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Self-similarity and fractional Brownian motions on Lie groups

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Abstract

The goal of this paper is to define and study a notion of fractional Brownian motion on a Lie group. We define it as at the solution of a stochastic differential equation driven by a linear fractional Brownian motion. We show that this process has stationary increments and satisfies a local self-similar property. Furthermore the Lie groups for which this self-similar property is global are characterized .

Key words: Fractional Brownian motion, Lie group.

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1 Introduction

Since the seminal works of Itô [13], Hunt [12], and Yosida [29], it is well known that the (left) Brownian motion on a Lie group $\mathbf{G}$ appears as the solution of a stochastic differential equation in the Stratonovitch sense

$$dX_t = \sum_{i=1}^{d} V_i(X_t) \circ dB_t^i, \ t \geq 0, \tag{1.1}$$

where $V_1, ..., V_d$ are left-invariant vector fields on $\mathbf{G}$ and where $(B_t)_{t\geq 0}$ is a Brownian motion; for further details on this, we also refer to [26]. In this paper we investigate the properties of the solution of an equation of the type (1.1) when the driving Brownian motion is replaced by a fractional Brownian motion with parameter H. When $H \neq 1/2$, fractional Brownian motion is neither a semimartingale neither a Markov process. Nevertheless, there have been numerous attempts to define a notion of differential equations driven by fractional Brownian motion. One dimensional differential equations can be solved using a Doss-Sussmann aproach for any values of the parameter H as in the work of Nourdin, [22]. The situation is quite different in the multidimensional case. When the Hurst parameter is greater than 1/2 existence and uniqueness of the solution are obtained by Zähle in [31] or Nualart-Rascânu in [24].

In the case $H < \frac{1}{2}$, since fractional Brownian motion with Hurst index H has a modification with sample paths α Hölder continuous for any $\alpha < H$, it falls into the framework of rough paths theory. Rough paths theory was introduced by Lyons in [19] and further developed in [20], see also [17] and the references therein. Let us give few words on it. Let x be a C^1 by steps path on $[0, 1]$ with values in $\mathbf{R}^\delta$. The geometric smooth functional over x of order p is $\mathbb{X} = (1, ..., X^p)$ where

$$X_{s,t}^i = \int_{s<s_1<...<s_i<t} dx_{s_1} \otimes \cdots \otimes dx_{s_i}, \quad 0 \leq s < t \leq 1, \quad i = 1, ..., p.$$

On geometric smooth functional Lyons consider the following Hölder distance

$$d_{\alpha,p}(X, Y) = \max_{i=1,...,[p]} \sup_{0\leq s<t\leq 1} \left(\frac{\left|X_{s,t}^i - Y_{s,t}^i\right|_{V^{\otimes i}}}{|t-s|^{i\alpha}} \right)^{1/i},$$

where $|.|$ is a compatible norm with the tensor product. The set of geometric functionals, $\Omega G(\mathbf{R}^\delta)$ is the closure of geometric smooth functional with respect to $d_{\alpha,[\frac{1}{\alpha}]}$. Let now $V_1, ..., V_\delta$ be $C^{[\frac{1}{\alpha}]+2}$ vector fields on $\mathbf{R}^n$, y be the solution of the ordinary equation

$$y(t) = y_0 + \sum_{i=1}^{\delta} \int_0^t V_i(y(t)) dx_t^i, \quad t \in [0, 1],$$

and Y be the geometric smooth functional over y. The Itô map is then defined on the set of geometric smooth functionals by $I_{V_1,...,V_\delta,y_0}(\mathbb{X}) = \mathbb{Y}$. The fundamental universal limit theorem of rough paths theory (see [20]), asserts that the Itô map is continuous with respect to $d_{\alpha,[\frac{1}{\alpha}]}$ and has an unique continuous extension to $\Omega G(\mathbf{R}^\delta)$ denoted again by $I_{V_1,...,V_\delta,y_0}$.

Let now x be an α Hölder continuous path on $[0, 1]$ taking its values in $\mathbf{R}^\delta$. Assume that there exists a geometric rough path over x, that is there exists a sequence (x^m) of paths C^1

by steps on $[0,1]$ such that the sequence $(\mathbb{X}^m)$ converges in $(\Omega G(\mathbf{R}^\delta), d_{\alpha,[\frac{1}{\alpha}]})$ to $\mathbb{X}$. Here $\mathbb{X}^m$ is the smooth rough path over x^m. Then, y the projection of $I_{V_1,\dots,V_\delta,y_0}(\mathbb{X})$ on $\mathbf{R}^d$, that is $y(t) = I_{V_1,\dots,V_d,y_0}(\mathbb{X}_{0,t})^1 + y_0, \quad t \in [0,1]$ is the unique solution of

$$y(t) = y_0 + \sum_{i=1}^{d} \int_0^t V_i(y(t)) \circ dx_t, \quad t \in [0,1],$$

in the sense of rough paths.

We can apply this to stochastic differential equations driven by fractional Brownian motions. Indeed, let now $B = (B^1, \dots, B^d)$ be a d dimensional fractional Brownian motion with Hurst parameter H, B^m be the linear interpolation of B along the dyadic subdivision of mesh m and $\mathbb{B}^m$ be the geometric smooth functional over B^m. In Coutin-Qian [8], it is proved that the sequence $(\mathbb{B}^m)$ converges in $(\Omega G(\mathbf{R}^d), d_{\alpha,[\frac{1}{\alpha}]})$ if and only if $H > 1/4$. Therefore a notion of solution is well-defined for $H > \frac{1}{4}$. If $H = \frac{1}{2}$, we recover stochastic differential equations driven by Brownian motions in Stratonovitch sense.

Let us observe that, for solving linear stochastic differential equations driven by fractional Brownian motions, an alternative approach based on the Skorohod integral ([6]) could be used (see [23]). But for geometric purposes, the pathwise approach given by rough paths theory is much more tractable, due to the simple form of the change of variable formula (the theory is invariant by the action of diffeomorphism groups). Actually, even in the case of Brownian motion, these are the Stratonovitch integrals that are used.

The paper is organized as follows.

In a first section, we show existence and uniqueness for the solution of an equation of the type (1.1) when $(B_t)_{t\geq 0}$ is a fractional Brownian motion with parameter $H > \frac{1}{4}$. The solution is shown to have stationary increments. We also check that the solution is invariant in law by isometries and finally, the Taylor development of the solution is obtained.

In the second section, we study the scaling properties of the solution. In the spirit of the notion of asymptotic self-similarity studied by Kunita [14], [15], we show that the fractional Brownian motion on the group is asymptotically self-similar with parameter H. After that, we characterize the groups for which the scaling property is global: such groups are necessarily simply connected, nilpotent and stratified; that is are Carnot groups.

To simplify the presentation of our results, we mainly worked in the setting of Lie groups of matrices. Nevertheless all our results extend to general Lie groups.

2 Fractional Brownian motion on a Lie group

Let us first recall that a d-dimensional fractional Brownian motion with Hurst parameter $H \in (0,1)$ is a Gaussian process B

$$B_t = (B_t^1, ..., B_t^d), \ t \geq 0,$$

where $B^1, ..., B^d$ are d independent centered Gaussian processes with covariance function

$$R(t,s) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right), \quad (s,t) \in [0,+\infty[^2.$$

It can be shown that such a process admits a continuous version whose paths are Hölder p continuous, $p < H$. Let us observe that for $H = \frac{1}{2}$, B is a d dimensional Brownian motion.

