Electronic Journal of Probability

Vol. 7 (2002) Paper No. 10, pages 1–21.

Journal URL
http://www.math.washington.edu/~ejpecp/
Paper URL
http://www.math.washington.edu/~ejpecp/EjpVol7/paper11.abs.html

DIFFUSION AND SCATTERING OF SHOCKS IN THE PARTIALLY ASYMMETRIC SIMPLE EXCLUSION PROCESS[1]

V. Belitsky
Instituto de Matematica e Estatistica, Universidade de São Paulo, Rua de Matão, 1010
CEP 05508-090, São Paulo, SP, Brazil
belitsky@ime.usp.br

G. M. Schütz
Institut für Festkörperforschung, Forschungszentrum Jülich, D-52425 Jülich, Germany
g.schuetz@fz-juelich.de

Abstract. We study the behavior of shocks in the asymmetric simple exclusion process on $\mathbb{Z}$ whose initial distribution is a product measure with a finite number of shocks. We prove that if the particle hopping rates of this process are in a particular relation with the densities of the initial measure then the distribution of this process at any time $t \geq 0$ is a linear combination of shock measures of the structure similar to that of the initial distribution. The structure of this linear combination allows us to interpret this result by saying that the shocks of the initial distribution perform continuous time random walks on $\mathbb{Z}$ interacting by the exclusion rule. We give explicit expressions for the hopping rates of these random walks. The result is derived with a help of quantum algebra technique. We made the presentation self-contained for the benefit of readers not acquainted with this approach, but interested in applying it in the study of interacting particle systems.

Keywords Asymmetric simple exclusion process, evolution of shock measures, quantum algebra.

AMS subject classification (2000) 60K35, 82C22, 82C20, 82C23.

Submitted to EJP on March 6, 2001. Final version accepted on February 21, 2002.

[1]SUPPORTED BY FAPESP, CNPQ AND CAPES-DAAD

1 Introduction

The Asymmetric Simple Exclusion Process (ASEP, for short) on $\mathbb{Z}$ is the name for the evolution of identical particles on $\mathbb{Z}$ that obey the following rules: *(1)* each particle performs a continuous time simple random walk on $\mathbb{Z}$; *(2)* when a particle hops to a site occupied by another particle, the former is immediately returned to the position from where it hopped (this property is usually called "the exclusion rule", another way of phrasing this rule is to say that a hopping attempt is rejected when the site where the particles tries to move is occupied.); *(3)* all the particles have the same hopping rate to the left (resp., right) which we denote by c_ℓ (resp., c_r). A formal definition of ASEP will be given in Section 3, where it is needed. More details in respect to the construction of the ASEP may be found in [L1, L2].

We study the evolution of shocks in the ASEP on $\mathbb{Z}$ that starts from a measure with a finite number of shocks (the shock measures considered here are explicitly defined in the statements of the theorems). We state (Theorems 1 and 2) that at any time $t \geq 0$, the distribution of this process is a linear combination of shock measures of a structure similar to that of the initial one. The probabilistic interpretation of the weights in this linear combinations allows us to say that the shocks (or better, the shock positions) perform continuous time random walks on $\mathbb{Z}$ interacting by the exclusion rule; the hopping rates of these random walks are given by (3) and (6) below. This phenomenon has been observed in Monte-Carlo simulations [BCFG, KSKS] for the case of a single shock.

We derive our results from the quantum algebra symmetry of the generator of the ASEP which arises from the integrability of the related six-vertex model in statistical mechanics (see [KR], [PS]). This approach is rarely used by probabilists who usually employ the technique based on the second class particle for studying shocks in ASEP (see [F1], [FKS], [FF] and references therein). In fact, this technique has been applied in [FFV] to the case of multiple shocks. The results from [FFV] have an asymptotic form ($time \to \infty$) and involve certain space/time change, while our results provide exact information for any $t \geq 0$ and for all finite lattice distances (for the second class particle to provide a detailed information on the shock measure, one usually needs that the process starts from its steady state, as one may see for example in [DJLS, DLS]; another advantage of our approach is that it applies to a multi-shock case as easily as to a single-shock). The results from [FFV] and ours complement one another in a manner that is worth being mentioned. Namely, if we apply the appropriate time-space rescaling to our results then we recover those from [FFV], and, moreover, the character of the interaction between shocks revealed in the present paper allows one to explain the phenomena observed in [FFV] through the time-space re-scale lenses. Note however, that our results require the particle hopping rates to be in a particular relation to the initial particle densities (the assumptions (1) and (4) from our theorems) while the phenomena observed in [FFV] do not depend on the initial conditions. This independence suggests that some basic features of the interaction between shocks that are described in our paper, persist also in general. Unfortunately, we can't state rigorously the counterpart of our results for the general case.

We feel that the study of ASEP could benefit from a popularization among probabilists of the quantum algebra technique (Remark 4 from this section justifies this statement). One of the objectives of our paper is, thus, to give a self contained presentation of this technique. In Note 3 (after (34)) we indicate why this technique works efficiently, at least in our case.

From a physical viewpoint a shock may be considered as a collective dynamical mode which describes the motion of many particles by just a single coordinate, viz. the 'shock position'. Macroscopically this position marks the sudden increase of the local density in space, as seen e.g. at the end of a traffic jam. Generally, such a description of a many-body system by just a few characteristic quantities (in this case: the shock positions) is an obvious necessity as one cannot possibly measure or handle the huge amount of data specifying the state of each particle in a macroscopic system. The art is the identification of the relevant collective modes and the description of their mutual interdependence in terms of mathematical equations (physical laws). These equations cannot normally be derived from the microscopic dynamics but have to be guessed and verified experimentally. Moreover, such a reduced description is usually not exact, but only approximately true to various degrees of accuracy. The ASEP is a nice and rare exception from this rule. The main results of our paper, the following two theorems, show that for certain families of shock densities the collective description is not an approximation, but actually exact. This feature, combined with the importance of shocks in non-equilibrium systems, provides a major physical motivation for our study.

Below in the text, we use the commonly accepted identification of the set $\{0,1\}^{\mathbb{Z}}$ with the set of all configurations of particles on $\mathbb{Z}$, satisfying the constraint "at most one particle per site", namely, an element $\eta \in \{0,1\}^{\mathbb{Z}}$ is called a configuration of particles and $\eta(i)$ is interpreted by saying that the site $i \in \mathbb{Z}$ is either occupied by a particle or empty.

Theorem 1. (The evolution of the shock in ASEP starting from a single-shock measure.) *For $k \in \mathbb{Z}$, let μ_k denote the product measure on $\{0,1\}^{\mathbb{Z}}$ with the density ρ_1 to the left of the site k (including k) and with the density ρ_2 elsewhere where ρ_1, ρ_2 are two arbitrary numbers from $(0,1)$. We call μ_k a shock measure with the shock at k. Consider the ASEP on $\mathbb{Z}$ with the particle hopping rates c_ℓ and c_r to the left and to the right respectively, satisfying*

$$\frac{\rho_2(1-\rho_1)}{\rho_1(1-\rho_2)} = \frac{c_r}{c_\ell} \tag{1}$$

Let $\mu_k(t)$ denote the distribution at time t of this ASEP, starting from μ_k. Then, for any $k \in \mathbb{Z}$,

$$\mu_k(t) = \sum_{i=-\infty}^{+\infty} p_t(i \,|\, k)\mu_i \tag{2}$$

where $p_t(i \,|\, k)$ is the probability that a particle that performs a continuous-time simple random walk on $\mathbb{Z}$ with the hopping rates

$$\delta_1 = \frac{1-\rho_1}{1-\rho_2} c_\ell \ \ \text{and} \ \ \delta_2 = \frac{1-\rho_2}{1-\rho_1} c_r \tag{3}$$

to the left and to the right respectively, is at the site i at time t, starting from the site k.

Remark 1. The numerical value of $p_t(i \,|\, k)$ is known:

$$p_t(i \,|\, k) = e^{-(\delta_1+\delta_2)t} \left(\frac{\delta_1}{\delta_2}\right)^{(k-i)/2} I_{k-i}(2\sqrt{\delta_1\delta_2}\, t)$$

where $I_j(\cdot)$ denotes the modified Bessel function.

Theorem 2. (The evolution of the shocks in ASEP starting from a measure with several shocks and extra particles at the shock positions.)
Let n be a positive integer and let $\rho_1, \ldots, \rho_{n+1}$ be $n+1$ real numbers from $(0,1)$ that satisfy the following relations

$$for\ some\ q, \qquad \frac{\rho_{i+1}(1-\rho_i)}{\rho_i(1-\rho_{i+1})} = q^2, \quad for\ all\ i = 1, \ldots, n \tag{4}$$

For arbitrary integers $k_1 < k_2 < \cdots < k_n$, let $\mu_{k_1 \cdots k_n}$ denote the product measure on $\{0,1\}^{\mathbb{Z}}$ with

the density ρ_1 on (∞, k_1);
the density ρ_i on $(k_{i-1}, k_i), i = 2, \ldots, n$;
the density ρ_{n+1} on $(k_n, +\infty)$;
the density 1 on the set $\{k_1, \ldots, k_n\}$.

We call $\mu_{k_1 \cdots k_n}$ shock measure with the shocks at $k_1, \ldots, k_n$. Consider the ASEP on $\mathbb{Z}$ with the particle hopping rates c_ℓ and c_r to the left and to the right respectively, that satisfy

$$\frac{c_r}{c_\ell} = q^2 \ for\ q\ from\ (4)$$

Let $\mu_{k_1 \cdots k_n}(t)$ denote the distribution of this ASEP at time t, starting from $\mu_{k_1 \cdots k_n}$. Then,

$$\mu_{k_1 \cdots k_n}(t) = \sum_{i_1 < i_2 < \cdots < i_n} p_t\left((i_1, \ldots, i_n) \,\middle|\, (k_1, \ldots, k_n)\right) \mu_{i_1 \cdots i_n} \tag{5}$$

where $p_t\left((i_1, \ldots, i_n) \,\middle|\, (k_1, \ldots, k_n)\right)$ has the following probabilistic interpretation: Consider n particles that perform continuous time random walks on $\mathbb{Z}$ and interact by the exclusion rule, where the hopping rates of the j-th particle (counting from the left) are

$$\delta_{j\ell} = \frac{1-\rho_j}{1-\rho_{j+1}} c_\ell \ \ and \ \ \delta_{jr} = \frac{1-\rho_{j+1}}{1-\rho_j} c_r \tag{6}$$

to the left and to the right respectively. Then, $p_t\left((i_1, \ldots, i_n) \,\middle|\, (k_1, \ldots, k_n)\right)$ is the probability of finding these particles at the sites $i_1, \ldots, i_n$ at time t, when their initial positions are $k_1, \ldots, k_n$, respectively.

