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Cauchy's principal value of local times of Lévy processes with no negative jumps via continuous branching processes

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Summary. Let X be a recurrent Lévy process with no negative jumps and n the measure of its excursions away from 0. Using Lamperti's connection [13] that links X to a continuous state branching process, we determine the joint distribution under n of the variables $C_T^+ = \int_0^T \mathbf{1}_{\{X_s>0\}} X_s^{-1} ds$ and $C_T^- = \int_0^T \mathbf{1}_{\{X_s<0\}} |X_s|^{-1} ds$, where T denotes the duration of the excursion. This provides a new insight on an identity of Fitzsimmons and Getoor [9] on the Hilbert transform of the local times of X. Further results in the same vein are also discussed.

Key words. Cauchy's principal value, Lévy process with no negative jumps, branching process.

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1 Introduction

This work originates from a striking result of Fitzsimmons and Getoor [9] on the Cauchy's principal value of the local times of certain Lévy processes, which we now recall. Let $X = (X_t : t \geq 0)$ be a real-valued Lévy process started at 0, which has local times $(L_t^x : x \in \mathbb{R}$ and $t \geq 0)$. This means that the occupation density formula

$$\int_0^t f(X_s)ds \;=\; \int_{-\infty}^{\infty} f(x)L_t^x dx$$

holds for all $t \geq 0$ and all Borel functions $f \geq 0$, a.s. We assume that the local times are sufficiently smooth in the space variable x, say a.s. locally Hölder continuous with some index $\rho > 0$ (see Barlow [1] for an explicit condition which entails the latter assumption). We can then define the so-called *Cauchy's principal value* of the local times

$$C_t \;=\; \int_0^{\infty} \left(L_t^x - L_t^{-x}\right) \frac{dx}{x} \;=\; \lim_{\varepsilon \to 0+} \int_0^t \mathbf{1}_{\{|X_s|>\varepsilon\}} \frac{ds}{X_s}\,.$$

In other words, $\pi^{-1}C_t$ coincides with the negative of the Hilbert transform of $L_t^{\cdot}$ evaluated at 0.

We then introduce $\sigma(\cdot) = \inf\{s \geq 0 : L_s^0 > \cdot\}$, the right-continuous inverse of the local time at level 0. Fitzsimmons and Getoor [9] have observed that when X is recurrent,

$$\left(C_{\sigma(t)} : t \geq 0\right) \text{ is a symmetric Cauchy process with parameter } \pi. \tag{1}$$

More precisely, (1) has first been shown by Biane and Yor [6] in the special case when X is a Brownian motion. Then (1) has been proven in [9] under the additional condition that X is symmetric; and that the symmetry condition can be dropped has been noted in [4]. The argument of [9] for establishing (1) is based on a combinatorial lemma on Euler numbers, whereas that of [4] relies on the Fourier analysis of the Feynman-Kac formula.

Making use of the well-known Lévy-Itô decomposition for a Cauchy process, it should be clear that (1) is essentially equivalent to the assertion that the process $\left(C_{\sigma(t)} - C_{\sigma(t-)} : t \geq 0\right)$ is a Poisson point process with characteristic measure $x^{-2}dx$ (i.e. the Lévy measure of a symmetric Cauchy process with parameter π). Let us denote by n the Itô measure of the excursions away from 0 of the Lévy process, and by $T = \inf\{t > 0 : X_t = 0\}$ the duration of an excursion [1]. A standard argument of excursion theory makes plain that $\left(C_{\sigma(t)} - C_{\sigma(t-)} : t \geq 0\right)$ is a Poisson point process with characteristic measure $n\left(C_T \in dx\right)$, where $C_T = \int_0^T ds/X_s$. In other words, (1) is essentially equivalent to

$$n\left(C_T \in dx\right) \;=\; x^{-2}dx\,. \tag{2}$$

It is remarkable that the identity (2) is independent of the distribution of the Lévy process, and it would be interesting to have a 'purely probabilistic' explanation of this feature.

In general, the major problem when one tries to prove (2) directly is that it seems that no known description of the excursion measure n is adequate to determine the law of C_T. In the

[1] By a slight common abuse of notation, we denote both the Lévy process and its generic excursion by X. The distinction will always be clear from the context.

