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Processes on Unimodular Random Networks*

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Abstract

We investigate unimodular random networks. Our motivations include their characterization via reversibility of an associated random walk and their similarities to unimodular quasi-transitive graphs. We extend various theorems concerning random walks, percolation, spanning forests, and amenability from the known context of unimodular quasi-transitive graphs to the more general context of unimodular random networks. We give properties of a trace associated to unimodular random networks with applications to stochastic comparison of continuous-time random walk.

Key words: Amenability, equivalence relations, infinite graphs, percolation, quasi-transitive, random walks, transitivity, weak convergence, reversibility, trace, stochastic comparison, spanning forests, sofic groups.

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1 Introduction

In the setting of infinite discrete graphs, the property of being a Cayley graph of a group is a strong form of "spatial homogeneity": many results not true for arbitrary graphs are true under this strong property. As we shall soon explain, weaker regularity properties sufficient for many results have been studied. In this paper, we turn to random graphs, investigating a notion of "statistical homogeneity" or "spatial stationarity" that we call a *unimodular random rooted network*. The root is merely a distinguished vertex of the network and the probability measure is on a certain space of rooted networks. In a precise sense, the root is "equally likely" to be any vertex of the network, even though we consider infinite networks. We shall show that many results known for deterministic graphs under previously-studied regularity conditions do indeed extend to unimodular random rooted networks.

Thus, a probabilistic motivation for our investigations is the study of stochastic processes under unimodularity. A second motivation is combinatorial: One often asks for asymptotics of enumeration or optimization problems on finite networks as the size of the networks tend to infinity. One can sometimes answer such questions with the aid of a suitable limiting infinite object. A survey of this approach is given by Aldous and Steele (2004). We call "random weak limit" the type of limit one considers; it is the limiting "view" from a uniformly chosen vertex of the finite networks. What limiting objects can arise this way? It has been observed before that the probabilistic objects of interest, unimodular random rooted networks, contain all the combinatorial objects of interest, random weak limits of finite networks. One open question is whether these two classes in fact coincide. An affirmative answer would have many powerful consequences, as we shall explain.

To motivate this by analogy, recall a simple fact about stationary sequences $\langle Y_i \rangle_{i\in\mathbb{Z}}$ of random variables. For each $n \geq 1$, let $\langle Y_{n,i} \rangle_{1\leq i\leq n}$ be arbitrary. Center it at a uniform index $U_n \in$

$\{1, 2, \ldots, n\}$ to get a bi-infinite sequence $\langle Y_{n,U_n+i} \rangle_{i \in \mathbb{Z}}$, interpreted arbitrarily outside its natural range. If there is a weak limit $\langle Y_i \rangle_{i \in \mathbb{Z}}$ as $n \to \infty$ of these randomly centered sequences, then the limit is stationary, and conversely any stationary sequence can be obtained trivially as such a limit.

By analogy, then, given a finite graph, take a uniform random vertex as root. Such a randomly rooted graph automatically has a certain property (in short, if mass is redistributed in the graph, then the expected mass that leaves the root is equal to the expected mass the arrives at the root) and in Section 2, we abstract this property as unimodularity. It is then immediate that any infinite random rooted graph that is a limit (in an appropriate sense that we call "random weak limit") of uniformly randomly rooted finite graphs will be unimodular, whereas the above question asks conversely whether any unimodular random rooted graph arises as a random weak limit of some sequence of randomly rooted finite graphs.

Additional motivation for the definition arises from random walk considerations. Given any random rooted graph, simple random walk induces a Markov chain on rooted graphs. Unimodularity of a probability measure μ on rooted graphs is equivalent to the property that a reversible stationary distribution for this chain is given by the root-degree biasing of μ, just as on finite graphs, a stationary distribution for simple random walk is proportional to the vertex degrees; see Section 4.

Let us return now to the case of deterministic graphs. An apparently minor relaxation of the Cayley graph property is the "transitive" property (that there is an automorphism taking any vertex to any other vertex). By analogy with the shift-invariant interpretation of stationary sequences, one might expect every transitive graph to fit into our set-up. But this is false. Substantial research over the last ten years has shown that the most useful regularity condition is that of a *unimodular transitive graph* (or, more generally, quasi-transitive). Intuitively, this is an unrooted transitive graph that can be given a random root in such a way that each vertex is equally likely to be the root. This notion is, of course, precise in itself for a finite graph. To understand how this is extended to infinite graphs, and then to unimodular random rooted graphs, consider a finite graph G and a function $f(x, y)$ of ordered pairs of vertices of G. Think of $f(x, y)$ as an amount of mass that is sent from x to y. Then the total mass on the graph G before transport equals the total after, since mass is merely redistributed on the graph. We shall view this alternatively as saying that for a randomly uniformly chosen vertex, the expected mass it receives is equal to the expected mass it sends out. This, of course, depends crucially on choosing the vertex uniformly and, indeed, characterizes the uniform measure among all probability measures on the vertices.

Consider now an infinite transitive graph, G. Since all vertices "look the same", we could just fix one, o, rather than try to choose one uniformly. However, a mass transport function f will not conserve the mass at o without some assumption on f to make up for the fact that o is fixed. Although it seems special at first, it turns out that a very useful assumption is that f is invariant under the diagonal action of the automorphism group of G. (For a finite graph that happened to have no automorphisms other than the identity, this would be no restriction at all.) This is still not enough to guarantee "conservation of mass", i.e., that

$$\sum_x f(o, x) = \sum_x f(x, o) \,, \tag{1.1}$$

but it turns out that (1.1) does hold when the automorphism group of G is unimodular. Here, "unimodular" is used in its original sense that the group admits a non-trivial Borel measure that is invariant under both left and right multiplication by group elements. We call G itself unimodular in that case; see Sections 2 and 3 for more on this concept. The statement that (1.1) holds under these assumptions is called the Mass-Transport Principle for G. If G is quasi-transitive, rather than transitive, we still have a version of (1.1), but we can no longer consider only one fixed vertex o. Instead, each orbit of the automorphism group must have a representative vertex. Furthermore, it must be weighted "proportionally to its frequency" among vertices; see Theorem 3.1. This principle was introduced to the study of percolation by Häggström (1997), then developed and exploited heavily by Benjamini, Lyons, Peres, and Schramm (1999b), hereinafter referred to as BLPS (1999b). Another way of stating it is that (1.1) holds in expectation when o is chosen randomly by an appropriate probability measure. If we think of o as the root, then we arrive at the notion of random rooted graphs, and the corresponding statement that (1.1) holds in expectation is a general form of the Mass-Transport Principle. This general form was called the "Intrinsic Mass-Transport Principle" by Benjamini and Schramm (2001b). We shall call a probability measure on rooted graphs unimodular precisely when this general form of the Mass-Transport Principle holds. We develop this in Section 2.

Thus, we can extend many results known for unimodular quasi-transitive graphs to our new setting of unimodular random rooted graphs, as noted by Benjamini and Schramm (2001b). As a bonus, our set-up allows the treatment of quasi-transitive graphs to be precisely parallel to that of transitive graphs, with no additional notation or thought needed, which had not always been the case previously.

To state results in their natural generality, as well as for technical convenience, we shall work in the setting of networks, which are just graphs with "marks" (labels) on edges and vertices. Mainly, this paper is organized to progress from the most general to the most specific models. An exception is made in Section 3, where we discuss random networks on fixed underlying graphs. This will not only help to understand and motivate the general setting, but also will be useful in deriving consequences of our general results.

Section 4 elaborates the comment above about reversible stationary distributions for random walk, discussing extremality and invariant σ-fields, speed of random walk, and continuous-time random walk and their explosions. Section 5 discusses a trace associated to unimodular random networks and comparison of return probabilities of different continuous-time random walks, which partially answers a question of Fontes and Mathieu. We then write out the extensions to unimodular random rooted graphs of results known for fixed graphs in the context of percolation (Section 6), spanning forests (Section 7) and amenability (Section 8). These extensions are in most (though not all) cases straightforward. Nevertheless, we think it is useful to list these extensions in the order they need to be proved so that others need not check the entire (sometimes long) proofs or chains of theorems from a variety of papers. Furthermore, we were required to find several essentially new results along the way.

In order to appreciate the scope of our results, we list many examples of unimodular probability measures in Section 9. In particular, there is a significant and important overlap between our theory and the theory of graphings of measure-preserving equivalence relations. This overlap is well known among a few specialists, but deserves to be made more explicit. We do that here in Example 9.9.

Among several open problems, we spotlight a special case of Question 2.4: Suppose we are given a partial order on the mark space and two unimodular probability measures, one stochastically dominating the other. That is, there is a monotone coupling of the two unimodular distributions that puts the networks on the same graphs, but has higher marks for the second network than for the first. Does this imply the existence of a *unimodular* monotone coupling? A positive answer would be of great benefit in a variety of ways.

Another especially important open question is Question 10.1, whether every unimodular probability measure is a limit of uniformly rooted finite networks. For example, in the case that the random rooted infinite network is just a Cayley graph (rooted, say, at the identity) with the edges marked by the generators, a positive answer to this question on finite approximation would answer a question of Weiss (2000), by showing that all finitely generated groups are "sofic", although this is contrary to the belief expressed by Weiss (2000). (Sofic groups were introduced, with a different definition, by Gromov (1999); see Elek and Szabó (2004) for a proof that the definitions are equivalent.) This would establish several conjectures, since they are known to hold for sofic groups: the direct finiteness conjecture of Kaplansky (1969) on group algebras (see Elek and Szabó (2004)), a conjecture of Gottschalk (1973) on "surjunctive" groups in topological dynamics (see Gromov (1999)), the Determinant Conjecture on Fuglede-Kadison determinants (see Elek and Szabó (2005)), and Connes's (Connes (1976)) Embedding Conjecture for group von Neumann algebras (see Elek and Szabó (2005)). The Determinant Conjecture in turn implies the Approximation Conjecture of Schick (2001) and the Conjecture of Homotopy Invariance of L^2-Torsion due to Lück (1994); see Chap. 13 of Lück (2002) for these implications and more information. Weiss (2000) gave another proof of Gottschalk's conjecture for sofic groups. One may easily extend that proof to show a form of Gottschalk's conjecture for all quasi-transitive unimodular graphs that are limits of finite graphs, but there are easy counterexamples for general transitive graphs.

Further discussion of the question on approximation by finite networks is given in Section 10. A positive answer would provide solid support for the intuition that the root of a unimodular random rooted network is equally likely to be any vertex. Section 10 also contains some variations that would result from a positive answer and some additional consequences for deterministic graphs.

The notion of weak convergence of rooted locally finite graphs or networks (needed to make sense of convergence of randomly rooted finite graphs to a limit infinite graph) has arisen before in several different contexts. Of course, the special case where the limit network is a Cayley diagram was introduced by Gromov (1999) and Weiss (2000). In the other cases, the limits provide examples of unimodular random rooted graphs. Aldous (1991) gives many examples of models of random finite trees which have an infinite-tree limit (and one such example, the limit of uniform random labeled trees being what is now called the Poisson-Galton-Watson tree, $\mathsf{PGW}^\infty(1)$, goes back to Grimmett (1980/81)). The idea that random weak limits of finite planar graphs of uniformly bounded degree provide an interesting class of infinite planar graphs was developed by Benjamini and Schramm (2001b), who showed that random walk on almost any such limit graph is recurrent. (Thus, such graphs do not include regular trees or hyperbolic graphs, other than trivial examples like $\mathbb{Z}$.) A specialization to random weak limits of plane triangulations was studied in more detail in interesting recent work of Angel and Schramm (2003) and Angel (2003).

Example 9.7 describes an infinite-degree tree, arising as a limit of weighted finite complete graphs. This example provides an interface between our setting and related ideas of "local weak convergence" and "the objective method in the probabilistic analysis of algorithms". A prototype is that the distribution of n random points in a square of area n converges in a natural sense as $n \to \infty$ to the distribution of a Poisson point process on the plane of unit intensity. One can ask whether solutions of combinatorial optimization problems over the n random points (minimum spanning tree, minimum matching, traveling salesman problem) converge to limits that are the solutions of analogous optimization problems over the Poisson point process in the whole plane. Example 9.7 can be regarded as a mean-field analogue of random points in the plane, and $n \to \infty$ limits of solutions of combinatorial optimization problems within this model have been studied using the non-rigorous cavity method from statistical physics. Aldous and Percus (2003) illustrate what can be done by non-rigorous means, while Aldous and Steele (2004) survey introductory rigorous theory.

The reader may find it helpful to keep in mind one additional example, a unimodular version of family trees of Galton-Watson branching processes; see also Example 10.2.

Example 1.1 (Unimodular Galton-Watson)**.** Let $\langle p_k\,;\ k \geq 0\rangle$ be a probability distribution on $\mathbb{N}$. Take two independent Galton-Watson trees with offspring distribution $\langle p_k\rangle$, each starting with one particle, the root, and join them by a new edge whose endpoints are their roots. Root the new tree at the root of the first Galton-Watson tree. This is **augmented Galton-Watson** measure, AGW . (If $p_0 \neq 0$, then we have the additional options to condition on either non-extinction or extinction of the joined trees.) Now bias by the reciprocal of the degree of the root to get **unimodular Galton-Watson** measure, UGW. In different language, Lyons, Pemantle, and Peres (1995) proved that this measure, UGW, is unimodular. Note that the mean degree of the root is

$$\overline{\deg}(\mathsf{UGW}) = \left(\sum_{k\geq 0} \frac{p_k}{k+1}\right)^{-1} . \tag{1.2}$$

2 Definitions and Basics

We denote a (multi-)graph G with vertex set V and undirected edge set E by $G = (\mathsf{V}, \mathsf{E})$. When there is more than one graph under discussion, we write $\mathsf{V}(G)$ or $\mathsf{E}(G)$ to avoid ambiguity. We denote the degree of a vertex x in a graph G by $\deg_G(x)$. **Simple random walk** on G is the Markov chain whose state space is V and whose transition probability from x to y equals the number of edges joining x to y divided by $\deg_G(x)$.

A **network** is a (multi-)graph $G = (\mathsf{V}, \mathsf{E})$ together with a complete separable metric space Ξ called the **mark space** and maps from V and E to Ξ. Images in Ξ are called **marks**. Each edge is given two marks, one associated to ("at") each of its endpoints. The only assumption on degrees is that they are finite. We shall usually assume that Ξ is Baire space $\mathbb{N}^{\mathbb{N}}$, since every uncountable complete separable metric space is Borel isomorphic to Baire space by Kuratowski's theorem (Theorem 15.10 of Royden (1988)). We generally omit mention of the mark maps from our notation for networks when we do not need them. For convenience, we consider graphs as special cases of networks in which all marks are equal to some fixed mark.

We now define ends in graphs. In the special case of a tree, an infinite path that starts at any vertex and does not backtrack is called a **ray**. Two rays are **equivalent** if they have infinitely many vertices in common. An equivalence class of rays is called an **end**. In a general infinite graph, G, an **end** of G is an equivalence class of infinite simple paths in G, where two paths are equivalent if for every finite $K \subset \mathsf{V}(G)$, there is a connected component of $G \setminus K$ that intersects both paths.

Let G be a graph. For a subgraph H, let its **(internal) vertex boundary** $\partial_{\mathsf{V}} H$ be the set of vertices of H that are adjacent to some vertex not in H. We say that G is **(vertex) amenable** if there exists a sequence of subsets $H_n \subset \mathsf{V}(G)$ with

$$\lim_{n\to\infty} \frac{|\partial_{\mathsf{V}} H_n|}{|\mathsf{V}(H_n)|} = 0\,,$$

where $|\bullet|$ denotes cardinality. Such a sequence is called a **Følner sequence**. A finitely generated group is **amenable** if its Cayley graph is amenable. For example, every finitely generated abelian group is amenable. For more on amenability of graphs and groups, see BLPS (1999b).

A **homomorphism** $\varphi : G_1 \to G_2$ from one graph $G_1 = (\mathsf{V}_1, \mathsf{E}_1)$ to another $G_2 = (\mathsf{V}_2, \mathsf{E}_2)$ is a pair of maps $\varphi_{\mathsf{V}} : \mathsf{V}_1 \to \mathsf{V}_2$ and $\varphi_{\mathsf{E}} : \mathsf{E}_1 \to \mathsf{E}_2$ such that φ_{V} maps the endpoints of e to the endpoints of $\varphi_{\mathsf{E}}(e)$ for every edge $e \in \mathsf{E}_1$. When both maps $\varphi_{\mathsf{V}} : \mathsf{V}_1 \to \mathsf{V}_2$ and $\varphi_{\mathsf{E}} : \mathsf{E}_1 \to \mathsf{E}_2$ are bijections, then φ is called an **isomorphism**. When $G_1 = G_2$, an isomorphism is called an **automorphism**. The set of all automorphisms of G forms a group under composition, denoted by $\mathrm{Aut}(G)$. The action of a group Γ on a graph G by automorphisms is said to be **transitive** if there is only one Γ-orbit in $\mathsf{V}(G)$ and to be **quasi-transitive** if there are only finitely many orbits in $\mathsf{V}(G)$. A graph G is **transitive** or **quasi-transitive** according as whether the corresponding action of $\mathrm{Aut}(G)$ is. For example, every Cayley graph is transitive. All the same terms are applied to networks when the maps in question preserve the marks on vertices and edges.