On $\mathbb{M}_D(\mathbf{R})$, the set of $D \times D$ real matrices , the application exp is defined by its series expansion :

$$\exp M = \sum_{n=0}^{\infty} \frac{1}{n!} M^n, \quad M \in \mathbb{M}_D(\mathbf{R}).$$

Let $\mathbf{G}$ be a finite-dimensional ($\dim \mathbf{G} = d$) connected and closed Lie group of matrices with Lie algebra $\mathfrak{g}$. That is

$$\mathfrak{g} = \{M \in \mathbb{M}_D(\mathbf{R}), \quad \exp tM \in \mathbf{G}, \ \ \forall t \in \mathbf{R}\}.$$

The set $\mathfrak{g}$ is a finite dimensional sub vector space of $\mathbb{M}_D(\mathbf{R})$, stable with respect to the Lie bracket,

$$[M,N] = MN - NM, \ \ (M,N) \in \mathbb{M}_D(\mathbf{R}).$$

It is also the tangent space of $\mathbf{G}$ at point $1_{\mathbf{G}}$. We consider a basis $(V_1, ..., V_d)$ of $\mathfrak{g}$.

If $(B_t^1, ..., B_t^d)_{t\geq 0}$ is a d-dimensional fractional Brownian motion in $\mathbf{R}^d$ with Hurst parameter $H \in (0,1)$. The process $B^{\mathfrak{g}}$ defined by

$$B_t^{\mathfrak{g}} = \sum_{i=1}^{d} B_t^i V_i, \ \ t \leq 0$$

shall be called the canonical fractional Brownian motion on $\mathfrak{g}$ with respect to the basis $(V_1, ..., V_d)$. Note that $B^{\mathfrak{g}}$ can be seen as a process taking its values in $\mathbf{R}^{D^2}$.

In the remainder of this section, we assume now $H > \frac{1}{4}$.

Theorem 2.1. *The equation*

$$dX_t = X_t \circ dB_t^{\mathfrak{g}}, \qquad X_0 = 1_{\mathbf{G}} \tag{2.2}$$

has a unique solution in $\mathbf{G}$ in the sense of rough paths of [20], denoted by $\mathbb{X} = (1, X^1, X^2, X^3)$. The first level of this solution $(X_t)_{t\geq 0} = (X_{0,t}^1)_{t\geq 0}$ satisfies for every $s \geq 0$, $(X_s^{-1} X_{t+s})_{t\geq 0} =^{law} (X_t)_{t\geq 0}$. The process $(X_t)_{t\geq 0}$ shall be called a left fractional Brownian motion with parameter H on $\mathbf{G}$ with respect to the basis $(V_1, ..., V_d)$.

Proof. We first show the existence and the uniqueness of a solution in $\mathbf{G}$ according to the proof of (35.8) of [26] in the Brownian case. Without loss of generality, we can work on the time interval $[0,1]$. First, according to Theorem 5 of [8], Equation (2.2) has a unique solution with finite p variation for $p > \frac{1}{H}$, taking its values in $\mathbb{M}_D(\mathbf{R})$. It is denoted by $\mathbb{X} = (1, X, X^2, X^3)$.

Secondly, let $B^{\mathfrak{g},m}$ be the sequence of linear interpolation of $B^{\mathfrak{g}}$ along the dyadic subdivision of mesh m; that is if $t_i^m = i2^{-m}$ for $i = 0, ..., 2^m$; then for $t \in [t_i^m, t_{i+1}^m)$,

$$B_t^{\mathfrak{g},m} = B_{t_{im}}^{\mathfrak{g}} + \frac{t - t_{im}}{t_{i+1}^m - t_i^m}(B_{t_{i+1}^m}^{\mathfrak{g}} - B_{t_i^m}^{\mathfrak{g}}).$$

Let us now denote X^m the solution of (2.2) where $B^{\mathfrak{g}}$ is replaced by $B^{\mathfrak{g},m}$, that is

$$dX_t^m = \sum_{k=1}^d X_t^m dB_t^{k,m} V_k.$$

It is easily seen that for $t \in [t_{n-1}^m, t_n^m)$, $n = 0, ..., 2^m - 1$,

$$X_t^m = \exp\left(2^m(t - t_{n-1}^m)\sum_{k=1}^d (B_{t_{n-1}^m}^k - B_{t_{n-1}^m}^k)V_k\right)\cdots\exp\left(\sum_{k=1}^d (B_{t_1^m}^k - B_{t_0^m}^k)V_k\right). \tag{2.3}$$

Since for $k = 1, ..., d$, $V_k \in \mathfrak{g}$ we have $\sum_{k=1}^d (B_{t_i^m}^k - B_{t_{i-1}^m}^k)V_k \in \mathfrak{g}, \quad i = 1, ..., 2^m$ and therefore $\exp\left(\sum_{k=1}^d (B_{t_i^m}^k - B_{t_{i-1}^m}^k)V_k\right) \in \mathbf{G}$. Thus, X^m takes its values in $\mathbf{G}$. Now, from [8], $\mathbb{X}^m$, the geometric functional build on X^m, converges to $\mathbb{X}$ for the distance of $1/p$ Hölder in $\mathbf{R}^{D^2}$:

$$\lim_{m\to\infty} \sup_{0\le s<t\le 1} \frac{\|X_{s,t}^{m,i} - X_{s,t}^i\|_i^{\frac{1}{i}}}{|t-s|^\alpha} = 0,$$

where $\|.\|_i$ is any norm on $\mathbb{M}_D(\mathbf{R})^{\otimes i}$. Since in finite dimension all norms are equivalents and since the group $\mathbf{G}$ is closed, we conclude that $X_{0,.}^1 = \lim_{m\to\infty} X_{0,.}^m$ takes is values in $\mathbf{G}$.

We now show that for every $s \ge 0$, the processes $((X_s)^{-1}X_{t+s}, \ t \ge 0)$ and $(X_t, \ t \ge 0)$ have the same law. Let us fix $s \ge 0$. Once time again, the idea is to use a linear interpolation along the dyadic subdivision of $[0,1]$ of mesh m and we keep the previous notations. First, let us observe that for $t \ge s$,

$$X_t = X_s + \int_s^t X_u \circ dB_u^{\mathfrak{g}}. \tag{2.4}$$

Let us now denote $(X_t^{m,s})_{s\le t\le s+1}$ the solution of (2.4) where $B^{\mathfrak{g}}$ is replaced by $(B_{s+t}^{\mathfrak{g},m})_{0\le t\le 1}$. Therefore, for $t \in [t_{n-1}^m, t_n^m)$,

$$X_{t+s}^{m,s} = X_s \exp\left(2^m(t - t_{n-1}^m)\sum_{k=1}^d (B_{s+t_n^m}^k - B_{s+t_{n-1}^m}^k)V_k\right)\cdots\exp\left(\sum_{k=1}^d (B_{s+t_1^m}^k - B_{s+t_0^m}^k)V_k\right).$$

By using the stationarity of the increments of the Euclidean fractional Brownian motion, we get therefore:

$$((X_s^m)^{-1}X_{t+s}^{m,s})_{0\le t\le 1} =^{law} (X_t^m)_{0\le t\le 1}.$$

Using the Wong-Zakai theorem (Theorem 5 of [8]) and passing to the limit, we obtain that for every $s \geq 0$, $(X_s^{-1}X_{t+s})_{t\geq 0} =^{law} (X_t)_{t\geq 0}$.