Brownian case, Biane and Yor [6] have been able to established (2) using decompositions of the excursion measure of Bessel processes and a connection between two Bessel processes of different dimensions based on a simple time-change. The main purpose of this work is to point out that there is another situation where this can be done, namely when the Lévy process has no negative jumps and zero Gaussian coefficient (and of course also, by an obvious symmetry, when the Lévy process has no positive jumps and zero Gaussian coefficient). Specifically, it is then easy to see that a typical excursion of X away from 0 first stays negative, then jumps across 0, and finally stays positive until it returns to 0. The joint distribution of the positive and negative parts of the excursion can be described explicitly, and this enables us to investigate the variables

$$C_T^+ = \int_0^T \mathbf{1}_{\{X_s>0\}} X_s^{-1} ds \quad , \quad C_T^- = \int_0^T \mathbf{1}_{\{X_s<0\}} |X_s|^{-1} ds \,.$$

The point in decomposing $C_T = C_T^+ - C_T^-$ into the positive and negative contributions, is thatwe are now dealing with a pair of positive functionals which appear in the work of Lamperti [13] connecting Lévy processes with no negative jumps to continuous state branching processes. This enables us to determine the joint distribution of C_T^+ and C_T^- under n (this method based on time-substitution is thus close to the original one used by Biane and Yor [6] in the Brownian case). We find in particular that the distribution of (C_T^+, C_T^-) depends on the law of the Lévy process though that of C_T does not.

The second section of this paper is devoted to preliminaries on Lévy processes with no negative jumps, continuous state branching processes and the Lamperti connection between the two. The framework is more general than that needed for our main purpose, but is relevant for other applications we have in mind. The result on the joint distribution of (C_T^+, C_T^-) under the excursion measure is stated and proved in section 3. Some related results, extensions and comments are presented in section 4.

2 Preliminaries

2.1 Lévy processes with no negative jumps

Throughout the rest of this paper, $X = (X_t : t \geq 0)$ denotes a Lévy process with no negative jumps (the trivial case when X is constant is implicitly excluded). For every $x \in \mathbb{R}$, we write $\mathbb{P}^x$ for the law of the Lévy process started at x, i.e. $\mathbb{P}^x$ is the distribution of $x + X$ under $\mathbb{P} = \mathbb{P}^0$. The so-called Laplace exponent $\psi : [0, \infty) \to (-\infty, \infty)$ is specified by the identity

$$\mathbb{E}^0 \left(\exp\{-\lambda X_t\} \right) = \exp\{t\psi(\lambda)\} \,, \qquad t, \lambda \geq 0$$

and can be expressed in the form

$$\psi(\lambda) = a\lambda + b\lambda^2 + \int_{(0,\infty)} \left(\mathrm{e}^{-\lambda x} - 1 + \lambda x \mathbf{1}_{\{x<1\}} \right) \Lambda(dx) \qquad (3)$$

where $a \in \mathbb{R}$, $b \geq 0$ and Λ is a measure on $(0, \infty)$ such that $\int_{(0,\infty)} (1 \wedge x^2)\, \Lambda(dx) < \infty$. One refers to (3) as the Lévy-Khintchine formula, and calls b the Gaussian coefficient and Λ the

Lévy measure. It is immediately checked that ψ is a strictly convex function, and that the mapping $\lambda \to \psi(\lambda)/\lambda$ is concave.

When X has non-decreasing sample paths, we say that X is a subordinator. This is equivalent to $\psi(\lambda) \leq 0$ for all $\lambda \geq 0$. When X is not a subordinator, the Laplace exponent ψ is ultimately increasing and tends to ∞. In both cases, we put

$$\Phi(0) \ = \ \inf\{\lambda \geq 0 : \psi(\lambda) > 0\}$$

with the usual convention that $\inf \emptyset = \infty$. In particular, $\Phi(0) = \infty$ if and only if X is a subordinator. When X is not a subordinator, then $\Phi(0)$ is the largest root of the equation $\psi(\lambda) = 0$ (either $\Phi(0) = 0$ is the only root, or $\Phi(0) > 0$ and then 0 and $\Phi(0)$ are the only roots). The condition $\Phi(0) > 0$ holds if and only if the Lévy process drifts to ∞, i.e. $\lim_{t\to\infty} X_t = \infty$ a.s.; see Corollary VII.2(ii) in [5].

