscr{F}$-measurable random variable, and $\epsilon > 0$, and letting ϵ going to 0. For α_-, the result is obtained by considering processes $X^{\xi,S,\epsilon}$ of the form $X^{\xi,S,\epsilon} := \xi \mathbf{1}_{(0\vee(S-\epsilon),S)}$. The result for $\alpha_\circ$ follows then from standard arguments. □

4.2 Proof of Proposition 2.1

We finally provide the proof of Proposition 2.1.

Proof. Observe that $\mathscr{S}^1(\mathbb{Q})$ is closed in the set $\tilde{\mathscr{S}}^1(\mathbb{Q})$ of all *làdlàg* $\mathscr{B}([0, T]) \otimes \mathscr{F}$-measurable processes X satisfying $\mathbb{E}^{\mathbb{Q}}[\|X\|_*] < \infty$. Using the Hahn-Banach theorem, we can find an extension $\tilde{\mu}$ of μ defined on $\tilde{\mathscr{S}}^1(\mathbb{Q})$, i.e. $\tilde{\mu}(X) = \mu(X)$ for $X \in \mathscr{S}^1(\mathbb{Q})$. Obviously, $\tilde{\mu}$ satisfies condition (**A1**) of Theorem 4.1. Thus it satisfies the representation of Theorem 4.1, and so does μ on $\mathscr{S}^\infty$.

Since $\mu(X) = 0$ for all *làdlàg* processes X such that $X = 0$ $\mathbb{Q}$-a.s., the measures α_-, $\alpha_\circ$ and α_+ admit Radon-Nykodim densities with respect to $\mathbb{P} \sim \mathbb{Q}$. We can thus find three $\mathbb{R}^d$-valued processes $\tilde{A}^-$,$\tilde{A}^\circ$ and $\tilde{A}^+$ with essentially bounded total variation satisfying for $X \in \mathscr{S}^\infty$:

$$\mu(X) = \mathbb{E}\left[\int_0^T X_{t-}\, d\tilde{A}_t^- + \int_0^T X_t\, d\tilde{A}_t^\circ + \int_0^T X_{t+}\, d\tilde{A}^+\right],$$

with $\tilde{A}_0^- = 0$ and $\tilde{A}_T^+ = \tilde{A}_{T-}^+$, $\tilde{A}^+$ and $\tilde{A}^-$ are pure jump processes. To conclude, it suffices to replace $\tilde{A}^-$ by its dual predictable projection A^-, and $\tilde{A}^\circ$,$\tilde{A}^+$ by their dual optional projections A° and A^+. One can always add the continuous parts of A^- and A^+ to A° to reduce to the case where A^- and A^+ coincide with pure jumps processes. □

5 The strong duality approach

In this section, we provide the proof of Theorem 2.1.

5.1 The closure property

The proof of Theorem 2.1 is based on a Hahn-Banach type argument and the following key closure property which is obtained by considering a probability measure $\mathbb{Q}_Z$ defined by $d\mathbb{Q}_Z/d\mathbb{P} := c_Z^{-1}\sum_{i\le d} Z_T^i$ with $c_Z := \mathbb{E}\left[\sum_{i\le d} Z_T^i\right]$, for some element Z of $\mathscr{Z}^s$. In the following Proposition, we also state a Fatou type closure property which will also be used in the proof of Theorem 2.1 in order to approximate elements of $\mathscr{S}_b$ by processes with essentially bounded supremum.

Proposition 5.1. (i) *For all $Z \in \mathscr{Z}^s$, $\widehat{\mathscr{C}}^0 \cap \mathscr{S}^1(\mathbb{Q}_Z)$ is closed in $\mathscr{S}^1(\mathbb{Q}_Z)$.*

(ii) *Let $a \in \mathbb{R}^d$ and let $(\widehat{C}^n)_{n\ge 1}$ be a sequence in $\widehat{\mathscr{C}}^0$ such that $\widehat{C}^n \succeq a$ for all $n \ge 1$ and $\|\widehat{C}^n - \widehat{C}\|_* \to 0$ in probability for some* làdlàg *optional process $\widehat{C}$ with values in $\mathbb{R}^d$. Then, $\widehat{C} \in \widehat{\mathscr{C}}^0$.*

Note that the last assertion is an immediate consequence of Lemma 8, Lemma 12 and Proposition 14 of [5], see also the proof of their Theorem 15. It is rather standard in this literature. However, the closure property in $\mathscr{S}^1(\mathbb{Q}_Z)$ is new and does not seem to have been exploited so far.