A locally compact group is called **unimodular** if its left Haar measure is also right invariant. In particular, every discrete countable group is unimodular. We call a graph G **unimodular** if $\mathrm{Aut}(G)$ is unimodular, where $\mathrm{Aut}(G)$ is given the weak topology generated by its action on G. Every Cayley graph and, as Soardi and Woess (1990) and Salvatori (1992) proved, every quasi-transitive amenable graph is unimodular. See Section 3 and BLPS (1999b) for more details on unimodular graphs.

A **rooted network** (G, o) is a network G with a distinguished vertex o of G, called the **root**. A **rooted isomorphism** of rooted networks is an isomorphism of the underlying networks that takes the root of one to the root of the other. We generally do not distinguish between a rooted network and its isomorphism class. When needed, however, we use the following notation to make these distinctions: G will denote a graph, $\overline{G}$ will denote a network with underlying graph G, and $[\overline{G}, o]$ will denote the class of rooted networks that are rooted-isomorphic to $(\overline{G}, o)$. We shall use the following notion introduced (in slightly different generalities) by Benjamini and Schramm (2001b) and Aldous and Steele (2004). Let $\mathcal{G}_*$ denote the set of rooted isomorphism classes of rooted *connected* locally finite networks. Define a metric on $\mathcal{G}_*$ by letting the distance between (G_1, o_1) and (G_2, o_2) be $1/(1+\alpha)$, where α is the supremum of those $r > 0$ such that there is some rooted isomorphism of the balls of (graph-distance) radius $\lfloor r \rfloor$ around the roots of G_i such that each pair of corresponding marks has distance less than $1/r$. It is clear that $\mathcal{G}_*$ is separable and complete in this metric. For probability measures μ, μ_n on $\mathcal{G}_*$, we write $\mu_n \Rightarrow \mu$ when μ_n converges weakly with respect to this metric.

For a probability measure μ on rooted networks, write $\overline{\deg}(\mu)$ for the *expectation* of the degree of the root with respect to μ. In the theory of measured equivalence relations (Example 9.9), this is twice the **cost** of the graphing associated to μ. Also, by the **degree** of μ we mean the *distribution* of the degree of the root under μ.

For a locally finite connected rooted network, there is a canonical choice of a rooted network in its rooted-isomorphism class. More specifically, there is a continuous map f from $\mathcal{G}_*$ to the space of networks on $\mathbb{N}$ rooted at 0 such that $f\big([G,o]\big) \in [G,o]$ for all $[G,o] \in \mathcal{G}_*$. To specify this, consider the following total ordering on rooted networks with vertex set $\mathbb{N}$ and root 0. First, total order $\mathbb{N} \times \mathbb{N}$ by the lexicographic order: $(i_1, j_1) \prec (i_2, j_2)$ if either $i_1 < i_2$ or $i_1 = i_2$ and $j_1 < j_2$. Second, the lexicographic order on Baire space Ξ is also a total order. We consider networks on $\mathbb{N}$ rooted at 0. Define a total order on such networks as follows. Regard the edges as oriented for purposes of identifying the edges with $\mathbb{N} \times \mathbb{N}$; the mark at i of an edge between i and j will be considered as the mark of the oriented edge (i, j). Suppose we are given a pair of networks on $\mathbb{N}$ rooted at 0. If they do not have the same edge sets, then the network that contains the smallest edge in their symmetric difference is deemed to be the smaller network. If they do have the same edge sets, but not all the vertex marks are the same, then the network that contains the vertex with the smaller mark on the least vertex where they differ is deemed the smaller network. If the networks have the same edge sets and the same vertex marks, but not all the edge marks are the same, then the network that contains the oriented edge with the smaller mark on the least oriented edge where they differ is deemed the smaller network. Otherwise, the networks are identical.

We claim that the rooted-isomorphism class of each locally finite connected network contains a unique smallest rooted network on $\mathbb{N}$ in the above ordering. This is its **canonical** representative. To prove our claim, consider only the networks in the class that have vertex set $\mathbb{N}$ and are rooted at 0. It is fairly easy to see that there is a smallest graph that supports a network in the class: the neighbors of 0 are $[1, n_1]$ for some $n_1 \geq 1$, the neighbors of 1, besides 0, are $[n_1+1, n_2]$ for some $n_2 \geq n_1$, the neighbors of 2, other than possibly 0 and 1, are $[n_2+1, n_3]$ for some $n_3 \geq n_2$, etc. In general, the neighbors of $k \geq 1$ that are larger than k are $[n_k+1, n_{k+1}]$ for some $n_{k+1} \geq n_k$. Write $n_0 := 0$. Let A_{-1} be the set of networks in the class that are on this smallest graph. All have the same mark at 0. Write $\psi(k)$ for the mark at a vertex k and $\psi(j,k)$ for the mark at the oriented edge (j,k). Define A_k recursively as follows. Given A_{k-1}, let A_k be the subset of networks in A_{k-1} such that the marks $\psi(j)$ are increasing for $j \in [n_k+1, n_{k+1}]$. Then let $B_{-1} := \bigcap_k A_k$. This is non-empty. Now define B_k recursively by letting B_k be the subset of B_{k-1} such that $j \mapsto \psi(k,j)$ is increasing on $[n_k+1, n_{k+1}]$. The set $\bigcap_k B_k$ is then a singleton, the desired canonical representative.

For a (possibly disconnected) network G and a vertex $x \in \mathsf{V}(G)$, write G_x for the connected component of x in G. If G is a network with probability distribution μ on its vertices, then μ induces naturally a distribution on $\mathcal{G}_*$, which we also denote by μ; namely, the probability of (G_x, x) is $\mu(x)$. More precisely, $\mu\big([G_x, x]\big) := \sum \big\{\mu(y)\,;\ y \in \mathsf{V}(G),\ (G_y, y) \in [G_x, x]\big\}$. For a finite network G, let $U(G)$ denote the distribution on $\mathcal{G}_*$ obtained this way by choosing a uniform random vertex of G as root. Suppose that G_n are finite networks and that μ is a probability measure on $\mathcal{G}_*$. We say the **random weak limit** of G_n is μ if $U(G_n) \Rightarrow \mu$. If $\mu\Big(\big\{[G,o]\big\}\Big) = 1$ for a fixed transitive network G (and (any) $o \in \mathsf{V}(G)$), then we say that the **random weak limit** of G_n is G.

As usual, call a collection $\mathcal{C}$ of probability measures on $\mathcal{G}_*$ **tight** if for each $\epsilon > 0$, there is a compact set $\mathcal{K} \subset \mathcal{G}_*$ such that $\mu(\mathcal{K}) > 1 - \epsilon$ for all $\mu \in \mathcal{C}$. Because $\mathcal{G}_*$ is complete, any tight collection has a subsequence that possesses a weak limit.

The class of probability measures μ that arise as random weak limits of finite networks is contained in the class of unimodular μ, which we now define. Similarly to the space $\mathcal{G}_*$, we define the space $\mathcal{G}_{**}$ of isomorphism classes of locally finite connected networks with an ordered pair of distinguished vertices and the natural topology thereon. We shall write a function f on $\mathcal{G}_{**}$ as $f(G, x, y)$.

Definition 2.1. Let μ be a probability measure on $\mathcal{G}_*$. We call μ **unimodular** if it obeys the **Mass-Transport Principle**: For all Borel $f : \mathcal{G}_{**} \to [0, \infty]$, we have

$$\int \sum_{x \in \mathsf{V}(G)} f(G, o, x)\, d\mu\big([G, o]\big) = \int \sum_{x \in \mathsf{V}(G)} f(G, x, o)\, d\mu\big([G, o]\big) . \tag{2.1}$$

Let $\mathcal{U}$ denote the set of unimodular Borel probability measures on $\mathcal{G}_*$.

Note that to define the sums that occur here, we choose a specific network from its rooted-isomorphism class, but which one we choose makes no difference when the sums are computed. We sometimes call $f(G, x, y)$ the amount of "mass" sent from x to y. The motivation for the name "unimodular" is two fold: One is the extension of the concept of unimodular automorphism groups of networks. The second is that the Mass-Transport Principle expresses the equality of two measures on $\mathcal{G}_{**}$ associated to μ, the "left" measure μ_{L} defined by

$$\int_{\mathcal{G}_{**}} f\, d\mu_{\mathrm{L}} := \int_{\mathcal{G}_*} \sum_{x \in \mathsf{V}(G)} f(G, o, x)\, d\mu\big([G, o]\big)$$

and the "right" measure μ_{R} defined by

$$\int_{\mathcal{G}_{**}} f\, d\mu_{\mathrm{R}} := \int_{\mathcal{G}_*} \sum_{x \in \mathsf{V}(G)} f(G, x, o)\, d\mu\big([G, o]\big) .$$

Thus, μ is unimodular iff $\mu_{\mathrm{L}} = \mu_{\mathrm{R}}$, which can also be expressed by saying that the left measure is absolutely continuous with respect to the right measure and has Radon-Nikodým derivative 1.

It is easy to see that any μ that is a random weak limit of finite networks is unimodular, as observed by Benjamini and Schramm (2001b), who introduced this general form of the Mass-Transport Principle under the name "intrinsic Mass-Transport Principle". The converse is open.

A special form of the Mass-Transport Principle was considered, in different language, by Aldous and Steele (2004). Namely, they defined μ to be **involution invariant** if (2.1) holds for those f supported on (G, x, y) with $x \sim y$. In fact, the Mass-Transport Principle holds for general f if it holds for these special f:

Proposition 2.2. *A measure is involution invariant iff it is unimodular.*

Proof. Let μ be involution invariant. The idea is to send the mass from x to y by single steps, equally spread among the shortest paths from x to y. For the proof, we may assume that $f(G, x, y) = 0$ unless x and y are at a fixed distance, say k, from each other, since any f is a sum of such f. Now write $L(G, x, y)$ for the set of paths of length k from x to y. Let $n_j(G, x, y; z, w)$

be the number of paths in $L(G,x,y)$ such that the jth edge goes from z to w. Define $f_j(G,z,w)$ for $1 \le j \le k$ and $z, w \in \mathsf{V}(G)$ by

$$f_j(G,z,w) := \sum_{x,y \in \mathsf{V}(G)} \frac{f(G,x,y) n_j(G,x,y;z,w)}{|L(G,x,y)|}\,.$$

Then $f_j(G,z,w) = 0$ unless $z \sim w$. Furthermore, $f_j(G,z,w) := f_j(G',z',w')$ if (G,z,w) is isomorphic to (G',z',w'). Thus, f_j is well defined and Borel on $\mathcal{G}_{**}$, whence involution invariance gives us

$$\int \sum_{x \in \mathsf{V}(G)} f_j(G,o,x)\, d\mu(G,o) = \int \sum_{x \in \mathsf{V}(G)} f_j(G,x,o)\, d\mu(G,o)\,.$$

On the other hand,

$$\sum_{x \in \mathsf{V}(G)} f(G,o,x) = \sum_{x \in \mathsf{V}(G)} f_1(G,o,x)\,,$$
$$\sum_{x \in \mathsf{V}(G)} f(G,x,o) = \sum_{x \in \mathsf{V}(G)} f_k(G,x,o)\,,$$

and for $1 \le j < k$, we have

$$\sum_{x \in \mathsf{V}(G)} f_j(G,x,o) = \sum_{x \in \mathsf{V}(G)} f_{j+1}(G,o,x)\,.$$

Combining this string of equalities yields the desired equation for f. ◻

Occasionally one uses the Mass-Transport Principle for functions f that are not nonnegative. It is easy to see that this use is justified when

$$\int \sum_{x \in \mathsf{V}(G)} |f(G,o,x)|\, d\mu(G,o) < \infty\,.$$

As noted by Oded Schramm (personal communication, 2004), unimodularity can be defined for probability measures on other structures, such as hypergraphs, while involution invariance is limited to graphs (or networks on graphs).

We shall sometimes use the following property of marks. Intuitively, it says that each vertex has positive probability to be the root.

Lemma 2.3 (Everything Shows at the Root)**.** *Suppose that μ is a unimodular probability measure on $\mathcal{G}_*$. Let ξ_0 be a fixed mark and Ξ_0 be a fixed Borel set of marks. If the mark of the root is a.s. ξ_0, then the mark of every vertex is a.s. ξ_0. If every edge incident to the root a.s. has its edge mark at the root in Ξ_0, then all edge marks a.s. belong to Ξ_0.*

Proof. In the first case, each vertex sends unit mass to each vertex with a mark different from ξ_0. The expected mass received at the root is zero. Hence the expected mass sent is 0. The second case is a consequence of the first, where we put the mark ξ_0 at a vertex when all the edge marks at that vertex lie in Ξ_0. ◻

When we discuss percolation in Section 6, we shall find it crucial that we have a unimodular coupling of the various measures (given by the standard coupling of Bernoulli percolation in this case). It would also be very useful to have unimodular couplings in more general settings. We now discuss what we mean.

Suppose that $\mathcal{R} \subseteq \Xi \times \Xi$ is a closed set, which we think of as a binary relation such as the lexicographic order on Baire space. Given two measures $\mu_1, \mu_2 \in \mathcal{U}$, say that μ_1 is **$\mathcal{R}$-related** to μ_2 if there is a probability measure ν, called an **$\mathcal{R}$-coupling of μ_1 to μ_2**, on rooted networks with mark space $\Xi \times \Xi$ such that ν is concentrated on networks all of whose marks lie in $\mathcal{R}$ and whose marginal given by taking the ith coordinate of each mark is μ_i for $i = 1, 2$. In particular, μ_1 and μ_2 can be coupled to have the same underlying rooted graphs.

It would be very useful to have a positive answer to the following question. Some uses are apparent in Section 5 and in Section 10, while others appear in Lyons (2005) and are hinted at elsewhere.

Question 2.4 (Unimodular Coupling)**.** Let $\mathcal{R} \subseteq \Xi \times \Xi$ be a closed set. If $\mu_1, \mu_2 \in \mathcal{U}$ and μ_1 is $\mathcal{R}$-related to μ_2, is there then a unimodular $\mathcal{R}$-coupling of μ_1 to μ_2?

The case where μ_i are amenable is established affirmatively in Proposition 8.6. However, the case where μ_i are supported by a fixed underlying non-amenable Cayley graph is open even when the marks take only two values. Here is a family of examples to illustrate what we do not know:

Question 2.5. Let T be the Cayley graph of $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ with respect to the generators a, b, c, which are all involutions. We label the edges with the generators. Fix three Borel symmetric functions f_a, f_b, f_c from $[0,1]^2$ to $[0,1]$. Also, fix an end ξ of T. Let $U(e)$ be i.i.d. Uniform$[0,1]$ random variables indexed by the edges e of T. For each edge e, let I_e be the two edges adjacent to e that lead farther from ξ and let J_e be the two other edges that are adjacent to e. Let $L(e)$ denote the Cayley label of e, i.e., a, b, or c. For an edge e and a pair of edges $\{e_1, e_2\}$, write $f\big(e, \{e_1, e_2\}\big) := f_{L(e)}\big(U(e_1), U(e_2)\big)$. Define $X(e) := f(e, I_e)$ and $Y(e) := \max\big\{f(e, I_e), f(e, J_e)\big\}$. Let ν be the law of (X, Y). Let μ_1 be the law of X and μ_2 be the law of Y. We use the same notation for the measures in $\mathcal{U}$ given by rooting T at the vertex corresponding to the identity of the group. Let $\mathcal{R}$ be $\le$ on $[0,1] \times [0,1]$. Since $X(e) \le Y(e)$ for all e, ν is an $\mathcal{R}$-coupling of μ_1 to μ_2. In addition, μ_2 is clearly $\mathrm{Aut}(T)$-invariant (recall that the edges are labeled), while the same holds for μ_1 since it is an i.i.d. measure. Thus, μ_i are both unimodular for $i = 1, 2$. On the other hand, ν is not $\mathrm{Aut}(T)$-invariant. Is there an invariant $\mathcal{R}$-coupling of μ_1 to μ_2? In other words, is there a unimodular $\mathcal{R}$-coupling of μ_1 to μ_2?

Another example concerns monotone coupling of the wired and free uniform spanning forests (whose definitions are given below in Section 7). This question was raised in Benjamini, Lyons, Peres, and Schramm (2001), hereinafter referred to as BLPS (2001); a partial answer was given by Bowen (2004). This is not the only interesting situation involving graph inclusion. To be more precise about this relation, for a map $\psi : \Xi \to \Xi$ and a network G, let $\psi(G)$ denote the network obtained from G by replacing each mark with its image under ψ. Given a Borel subset $\Xi_0 \subseteq \Xi$ and a network G, call the subnetwork consisting of those edges both of whose edge marks lie in Ξ_0 the **Ξ_0-open subnetwork** of G. If μ and μ' are two probability measures on rooted networks, let us say that μ is **edge dominated** by μ' if there exists a measure ν on $\mathcal{G}_*$, a Borel subset $\Xi_0 \subseteq \Xi$, and Borel functions $\psi, \psi' : \Xi \to \Xi$ such that if (G', o) denotes a network with law ν and (G, o) the component of o in the Ξ_0-open subnetwork, then $\big(\psi(G), o\big)$ has law μ and $\big(\psi'(G'), o\big)$ has law μ'. If the measure ν can be chosen to be unimodular, then we say that μ is **unimodularly edge dominated** by μ'. As a special case of Question 2.4, we do not know whether the existence of such a measure ν that is not unimodular implies the existence of ν that is unimodular when μ and μ' are both unimodular themselves.

3 Fixed Underlying Graphs

Before we study general unimodular probability measures, it is useful to examine the relationship between unimodularity in the classical sense for graphs and unimodularity in the sense investigated here for random rooted network classes.