2.2 Continuous state branching processes

Continuous state branching processes form a class of Markov processes that appear as limit of integer valued branching processes. They have been introduced by Jirina [11] and studied by many authors including Bingham [7], Grey [10], Lamperti [13, 14], Kawazu and Watanabe [12], Silverstein [17] *etc...* Specifically, let $(Z_t : t \geq 0)$ be a càdlàg process with values in $[0, \infty]$, where 0 and ∞ are two cemetery points. More precisely, if we introduce the absorption time

$$\alpha \ = \ \inf\{t \geq 0 : Z_t = 0\}$$

and the explosion time

$$\zeta \ = \ \inf\{t \geq 0 : Z_t = \infty\} \ ,$$

then $Z_t = 0$ for every $t \geq \alpha$ and $Z_t = \infty$ for every $t \geq \zeta$. Next, let $\mathbf{P} = (\mathbf{P}^x : x \in [0, \infty])$ be a family of probability measures such that $\mathbf{P}^x(Z_0 = x) = 1$, and let ψ be given in the form (3). We say that $(Z, \mathbf{P})$ is a continuous state branching process with branching mechanism ψ if it is a Markov process whose semigroup is specified by

$$\mathbf{E}^x\left(\exp\{-\lambda Z_t\}\right) \ = \ \exp\{-x v_\lambda(t)\} \ , \qquad \lambda \geq 0$$

where $v_\lambda : [0, \infty) \to [0, \infty]$ solves the integral equation

$$v_\lambda(t) + \int_0^t \psi(v_\lambda(s))ds \ = \ \lambda \tag{4}$$

(here and in the sequel, we use the convention $\mathrm{e}^{-\lambda \cdot \infty} = 0$). Plainly, (4) can be solved explicitly in terms of ψ, and this readily yields the law of the absorption time (see Grey [10]).

Lemma 1 (i) *If $\Phi(0) = \infty$ or if $\int^\infty d\lambda/\psi(\lambda) = \infty$, then $\mathbf{P}^x(\alpha < \infty) = 0$ for every $x > 0$.*
(ii) *If $\Phi(0) < \infty$ and $\int^\infty d\lambda/\psi(\lambda) < \infty$, then put*

$$f(t) \ = \ \int_t^\infty \frac{d\lambda}{\psi(\lambda)} \, , \qquad t \in (\Phi(0), \infty) \, .$$

The mapping $f : (\Phi(0), \infty) \to (0, \infty)$ *is bijective; we write* $\varphi : (0, \infty) \to (\Phi(0), \infty)$ *for the inverse mapping. We then have for every* $x, t > 0$

$$\mathbf{P}^x(\alpha < t) = \exp\{-x\varphi(t)\} .$$

Similarly, the law of the explosion time is given by the following:

Lemma 2 (i) *If* $\Phi(0) = 0$ *or if* $-\int_{0+} d\lambda/\psi(\lambda) = \infty$*, then* $\mathbf{P}^x(\zeta < \infty) = 0$ *for every* $x \geq 0$*.*
(ii) *If* $\Phi(0) > 0$ *and* $-\int_{0+} d\lambda/\psi(\lambda) < \infty$*, then put*

$$g(t) = -\int_0^t \frac{d\lambda}{\psi(\lambda)}, \qquad t \in (0, \Phi(0)) .$$

The mapping $g : (0, \Phi(0)) \to (0, \infty)$ *is bijective; we write* $\gamma : (0, \infty) \to (0, \Phi(0))$ *for the inverse mapping. We then have for every* $x, t > 0$

$$\mathbf{P}^x(\zeta > t) = \exp\{-x\gamma(t)\} .$$

2.3 Lamperti's time substitution

Lamperti [13] observed that the continuous state branching process with branching mechanism ψ is connected to the Lévy process with no negative jumps and Laplace exponent ψ by a simple time-change. Specifically, consider the Lévy process started at $x > 0$ (i.e. work under $\mathbb{P}^x$) and write

$$T = \inf\{t > 0 : X_t = 0\}$$

for the first return time to 0. Next introduce the clock

$$C_t = \int_0^t \frac{ds}{X_s}, \qquad t \in [0, T) .$$

We then put

$$\eta_t = \begin{cases} s & \text{if } C_s = t \text{ for some } s \in [0, T) \\ T & \text{otherwise.} \end{cases}$$