In order to prove Proposition 5.1, we start with an easy Lemma which essentially follows from arguments used in the proof of Lemma 8 in [5].

Lemma 5.1. *Fix $\widehat{C} \in \widehat{\mathscr{C}}^0 \cap \mathscr{S}^1(\mathbb{Q}_Z)$ for some $Z \in \mathscr{Z}^s$ and $\widehat{V} \in \widehat{\mathscr{V}}^0$ such that $\widehat{V} \succeq \widehat{C}$. Then, $Z\widehat{V}$ is a supermartingale. Moreover,*

$$\mathbb{E}\left[\int_0^T Z_s\,\dot{\widehat{V}}{}_s^c d\mathrm{Var}_s(\widehat{V}^c) + \sum_{s\le T} Z_{s-}\,\Delta\widehat{V}_s + \sum_{s<T} Z_s\,\Delta^+\widehat{V}_s\right] \ge \mathbb{E}\left[Z_T\widehat{V}_T\right] .$$

Proof. Since $Z_t \in \widehat{K}_t^*$ and $\widehat{V}_t - \widehat{C}_t \in \widehat{K}_t$ for all $t \le T$ $\mathbb{P}-$a.s., it follows that $Z_t\widehat{V}_t \ge Z_t\widehat{C}_t$ for all $t \le T$ $\mathbb{P}-$a.s. and therefore, by the martingale property of Z,

$$Z_t\widehat{V}_t \ge \mathbb{E}\left[Z_T\widehat{C}_t \mid \mathscr{F}_t\right] \ge -\,\mathbb{E}\left[\|Z_T\| \sup_{s\in[0,T]} \|\widehat{C}_s\| \mid \mathscr{F}_t\right] \text{ for all } t \le T\ \mathbb{P}-\text{a.s.} \tag{5.1}$$

Since $\widehat{C} \in \mathscr{S}^1(\mathbb{Q}_Z)$, the right-hand side term is a martingale. Moreover, a direct application of the integration by parts formula yields

$$Z_t\widehat{V}_t \;=\; \int_0^t \widehat{V}_s dZ_s + \int_0^t Z_s\dot{\widehat{V}}{}_s^c\, d\mathrm{Var}_s(\widehat{V}^c) + \sum_{s\le t} Z_{s-}\,\Delta\widehat{V}_s + \sum_{s<t} Z_s\,\Delta^+\widehat{V}_s\,.$$

We now observe that the definitions of $\mathscr{Z}^s$ and $\widehat{\mathscr{V}}^0$ imply that the three last integrals on the right-hand side are equal to non-increasing processes. In view of (5.1), this implies that the local martingale $(\int_0^t \widehat{V}_s dZ_s)_{t\le T}$ is bounded from below by a martingale and is therefore a supermartingale. Similarly, $Z\widehat{V}$ is a local super-martingale which is bounded from below by a martingale and is therefore a super-martingale. The proof is concluded by taking the expectation in both sides of the previous inequality applied to $t = T$. □

To complete the proof, we shall appeal to the following alternative representation of the set $\widehat{\mathscr{V}}^0$ which is proved in [5, Lemma 8] under our standing assumption $\mathscr{Z}^s \neq \emptyset$.

Proposition 5.2. *Let $\widehat{V}$ be a $\mathbb{R}^d$-valued predictable process with $\mathbb{P}$-a.s. finite total variation such that $\widehat{V}_0 = 0$. Then, $\widehat{V} \in \widehat{\mathscr{V}}^0$ if and only if*

$$\widehat{V}_\tau - \widehat{V}_\sigma \in -\widehat{K}_{\sigma,\tau} \ \ \mathbb{P}-a.s. \ \textit{for all stopping times } \sigma \leq \tau \leq T \ , \tag{5.2}$$

with

$$\widehat{K}_{\sigma,\tau}(\omega) := \overline{conv}\left(\bigcup_{\sigma(\omega)\leq t<\tau(\omega)} \widehat{K}_t(\omega)\ ,\ 0\right)$$

where $\overline{conv}$ denotes the closure in $\mathbb{R}^d$ of the convex envelope.