$\square$

Remark 2.2. *In the same way, we call the solution of the differential equation*

$$dX_t = \circ dB_t^{\mathfrak{g}} X_t, \qquad X_0 = \mathbf{1_G}.$$

a right fractional Brownian motion on $\mathbf{G}$. *It is easily seen that if* $(X_t)_{t\geq 0}$ *is a left fractional Brownian motion on* $\mathbf{G}$, *then* $(X_t^{-1})_{t\geq 0}$ *is a right fractional Brownian motion on* $\mathbf{G}$.

Let us now turn to some examples.

Example 2.3. *The first basic example is the circle. Let*

$$\mathbf{S}^1 = \{z \in \mathbb{C}, \mid z \mid = 1\}.$$

The Lie algebra of $\mathbf{S}^1$ *is* $\mathbb{R}$ *and is generated by* $\frac{\partial}{\partial \theta}$ *and the fractional Brownian motion on* $\mathbf{S}^1$ *is given by*

$$X_t = e^{iB_t}, \quad t \geq 0,$$

where $(B_t)_{t\geq 0}$ *is a fractional Brownian motion on* $\mathbb{R}$.

Example 2.4. *Let us consider the Lie group* $\mathbf{SO}(3)$, *i.e. the group of* 3×3, *real, orthogonal matrices of determinant* 1. *Its Lie algebra* , $\mathfrak{so}(3)$, *consists of* 3×3, *real, skew-adjoint matrices. A basis of* $\mathfrak{so}(3)$ *is formed by*

$$V_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; V_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \; V_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

A left fractional Brownian motion on $\mathbf{SO}(3)$ *is therefore given by the solution of the linear equation*

$$dX_t = X_t \circ \begin{pmatrix} 0 & dB_t^1 & dB_t^3 \\ -\,dB_t^1 & 0 & dB_t^2 \\ -\,dB_t^3 & -dB_t^2 & 0 \end{pmatrix}, \quad X_0 = Id_{\mathbf{R}^3}.$$

This notion of fractional Brownian motion on a Lie group is invariant by isometries, so that the law is invariant by an orthonormal change of basis. More precisely, let us consider the scalar product on $\mathfrak{g}$ that makes the basis $V_1, ..., V_d$ orthonormal. This scalar product defines a Riemannian structure on $\mathbf{G}$ for which the left action is an action by isometries. We have the following proposition:

Proposition 2.5. *Let* $\Psi : \mathbf{G} \to \mathbf{G}$ *be a Lie group morphism such that* $d\Psi_{1_\mathbf{G}}$ *(differential of* Ψ *at* $1_\mathbf{G}$*) is an isometry and let* $(X_t)_{t\geq 0}$ *be the left fractional Brownian motion on* $\mathbf{G}$ *as defined in Theorem 2.1. We have:*

$$(\Psi(X_t))_{t\geq 0} =^{law} (X_t)_{t\geq 0}.$$

Proof. As in the proof of Theorem 2.1., let us introduce the approximation defined for $t \in [t^m_{n-1}, t^m_n)$, $n = 0, ..., 2^m - 1$, by

$$X^m_t = \exp\left(2^m(t - t^m_{n-1})\sum_{k=1}^d (B^k_{t^m_{n-1}} - B^k_{t^m_{n-1}})V_k\right)\cdots\exp\left(\sum_{k=1}^d (B^k_{t^m_1} - B^k_{t^m_0})V_k\right).$$

Since Ψ is a Lie group morphism, we have for every $u \in \mathfrak{g}$, $\Psi(e^u) = e^{d\Psi_{1_{\mathbf{G}}}(u)}$, therefore for $t \in [t^m_{n-1}, t^m_n)$, $n = 0, ..., 2^m - 1$,

$$\Psi(X^m_t) = \exp\left(2^m(t - t^m_{n-1})\sum_{k=1}^d (B^k_{t^m_{n-1}} - B^k_{t^m_{n-1}})d\Psi_{1_{\mathbf{G}}}(V_k)\right)\cdots\exp\left(\sum_{k=1}^d (B^k_{t^m_1} - B^k_{t^m_0})d\Psi_{1_{\mathbf{G}}}(V_k)\right).$$

Thus, because of the orthogonal invariance of the Euclidean fractional Brownian motion,

$$(\Psi(X^m_t))_{t\geq 0} =^{law} (X^m_t)_{t\geq 0},$$

and the result follows from Theorem 5 of [8]. $\square$

Remark 2.6. *If* $\mathbf{G}$ *is compact then there exists a bi-invariant Riemann metric and so, if* $(X_t)_{t\geq 0}$ *denotes a left fractional Brownian motion for this bi-invariant metric, from the previous proposition, we get that for every* $g \in \mathbf{G}$,

$$(gX_tg^{-1})_{t\geq 0} =^{law} (X_t)_{t\geq 0}.$$

If the group $\mathbf{G}$ is nilpotent then we have a closed formula for the left fractional Brownian motion on $\mathbf{G}$ that extends the well-known formula for the Brownian motion on a nilpotent group (see by e.g. [1], [5] or [28]).

Let us introduce some notations: For $k \geq 1$,

- $$\Delta^k[s,t] = \{(t_1, ..., t_k) \in [s,t]^k, t_1 < ... < t_k\}, \quad s < t;$$
- If $I = (i_1, ...i_k) \in \{1, ..., d\}^k$ is a word with length k,
$$\int_{\Delta^k[0,t]} \circ dB^I = \int_{0<t_1<...<t_k\leq t} \circ dB^{i_1}_{t_1} ... \circ dB^{i_k}_{t_k};$$
- We denote $\mathfrak{S}_k$ the group of the permutations of the index set $\{1, ..., k\}$ and if $\sigma \in \mathfrak{S}_k$, we denote for a word $I = (i_1, ..., i_k)$, $\sigma \cdot I$ the word $(i_{\sigma(1)}, ..., i_{\sigma(k)})$;
- If $I = (i_1, ..., i_k) \in \{1, ..., d\}^k$ is a word, we denote by V_I the Lie commutator defined by
$$V_I = [V_{i_1}, [V_{i_2}, ..., [V_{i_{k-1}}, V_{i_k}]...];$$
- If $\sigma \in \mathfrak{S}_k$, we denote $e(\sigma)$ the cardinality of the set
$$\{j \in \{1, ..., k-1\}, \sigma(j) > \sigma(j+1)\};$$

- Finally, if $I=(i_1,...,i_k)\in\{1,...,d\}^k$ is a word

$$\Lambda_I(B)_t=\sum_{\sigma\in\mathfrak{S}_k}\frac{(-1)^{e(\sigma)}}{k^2\begin{pmatrix}k-1\\ e(\sigma)\end{pmatrix}}\int_{\Delta^k[0,t]}\circ dB^{\sigma^{-1}\cdot I}.$$