Remark 2. The particles that are put at the shock positions of $\mu_{k_1 \cdots k_n}$ (that is, at the sites $k_1, \ldots, k_n$) are not the second class particles in the sense of [FKS] and [FFV].

Remark 3. The numerical values of the transition probabilities

$$p_t\left((i_1, \ldots, i_n) \,\middle|\, (k_1, \ldots, k_n)\right)$$

can be calculated using the Bethe ansatz along the lines indicated in [S3].

Remark 4. We note that a result similar to Theorems 1 and 2 can be obtained for the ASEP on a finite lattice with open boundary conditions where particles enter and exit the system with appropriately chosen rates [K]. The relationship (1) has also appeared in [DLS] where it was observed that for this choice of densities the steady state measures to the right and left resp. of a second-class particle are Bernoulli at all finite lattice distances, not just asymptotically. Our result relates the origin of this relation to the quantum algebra symmetry and furthermore suggests that a similar statement is true for all times and could be extended to a system with several second-class particles with intermediate densities of first-class particles given by (4). In order to prove this conjecture one would have to prove the analog of Theorem 2 with first-class particles on sites $k_1 \cdots k_n$ replaced by second-class particles. We do not know whether an analog of Theorem 2 would hold, if $\mu_{k_1 \cdots k_n}$ were a shock measure without extra particles situated at the shock positions.

Let us now indicate the main steps of the proof of Theorem 1 and the technique employed in each of these steps.

Step 1. We fix arbitrarily $m \in \mathbb{N}$ and we consider the ASEP on the finite lattice $\mathcal{M} := \{-m+1, -m+2, \ldots, m\}$ with the particle hopping rates c_ℓ and c_r to the left and to the right, respectively; in this process the particles are prohibited to exit from $\mathcal{M}$. For $k \in \mathcal{M}$, let μ_k^m denote the product measure on $\{0,1\}^{\mathcal{M}}$ with the density ρ_1 on the sites $-m+1, \ldots, k$ and the density ρ_2 on the sites $k+1, \ldots, m$, and let $\mu_k^m(t)$ denote the distribution of the ASEP on $\mathcal{M}$ at time t, starting from μ_k^m. In Step 1, we derive a system of differential-difference equations (see (8) below) that involve the functions from the family $\{\mu_k^m(s), s \geq 0\}_{k \in \mathcal{M}}$. This system is almost closed with respect to these functions, but not completely. This incompleteness is caused by the "boundary effect", that is the prohibition of the particle jumps outside of $\mathcal{M}$.

In **Step 2**, we consider the $m \to \infty$ limit of the system of equations obtained in Step 1. We show that in this limit, the boundary effect disappears and that the system "tends" to a particular system of differential-difference equations. This system has the same structure as the system of the differential-difference equations involving the functions $\{p_\cdot(i \,|\, k),\, k, i \in \mathbb{Z}\}$ defined above. This equivalence leads then to the desired result.

The proofs of Step 2 use some basic facts from the theory of differential equations and certain classical tools from the field of Interacting Particle Systems (exactly to say, the coupling and the convergence theorems) which one may find in [L1]. The proofs of Step 1 use a quantum algebra symmetry ([KR]) of the generator of the ASEP which follows from its relationship to a Heisenberg-type quantum spin chain. The description of this technique and its application to the interacting particle systems may be found in [S2]; the present paper contains a short review. In our arguments, we combine the classical tools with a recent result of Schütz (see (29) below or eq. (3.8) in [S1]) which suggested to us that the distribution of ASEP, starting from a shock measure, may be calculated explicitly for any time $t \geq 0$. We also note that (29) is established in [S1] using the assumption (1); we could not extend our results to the cases that violate this assumption.

The proof of Theorem 2 is very similar to that of Theorem 1. There, certain generalizations of the relation (29) are used in the place of (29). These generalizations have been derived by Schütz in [S1] (see eq. (3.10) there).

2 Proof of Theorem 1

2.1. Notations. Column vectors are denoted by $|\cdot\rangle$ and row vectors are denoted by $\langle\cdot|$. "T" means the transposition operation on both vectors and matrices. The tensor product is denoted by $\otimes$, and $A^{\otimes k}$ denotes the k-fold tensor product of A.

2.2. The vector representation of the space $\{0,1\}^{\mathcal{M}}$ and the measures on it. Recall $\mathcal{M} := \{-m+1,\dots,m-1,m\}$. A configuration $\eta \in \{0,1\}^{\mathcal{M}}$ will be represented by the vector $|\eta\rangle \in \mathbb{R}^{2^{2m}}$ which is defined in the following manner

$$|\eta\rangle = |v_{-m+1}\rangle \otimes |v_{-m+2}\rangle \otimes \cdots \otimes |v_m\rangle \tag{7}$$

where for each $i = -m+1,\dots,m$, $|v_i\rangle = (0,1)^T$, if the i-th site in the configuration η contains a particle, and $|v_i\rangle = (1,0)^T$, otherwise. Observe that for any configuration η, the corresponding vector $|\eta\rangle$ has 1 at one of its coordinates and 0 at all others. We use this fact to enumerate the elements of $\{0,1\}^{\mathcal{M}}$: for $i = 1,2,\dots,2^{2m}$, η_i will denote the configuration such that $|\eta_i\rangle$ has 1 at the i-the coordinate. A measure μ on $\{0,1\}^{\mathcal{M}}$ will be then represented by the column vector $|\mu\rangle := (\mu_1,\dots,\mu_{2^{2m}})^T \in \mathbb{R}^{2^{2m}}$, such that μ_i is the μ-measure of the configuration η_i. (Here, and only here, μ_i means the i-th component of the vector $|\mu\rangle$, and should not be confused with μ_k from Theorem 1.) We remark that in the vector formalism set up here it is often useful to embed $\mathbb{R}^{2^{2m}}$ in $\mathbb{C}^{2^{2m}}$ and hence consider vectors as elements of the complex vector space [S2]. In the present context, however, only the real vector space will be used.

2.3. Lemma 1. (The main relation that yields the proof of Thm. 1) *Let m be an arbitrarily fixed natural number and let $\mathcal{M} := \{-m+1,-m+2,\dots,m\}$. For a measure μ on $\{0,1\}^{\mathcal{M}}$, let $\bar{\mu}$ denote the measure that coincides with μ on $\{0,1\}^{\mathcal{M}\setminus\{-m+1\}}$ and that is concentrated on the configurations that have a particle at the site $-m+1$ (that is, $\bar{\mu}\big[\eta \in \{0,1\}^{\mathcal{M}} : \eta(-m+1) = 1\big] = 1$); also let $\hat{\mu}$ denote the measure that coincides with μ on $\{0,1\}^{\mathcal{M}\setminus\{m\}}$ and that is concentrated on the configurations that have a particle at the site m. Let c_ℓ and c_r be two arbitrarily fixed positive real numbers. For a measure μ on $\{0,1\}^{\mathcal{M}}$ let $\mu(t)$ denote the distribution at time t of ASEP on $\mathcal{M}$ with the particle hopping rates c_ℓ and c_r to the left and to the right, respectively, given that the initial distribution is μ. Let ρ_1,ρ_2 relate to c_ℓ, c_r by (1). For $k = -m,\dots,m$, let μ_k^m denote the product measure on $\{0,1\}^{\mathcal{M}}$ with the density ρ_1 on the sites $-m+1,\dots,k$ and the density ρ_2 on the sites $k+1,\dots,m$ (to avoid possible confusions, let us state explicitly that μ_{-m}^m (resp., μ_m^m) is the product measure on $\{0,1\}^{\mathcal{M}}$ with the density ρ_2 (resp., ρ_1)). Then*

$$\frac{d}{dt}|\mu_{-m}^m(t)\rangle = -\rho_2(c_r-c_\ell)\left(|\bar{\mu}_{-m}^m(t)\rangle - |\hat{\mu}_{-m}^m(t)\rangle\right) \tag{8a}$$

$$\begin{aligned}\frac{d}{dt}|\mu_k^m(t)\rangle &= \delta_1|\mu_{k-1}^m(t)\rangle + \delta_2|\mu_{k+1}^m(t)\rangle - (c_r+c_\ell)|\mu_k^m(t)\rangle \\ &\quad - (c_r-c_\ell)\left(\rho_1|\bar{\mu}_k^m(t)\rangle - \rho_2|\hat{\mu}_k^m(t)\rangle\right), \quad k = -m+1,\dots,m-1\end{aligned} \tag{8b}$$

$$\frac{d}{dt}|\mu_m^m(t)\rangle = -\rho_1(c_r-c_\ell)\left(|\bar{\mu}_m^m(t)\rangle - |\hat{\mu}_m^m(t)\rangle\right) \tag{8c}$$

where the constants δ_1 and δ_2 have been defined in (3).