Then the time-changed process [2] $X \circ \eta$ is a continuous state branching process started at x, with branching mechanism ψ. In the sequel, it will be convenient to identify $Z = X \circ \eta$ and $\mathbb{P}^x = \mathbf{P}^x$. In particular, one immediately derives the following formulas for the absorption and explosion times

$$\alpha = \begin{cases} C_T & \text{if } T < \infty \\ \infty & \text{if } T = \infty \end{cases} , \qquad \zeta = \begin{cases} C_\infty & \text{if } T = \infty \\ \infty & \text{if } T < \infty \end{cases} .$$

As a consequence of Lemmas 1 and 2 we get

[2] Because $T = \infty$ can only occur if $\lim_{t\to\infty} X_t = \infty$ a.s., the notation $X \circ \eta$ is never ambiguous.

Corollary 3 (i) *If* $\Phi(0) = \infty$ *or if* $\int^\infty d\lambda/\psi(\lambda) = \infty$*, then* $\mathbb{P}^x(C_T < \infty, T < \infty) = 0$ *for every* $x > 0$*. Otherwise, in the notation of Lemma 1, we have for every* $x, t > 0$

$$\mathbb{P}^x(C_T < t, T < \infty) \;=\; \exp\{-x\varphi(t)\}\,.$$

(ii) *If* $\Phi(0) = 0$ *or if* $-\int_{0+} d\lambda/\psi(\lambda) = \infty$*, then* $\mathbb{P}^x(C_\infty < \infty, T = \infty) = 0$ *for every* $x \geq 0$*. Otherwise, in the notation of Lemma 2, we have for every* $x, t > 0$

$$\mathbb{P}^x(C_\infty < t, T = \infty) \;=\; 1 - \exp\{-x\gamma(t)\}\,.$$

3 Main result

We now assume that the Lévy process is recurrent, which is equivalent to

$$\lim_{\lambda\to 0+} \psi(\lambda)/\lambda \;=\; 0\,, \tag{5}$$

see for instance Corollary VII.2(ii) in [5]. In particular $\Phi(0) = 0$ (because ψ is strictly convex). Due to Corollary 3(i), we also suppose that

$$\int^\infty \frac{d\lambda}{\psi(\lambda)} \;<\; \infty \tag{6}$$

since otherwise the functional C_T can clearly not be defined under the excursion measure. Because ψ is increasing, (6) forces $\lim_{\lambda\to\infty} \psi(\lambda)/\lambda = \infty$, which is the necessary and sufficient condition for 0 to be regular for itself (see Corollary VII.5 in [5]). This ensures the existence of a continuous everywhere positive resolvent density $u^q : (-\infty, \infty) \to (0, \infty)$, i.e. for every $q > 0$:

$$\mathbb{E}^0\left(\int_0^\infty e^{-qt} f(X_t)dt\right) \;=\; \int_{-\infty}^\infty f(x)u^q(x)dy \tag{7}$$

for every Borel function $f \geq 0$; see Theorem II.19 in [5].

The fact that 0 is regular for itself also enables us to define the local time at level 0 as the occupation density at 0:

$$L_t^0 \;=\; \lim_{\varepsilon\to 0+} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{|X_s|<\varepsilon\}} ds\,, \tag{8}$$

see Proposition V.2 in [5]. Recall that σ stands for the inverse local time at 0,

$$\sigma(t) \;=\; \inf\left\{s \geq 0 : L_s^0 > t\right\}\,, \qquad t \geq 0\,.$$

Because X is recurrent, we have $\sigma(t) < \infty$ a.s., and the process σ is a subordinator with

$$\mathbb{E}\left(\exp\{-q\sigma(t)\}\right) \;=\; \exp\{-t/u^q(0)\}\,, \tag{9}$$

see Proposition V.4(ii) in [5].