Proposition 2.7. *Assume that* $\mathbf{G}$ *is a nilpotent group then:*

$$X_t=\exp\left(\sum_{k=1}^{+\infty}\sum_{I\in\{1,\dots,d\}^k}\Lambda_I(B)_tV_I\right),\quad t\geq 0,$$

where the above sum is actually finite and where $(X_t)_{t\geq 0}$ *is the left fractional Brownian motion defined as in Theorem 2.1.*

Proof. Let $B^{\mathfrak{g},m}$ be the sequel of linear interpolation of $B^{\mathfrak{g}}$ along the dyadic subdivision of mesh m. Let us now denote X^m the solution of (2.2) where $B^{\mathfrak{g}}$ is replaced by $B^{\mathfrak{g},m}$. As already seen in (2.3) for $t\in[t^m_{n-1},t^m_n)$,

$$X^m_t=\exp\left(2^m(t-t^m_{n-1})\sum_{k=1}^{d}(B^k_t-B^k_{t^m_{n-1}})V_k\right)\cdots\exp\left(\sum_{k=1}^{d}(B^k_{t^m_1}-B^k_{t^m_0})V_k\right).$$

Now we use the Baker-Campbell-Hausdorff formula in nilpotent Lie groups (see [1], [5], [27]) to write the previous product of exponentials under the form

$$X^m_t=\exp\left(\sum_{k=1}^{+\infty}\sum_{I=(i_1,\dots,i_k)}\Lambda_I(B^m)_tV_I\right),$$

where

$$\Lambda_I(B^m)_t=\sum_{\sigma\in\mathfrak{S}_k}\frac{(-1)^{e(\sigma)}}{k^2\begin{pmatrix}k-1\\ e(\sigma)\end{pmatrix}}\int_{\Delta^k[0,t]}\circ dB^{m,\sigma^{-1}\cdot I}.$$

>From [8] Theorem 2, in the distance of p variation, with $p>\frac{1}{H}$, and if the length of the word I is less than 3,

$$(\Lambda_I(B^m)_t)_{t\leq 0}\to_{m\to+\infty}\Lambda_I(B)_{t\leq 0}. \tag{2.5}$$

By using Theorem 3.1.3 of [20], the convergence in (2.5) can actually be extended to words of any length. Therefore from [8],

$$X_t=\exp\left(\sum_{k=1}^{+\infty}\sum_{I=(i_1,\dots,i_k)}\Lambda_I(B)_tV_I\right),\ t\geq 0.$$

$\square$

Remark 2.8. *Since* exp *is a local diffeomorphism, there exists a positive strictely random variable* T *such that Proposition 2.7 may be true on non nilpotent group for* $t\leq T$ *(see [3] and [5] for the Brownian case).*

Example 2.9. *In a two-step nilpotent group, that if is all brackets with length more than two are zero, we have therefore*

$$X_t = \exp\left(\sum_{i=1}^{d} B_t^i V_i + \frac{1}{2}\sum_{1\le i<j\le d}\left(\int_0^t B_s^i \circ dB_s^j - B_s^j \circ dB_s^i\right)[V_i, V_j]\right), \quad t \le 0.$$

Example 2.10. *The Heisenberg group $\mathbb{H}$ is the set of 3×3 matrices:*

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbf{R}.$$

The Lie algebra of $\mathbb{H}$ is spanned by the matrices

$$D_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; D_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } D_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for which the following equalities hold

$$[D_1, D_2] = D_3, \; [D_1, D_3] = [D_2, D_3] = 0.$$

Consider now the solution of the equation

$$dX_t = X_t(D_1 dB_t^1 + D_2 dB_t^2 + D_3 dB_t^3), \quad X_0 = 1,$$

where $(B_t^1, B_t^2, B_t^3)_{t\ge 0}$ is a three-dimensional fractional Brownian motion with Hurst parameter $H > \frac{1}{4}$. It is easily seen that

$$X_t = \begin{pmatrix} 1 & B_t^1 & \frac{1}{2}\left(B_t^1 B_t^2 + 2B_t^3 + \int_0^t B_s^1 \circ dB_s^2 - B_s^2 \circ dB_s^1\right) \\ 0 & 1 & B_t^2 \\ 0 & 0 & 1 \end{pmatrix}, \quad t \ge 0$$

is a fractional Brownian motion on $\mathbb{H}$.

We conclude this section with the Taylor development of fractional Brownian motion on a Lie group, as in [2]. Let us observe that similar type of development is obtained in [10].

Proposition 2.11. *Under assumptions of Theorem 2.1, almost surely,*

$$X_t = 1 + \sum_{k=1}^{\infty} \sum_{I\in\{1,\dots,d\}^k} \int_{\Delta^k[0,t]} \circ dB^I V_{i_1} \dots . V_{ik},$$

where the convergence of the series of functions of the right member holds for the p variation topology and the uniform norm of function on compact of $\mathbf{R}^+$.

Proof. We work on $[0,1]$ without loss of generality. Let $B^{\mathfrak{g},m}$ denote again the sequel of linear interpolation of $B^{\mathfrak{g}}$ along the dyadic subdivision of mesh m; that is if $t_i^m = i2^{-m}$ for $i = 0, ..., 2^m$; then for $t \in [t_i^m, t_{i+1}^m)$,

$$B_t^{\mathfrak{g},m} = B_{t_i^m}^{\mathfrak{g}} + \frac{t - t_{i^m}}{t_{i+1}^m - t_i^m}(B_{t_{i+1}^m}^{\mathfrak{g}} - B_{t_i^m}^{\mathfrak{g}}).$$

Let $p > 1/H$, according to Theorem 4 of [8], $\left(1, (\int_{\Delta^k[s,t]} dB^{m,I})_{I,|I|\leq 3}\right)_{0\leq s\leq t\leq 1}$ converges in the distance of the p variation to $\left(1, (\int_{\Delta^k[s,t]} dB^{m,I})_{I,|I|\leq 3}\right)_{0\leq s\leq t\leq 1}$. That means that, almost surely,

$$\sup_{\Pi} \sum_j \sum_{k=0}^{3} \sum_{I,|I|=k} \left| \int_{\Delta^k[t_{j-1},t_j]} dB^{m,I} - \int_{\Delta^k[t_{j-1},t_j]} \circ dB^I \right|^{p/k},$$

converges to 0 when m goes to infinity, where $\sup_\pi$ runs over all finite subdivisions of $[0,1]$. According to Proposition 3.3.3 of [20] page 51, almost surely there exists a subsequence (m_l) and a control ω such that for all $0 \leq s \leq t \leq 1$, I word of length $|I| \leq 3$, $k \in \mathbb{N}$,

$$\left| \int_{\Delta^k[s,t]} dB^{m_l,I} \right| \leq \frac{1}{\beta(k:p)!}\omega(s,t)^{k/p},$$
$$\left| \int_{\Delta^k[s,t]} \circ dB^I \right| \leq \frac{1}{\beta(k:p)!}\omega(s,t)^{k/p},$$
$$\left| \int_{\Delta^k[s,t]} dB^{m_l,I} - \int_{\Delta^k[s,t]} \circ dB^I \right| \leq \frac{1}{\beta(k:p)!}2^{-l}\omega(s,t)^{k/p},$$

where $k : p = [k/p]$ and $\beta > p^2(1 + \sum_{r=3}^{\infty}(\frac{2}{r-2})^{([p]+1)/p})$. Recall that ω is a control if and only if ω is continuous, super-additive on $\{0 \leq s \leq t \leq 1\}$ taking its values in $[0, +\infty[$ and $\omega(t,t) = 0,\ t \in [0,1]$. Moreover according Theorem 3.1.3, for all $0 \leq s \leq t \leq 1$, I word, $k \in \mathbb{N}$,

$$\begin{aligned}
&\left| \int_{\Delta^k[s,t]} dB^{m_l,I} \right| \leq \frac{1}{\beta(k:p)!}\omega(s,t)^{k/p},\\
&\left| \int_{\Delta^k[s,t]} \circ dB^I \right| \leq \frac{1}{\beta(k:p)!}\omega(s,t)^{k/p},\\
&\left| \int_{\Delta^k[s,t]} dB^{m_l,I} - \int_{\Delta^k[s,t]} \circ dB^I \right| \leq \frac{1}{\beta(k:p)!}2^{-l}\omega(s,t)^{k/p}.
\end{aligned} \tag{2.6}$$

