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Excited Random Walk on Trees

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Abstract: We consider a nearest-neighbor stochastic process on a rooted tree $\mathcal{G}$ which goes toward the root with probability $1-\varepsilon$ when it visits a vertex for the first time. At all other times it behaves like a simple random walk on $\mathcal{G}$. We show that for all $\varepsilon \geq 0$ this process is transient. Also we consider a generalization of this process and establish its transience in *some* cases.

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1 Introduction

Consider a graph G. Following [2], we define the Excited Random Walk (ERW) on G, as a process X_n on the vertices of $\mathcal{G}$ which behaves as a simple random walk on G (that is, it goes to each of its neighbors with the same probability), provided the current vertex has been visited before. However, when the ERW visits a vertex for the first time, the walk (and the vertex) becomes "excited" and the walk's transitional probabilities are different there. The choice of the law on the excited vertices may vary, but we can assume that the walk has some preferred direction.

In [2], the ERW on $\mathbb{Z}^d$ was considered. The rules of their walk require it on the excited vertices to step right with the probability $(1+\varepsilon)/(2d)$, left with the probability $(1-\varepsilon)/(2d)$, and in other directions with probability $1/(2d)$. On not excited vertices, the ERW jumps as a simple random walk. It was shown in [2] that for $d \geq 2$ and any $\varepsilon > 0$ such a process is transient, by which we understand that the walk visits every vertex at most finitely often.

The idea of the ERW is probably related to [5], and in general the ERW belongs to a class of path-dependent process, including notoriously hard to analyze reinforced random walks (RRW). More references can be found in [2]; also for a quite comprehensive review of the RRWs see [10].

Consider an infinite tree $\mathcal{G}$ rooted at v_0 with the property that each vertex, except possibly the root, is incident to at least three vertices. (The simplest example is the regular binary tree.) Fix an $\varepsilon > 0$. We define the ERW on a tree $\mathcal{G}$ as the process which goes toward the root with probability $1-\varepsilon$ when it visits a vertex v for the first time and goes to every neighbor of v with equal probability from already visited vertices. In a sense, this walk can be viewed as a vertex analogue of the once edge-reinforced random walk (once edge RRW), studied in [9]. This is a nearest-neighbor walk on a regular tree, with transition probabilities proportional to "weights" of the edges. Initially all edges have weight 1, and the weight of the ever traversed edge is c for some fixed constant $c > 1$. It is shown in [9] that once edge RRW on a tree is recurrent for all values of $c \geq 1$.

Alternatively, the ERW on a tree can be represented as a RRW of *sequence* type (see [4], [11]). The sequence type edge RRW is defined as a nearest-neighbor process on the vertices of a graph G, which jumps across each edge with the probability proportional to that the *weight* of that edge. In turn, the weight of a particular edge equals w_k whenever the walk has traversed this edge exactly k times, in either direction. On a $b-$ary regular tree, the sequence of weights $w_0 = 1$, $w_2 = w_3 = \cdots = c > 0$ gives the once edge RRW; and the sequence of weights $w_0 = 1$, $w_1 = b(\varepsilon^{-1} - 1)$, $w_2 = w_3 = \cdots = 1$ corresponds to an ERW. In [11] a number of properties of the edge RRW on the $d-$dimensional integer lattice has been established.

In our paper, we will show that on any tree satisfying the condition above (about the number of edges attached to a vertex), and for all $\varepsilon \geq 0$ the ERW is transient. Intuitively it seems "clear" that for the ERW it is "easier" to run to infinity than for the once edge RRW, however, as usual for this type of problems, it was not possible to construct a rigorous coupling.

In Section 2 we provide rigorous definitions as well as the proofs for the case $\varepsilon > 0$. The case $\varepsilon = 0$ is studied in Section 3 alongside with a generalization of the ERW, which we call *Digging Random Walk*. Final comments are presented in Section 4.

2 The proofs

Let X_n be a nearest-neighbor walk on the vertices of $\mathcal{G}$. As it was mentioned above, the vertex $v \neq v_0$ is called *excited* at time n if $X_n = v$ and for all $m < n$ $X_m \neq v$. The root v_0 is never excited by convention.