Given a graph G and a vertex $x \in \mathsf{V}(G)$, write $\mathsf{Stab}(x) := \{\gamma \in \mathrm{Aut}(G) ;\ \gamma x = x\}$ for the **stabilizer subgroup** of x. Also, write $[x] := \mathrm{Aut}(G)x$ for the orbit of x. Recall the following principle from BLPS (1999b):

MASS-TRANSPORT PRINCIPLE. *If $G = (\mathsf{V}, \mathsf{E})$ is any graph, $f : \mathsf{V} \times \mathsf{V} \to [0, \infty]$ is invariant under the diagonal action of $\mathrm{Aut}(G)$, and $o, o' \in \mathsf{V}$, then*

$$\sum_{z \in [o']} f(o, z)|\mathsf{Stab}(o')| = \sum_{y \in [o]} f(y, o')|\mathsf{Stab}(y)| \,.$$

Here, $|\bullet|$ denotes Haar measure on $\mathrm{Aut}(G)$, although we continue to use this notation for cardinality as well. Since $\mathsf{Stab}(x)$ is compact and open, $0 < |\mathsf{Stab}(x)| < \infty$. As shown in Schlichting (1979) and Trofimov (1985),

$$|\mathsf{Stab}(x)y|/|\mathsf{Stab}(y)x| = |\mathsf{Stab}(x)|/|\mathsf{Stab}(y)| \,; \,. \tag{3.1}$$

It follows easily that G is unimodular iff

$$|\mathsf{Stab}(x)y| = |\mathsf{Stab}(y)x| \tag{3.2}$$

whenever x and y are in the same orbit.

Theorem 3.1 (Unimodular Fixed Graphs)**.** *Let G be a fixed connected graph. Then G has a random root that gives a unimodular measure iff G is a unimodular graph with*

$$c := \sum_i |\mathsf{Stab}(o_i)|^{-1} < \infty \,, \tag{3.3}$$

where $\{o_i\}$ is a complete orbit section. In this case, there is only one such measure μ on random rooted graphs from G and it satisfies

$$\mu([G, x]) = c^{-1}|\mathsf{Stab}(x)|^{-1} \tag{3.4}$$

for every $x \in \mathsf{V}(G)$.

Of course, a similar statement holds for fixed networks. An example of a graph satisfying (3.3), but that is not quasi-transitive, is obtained from the random weak limit of balls in a 3-regular tree. That is, let $\mathsf{V} := \mathbb{N} \times \mathbb{N}$. Join (m, n) by edges to each of $(2m, n-1)$ and $(2m+1, n-1)$ for $n \geq 1$. The result is a tree with only one end and $\big|\mathsf{Stab}\big((m, n)\big)\big| = 2^n$.

Proof. Suppose first that G is unimodular and that $c < \infty$. Define μ by

$$\forall i \;\; \mu([G, o_i]) := c^{-1} |\mathsf{Stab}(o_i)|^{-1} .$$

To show that μ is unimodular, let $f : \mathcal{G}_{**} \to [0, \infty]$ be Borel. Since we are concerned only with the graph G, we shall write f instead as a function $f : \mathsf{V} \times \mathsf{V} \to [0, \infty]$ that is $\mathrm{Aut}(G)$-invariant. Then

$$\begin{aligned}
\int \sum_x f(o, x)\, d\mu(G, o) &= c^{-1} \sum_i \sum_x f(o_i, x) |\mathsf{Stab}(o_i)|^{-1} \\
&= c^{-1} \sum_i |\mathsf{Stab}(o_i)|^{-1} \sum_j |\mathsf{Stab}(o_j)|^{-1} \sum_{x \in [o_j]} f(o_i, x) |\mathsf{Stab}(o_j)| \\
&= c^{-1} \sum_i |\mathsf{Stab}(o_i)|^{-1} \sum_j |\mathsf{Stab}(o_j)|^{-1} \sum_{y \in [o_i]} f(y, o_j) |\mathsf{Stab}(y)| \\
&\qquad\qquad [\text{by the Mass-Transport Principle for } G] \\
&= c^{-1} \sum_i |\mathsf{Stab}(o_i)|^{-1} \sum_j |\mathsf{Stab}(o_j)|^{-1} \sum_{y \in [o_i]} f(y, o_j) |\mathsf{Stab}(o_i)| \\
&\qquad\qquad [\text{by unimodularity of } G] \\
&= c^{-1} \sum_j \sum_y f(y, o_j) |\mathsf{Stab}(o_j)|^{-1} \\
&= \int \sum_y f(y, o)\, d\mu(G, o) .
\end{aligned}$$

Since μ satisfies the Mass-Transport Principle, it is unimodular.

Conversely, suppose that μ is a unimodular probability measure on rooted versions of G. To see that G is unimodular, consider any u, v in the same orbit. Define

$$\mu\big([x]\big) := \mu\big([G, x]\big) .$$

We first show that $\mu\big([u]\big) > 0$. Every graph isomorphic to G has a well-defined notion of vertices of type $[u]$. Let each vertex x send mass 1 to each vertex of type $[u]$ that is nearest to x. This is a Borel function on $\mathcal{G}_{**}$ if we transport no mass on graphs that are not isomorphic to G. The expected mass sent is positive, whence so is the expected mass received. Since only vertices of type $[u]$ receive mass, it follows that $\mu\big([u]\big) > 0$, as desired.

Let $f(x, y) := \mathbf{1}_{\Gamma_{u,x} v}(y)$, where $\Gamma_{u,x} := \{\gamma \in \mathrm{Aut}(G)\,;\; \gamma u = x\}$. Note that $y \in \Gamma_{u,x} v$ iff $x \in \Gamma_{v,y} u$. It is straightforward to check that f is diagonally invariant under $\mathrm{Aut}(G)$. Note that

$$|\mathsf{Stab}(x) y| \mathbf{1}_{[x]}(o) = |\Gamma_{x,o} y|$$

for all $x, y, o \in \mathsf{V}(G)$. Therefore, we have

$$
\begin{aligned}
|\mathsf{Stab}(u)v|\mu([u]) &= \int |\Gamma_{u,o}v| \, d\mu(G,o) = \int \sum_x \mathbf{1}_{\Gamma_{u,o}v}(x) \, d\mu(G,o) \\
&= \int \sum_x f(o,x) \, d\mu(G,o) = \int \sum_x f(x,o) \, d\mu(G,o) \\
&\qquad\qquad\qquad\text{[by the Mass-Transport Principle for } \mu] \\
&= \int \sum_x \mathbf{1}_{\Gamma_{u,x}v}(o) \, d\mu(G,o) = \int \sum_x \mathbf{1}_{\Gamma_{v,o}u}(x) \, d\mu(G,o) \\
&= \int |\Gamma_{v,o}u| \, d\mu(G,o) = |\mathsf{Stab}(v)u|\mu([v]) \, .
\end{aligned}
$$

That is,

$$
|\mathsf{Stab}(u)v|\mu([u]) = |\mathsf{Stab}(v)u|\mu([v]) \, . \tag{3.5}
$$

Since u and v are in the same orbit, we have $[u] = [v]$, so $\mu([u]) = \mu([v])$. Since $\mu([u]) > 0$, we obtain (3.2). That is, G is unimodular. Comparison of (3.5) with (3.1) shows (3.4). ☐

Automorphism invariance for random unrooted networks on fixed underlying graphs is also closely tied to unimodularity of random rooted networks. Here, we shall need to distinguish between graphs, networks, and isomorphism classes of rooted networks. Recall that $\overline{G}$ denotes a network whose underlying graph is G and $[\overline{G}, o]$ denotes an equivalence class of networks $\overline{G}$ on G with root o.

Let G be a fixed connected unimodular graph satisfying (3.3). Fix a complete orbit section $\{o_i\}$ of $\mathsf{V}(G)$. For a graph G' and $x \in \mathsf{V}(G')$, $z \in \mathsf{V}(G)$, let $\Phi(x, z)$ be the set of rooted isomorphisms, if any, from (G', x) to (G, z). When non-empty, this set carries a natural probability measure, $\lambda'_{(G',x;z)}$ arising from the Haar probability measure on $\mathsf{Stab}(z)$. When $\Phi(x, z) = \emptyset$, let $\lambda'_{(G',x,z)} := 0$. Define

$$
\lambda_{(G',x)} := \sum_{z \in \mathsf{V}(G)} \lambda'_{(G',x;z)} \, .
$$

This is the analogue for isomorphisms from G' to G of Haar measure on $\mathrm{Aut}(G)$. In particular, any $\gamma \in \mathrm{Aut}(G)$ pushes forward $\lambda'_{(G',x,z)}$ to $\lambda'_{(G',x,\gamma z)}$.

For a graph G' isomorphic to G and $x \in \mathsf{V}(G')$, let $\tau(G', x) := o_i$ for the unique o_i for which $\Phi(x, o_i) \neq \emptyset$. Note that $\lambda_{(G',x)} = \lambda_{(G',y)}$ when $\tau(G', x) = \tau(G', y)$.

Every probability measure μ on $\mathcal{G}_*$ that is concentrated on network classes whose underlying graph is G induces a probability measure λ_μ on unrooted networks on G:

$$
\lambda_\mu(\mathcal{A}) := \int \int_{\Phi\left(o,\tau(G',o)\right)} \mathbf{1}_{\mathcal{A}}(\phi \overline{G}') \, d\lambda_{(G',o)}(\phi) \, d\mu([\overline{G}', o])
$$

for Borel sets $\mathcal{A}$ of networks on G. It is easy to see that this is well defined (the choice of $(\overline{G}', o)$ in its equivalence class not mattering).

Theorem 3.2 (Invariance and Unimodularity)**.** *Let G be a fixed connected unimodular graph satisfying* (3.3)*. Let ν be an* $\mathrm{Aut}(G)$*-invariant probability measure on unrooted networks whose underlying graph is G. Then randomly rooting the network as in* (3.4) *gives a measure $\mu \in \mathcal{U}$.*

Conversely, let $\mu \in \mathcal{U}$ be supported on networks whose underlying graph is G. Then λ_μ is $\mathrm{Aut}(G)$*-invariant.*

Proof. The first part of the theorem is proved just as is the first part of Theorem 3.1, so we turn to the second part. Let $\gamma_0 \in \mathrm{Aut}(G)$ and F be a bounded Borel-measurable function of networks on G. Invariance of λ_μ means that $\int F(\overline{G})\,d\lambda_\mu(\overline{G}) = \int F(\gamma_0\overline{G})\,d\lambda_\mu(\overline{G})$. To prove that this holds, let

$$f(\overline{G}', x, y) := \int_{\substack{\Phi\left(x,\tau(G',y)\right)\\ \cap\Phi\left(y,\gamma_0\tau(G,y)\right)}} F(\phi\overline{G}')\,d\lambda_{(G',y)}(\phi)\,.$$

It is straightforward to check that f is well defined and Borel on $\mathcal{G}_{**}$. Therefore, unimodularity of μ gives

$$\begin{aligned}
\int F(\gamma_0\overline{G})\,d\lambda_\mu(\overline{G}) &= \int\int_{\Phi\left(o,\tau(G',o)\right)} F(\gamma_0\phi\overline{G}')\,d\lambda_{(G',o)}(\phi)\,d\mu([\overline{G}',o]) \\
&= \int\int_{\Phi\left(o,\gamma_0\tau(G',o)\right)} F(\phi\overline{G}')\,d\lambda_{(G',o)}(\phi)\,d\mu([\overline{G}',o]) \\
&= \int \sum_{x\in\mathsf{V}(G')} \int_{\substack{\Phi\left(x,\tau(G',o)\right)\\ \cap\Phi\left(o,\gamma_0\tau(G',o)\right)}} F(\phi\overline{G}')\,d\lambda_{(G',o)}(\phi)\,d\mu([\overline{G}',o]) \\
&= \int \sum_{x\in\mathsf{V}(G')} f(\overline{G}',x,o)\,d\mu([\overline{G}',o]) \\
&= \int \sum_{x\in\mathsf{V}(G')} f(\overline{G}',o,x)\,d\mu([\overline{G}',o]) \\
&= \int \sum_{x\in\mathsf{V}(G')} \int_{\substack{\Phi\left(o,\tau(G',x)\right)\\ \cap\Phi\left(x,\gamma_0\tau(G',x)\right)}} F(\phi\overline{G}')\,d\lambda_{(G',x)}(\phi)\,d\mu([\overline{G}',o]) \\
&= \int \sum_{x\,;\,\tau(G',x)=\tau(G',o)} \int_{\substack{\Phi\left(o,\tau(G',x)\right)\\ \cap\Phi\left(x,\gamma_0\tau(G',x)\right)}} F(\phi\overline{G}')\,d\lambda_{(G',x)}(\phi)\,d\mu([\overline{G}',o]) \\
&= \int \sum_{x\,;\,\tau(G',x)=\tau(G',o)} \int_{\substack{\Phi\left(o,\tau(G',o)\right)\\ \cap\Phi\left(x,\gamma_0\tau(G',o)\right)}} F(\phi\overline{G}')\,d\lambda_{(G',o)}(\phi)\,d\mu([\overline{G}',o]) \\
&= \int \sum_{x\in\mathsf{V}(G')} \int_{\substack{\Phi\left(o,\tau(G',o)\right)\\ \cap\Phi\left(x,\gamma_0\tau(G',o)\right)}} F(\phi\overline{G}')\,d\lambda_{(G',o)}(\phi)\,d\mu([\overline{G}',o]) \\
&= \int\int_{\Phi\left(o,\tau(G',o)\right)} F(\phi\overline{G}')\,d\lambda_{(G',o)}(\phi)\,d\mu([\overline{G}',o]) \\
&= \int F(\overline{G})\,d\lambda_\mu(G)\,.
\end{aligned}$$

□

Remark 3.3. As this section shows, unimodular quasi-transitive graphs are special cases of unimodular rooted networks. However, sometimes one is interested in random networks on a graph G that are not necessarily invariant under the full group $\mathrm{Aut}(G)$, but only under some subgroup, $\Gamma \subset \mathrm{Aut}(G)$. This is common when G is a Cayley graph of Γ. In this case, we could mark the edges by the generators they represent; that is, if $x, y \in \Gamma$ and $y = xa$ with a one of the generators used to form G, then we can mark the edge $[x, y]$ at x by a. This makes the full automorphism group of the network $\overline{G}$ equal to Γ, rather than to $\mathrm{Aut}(G)$. The theory here then goes through with only a complication of notation. However, given any graph G and any closed subgroup $\Gamma \subset \mathrm{Aut}(G)$ that acts quasi-transitively on G, we do not know whether it is possible to mark the edges and vertices of G to get a network whose automorphism group is equal to Γ. Yet, the theory for quasi-transitive subgroups is the same; see BLPS (1999b).

4 Random Walks and Extremality

Random walks on networks, besides being of intrinsic interest, form an important tool for studying networks. A random walk is most useful when it has a stationary measure, in other words, when the distribution of (G, w_0) is the same as the distribution of (G, w_1), where w_0 is the initial location of the random walk and w_1 is the next location of the random walk.

Consider simple random walk on a random graph chosen by a unimodular probability measure μ on rooted graphs, where we start the random walk at the root. Just as for finite graphs, we do not expect μ to be stationary for the random walk; rather, we get a stationary measure by biasing μ by the degree of the root. The fact that this measure is stationary follows from the definition of involution invariance; in fact, the definition is precisely that simple random walk is reversible, i.e., that the distribution of $\big((G, w_0), (G, w_1)\big)$ is the same as the distribution of $\big((G, w_1), (G, w_0)\big)$, where (G, w_0) has distribution μ biased by the degree of the root and w_1 is a uniform random neighbor of the root.[1] If $\overline{\deg}(\mu) < \infty$, then we can normalize the biased measure to obtain a probability measure.

In particular, recall from Example 1.1 the definition of the augmented Galton-Watson measure AGW. In Lyons, Pemantle, and Peres (1995), it was remarked in reference to the stationarity of AGW for simple random walk that "unlike the situation for finite graphs, there is no biasing in favor of vertices of large degree". However, we now see that contrary to this remark, the situations of finite graphs and AGW are, in fact, parallel. That is because the biasing by the degree has already been made part of the probability measure AGW. The correct comparison of the uniform measure on vertices of finite graphs is to the unimodular Galton-Watson probability measure on trees, UGW, because it is for this measure that "all vertices are equally likely to be the root".