Then the processes Y^l, $l \in \mathbb{N}$, and Y,

$$Y_t = 1 + \sum_{k=1}^{\infty} \sum_{I=(i_1,...,i_k)} \int_{\Delta^k[0,t]} \circ dB^I V_{i_1} \cdots\cdot V_{ik},$$
$$Y_t^l = 1 + \sum_{k=1}^{\infty} \sum_{I=(i_1,...,i_k)} \int_{\Delta^k[0,t]} dB^{m_l,I} V_{i_1} \cdots\cdot V_{ik}, \quad t \in [0,1],$$

are well defined. First, since $B^{\mathfrak{g},m}$ is C^1 by piece, Y^l is the solution of

$$dX_t = X_t dB^{\mathfrak{g},m_l}, \quad X_0 = 1,$$

and therefore $Y^l = X^{m_l}, \quad l \in \mathbb{N}$. Secondly, thanks to Theorem 5 of [8], $X^{m_l} = Y^l$ converges to X in the distance of p variation, for $p > 1/H$. We conclude that $Y = X$, almost surely. □

3 Self-similarity of a fractional Brownian motion on a Lie group

Recall that for the Euclidean fractional Brownian motion, we have

$$(B^1_{ct}, ..., B^d_{ct})_{t\geq 0} =^{law} (c^H B^1_t, ..., c^H B^d_t)_{t\geq 0}$$

This property is called the scaling property of the fractional Brownian motion. In this section, we are going to study scaling properties of fractional Brownian motions on a Lie group.

As in the previous section, let $\mathbf{G}$ be a connected Lie and closed group (of matrices) with Lie algebra $\mathfrak{g}$. Let $V_1, ..., V_d$ be a basis of $\mathfrak{g}$ and denote by $(X_t)_{t\geq 0}$ the solution of the equation

$$dX_t = X_t \left(\sum_{i=1}^d V_i \circ dB^i_t \right), \quad X_0 = 1_{\mathbf{G}},$$

where $(B_t)_{t\geq 0}$ is a d-dimensional fractional Brownian with Hurst parameter $H > \frac{1}{4}$.

3.1 Local self-similarity

First, we notice that for $(X_t)_{t\geq 0}$ we always have an asymptotic scaling property in the following sense:

Proposition 3.1. *Under the assumption of Theorem 2.1, when $c \to 0$, $c > 0$, the sequence of processes $\left(\frac{1}{c}(X_{ct} - 1_{\mathbf{G}})\right)_{0\leq t}$ converges in law to β a one dimensional Brownian motion. Moreover, if $f : \mathbf{G} \to \mathbb{R}$ is a C^2 map such that $\sum_{i=1}^d (V_i f)(1_{\mathbf{G}})^2 \neq 0$; then, when $c \to 0$, $c > 0$, the sequence of processes $\left(\frac{1}{c^H}\left(f(X_{ct}) - f(1_{\mathbf{G}})\right)\right)_{0\leq t\leq 1}$ converges in law to $(a\beta_t)_{0\leq t\leq 1}$ where $(\beta_t)_{t\geq 0}$ is a one-dimensional fractional Brownian motion and*

$$a = \sqrt{\sum_{i=1}^d (V_i f)(1_{\mathbf{G}})^2}.$$

Proof. >From Proposition 2.11,

$$\frac{X_{ct} - 1_G}{c^H} = \frac{1}{c^H} B^{\mathfrak{g}}_{ct} + \frac{1}{c^H} R(ct) \quad t \geq 0,$$

where the remainder term R is given by

$$R(t) = \sum_{k=2}^{\infty} \sum_{I \in \{1,...,d\}^k} \int_{\Delta^k[0,t]} \circ dB^I V_{i_1} ... V_{i_k} \in \mathfrak{g},$$

Let us observe that the convergence of the above series stems from (2.6). Using the scaling property of fractional Brownian motion we get that $(c^{-H}B^{\mathfrak{g}}_{ct}, c^{-H}R(ct))_{t\geq 0}$ has the same law as

$$\left(B^{\mathfrak{g}}_t, c^H \sum_{k=2}^{\infty} c^{(k-2)H} \sum_{I=(i_1,...,i_k)} \int_{\Delta^k[0,t]} dB^I V_{i_1}...V_{i_k}\right)_{t\leq 0}.$$

Again, from (2.6), almost surely,

$$\lim_{c\to 0} \sup_{t\in[0,1]} \|c^H \sum_{k=2}^{\infty} c^{(k-2)H} \sum_{I=(i_1,...,i_k)} \int_{\Delta^k[0,t]} dB^I V_{i_1}...V_{i_k}\|_{\mathfrak{g}} = 0.$$

Thus, $(c^{-H}B^{\mathfrak{g}}_{ct}, c^{-H}R(ct))_{t\geq 0}$ converges in law to $(B^{\mathfrak{g}}_t, 0)_{t\geq 0}$ when c goes to 0 and $(\frac{X_{ct}-1_G}{c^H})_{t\geq 0}$ converges in law to $(B^{\mathfrak{g}}_t)_{t\geq 0}$ when c goes to 0.

Now, by using the Taylor expansion of f between X_t and $1_{\mathbf{G}}$, two points of $\mathbb{M}(\mathbf{R}^D)$ seen as $\mathbf{R}^{\otimes D^2}$,

$$\frac{f(X_{ct}) - f(1_{\mathbf{G}})}{c^H} = \sum_{i,j=1}^{D} \frac{\partial f}{\partial x_{i,j}}(1_G)\frac{(X_{ct}-1)^{i,j}}{c^H} + \sum_{i,j,k,l=1}^{D} c^{-2H} R^{i,j}(ct) R^{k,l}(ct) \int_0^1 \frac{\partial f^2}{\partial x_{k,l}}(\theta X_s) d\theta.$$