Let $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$ be the sigma-algebra generated by the path of the X_n. For a vertex $v \in \mathcal{G}$, let $N(v)$ denote the set of the adjacent vertices, and let $|N(v)|$ be the cardinality of this set. If $v \neq v_0$, let $A(v)$ be the "parent" vertex in $N(v)$ which lies on the shortest path connecting v to v_0. Fix an $\varepsilon \geq 0$. Suppose that for each $w \neq v_0$ there is a positive constant

$\varepsilon_w \geq 0$ such that for any $v \neq v_0$

$$\sum_{w:\, w \text{ is a child of } v} \varepsilon_w = \varepsilon.$$

Define the transitional probabilities of the ERW as

$$\mathbb{P}(X_{n+1} = w \,|\, X_n = v, \mathcal{F}_n) = \begin{cases} 1/|N(v)|, & \text{if } w \in N(v) \text{ and } v \text{ is not excited} \\ 1-\varepsilon, & \text{if } w = A(v) \in N(v) \text{ and } v \text{ is excited} \\ \varepsilon_w, & \text{if } w \in N(v)\backslash A(v) \text{ and } v \text{ is excited} \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that $X_0 = v_0$, and let $\mathbb{P}$ denote the law of the stochastic process X_n.

Let us also define a modification of the ERW. Suppose $\mathcal{G}_1$ is a finite subtree of $\mathcal{G}$ containing the root. Let the ERW($\mathcal{G}_1$) be the process identical to the ERW, except that at vertices of $\mathcal{G}_1$ the walk goes to each of its neighbors with the same probability, *even if* this vertex is visited for the first time. One can think of the ERW($\mathcal{G}_1$) as of the ERW started in the past and conditioned on the fact that by time 0 its range is exactly $\mathcal{G}_1$, and X_n is at the root at this time. Thus the ERW described above coincides with the ERW($\emptyset$). Let $\mathbb{P}_{\mathcal{G}_1}$ denote the law of the ERW($\mathcal{G}_1$).

¿From now on throughout this section we assume that $\varepsilon \neq 0$ (the case $\varepsilon = 0$ is studied in Section 3).

Let

$$A_n = \{\exists k \geq 1 : \; |X_k| = n \text{ and } 0 < |X_i| < n \text{ for all } 1 \leq i < k\} \tag{2.1}$$

be the event that the ERW($\mathcal{G}_1$) started at the root hits level n before returning to the root. (As usual, for a vertex $v \in \mathcal{G}$, $|v|$ denotes the number of edges in the shortest path connecting v and the root, with $|v_0| = 0$).

Since it is straightforward how to establish the transience of the ERW when $\varepsilon > 1/2$, throughout the rest of the proof we assume that $0 < \varepsilon \leq 1/2$.

Set

$$\alpha_n = \inf_{\text{all } \mathcal{G}_1} \mathbb{P}_{\mathcal{G}_1}(A_n). \tag{2.2}$$

Note that since the event A_n depends only on finitely many vertices of $\mathcal{G}$, $\alpha_n > 0$ for any $n \geq 1$. Also, since $A_{n+1} \subset A_n$, $\{\alpha_n\}$ is a decreasing sequence.

First, we obtain a crude lower bound on α_n, which will be improved later.

Lemma 1 *There exists a $C_1 = C_1(\varepsilon) > 0$ such that for all $n \geq 2$*

$$\alpha_n \geq \frac{C_1}{n^{3/2}}. \tag{2.3}$$

Proof. On the event A_n, let $k = \inf\{i : |X_i| = n\}$, and let $v_n = X_k$ be the first visited vertex on level n. Also, let $l = (v_0, v_1, \ldots, v_{n-1}, v_n)$ be the unique self-avoiding path connecting the root v_0 to v_n. Then all vertices of l have been visited by time k. Let $\tau = \inf\{i \geq 1 : |X_i| = 0\}$ and

$$B = \{\forall i \in [k+1, \tau) : X_i \in l \backslash \{v_n\} \text{ and } X_{i+1} \notin l,\ \exists j > i : X_j \in l\}$$

be the event that (before returning to the root if this ever happens) the walk always returns to path l after leaving it for another subtree. Since $\mathbb{P}_{\mathcal{G}_1}(A_{n+1}) = \mathbb{P}_{\mathcal{G}_1}(A_{n+1} \,|\, A_n)\mathbb{P}_{\mathcal{G}_1}(A_n)$ and $\mathbb{P}_{\mathcal{G}_1}(A_{n+1} \,|\, B^c, A_n) = 1$ for any $\mathcal{G}_1$, to get the lower bound on α_{n+1} it suffices to compute the probability of A_{n+1} conditioned on A_n and B.