2.4. How Theorem 1 follows from Lemma 1. For $m \in \mathbb{N}$, let us consider the following system of differential-difference equations involving vector-functions $\{|\nu_k^m(t)\rangle, t \geq 0\}_{k\in\mathcal{M}}$

$(|\nu_k^m(t)\rangle \in \mathbb{R}^{2^{2m}}, \forall\, k,m,t)$:

$$\frac{d}{dt}|\nu_{-m}^m(t)\rangle = \delta_2|\nu_{-m+1}^m(t)\rangle - \delta_1|\nu_{-m}^m(t)\rangle \tag{9a}$$

$$\frac{d}{dt}|\nu_k^m(t)\rangle = \delta_1|\nu_{k-1}^m(t)\rangle + \delta_2|\nu_{k+1}^m(t)\rangle - (\delta_1+\delta_2)|\nu_k^m(t)\rangle, \quad k=-m+1,\dots,m-1 \tag{9b}$$

$$\frac{d}{dt}|\nu_m^m(t)\rangle = \delta_1|\nu_{m-1}^m(t)\rangle - \delta_2|\nu_m^m(t)\rangle \tag{9c}$$

with the initial conditions

$$|\nu_k^m(0)\rangle = |\mu_k^m\rangle, \quad k=-m,\dots,m \tag{10}$$

Let us look for the solution of (9),(10) in the form

$$|\nu_k^m(t)\rangle = \sum_{i=-m}^{m} y_t^m(k\,|\,i)|\mu_i^m\rangle, \quad k=-m,\dots,m \tag{11}$$

Plugging (11) in (9), we obtain a system of $(2m+1)^2$ equations involving the scalar functions $\{y_t^m(k\,|\,i), t\ge 0\}_{i,k=-m\cdots m}$. These equations may be separated into $(2m+1)$ mutually independent systems (this means that any scalar function appears solely in one of these $2m+1$ systems). These systems are indexed by $i=-m,\dots,m$ and the i-th of them has the following form:

$$\frac{d}{dt}y_t^m(-m\,|\,i) = \delta_2 y_t^m(-m+1\,|\,i) - \delta_1 y_t^m(-m\,|\,i) \tag{12a}$$

$$\frac{d}{dt}y_t^m(k\,|\,i) = \delta_1 y_t^m(k-1\,|\,i) + \delta_2 y_t^m(k+1\,|\,i) - (\delta_1+\delta_2)y_t^m(k\,|\,i), \quad k=-m+1,\dots,m-1 \tag{12b}$$

$$\frac{d}{dt}y_t^m(m\,|\,i) = \delta_1 y_t^m(m-1\,|\,i) - \delta_2 y_t^m(m\,|\,i) \tag{12c}$$

The initial condition for the i-th system is easily obtained from (10) and (11). It has the following form:

$$y_0^m(i\,|\,i) = 1, \qquad y_0^m(k\,|\,i) = 0, \text{ when } k\ne i \tag{13}$$

$y_t^m(k\,|\,i)$ that satisfies (12) and (13) is known to have the following probabilistic interpretation ([F], Chapter XVIII, §5): it is the probability that a particle that performs a continuous time simple random walk on $-m,\dots,m$ with the hopping rates δ_1 and δ_2 to the right and to the left, respectively, will be at the site k at time t, starting from the site i. Let $p_t(i\,|\,k)$ be as defined in Theorem 1. Clearly, $y_t^m(k\,|\,i)\to p_t(i\,|\,k)$ as $m\to\infty$, while t,i and k are arbitrary but fixed. Thus, from (11), we conclude that

$$\mathrm{Var}\left(\nu_k^m(t)\,,\ \sum_{i=-m}^{m} p_t(i\,|\,k)\mu_i^m\right) \to 0 \text{ as } m\to\infty \text{ for any fixed } k \text{ and } t \tag{14}$$

where Var means the variational distance between measures that correspond to the vectors $|\nu_k^m(t)\rangle$ and $\sum_{i=-m}^m p_t(i\,|\,k)|\mu_i^m\rangle$.

The rest of our argument is not rigorous but may be made so by a standard technique. We believe that a reader acquainted with the classical methods used in the field of interacting particle systems has a clear idea of how it may be made. However, for the sake of completeness of our proofs, we shall present the formalization of this argument in a separate section (see Section 4).

Suppose that $\mu_k^m(t)$ has a limit measure on $\{0,1\}^{\mathbb{Z}}$ (as $m \to \infty$ while k and t are kept fixed) that we denote by $\mu_k^\infty(t)$. Intuitively, the family $\{\mu_k^\infty(\cdot)\}_{k\in\mathbb{Z}}$ must then satisfy the system of differential-difference equations obtained by taking the $m \to \infty$ limit of (8). Let us "derive" this limit system. Since $\mathbb{Z}$ "has no boundaries" then *(i)* both $\hat{\mu}_k^\infty(\cdot)$ and $\bar{\mu}_k^\infty(\cdot)$ must be substituted by $\mu_k^\infty(\cdot)$; and *(ii)* the equations (8a) and (8c) should not exist while the equations (8b) should hold for all $k \in \mathbb{Z}$. Moreover, due to *(i)* and because $(c_r + c_\ell) + (c_r - c_\ell)(\rho_1 - \rho_2) = \delta_1 + \delta_2$, the equation (8b) acquires the following form:

$$\frac{d}{dt}|\mu_k^\infty(t)\rangle = \delta_1|\mu_{k-1}^\infty(t)\rangle + \delta_2|\mu_{k+1}^\infty(t)\rangle - (\delta_1+\delta_2)|\mu_k^\infty(t)\rangle \tag{15}$$

Thus, the limit system consists of the equations (15) indexed by $k \in \mathbb{Z}$.

By analogy with $\mu_k^\infty(t)$, let us now define $\nu_k^\infty(t) := \lim_{m\to\infty}\nu_k^m(t)$. A similar reasoning shows that $\{\nu_k^\infty(\cdot)\}_{k\in\mathbb{Z}}$ satisfies (15) for any $k \in \mathbb{Z}$, with μ being substituted by ν throughout. Since also $\nu_k^\infty(0) = \mu_k^\infty(0), \forall k$ (as it follows directly from our constructions) then $\nu_k^\infty(t) = \mu_k^\infty(t), \forall k$ and $\forall t \geq 0$. Thus, $\mathrm{Var}\left(\mu_k^\infty(t)\,,\,\sum_{i=-\infty}^{\infty} p_t(i\,|\,k)\mu_i^\infty\right) = \mathrm{Var}\left(\nu_k^\infty(t)\,,\,\sum_{i=-\infty}^{\infty} p_t(i\,|\,k)\mu_i^\infty\right)$. But the latter is 0 due to (14). Thus, (2) follows.

2.5. Preparing for the proof of Lemma 1. More notations. Let $|\eta|$ denote the number of particles in $\eta \in \{0,1\}^{\mathcal{M}}$. For $n = 0, 1, \ldots, 2m$, we define

$$|\mathbf{n}\rangle := \sum_{\eta\,:\,|\eta|=n} |\eta\rangle \in \mathbb{R}^{2^{2m}} \tag{16}$$

$|\mathbf{n}\rangle$ is the (vector) sum of all (vector) configurations that have exactly n particles. One particular case of the above definition will be frequently used. It is the vector $|\mathbf{0}\rangle = (1,0,0,\ldots,0)^T$ that corresponds to the configuration η_1 that has no particles. We draw the reader attention to the fact that $|\mathbf{1}\rangle$ should be distinguished from $\mathbb{I}$; the latter is a 2 by 2 matrix defined in (22) below. We also define the vector $|\mathbf{s}\rangle$ by the first equality in (17) below; the alternative expressions of $|\mathbf{s}\rangle$ given in (17) will be used in our arguments.

$$|\mathbf{s}\rangle := (1,1,\ldots,1)^T = \begin{pmatrix}1\\1\end{pmatrix}^{\otimes 2m} = \sum_{\eta\in\{0,1\}^{\mathcal{M}}} |\eta\rangle = \sum_{n=0}^{2m} |\mathbf{n}\rangle \in \mathbb{R}^{2^{2m}} \tag{17}$$

By uppercase Latin letters we shall denote 2^{2m}-by-2^{2m} matrices with real entries. The matrices that appear in our arguments will be usually written as a tensor product of 2-by-2 matrices. A 2-by-2 matrix will be denoted by a lower-case Latin letter. We shall usually refer to matrices as operators.

2.6. Preparing for the proof of Lemma 1. An example. Let $\mathcal{M} = \{0,1\}$. The vector $(1,0)^T \otimes (1,0)^T = (1,0,0,0)^T$ represents the configuration in which both sites 0 and 1 are empty. This vector is denoted by $|\mathbf{0}\rangle$ and by $|\eta_1\rangle$, in our notations. The vector $(1,0)^T \otimes (0,1)^T = (0,1,0,0)^T$ represents the configuration η in which the site 1 contains a particle and the site 0 does not. Interchanging the values of η at 0 and 1 gives the configuration which is represented by the vector $(0,0,1,0)^T$. Thus, $|\mathbf{1}\rangle = (0,1,1,0)^T$, while $|\mathbf{2}\rangle = (0,0,0,1)^T$; the latter is also the vector corresponding to the configuration in which both sites of $\mathcal{M}$ are occupied by particles. The product measure with the density ρ_1 at the site 0 and the density ρ_2 at the site 1, is then represented by the vector

$$(1-\rho_1,\rho_1)^T \otimes (1-\rho_2,\rho_2)^T = ((1-\rho_1)(1-\rho_2),(1-\rho_1)\rho_2,\rho_1(1-\rho_2),\rho_1\rho_2)^T \tag{18}$$

For two arbitrary positive numbers z and q, let us introduce

$$Q_1 := \begin{pmatrix} 1 & 0 \\ 0 & q^{-2} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad Z := \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}^{\otimes 2}$$

It may be checked straightforwardly that $Q_1 Z|\mathbf{s}\rangle \equiv Q_1\, Z\,(1,1,1,1)^T = (1,zq^{-2})^T \otimes (1,z)^T$. Thus, there exists a constant $c = c(\rho_1,\rho_2,z,q)$ such that (18) $= c^{-1}Q_1\, Z\,|\mathbf{s}\rangle$. The last equality is an illustration of the general relation (21) to be defined and used below.

2.7. Preparing for the proof of Lemma 1. The vector representation of a shock measure. We define the constants

$$q := \sqrt{\frac{c_r}{c_\ell}}, \quad z := \rho_2/(1-\rho_2), \quad \alpha := \frac{1+z}{1+zq^{-2}}, \quad \beta := (1+z)^{-m}(1+zq^{-2})^{-m} \tag{19}$$

and the operators

$$Z := \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}^{\otimes 2m}, Q_k := \begin{pmatrix} 1 & 0 \\ 0 & q^{-2} \end{pmatrix}^{\otimes (k+m)} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{\otimes (m-k)}, \quad k = -m,\dots,m. \tag{20}$$

It is straightforward to verify that

$$|\mu_k^m\rangle = \beta\alpha^k Q_k Z|\mathbf{s}\rangle, \quad k=-m,\dots,m \tag{21}$$

where μ_k^m has been defined in the formulation of Lemma 1.