Since, $(c^{-H}B^{\mathfrak{g}}_{ct}, c^{-H}R(ct))_{t\geq 0}$ converges in law to $(B^{\mathfrak{g}}_t, 0)_{t\geq 0}$ when c goes to 0, and $\frac{\partial^2 f}{\partial x_{i,j}\partial x_{k,l}}$ are continuous, we deduce that when $c \to 0$, $c > 0$, the sequence of processes $\left(\frac{1}{c^H}\left(f(X_{ct}) - f(1_{\mathbf{G}})\right)\right)_{0\leq t\leq 1}$ converges in law to $(a\beta_t)_{0\leq t\leq 1}$ where $(\beta_t)_{t\geq 0}$ is a one-dimensional fractional Brownian motion and

$$a = \sqrt{\sum_{i=1}^{d} (V_i f)(1_{\mathbf{G}})^2}.$$

□

Remark 3.2. *Slightly more generally, by using the Taylor expansion proved in (2.6), we obtain in the same way: Let $f : \mathbf{G} \to \mathbb{R}$ be a smooth map such that there exist $k \geq 1$ and $(i_1, ..., i_k) \in \{1, ..., d\}^k$ that satisfy*

$$(V_{i_1} \cdots V_{i_k} f)(1_{\mathbf{G}}) \neq 0.$$

Denote n the smallest k that satisfies the above property. Then, when $c \to 0$, $c > 0$, the sequence of processes $\left(\frac{1}{c^{nH}}\left(f(X_{ct}) - f(1_{\mathbf{G}})\right)\right)_{0\leq t\leq 1}$ converges in law to $(\beta_t)_{0\leq t\leq 1}$ where $(\beta_t)_{t\geq 0}$ is a process such that

$$(\beta_{ct})_{t\geq 0} =^{law} (c^{nH}\beta_t)_{t\geq 0}.$$

3.2 Global self-similarity

Despite the local self-similar property, as we will see, in general there is no global scaling property for the fractional Brownian motion on a Lie group. Let us first briefly discuss what should be a good notion of scaling in a Lie group (see also [14] and [15]). If we can find a family a map Δ_c, $c > 0$, such that

$$(X_{ct})_{t\geq 0} =^{law} (\Delta_c X_t)_{t\geq 0},$$

first of all it is natural to require that the map $c \to \Delta_c$ is continuous and $\lim_{c\to 0}\Delta_c = 1_{\mathbf{G}}$. Then by looking at $(X_{c_1c_2t})_{t\geq 0}$, we will also naturally ask that $\Delta_{c_1c_2} = \Delta_{c_1}\circ\Delta_{c_2}$. Finally, since $(X_s^{-1}X_{t+s})_{t\geq 0} =^{law} (X_t)_{t\geq 0}$, we will also ask that Δ_c is a Lie group automorphism.

The following theorem shows that the existence of such a family Δ_c on $\mathbf{G}$ only holds if the group is $(\mathbf{R}^d,+)$. This is partly due to the following lemma of Lie group theory that says that the existence of a dilation on $\mathbf{G}$ imposes strong topological and algebraic restrictions:

Lemma 3.3. *(See [16]) Assume that there exists a Lie group automorphism $\Psi : \mathbf{G} \to \mathbf{G}$ such that $d\Psi_{1_{\mathbf{G}}}$ (differential of Ψ at $1_{\mathbf{G}}$) has all its eigenvalues of modulus > 1, then $\mathbf{G}$ is a simply connected nilpotent Lie group.*

We can now show:

Theorem 3.4. *There exists a family of Lie group automorphisms $\Delta_t : \mathbf{G} \to \mathbf{G}$, $t > 0$, such that:*

1. *The map $t \to \Delta_t$ is continuous and $\lim_{t\to 0}\Delta_t = 1_{\mathbf{G}}$;*
2. *For $t_1, t_2 \geq 0$, $\Delta_{t_1t_2} = \Delta_{t_1}\circ\Delta_{t_2}$;*
3. *$(X_{ct})_{t\geq 0} =^{law} (\Delta_c X_t)_{t\geq 0}$;*

if and only if the group $\mathbf{G}$ is isomorphic to $(\mathbf{R}^d,+)$.

Proof. If $\mathbf{G}$ is isomorphic to $(\mathbf{R}^d,+)$, $(X_t)_{t\geq 0}$ is a Euclidean fractional Brownian motion and the result is trivial.

We prove now the converse statement. Let us first show that the existence of the family $(\Delta_t)_{t>0}$ implies that $\mathbf{G}$ is a simply connected nilpotent Lie group.

Let us denote

$$\delta_c = d\Delta_c(1_{\mathbf{G}}),$$

the differential map of Δ_c at $1_{\mathbf{G}}$ and observe that δ_c is a Lie algebra automorphism $\mathfrak{g} \to \mathfrak{g}$, that satisfies $\Delta_c(\exp M) = \exp\delta_c(M)$, $\quad M \in \mathfrak{g}$. The map $f : t \to \delta_{e^t}$ is a map from $\mathbf{R}$ onto the set of linear maps $\mathfrak{g} \to \mathfrak{g}$ that is continuous. We have furthermore the property

$$f(t+s) = f(t)f(s).$$

Consequently there exists a linear map $\phi : \mathfrak{g} \to \mathfrak{g}$ such that

$$\delta_c = \mathrm{Exp}(\phi \ln c),\ c > 0,$$

where Exp denotes here the exponential of linear maps (and not the exponential map $\mathfrak{g} \to \mathbf{G}$ which is denoted exp). Since δ_c is a linear application on the finite dimensional vector space $\mathfrak{g}$, for $c^* > 1$ close enough from 1,

$$\phi \ln c^* = Ln(I_{\mathcal{L}(\mathfrak{g})} + \delta_{c^*} - I_{\mathcal{L}(\mathfrak{g})}),$$

where $\mathcal{L}(\mathfrak{g})$ is the set of linear functions from $\mathfrak{g}$ into itself, $I_{\mathcal{L}(\mathfrak{g})}$ is the identity map from $\mathfrak{g}$ into itself, and for $\varphi \in \mathcal{L}(\mathfrak{g})$ in a neighbour of 0 $Ln(I_{\mathcal{L}(\mathfrak{g})} + \varphi) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\varphi^{\circ(n)}$. If α is an eigenvalue of δ_{c^*}, then $\ln \alpha$ is an eigenvalue of $\phi \ln c^*$. Let us furthermore observe that if $\lambda \in \mathbf{Sp}(\phi)$ is an eigenvalue of ϕ, then $e^{\lambda \ln c}$ is an eigenvalue of δ_c, therefore $\Re\lambda > 0$ because $\lim_{c\to 0} \Delta_c = 1_{\mathbf{G}}$ which implies $\lim_{c\to 0} \delta_c = 0$. Therefore, $|\alpha| > 1$ from Lemma 3.3 with $\Psi = \delta_{c^*}$, $\mathbf{G}$ has to be a simply connected nilpotent Lie group.