Firstly, observe that after reaching level n, on the next step the walk either goes directly to level $n+1$ with probability ε or with probability $1-\varepsilon$ goes to v_{n-1} (unless $v_n \in \mathcal{G}_1$, which will make the first probability at least $\frac{1}{3}$). Further in the proof we assume that $v_n \notin \mathcal{G}_1$, as this case can be handled in the same way.

Secondly, if $X_{k+1} = v_{n-1}$, then the vertex v_n will not be excited when (and if) the walk visits v_n in the future. We consider the walk restricted to l and conditioned on B, whence from each v_i it goes to either v_{i-1} or v_{i+1} with probability $\frac{1}{2}$, except the root and v_n (see also the proof of Theorem 1). ¿From the latter the ERW goes to level $n+1$ with probability no less than $\frac{2}{3}$, since v_n has at least two children by the condition imposed on the tree in the introduction. Consequently, the probability that the ERW($\mathcal{G}_1$) started at v_{n-1} will visit level $n+1$ before the root is at least

$$\frac{n-1}{(n-1)+1+1/2} = \frac{n-1}{n+1/2}$$

(to get this probability one can use e.g. the electrical networks arguments – see [7]). Hence for $n \geq 2$

$$
\begin{aligned}
\alpha_{n+1} &\geq \alpha_n \left[\varepsilon + (1-\varepsilon)\frac{n-1}{n+1/2}\right] \geq \alpha_n \left[1 - \frac{3/2}{n+1/2}\right] \\
&\geq \alpha_n \exp\left(-\frac{3/2}{n} - \frac{3}{n^2}\right),
\end{aligned}
$$

where we used the inequality $\log(1-x) > -x - x^2$ for $x < 2/3$. Iterating this for $n = 2, 3, \ldots$ and taking into account the fact that both $\log(N) - \sum_{n=1}^{N} 1/(n+1)$ and $\sum_{n=1}^{N} 1/n^2$ converge to constants, we obtain (2.3). ∎

Next, we need the following elementary lemma about the simple random walk on $\mathbb{Z}$, which can be derived, for example from [1]. For the purpose of exposition, below we present its short proof.

Lemma 2 *Let S_i, $i \geq 0$, be a simple random walk on $\mathbb{Z}$, i.e. $S_{i+1} - S_i$ are iid random variables equal to ± 1 with probability $\frac{1}{2}$. Suppose $S_0 = 0$. Then for any $0 < \delta < 1/2$ and $\gamma > 0$ there is a $C_2 = C_2(\delta, \gamma) > 0$ such that*

$$
\mathbb{P}(\max_{0 \leq i \leq N} S_i > \gamma N^{1/2+\delta}) \leq C_2 N^{1/2} \exp(-\gamma^2 N^{2\delta}/2). \tag{2.4}
$$

Proof. From the reflection principle ([3], Chapter 1.4) it follows that for any positive integer a,

$$
\mathbb{P}(\max_{0 \leq i \leq N} S_i > a) \leq 2\mathbb{P}(S_N > a) = 2\sum_{j>a} \mathbb{P}(S_N = j). \tag{2.5}
$$

On the other hand, if $N = 2m$ and $j = 2k > 0$

$$
\mathbb{P}(S_N = j) = \mathbb{P}(S_{2m} = 2k) = \frac{(2m)!}{(m-k)!(m+k)!} \times \frac{1}{2^{2m}}
$$

which decreases as $k \geq 0$ goes up. Therefore, for $k > k_0 = \lfloor \gamma/2\, N^{1/2+\delta} \rfloor$

$$
\begin{aligned}
\mathbb{P}(S_{2m} = 2k) &\leq \mathbb{P}(S_{2m} = 2k_0) \qquad (2.6) \\
&\sim \frac{1}{\sqrt{\pi m}} \left(1 - \frac{k_0^2}{m^2}\right)^{-m} \left(1 + \frac{k_0}{m}\right)^{-k_0-0.5} \left(1 - \frac{k_0}{m}\right)^{k_0-0.5}
\end{aligned}
$$