Proof of Proposition 5.1. As already mentioned, the last assertion is an immediate consequence of Lemma 8, Lemma 12 and Proposition 14 of [5], see also the proof of their Theorem 15. We now prove the first one which is obtained by very similar arguments. Let $(\widehat{C}^n)_{n\geq 1}$ be a sequence in $\widehat{\mathscr{C}}^0 \cap \mathscr{S}^1(\mathbb{Q}_Z)$ that converges to some $\widehat{C}$ in $\mathscr{S}^1(\mathbb{Q}_Z)$. After possibly passing to a subsequence, we may assume that the convergence holds a.s. uniformly in $t \leq T$. Let $(\widehat{V}^n)_{n\geq 1}$ be a sequence in $\widehat{\mathscr{V}}^0$ such that $\widehat{V}^n \succeq \widehat{C}^n$ for all $n \geq 1$. It follows from the same arguments as in the proof of Lemma 12 in [5] that there is $\tilde{\mathbb{Q}}_Z \sim \mathbb{P}$, which depends only on Z, such that

$$\begin{aligned}&\mathbb{E}\left[-\int_0^T Z_s\,\dot{\widehat{V}}{}^{nc}_s d\mathrm{Var}_s(\widehat{V}^{nc}) - \sum_{s\leq T} Z_{s-}\,\Delta\widehat{V}^n_s - \sum_{s<T} Z_s\,\Delta^+\widehat{V}^n_s\right]\\ &\geq \mathbb{E}^{\tilde{\mathbb{Q}}_Z}\left[\mathrm{Var}_T(\widehat{V}^n)\right]\ .\end{aligned}$$

In view of Lemma 5.1, this implies that $\mathbb{E}^{\tilde{\mathbb{Q}}_Z}\left[\mathrm{Var}_T(\widehat{V}^n)\right] \leq -\mathbb{E}\left[Z_T\widehat{V}^n_T\right]$ for all $n \geq 1$. We now observe that $\widehat{V}^n_T - \widehat{C}^n_T \in \widehat{K}_T$ $\mathbb{P}-$a.s. implies that

$$-\mathbb{E}\left[Z_T\widehat{V}^n_T\right] \leq -\mathbb{E}\left[Z_T\widehat{C}^n_T\right] \leq\ c_Z\ \mathbb{E}^{\mathbb{Q}_Z}\left[\|\widehat{C}^n_T\|\right]\ .$$

Since $\widehat{C}^n_T$ converges to $\widehat{C}_T$ in $L^1(\mathbb{Q}_Z)$, the right-hand side of the latter inequality is uniformly bounded and so is the quantity $\mathbb{E}^{\tilde{\mathbb{Q}}_Z}\left[\mathrm{Var}_T(\widehat{V}^n)\right]$. It thus follows from Proposition 14 in [5] that, after possibly passing to convex combinations, we can assume that, $\mathbb{P}-$a.s., $(\widehat{V}^n)_{n\geq 1}$ converges pointwise on $[0,T]$ to a predictable process $\widehat{V}$ with finite variations. The pointwise convergence ensures that $\widehat{V} \succeq \widehat{C}$. By Proposition 5.2, $\widehat{V}^n$ satisfies (5.2) for all $n \geq 1$, and it follows from the pointwise convergence that $\widehat{V}$ satisfies (5.2) too. We can then conclude from Proposition 5.2 that $\widehat{V} \in \widehat{\mathscr{V}}^0$. □

5.2 Proof of Theorem 2.1

We can now prove the super-hedging Theorem. We split the proof in several Propositions.

Proposition 5.3. $\widehat{\mathscr{C}}^0_b \subset \mathscr{D}^\diamond$.