Our final assumption on the Lévy process is that its Gaussian component is zero,

$$b \;=\; 0\,. \tag{10}$$

Together with (5), this entails that the Lévy-Khintchine formula can be re-written as

$$\psi(\lambda) \;=\; \int_{(0,\infty)} \left(\mathrm{e}^{-\lambda x} - 1 + \lambda x\right) \Lambda(dx) \;=\; \lambda \int_0^\infty \left(1 - \mathrm{e}^{-\lambda x}\right) \overline{\Lambda}(x) dx\,, \tag{11}$$

where $\overline{\Lambda}(x) = \Lambda((x,\infty))$ denotes the tail of the Lévy measure.

The Itô's measure n of the excursions away from 0 of the Lévy process (corresponding to the local time that has just been specified by (8)), is supported by the set of excursions that first stay negative then jump across 0, and finally stay positive until they return to 0 (see [3] on page 1467). Denote by $j = \inf\{t > 0 : X_t > 0\}$ the unique instant when the excursion jumps across 0. We thus have on a set of full n-measure

$$C_T^+ \;=\; \int_0^T \mathbf{1}_{\{X_s>0\}} \frac{ds}{X_s} \;=\; \int_j^T \frac{ds}{X_s} \quad \text{and} \quad -\,C_T^- \;=\; \int_0^T \mathbf{1}_{\{X_s<0\}} \frac{ds}{X_s} \;=\; \int_0^j \frac{ds}{X_s}\,.$$

Recall that the function φ has been defined in Lemma 1. We are now able to state the main result of this note.

Theorem 4 *Suppose that (5), (6) and (10) hold. The distribution of the pair* (C_T^+, C_T^-) *under the excursion measure is given by*

$$n\left(C_T^+ > t, C_T^- > s\right) \;=\; \frac{\psi(\varphi(t))}{\varphi(t)} + \frac{\psi(\varphi(s))}{\varphi(s)} - \frac{\psi(\varphi(t)) - \psi(\varphi(s))}{\varphi(t) - \varphi(s)} \qquad (s,t > 0)\,.$$

The proof of Theorem 4 relies on the following description of the excursion decomposed at the jump time j across 0. Recall that Λ denotes the Lévy measure.

Lemma 5 (i) *For every* $y > 0$ *and* $x \in [0,y]$*, we have*

$$n\left(X_j \in dx, X_j - X_{j-} \in dy\right) \;=\; dx\Lambda(dy)\,.$$

(ii) *Under the conditional probability measure* $n\left(\cdot \mid X_j = x, X_j - X_{j-} = y\right)$*, the processes*

$$\left(X_{j+t} : 0 \le t < T - j\right) \quad \textit{and} \quad \left(-X_{(j-t)-} : 0 \le t < j\right)$$

are independent. The first has the same law as $(X_t : 0 \le t < T)$ *under* $\mathbb{P}^x$*, and the second has the same law as* $(X_t : 0 \le t < T)$ *under* $\mathbb{P}^{y-x}$*.*

Proof: The second part of the statement just rephrases Lemma 1 in [3]. The remark on page 1470 in [3] shows that

$$n\left(X_j \in dx, X_j - X_{j-} \in dy\right) \;=\; k dx\Lambda(dy) \tag{12}$$

for some constant k which depends on the normalization of the local time. So all that we have to check is that the choice (8) for the local time implies $k = 1$.

To this end, take an arbitrary Borel function $h \ge 0$ that vanishes on $(-\infty, 0]$, and recall (7). A standard application of excursion theory yields for every $q > 0$

$$\begin{aligned} \int_{-\infty}^\infty h(y) u^q(y) dy \;&=\; \mathbb{E}\left(\int_0^\infty \mathrm{e}^{-qt} h(X_t) dt\right) \\ &=\; \mathbb{E}\left(\int_0^\infty \mathrm{e}^{-q\sigma(t)} dt\right) n\left(\int_0^T \mathrm{e}^{-qt} h(X_t) dt\right) \\ &=\; u^q(0) n\left(\int_0^T \mathrm{e}^{-qt} h(X_t) dt\right)\,, \end{aligned}$$

using (9) in the last line. As $u^q(y) \le u^q(0)$, and, since X is recurrent, $u^q(y) \sim u^q(0)$ as $q \to 0+$ (see e.g. Corollary II.18 in [5]), we conclude by dominated convergence that