where we used the Stirling's formula (see e.g. [8] Chapter 2.1) for large k_0 and m, and $x_n \sim y_n$ means that $x_n/y_n \to 1$. The RHS of (2.6) asymptotically equals

$$\frac{\exp(-\gamma^2 N^{2\delta})/2}{\sqrt{\pi N/2}}.$$

Plugging this into (2.5), and observing that $\mathbb{P}(S_N = j) = 0$ for odd j's and for $j > N$, completes the proof for large even N. Also, we can choose C_2 so large, that (2.4) will hold for *all* even $N \geq 2$. The proof in the case when N is odd is similar. ∎

Next we improve the lower bound obtained in Lemma 1.

Theorem 1 *There exists a positive*

$$\alpha := \lim_{n\to\infty} \alpha_n > 0.$$

Proof. First, the limit trivially exists since $\alpha_n > 0$ is a decreasing sequence as mentioned above. Thus we only need to show that it is $\neq 0$.

Let B, v_n, and k be the same as the proof of Lemma 1. Let V_{n+1} be the set of vertices which are the children of v_n on the tree $\mathcal{G}$. Recall that $\mathbb{P}_{\mathcal{G}_1}(A_{n+1} \mid B^c, A_n) = 1$, whence $\mathbb{P}_{\mathcal{G}_1}(A_{n+1} \mid A_n) \geq \mathbb{P}_{\mathcal{G}_1}(A_{n+1} \mid A_n, B)$. We will show that this probability is very close to 1. To avoid cumbersome notations involving the integer part sign $\lfloor\cdot\rfloor$, without loss of generality suppose that n is divisible by 4.

On the event $A_n \cap B$, let

$$\begin{aligned} \tau &= \inf\{i > k :\ X_i = v_{n/2}\}, \\ \eta &= \inf\{i > k :\ X_i \in V_{n+1}\}. \end{aligned}$$

Obviously,

$$\mathbb{P}_{\mathcal{G}_1}(A_{n+1} \mid A_n, B) \geq \mathbb{P}_{\mathcal{G}_1}(A_{n+1} \text{ and } \{\tau < \eta\} \mid A_n, B)$$

with the convention for the stopping times that $\infty = \infty$.

Next, conditioned on $A_n \cap B \cap \{\tau < \eta\}$ let $\nu(0) \equiv \tau, \nu(1), \nu(2)\ldots$ be the times of the consecutive steps of the walk on l, that is for $m \geq 1$

$$\nu(m) = \inf\{i > \nu(m-1) : \; X_i \in l, \; X_i \neq X_{\nu(m-1)}\},$$

so that $X_{\nu(m)}$ is an embedded random walk on l. Moreover, as long as $\nu(m)$ is finite, $X_{\nu(m)}$ is isomorphic to a a simple random walk on $\mathbb{Z}$. Set

$$\kappa = \min\{m > 0 : \; X_{\nu(m)} = v_{n/4} \text{ or } X_{\nu(m)} = v_{3n/4}\}.$$

Fix some small $\delta > 0$ and let $N = \lfloor n^{2-\delta} \rfloor$. Then

$$\mathbb{P}(\kappa \leq N \,|\, A_n, B, \tau < \eta) = \mathbb{P}(\max_{i=1,\ldots,N} |S_i| > n/4) \tag{2.7}$$

where S_i is a simple random walk on $\mathbb{Z}$ with $S_0 = 0$. By symmetry, and by using Lemma 2 we obtain

$$\begin{aligned} \mathbb{P}(\max_{i=1,\ldots,N} |S_i| > n/4) &\leq 2\mathbb{P}(\max_{i=1,\ldots,N} S_i > n/4) \\ &\leq 2\mathbb{P}\left(\max_{i=1,\ldots,N} S_i > \tfrac{1}{4} N^{\frac{1}{2}+\frac{\delta}{4-2\delta}}\right) \\ &\leq 2C_2(\delta/(4-2\delta), 1/4) N^{1/2} \exp\left(-\tfrac{1}{32} N^{\frac{\delta}{2-\delta}}\right) \\ &= 2C_2 \exp\left(-n^\delta[1+o(1)]/32\right) =: r_n \end{aligned} \tag{2.8}$$