2.8. Preparing for the proof of Lemma 1. The matrix representation of the dynamics of ASEP. We introduce four matrices

$$s^+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad s^- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad p := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbb{1} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{22}$$

and then, for each $k = -m+1, -m+2, \dots, m$, we introduce

$$S_k^\pm := \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes s^\pm \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}, \text{ and } P_k := \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes p \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$$

where a non-identity matrix appears at the $(m+k)$-th position, counting from the left to the right. Notice that $(1,0)s^+ = (0,1)p = (0,1)$, $(0,1)s^- = (1,0)$, $(1,0)p = (1,0)s^- = (0,1)s^+ = (0,0)$; and analogously, $s^+(0,1)^T = (1,0)^T$, $s^-(1,0)^T = p(0,1)^T = (0,1)^T$, $p(1,0)^T = s^+(1,0)^T = s^-(0,1)^T = (0,0)$. These equations and the representation (7) yield the following properties of $S_k^\pm$ and P_k: If a configuration η does not have a particle at the site k then $|\eta'\rangle := S_k^-|\eta\rangle$ corresponds to the configuration that coincides with η on $\mathcal{M}\setminus\{k\}$ and has a particle at k; if to the contrary η has a particle at k then $S_k^-|\eta\rangle = 0$ (the latter means the vector with all components equal to 0; it is not $|\mathbf{0}\rangle$). If a configuration η has a particle at the site k then $|\eta^*\rangle := S_k^+|\eta\rangle$ corresponds to the configuration that coincides with η on $\mathcal{M}\setminus\{k\}$ and does not have a particle at k; if to the contrary η does not have a particle at k then $S_k^+|\eta\rangle = 0$. Using the equality $(S_k^+)^T = S_k^-$ and the above notations, we have that if η does not have a particle at k then $\langle\eta|S_k^+ = \langle\eta'|$ and $\langle\eta|S_k^- = 0$, while if η has a particle at k then $\langle\eta|S_k^+ = 0$ and $\langle\eta|S_k^- = \langle\eta^*|$. Accordingly, S_k^- and S_k^+ are called *the particle creating/annihilating operators*. The operator P_k is called *the number operator*; when applied to $|\eta\rangle$, it returns $|\eta\rangle$, if there was a particle at the site k, and results in 0 otherwise.

Let $I := \mathbf{I}^{\otimes 2m}$ denote the identity matrix. Let us now define (everywhere below, $c(\ell) \equiv c_\ell$ and $c(r) \equiv c_r$)

$$H_{c(\ell),c(r)} := -\left[\sum_{i=-m+1}^{m-1} c_r\left\{S_i^+S_{i+1}^- - P_i(I-P_{i+1})\right\} + c_\ell\left\{S_i^-S_{i+1}^+ - (I-P_i)P_{i+1}\right\}\right] \tag{23}$$

Let $\mu(0)$ be an arbitrary distribution on $\{0,1\}^{\mathcal{M}}$ and let $|\mu(0)\rangle$ be its vector representation as defined above. Denote then by $|\mu(t)\rangle$ the vector representation of the distribution of ASEP at time t, starting from $\mu(0)$. Using a standard argument (see for example Chpt. XVII of Feller [F]), one finds that for all $t \geq 0$,

$$\frac{d}{dt}|\mu(t)\rangle = -H_{c(\ell),c(r)}|\mu(t)\rangle \text{ and consequently } |\mu(t)\rangle = e^{-tH_{c(\ell),c(r)}}|\mu(0)\rangle \tag{24}$$

The equations (24) express the relation of $H_{c(\ell),c(r)}$ to the dynamics of ASEP. The operator $H_{c(\ell),c(r)}$ is called the Hamiltonian of ASEP on $\mathcal{M}$.

2.9. Lemma 2. (An auxiliary result to be used in the proof of Lemma 1.)

$$\left(H_{c(\ell),c(r)}\right)^T = H_{c(r),c(\ell)} + (c_r - c_\ell)\left(P_{-m+1} - P_m\right) \;\; \textit{for all } c_r, c_\ell > 0 \textit{ and all } m \in \mathbb{N}.$$

Proof. It follows from the definitions that $\forall k \in \mathcal{M}$, $\left(S_k^{\pm}\right)^T = S_k^{\mp}$, and $(P_k)^T = P_k$. Using these relations, an explicit expression for $\left(H_{c(\ell),c(r)}\right)^T$ is easily obtained from (23). The lemma follows then by straightforward calculations. $\square$

2.10. Proof of Lemma 1. Let us define the following operators

$$S_k^+(q) := \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}^{\otimes k+m-1} \otimes s^+ \otimes \begin{pmatrix} 1 & 0 \\ 0 & q^{-1} \end{pmatrix}^{\otimes m-k}, k \in \mathcal{M} \tag{25}$$

and introduce the following shorthand notations

$$D_k^+ := \sum_{j=-m+1}^{k} S_j^+(q), \quad k \in \mathcal{M} \tag{26}$$

Let us now state three properties (eqs. (28), (27) and (29) below) that will be used in our arguments. The first of them is the following commutation relation (27):

$$D_m^+ H_{c(r),c(\ell)} = H_{c(r),c(\ell)} D_m^+ \tag{27}$$

The proof of (27) may be found in [S1] (see eq. 2.12 there). It was noted there that this relation had been known before in a slightly different form [KR]. This relation (together with one for an analogous operator D_m^- not used here) and particle number conservation of the process reflects a non-abelian invariance property of $H_{c(r),c(\ell)}$ which is known as quantum algebra symmetry. In the limit $q \to 1$ of symmetric hopping, this becomes the well-known symmetry under the Lie algebra $SU(2)$ of the symmetric exclusion process. This symmetry is the algebraic origin [S2] for the probabilistic duality relations [L1] in symmetric exclusion.

To state the second property, we need more definitions: for $q > 0$ and $n \in \mathbb{Z}^+$, we define $[n]_q := (q^n - q^{-n})/(q - q^{-1})$ and $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$; we also set $[0]_q! := 1$. Using these definitions and the creating/annihilating properties of $s^\pm$ (see the text above the equation (23)) one can derive by straightforward calculations that

$$\langle \mathbf{n}| = \frac{1}{[n]_q!}\langle \mathbf{0}|(D_m^+)^n, \qquad \text{for all } n = 0, 1, \ldots, 2m \tag{28}$$

The third property has been established in [S1] (see eq. 3.8 there). It has been used there to derive certain duality relations for the ASEP, generalizing those for symmetric exclusion. It states that

$$\frac{1}{[n]_q!}\left[(D_m^+)^n, Q_k\right] = \frac{q^{-n+1}(q^{-2}-1)}{[n-1]_q!} Q_k D_k^+ (D_m^+)^{n-1}, \text{ for all } k \in \mathcal{M}, n = 1, \ldots, 2m \tag{29}$$

where $\left[A, B\right] := AB - BA$ is the commutator of the operators A and B.

Note 1. It is because of the form of the relation (29) that we shall first calculate $-\langle \mu_k^m | H^T_{c(r),c(\ell)}$ and then pass to $-H_{c(\ell),c(r)}|\mu_k^m\rangle$ with the aid of Lemma 2. An alternative way would be to derive the expression for $\left[Q_k, (D_m^-)^n\right]$ from (29) and then use it in the direct calculation of $-H_{c(\ell),c(r)}|\mu_k^m\rangle$.

Note 2. Since (26) and consequently (29) are not defined for $k = -m$, the argument employed in the derivation of (8a) will differ from that of (8b) and (8c).

We shall also need the following two relations that may be verified straightforwardly:

$$\langle \mathbf{0}|Q_k = \langle \mathbf{0}|, \text{ for all } k = -m, \ldots, m, \qquad \langle \mathbf{0}|H_{a,b} = 0, \text{ for all } a, b > 0 \tag{30}$$

Using (28), (29) and the first relation from (30), we conclude that for every $n = 1, \ldots, 2m$,

$$\begin{aligned} \langle \mathbf{n}|Q_k =& \frac{1}{[n]_q!}\langle \mathbf{0}| \left[(D_m^+)^n, Q_k\right] + \frac{1}{[n]_q!}\langle \mathbf{0}|Q_k(D_m^+)^n \\ =& \frac{q^{-n+1}(q^{-2}-1)}{[n-1]_q!}\langle \mathbf{0}|D_k^+(D_m^+)^{n-1} + \frac{1}{[n]_q!}\langle \mathbf{0}|(D_m^+)^n, \quad \forall k \in \mathcal{M} \end{aligned} \tag{31}$$

Applying $-H_{c(r),c(\ell)}$ to (31) and using then (27) and the second relation from (30), we get that for each $n = 1, \ldots, 2m$ and for each $k \in \mathcal{M}$,

$$-\langle \mathbf{n}|Q_k H_{c(r),c(\ell)} = -\frac{q^{-n+1}(q^{-2}-1)}{[n-1]_q!}\langle \mathbf{0}|D_k^+ H_{c(r),c(\ell)}(D_m^+)^{n-1} \tag{32}$$

Notice that we have applied $-H_{c(r),c(\ell)}$. The passage to $H_{c(\ell),c(r)}$ will be then done later with the help of Lemma 2. For an arbitrarily fixed $k \in \mathcal{M}$ let us consider separately

$$-\langle \mathbf{0}|D_k^+ H_{c(r),c(\ell)} = -\sum_{j=-m+1}^{k} \langle \mathbf{0}|S_j^+(q) H_{c(r),c(\ell)} \tag{33}$$

Here $\langle \mathbf{0}|S_j^+(q)$ is just the row vector that corresponds to the configuration that has a single particle which occupies the site j; let $\langle \mathbf{1}_j|$ denote the vector that corresponds to this configuration, $j \in \mathcal{M}$. The result of the action of $H_{c(r),c(\ell)}$ from the right on $\langle \mathbf{1}_j|$ is easily obtained from the properties of $S_i^\pm$'s and P_i's described in the text right before the equation (23). They are as follows:

$$-\langle \mathbf{1}_{-m+1}|H_{c(r),c(\ell)} = c_r\langle \mathbf{1}_{-m+2}| - c_\ell\langle \mathbf{1}_{-m+1}| \tag{34a}$$

$$-\langle \mathbf{1}_j|H_{c(r),c(\ell)} = c_\ell\langle \mathbf{1}_{j-1}| + c_r\langle \mathbf{1}_{j+1}| - (c_r + c_\ell)\langle \mathbf{1}_j|, \;\; j = -m+2, \ldots, m-1 \tag{34b}$$

$$-\langle \mathbf{1}_m|H_{c(r),c(\ell)} = c_\ell\langle \mathbf{1}_{m-1}| - c_r\langle \mathbf{1}_m| \qquad (34c) \quad (1)$$

Note 3. The relations (34) are the cornerstone in the derivation of the lemma assertion. The form of these relation suggests the following informal remark on the place and role of the quantum algebra technique in our arguments: it allows us to deduce the result of the action of a Hamiltonian on a measure from the result of its action on configurations that contain a single particle.