Proof. Let $\widehat{C}$ be a *làdlàg* optional process such that $\widehat{C} \succeq a$ for some $a \in \mathbb{R}^d$, and let $\widehat{V} \in \widehat{\mathscr{V}}^0$ be such that $\widehat{V} \succeq \widehat{C}$. Since, by Lemma 2.1, $A = (A^-, A^\circ, A^+) \in \mathscr{D} \subset \mathscr{R}_{\widehat{K}}$, we have

$$\int_0^T (\widehat{C}_{t-} - \widehat{V}_{t-})dA^-_t + \int_0^T (\widehat{C}_t - \widehat{V}_t)dA^\circ_t + \int_0^T (\widehat{C}_{t+} - \widehat{V}_{t+})dA^+_t \leq 0\ \ \mathbb{P}-\text{a.s.}$$

Thus, it suffices to show that

$$\mathbb{E}\left[\int_0^T \widehat{V}_{t-}\, dA_t^- + \int_0^T \widehat{V}_t\, dA_t^\circ + \int_0^T \widehat{V}_{t+}\, dA_t^+\right] \leq 0\,.$$

For $\widehat{V}$ with an essentially bounded total variation, this is a direct consequence of **(C2)** in the definition of $\mathscr{D}$. In the general case, we observe that $\widehat{V} \succeq \widehat{C} \succeq a \in \mathbb{R}^d$ implies that $\widehat{V}$ can be approximated from below (in the sense of $\succeq$) by the sequence $(\widehat{V}^n)_{n\geq 1}$ defined by $\widehat{V}^n := \widehat{V}\mathbf{1}_{[[0,\tau_n]]} + a\ \mathbf{1}_{]]\tau_n,T]]}$ where $(\tau_n)_{n\geq 1}$ is a localizing sequence of stopping times for $\mathrm{Var}(\widehat{V})$, so that $\tau_n \uparrow \infty$. The existence of this sequence is justified by the fact that $\mathrm{Var}(\widehat{V})$ is predictable and almost surely finite, thus locally bounded. Observe that $\widehat{V}^n \in \widehat{\mathscr{V}}^0$ and has essentially bounded variation, we thus have

$$\mathbb{E}\left[\int_0^T \widehat{V}^n_{t-}\, dA_t^- + \int_0^T \widehat{V}^n_t\, dA_t^\circ + \int_0^T \widehat{V}^n_{t+}\, dA_t^+\right] \leq 0\,.$$

Since $\widehat{V}^n \succeq a$ for all $n \geq 1$ and $A \in \mathscr{R}_{\widehat{K}}$, each integral in the expectation is bounded from below, uniformly in n, by an integrable random variable which depends only on A and a. Since $\widehat{V}^n \to \widehat{V}$ uniformly on compact sets, $\mathbb{P}-$a.s., we can conclude by appealing to Fatou's Lemma. $\square$

We now prove the converse implication. To this purpose, we shall appeal to our two key results: Proposition 5.1 and Proposition 2.1.

Proposition 5.4. $\widehat{\mathscr{C}}^0_b \supset \mathscr{D}^\diamond$.

Proof. Fix $\widehat{C} \in \mathscr{D}^\diamond$ such that $\widehat{C} \succeq a$.

Step 1. We first consider the case where $\widehat{C} \in \mathscr{S}^\infty$. Assume that $\widehat{C}$ does not belong to the convex cone $\widehat{\mathscr{C}}^0$. Fix $Z \in \mathscr{Z}^s$ and observe that $\widehat{C} \notin \widehat{\mathscr{C}}^0 \cap \mathscr{S}^1(\mathbb{Q}_Z)$. The latter being closed in $\mathscr{S}^1(\mathbb{Q}_Z)$, see Proposition 5.1, it follows from the Hahn-Banach separation theorem that we can find μ in the dual of $\mathscr{S}^1(\mathbb{Q}_Z)$ such that $\mu(X) \leq c < \mu(\widehat{C})$ for all $X \in \widehat{\mathscr{C}}^0 \cap \mathscr{S}^\infty$, for some real c. Since $\widehat{\mathscr{C}}^0$ is a cone, it is clear that $c = 0$. Thus,

$$\sup_{X\in\widehat{\mathscr{C}}^0\cap\mathscr{S}^\infty} \mu(X) \leq 0 < \mu(\widehat{C})\,. \tag{5.3}$$