More generally, we can consider stationarity of random walk in a random environment with random scenery. Here, if the graph underlies a network, the marks are not restricted to play a passive role, but may, in fact, determine the transition probabilities (as in Section 5) and provide a scenery for the random walk. That is, a Borel function $p : \mathcal{G}_{**} \to [0, 1]$, written as $p : (G, x, y) \mapsto p_G(x, y)$, such that $\sum_{y \in \mathsf{V}} p_G(x, y) = 1$ for all vertices x is called an **environment**. A Borel map $\nu : \mathcal{G}_* \to (0, \infty)$, written $\nu : (G, x) \mapsto \nu_G(x)$, is called an **initial bias**. It is called

[1] Note that the degree times counting measure is reversible on every graph, regardless of unimodularity of the measure on rooted graphs.

p**-stationary** if for all G, the measure ν_G is stationary for the random walk on G given by the environment p_G. Write $\mathcal{P}_*$ for the set of (equivalence classes of) pairs $\big((G, w_0), \langle w_n\, ;\ n \ge 0\rangle\big)$ with $(G, w_0) \in \mathcal{G}_*$ and $w_n \in \mathsf{V}(G)$. Let $\widehat{\mu}$ denote the distribution on $\mathcal{P}_*$ of the trajectory of the Markov chain determined by the environment starting at o with initial distribution equal to μ biased by $\nu_G(o)$. That is, if $\theta_{(G,o)}$ denotes the probability measure on $\mathcal{P}_*$ determined by the environment on G with initial vertex $w_0 = o$, then for all events $\mathcal{B}$, we have

$$\widehat{\mu}(\mathcal{B}) := \int_{\mathcal{G}_*} \theta_{(G,o)}(\mathcal{B}) \nu_G(o)\, d\mu(G, o)\,.$$

Let $\mathcal{I}$ denote the σ-field of events (in the Borel σ-field of $\mathcal{G}_*$) that are invariant under non-rooted isomorphisms. To avoid possible later confusion, note that this does not depend on the measure μ, so that even if there are no non-trivial non-rooted isomorphisms μ-a.s., the σ-field $\mathcal{I}$ is still not equal (mod 0) to the σ-field of μ-measurable sets. It is easy to see that for any $\mu \in \mathcal{U}$ and $\mathcal{A} \in \mathcal{I}$ with $\mu(\mathcal{A}) > 0$, the probability measure $\mu(\,\bullet \mid \mathcal{A})$ is also unimodular. Define the **shift** $\mathcal{S} : \mathcal{P}_* \to \mathcal{P}_*$ by

$$\mathcal{S}\big((G, w_0), \langle w_n\rangle\big) := \big((G, w_1), \langle w_{n+1}\rangle\big)\,.$$

The following extends Theorem 3.1 of Lyons and Schramm (1999b); the proof is essentially the same.

Theorem 4.1 (Random Walk in a Random Environment and Random Scenery)**.** *Let μ be a unimodular probability measure on $\mathcal{G}_*$. Let $p_\bullet(\bullet)$ be an environment and $\nu_\bullet(\bullet)$ be an initial bias that is p-stationary. Let $\widehat{\mu}$ be the corresponding measure on trajectories. Then $\widehat{\mu}$ is stationary for the shift. If p is also reversible with respect to $\nu_\bullet(\bullet)$, then $\widehat{\mu}$ is reversible, in other words, for all events $\mathcal{A}$, $\mathcal{B}$, we have*

$$\widehat{\mu}[(G, w_0) \in \mathcal{A}, (G, w_1) \in \mathcal{B}] = \widehat{\mu}[(G, w_1) \in \mathcal{A}, (G, w_0) \in \mathcal{B}]\,.$$

If

$$\int \nu_G(o)\, d\mu(G, o) = 1\,, \tag{4.1}$$

then $\widehat{\mu}$ is a probability measure.

Proof. The reversibility was not mentioned in prior work, so we give that proof here. Assuming that p is ν-reversible, we have

$$\begin{aligned}\widehat{\mu}[(G, w_0) \in \mathcal{A}, (G, w_1) \in \mathcal{B}] &= \mathbf{E}\big[\sum_{x \in \mathsf{V}(G)} \mathbf{1}_{\mathcal{A}}(G, o)\nu_G(o)p_G(o, x)\mathbf{1}_{\mathcal{B}}(G, x)\big] \\ &= \mathbf{E}\big[\sum_{x \in \mathsf{V}(G)} \mathbf{1}_{\mathcal{A}}(G, o)\nu_G(x)p_G(x, o)\mathbf{1}_{\mathcal{B}}(G, x)\big]\,.\end{aligned}$$

The Mass-Transport Principle now gives that this

$$= \mathbf{E}\big[\sum_{x \in \mathsf{V}(G)} \mathbf{1}_{\mathcal{A}}(G, x)\nu_G(o)p_G(o, x)\mathbf{1}_{\mathcal{B}}(G, o)\big] = \widehat{\mu}[(G, w_1) \in \mathcal{A}, (G, w_0) \in \mathcal{B}]\,.$$

$\square$

Remark 4.2. This theorem is made more useful by noticing that for any $\mu \in \mathcal{U}$, there is a choice of $p_\bullet(\bullet)$ and $\nu_\bullet(\bullet)$ that satisfies all the hypotheses, including (4.1). For example, if $F_G(x)$ denotes $\sum_{y\sim x} 1/\deg_G(y)$, then let $p_G(x,y) := 1/[F_G(x)\deg_G(y)]$ and $\nu_G(x) := Z^{-1}F_G(x)/\deg_G(x)$, where

$$Z := \int F_G(o)/\deg_G(o)\, d\mu(G,o)\,.$$

It is clear that p is an environment. Since $F_G(o) \le \sum_{y\sim x} 1 = \deg_G(o)$, we also have that $Z < \infty$, so that ν is a p-stationary initial bias and p is ν-reversible.

Given a network with positive edge weights and a time $t > 0$, form the **transition operator** P_t for continuous-time random walk whose rates are the edge weights; in the case of unbounded weights (or degrees), we take the minimal process, which dies after an explosion. That is, if the entries of a matrix A indexed by the vertices are equal off the diagonal to the negative of the edge weights and the diagonal entries are chosen to make the row sums zero, then $P_t := e^{-At}$; in the case of unbounded weights, we take the self-adjoint extension of A corresponding to the minimal process. The matrix A is called the **infinitesimal generator** or the **Laplacian** of the network.

Corollary 4.3. *Suppose that $\mu \in \mathcal{U}$ is carried by networks with non-negative edge weights such that the corresponding continuous-time Markov chain has no explosions a.s. Then μ is stationary and reversible.*

Proof. Fix $t > 0$ and let $p_G(x,y) := P_t(x,y)$. It is well known that p is reversible with respect to the uniform measure $\nu_G \equiv 1$. Thus, Theorem 4.1 applies. ◻

We can also obtain a sufficient condition for lack of explosions:

Corollary 4.4. *Suppose that $\mu \in \mathcal{U}$ is carried by networks with non-negative edge weights $c_G(e)$ such that $Z := \mathbf{E}[\sum_{x\sim o} c_G(o,x)] < \infty$. Then the corresponding continuous-time Markov chain has no explosions.*

Proof. In this case, consider the discrete-time Markov chain corresponding to these weights. It has a stationary probability measure arising from the choice $\nu_G(x) := \sum_{y\sim o} c_G(x,y)/Z$. It is well known that explosions occur iff

$$\sum_{n\ge 0} \nu_G(w_n)^{-1} < \infty$$

with positive probability. However, stationarity guarantees that this sum is infinite a.s. (by the Poincaré recurrence theorem). ◻

Remark 4.5. It is possible for explosions to occur: For example, consider the uniform spanning tree T in $\mathbb{Z}^2$ (see BLPS (2001)). The only fact we use about T is that it has one end a.s. and has an invariant distribution. Let $c_G(e) := 0$ for $e \notin T$ and $c_G(e) := 2^{f(e)}$ when $e \in T$ and $f(e)$ is the number of vertices in the finite component of $T \setminus e$. Then it is easy to verify that the corresponding continuous-time Markov chain explodes a.s.

Furthermore, explosions may occur on a fixed transitive graph that is not unimodular, even if the condition in Corollary 4.4 is satisfied. To see this, let ξ be a fixed end of a regular tree T of degree 3. Thus, for every vertex x in T, there is a unique ray $x_\xi := \langle x_0 = x, x_1, x_2, \ldots\rangle$

starting at x such that x_ξ and y_ξ differ by only finitely many vertices for any pair x, y. Call x_1 the **ξ-parent** of x, call x a **ξ-child** of x_1, and call x_2 the **ξ-grandparent** of x. Let G be the graph obtained from T by adding the edges (x, x_2) between each x and its ξ-grandparent. Then G is a transitive graph, first mentioned by Trofimov (1985). In fact, every automorphism of G fixes ξ. Now consider the following random weights on G. Put weight 0 on every edge in G that is not in T. For each vertex of G, declare open the edge to precisely one of its two ξ-children, chosen uniformly and independently for different vertices. The open components are rays. Let the weight of every edge that is not open also be 0. If an edge (x, y) between a vertex x and its ξ-parent y is open and y is at distance n from the beginning of the open ray containing (x, y), then let the weight of the edge be $(3/2)^n$. Since this event has probability $1/2^{n+1}$, the condition of Corollary 4.4 is clearly satisfied. It is also clear that the Markov chain explodes a.s.

The class $\mathcal{U}$ of unimodular probability measures on $\mathcal{G}_*$ is clearly convex. An element of $\mathcal{U}$ is called **extremal** if it cannot be written as a convex combination of other elements of $\mathcal{U}$. We shall show that the extremal measures are those for which $\mathcal{I}$ contains only sets of measure 0 or 1. Intuitively, they are the extremal measures for unrooted networks since the distribution of the root is forced given the distribution of the unrooted network. For example, one may show that UGW is extremal when conditioned on non-extinction. First, we show the following ergodicity property, analogous to Theorem 5.1 of Lyons and Schramm (1999a). Recall that a σ-field is called **μ-trivial** if all its elements have measure 0 or 1 with respect to μ.

Theorem 4.6 (Ergodicity)**.** *Let μ be a unimodular probability measure on $\mathcal{G}_*$. Let $\mathrm{p}_\bullet(\bullet)$ be an environment that satisfies*

$$\forall G \;\; \forall x, y \in \mathsf{V}(G) \qquad x \sim y \Longrightarrow p_G(x, y) > 0 \tag{4.2}$$

and $\nu_\bullet(\bullet)$ be an initial bias that is p-stationary and satisfies (4.1). Let $\widehat{\mu}$ be the corresponding probability measure on trajectories. If $\mathcal{I}$ is μ-trivial, then every event that is shift invariant is $\widehat{\mu}$-trivial. More generally, the events $\mathcal{B}$ in the $\widehat{\mu}$-completion of the shift-invariant σ-field are those of the form

$$\mathcal{B} = \big\{ \big((G, o), w\big) \in \mathcal{P}_* \, ; \; (G, o) \in \mathcal{A} \big\} \triangle \, \mathcal{C} \tag{4.3}$$

for some $\mathcal{A} \in \mathcal{I}$ and some event $\mathcal{C}$ with $\widehat{\mu}(\mathcal{C}) = 0$.

Proof. Let $\mathcal{B}$ be a shift-invariant event. As in the proof of Theorem 5.1 of Lyons and Schramm (1999a), we have $\theta_{(G,o)}(\mathcal{B}) \in \{0, 1\}$ μ-a.s. The set $\mathcal{A}$ of (G, o) where this probability equals 1 is in $\mathcal{I}$ by (4.2), and a little thought reveals that (4.3) holds for some $\mathcal{C}$ with $\widehat{\mu}(\mathcal{C}) = 0$. If $\mathcal{I}$ is μ-trivial, then $\mu(\mathcal{A}) \in \{0, 1\}$, whence $\widehat{\mu}(\mathcal{B}) \in \{0, 1\}$ as desired. Conversely, every event $\mathcal{B}$ of the form (4.3) is clearly in the $\widehat{\mu}$-completion of the shift-invariant σ-field. ∎

We may regard the space $\mathcal{P}_*$ as the space of sequences of rooted networks, where all roots belong to the same network. Thus, $\mathcal{P}_*$ is the natural trajectory space for the Markov chain with the transition probability from (G, x) to (G, y) given by $p_G(x, y)$. With this interpretation, Theorem 4.6 says that this Markov chain is ergodic when $\mathcal{I}$ is μ-trivial. The next theorem says that this latter condition is, in turn, equivalent to extremality of μ.

Theorem 4.7 (Extremality)**.** *A unimodular probability measure μ on $\mathcal{G}_*$ is extremal iff $\mathcal{I}$ is μ-trivial.*

Proof. Let $\mathcal{A} \in \mathcal{I}$. If $\mathcal{A}$ is not μ-trivial, then we may write μ as a convex combination of μ conditioned on $\mathcal{A}$ and μ conditioned on the complement of $\mathcal{A}$. Each of these two new probability measures is unimodular, yet distinct, so μ is not extremal.

Conversely, suppose that $\mathcal{I}$ is μ-trivial. Choose an environment and stationary initial bias that satisfy (4.1) and (4.2), as in Remark 4.2. Let $\mathcal{A}$ be an event of $\mathcal{G}_*$. Let α be the function on $\mathcal{P}_*$ that gives the frequency of visits to $\mathcal{A}$:

$$\alpha\big((G, w_0), \langle w_n \rangle)\big) := \liminf_{N\to\infty} \frac{1}{N} |\{n \le N\,;\ (G, w_n) \in \mathcal{A}\}|\,.$$

Theorem 4.1 allows us to apply the ergodic theorem to deduce that $\int \alpha\, d\widehat{\mu} = (\nu\mu)(\mathcal{A})$, where $\nu\mu$ stands for the measure $d(\nu\mu)(G, o) = \nu_G(o) d\mu(G, o)$. On the other hand, α is a shift-invariant function, which, according to Theorem 4.6, means that α is a constant $\widehat{\mu}$-a.s. Thus, we conclude that $\alpha = (\nu\mu)(\mathcal{A})$ $\widehat{\mu}$-a.s. Consider any non-trivial convex combination of two unimodular probability measures, μ_1 and μ_2, that gives μ. Then $\widehat{\mu}$ is a (possibly different) convex combination of $\widehat{\mu}_1$ and $\widehat{\mu}_2$. The above applies to each of $\widehat{\mu}_i$ $(i = 1, 2)$ and the associated probability measures $a_i\nu\mu_i$, where $a_i := \big(\int \nu_G(o)\, d\mu_i(G, o)\big)^{-1}$. Therefore, we obtain that $a_1(\nu\mu_1)(\mathcal{A}) = (\nu\mu)(\mathcal{A}) = a_2(\nu\mu_2)(\mathcal{A})$. Since this holds for all $\mathcal{A}$, we obtain $a_1(\nu\mu_1) = a_2(\nu\mu_2)$. Since μ_1 and μ_2 are probability measures, this is the same as $\mu_1 = \mu_2$, whence μ is extremal. ☐

We define the **speed** of a path $\langle w_n \rangle$ in a graph G to be $\lim_{n\to\infty} \mathrm{dist}_G(w_0, w_n)/n$ when this limit exists, where dist_G indicates the distance in the graph G.

The following extends Lemma 4.2 of Benjamini, Lyons, and Schramm (1999).

Proposition 4.8 (Speed Exists)**.** *Let μ be a unimodular probability measure on $\mathcal{G}_*$ with an environment and stationary initial distribution $\nu_\bullet(\bullet)$ with $\int \nu_G(o)\, d\mu(G, o) = 1$, so that the associated random walk distribution $\widehat{\mu}$ is a probability measure. Then the speed of random walk exists $\widehat{\mu}$-a.s. and is equal $\widehat{\mu}$-a.s. to an $\mathcal{I}$-measurable function. The same holds for simple random walk when $\overline{\deg}(\mu) < \infty$.*

Proof. Let $f_n\big((G, o), w\big) := \mathrm{dist}_G\big(w(0), w(n)\big)$. Clearly

$$f_{n+m}\big((G, o), w\big) \le f_n\big((G, o), w\big) + f_m\big(\mathcal{S}^n\big((G, o), w\big)\big)\,,$$

so that the Subadditive Ergodic Theorem ensures that the speed $\lim_{n\to\infty} f_n\big((G, o), w\big)/n$ exists $\widehat{\mu}$-a.s. Since the speed is shift invariant, Theorem 4.6 shows that the speed is equal $\widehat{\mu}$-a.s. to an $\mathcal{I}$-measurable function. The same holds for simple random walk since it has an equivalent stationary probability measure (degree biasing) when $\overline{\deg}(\mu) < \infty$. ☐

In the case of simple random walk on trees, we can actually calculate the speed:

Theorem 4.9 (Speed on Trees)**.** *Let $\mu \in \mathcal{U}$ be concentrated on infinite trees. If μ is extremal and $\overline{\deg}(\mu) < \infty$, then the speed of simple random walk is $\widehat{\mu}$-a.s.*

$$\int \frac{\deg_G(o) - 2}{\deg_G(o)} d\mu(G, o)\,. \tag{4.4}$$

This is positive iff $\overline{\deg}(\mu) > 2$.

Proof. Given a rooted tree (G, o) and $x \in \mathsf{V}(G)$, write $|x|$ for the distance in G between o and x. The speed of a path $\langle w_n \rangle$ is the limit

$$\lim_{n\to\infty} \frac{1}{n}|w_n| = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\big(|w_{k+1}| - |w_k|\big)\,.$$

Now the strong law of large numbers for martingale differences (Feller (1971), p. 243) gives

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\big(|w_{k+1}| - |w_k|\big) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\mathbf{E}\big[|w_{k+1}| - |w_k| \bigm| \langle w_i\,;\ i \le k\rangle\big] \quad \text{a.s.}$$

Provided $w_k \ne o$, the kth term on the right equals

$$\frac{\deg_G w_k - 2}{\deg_G w_k}\,. \tag{4.5}$$

Since G is a.s. infinite, $w_k = o$ for only a set of k of density 0 a.s., whence the speed equals

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\frac{\deg_G w_k - 2}{\deg_G w_k}\,.$$

Since this is the limit of averages of an ergodic stationary sequence, the ergodic theorem tells us that it converges a.s. to the mean of an element of the sequence, which is (4.4).

According to the Cauchy-Schwarz inequality,

$$\int \frac{\deg_G(o) - 2}{\deg_G(o)} d\mu(G, o) \int \deg_G(o) d\mu(G, o) \ge \int \big(\deg_G(o) - 2\big) d\mu(G, o)\,,$$

which proves half of the last sentence. The other half follows from the trivial inequality

$$\int \frac{\deg_G(o) - 2}{\deg_G(o)} d\mu(G, o) \le \int \big(\deg_G(o) - 2\big) d\mu(G, o)\,.$$

$\square$

When we study percolation, the following consequence will be useful.