$$n\left(\int_0^T \mathbf{1}_{\{X_t \in dy\}} dt\right) = dy\,.$$

On the other hand, let $W : [0,\infty) \to [0,\infty)$ denote the scale function, that is the unique continuous function with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\lambda y} W(y) dy = 1/\psi(\lambda)\,, \qquad \lambda > 0\,;$$

see section VII.2 in [5]. It is known that for every $x > 0$

$$\mathbb{E}^x\left(\int_0^T h(X_s) ds\right) = \int_0^\infty h(y)\left(W(y) - \mathbf{1}_{\{x<y\}} W(y-x)\right) dy\,,$$

see [5] on page 207. We thus get from the description (ii) of the excursion measure and (12) that for every $y > 0$

$$\begin{aligned} n\left(\int_0^T \mathbf{1}_{\{X_t \in dy\}} dt\right) &= \int_{[0,\infty)} n(X_j \in dx) \mathbb{E}^x \left(\int_0^T \mathbf{1}_{\{X_t \in dy\}} dt\right) \\ &= k\left(\int_0^\infty \overline{\Lambda}(x)\left(W(y) - \mathbf{1}_{\{x<y\}} W(y-x)\right) dx\right) dy\,. \end{aligned}$$

Putting $\overline{\overline{\Lambda}}(x) = \int_x^\infty \overline{\Lambda}(s) ds$, we have

$$\int_0^\infty \overline{\Lambda}(x)\left(W(y) - \mathbf{1}_{\{x<y\}} W(y-x)\right) dx = \int_0^y \overline{\overline{\Lambda}}(y-x) dW(x)\,.$$

The Laplace transform of the measure dW is $\lambda/\psi(\lambda)$, and it is plain from the Lévy-Khintchine formula that the Laplace transform of $\overline{\overline{\Lambda}}$ is $\psi(\lambda)/\lambda^2$. We deduce by Laplace inversion that the last displayed quantity equals 1 for a.e. $y > 0$. We conclude that

$$n\left(\int_0^T \mathbf{1}_{\{X_t \in dy\}} dt\right) = k dy\,,$$

and thus $k = 1$. ■

Lemma 6 *For every $\lambda, \mu > 0$, we have*

$$n\left(1 - \exp\{-\lambda X_j + \mu X_{j-}\}\right) = \frac{\psi(\lambda) - \psi(\mu)}{\lambda - \mu}\,.$$

Proof: By Lemma 5(i), the left-hand side in the statement equals

$$\begin{aligned} &\int_{(0,\infty)} \Lambda(dy) \int_0^y dx \left(1 - \mathrm{e}^{-\lambda x - \mu(y-x)}\right) \\ =\ &\int_{(0,\infty)} \Lambda(dy)\left(y - \frac{\mathrm{e}^{-\mu y} - \mathrm{e}^{-\lambda y}}{\lambda - \mu}\right) \\ =\ &\frac{1}{\lambda - \mu} \int_{(0,\infty)} \Lambda(dy)\left(\left(\mathrm{e}^{-\lambda y} - 1 + \lambda y\right) - \left(\mathrm{e}^{-\mu y} - 1 + \mu y\right)\right)\,, \end{aligned}$$

and the claim follows from the Lévy-Khintchine's formula (11). ■

The proof of Theorem 4 is now straightforward.

Proof of Theorem 4 : By Lemma 5(ii), we have

$$n\left(C_T^+ > t, C_T^- > s\right) = \int_{0<x\leq y} n\left(X_j \in dx, X_j - X_{j-} \in dy\right) \mathbb{P}^x(C_T > t)\mathbb{P}^{y-x}(C_T > s)\,.$$

When then apply Corollary 3(i) (recall that X is recurrent, so that $T < \infty$ a.s.) to express the right-hand side as

$$\begin{aligned}&\int_{0<x\leq y} n\left(X_j \in dx, X_j - X_{j-} \in dy\right)\left(1 - \exp\{-x\varphi(t)\}\right)\left(1 - \exp\{-(y-x)\varphi(s)\}\right)\\ = \;& n\left(\left(1 - \exp\{-\varphi(t)X_j\}\right) + \left(1 - \exp\{\varphi(s)X_{j-}\}\right) - \left(1 - \exp\{-\varphi(t)X_j + \varphi(s)X_{j-}\}\right)\right)\,.\end{aligned}$$