since $n \geq N^{1/(2-\delta)} = N^{1/2+\delta/(4-2\delta)}$. Therefore, $\kappa \geq N$ with probability at least $1 - r_n$. On the other hand, every time X_i visits vertex $v \in \{v_{n/4+1}, \ldots, v_{3n/4-1}\} \subset l$ at time j, with probability at least $\frac{1}{3}$ it goes to a child of v, say v', such that $v' \notin l$. Consider the walk on a subtree $\mathcal{G}(v')$ rooted at v'. ¿From v', with probability at least $\varepsilon\alpha_{3n/4}$ it reaches level $3n/4$ of the subtree $\mathcal{G}(v)$ before returning to v. It is essential that this lower bound is independent of the past of the process X_i, i.e., $\{X_i, i < j\}$. Indeed, if the walk visited $\mathcal{G}(v')$ before j, some of its vertices will not be excited when visited after time j. However, this is not a problem because of the way α_n has been defined in (2.2).

At the same time, if the walk on $\mathcal{G}(v')$ reaches level $3n/4$, since $(n/4 + 1) + 3n/4 = n + 1$, it means that the event A_{n+1} has occurred. The formulae

(2.7) and (2.8) imply

$$\mathbb{P}(A_{n+1} \mid A_n, B, \tau < \eta) \geq (1 - r_n)\mathbb{P}(A_{n+1} \mid A_n, B, \tau < \eta, \kappa > N)$$

while

$$\begin{aligned}
\mathbb{P}(A_{n+1} \mid A_n, B, \tau < \eta,\, \kappa > N) &\geq 1 - \left[1 - \frac{1}{3}\varepsilon\alpha_{3n/4}\right]^{N+1} \\
&\geq 1 - \left[1 - \frac{C_1\varepsilon(4/3)^{3/2}}{3n^{3/2}}\right]^{n^{2-\delta}} \qquad (2.9)\\
&> 1 - \exp\left(-\frac{C_1\varepsilon}{2}n^{1/2-\delta}\right) =: 1 - s_n
\end{aligned}$$

for large n. Choose $\delta \in (0, 1/2)$. Then, finally,

$$\alpha_{n+1} \geq \alpha_n(1 - r_n)(1 - s_n)$$

and since $\sum_n r_n < \infty$ and $\sum_n s_n < \infty$, this yields

$$\alpha = \lim \alpha_n \geq \alpha_{n_0} \prod_{n=n_0}^{\infty} (1 - r_n)(1 - s_n) > 0.$$

■

Corollary 1 *The ERW is transient a.s.*

Proof. Every time the ERW visits a new vertex v, independently of the past the walk leaves to a child of v and never returns to v with probability at least $\varepsilon\alpha > 0$. Thus the corollary follows e.g. from the conditional Borel-Cantelli lemma ([8], Chapter 4.3). ■

3 Digging random walk

In this section we consider the case $\varepsilon = 0$. However, we would like to view this as a special case of the process, which we call a *Digging Random Walk*

(DRW) with the transitional probabilities defined below. Let $r \geq 1$ be a positive integer.The DRW on a tree $\mathcal{G}$ is a nearest-neighbor random walk with the transitional probabilities

$$\tilde{\mathbb{P}}(X_{n+1} = w \,|\, X_n = v, \mathcal{F}_n)$$

$$= \begin{cases} 1/|N(v)|, & \text{if } w \in N(v) \text{ and either } v = v_0 \textit{ or } v \text{ has been visited} \\ & \text{ at least } r \text{ times up to time } n-1 \\ 1, & \text{if } w = A(v) \in N(v) \text{ and } v \text{ has been visited} < r \text{ times} \\ 0, & \text{otherwise} \end{cases}$$

where $N(v)$, $A(v)$ and $\mathcal{F}_n$ are defined in the introduction. Also we suppose that $X_0 = v_0$ and at the root the walk is always equally likely to go to each of its neighbors. We say that a vertex of $\mathcal{G}$ is *excited*, if it was visited by the walk for the k^{th} time, $k < r$. From an excited vertex the walk always goes back to where it came from. From a not excited vertex the walk jumps as a simple random walk on $\mathcal{G}$.