From (32), (33), (34), we conclude that when $k = -m+2, \ldots, m-1$, we have for each $n = 1, \ldots, 2m$,

$$-\langle \mathbf{n}|Q_k H_{c(r),c(\ell)} = \frac{q^{-n+1}(q^{-2}-1)}{[n-1]_q!}\langle \mathbf{0}| \left(c_\ell D_{k-1}^+ + c_r D_{k+1}^+ - (c_\ell + c_r) D_k^+\right) (D_m^+)^{n-1} \tag{35}$$

We now add $0 = \frac{1}{[n]_q!}\{c_\ell + c_r - (c_\ell + c_r)\}\langle \mathbf{0}|(D_m^+)^n$ to the right hand side of (35) and using then (31), derive that for $n = 1, \ldots, 2m$ and $k = -m+2, \ldots, m-1$,

$$-\langle \mathbf{n}|Q_k H_{c(r),c(\ell)} = c_\ell\langle \mathbf{n}|Q_{k-1} + c_r\langle \mathbf{n}|Q_{k+1} - (c_\ell + c_r)\langle \mathbf{n}|Q_k \tag{36b}$$

By a similar reasoning (and using that Q_{-m} is just the identity operator), we have that for each $n = 1, \ldots, 2m$,

$$-\langle \mathbf{n}|Q_{-m+1} H_{c(r),c(\ell)} = c_r\langle \mathbf{n}|Q_{-m+2} + c_\ell Q_{-m}\langle \mathbf{n}| - (c_\ell + c_r)\langle \mathbf{n}|Q_{-m+1} \tag{36a}$$

$$-\langle \mathbf{n}|Q_m H_{c(r),c(\ell)} = 0 \tag{36c}$$

Notice that we have established (36a-c) for $n = 1, 2, \ldots, 2m$. However, they are true also for $n = 0$ which follows directly from (30).

The equations (36) and the following relations (that may be obtained by straightforward calculations)

$$Q_k Z = Z Q_k, Q_k^T = Q_k, \; \forall\, k = -m, \ldots, m; \quad Z^T = Z; \quad H_{a,b} Z = Z H_{a,b} \; \forall\, a, b > 0 \tag{37}$$

will be used now to express $-\langle \mu_k^m | H_{c(r),c(\ell)}$. For $k=-m+1,\dots,m-1$, we have

$$
\begin{aligned}
-\langle \mu_k^m | H_{c(r),c(\ell)} =& -\beta\alpha^k \left\{ \sum_{n=0}^{2m} \langle \mathbf{n} | Q_k H_{c(r),c(\ell)} \right\} Z \\
=& c_\ell \alpha\beta\alpha^{k-1} \left\{ \sum_{n=0}^{2m} \langle \mathbf{n} | Q_{k-1} Z \right\} \\
&+ c_r \alpha^{-1} \beta\alpha^{k+1} \left\{ \sum_{n=0}^{2m} \langle \mathbf{n} | Q_{k+1} Z \right\} \\
&- (c_\ell + c_r)\beta\alpha^k \left\{ \sum_{n=0}^{2m} \langle \mathbf{n} | Q_k Z \right\} \\
=& \delta_1 \langle \mu_{k-1}^m | + \delta_2 \langle \mu_{k+1}^m | - (c_\ell + c_r) \langle \mu_k^m |
\end{aligned}
\tag{38}
$$

where we have used (17) and (21) in the first and in the last equation in (38), while the second equation in (38) is based on (36).

We are now in a position to prove (8b). Indeed, for $k \neq -m, m$, we have

$$
\begin{aligned}
d/dt |\mu_k^m(t)\rangle =& e^{-H_{c(\ell),c(r)}t} \left(-H_{c(\ell),c(r)} |\mu_k^m\rangle \right) \\
=& e^{-H_{c(\ell),c(r)}t} \left\{ -\left(\langle \mu_k^m | H_{c(r),c(\ell)} \right)^T - (c_r - c_\ell)\left(P_{-m+1} - P_m\right)|\mu_k^m\rangle \right\} \\
=& \text{ r.h.s. of } (8b)
\end{aligned}
\tag{zachem}
$$

where the second equality is based on Lemma 2 and the facts that $P_{-m+1}^T = P_{-m+1}$, $P_m^T = P_m$, while the last equality follows from (38) and from $P_{-m+1}|\mu_k^m\rangle = \rho_1 |\bar{\mu}_k^m\rangle$ and $P_m |\mu_k^m\rangle = \rho_2 |\hat{\mu}_k^m\rangle$; note that both in the first and in the last passages in (39), we employed the relations (24) and the notational convention $|\mu_k^m\rangle = |\mu_k^m(0)\rangle$.

The equation (8c) follows by a similar argument from (36c). To derive (8a), we first note that

$$
-\langle \mu_{-m}^m | H_{c(r),c(\ell)} = \beta\alpha^m \left(\sum_{n=0}^{2m} \langle \mathbf{n} | \right) H_{c(r),c(\ell)} = \beta\alpha^m \left(\sum_{n=0}^{2m} \langle \mathbf{0} | \frac{(D_m^+)^n}{[n]_q!} \right) H_{c(r),c(\ell)} = 0
$$

because of the commutation relation (27) and the second relation in (30). Thus, by Lemma 2,

$$
-H_{c(\ell),c(r)} |\mu_{-m}^m\rangle = -(c_r - c_\ell)\left(P_{-m+1} - P_m\right)|\mu_{-m}^m\rangle = -\rho_2 (c_r - c_\ell)\left(|\bar{\mu}_{-m}^m\rangle - |\hat{\mu}_{-m}^m\rangle \right)
$$

This leads to (8a) by the argument similar to (39). □

3 Formal Derivation of Theorem 1 from Lemma 1

The definitions of the terms that are used but not defined in the present section, may be found in [L1].

Let $\mathcal{C}$ denote the set of bounded cylinder functions $\{0,1\}^{\mathbb{Z}} \to \mathbb{R}$. Let L denote the operator defined on $\mathcal{C}$ as follows:

$$
(Lf)(\eta) = \sum_{x\in\mathbb{Z}} c_r \left[f(\eta^{x,x+1}) - f(\eta) \right] + \sum_{x\in\mathbb{Z}} c_\ell \left[f(\eta^{x-1,x}) - f(\eta) \right], \;\; \forall f \in \mathcal{C}, \; \forall \eta \in \{0,1\}^{\mathbb{Z}} \tag{40}
$$

where $\eta^{x,y}$ denotes the configuration that coincides with η on $\mathbb{Z} \setminus \{x,y\}$ and $\eta^{x,y}(x) = \eta(y), \eta^{x,y}(y) = \eta(x)$. For $m \in \mathbb{N}$, let $\mathcal{L}_m$ be the operator defined on $\mathcal{C}$ in the following way: $\mathcal{L}_m f$ is given by the right hand side of (40) with the first sum taken over $x = -m+1, \dots, m-1$ and the second sum taken over $x = -m+2, \dots, m$. The operator L is known to be the pre-generator of ASEP on $\mathbb{Z}$ defined in Section 1. The Markov process with the state space $\{0,1\}^{\mathbb{Z}}$ whose pre-generator is $\mathcal{L}_m$ will be denoted by ASEP$_m$. Let $S_m(t)\mu$ and $S(t)\mu$ denote the distribution of respectively, ASEP$_m$ and ASEP on $\mathbb{Z}$ at time t, when the initial distribution is μ. It may be verified straightforwardly from the definitions that $\mathcal{L}_m f \to Lf$ as $m \to \infty$ for any cylinder f. Thus, Corollary 3.14 from [L1] implies that $\lim_{m\to\infty} S_m(t)\mu = S(t)\mu$ for any fixed t and μ. Consequently, Theorem 1 is established as soon as we prove that $S_m(t)\mu_k \to \sum_{i=-\infty}^{+\infty} p_t(i\,|\,k)\mu_i$ for all $t \geq 0$ and $k \in \mathbb{Z}$, as $m \to \infty$. By the definition of the weak convergence, the latter is equivalent to

$$\int g\, d\,(S_m(t)\mu_k) \;\to\; \int g\, d\left(\sum\nolimits_{i=-\infty}^{+\infty} p_t(i\,|\,k)\mu_i\right) \text{ as } m \to \infty, \text{ for all } k \in \mathbb{Z},\ t \geq 0,\ g \in \mathcal{C} \quad (41)$$

For arbitrary $g \in \mathcal{C}$ and $m \in \mathbb{N}$, let g^m denote the function $\{0,1\}^{\mathcal{M}} \to \mathbb{R}$ (recall $\mathcal{M} := \{-m+1, \dots, m-1\}$) defined in the following manner: Denote by $[-N,N]$ the support of g, that is, N is the minimal positive integer such that $g(\eta) = g(\zeta)$ provided η and ζ coincide on $[-N,N]$. Then, if $N \leq m$ we define g^m as the restriction of g to $\{0,1\}^{\mathcal{M}}$, while if $N > m$ we set $g^m \equiv 0$. Define then $\langle g^m|$ as the vector which i-th coordinate is $g^m(\eta_i)$, $i = 1, 2, \dots, 2^{2m}$.