We deduce from Proposition 2.7 that

$$X_t = \exp\left(\sum_{k=1}^{+\infty} \sum_{I\in\{1,\dots,d\}^k} \Lambda_I(B)_t V_I\right), \quad t \geq 0,$$

where the above sum is actually finite and

$$\Lambda_I(B)_t = \sum_{\sigma\in\mathfrak{S}_k} \frac{(-1)^{e(\sigma)}}{k^2 \begin{pmatrix} k-1 \\ e(\sigma) \end{pmatrix}} \int_{\Delta^k[0,t]} \circ dB^{\sigma^{-1}\cdot I}. \tag{3.7}$$

Due to the assumption that

$$(X_t)_{t\geq 0} =^{law} (\Delta_c X_{\frac{t}{c}})_{t\geq 0},$$

we deduce that

$$\left(\exp\left(\sum_{k=1}^{+\infty} \sum_{I\in\{1,\dots,d\}^k} \Lambda_I(B)_t V_I\right)\right)_{t\geq 0} =^{law} \left(\exp\left(\sum_{k=1}^{+\infty} \sum_{I\in\{1,\dots,d\}^k} \Lambda_I(B)_{\frac{t}{c}} (\delta_c V)_I\right)\right)_{t\geq 0}.$$

But since the group $\mathbf{G}$ is nilpotent and simply connected the exponential map is a diffeomorphism, therefore

$$\left(\sum_{k=1}^{+\infty} \sum_{I\in\{1,\dots,d\}^k} \Lambda_I(B)_t V_I\right)_{t\geq 0} =^{law} \left(\sum_{k=1}^{+\infty} \sum_{I\in\{1,\dots,d\}^k} \Lambda_I(B)_{\frac{t}{c}} (\delta_c V)_I\right)_{t\geq 0}.$$

Let us now observe that due to the scaling property of the fractional Brownian motion

$$\left(\sum_{k=1}^{+\infty} \sum_{I\in\{1,\dots,d\}^k} \Lambda_I(B)_{\frac{t}{c}} (\delta_c V)_I\right)_{t\geq 0} =^{law} \left(\sum_{k=1}^{+\infty} \sum_{I\in\{1,\dots,d\}^k} \Lambda_I(B)_t \frac{1}{c^{H|I|}} \delta_c V_I\right)_{t\geq 0},$$

where $|\ I\ |$ is the length of the word I. Thus, for every $c > 0$,

$$\left(\sum_{k=1}^{+\infty} \sum_{I\in\{1,\dots,d\}^k} \Lambda_I(B)_t V_I\right)_{t\geq 0} =^{law} \left(\sum_{k=1}^{+\infty} \frac{1}{c^{kH}} \delta_c \left(\sum_{I\in\{1,\dots,d\}^k} \Lambda_I(B)_t V_I\right)\right)_{t\geq 0}$$

Let us now observe that $V_1, \dots, V_d$ is a basis of $\mathfrak{g}$, therefore all commutators are linear combinations of the V_i's and

$$V_I = \sum_{i=1}^{d} \alpha_I^i V_i, \quad I \in \{1, \dots, d\}^k, \quad k \in \mathbb{N}^*,$$

$$\delta_c(V_j) = \sum_{i=1}^{d} \delta_c^{i,j} V_i, \quad j = 1, \dots, d.$$

The projections on V_j, $j = 1, ..., d$, are continuous linear maps (since $\mathfrak{g}$ has a finite dimension), therefore

$$\left(\sum_{k=1}^{N} \sum_{I \in \{1,...,d\}^k} \Lambda_I(B)_t \alpha_I^j \right)_{t \geq 0, j=1,...,d}$$

has the same law as

$$\left(\sum_{k=1}^{N} c^{kH} \sum_{I \in \{1,...,d\}^k} \Lambda_I(B)_t \sum_{i=1}^{d} \alpha_I^j \delta_c^{i,j} \right)_{t \geq 0},$$

where N is the degree of nilpotence of $\mathbf{G}$. We take now expectation on both sides and observe that the maps $t \mapsto \mathbb{E}(\sum_{k=1}^{N} c^{kH} \sum_{I=(i_1,...,i_k),\ i_1 \leq ... \leq i_k} \Lambda_I(B)_t)$, $j = 1, ..., d$ are polynomial in t^{2H}, with coefficient associated to t^{2H} given by

$$1 = c^{-2H} \sum_{i=1}^{d} (\delta_c^{i,j})^2, \quad j = 1, ..., d.$$

Since the family of linear applications $(c^{-H}\delta_c)_{c>0}$ is bounded, there exists a subsequence $(c_l)_{l \in \mathbb{N}}$ and a matrix $(\tilde{\alpha}_i^j)_{i,j=1,..,d}$ such that $\lim_{l \to \infty} c_l = \infty$, and for $i, j = 1, ..., d$, $\lim_{l \to \infty} c_l^{-H} \delta_{c_l}^{i,j} = \tilde{\alpha}_i^j$. Then,

$$\left(\sum_{k=1}^{N} c_l^{kH} \sum_{I \in \{1,...,d\}^k} \Lambda_I(B)_t \sum_{i=1}^{d} \alpha_I^j \delta_{c_l}^{i,j} \right)_{t \geq 0, \quad j=1,...,d},$$

converges in law to the Gaussian process $(\sum_{i=1}^{d} B_t^i \tilde{\alpha}_i^j)_{t \geq 0, j=1,...,d}$. For $j = 1, ..., d$, by using the scaling property of fractional Brownian motion, we observe that the map

$$\mathbb{E}\left(\left(\sum_{k=1}^{N} \sum_{I \in \{1,...,d\}^k} \Lambda_I(B)_t \alpha_I^j \right)^2 \right)$$

is a polynomial with degree 1 in t^{2H}. The leading coefficient of t^{4H} is given by

$$\mathbb{E}\left(\left(\sum_{I=(i_1,i_2)} \Lambda_I(B)_1 \alpha_I^j \right)^2 \right).$$

We conclude therefore

$$\mathbb{E}\left(\left(\sum_{I=(i_1,i_2)} \Lambda_I(B)_1 \alpha_I^j \right)^2 \right) = 0.$$

From the support theorem of [7] (Proposition 3), we conclude that for all $I = (i_1, i_2), , j = 1, ..., d$, $\alpha_I^j = 0$, that is $V_I = [V_{i_1}, V_{i_2}] = 0$. Therefore all the brackets have to be 0, that is $\mathbf{G}$ is commutative. We can now conclude that $\mathbf{G}$ is isomorphic to $(\mathbb{R}^d, +)$, because so are all simply connected and commutative Lie groups. □

Remark 3.5. *The previous theorem in particular applies to the case of a Brownian motion, that corresponds to* $H = \frac{1}{2}$.

If we relax the assumption that the family $(V_1, ..., V_d)$ forms a basis of the Lie algebra $\mathfrak{g}$, we can have a global scaling property in slightly more general groups than the commutative ones. Let us first look at one example.

The Heisenberg group $\mathbb{H}$ is the set of 3×3 matrices:

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbf{R}.$$

The Lie algebra of $\mathbb{H}$ is spanned by the matrices

$$D_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ D_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } D_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for which the following equalities hold

$$[D_1, D_2] = D_3, \ [D_1, D_3] = [D_2, D_3] = 0.$$

Consider now the solution of the equation

$$dX_t = X_t(D_1 \circ dB_t^1 + D_2 \circ dB_t^2), \quad X_0 = 1,$$

where (B_t^1, B_t^2) is a two-dimensional fractional Brownian motion with Hurst parameter $H > \frac{1}{4}$. It is easily seen that

$$X_t = \begin{pmatrix} 1 & B_t^1 & \frac{1}{2}\left(B_t^1 B_t^2 + \int_0^t B_s^1 \circ dB_s^2 - B_s^2 \circ dB_s^1\right) \\ 0 & 1 & B_t^2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore $(X_{ct})_{t \geq 0} =^{law} (\Delta_c X_t)_{t \geq 0}$, where Δ_c is defined by

$$\Delta_c \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & c^H x & c^{2H} z \\ 0 & 1 & c^H y \\ 0 & 0 & 1 \end{pmatrix}.$$

In that case, we thus have a global scaling property whereas $\mathbb{H}$ is of course not commutative but step-two nilpotent. Actually, we shall have a global scaling property in the Lie groups that are called the Carnot groups. Let us recall the definition of a Carnot group.