Moreover, by Proposition 2.1, there is a process $A := (A^-, A^\circ, A^+) \in \mathscr{R}$ such that

$$\mu(X) = \mathbb{E}\left[\int_0^T X_{t-}\, dA_t^- + \int_0^T X_t\, dA_t^\circ + \int_0^T X_{t+}\, dA_t^+\right] \quad \text{for all } X \in \mathscr{S}^\infty\,. \tag{5.4}$$

Since $\widehat{C} \in \mathscr{S}^\infty$, it thus suffices to show that $A \in \mathscr{D}$ to obtain a contradiction.

We first note that $X = -\xi$ belongs to $\widehat{\mathscr{C}}^0 \cap \mathscr{S}^\infty$ for all process $\xi \in \mathscr{S}^\infty$ satisfying $\xi \succeq 0$. Thus, A satisfies **(C1)**. Since it also has to satisfy **(C2)**, this implies that $A \in \mathscr{D}$, which leads to a contradiction.

Step 2. We conclude the proof by considering the case where $\widehat{C}$ is not bounded but only satisfies $\widehat{C} \succeq a$ for some $a \in \mathbb{R}^d$. Define the bounded process $\widehat{C}^n := \widehat{C}\mathbb{1}_{\{\|\widehat{C}\|\leq n\}} + a\ \mathbb{1}_{\{\|\widehat{C}\|>n\}}$ for $n \geq 1$.

Observing that $\widehat{C} \succeq \widehat{C}^n$ for all $n \geq 1$ and recalling that $\widehat{C} \in \mathscr{D}^\diamond$, we deduce from Lemma 2.1 ($\mathscr{D} \subset \mathscr{R}_{\widehat{K}}$) that

$$\mathbb{E}\left[\int_0^T \widehat{C}^n_{t-}\, dA^-_t + \int_0^T \widehat{C}^n_t\, dA^\circ_t + \int_0^T \widehat{C}^n_{t+}\, dA^+_t\right] \leq 0$$

for all $A = (A^-, A^\circ, A^+) \in \mathscr{D}$ and $n \geq 1$. It follows from Step 1 that $\widehat{C}^n \in \widehat{\mathscr{C}}^0$ for all $n \geq 1$. Since $\widehat{C}^n \succeq a$ for all $n \geq 1$ and $\|\widehat{C}^n - \widehat{C}\|_* \to 0$ $\mathbb{P}-$a.s., it follows from Proposition 5.1 (ii) that $\widehat{C} \in \widehat{\mathscr{C}}^0$ too. □

We now conclude the proof of Theorem 2.1.

Proposition 5.5. $(\widehat{\mathscr{C}}^0_b)^\diamond = \mathscr{D}$.

Proof. By definition of $\mathscr{D}$, we have $(\widehat{\mathscr{C}}^0_b)^\diamond \subset \mathscr{D}$. Moreover, it follows from Proposition 5.3 that $\widehat{\mathscr{C}}^0_b \subset \mathscr{D}^\diamond$ which implies that $\mathscr{D} \subset (\mathscr{D}^\diamond)^\diamond \subset (\widehat{\mathscr{C}}^0_b)^\diamond$. □

A Appendix

Proof of Lemma 2.1. We only prove that (i)-(iii) holds if **(C1)** is satisfied. The converse is obvious. Let ξ be any bounded optional *làdlàg* process such that $-\xi \succeq 0$. Given $B \in \mathscr{F}$, let λ be the optional projection of $\mathbf{1}_B$. Note that it is *càdlàg*, since $\mathbf{1}_B(\omega)$ is constant for each ω, and that the process λ_- coincides with the predictable projection of $\mathbf{1}_B$, see Chapter V in [8]. We then set $\tilde{\xi} := \lambda\xi$. We remark that $\tilde{\xi}$ is the optional projection of $\mathbf{1}_B\xi$, since ξ is optional, and that $\tilde{\xi}_-$ is the predictable projection of $\mathbf{1}_B\xi_-$, since ξ_- is predictable. Since the set valued process $\widehat{K}$ is a cone and λ takes values in $[0,1]$, we have $-\tilde{\xi} \succeq 0$. Moreover, since A^- is predictable (resp. A°,A^+ are optional), it follows that the induced measure commutes with the predictable projection (resp. the optional projection), see e.g. Theorem 3 Chapter I in [21]. Applying **(C1)** to $\tilde{\xi}$ thus implies that