Let now $|\mu_k^m(t)\rangle$ be as defined in Lemma 1. Notice that the particles in ASEP$_m$ evolve on $\mathcal{M}$ by the exclusion rules while the particles outside of $\mathcal{M}$ are intact all the time. Thus,

$$\int g\, d\,(S_m(t)\mu_k) \;=\; \langle g^m, \mu_k^m(t)\rangle \text{ when } m > N \quad (42)$$

Recall from Section 2.4 the constructions of the measures $\{|\nu_i^m(\cdot)\rangle, i = -m, \dots, m\}$. The relation (14) derived there yields

$$\langle g^m, \nu_k^m(t)\rangle \to \int g\, d\left(\sum\nolimits_{i=-\infty}^{+\infty} p_t(i\,|\,k)\mu_i\right) \text{ as } m \to \infty \text{ for all } k \in \mathbb{Z},\ t \geq 0,\ g \in \mathcal{C} \quad (43)$$

Combining (43), (41) and (42), we conclude that Theorem 1 follows from the relation

$$\lim_{m\to\infty} \langle g^m, \mu_k^m(t)\rangle = \lim_{n\to\infty} \langle g^m, \nu_k^m(t)\rangle, \text{ for all } k \in \mathbb{Z}, t \geq 0,\, g \in \mathcal{C} \quad (44)$$

A heuristic justification of (44) has been presented in Section 2.4 after (14). Here we shall prove it rigorously. To this end we shall need the following

Lemma 3. (An auxiliary result for the proof of (44).) *Let $t \geq 0$ and $g \in \mathcal{C}$ be arbitrarily fixed and let N be determined by g as described in the text above. For $m \in \mathbb{N}$, let $\mu_i^m(\cdot)$, $\bar{\mu}_i^m(\cdot)$, $\hat{\mu}_i^m(\cdot)$, $i = -m, \dots, m$, be as defined in Lemma 1. Define then*

$$\begin{gathered}\bar{\varepsilon}_{g,i}^m(s) := \langle g^m, \bar{\mu}_i^m(s)\rangle - \langle g^m, \mu_i^m(s)\rangle \text{ and } \hat{\varepsilon}_{g,i}^m(s) := \langle g^m, \hat{\mu}_i^m(s)\rangle - \langle g^m, \mu_i^m(s)\rangle, \\ s \geq 0,\ i = -m, \dots, m\end{gathered} \quad (45)$$

Then, there is $m_0 = m_0(g,t)$ such that for appropriate positive constants $C = C(g,t)$ and $c = c(g,t)$ it holds that

$$\left|\bar{\varepsilon}^m_{g,i}(s)\right| \le C\, e^{-cs}\, \frac{(cs)^{m-N}}{(m-N)!}, \text{ and } \left|\hat{\varepsilon}^m_{g,i}(s)\right| \le C\, e^{-cs}\, \frac{(cs)^{m-N}}{(m-N)!}, \ \forall m \ge m_0, \ s \in [0,t] \tag{46}$$

Proof. We shall prove the first inequality in (46); the other one may be proved analogously.

Let $\{\mathcal{N}^r_i(s), s \in [0,t]\}_{i\in\mathbb{Z}}$ and $\{\mathcal{N}^\ell_i(s), s \in [0,t]\}_{i\in\mathbb{Z}}$ be two families of independent Poisson point processes (PPP), such that for each $i \in \mathbb{Z}$, the intensities of $\mathcal{N}^r_i(\cdot)$ and $\mathcal{N}^\ell_i(\cdot)$ are respectively, c_r and c_ℓ (by "independent" here we mean that all the processes are independent between themselves and of everything else, in particular, of μ^m_k). Let Ω denote the state space of the realizations of these processes and let $\mathbf{P}_\Omega$ be the probability measure on Ω.

Let us define the evolution of particles on $\mathcal{M}$ by the following rule: The configuration of particles changes at time $\tau \in [0,t]$ if and only if either the conditions (a)-(b) or the conditions (a′)-(b′) are satisfied.

(a) For some $j \in \mathcal{M} \setminus \{m\}$ the process $\mathcal{N}^r_j(\cdot)$ changes its value at time τ.

(b) There is a particle at j and there is no particle at $j+1$ in the configuration at time τ^-.

(a′) For some $j \in \mathcal{M} \setminus \{-m+1\}$ the process $\mathcal{N}^\ell_j(\cdot)$ changes its value at time τ.

(b′) There is a particle at j and there is no particle at $j-1$ in the configuration at time τ^-.

The particle from j will be moved at time τ to $j+1$, if (a)-(b) holds, and to $j-1$, if (a′)-(b′) holds. Let $\zeta^\eta_s(\omega)$ denote the configuration of particles that is obtained at time s on the realization $\omega \in \Omega$ from the initial configuration $\eta \in \{0,1\}^{\mathcal{M}}$ using the rules introduced above. The measure $\mathbf{P}_\Omega$ induces the distribution of ζ^η_t on $\{0,1\}^{\mathbb{Z}}$ which is known to coincide with the distribution of ASEP on $\mathcal{M}$ at time t, starting from η. For the justification, see the parts of [L1] related to the "graphical representation of Interacting Particle Systems".

For an arbitrary $\eta \in \{0,1\}^{\mathbb{Z}}$ let $\bar{\eta}$ denote the configuration that has a particle at m and coincides with η on $\mathcal{M} \setminus \{m\}$. Take any η that does not have a particle at m, i.e. $\eta(m) = 0$. It may be verified straightforwardly that for each $s \in [0,t]$ and each ω, the configurations $\zeta^\eta_s(\omega)$ and $\zeta^{\bar{\eta}}_s(\omega)$ coincide at all but one site. We denote this site by $X_s(\eta,\omega)$. Let us postulate that $X_s(\eta,\omega) \equiv m$, if η has a particle at m.

We now construct another process that will be used to estimate $X_t(\eta,\omega)$. Consider a single particle that moves on $(-\infty, m]$ by the following rules: At time 0 it is at m. It jumps from j to $j-1$ at time τ if and only if its position at τ^- is j and either $\mathcal{N}^r_j(\cdot)$ or $\mathcal{N}^\ell_j(\cdot)$ changes its value at time τ. Let $Y_s(\omega)$ denote the position of this particle at time $s \in [0,t]$ on the realization $\omega \in \Omega$. From our constructions,

$$X_s(\eta,\omega) \ge Y_s(\omega), \qquad \text{for all } \eta \in \{0,1\}^{\mathbb{Z}} \text{ for all } s \in [0,t] \text{ and all } \omega \in \Omega \tag{47}$$

Notice that the hopping times of the particle that determines the process Y, form a Poisson Point Process with the intensity $c_r + c_\ell$. Recall that this particles starts from m. Assuming that

$m > N$ we conclude that for this particle to get to the left of the site N at time s, it is necessary that this PPP changes its value at least $m - N$ times. Thus,

$$\mathbf{P}\left[Y_s(\omega) \leq N\right] = e^{-(c_r+c_\ell)s} \sum_{j=m-N}^{\infty} \frac{((c_r+c_\ell)s)^j}{j!} \tag{48}$$

Using our constructions, we now can write that for any $s \in [0,t]$ and any $k = -m, \dots, m$ (below $|||g^m|||$ means the sup norm of g^m),

$$\begin{aligned}
&\left|\int g^m d\bar{\mu}_k^m(s) - \int g^m d\mu_k^m(s)\right| = \left|\mathbf{E}_\Omega \int g^m \left(\zeta_s^{\bar{\eta}}(\omega)\right) - g^m \left(\zeta_s^{\eta}(\omega)\right) \mu_k^m(d\eta)\right| \\
&\leq 2|||g^m||| \int \mathbf{P}_\Omega \left[X_s(\eta,\omega) \in [-N,N]\right] \\
&\leq 2|||g^m|||\, \mathbf{P}_\Omega \left[Y_s \leq N\right] \leq 2|||g^m|||\, e^{-(c_r+c_\ell)s} \frac{((c_r+c_\ell)s)^{m-N}}{(m-N)!} \left[\frac{1}{1-t/(m-N)}\right]
\end{aligned}$$

when m is such that $\frac{t}{m-N} < 1$. The last inequality leads to (46). □

We shall now prove (44) which, according to the argument presented above, completes the proof of Theorem 1.

For $m \in \mathbb{N}$, let us define $\overrightarrow{\langle g^m, \mu^m(s)\rangle}$ and $\overrightarrow{\langle g^m, \nu^m(s)\rangle}$ as $(2m+1)$-dimensional vectors whose i-th coordinate is $\langle g^m, \mu_i^m(s)\rangle$ and $\langle g^m, \nu_i^m(s)\rangle$ respectively ($i = -m, \dots, m$). Then, the equations (9), (10) yield

$$\frac{d}{ds}\overrightarrow{\langle g^m, \nu^m(s)\rangle} = A\,\overrightarrow{\langle g^m, \nu^m(s)\rangle}, s \geq 0; \quad \langle g^m, \nu_i^m(0)\rangle = \langle g^m, \mu_i^m\rangle, i = -m, \dots, m \tag{49}$$

where A is a three-diagonal $(2m+1) \times (2m+1)$ matrix which has $-(\delta_1 + \delta_2)$ on the main diagonal, $\delta_1$above this diagonal and δ_2 below it. Similarly, the equations (8) from Lemma 1, the definitions of $\bar{\mu}$ and $\hat{\mu}$, and the notations introduced in (45) yield

$$\frac{d}{ds}\overrightarrow{\langle g^m, \mu^m(s)\rangle} = B\,\overrightarrow{\langle g^m, \mu^m(s)\rangle} + \vec{\epsilon}(s), s \geq 0; \quad \langle g^m, \mu_i^m(0)\rangle = \langle g^m, \mu_i^m\rangle, i = -m, \dots, m \tag{50}$$

where B is a three-diagonal $(2m+1) \times (2m+1)$ matrix which coincides with A everywhere except for the first and the last lines where its entries are 0, and $\vec{\epsilon}(s)$ is a $(2m+1)$-dimensional vector $(\epsilon_{-m}(s), \dots, \epsilon_m(s))$ with the following structure

$$\begin{aligned}
\epsilon_{-m}(s) &= -\rho_2(c_r - c_\ell)(\bar{\varepsilon}_{g,-m}^m(s) - \hat{\varepsilon}_{g,-m}^m(s)) \\
\epsilon_i(s) &= -\rho_1(c_r - c_\ell)\bar{\varepsilon}_{g,i}^m(s) + \rho_2(c_r - c_\ell)\hat{\varepsilon}_{g,i}^m(s), \quad i = -m+1, \dots, m-1 \\
\epsilon_m(s) &= -\rho_1(c_r - c_\ell)(\bar{\varepsilon}_{g,m}^m(s) - \hat{\varepsilon}_{g,m}^m(s))
\end{aligned}$$

The solutions of (49) and (50) have the forms $e^{As}\overrightarrow{\langle g^m, \nu^m(0)\rangle}$ and $e^{Bs}\overrightarrow{\langle g^m, \mu^m(0)\rangle} + \int_0^s e^{B(s-u)}\vec{\epsilon}(u)du$ respectively. The structure of these solutions, the coincidence of the initial conditions of both systems, the relation between A and B as indicated above, and the estimate on $\vec{\epsilon}(\cdot)$ provided by Lemma 3, altogether yield (44). Thus, the proof of Theorem 1 is completed.

4 Proof of Theorem 2

The proof will be presented for the double shock case ($n = 2$). Thus, throughout the proof $q, c_\ell, c_r, \rho_1, \rho_2, \rho_3$ are supposed to be fixed and satisfy the assumptions of the theorem.