One can think of this process as of a rodent exploring tunnels, but it has to visit each junction at least r times ("to dig" through it) before it can go any further.

Once again, the DRW on any tree corresponds to the sequence type edge RRW defined in [11] with the weight sequence $w_1 = w_3 = \cdots = w_{2r-1} = \infty$ and the rest of weights $= 1$. Also, the ERW with $\varepsilon = 0$ is identical to the DRW with $r = 1$.

Theorem 2 *Suppose that $r \in \{1, 2\}$. Then the DRW is transient a.s.*

Proof. Basically we will "recycle" the proofs for the ERW with some modifications. To sketch these modifications and to simplify the exposition, throughout this proof we assume that $\mathcal{G}$ is the binary rooted tree, as the same proof more or less verbatim can be translated to all the trees with the properties described in the introduction. Also suppose $r = 2$ (the case $r = 1$ is similar).

First, the events A_n described in (2.1) now cannot occur. Instead, given $X_0 = v_0$, we define the events

$$\tilde{A}_n = \{\exists k \geq 1 : \ |X_k| = n \text{ and } 0 < |X_i| < n \text{ for all } k_1 \leq i < k\}$$
$$\text{where } k_1 \text{ is the time of the } r+1^{st} \text{ visit to } N(v_0)$$

Then the events $\tilde{A}_n$ are non-trivial, as with a positive probability the walk jumps r times to the same child v_1 of v_0 and then it can with a positive probability not to return to v_0 before reaching level n. Similarly to the ERW($\mathcal{G}_1$), we define DRW($\mathcal{G}_1$), the measures $\tilde{\mathbb{P}}$ and $\tilde{\mathbb{P}}_{\mathcal{G}_1}$ corresponding to the DRW and DRW($\mathcal{G}_1$) respectively, and

$$\tilde{\alpha}_n = \inf_{\text{all } \mathcal{G}_1} \tilde{\mathbb{P}}_{\mathcal{G}_1}(\tilde{A}_n).$$

Suppose that we could show that

$$\tilde{\alpha}_n \geq \frac{\tilde{C}_1}{n^{2-\nu}} \tag{3.10}$$

for some $0 < \nu < 2$. Then the proof of the Theorem 1 (and therefore Corollary 1) will be applicable for the DRW with the only exception that the RHS of (2.9) is replaced by

$$\begin{aligned} & 1 - \left[1 - \frac{2}{3^{r+1}}\tilde{\alpha}_{3n/4}\right]^{N+1} \\ > \ & 1 - \left[1 - \frac{2\tilde{C}_1(4/3)^{2-\nu}}{3^{r+1}n^{2-\nu}}\right]^{n^{2-\delta}} \\ > \ & 1 - \exp\left(-\frac{\tilde{C}_1}{3^{r+1}n^{\nu-\delta}}\right) =: 1 - \tilde{s}_n \end{aligned}$$

since from a vertex v described in the proof of Theorem 1, the walk makes $r+1$ consecutive steps to the same v' and back, and one more step to a child of v' with probability $(\frac{1}{3})^r \times \frac{2}{3}$. Consequently, we can consider the DRW($\mathcal{G}_1$) on the subtree rooted at v' and use the definition of $\tilde{\alpha}_n$ to obtain the above inequality. Choosing $\delta \in (0, \nu)$ would finish the proof.

Thus, it suffices to show (3.10) for some $\nu > 0$. We proceed along the lines of the proof of Lemma 1. Let k be the first time when the walk hits level n at some vertex v_n and let v'_n be the other (yet unvisited) child of v_{n-1}. Let $q_{i,j}$ be the (conditional, given the past) probability that the walk will hit the root before level $n+1$ after time m, given that $X_m = v_n$ and that the vertex v_n (v'_n resp.) has been visited by time m exactly i (j resp.) times[2]. Then, taking into account that $q_{i,j} = q_{r+1,j}$ when $i > r$, and using the symmetry between v'_n and v_n, we have