Definition 3.6. *A Carnot group of step (or depth)* N *is a simply connected Lie group* $\mathbb{G}$ *whose Lie algebra can be written*

$$\mathcal{V}_1 \oplus ... \oplus \mathcal{V}_N,$$

where

$$[\mathcal{V}_i, \mathcal{V}_j] = \mathcal{V}_{i+j}$$

and

$$\mathcal{V}_s = 0, \ \textit{for } s > N.$$

Example 3.7. *Consider the set $\mathbb{H}_n = \mathbb{R}^{2n} \times \mathbb{R}$ endowed with the group law*

$$(x, \alpha) \star (y, \beta) = \left(x + y, \alpha + \beta + \frac{1}{2}\omega(x, y) \right),$$

where ω is the standard symplectic form on $\mathbb{R}^{2n}$, that is

$$\omega(x, y) = x^t \begin{pmatrix} 0 & -\mathbf{I}_n \\ \mathbf{I}_n & 0 \end{pmatrix} y.$$

On $\mathfrak{h}_n$ the Lie bracket is given by

$$[(x, \alpha), (y, \beta)] = (0, \omega(x, y)),$$

and it is easily seen that

$$\mathfrak{h}_n = \mathcal{V}_1 \oplus \mathcal{V}_2,$$

where $\mathcal{V}_1 = \mathbb{R}^{2n} \times \{0\}$ and $\mathcal{V}_2 = \{0\} \times \mathbb{R}$. Therefore $\mathbb{H}_n$ is a Carnot group of depth 2 and observe that $\mathbb{H}_1$ is isomorphic to the Heisenberg group.

Notice that the vector space $\mathcal{V}_1$, which is called the basis of $\mathbb{G}$, Lie generates $\mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra of $\mathbb{G}$. Since $\mathbb{G}$ is step N nilpotent and simply connected, the exponential map is a diffeomorphism. On $\mathfrak{g}$ we can consider the family of linear operators $\delta_t : \mathfrak{g} \to \mathfrak{g}$, $t \geq 0$ which act by scalar multiplication t^i on $\mathcal{V}_i$. These operators are Lie algebra automorphisms due to the grading. The maps δ_t induce Lie group automorphisms $\Delta_t : \mathbb{G} \to \mathbb{G}$ which are called the canonical dilations of $\mathbb{G}$. Let us now take a basis $U_1, ..., U_d$ of the vector space $\mathcal{V}_1$. The vectors U_i's can be seen as left invariant vector fields on $\mathbb{G}$ so that we can consider the following stochastic differential equation on $\mathbb{G}$:

$$dY_t = \sum_{i=1}^{d} \int_0^t U_i(Y_s) \circ dB_s^i, \ t \geq 0, \tag{3.8}$$

which is easily seen to have a unique solution associated with the initial condition $Y_0 = 1_{\mathbb{G}}$. We have then the following global scaling property:

Proposition 3.8.

$$(Y_{ct})_{t \geq 0} =^{law} (\Delta_{c^H} Y_t)_{t \geq 0}.$$

Proof. We keep the notations introduced before the proof of Theorem 2.7. From Theorem 2.7, we have

$$Y_t = \exp\left(\sum_{k=1}^{N} \sum_{I \in \{1,...,d\}^k} \Lambda_I(B)_t U_I \right), \ t \geq 0.$$

Therefore,

$$(Y_{ct})_{t \geq 0} =^{law} \left(\exp\left(\sum_{k=1}^{N} c^{H|I|} \sum_{I \in \{1,...,d\}^k} \Lambda_I(B)_t U_I \right) \right)_{t \geq 0}.$$

Since

$$\exp\left(\sum_{k=1}^{N} c^{H|I|} \sum_{I\in\{1,...,d\}^k} \Lambda_I(B)_t U_I\right) = \exp\left(\sum_{k=1}^{N} \sum_{I\in\{1,...,d\}^k} \Lambda_I(B)_t (\delta_{c^H} U_I)\right)$$
$$= \Delta_{c^H} \exp\left(\sum_{k=1}^{N} \sum_{I\in\{1,...,d\}^k} \Lambda_I(B)_t U_I\right),$$

we conclude

$$(Y_{ct})_{t\geq 0} =^{law} (\Delta_{c^H} Y_t)_{t\geq 0}.$$

□

The previous proposition admits a counterpart.

Theorem 3.9. *Let* $\mathbf{G}$ *be a connected Lie group (of matrices) with Lie algebra* $\mathfrak{g}$. *Let* $V_1, ..., V_d$ *be a family of* $\mathfrak{g}$. *Consider now the solution of the equation*

$$dX_t = X_t \left(\sum_{i=1}^{d} V_i \circ dB_t^i\right), \quad X_0 = 1_{\mathbf{G}}.$$

Assume that there exists a family of Lie group automorphisms $\Delta_t : \mathbf{G} \to \mathbf{G}$, $t > 0$, *such that:*

1. *The map* $t \to \Delta_t$ *is continuous and* $\lim_{t\to 0} \Delta_t = 1_{\mathbf{G}}$*;*
2. *For* $t_1, t_2 \geq 0$, $\Delta_{t_1 t_2} = \Delta_{t_1} \circ \Delta_{t_2}$*;*
3. $(X_{ct})_{t\geq 0} =^{law} (\Delta_c X_t)_{t\geq 0}$*;*

Then the Lie subgroup $\mathbf{H}$ *that is generated by* $e^{V_1}, ..., e^{V_d}$ *is a Carnot group.*

Proof.

We can readily follow the lines of the proof of Theorem 3.4, so that we do not enter into details. First we obtain that $\mathbf{H}$ has to be a simply connected nilpotent group and that for every $c > 0$,

$$\left(\sum_{k=1}^{+\infty} \sum_{I=(i_1,...,i_k),\ i_1\leq ...\leq i_k} \Lambda_I(B)_t V_I\right)_{t\geq 0} =^{law} \left(\sum_{k=1}^{+\infty} \frac{1}{c^{kH}} \delta_c \left(\sum_{I=(i_1,...,i_k),\ i_1\leq ...\leq i_k} \Lambda_I(B)_t V_I\right)\right)_{t\geq 0},$$

where δ_c is the differential map of Δ_c at $1_{\mathbf{G}}$. For $k \geq 1$, we denote $\mathcal{V}_k$ the linear space generated by the set of commutators:

$$\{V_I, | I |= k\}.$$

By letting $c \to +\infty$ and $c \to 0$ and by using the support theorem of [7], we obtain that $c^{-kH}\delta_c$ is bounded on $\mathcal{V}_k$ for $c \to +\infty$ and $c \to 0$. Since $c^{-kH}\delta_c = \mathrm{Exp}((\phi - kH\mathbf{Id})\ln c)$, for some matrix ϕ, we conclude

$$\mathfrak{h} = \bigoplus_{k=1}^{+\infty} \mathcal{V}_k,$$

where $\mathfrak{h}$ is the Lie algebra of $\mathbf{H}$. This proves that $\mathbf{H}$ is a Carnot group. □

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