We conclude with Lemma 6. ■

4 Comments

1. It is important to check that (2) can be recovered from Theorem 4. Specifically, for every $t > 0$, we can express $n\left(C_T > t\right)$ as

$$n\left(C_T^+ - C_T^- > t\right) = \int_0^\infty n\left(C_T^+ > t+s, C_T^- \in ds\right) = \int_0^\infty \left(\mathtt{A}(s) + \mathtt{B}(s) - \mathtt{C}(s)\right) ds$$

with $\mathtt{A}(s)ds = d\left(\psi(\varphi(s))/\varphi(s)\right)$ and

$$\mathtt{B}(s) = \frac{\psi(\varphi(t+s)) - \psi(\varphi(s))}{\left(\varphi(t+s) - \varphi(s)\right)^2}\psi((\varphi(s)) \quad , \quad \mathtt{C}(s) = \frac{\psi'((\varphi(s))\psi(\varphi(s))}{\varphi(t+s) - \varphi(s)}$$

(note that $\varphi' = \psi \circ \varphi$). An integration by parts shows that for every $0 < \varepsilon < \kappa$

$$\int_\varepsilon^\kappa \mathtt{B}(s)ds = \int_\varepsilon^\kappa \mathtt{C}(s)ds + \left[\frac{\psi(\varphi(s))}{\varphi(t+s) - \varphi(s)}\right]_\varepsilon^\kappa,$$

so that $n\left(C_T > t\right) = \lim_{\kappa\to\infty} \mathtt{D}(\kappa) - \lim_{\varepsilon\to 0+} \mathtt{D}(\varepsilon)$ with

$$\mathtt{D}(s) = \frac{\varphi(t+s)\psi(\varphi(s))}{\varphi(s)\left(\varphi(t+s) - \varphi(s)\right)}\,.$$

We let ε tend to $0+$, so $\varphi(\varepsilon) \to \infty$ and $\varphi(t+\varepsilon) \to \varphi(t) < \infty$. Since $\psi(\lambda) = o(\lambda^2)$ as $\lambda \to \infty$ (this is seen from the Lévy-Khintchine's formula (11)), we deduce that $\mathtt{D}(\varepsilon) \to 0$.

We next let κ go to ∞; both $\varphi(\kappa)$ and $\varphi(t+\kappa)$ tend to $0+$. By the mean value theorem, there exists some $\lambda_\kappa \in [\varphi(t+\kappa), \varphi(\kappa)]$ such that $\varphi(\kappa) - \varphi(t+\kappa) = \psi(\lambda_\kappa)t$. We deduce from (5) that $\varphi(\kappa) - \varphi(t+\kappa) = o(\varphi(\kappa))$, so

$$\varphi(t+\kappa) \sim \varphi(\kappa) \sim \lambda_\kappa\,.$$

Finally

$$\mathtt{D}(\kappa) \sim \frac{\psi(\varphi(\kappa))}{t\psi(\lambda_\kappa)} \sim \frac{1}{t}$$

and we conclude that $n\,(C_T > t) = 1/t$. By same calculation as above, we have $n\,(C_T < -t) = 1/t$, and (2) is checked.

2. Write $S_\cdot = \sup_{0\le s\le \cdot} X_s$ for the supremum of X. The process $R^{(s)} = S - X$ is known as the process reflected at the supremum; it has the strong Markov property and only takes nonnegative values. We write $n^{(s)}$ for the excursion measure of $R^{(s)}$ away from 0, and $\tau = \inf\{t > 0 : R^{(s)}_t = 0\}$ for the duration of the excursion. Assuming that (5), (6), and (10) hold, an excursion always has a finite duration and ends by a jump at time τ. One has the following description of $n^{(s)}$ (see Corollary 1 in [3]): $n^{(s)}(R^{(s)}_{\tau-} \in dx) = \overline{\Lambda}(x)dx$, and conditionally on $R^{(s)}_{\tau-} = x$, the reversed excursion $\left(R^{(s)}_{(\tau-t)-} : 0 \le t < \tau\right)$ has the same distribution as $(X_t : 0 \le t < T)$ under $\mathbb{P}^x$. The same calculation as for Theorem 4 then yields for every $t > 0$

$$n^{(s)}\left(\int_0^\tau \frac{dv}{R^{(s)}_v} > t\right) = \frac{\psi(\varphi(t))}{\varphi(t)}\,.$$