$$\begin{aligned}
q_{10} &\leq \frac{1}{2n} + \frac{1}{2}(q_{11} + q_{20}) + O(n^{-2}) \\
q_{11} &\leq \frac{1}{2n} + q_{21} + O(n^{-2}) \\
q_{12} = q_{21} &\leq \frac{1}{2n} + \frac{1}{2}(q_{31} + q_{22}) + O(n^{-2}) \\
q_{13} &\leq \frac{1}{2n} + \frac{1}{2}(q_{31} + q_{23}) + O(n^{-2}) \\
q_{20} &\leq \frac{1}{2n} + \frac{1}{2}(q_{30} + q_{12}) + O(n^{-2})
\end{aligned}$$

$$\begin{aligned}
q_{22} &\leq \frac{1}{2n} + q_{32} + O(n^{-2}) \\
q_{23} &\leq \frac{1}{2n} + q_{33} + O(n^{-2}) \\
q_{30} &\leq \frac{1}{3}\left[\frac{1}{2n} + \frac{1}{2}(q_{30} + q_{13})\right] + O(n^{-2}) \\
q_{31} &\leq \frac{1}{3}\left[\frac{1}{2n} + \frac{1}{2}(q_{31} + q_{23})\right] + O(n^{-2}) \\
q_{32} &\leq \frac{1}{3}\left[\frac{1}{2n} + \frac{1}{2}(q_{32} + q_{33})\right] + O(n^{-2}) \\
q_{33} &\leq \frac{1}{3}\frac{3}{4n} + O(n^{-2}) = \frac{1}{4n} + O(n^{-2}).
\end{aligned}$$

Again, to compute some of the probabilities we used the electrical network arguments, in the same spirit as e.g. in [6]. For the purpose of exposition, let

[2] to be rigorous, one has to define $q_{i,j}$ also as the infimum over all possible $\mathcal{G}_1$'s

us explain how to obtain the first of these inequalities, for q_{10}. When the walk hits v_n, it has to go back to its parent v_{n-1}. Then, with probability at most $\frac{1/2}{1/2+(n-1)} = \frac{1}{2n-1} = \frac{1}{2n} + O(n^{-2})$ the walk reaches the root before returning to the set $\{v_n, v_n'\}$. On the other hand, if the walk hits this set first, then it is equally likely to be at v_n' or v_n. In the first case, the probability to hit the root is bounded by q_{11} and in the second case it is bounded by q_{20}. The rest of the inequalities is obtained in the same way.

Solving this system of inequalities, we conclude

$$q_{10} \leq \frac{1.89}{n} + O(n^{-2})$$

whence

$$\tilde{\alpha}_{n+1} \geq \tilde{\alpha}_n \left[1 - \frac{1.9}{n}\right] \geq \tilde{\alpha}_n \exp\left(-\frac{1.9}{n} - O(n^{-2})\right),$$

yielding (3.10) with $\nu = 0.1$. ∎

Unfortunately, similar arguments fail when $r \geq 3$.

4 Other possible results

Repeating the arguments from [9] one can demonstrate that the ERW (and perhaps DRW for $r = 1, 2$) go to infinity approximately at a constant speed and to establish that $|X_n|$ also satisfies the invariance principle – see Theorems 2-3 in [9]. Let us briefly sketch what needs to be changed in their proofs.

First of all, ERW will have the same *cut-times* as the once edge RRW (see Section 3 of [9]), since their Lemma 1 and inequality (2.6) and later (3.12) hold verbatim for the ERW, with their constant $b/(b+c)$ replaced by ε. Their Lemmas 7 and 8 also hold, with the RHS of inequality (2.40) (and hence (3.10)) replaced by the similar expression $1 - 2C_2 \exp\{-C_3 n^{1/4}\}$ resulting from our inequalities (2.8) and (2.9) with $\delta = 1/4$. In turn, this yields different constants K_{14} and δ in the proof of Lemma 7 and different

q in the proof of Lemma 8 (the equation (3.13) remains unchanged, but the one just above it will have a different RHS, in accordance with inequalities (2.8) and (2.9) of our paper). The rest of the proof of Lemma 8 is identical. And the proofs of Theorems 2 and 3 of [9] are based on these two Lemmas – see the discussion above Lemma 7.

The most significant question left open by our paper, is what happens with the DRW for $r \geq 3$. We conjecture that the DRW is recurrent for *all* integer $r \geq 0$.

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