Proposition 4.10 (Comparison of Transience on Trees)**.** *Suppose $\mu \in \mathcal{U}$ is concentrated on networks whose underlying graphs are trees that are transient for simple random walk. Suppose that the mark space is $(0, \infty)$, that marks $\psi(\bullet, \bullet)$ on edges are the same at both endpoints, that the environment is $p_G(x, y) := \psi(x, y)/\nu_G(x)$, where $\nu_G(x) := \sum_{y\sim x}\psi(x, y)$, and that $\int \nu_G(o)\, d\mu(G, o) = 1$, so that the associated random walk distribution $\widehat{\mu}$ is a probability measure. Then random walk is also transient with respect to the environment $p_\bullet(\bullet)$ μ-a.s.*

Proof. Let $\mathcal{A}$ be the set of $p_\bullet(\bullet)$-recurrent networks. Suppose that $\mu(\mathcal{A}) > 0$. By conditioning on $\mathcal{A}$, we may assume without loss of generality that $\mu(\mathcal{A}) = 1$. By Theorem 6.2 and the recurrence of simple random walk on trees with at most two ends, we have $\overline{\deg}(\mu) > 2$. Let $\epsilon > 0$ be sufficiently small that

$$\int |\{x \sim o\,;\ \psi(o, x) \ge \epsilon\}|\, d\mu(G, o) > 2\,.$$

Since finite trees have average degree strictly less than 2, is follows that the subnetwork (G_ϵ, o), defined to be the connected component of o formed by the edges with marks at least ϵ, is infinite with positive probability. Let μ' be the law of (G_ϵ, o) when (G, o) has the law μ, and conditioned on the event $\mathcal{B}$ that (G_ϵ, o) is infinite. Then $\mu' \in \mathcal{U}$ and $\overline{\deg}(\mu') > 2$. By Proposition 4.9, simple random walk has positive speed μ'-a.s., so, in particular, is transient a.s. Now simple random walk is the walk corresponding to all edge weights in G_ϵ being, say, $1/\epsilon$. Rayleigh's monotonicity principle (Doyle and Snell (1984) or Lyons with Peres (2008)) now implies that random walk is transient $\widehat{\mu}$-a.s. on $\mathcal{B}$. Thus, $\mathcal{A} \cap \mathcal{B} = \emptyset$. This contradicts our initial assumption that $\mu(\mathcal{A}) = 1$. $\square$

The converse of Proposition 4.10 is not true, as there are transient reversible random walks on 1-ended trees (see Example 9.2 for an example of such graphs; weights can be defined appropriately). Also, Proposition 4.10 does not extend to arbitrary networks, as one may construct an invariant network on $\mathbb{Z}^3$ that gives a recurrent random walk.

Given two probability measures μ and μ' on rooted networks and one of the standard notions of product networks, one can define the **independent product** $\mu \boxtimes \mu'$ of the two measures by choosing a network from each measure independently and taking their product, rooted at the ordered pair of the original roots.

Proposition 4.11 (Product Networks)**.** *Let μ and μ' be two unimodular probability measures on $\mathcal{G}_*$. Then their independent product $\mu \boxtimes \mu'$ is also unimodular. If μ and μ' are both extremal, then so is $\mu \boxtimes \mu'$.*

Proof. Let G_n and G'_n be finite connected networks whose random weak limits are μ and μ', respectively. Then $G_n \times G'_n$ clearly has random weak limit $\mu \boxtimes \mu'$, whence the product is unimodular. Now suppose that both μ and μ' are extremal. Let $\mathcal{A} \in \mathcal{I}$. Then $\mathcal{A}_{(G',o')} := \{(G, o)\,;\ (G, o) \times (G', o') \in \mathcal{A}\} \in \mathcal{I}$ since $\mathrm{Aut}(G) \times \mathrm{Aut}(G') \subseteq \mathrm{Aut}(G \times G')$. Therefore, $\mu\mathcal{A}_{(G',o')} \in \{0, 1\}$. On the other hand, $\mathcal{A}_{(G',o')} = \mathcal{A}_{(G',o'')}$ for all $o'' \in \mathsf{V}(G')$ because $\mathcal{A} \in \mathcal{I}$. Therefore, $\mathcal{B} := \{(G', o')\,;\ \mu\mathcal{A}_{(G',o')} = 1\} \in \mathcal{I}$, whence $\mu'\mathcal{B} \in \{0, 1\}$. Hence Fubini's theorem tells us that $(\mu \boxtimes \mu')(\mathcal{A}) \in \{0, 1\}$, as desired. $\square$

Remark 4.12. Another type of product that can sometimes be defined does not always produce an extremal measure from two extremal measures. That is, suppose that μ and μ' are two extremal unimodular probability measures on $\mathcal{G}_*$ that, for simplicity, we assume are concentrated on networks with a fixed underlying transitive graph, G. Let μ'' be the measure on networks given by taking a fixed root, o, and choosing the marks as (ψ, ψ'), where ψ gives a network with law μ, ψ' gives a network with law μ', and ψ, ψ' are independent. Then it may be that μ'' is not extremal. For an example, consider the following. Fix an irrational number, α. Given $x \in [0, 1]$, form the network G_x on the integer lattice graph by marking each integer n with the indicator that the fractional part of $n\alpha + x$ lies in $[0, 1/2]$. Let μ be the law of $(G_x, 0)$ when x is chosen uniformly. Then μ is unimodular and extremal (by ergodicity of Lebesgue measure with respect to rotation by α), but if $\mu' = \mu$ and μ'' is the associated measure above, then μ'' is not extremal since when the marks come from $x, y \in [0, 1]$, the fractional part of $x - y$ is $\mathcal{I}$-measurable.

It may be useful to keep in mind the vast difference between stationarity and reversibility in this context. For example, let T be a 3-regular tree and ζ be an end of T. Mark each edge by two independent random variables, one that is uniform on $[0, 1]$ and the other uniform on $[1, 2]$, with the latter one at its endpoint closer to ζ and with all these random variables mutually independent for different edges. Then simple random walk is stationary in this scenery, but not reversible, even though T is a Cayley graph.

5 Trace and Stochastic Comparison

There is a natural trace associated to every measure in $\mathcal{U}$. This trace is useful for making various comparisons. We illustrate this by extending results of Pittet and Saloff-Coste (2000) and Fontes and Mathieu (2006) on return probabilities of continuous-time random walks.

Suppose that μ is a unimodular probability measure on $\mathcal{G}_*$. Consider the Hilbert space $H := \int^{\oplus} \ell^2\big(\mathsf{V}(G)\big)\, d\mu(G,o)$, a direct integral (see, e.g., Nielsen (1980) or Kadison and Ringrose (1997), Chapter 14). Here, we always choose the canonical representative for each network, which, recall, is a network on the vertex set $\mathbb{N}$. The space H is defined as the set of (μ-equivalence classes of) μ-measurable functions f defined on canonical rooted networks (G,o) that satisfy $f(G,o) \in \ell^2(\mathbb{N})$ and $\int \|f(G,o)\|^2\, d\mu(G,o) < \infty$. We write $f = \int^{\oplus} f(G,o)\, d\mu(G,o)$. The inner product is given by $(f,g) := \int \big(f(G,o), g(G,o)\big)\, d\mu(G,o)$. Let $T : (G,o) \mapsto T_{G,o}$ be a measurable assignment of bounded linear operators on $\ell^2\big(\mathsf{V}(G)\big) = \ell^2(\mathbb{N})$ with finite supremum of the norms $\|T_{G,o}\|$. Then T induces a bounded linear operator $T := T^\mu := \int^{\oplus} T_{G,o}\, d\mu(G,o)$ on H via

$$T^\mu : \int^{\oplus} f(G,o)\, d\mu(G,o) \mapsto \int^{\oplus} T_{G,o} f(G,o)\, d\mu(G,o)\,.$$

The norm $\|T^\mu\|$ of T^μ is the μ-essential supremum of $\|T_{G,o}\|$. Identify each $x \in \mathsf{V}(G)$ with the vector $\mathbf{1}_{\{x\}} \in \ell^2\big(\mathsf{V}(G)\big)$. Let $\mathsf{Alg} = \mathsf{Alg}(\mu)$ be the von Neumann algebra of (μ-equivalence classes of) such maps T that are equivariant in the sense that for all network isomorphisms $\phi : G_1 \to G_2$, all $o_1, x, y \in \mathsf{V}(G_1)$ and all $o_2 \in \mathsf{V}(G_2)$, we have $(T_{G_1,o_1}x, y) = (T_{G_2,o_2}\phi x, \phi y)$. For $T \in \mathsf{Alg}$, we have in particular that $T_{G,o}$ depends on G but not on the root o, so we shall simplify our notation and write T_G in place of $T_{G,o}$. For simplicity, we shall even write T for T_G when no confusion can arise. Recall that if S and T are self-adjoint operators on a Hilbert space H, we write $S \le T$ if $(Su,\, u) \le (Tu,\, u)$ for all $u \in H$. We claim that

$$\mathrm{Tr}(T) := \mathrm{Tr}_\mu(T) := \mathbf{E}\big[(T_G o, o)\big] := \int (T_G o, o)\, d\mu(G,o)$$

is a **trace** on Alg, i.e., $\mathrm{Tr}(\bullet)$ is linear, $\mathrm{Tr}(T) \ge 0$ for $T \ge 0$, and $\mathrm{Tr}(ST) = \mathrm{Tr}(TS)$ for $S, T \in \mathsf{Alg}$. Linearity of Tr is obvious. Also, the second property is obvious since the integrand is nonnegative for $T \ge 0$. The third property follows from the Mass-Transport Principle: We have

$$\begin{aligned}
\mathbf{E}\big[(STo,o)\big] &= \mathbf{E}\big[(To, S^*o)\big] = \mathbf{E}\Bigg[\sum_{x \in \mathsf{V}(G)} (To,x)(x,S^*o)\Bigg] \\
&= \mathbf{E}\Bigg[\sum_{x \in \mathsf{V}(G)} (To,x)(Sx,o)\Bigg] = \mathbf{E}\Bigg[\sum_{x \in \mathsf{V}(G)} (Tx,o)(So,x)\Bigg] \\
&= \mathbf{E}\big[(TSo,o)\big]\,.
\end{aligned}$$

In order to justify this use of the Mass-Transport Principle, we check absolute integrability:

$$\mathbf{E}\Big[\sum_{x\in\mathsf{V}(G)}|(To,x)(x,S^*o)|\Big] \le \bigg(\mathbf{E}\Big[\sum_{x\in\mathsf{V}(G)}|(To,x)|^2\Big]\mathbf{E}\Big[\sum_{x\in\mathsf{V}(G)}|(x,S^*o)|^2\Big]\bigg)^{1/2}$$
$$= \bigg(\mathbf{E}\big[\|To\|^2\big]\mathbf{E}\big[\|S^*o\|^2\big]\bigg)^{1/2} \le \|T\|\cdot\|S\| < \infty\,.$$

A general property of traces that are finite and normal, as ours is, is that if $S \le T$, then $\mathrm{Tr}f(S) \le \mathrm{Tr}f(T)$ for any increasing function $f:\mathbb{R}\to\mathbb{R}$. One proof is as follows. First, if $f \ge 0$ and T is self-adjoint, then $f(T) \ge 0$. Second, if $S, T \ge 0$, then $\mathrm{Tr}(ST) = \mathrm{Tr}\big(S^{1/2}TS^{1/2}\big) \ge 0$ since $S^{1/2}TS^{1/2} = (T^{1/2}S^{1/2})^*(T^{1/2}S^{1/2}) \ge 0$. Third, if f is an increasing polynomial and $S \le T$, then

$$\frac{d}{dz}\Big(\mathrm{Tr}f\big(S+z(T-S)\big)\Big) = \mathrm{Tr}\Big(f'\big(S+z(T-S)\big)(T-S)\Big) \ge 0$$

for $z \ge 0$ since $f' \ge 0$, $S+z(T-S)$ is self-adjoint, and $0 \le T-S$. This shows that with these restrictive hypotheses, $\mathrm{Tr}f(S) \le \mathrm{Tr}f(T)$. Fourth, any monotone increasing function can be approximated by an increasing polynomial. This shows the result in general. See Brown and Kosaki (1990), pp. 6–7 for another proof. (They stated the result only for continuous f with $f(0) = 0$ because they dealt with more general traces and operators, but such restrictions are not needed in our situation.)

Recall from Section 4 that given a network with positive edge weights and a time $t > 0$, we form the transition operator P_t for continuous-time random walk whose rates are the edge weights; in the case of unbounded weights (or degrees), we take the minimal process, which dies after an explosion. If A is the Laplacian of the network, then $P_t := e^{-At}$.

Theorem 5.1 (Return Probabilities)**.** *Let the mark space be $\mathbb{R}^+$ and let $\mathcal{R}$ be $\le$. Let $\mu_i \in \mathcal{U}$ have edge weights that are the same at both ends of each edge ($i = 1, 2$). Suppose that there is a unimodular $\mathcal{R}$-coupling ν of μ_1 to μ_2. Let $P_t^{(i)}$ be the transition operators corresponding to the edge weights ($i = 1, 2$). Then*

$$\int P_t^{(1)}(o,o)\,d\mu_1(G,o) \ge \int P_t^{(2)}(o,o)\,d\mu_2(G,o)$$

for all $t > 0$.

Proof. Suppose first that the weights are bounded a.s. The assumptions tell us that the infinitesimal generators $A^{(i)} \in \mathsf{Alg}$ satisfy $A^{(1)} \le A^{(2)}$, so that for all $t > 0$, we have $-A^{(1)}t \ge -A^{(2)}t$. Therefore $\int P_t^{(1)}(o,o)\,d\mu_1(G,o) = \mathrm{Tr}_\nu(e^{-A^{(1)}t}) \ge \mathrm{Tr}_\nu(e^{-A^{(2)}t}) = \int P_t^{(2)}(o,o)\,d\mu_2(G,o)$.

In the case of unbounded weights, note that $P_t^{(i)}(o,o) = \lim_{n\to\infty} P_t^{(i,n)}(o,o)$ for $i = 1, 2$, where the superscript n refers to truncating all weights larger than n. (This is because we are looking at the minimal process, which is reversible.) Thus, the result follows from the bounded case, which we have proved. ☐

Theorem 5.1 extends a result of Fontes and Mathieu (2006), who proved it in the case of $\mathbb{Z}^d$ for processes without explosions. Pittet and Saloff-Coste (2000), Lemma 3.1, prove an analogous comparison result for Cayley graphs, but with different assumptions on the pairs of rates (which are deterministic for them). The case of Theorem 5.1 specialized to finite networks was proved earlier by Benjamini and Schramm; see Theorem 3.1 of Heicklen and Hoffman (2005).

This theorem also gives a partial answer to a question of Fontes and Mathieu, who asked whether the same holds when μ_i are supported on a single Cayley graph, are invariant under the group action, and are $\mathcal{R}$-related. For example, the theorem shows that it holds when the networks (which are the environments for the random walks) are given by i.i.d. edge marks, since in such a case, it is trivial that being $\mathcal{R}$-related implies the existence of a unimodular $\mathcal{R}$-coupling. Also, in the amenable case, the existence of a unimodular $\mathcal{R}$-coupling follows from the existence of an $\mathcal{R}$-coupling, as is proved in Proposition 8.6; in the case of fixed amenable transitive graphs, this is a well-known averaging principle.

Question 5.2. Does Theorem 5.1 hold without the assumption of a unimodular $\mathcal{R}$-coupling, but just an $\mathcal{R}$-coupling?

This question asks whether we can compare the traces from two different von Neumann algebras. One situation where we can do this is as follows.

If μ_1 and μ_2 are probability measures on $\mathcal{G}_*$, then a probability measure ν on $\mathcal{G}_* \times \mathcal{G}_*$ whose coordinate marginals are μ_1 and μ_2 is called a **monotone graph coupling** of μ_1 and μ_2 if ν is concentrated on pairs of rooted networks $\big((G_1, o), (G_2, o)\big)$ that share the same roots and satisfy $\mathsf{V}(G_1) \subseteq \mathsf{V}(G_2)$. In this instance, let V_1 be the inclusion of $\ell^2\big(\mathsf{V}(G_1)\big)$ in $\ell^2\big(\mathsf{V}(G_2)\big)$.

When there is a unimodular coupling (as in Theorem 5.1), the following result is easy. The fact that it holds more generally is useful.