First of all, for each $k \in \mathcal{M} \setminus \{-m+1\}$, we define

$$\tilde{Q}_k := \frac{Q_k - Q_{k-1}}{q^{-2} - 1} = \begin{pmatrix} 1 & 0 \\ 0 & q^{-2} \end{pmatrix}^{\otimes(k+m-1)} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{\otimes(m-k)} \tag{51}$$

and we also define $\tilde{Q}_{-m+1} := p \otimes \mathbf{I}^{\otimes(2m-1)}$ (using the notations (22)). Let $|\mathbf{2}_{k_1k_2}\rangle$ denote the vector corresponding to the configuration that has exactly two particles situated at the sites k_1 and k_2. We shall use the following relation:

$$\langle \mathbf{n}|\tilde{Q}_{k_1}\tilde{Q}_{k_2} = \frac{q^{-n+2}}{[n-2]_q!}\langle \mathbf{2}_{k_1k_2}| \left(D_m^+\right)^{n-2}, \quad -m+1 \le k_1 < k_2 \le m,\ n = 2, \dots, 2m \tag{52}$$

It is a particular case of a general relation

$$\langle \mathbf{n}|\tilde{Q}_{k_1}\dots\tilde{Q}_{k_j} = \frac{q^{-j(n-1)}}{[n-j]_q!}\langle \mathbf{j}_{k_1\dots k_j}| \left(D_m^+\right)^{n-j}, \ -m+1 \le k_1 < \dots < k_j \le m, n = j, \dots, 2m$$

where $\langle \mathbf{j}_{k_1\dots k_j}|$ denotes the vector that corresponds to the configuration from $\{0,1\}^{\mathcal{M}}$ that has exactly j particles located at sites $k_1, \dots, k_j$. The latter relation has been established in [S1] (see eq. (3.10) there). It follows directly from (29) and (28).

Let now $\mu^m_{k_1k_2}$ denote the product measure on $\{0,1\}^{\mathcal{M}}$ with the density ρ_1 at the sites $-m+1, \dots, k_1-1$, the density ρ_2 at $k_1+1, \dots, k_2-1$, the density ρ_3 at $k_2+1, \dots, m$, and the density 1 at k_1 and k_2. It may be shown straightforwardly that

$$|\mu^m_{k_1k_2}\rangle = (zq^{-4}+1)^{-m+1-k_1}(zq^{-2})^{-1}(1+zq^{-2})^{-k_2+k_1+1}z^{-1}(1+z)^{m+k_2}\tilde{Q}_{k_1}\tilde{Q}_{k_2}Z|\mathbf{s}\rangle \tag{53}$$

where $z = \rho_3/(1-\rho_3)$ and Z is defined as in (20). Put $\alpha := (1+zq^{-2})/(1+zq^{-4})$, $\gamma := (1+z)/(1+zq^{-2})$ and let β^* be such that $\alpha^{k_1}\gamma^{k_2}\beta^*$ is the coefficient in the right hand side of the above equality; notice that β^* does not depend on either k_1 or k_2. Notice also that $|\mathbf{s}\rangle$ may be substituted by $\sum_{n=2}^m |\mathbf{n}\rangle$ because $\tilde{Q}_{k_1}\tilde{Q}_{k_2}$ has two projectors and as a consequence of this, the result of the application of $\tilde{Q}_{k_1}\tilde{Q}_{k_2}$ on $|\mathbf{0}\rangle$ or $|\mathbf{1}\rangle$ or $|\mathbf{2}\rangle$ is 0. These facts will be used in the calculations below.

For the case when $k_1 \neq -m+1$, $k_2 \neq m$, and k_1 and k_2 are not two neighboring sites, we have:

$$
\begin{aligned}
&-\langle \mu^m_{k_1k_2}|H_{c(r),c(\ell)} = -\beta^*\alpha^{k_1}\gamma^{k_2}\left(\sum_{n=2}^{2m}\langle \mathbf{n}|\right)\tilde{Q}^T_{k_1}\tilde{Q}^T_{k_2}Z^T H_{c(r),c(\ell)} \\
&= -\beta^*\alpha^{k_1}\gamma^{k_2}\left(\sum_{n=2}^{2m}\langle \mathbf{n}|\tilde{Q}_{k_1}\tilde{Q}_{k_2}H_{c(r),c(\ell)}\right)Z \\
&= -\beta^*\alpha^{k_1}\gamma^{k_2}\left(\sum_{n=2}^{2m}\frac{q^{-n+2}}{[n-2]_q!}\langle \mathbf{2}_{k_1k_2}|\left(D^+_m\right)^{n-2}H_{c(r),c(\ell)}\right)Z \\
&= -\beta^*\alpha^{k_1}\gamma^{k_2}\sum_{n=2}^{2m}\left\{\langle \mathbf{2}_{k_1k_2}|H_{c(r),c(\ell)}\right\}\frac{q^{-n+2}}{[n-2]_q!}\left(D^+_m\right)^{n-2}Z \qquad (54)\\
&= \beta^*\alpha^{k_1}\gamma^{k_2}\sum_{n=2}^{2m}\{c_\ell\langle \mathbf{2}_{k_1-1,k_2}| + c_\ell\langle \mathbf{2}_{k_1,k_2-1}| + c_r\langle \mathbf{2}_{k_1+1,k_2}| + c_r\langle \mathbf{2}_{k_1,k_2+1}| \\
&\qquad\qquad -(2c_\ell+2c_r)\langle \mathbf{2}_{k_1,k_2}|\}\times\frac{q^{-n+2}}{[n-2]_q!}(D^+_m)^{n-2}Z \\
&= \beta^*\alpha^{k_1}\gamma^{k_2}\sum_{n=2}^{2m}\langle \mathbf{n}|\Big(c_\ell\tilde{Q}_{k_1-1}\tilde{Q}_{k_2}Z + c_\ell\tilde{Q}_{k_1}\tilde{Q}_{k_2-1}Z + c_r\tilde{Q}_{k_1+1}\tilde{Q}_{k_2}Z + c_r\tilde{Q}_{k_1}\tilde{Q}_{k_2+1}Z \\
&\qquad\qquad -(2c_\ell+2c_r)\tilde{Q}_{k_1}\tilde{Q}_{k_2}Z\Big) \\
&= c_\ell\alpha\langle \mu^m_{k_1-1,k_2}| + c_\ell\gamma\langle \mu^m_{k_1,k_2-1}| + c_r\alpha^{-1}\langle \mu^m_{k_1+1,k_2}| + c_r\gamma^{-1}\langle \mu^m_{k_1,k_2+1}| - (2c_\ell+2c_r)\langle \mu^m_{k_1k_2}| \qquad (2)
\end{aligned}
$$

In the first and the last equalities in (54), we use (53); the second equality is based on (37); the third and the sixth ones use (52); the fourth equality employs (27). In the fifth equality in (54), $\langle \mathbf{2}_{k_1k_2}|H_{c(r),c(\ell)}$ is substituted by its expression obtained using the expression of $H_{c(r),c(\ell)}$ via $S^\pm_i, P_i, i \in \mathcal{M}$, and the results of action of these operators on $\langle \mathbf{2}_{k_1k_2}|$; these results follow from the properties of these operators as explained in the text before equation (23). Using (54) and Lemma 2, we get

$$
\begin{aligned}
d/dt|\mu^m_{k_1k_2}(t)\rangle =& e^{-H_{c(\ell),c(r)}t}\left(-H_{c(\ell),c(r)}|\mu^m_{k_1k_2}\rangle\right) \\
=& e^{-H_{c(\ell),c(r)}t}\Big\{-\left(\langle \mu^m_{k_1k_2}|H_{c(r),c(\ell)}\right)^T - (c_r-c_\ell)\left(P_{-m+1}-P_m\right)|\mu^m_{k_1k_2}\rangle\Big\} \\
=& c_\ell\alpha|\mu^m_{k_1-1,k_2}\rangle + c_\ell\gamma|\mu^m_{k_1,k_2-1}\rangle + c_r\alpha^{-1}|\mu^m_{k_1+1,k_2}\rangle + c_r\gamma^{-1}|\mu^m_{k_1,k_2+1}\rangle \\
&-(2c_\ell+2c_r)|\mu^m_{k_1k_2}\rangle - (c_r-c_\ell)\left(\rho_1|\bar{\mu}^m_{k_1k_2}\rangle - \rho_3|\hat{\mu}^m_{k_1k_2}\rangle\right)
\end{aligned} \qquad (55)
$$

where the operations $\bar{\cdot}$ and $\hat{\cdot}$ have been defined in Lemma 1.

In order to treat the case when still $k_1 \neq -m+1$ and $k_2 \neq m$, but now k_1 and k_2 are two neighboring sites of $\mathcal{M}$, we modify appropriately the argument that has been used to get (55). The only modification necessary is in the fifth equality in (54). There, $-\langle \mathbf{2}_{k_1k_2}|H_{c(r),c(\ell)}$ is given by the expression $c_\ell\langle \mathbf{2}_{k_1-1,k_2}| + c_r\langle \mathbf{2}_{k_1,k_2+1}| - (c_r+c_\ell)\langle \mathbf{2}_{k_1k_2}|$. This expression differs from its counterpart because in the nearest-neighbor case the particle from k_1 cannot hop on the particle at k_2 and vice versa. The final result is

$$
\begin{aligned}
d/dt|\mu^m_{k_1k_2}(t)\rangle =& c_\ell\alpha|\mu^m_{k_1-1,k_2}\rangle + c_r\gamma^{-1}|\mu^m_{k_1,k_2+1}\rangle - (c_\ell+c_r)|\mu^m_{k_1k_2}\rangle \\
&-(c_r-c_\ell)\left(\rho_1|\bar{\mu}^m_{k_1k_2}\rangle - \rho_3|\hat{\mu}^m_{k_1k_2}\rangle\right)\}
\end{aligned} \qquad (56)
$$

The expressions for $d/dt|\mu^m_{k_1k_2}(t)\rangle$ provided by (55) and (56) lead to (5) in the same manner as (8) has led to (2). The main steps of this argument are presented below.