$$\begin{aligned} 0 \quad &\geq \quad \mathbb{E}\left[\int_0^T \lambda_{t-}\xi_{t-}\, dA^-_t + \int_0^T \lambda_t\xi_t\, dA^\circ_t + \int_0^T \lambda_t\xi_{t+}\, dA^+_t\right] \\ &= \quad \mathbb{E}\left[\mathbf{1}_B\left(\int_0^T \xi_{t-}\, dA^-_t + \int_0^T \xi_t\, dA^\circ_t + \int_0^T \xi_{t+}\, dA^+_t\right)\right] . \end{aligned}$$

By the arbitrariness of B, this shows that the *càdlàg* process X defined by

$$X := \int_0^\cdot \xi_{t-}\, dA^-_t + \int_0^\cdot \xi_t\, dA^\circ_t + \int_0^\cdot \xi_{t+}\, dA^+_t$$

satisfies $X_T \leq 0$ $\mathbb{P}-$a.s. Moreover, replacing ξ by $\xi\mathbf{1}_{(s+\varepsilon,t+\varepsilon\wedge T]}$ for $s < t \leq T$, $\varepsilon > 0$, and sending $\varepsilon \to 0$ shows that X is non-decreasing (recall (iii) of the definition of $\mathscr{R}$). In particular, its continuous part is non-decreasing, see e.g. Chapter VII in [20]. Since A^- and A^+ are purely discontinuous, this implies that the continuous part of $\int_0^\cdot \xi_t dA^\circ_t$ is non-decreasing. Letting $A^{\circ c}$ denote the continuous part of A°, we thus deduce that

$$\xi\, \dot{A}^{\circ c} \leq 0 \;\; d\mathrm{Var}(A^{\circ c}) \otimes \mathbb{P} - a.e \tag{A.1}$$

We now replace ξ by $\tilde{\xi} := \xi \mathbf{1}_{]]\tau,\tau_h[[}$ where τ is some stopping time with values in $[0,T)$ and $\tau_h := (\tau+h)\wedge T$ for some $h>0$. The same argument as above shows that

$$\int_{]]\tau,\tau_h]]} \xi_{t-}\, dA^-_t + \int_{]]\tau,\tau_h[[} \xi_t\, dA^\circ_t + \int_{[[\tau,\tau_h[[} \xi_{t+}\, dA^+_t \le 0 \ \ \mathbb{P}-\text{a.s.}$$

For $h\to 0$, this leads to

$$\xi_{\tau+}\, \Delta A^+_\tau \le 0 \ \ \mathbb{P}-\text{a.s.} \tag{A.2}$$

for all stopping times τ with values in $[0,T)$. Arguing as above with ξ replaced by $\tilde{\xi} := \xi \mathbf{1}_{]]\tau_n,\tau[[}$, where τ is a predictable stopping time with values in $(0,T]$ and $(\tau_n)_{n\ge 1}$ is an announcing sequence for τ, leads to

$$\xi_{\tau-}\, \Delta A^-_\tau \le 0 \ \ \mathbb{P}-\text{a.s.} \tag{A.3}$$

Finally, we replace ξ by $\tilde{\xi} := \xi \mathbf{1}_{[[\tau]]}$ to obtain

$$\xi_\tau\, \Delta A^\circ_\tau \le 0 \ \ \mathbb{P}-\text{a.s.} \tag{A.4}$$

for all stopping times τ with values in $[0,T]$. Since the cone valued process $\widehat{K}$ is generated by a family of *càdlàg* adapted processes, which we can always assume to be bounded, (A.1), (A.2), (A.3), (A.4) and (iii) of the definition of $\mathscr{R}$ imply the required result. □

The following prepares for the proof of Lemma 2.2.