Proposition 5.3 (Trace Comparison)**.** *Let ν be a monotone graph coupling of two unimodular probability measures μ_1 and μ_2. Let $T^{(i)} \in \mathsf{Alg}(\mu_i)$ be self-adjoint with*

$$T^{(1)}_{G_1} \le V_1^* T^{(2)}_{G_2} V_1 \tag{5.1}$$

for ν-almost all pairs $\big((G_1, o), (G_2, o)\big)$. Then $\mathrm{Tr}_{\mu_1}(T^{(1)}) \le \mathrm{Tr}_{\mu_2}(T^{(2)})$. If in addition for ν-almost all pairs $\big((G_1, o), (G_2, o)\big)$ and for all $x \in \mathsf{V}(G_1)$ we have

$$\deg_{G_1}(x) < \deg_{G_2}(x) \quad \Longrightarrow \quad \big(T^{(1)}_{G_1} x,\, x\big) < \big(T^{(2)}_{G_2} x,\, x\big)\,, \tag{5.2}$$

then either

$$\mathrm{Tr}_{\mu_1}(T^{(1)}) < \mathrm{Tr}_{\mu_2}(T^{(2)}) \tag{5.3}$$

or

$$\mathsf{V}(G_1) = \mathsf{V}(G_2)\,,\ \mathsf{E}(G_1) = \mathsf{E}(G_2)\,, \quad \textit{and} \ \ T^{(1)}_{G_1} = T^{(2)}_{G_2} \quad \nu\text{-a.s.} \tag{5.4}$$

Proof. Suppose that (5.1) holds. The fact that $\mathrm{Tr}_{\mu_1}(T^{(1)}) \le \mathrm{Tr}_{\mu_2}(T^{(2)})$ is an immediate consequence of the definition of trace and of the hypothesis:

$$\begin{aligned}
\mathrm{Tr}_{\mu_1}(T^{(1)}) &= \int \big(T^{(1)}_G o,\, o\big)\, d\mu_1(G, o) = \int \big(T^{(1)}_{G_1} o,\, o\big)\, d\nu\big((G_1, o), (G_2, o)\big) \\
&\le \int \big(V_1^* T^{(2)}_{G_2} V_1 o,\, o\big)\, d\nu\big((G_1, o), (G_2, o)\big) \\
&= \int \big(T^{(2)}_{G_2} o,\, o\big)\, d\nu\big((G_1, o), (G_2, o)\big) = \mathrm{Tr}_{\mu_2}(T^{(2)})\,.
\end{aligned}$$

Suppose that equality holds in this inequality, i.e., (5.3) fails. Then

$$\big(T^{(1)}_{G_1} o,\, o\big) = \big(T^{(2)}_{G_2} o,\, o\big) \quad \nu\text{-a.s.}$$

Of course, we also have by hypothesis that ν-a.s.,

$$\big(T^{(1)}_{G_1} x,\, x\big) \le \big(T^{(2)}_{G_2} x,\, x\big) \tag{5.5}$$

for all $x \in \mathsf{V}(G_1)$. Assume now that (5.2) holds. We shall prove that (5.4) holds. First, we claim that ν-a.s.,

$$\mathsf{V}(G_1) = \mathsf{V}(G_2),\ \ \mathsf{E}(G_1) = \mathsf{E}(G_2)\,, \tag{5.6}$$

and

$$\big(T^{(1)}_{G_1} x,\, x\big) = \big(T^{(2)}_{G_2} x,\, x\big) \tag{5.7}$$

for all $x \in \mathsf{V}(G_2)$. If not, let k be the smallest integer such that with positive ν-probability, there is a vertex x at G_1-distance k from o where (5.7) does not hold. Such a k exists by virtue of (5.2). Consider $\big(T^{(i)}_{G_i} x,\, x\big)$ as part of the mark at x. According to Theorem 4.1, the random walk on G_i given in Remark 4.2 yields a shift-stationary measure $\widehat{\mu}_i$ on trajectories $\big((G_i, w_0), \langle w_n\,;\; n \ge 0\rangle\big)$ for each i. In particular, the distribution of $\big(T^{(i)}_{G_i} w_k,\, w_k\big)$ is the same as that of $\big(T^{(i)}_{G_i} w_0,\, w_0\big)$. Now the latter is the same for $i = 1$ as for $i = 2$ (since $w_0 = o$). Note that for all x at distance less than k from o, we have $\deg_{G_1}(x) = \deg_{G_2}(x)$ by (5.2). Thus, the walks may be coupled together up to time k, whence the distribution of $\big(T^{(i)}_{G_i} w_k,\, w_k\big)$ is not the same for $i = 1$ as for $i = 2$ in light of (5.5) and choice of k. This is a contradiction. It follows that $\deg_{G_1}(x) = \deg_{G_2}(x)$ for all $x \in \mathsf{V}(G_1)$, whence (5.6) and (5.7) hold.

Now $T := T^{(2)}_{G_2} - T^{(1)}_{G_1} \ge 0$ is self-adjoint. It follows that for any $x, y \in \mathsf{V}(G_2)$ and any complex number α of modulus 1,

$$0 \le \big(T(\alpha x + y),\, \alpha x + y\big) = 2\Re\big\{\alpha(Tx,\, y)\big\}\,,$$

whence $\big(Tx,\, y\big) = 0$. That is, $T = 0$ and (5.4) holds. ☐

6 Percolation

We now begin our collection of extensions of results that are known for unimodular fixed graphs. For most of the remainder of the paper, we consider graphs without marks, or, equivalently, with constant marks, except that marks are used as explained below to perform percolation on the given graphs. We begin this section on percolation with some preliminary results on expected degree.

Theorem 6.1. (Minimal Expected Degree). *If μ is a unimodular probability measure on $\mathcal{G}_*$ concentrated on infinite graphs, then $\overline{\deg}(\mu) \ge 2$.*

This is proved exactly like Theorem 6.1 of BLPS (1999b) is proved. In the context of equivalence relations, this is well known and was perhaps first proved by Levitt (1995).

Theorem 6.2 (Degree Two)**.** *If μ is a unimodular probability measure on $\mathcal{G}_*$ concentrated on infinite graphs, then $\overline{\deg}(\mu) = 2$ iff μ-a.s. G is a tree with at most 2 ends.*

The proof is like that of Theorem 7.2 of BLPS (1999b).

Proposition 6.3 (Limits of Trees)**.** *If G_n are finite trees with random weak limit μ, then $\overline{\deg}(\mu) \le 2$ and μ is concentrated on trees with at most 2 ends.*

Proof. Since $\overline{\deg}\big(U(G_n)\big) < 2$, we have $\overline{\deg}(\mu) \le 2$. The remainder follows from Theorem 6.2. ∎

We now discuss what we mean by percolation on a random rooted network. Given a probability measure $\mu \in \mathcal{U}$, we may wish to randomly designate some of the edges of the random network "open". For example, in Bernoulli(p) bond percolation, each edge is independently open with probability p. Recall that in Section 10 we have added a second independent uniform $[0,1]$ coordinate to the edge marks to form $\mu^{\mathrm{B}} \in \mathcal{U}$, the standard coupling of Bernoulli percolation on μ. For $p \in [0,1]$, one can define **Bernoulli(p) bond percolation on** μ as the measure μ_p^{B} that replaces the second coordinate of each edge mark by "open" if it is at most p and by "closed" otherwise.

Note that if $\phi : \Xi \times [0,1] \to \Xi$ is the projection onto the first coordinate, then $\mu = \mu_p^{\mathrm{B}} \circ \phi^{-1}$. We have changed the mark space, but a fixed homeomorphism would bring it back to Ξ. Thus, more generally, if $\psi : \Xi \to \Xi$ is Borel, then we call μ a **percolation** on $\mu \circ \psi^{-1}$.

Definition 6.4. Let $G = \big(\mathsf{V}(G), \mathsf{E}(G)\big)$ be a graph. Given a configuration $A \in \{0,1\}^{\mathsf{E}(G)}$ and an edge $e \in \mathsf{E}(G)$, denote $\Pi_e A$ the element of $\{0,1\}^{\mathsf{E}(G)}$ that agrees with A off of e and is 1 on e. For $\mathcal{A} \subset \{0,1\}^{\mathsf{E}(G)}$, we write $\Pi_e \mathcal{A} := \{\Pi_e A ;\ A \in \mathcal{A}\}$. For bond percolation, call an edge "closed" if it is marked "0" and "open" if it is marked "1". A bond percolation process $\mathbf{P}$ on G is **insertion tolerant** if $\mathbf{P}(\Pi_e \mathcal{A}) > 0$ for every $e \in \mathsf{E}(G)$ and every Borel $\mathcal{A} \subset \{0,1\}^{\mathsf{E}(G)}$ satisfying $\mathbf{P}(\mathcal{A}) > 0$. The primary subtlety in extending this notion to percolation on unimodular random networks is that it may not be possible to pick an edge measurably from a rooted-automorphism-invariant set of edges. Thus, we shall make an extra assumption of distinguishability with marks. That is, a percolation process $\mathbf{P}$ on a unimodular probability measure on $\mathcal{G}_*$ is **insertion tolerant** if $\mathbf{P}$-a.s. there is no nontrivial rooted isomorphism of the marked network and for any event $\mathcal{A} \subseteq \{(A, G) ;\ A \subseteq \mathsf{E}(G),\ G \in \mathcal{G}_*\}$ with $\mathbf{P}(\mathcal{A}) > 0$ and any Borel function $e : G \mapsto e(G) \in \mathsf{E}(G)$ defined on $\mathcal{G}_*$, we have $\mathbf{P}(\Pi_e \mathcal{A}) > 0$.

For example, Bernoulli(p) bond percolation is insertion tolerant when $p \in (0,1]$.

We call a connected component of open edges (and their endpoints) a **cluster**. Given a rooted graph (G, o), define

$$p_{\mathrm{c}}(G, o) := \sup\{p ;\ \text{Bernoulli}(p) \text{ percolation on } G \text{ has no infinite clusters a.s.}\} .$$

Clearly p_{c} is $\mathcal{I}$-measurable, so if $\mu \in \mathcal{U}$ is extremal, then there is a constant $p_{\mathrm{c}}(\mu)$ such that $p_{\mathrm{c}}(G, o) = p_{\mathrm{c}}(\mu)$ for μ-a.e. (G, o).

Example 6.5. Even if $\mu \in \mathcal{U}$ satisfies $\overline{\deg}(\mu) < \infty$, it does not necessarily follow that $p_{\mathrm{c}}(G) > 0$ for μ-a.e. (G, o). For example, let $p_k := 1/[k(k+1)]$ for $k \ge 1$ and $p_0 := 0$. Let UGW be the corresponding unimodular Galton-Watson measure (see Example 1.1). Then $\overline{\deg}(\mathsf{UGW}) = 6/(12 - \pi^2)$ by (1.2), but since $\sum k p_k = \infty$, we have $p_{\mathrm{c}}(G) = 0$ a.s. by Lyons (1990).

A more elaborate example shows that no stochastic bound on the degree of the root, other than uniform boundedness, implies $p_{\mathrm{c}}(\mu) > 0$:

Example 6.6. Given $a_n > 0$, we shall construct $\mu \in \mathcal{U}$ such that $\mu[\deg_G(o) \geq n] < a_n$ for all large n and $p_c(G) = 0$ for μ-a.e. (G, o). We may assume that $\sum a_n < \infty$. Consider the infinite tree T of degree 3 and $o \in \mathsf{V}(T)$. Let $j \geq 2$ and set $p_j := \frac{1}{2} + \frac{1}{j}$. Consider supercritical Bernoulli(p_j) bond percolation on T, so that

$$\theta_j := \mathbf{P}[o \text{ belongs to an infinite component}] > 0 \,.$$

Define $q_j < 1$ by $p_j q_j = \frac{1}{2} + \frac{1}{2j}$, so that the fragmentation of an infinite p_j-component by an independent q_j percolation process will still contain infinite components. Let N_j be the smallest integer with $(1 - \frac{1}{j^2})^{N_j} < 1 - q_j$. Finally, choose some sequence $1 > r_j \downarrow 0$ sufficiently fast. We label the edges of T by the following operations, performed independently for each $j \geq 2$.

Take the infinite components of Bernoulli(p_j) bond percolation on T. "Thin" by retaining each component independently with probability r_j and deleting other components. For each edge e in the remaining components, let $L_j(e) := N_j$, while $L_j(e) := 1$ for deleted edges e.

Now let $L(e) := \sup_j L_j(e)$. Since $r_j \to 0$ fast, $L(e) < \infty$ for all e a.s. Consider the graph G obtained by replacing each edge e of T by $L(e)$ parallel chains of length 2. To estimate $\deg_G(o)$, note that the chance that o is incident to an edge e with $L_j(e) = N_j$ (thus contributing at most $3N_j$ to the degree) equals $r_j \theta_j$. Thus, by choice of $\langle r_j \rangle$, we can make the root-degree distribution have tail probabilities eventually less than a_n. Now consider Bernoulli($1/j$) bond percolation on G. For an edge e of T which is replaced by N_j chains of G, the chance of percolating across at least one of these N_j chains is (by definition of N_j) larger than q_j. Thus the percolation clusters on G dominate the q_j-percolation clusters on the retained components of the original infinite p_j-percolation clusters on T, and as observed above must therefore contain infinite components.

To see how to make this into a probability measure in $\mathcal{U}$, note that L is an invariant random network on T. Therefore, we obtain a probability measure in $\mathcal{U}$ by Theorem 3.2. We may now use the edge labels to replace an edge labeled n by n parallel chains of length 2, followed by a suitable re-rooting as in Example 9.8 (below). This gives a new probability measure in $\mathcal{U}$ that has the property desired, since the expected degree of the root is finite by the hypothesis $\sum_n a_n < \infty$. Also, the re-rooting stochastically decreases the degree since it introduces roots of degree 2.

The following extends a well-known result of Häggström and Peres (1999). The proof is the same.

Theorem 6.7 (Uniqueness Monotonicity and Merging Clusters)**.** *Let μ be a unimodular probability measure on $\mathcal{G}_*$. Let $p_1 < p_2$ and $\mathbf{P}_i$ $(i = 1, 2)$ be the corresponding Bernoulli(p_i) bond percolation processes on μ. If there is a unique infinite cluster $\mathbf{P}_1$-a.s., then there is a unique infinite cluster $\mathbf{P}_2$-a.s. Furthermore, in the standard coupling of Bernoulli percolation processes, if μ is extremal, then μ-a.s. for all p_1, p_2 satisfying $p_c(\mu) < p_1 < p_2 \leq 1$, every infinite p_2-cluster contains an infinite p_1-cluster.*

As a consequence, for extremal $\mu \in \mathcal{U}$, there is a constant $p_u(\mu)$ such that for any $p > p_u(\mu)$, we have $\mathbf{P}_p$-a.s., there is a unique infinite cluster, while for any $p < p_u(\mu)$, we have $\mathbf{P}_p$-a.s., there is not a unique infinite cluster.

Every unimodular probability measure on $\mathcal{G}_*$ can be written as a Choquet integral of extremal measures. In the following, we refer to these extremal measures as "extremal components".

Lemma 6.8. *If* $\mathbf{P}$ *is an insertion-tolerant percolation on a unimodular random network that is concentrated on infinite graphs, then almost every extremal component of* $\mathbf{P}$ *is insertion tolerant.*

Proof. The proof can be done precisely as that of Lemma 1 of Gandolfi, Keane, and Newman (1992), but by using the present Theorems 4.1, 4.6, and 4.7, as well as Remark 4.2 to replace the use of a measure-preserving transformation in Gandolfi, Keane, and Newman (1992) by the shift on trajectories of a stationary Markov chain. The fact that $\mathbf{P}$ is concentrated on infinite graphs is used to deduce that the corresponding Markov chain is not positive recurrent, whence the asymptotic frequency of visits to any given neighborhood of the root is 0. ☐

Corollary 6.9 (Number of Infinite Clusters)**.** *If* $\mathbf{P}$ *is an insertion-tolerant percolation on a unimodular random network, then* $\mathbf{P}$*-almost surely, the number of infinite clusters is* $0, 1$ *or* ∞*.*

The proof is standard for the extremal components; cf. Newman and Schulman (1981).

The following extends Häggström and Peres (1999) and Proposition 3.9 of Lyons and Schramm (1999a). The proof is parallel to that of the latter.

Proposition 6.10. *Let* $\mathbf{P}$ *be a percolation on a unimodular random network. Then* $\mathbf{P}$*-a.s. each infinite cluster that has at least* 3 *ends has no isolated ends.*

The following corollary is proved just like Proposition 3.10 of Lyons and Schramm (1999a). There is some overlap with Theorem 3.1 of Paulin (1999).

Corollary 6.11 (Many Ends)**.** *Let* $\mathbf{P}$ *be an insertion-tolerant percolation on a unimodular random network. If there are infinitely many infinite clusters* $\mathbf{P}$*-a.s., then* $\mathbf{P}$*-a.s. every infinite cluster has continuum many ends and no isolated end.*

The following extends Lemma 7.4 and Remark 7.3 of BLPS (1999b) and is proved similarly.

Lemma 6.12 (Subforests)**.** *Let* $\mathbf{P}$ *be a percolation on a unimodular random network. If* $\mathbf{P}$*-a.s. there is a component of the open subgraph* ω *with at least three ends, then there is a percolation* $\mathfrak{F}$ *on* ω *whose components are trees such that a.s. whenever a component* K *of* ω *has at least three ends, there is a component of* $K \cap \mathfrak{F}$ *that has infinitely many ends and has* $p_c < 1$*.*

The following extends Proposition 3.11 of Lyons and Schramm (1999a) and is proved similarly (using the preceding Lemma 6.12).

Proposition 6.13 (Transient Subtrees)**.** *Let* $\mathbf{P}$ *be an insertion-tolerant percolation on a unimodular random network. If there are* $\mathbf{P}$*-almost surely infinitely many infinite clusters, then* $\mathbf{P}$*-a.s. each infinite cluster is transient and, in fact, contains a transient tree.*

In order to use this, we shall use the comparison of simple to network random walks given in Proposition 4.10.

Definition 6.14. A percolation process $\mathbf{P}$ on a unimodular probability measure on $\mathcal{G}_*$ has **indistinguishable infinite clusters** if for any event $\mathcal{A} \subseteq \{(A, (G, o))\,;\ A \in \{0,1\}^{\mathsf{V}(G)} \times \{0,1\}^{\mathsf{E}(G)},\ (G, o) \in \mathcal{G}_*\}$ that is invariant under non-rooted isomorphisms, almost surely, for all infinite clusters C of the open subgraph ω, we have $(C, \omega) \in \mathcal{A}$, or for all infinite clusters C, we have $(C, \omega) \notin \mathcal{A}$.