We consider the following system of differential-difference equations involving vector-functions $\{|\nu^m_{k_1k_2}(\cdot)\rangle,\ -m+1 \le k_1 < k_2 \le m\}$ (below $k_1 \sim k_2$ means that k_1 and k_2 are two neighboring sites):

$$\frac{d}{dt}|\nu^m_{k_1k_2}(t)\rangle = \delta_{1r}|\nu^m_{k_1-1,k_2}\rangle + \delta_{2r}|\nu^m_{k_1,k_2-1}\rangle + \delta_{1\ell}|\nu^m_{k_1+1,k_2}\rangle + \delta_{2\ell}|\mu^m_{k_1,k_2+1}\rangle$$
$$- (\delta_{1\ell} + \delta_{1r} + \delta_{2\ell} + \delta_{2r})|\nu^m_{k_1k_2}\rangle,\ \text{if } k_1 \not\sim k_2, k_1 \ne -m+1, k_2 \ne m \tag{57a}$$

$$\frac{d}{dt}|\nu^{k_1k_2}(t)\rangle = \delta_{1r}|\nu^m_{k_1-1,k_2}\rangle + \delta_{2\ell}|\nu^m_{k_1,k_2+1}\rangle - (\delta_{1\ell} + \delta_{2r})|\nu^m_{k_1k_2}\rangle,$$
$$\text{if } k_1 \sim k_2, k_1 \ne -m+1, k_2 \ne m \tag{57b}$$

$$\frac{d}{dt}|\nu^m_{-m+1,k_2}(t)\rangle = \delta_{1\ell}|\nu^m_{-m+2,k_2}(t)\rangle + \delta_{2\ell}|\nu^m_{-m+1,k_2+1}(t)\rangle + \delta_{2r}|\nu^m_{-m+1,k_2-1}(t)\rangle$$
$$- (\delta_{2\ell} + \delta_{2r} + \delta_{1r})|\nu^m_{-m+1,k_2}(t)\rangle,\ \text{if } -m+1 \not\sim k_2, k_2 \ne m \tag{57c}$$

$$\frac{d}{dt}|\nu^m_{k_1,m}(t)\rangle = \delta_{1\ell}|\nu^m_{k_1+1,m}(t)\rangle + \delta_{1r}|\nu^m_{k_1-1,m}(t)\rangle + \delta_{2r}|\nu^m_{k_1,m-1}(t)\rangle$$
$$- (\delta_{1\ell} + \delta_{1r} + \delta_{2\ell})|\nu^m_{k_1m}(t)\rangle, \qquad \text{if } k_1 \not\sim m, k_1 \ne -m+1 \tag{57d}$$

$$\frac{d}{dt}|\nu^m_{-m+1,m}(t)\rangle = \delta_{1\ell}|\nu^m_{-m+2,m}(t)\rangle + \delta_{2r}|\nu^m_{-m+1,m-1}(t)\rangle - (\delta_{1r} + \delta_{2\ell})|\nu^m_{-m+1,m}(t)\rangle \tag{57e}$$

$$\frac{d}{dt}|\nu^m_{-m+1,-m+2}(t)\rangle = \delta_{2\ell}|\nu^m_{-m+1,-m+3}(t)\rangle - \delta_{2r}|\nu^m_{-m+1,-m+2}(t)\rangle \tag{57f}$$

$$\frac{d}{dt}|\nu^m_{m-1,m}(t)\rangle = \delta_{1r}|\nu^m_{m-2,m}(t)\rangle - \delta_{1\ell}|\nu^m_{m-2,m}(t)\rangle \tag{57g}$$

with the initial conditions

$$|\nu^m_{k_1k_2}(0)\rangle = |\mu^m_{k_1k_2}\rangle, \quad -m+1 \le k_1 < k_2 \le m \tag{58}$$

Let us look for the solution of (57),(58) in the form

$$|\nu^m_{k_1k_2}(t)\rangle = \sum_{-m+1\le i<j\le m} y^m_t(k_1,k_2\,|\,i,j)|\mu^m_{ij}\rangle, -m+1 \le k_1 < k_2 \le m \tag{59}$$

The system of differential difference equations involving the functions y which is obtained when (59) is plugged in (57), may be divided into $2m(2m-1)/2$ "independent" systems that we index by (i,j), $-m+1 \le i < j \le m$. For each (i,j), the system with this index contains solely the functions from the family $\{y^m_\cdot(k_1,k_2\,|\,i,j), -m+1 \le i < j \le m\}$, and moreover, none of the functions from this family appears in any other system. This separation of functions y among systems suggested us to call the systems "independent". The probabilistic interpretation of the solution of the (i,j)-th system with the appropriate initial condition (obtained by plugging (59) into (58)) is known: $y^m_t(k_1,k_2\,|\,i,j)$ is the probability that two particles, starting from i and j, will be at the sites k_1 and k_2, respectively, at time t, where the particles interact by the exclusion rule, and the particle that started at i (resp., j) performs a simple continuous time random walk on $\mathcal{M}$ with reflecting boundaries, and the hopping rates $\delta_{1\ell}$ (resp., $\delta_{2\ell}$) to the left and δ_{1r} (resp., δ_{2r}) to the right. Thus, we have

$$\mathrm{Var}\left(\nu^m_{k_1k_2}(t)\,,\ \sum_{-m+1\le i<j\le m} p_t(i,j\,|\,k_1,k_2)\mu^m_{ij}\right) \to 0 \text{ as } m\to\infty \text{ for any fixed } k_1,k_2 \text{ and } t \tag{60}$$

which is the analogue of (14). The relation (60) leads to (5) in the same way as (14) led to (2). The rigorous argument that concludes (5) from (60) will not be presented here because it is similar to the argument of Section 3 that derived (2) from (14). What we shall present however, is the heuristics behind this argument. It is very similar to that of Section 2.4. The program is to show that the families $\{\mu^{\infty}_{k_1k_2}(t), -\infty < k_1 < k_2 < \infty\}$ and $\{\nu^{\infty}_{k_1k_2}(t), -\infty < k_1 < k_2 < \infty\}$ satisfy the same system of differential-difference equations with the same initial conditions, where $\mu^{\infty}_{k_1k_2}(t) := \lim_{m\to\infty} \mu^{m}_{k_1k_2}(t)$ and $\nu^{\infty}_{k_1k_2}(t) := \lim_{m\to\infty} \nu^{m}_{k_1k_2}(t)$, because this would allow us to substitute $\nu^{\infty}_{k_1k_2}(t)$ by $\mu^{\infty}_{k_1k_2}(t)$ in (60). As for the coincidence of the initial conditions, it is provided by our choice (58). Let us check the coincidence of the limit of the system (55), (56) with the limit of the system (57). Since the lattice $\mathbb{Z}$, which is the $m \to \infty$ limit of the lattice $\mathcal{M}$, does not have boundary sites, then the equations (57c-e) disappear. Thus, ν^{∞} satisfy (57a-b). Because of the absence of the boundary sites in $\mathbb{Z}$, the equations for $d/dt|\mu^{m}_{-m+1,k_2}(t)\rangle$, for $d/dt|\mu^{m}_{k_1,m}(t)\rangle$, and for $d/dt|\mu^{m}_{-m+1,m}(t)\rangle$ do not have any influence on the form of the system satisfied by μ^{∞} (for this reason these equations have not even been derived). For the same reason, $\lim_{m\to\infty} \bar{\mu}^{m}_{k_1k_2}(t) = \mu^{\infty}_{k_1k_2}(t)$ and $\lim_{m\to\infty} \hat{\mu}^{m}_{k_1k_2}(t) = \mu^{\infty}_{k_1k_2}(t)$. Using these relations and a simple algebra, one easily derives that as $m \to \infty$, the equations (55) and (56) acquire the form of respectively, (57a) and (57b) with everywhere ν being substituted by μ. This completes our program. Once more, we draw a reader's attention to the fact that the argument just presented may be formalized using the ideas and the technique from Section 3.

References

[BCFG] A. Boldrighini, C. Cosimi, A. Frigio, M. Grasso-Nunes, *Computer simulations of shock waves in the completely asymmetric simple exclusion process*, J. Stat. Phys. **55**, (1989), 611-623.

[DJLS] B. Derrida, S. A. Janowsky, J. L. Lebowitz, E. R. Speer, *Exact solution of the totally asymmetric simple exclusion process: Shock profiles*, J. Stat. Phys. **73** (1993), 813-842.

[DLS] B. Derrida, J. L. Lebowitz, E. R. Speer, *Shock profiles in the asymmetric simple exclusion process in one dimension*, J. Stat. Phys. **89** (1997), 135-167.

[F] W. Feller, *Introduction to the probability theory and its applications.* Vol. I, Wiley P, 1968 (3-d edition).

[F1] P. A. Ferrari, *Shocks in one-dimensional processes with drift.* In *Probability and Phase Transition.* (Ed. G. Grimmett), Cambridge, 1993.

[FF] P. A. Ferrari, L. R. G. Fontes, *Shock fluctuations in the asymmetric simple exclusion process*, Probab. Theor. Rel. Fields **99**, (1994), 305-319.

[FFV] P. A. Ferrari, L. R. G. Fontes and M. E. Vares, *The asymmetric simple exclusion model with multiple shocks.* Ann. Inst. Henri Poincaré, Probabilités et Statistiques 36, 2 (2000) 109-126.

[FKS] P. A. Ferrari, C. Kipnis, S. Saada, *Microscopic structure of traveling waves in the asymmetric simple exclusion process*, Ann. Prob., **19**, No. 1 (1991), pp. 226-244.

[KR] A. N. Kirillov, N. Yu. Reshetikhin, in: *Proceedings of the 1988 Luminy Conference on Infinite-Dimensional Lie Algebras and Groups*, V. G. Kac (ed.), (World Scientific, Singapore, 1988).

[KSKS] A. B. Kolomeisky, G. M. Schütz, E. B. Kolomeisky and J. P. Straley, *Phase diagram of one-dimensional driven lattice gases with open boundaries.* J. Phys. A **31** (1998), 6911 - 6919.

[K] K. Krebs, Ph.D. thesis, University of Bonn, 2001.

[L1] T. M. Liggett, *Interacting Particle Systems* Springer, Berlin (1985).

[L2] T. M. Liggett, *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes* Springer, Berlin (1999).

[PS] V. Pasquier and H. Saleur, *Common structures between finite systems and conformal field theories through quantum groups*, Nucl. Phys. B **330** (1990), 523–556.

[S1] G. M. Schütz, *Duality relations for asymmetric exclusion process.* J. Stat. Phys. **86**, Nos. 5/6 (1997), 1265 - 1287.

[S2] G. M. Schütz, Exactly solvable models for many-body systems far from equilibrium, *Phase Transitions and Critical Phenomena.* Vol. 19, Eds. C. Domb and J. Lebowitz, Academic Press, London, (2000).

[S3] G. M. Schütz, *Exact solution of the master equation for the asymmetric exclusion process.* J. Stat. Phys. **88** (1997), 427 - 445.