Lemma A.1. *Let $\widehat{V}$ be an element $\widehat{\mathscr{V}}^0$ with essentially bounded total variation. Then,*

$$(\widehat{V}|A) = \mathbb{E}\left[\sum_{t\le T} \bar{A}^-_t \Delta \widehat{V}_t + \int_0^T \bar{A}^+_t d\widehat{V}^c_t + \sum_{t<T} \bar{A}^+_t \Delta^+ \widehat{V}_t\right] \ , \ \forall\, A := (A^-, A^\circ, A^+) \in \mathscr{R} \ .$$

Proof. By Fubini's theorem and the continuity of $\widehat{V}^c$,

$$\begin{aligned}
\int_0^T \widehat{V}_{t-}\, dA^-_t &= \int_0^T \left(\int_0^t d\widehat{V}^c_s + \sum_{s<t} (\Delta \widehat{V}_s + \Delta^+ \widehat{V}_s) \right) dA^-_t \\
&= \int_0^T (A^-_T - A^-_t) d\widehat{V}^c_t + \sum_{t<T} (A^-_T - A^-_t)\left(\Delta \widehat{V}_t + \Delta^+ \widehat{V}_t\right) \ .
\end{aligned}$$

Similarly,

$$\int_0^T \widehat{V}_t\, dA^\circ_t = \int_0^T (A^\circ_T - A^\circ_t) d\widehat{V}^c_t + \sum_{t\le T} (A^\circ_T - A^\circ_{t\_}) \Delta \widehat{V}_t + \sum_{t<T} (A^\circ_T - A^\circ_t) \Delta^+ \widehat{V}_t$$

and

$$\int_0^T \widehat{V}_{t+}\, dA^+_t = \int_0^T (A^+_T - A^+_{t-}) d\widehat{V}^c_t + \sum_{t\le T} (A^+_T - A^+_{t\_})\left(\Delta \widehat{V}_t + \Delta^+ \widehat{V}_t\right) \ .$$

This shows hat

$$(A|V] \quad = \quad \mathbb{E}\left[\sum_{t\le T}\delta A^-_t\,\Delta\widehat{V}_t+\int_0^T \delta A^+_t\,d\widehat{V}^c_t+\sum_{t<T}\delta A^+_t\,\Delta^+\widehat{V}_t\right]\,.$$

The proof is concluded by replacing δA^- (resp. δA^+) by its dual predictable (resp. optional) projection, which is made possible by the special measurability of the variations of $\widehat{V}$, see Definition 1.1. □

Proof of Lemma 2.2. First assume that **(C2)** holds. Then, it follows from Lemma A.1 that:

$$\mathbb{E}\left[\sum_{t\le T}\bar{A}^-_t\,\Delta\widehat{V}_t+\int_0^T \bar{A}^+_t\,d\widehat{V}^c_t+\sum_{t<T}\bar{A}^+_t\,\Delta^+\widehat{V}_t\right]\le 0\,, \tag{A.5}$$

for all $\widehat{V}\in\widehat{\mathscr{V}}^0\cap\mathscr{S}^\infty$ with essentially bounded total variation. It thus follows from Definition 1.1 that $\mathbb{E}\left[\bar{A}^-_\tau\xi\right]\le 0$ for all predicable stopping times $\tau\le T$ $\mathbb{P}-$a.s. and bounded $\mathscr{F}_{\tau-}$-measurable ξ taking values in $-\widehat{K}_{\tau-}$ $\mathbb{P}-$a.s. Similarly, $\mathbb{E}\left[\bar{A}^+_\tau\xi\right]\le 0$ for all stopping times $\tau<T$ $\mathbb{P}-$a.s. and bounded $\mathscr{F}_\tau$-measurable ξ taking values in $-\widehat{K}_\tau$ $\mathbb{P}-$a.s. Observe that $\bar{A}^+_T=0\in\widehat{K}^*_T$ since $\Delta A^+_T=0$. Recalling the definition of $\widehat{K}$ in terms of its generating family based on Π, this implies that (i) and (ii) are satisfied. Conversely, if A satisfies (i) and (ii), then (A.5) holds for all $\widehat{V}\in\widehat{\mathscr{V}}^0$ with essentially bounded total variation and we deduce from Lemma A.1 that **(C2)** holds. □