3. Write $I_\cdot = \inf_{0\le s\le \cdot} X_s$ for the infimum process of X. The process $R^{(i)} = X - I$ has again the strong Markov property and takes only nonnegative values; it is referred to as the process reflected at the infimum. We write $n^{(i)}$ for the excursion measure of $R^{(i)}$ away from 0, and $\tau = \inf\{t > 0 : R^{(i)}_s = 0\}$ for the duration of the excursion. Suppose that (5) holds. It can be proved that $n^{(i)}$ is the weak limit as $x \to 0+$ of the finite measure $x^{-1}\mathbb{P}^x\,(\cdot \circ \mathtt{k}_T)$, where $\mathtt{k}_T$ denotes the killing operator at T, the first hitting time of 0. See Chaumont [8] on page 19. It is then easy to deduce from Corollary 3(i) that, when (6) holds, then for every $t > 0$

$$n^{(i)}\left(\int_0^\tau \frac{ds}{R^{(i)}_s} > t\right) = \varphi(t)\,.$$

4. Corollary 3(ii), which is related to the explosion time of the continuous state branching process, has also some interesting consequences. Suppose here that X is a subordinator started from 0 (so $\Phi(0) = \infty$ and we work under $\mathbb{P} = \mathbb{P}^0$). The Stieltjes transform $\mathcal{S}_X$ of X is given by

$$\mathcal{S}_X(y) = \int_0^\infty \frac{ds}{X_s + y}\,, \qquad y > 0\,.$$

We deduce from Corollary 3(ii) that $\mathcal{S}_X(y) \equiv \infty$ a.s. for every $y > 0$ if $-\int_{0+} d\lambda/\psi(\lambda) = \infty$, and otherwise

$$\mathbb{P}\,(\mathcal{S}_X(y) > t) = \exp\{-y\gamma(t)\}\,.$$

5. Suppose that X drifts to ∞ but is not a subordinator, i.e. $\Phi(0) \in (0, \infty)$. For every $x > 0$, one can plainly define the conditional law $\mathbb{Q}^x = \mathbb{P}^x(\cdot \mid T = \infty)$ of the Lévy process started at x and conditioned to stay positive. It should be clear from Corollary 3(ii) that $\int_0^\infty ds/X_s = \infty$ $\mathbb{Q}^x$-a.s. if $-\int_{0+} d\lambda/\psi(\lambda) = \infty$ and otherwise

$$\mathbb{Q}^x\left(\int_0^\infty \frac{ds}{X_s} < t\right) = \frac{1 - \exp\{-x\gamma(t)\}}{1 - \exp\{-x\Phi(0)\}}\,. \tag{13}$$

We now assume moreover that $-\int_{0+} d\lambda/\psi(\lambda) < \infty$ and that 0 is regular for $(0,\infty)$; the later is equivalent to $\lim_{\lambda\to\infty} \psi(\lambda)/\lambda = \infty$ (see Corollary VII.5 in [5]). It is then easy to check that $\mathbb{Q}^x$ has a limit as $x \to 0+$. The limit law will be denoted by $\mathbb{Q}$, it can be viewed as the distribution of the Lévy process started from 0 and conditioned to stay nonnegative. See [2] for details. The law $\mathbb{Q}$ appears in particular in the decomposition of X at the instant of its overall infimum, see Millar [16]. It is easy to deduce from (13) that

$$\mathbb{Q}\left(\int_0^\infty \frac{ds}{X_s} < t\right) = \frac{\gamma(t)}{\Phi(0)}.$$

6. We refer to Bingham [7] for numerous interesting applications of the result of Lamperti to continuous state branching processes. We also mention that a different connection linking Lévy processes with no negative jumps to continuous state branching processes has recently been pointed out by Le Gall and Le Jan [15].

The reader will find complete a account on the Cauchy's principal value of the Brownian local times in Yamada [18], Yor [19] and the references therein.

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