The following extends Theorem 3.3 of Lyons and Schramm (1999a) and is proved similarly using the preceding results: For example, instead of the use of delayed simple random walk by Lyons and Schramm (1999a), we use the network random walk in Remark 4.2. This is a reversible random walk corresponding to edge weights $(x,y) \mapsto 1/[(\deg x)(\deg y)]$. It is transient by Propositions 6.13 and 4.10, combined with Rayleigh's monotonicity principle.

Theorem 6.15 (Indistinguishable Clusters)**.** *If* $\mathbf{P}$ *is an insertion-tolerant percolation on a unimodular random network, then* $\mathbf{P}$ *has indistinguishable infinite clusters.*

Among the several consequences of this result is the following extension of Theorem 4.1 of Lyons and Schramm (1999a), proved similarly.

Theorem 6.16 (Uniqueness and Long-Range Order)**.** *Let* $\mathbf{P}$ *be an insertion-tolerant percolation on a unimodular random network,* μ*. If* $\mathbf{P}$ *is extremal and there is more than one infinite cluster* $\mathbf{P}$*-a.s., then* μ*-a.s.,*

$$\inf\big\{\mathbf{P}[\textit{there is an open path from } x \textit{ to } y \mid G]\,;\ x,y \in \mathsf{V}(G)\big\} = 0\,.$$

The following extends Theorem 6.12 of Lyons and Schramm (1999a) and is proved similarly.

Theorem 6.17 (Uniqueness in Products)**.** *Suppose that* μ*,* μ_1*, and* μ_2 *are extremal unimodular probability measures on* $\mathcal{G}_*$*, with* μ *supported on infinite graphs and* μ_1 *a percolation on* μ_2*. Then* $p_{\mathrm{u}}(\mu \boxtimes \mu_1) \ge p_{\mathrm{u}}(\mu \boxtimes \mu_2)$*. In particular,* $p_{\mathrm{u}}(\mu) \ge p_{\mathrm{u}}(\mu \boxtimes \mu_2)$*.*

More results on percolation will be presented in Section 8.

7 Spanning Forests

An interesting type of percolation other than Bernoulli is given by certain random forests. There are two classes of such random forests that have been widely studied, the uniform ones and the minimal ones.

We first discuss the uniform case. Given a finite connected graph, G, let $\mathsf{UST}(G)$ denote the uniform measure on spanning trees on G. Pemantle (1991) proved a conjecture of Lyons, namely, that if an infinite connected graph G is exhausted by a sequence of finite connected subgraphs G_n, then the weak limit of $\langle \mathsf{UST}(G_n)\rangle$ exists. However, it may happen that the limit measure is not supported on trees, but on forests. This limit measure is now called the **free (uniform) spanning forest** on G, denoted FSF or FUSF. If G is itself a tree, then this measure is trivial, namely, it is concentrated on $\{G\}$. Therefore, Häggström (1998) introduced another limit that had been considered on $\mathbb{Z}^d$ more implicitly by Pemantle (1991) and explicitly by Häggström (1995), namely, the weak limit of the uniform spanning tree measures on G_n^*, where G_n^* is the graph G_n with its boundary identified ("wired") to a single vertex. As Pemantle (1991) showed, this limit also always exists on any graph and is now called the **wired (uniform) spanning forest**, denoted WSF or WUSF. It is clear that both FSF and WSF are concentrated on the set of spanning forests[2] of G that are **essential**, meaning that all their trees are infinite. Both FSF and WSF are important in their own right; see Lyons (1998) for a survey and BLPS (2001) for a comprehensive treatment.

[2]By a "spanning forest", we mean a subgraph without cycles that contains every vertex.

In all the above, one may work more generally with a weighted graph, where the graph has positive weights on its edges. In that case, UST stands for the measure such that the probability of a spanning tree is proportional to the product of the weights of its edges. The above theorems continue to hold and we use the same notation for the limiting measures.

Most results known about the uniform spanning forest measures hold for general graphs. Some, however, require extra hypotheses such as transitivity and unimodularity. We extend some of these latter results here.

Given μ, taking the wired uniform spanning forest on each graph gives a percolation that we denote $\mathsf{WUSF}(\mu)$. Our first result shows, among other things, that the kind of limit considered in this paper, i.e., random weak convergence, gives another natural way to define WSF. It might be quite useful to have an explicit description of measures on finite graphs whose random weak limit is the *free* spanning forest.

Proposition 7.1 (UST Limits)**.** *If μ is a unimodular probability measure on infinite networks in $\mathcal{G}_*$, then $\overline{\deg}\big(\mathsf{WUSF}(\mu)\big) = 2$. If G_n are finite connected networks whose random weak limit is μ, then $\mathsf{UST}(G_n) \Rightarrow \mathsf{WUSF}(\mu)$. More generally, if μ_n are unimodular probability measures on $\mathcal{G}_*$ with $\mu_n \Rightarrow \mu$, then $\mathsf{WUSF}(\mu_n) \Rightarrow \mathsf{WUSF}(\mu)$.*

Proof. Suppose first that μ is concentrated on recurrent networks. Then Wilson's algorithm gives both $\mathsf{UST}(G_n)$ and $\mathsf{WUSF}(\mu)$. Hence it is clear that $\mathsf{UST}(G_n) \Rightarrow \mathsf{WUSF}(\mu)$. Since $\overline{\deg}(\mathsf{UST}(G_n)) = 2 - 2/|\mathsf{V}(G_n)| \le 2$, this limit relation in combination with Fatou's lemma shows that $\overline{\deg}(\mathsf{WUSF}(\mu)) \le 2$, whence equality results from Theorem 6.1.

Suppose next that μ is concentrated on transient networks. Then the proof of Theorem 6.5 of BLPS (2001) gives the same result.

Finally, if μ is concentrated on neither recurrent nor transient networks, then we may write μ as a mixture of two unimodular measures that are concentrated on recurrent or on transient networks and apply the preceding.

This proves the first sentence. The second sentence is a special case of the third, so we prove the third.

If $\mu_n \Rightarrow \mu$, then $\mathsf{UST}_R(\mu_n) \Rightarrow \mathsf{UST}_R(\mu)$ as $n \to \infty$ and the intersection of $\mathsf{WUSF}(\mu_n)$ with the ball of radius R stochastically dominates $\mathsf{UST}_R(\mu_n)$, so that every weak limit point of $\langle \mathsf{WUSF}(\mu_n) \rangle$ stochastically dominates $\mathsf{WUSF}(\mu)$. However, since they all have expected degree 2 by what we have just shown, they are equal. ∎

We next show that the trees of the WSF have only one end a.s. The first theorem of this type was proved by Pemantle (1991). His result was completed and extended in Theorem 10.1 of BLPS (2001), which dealt with the transitive unimodular case. The minor modifications needed for the quasi-transitive unimodular case were explained by Lyons (2005). Another extension is given by Lyons, Morris, and Schramm (2007), who showed that for graphs with a "reasonable" isoperimetric profile, each tree has only one end WSF-a.s.

Theorem 7.2 (One End)**.** *If μ is a unimodular probability measure on $\mathcal{G}_*$ that is concentrated on transient networks with bounded degree, then $\mathsf{WUSF}(\mu)$-a.s., each tree has exactly one end.*

Proof. The proof is essentially the same as in BLPS (2001), with the following modifications. In the proof of Theorem 10.3 of BLPS (2001), which is the case where there is only one tree a.s., we replace x and y there by o and X_n, where $\langle X_n \rangle$ is the simple random walk starting from the root; to use stationarity, we bias the underlying network by the degree of the root. This gives a measure equivalent to μ, so that almost sure conclusions for it hold for μ as well. The stationarity and reversibility give that the probability that the random walk from o ever visits X_n is equal to the probability that a random walk from X_n ever visits o. By transience, this tends to 0 as $n \to \infty$, which allows the proof of BLPS (2001) to go through. [Here, we needed finite expected degree to talk about probability since we used the equivalent probability measure of biasing by the degree.]

In the proof of Theorem 10.4 of BLPS (2001), the case when there is more than one tree a.s., we need the degrees to be bounded for the displayed equality on p. 36 of BLPS (2001) to hold up to a constant factor. $\square$

For our proof, we had to assume transience; there is presumably an extension to the recurrent case, which would say that the number of ends $\mathsf{WUSF}(\mu)$-a.s. is the same as the number of ends μ-a.s. when μ is concentrated on recurrent networks. Also, presumably the assumption that the degrees are bounded is not needed. In any case, our result here goes beyond what has been done before and gives a partial answer to Question 15.4 of BLPS (2001); removing the assumption of bounded degrees would completely answer that question. It also applies, e.g., to transient clusters of Bernoulli percolation; see Grimmett, Kesten, and Zhang (1993) and Benjamini, Lyons, and Schramm (1999) for sufficient conditions for transience.

We now prove analogous results for the other model of spanning trees, the minimal ones. Given a finite connected graph, G, and independent uniform $[0, 1]$ random variables on its edges, the spanning tree that minimizes the sum of the labels of its edges has a distribution denoted $\mathsf{MST}(G)$, the **minimal spanning tree** measure on G. If G is infinite, there are two analogous measures, as in the uniform case. They can be defined by weak limits, but also directly (and by pointwise limits). Namely, given independent uniform $[0, 1]$ edge labels, remove all edges whose label is the largest in some cycle containing that edge. The remaining edges form the **free minimal spanning forest**, FMSF. If one also removes the edges e both of whose endpoints belong to infinite paths of edges that are all labeled smaller than e is, then the resulting forest is called the **wired minimal spanning forest**, WMSF.

The following is analogous to Proposition 7.1 above and is proved similarly to it and part of Theorem 3.12 of Lyons, Peres, and Schramm (2006), using Theorem 8.11 below. Parts of it were also proved by Aldous and Steele (2004).

Proposition 7.3 (MST Limits)**.** *If μ is a unimodular probability measure on infinite networks in $\mathcal{G}_*$, then $\overline{\deg}\big(\mathsf{WMSF}(\mu)\big) = 2$. If G_n are finite connected networks whose random weak limit is μ, then $\mathsf{MST}(G_n) \Rightarrow \mathsf{WMSF}(\mu)$. More generally, if μ_n are unimodular probability measures on $\mathcal{G}_*$ with $\mu_n \Rightarrow \mu$, then $\mathsf{WMSF}(\mu_n) \Rightarrow \mathsf{WMSF}(\mu)$.*

The following extends a result of Lyons, Peres, and Schramm (2006), which in turn extends a result of Alexander (1995), who proved this in fixed Euclidean lattices. Our proof follows slightly different lines. For information on when the hypothesis is satisfied, see Theorem 8.11 below.

Theorem 7.4 (One End)**.** *If μ is an extremal unimodular probability measure on infinite networks in $\mathcal{G}_*$ and there is $\mathbf{P}_{p_c(\mu)}$-a.s. no infinite cluster, then $\mathsf{WMSF}(\mu)$-a.s., each tree has exactly one end.*

Proof. By the first part of Proposition 7.3 and by Theorem 6.2, each tree has at most 2 ends $\mathsf{WMSF}(\mu)$-a.s. Suppose that some tree has 2 ends with positive probability. A tree with precisely two ends has a **trunk**, the unique bi-infinite simple path in the tree. By the definition of WMSF, the labels on a trunk cannot have a maximum. By the Mass-Transport Principle, the limsup in one direction must equal the limsup in the other direction, since otherwise we could identify the one edge that has label larger than the average of the two limsups and is the last such edge in the direction from the larger limsup to the smaller limsup. Let p be this common limsup. By the preceding, all the labels on the trunk are strictly less than p. The root belongs to the trunk with positive probability. Assume this happens. Then the root belongs to an infinite p-cluster (the one containing the trunk). Now invasion from the root will fill (the vertices of) this p-cluster and is part of the tree containing the root (see Lyons, Peres, and Schramm (2006)), whence the tree contains (the vertices of) the entire p-cluster of the root. By Theorem 6.7, the tree therefore also contains an infinite p'-cluster for every $p' \in (p_c(\mu), p)$ if $p > p_c(\mu)$. Let x be a vertex in the tree that is in an infinite p'-cluster C for $p' := (p_c(\mu) + p)/2$. Now invasion from x has a finite symmetric difference with invasion from o (see Lyons, Peres, and Schramm (2006)), invasion from x does not leave C, and invasion from o fills the trunk. It follows from the definition of p that $p' \geq p$. That is, $p = p_c(\mu)$. Therefore, there is $\mathbf{P}_{p_c(\mu)}$-a.s. an infinite cluster. ☐

8 Amenability and Nonamenability

Recall that a graph G is (vertex) amenable iff there is a sequence of subsets $H_n \subset \mathsf{V}(G)$ with

$$\lim_{n\to\infty} \frac{|\partial_{\mathsf{V}} H_n|}{|\mathsf{V}(H_n)|} = 0\,,$$

where $|\bullet|$ denotes cardinality.

Amenability, originally defined for groups, now appears in several areas of mathematics, including probability theory and ergodic theory. Its presence provides many tools one is used to from $\mathbb{Z}$ actions, yet its absence also provides a powerful threshold principle. There are many equivalent definitions of amenability. We choose one that is not standard, but is useful for our probabilistic purposes. We show that it is equivalent to other definitions. Then we shall illustrate its uses.

Definition 8.1. Let $\mathsf{prj} : \Xi \to \Xi$ be the composition of a homeomorphism of Ξ with Ξ^2 followed by the projection onto the first coordinate. If a rooted network (G, o) is understood, then for a subset $\Xi_0 \subseteq \Xi$ and vertex x, the Ξ_0**-component of** x is the set of vertices that can be reached from x by edges both of whose marks lie in Ξ_0. Write $K(\Xi_0)$ for the Ξ_0-component of the root. For a probability measure μ on rooted graphs, denote by $\mathsf{FC}(\mu)$ the class of percolations on μ that have only finite components. That is, $\mathsf{FC}(\mu)$ consists of pairs (ν, Ξ_0) such that ν is a unimodular probability measure on $\mathcal{G}_*$, $\Xi_0 \subseteq \Xi$ is Borel, $\mu = \nu \circ \mathsf{prj}^{-1}$, and $K(\Xi_0)$ is finite ν-a.s. (By Lemma 2.3, all Ξ_0-components are then finite ν-a.s.) For $\Xi_0 \subseteq \Xi$ and $x \in \mathsf{V}(G)$, write

$$n(x, \Xi_0) := |\{y \in \mathsf{V}(G)\,;\ (x, y) \in \mathsf{E}(G),\ \text{some edge mark of } (x, y) \text{ is } \notin \Xi_0\}|\,.$$

For $K \subset \mathsf{V}(G)$, define

$$n(K, \Xi_0) := \sum_{x\in K} n(x, \Xi_0)\,.$$

Define

$$\iota_{\mathsf{E}}(\mu) := \inf \left\{ \int \frac{n\big(K(\Xi_0), \Xi_0\big)}{|K(\Xi_0)|} d\mu'(G, o) \, ; \; (\mu', \Xi_0) \in \mathsf{FC}(\mu) \right\} .$$

Call μ **amenable** if $\iota_{\mathsf{E}}(\mu) = 0$. Define

$$\overline{\deg}(\mu', \Xi_0) := \int \big[\deg_G(o) - n(o, \Xi_0) \big] \, d\mu'(G, o) \, ,$$

the expected degree in the Ξ_0-component of the root, and

$$\alpha(\mu) := \sup \big\{ \overline{\deg}(\mu', \Xi_0) \, ; \; (\mu', \Xi_0) \in \mathsf{FC}(\mu) \big\} .$$

Of course, neither $\iota_{\mathsf{E}}(\mu)$ nor $\alpha(\mu)$ depends on the choice of homeomorphism in prj. Furthermore, these quantities depend only on the probability measure on the underlying graphs of the networks, not on the marks.

This definition of amenability is justified in three ways: It agrees with the usual definition of amenability for fixed unimodular quasi-transitive graphs by Theorems 5.1 and 5.3 of BLPS (1999b); it agrees with the usual notion of amenability for equivalence relations by Theorem 8.5 below; and it allows us to extend to non-amenable unimodular random rooted graphs many theorems that are known for non-amenable unimodular fixed graphs, as we shall see.

We say that a graph G is **anchored amenable** if there is a sequence of subsets $H_n \subset \mathsf{V}(G)$ such that $\bigcap_n H_n \neq \emptyset$, each H_n induces a connected subgraph of G, and

$$\lim_{n \to \infty} \frac{|\partial_{\mathsf{V}} H_n|}{|\mathsf{V}(H_n)|} = 0 \, .$$

The relationship between amenability of μ and amenability or anchored amenability of μ-a.e. graph is as follows. The first clearly implies the third, which implies the second, but the third does not imply the first. Indeed, take a 3-regular tree and randomly subdivide its edges by a number of vertices whose distribution does not have a finite exponential tail. Chen and Peres (2004) show that the result is anchored amenable a.s. However, there is an appropriate unimodular version if the subdividing distribution has finite mean (see Example 2.4.4 of Kaimanovich (1998) or Example 9.8 below), and it is non-amenable by Corollary 8.10 below.

In order to work with this definition, we shall need some easy facts.