References

[1] Bismut J.-M. (1979). Temps d'arrêt optimal, quasi-temps d'arrêt et retournement du temps. *Ann. Probab.* 7, 933-964. MR0548890

[2] Bouchard B. and H. Pham (2004). Wealth-Path Dependent Utility Maximization in Incomplete Markets. *Finance and Stochastics*, 8 (4), 579-603. MR2212119

[3] Bouchard B. and E. Temam (2005). On the Hedging of American Options in Discrete Time Markets with Proportional Transaction Costs. *Electronic Journal of Probability*, 10, 746-760. MR2164029

[4] Bouchard B., N. Touzi and A. Zeghal (2004). Dual Formulation of the Utility Maximization Problem : the case of Nonsmooth Utility. *The Annals of Applied Probability*, 14 (2), 678-717. MR2052898

[5] Campi L. and W. Schachermayer (2006). A super-replication theorem in Kabanov's model of transaction costs. *Finance and Stochastics*, 10(4), 579-596. MR2276320

[6] Chalasani P. and S. Jha (2001). Randomized stopping times and American option pricing with transaction costs. *Mathematical Finance*, 11(1), 33-77. MR1807848

[7] Delbaen F. and W. Shachermayer (1998). The fundamental theorem of asset pricing for unbounded stochastic processes. *Math. Annalen* , **312**, 215-250. MR1671792

[8] Dellacherie C. (1972). *Capacités et processus stochastiques*. Springer-Verlag. MR0448504

[9] Denis E. , D. De Vallière and Y. Kabanov (2008). Hedging of american options under transaction costs. *preprint.*

[10] El Karoui N. (1979). Les aspects probabilistes du contrôle stochastique. *Ecole d'Eté de Probabilités de Saint Flour* IX, Lecture Notes in Mathematics 876, Springer Verlag.

[11] El Karoui N. (1982). Une propriété de domination de l'enveloppe de Snell des semimartingales fortes. *Sém. prob. Strasbourg,* 16, 400-408.

[12] Kabanov Y. and G. Last (2002). Hedging under transaction costs in currency markets: a continuous time model. *Mathematical Finance,* 12, 63-70. MR1883786

[13] Kabanov Y. and C. Stricker (2002). Hedging of contingent claims under transaction costs. *Advances in Finance and Stochastics.* Eds. K. Sandmann and Ph. Schönbucher, Springer, 125-136. MR1929375

[14] Karatzas I. and S. G. Kou (1998). Hedging American contingent claims with constrained portfolios. *Finance and Stochastics,* 2, 215-258. MR1809521

[15] Karatzas I. and S. E. Shreve (1991). *Brownian motion and stochastic calculus.* Springer Verlag, Berlin. MR1121940

[16] Karatzas I. et S.E. Shreve (1998), *Methods of Mathematical Finance*, Springer Verlag. MR1640352

[17] Kindler J. (1983). A simple proof of the Daniell-Stone representation theorem. *Amer. Math. Monthly*, 90 (3), 396-397. MR0707155

[18] Kramkov D. (1996). Optional decomposition of supermartingales and hedging in incomplete security markets. *Probability Theory and Related Fields*, 105 (4), 459-479. MR1402653

[19] Kramkov D. and W. Schachermayer (1999). The Asymptotic Elasticity of Utility Functions and Optimal Investment in Incomplete Markets. *Annals of Applied Probability,* **9** 3, 904 - 950. MR1722287

[20] Meyer P.A. (1966). *Probabilités et potentiel.* Hermann, Paris. MR0205287

[21] Meyer P.A. (1976). Un cours sur les intégrales stochastiques. *Sém. prob. Strasbourg,* 10, 245-400. MR0501332

[22] Rasonyi M. (2003). A remark on the superhedging theorem under transaction costs. *Séminaire de Probabilités XXXVII,* Lecture Notes in Math., 1832, Springer, Berlin-Heidelberg-New York, 394-398. MR2053056

[23] Schachermayer W. (2004). The Fundamental Theorem of Asset Pricing under Proportional Transaction Costs in Finite Discrete Time. *Mathematical Finance*, 14 (1), 19-48. MR2030834