Lemma 8.2. *If μ is a unimodular probability measure on $\mathcal{G}_*$ and $(\mu', \Xi_0) \in \mathsf{FC}(\mu)$, then*

$$\int \frac{n\big(K(\Xi_0), \Xi_0\big)}{|K(\Xi_0)|} d\mu' = \int n(o, \Xi_0) \, d\mu' \, .$$

Proof. Let K_x be the Ξ_0-component of x. Let each vertex x send mass $n(y, \Xi_0)/|K_x|$ to each $y \in K_x$. Then the left-hand side is the expected mass sent from the root and the right-hand side is the expected mass received by the root. ☐

Proposition 8.3. *If μ is a unimodular probability measure on $\mathcal{G}_*$, then*

$$\iota_{\mathsf{E}}(\mu) + \alpha(\mu) = \overline{\deg}(\mu) \, . \tag{8.1}$$

Therefore, if $\overline{\deg}(\mu) < \infty$, then μ is amenable iff $\alpha(\mu) = \overline{\deg}(\mu)$.

Proof. This is obvious from Lemma 8.2. ☐

Lemma 8.4. *Let $\mu, \nu \in \mathcal{U}$ with ν a percolation on μ, that is, there is some Borel $\psi : \Xi \to \Xi$ such that $\mu = \nu \circ \psi^{-1}$. Let κ be a regular conditional probability measure of μ with respect to the σ-field generated by ψ, i.e., a disintegration of ν with respect to ψ, with $\kappa_{(G,o)}$ being the probability measure on the fiber over (G, o). Let $h : \mathcal{G}_{**} \to [0, 1]$ be Borel and symmetric: $h(G, x, y) = h(G, y, x)$. Define $k(G, x, y) := \int h(G, x, y)\, d\kappa_{(G,x)}$. Then there is a symmetric Borel λ such that for μ-a.e. (G, o) and all $x \in \mathsf{V}(G)$, we have $k(G, o, x) = \lambda(G, o, x)$.*

Proof. It suffices to show that for all $f : \mathcal{G}_{**} \to [0, 1]$, we have

$$\int \sum_{x \in \mathsf{V}(G)} k(G, o, x) f(G, o, x)\, d\mu(G, o) = \int \sum_{x \in \mathsf{V}(G)} k(G, x, o) f(G, o, x)\, d\mu(G, o)\,,$$

since this shows symmetry of k a.e. with respect to the left measure μ_{L}. To see that this equation holds, observe that

$$\begin{aligned}
\int \sum_{x \in \mathsf{V}(G)} k(G, o, x) f(G, o, x)\, d\mu(G, o) &= \int\int \sum_{x \in \mathsf{V}(G)} h(G, o, x) f(G, o, x)\, d\kappa_{(G,o)}\, d\mu(G, o) \\
&= \int\int \sum_{x \in \mathsf{V}(G)} h(G, x, o) f(G, x, o)\, d\kappa_{(G,o)}\, d\mu(G, o) \\
&\qquad \text{[by the Mass-Transport Principle for } \nu] \\
&= \int\int \sum_{x \in \mathsf{V}(G)} h(G, o, x) f(G, o, x)\, d\kappa_{(G,x)}\, d\mu(G, o) \\
&\qquad \text{[by the Mass-Transport Principle for } \mu] \\
&= \int\int \sum_{x \in \mathsf{V}(G)} h(G, x, o) f(G, o, x)\, d\kappa_{(G,x)}\, d\mu(G, o) \\
&\qquad \text{[by symmetry of } h] \\
&= \int \sum_{x \in \mathsf{V}(G)} k(G, x, o) f(G, o, x)\, d\mu(G, o)\,.
\end{aligned}$$

☐

We now prove some properties that are equivalent to amenability. Most of these are standard in the context of equivalence relations. With appropriate modifications, these equivalences hold with a weakening of the assumption of unimodularity. They are essentially due to Connes, Feldman, and Weiss (1981) and Kaimanovich (1997), although (ii) seems to be new.

Theorem 8.5 (Amenability Criteria)**.** *Let μ be a unimodular probability measure on $\mathcal{G}_*$ with $\overline{\deg}(\mu) < \infty$. The following are equivalent:*

(i) μ is amenable;

(ii) there is a sequence of Borel functions $\lambda_n : \mathcal{G}_{**} \to [0,1]$ such that for all $(G,x,y) \in \mathcal{G}_{**}$ and all n, we have

$$\lambda_n(G,x,y) = \lambda_n(G,y,x) \tag{8.2}$$

and for μ-a.e. (G,o), we have

$$\sum_{x \in \mathsf{V}(G)} \lambda_n(G,o,x) = 1 \tag{8.3}$$

and

$$\lim_{n\to\infty} \sum_{x \in \mathsf{V}(G)} \sum_{y \sim x} |\lambda_n(G,o,x) - \lambda_n(G,o,y)| = 0\,; \tag{8.4}$$

(iii) there is a sequence of Borel functions $\lambda_n : \mathcal{G}_{**} \to [0,1]$ such that for μ-a.e. (G,o),

$$\sum_{x \in \mathsf{V}(G)} \lambda_n(G,o,x) = 1$$

and

$$\lim_{n\to\infty} \sum_{y \sim o} \sum_{x \in \mathsf{V}(G)} |\lambda_n(G,o,x) - \lambda_n(G,y,x)| = 0\,;$$

(iv) μ is hyperfinite, meaning that there is a unimodular measure ν on $\mathcal{G}_*$, an increasing sequence of Borel subsets $\Xi_n \subseteq \Xi$, and a Borel function $\psi : \Xi \to \Xi$ such that if G denotes a network with law ν and G_n the subnetwork consisting of those edges both of whose edge marks lie in Ξ_n, then $\psi(G)$ has law μ, all components of G_n are finite, and $\bigcup_n \Xi_n = \Xi$.

Proof. Assume from now on that μ is carried by networks with distinct marks. We shall use the following construction. Suppose that $(\nu, \Xi_0) \in \mathsf{FC}(\mu)$. Let κ be a regular conditional probability measure of μ with respect to the σ-field generated by prj. By Lemma 8.4, there is a Borel symmetric $\lambda : \mathcal{G}_{**} \to [0,1]$ such that for μ-a.e. (G,o) and $x \in \mathsf{V}(G)$, we have

$$\lambda(G,o,x) = \int \mathbf{1}_{\{x \in K(\Xi_0)\}} / |K(\Xi_0)| \, d\kappa_{(G,o)}\,. \tag{8.5}$$

Clearly,

$$\sum_{x \in \mathsf{V}(G)} \lambda(G,o,x) = 1 \tag{8.6}$$

for μ-a.e. (G,o). For μ-a.e. (G,o), we have

$$\begin{aligned}
&\sum_{x \in \mathsf{V}(G)} \sum_{y \sim x} |\lambda(G,o,x) - \lambda(G,o,y)| \\
&\qquad\qquad \le \sum_{x \in \mathsf{V}(G)} \sum_{y \sim x} \int |\mathbf{1}_{\{x \in K(\Xi_0)\}} - \mathbf{1}_{\{y \in K(\Xi_0)\}}| / |K(\Xi_0)| \, d\kappa_{(G,o)} \\
&\qquad\qquad \le \int \frac{2n\big(K(\Xi_0), \Xi_0\big)}{|K(\Xi_0)|} \, d\kappa_{(G,o)}\,.
\end{aligned} \tag{8.7}$$

Now if (i) holds, then we may choose $(\mu_n, \Xi_n) \in \mathsf{FC}(\mu)$ such that

$$\int \sum_n \frac{n\big(K(\Xi_n), \Xi_n\big)}{|K(\Xi_n)|} \, d\mu_n < \infty \,.$$

Let $\kappa^{(n)}$ and λ_n be as above (but for μ_n). Then by (8.7), we have

$$\begin{aligned}\sum_n \int \sum_{x \in \mathsf{V}(G)} \sum_{y \sim x} \big|\lambda_n(G,o,x) - \lambda_n(G,o,y)\big| \, d\mu(G,o) \\ \le \sum_n \int \int \frac{2n\big(K(\Xi_n), \Xi_n\big)}{|K(\Xi_n)|} \, d\kappa^{(n)}_{(G,o)} \, d\mu(G,o) \\ = \sum_n \int \frac{2n\big(K(\Xi_n), \Xi_n\big)}{|K(\Xi_n)|} \, d\mu_n < \infty \,,\end{aligned}$$

which shows that (ii) holds.

Next, suppose that (ii) holds. Then by the bounded convergence theorem, we have

$$\lim_{n \to \infty} \int \sum_{x \in \mathsf{V}(G)} \sum_{y \sim x} |\lambda_n(G,o,x) - \lambda_n(G,o,y)| \; d\mu(G,o) = 0 \,.$$

On the other hand, the Mass-Transport Principle and (8.2) show that this integral is the same as

$$\int \sum_{y \sim o} \sum_{x \in \mathsf{V}(G)} |\lambda_n(G,o,x) - \lambda_n(G,y,x)| \; d\mu(G,o) \,.$$

Therefore, by taking a subsequence if necessary, we achieve (iii).

Next, suppose that (iii) holds. Then we may define a sequence similar to λ_n on the corresponding equivalence relation (see Example 9.9), which implies that the equivalence relation is amenable by Kaimanovich (1997), and hence hyperfinite by a theorem of Connes, Feldman, and Weiss (1981). (Another proof of the latter theorem was sketched by Kaimanovich (1997), with more details given by Kechris and Miller (2004)). Translating the definition of hyperfinite equivalence relation to rooted networks gives (iv).

Finally, that (iv) implies (i) is an immediate consequence of Lebesgue's Dominated Convergence Theorem, our assumption that $\overline{\deg}(\mu) < \infty$, and Lemma 8.2. ☐

Now we show how to produce unimodular networks from non-unimodular ones on amenable measures, just as we can produce invariant measures from non-invariant ones on amenable groups. We illustrate in the context of couplings.

Proposition 8.6 (Coupling From Amenability)**.** *Let $\mathcal{R} \subseteq \Xi \times \Xi$ be a closed set. If $\mu_1, \mu_2 \in \mathcal{U}$ are amenable and μ_1 is $\mathcal{R}$-related to μ_2, then there is a unimodular $\mathcal{R}$-coupling of μ_1 to μ_2.*

Proof. Let ν be an $\mathcal{R}$-coupling of μ_1 to μ_2. Let λ_n be as in Theorem 8.5(ii) for $(\mu_1 + \mu_2)/2$. Define the measures ν_n by

$$\nu_n(\mathcal{B}) := \int \sum_{x \in \mathsf{V}(G)} \lambda_n(G,o,x) \mathbf{1}_{\{(G,x) \in \mathcal{B}\}} \, d\nu(G,o)$$

for Borel $\mathcal{B} \subseteq \mathcal{G}_*$ (with mark space $\Xi \times \Xi$). Then ν_n is a probability measure by (8.3). Since ν is carried by networks all of whose marks are in $\mathcal{R}$, so is ν_n. If $\mathcal{B}$ is an event that specifies only the first coordinates of the marks, i.e., $\mathcal{B} = \mathcal{B}_1 \times 2^\Xi$ for some $\mathcal{B}_1$, then

$$\begin{aligned}
\nu_n(\mathcal{B}) &= \int \sum_{x \in \mathsf{V}(G)} \lambda_n(G,o,x) \mathbf{1}_{\{(G,x)\in\mathcal{B}\}}\, d\nu(G,o) \\
&= \int \sum_{x \in \mathsf{V}(G)} \lambda_n(G,o,x) \mathbf{1}_{\{(G,x)\in\mathcal{B}_1\}}\, d\mu_1(G,o) \\
&= \int \sum_{x \in \mathsf{V}(G)} \lambda_n(G,x,o) \mathbf{1}_{\{(G,o)\in\mathcal{B}_1\}}\, d\mu_1(G,o) \\
&= \int \mathbf{1}_{\{(G,o)\in\mathcal{B}_1\}}\, d\mu_1(G,o) \\
&= \mu_1(\mathcal{B})
\end{aligned}$$

by the Mass-Transport Principle, (8.2), and (8.3). Likewise, if $\mathcal{B}$ is an event that specifies only the second coordinates of the marks, then $\nu_n(\mathcal{B}) = \mu_2(\mathcal{B})$. Thus, ν_n is an $\mathcal{R}$-coupling of μ_1 to μ_2.

Now by definition of ν_n, we have

$$\int f(G,o)\, d\nu_n(G,o) = \int \sum_{x \in \mathsf{V}(G)} \lambda_n(G,o,x) f(G,x)\, d\nu(G,o)$$

for every Borel $f : \mathcal{G}_* \to [0,\infty]$. Therefore, for every Borel $h : \mathcal{G}_{**} \to [0,1]$ with $h(G,x,y) = 0$ unless $x \sim y$, we have

$$\begin{aligned}
\int \sum_{y \in \mathsf{V}(G)} h(G,o,y)\, d\nu_n(G,o) &= \int \sum_{x \in \mathsf{V}(G)} \lambda_n(G,o,x) \sum_{y \in \mathsf{V}(G)} h(G,x,y)\, d\nu_n(G,o) \\
&= \int \sum_{y \in \mathsf{V}(G)} \sum_{x \in \mathsf{V}(G)} h(G,x,y) \lambda_n(G,o,x)\, d\nu(G,o)
\end{aligned}$$

and

$$\begin{aligned}
\int \sum_{y \in \mathsf{V}(G)} h(G,y,o)\, d\nu_n(G,o) &= \int \sum_{x \in \mathsf{V}(G)} \lambda_n(G,o,x) \sum_{y \in \mathsf{V}(G)} h(G,y,x)\, d\nu_n(G,o) \\
&= \int \sum_{y \in \mathsf{V}(G)} \sum_{x \in \mathsf{V}(G)} h(G,x,y) \lambda_n(G,o,y)\, d\nu(G,o)\,,
\end{aligned}$$

where, in the last step, we have interchanged x and y. Therefore,

$$\begin{aligned}
&\left| \int \sum_{y \in \mathsf{V}(G)} h(G,o,y)\, d\nu_n(G,o) - \int \sum_{y \in \mathsf{V}(G)} h(G,y,o)\, d\nu_n(G,o) \right| \\
&\qquad\qquad \le \int \sum_{y \in \mathsf{V}(G)} \sum_{x \sim y} |\lambda_n(G,o,x) - \lambda_n(G,o,y)|\, d\nu(G,o) \\
&\qquad\qquad = \int \sum_{y \in \mathsf{V}(G)} \sum_{x \sim y} |\lambda_n(G,o,x) - \lambda_n(G,o,y)|\, d\mu_1(G,o)\,,
\end{aligned}$$

which tends to 0 by (8.4). Thus, any limit point of ν_n is involution invariant and, since $\mathcal{R}$ is closed, is an $\mathcal{R}$-coupling of μ_1 to μ_2. $\square$

Proposition 8.7 (Recurrence Implies Amenability)**.** *If $\mu \in \mathcal{U}$ and simple random walk is μ-a.s. recurrent, then μ is amenable.*

Proof. Consider the "lazy" simple random walk that moves nowhere with probability 1/2 and otherwise moves to a random neighbor, like simple random walk. It is recurrent by hypothesis and aperiodic by construction. Let $\lambda_n(G, o, x)$ be the probability that lazy simple random walk on G starting from o will be at x at time n. By Orey (1962) and recurrence, the functions λ_n satisfy property (iii) of Theorem 8.5, whence μ is amenable. $\square$

The following is proved similarly to Remark 6.2 of BLPS (2001).

Proposition 8.8 (Forests in Amenable Networks)**.** *If $\mu \in \mathcal{U}$ is amenable and $\Xi_0 \subseteq \Xi$ is such that the Ξ_0-open subgraph $\mathfrak{F}$ of (G, o) is a forest μ-a.s., then the expected degree of o in $\mathfrak{F}$ is at most 2.*

The following extends Theorem 5.3 of BLPS (1999b). In the following, we say that $\mathbf{P}$ is a **percolation** on μ that **gives** subgraphs $\mathcal{A}$ a.s. if there is a Borel function $\psi : \Xi \to \Xi$ such that $\mu = \mathbf{P} \circ \psi^{-1}$ and there is a Borel subset $\Xi_0 \subseteq \Xi$ such that if $G(\Xi_0)$ denotes the Ξ_0-open subnetwork of G, then $\psi\big(G(\Xi_0)\big) \in \mathcal{A}$ for $\mathbf{P}$-a.e. G.

Theorem 8.9. *Let $\mu \in \mathcal{U}$ with $\overline{\deg}(\mu) < \infty$. The following are equivalent:*

- *(i) μ is amenable;*
- *(ii) there is a percolation $\mathbf{P}$ on μ that gives spanning trees with at most 2 ends a.s.;*
- *(iii) there is a percolation $\mathbf{P}$ on μ that gives non-empty connected subgraphs ω that satisfy $p_c(\omega) = 1$ a.s.*

Proof. The proof that (i) implies (ii) is done as for Theorem 5.3 of BLPS (1999b), but uses Propositions 8.8 and 6.3. That (ii) implies (iii) is obvious. The proof that (iii) implies (i) follows the first part of the proof of Theorem 1.1 in Benjamini, Lyons, Peres, and Schramm (1999a). $\square$

Corollary 8.10 (Amenable Trees)**.** *A unimodular probability measure μ on infinite rooted trees is amenable iff $\overline{\deg}(\mu) = 2$ iff μ-a.s. G has 1 or 2 ends.*

Proof. Combine Theorem 8.9 with Theorem 6.2. $\square$

The next result was proved for non-amenable unimodular transitive graphs in BLPS (1999b) with a more direct proof in Benjamini, Lyons, Peres, and Schramm (1999a). This extension is proved similarly. Presumably, the hypothesis that μ is non-amenable can be replaced by the assumption that $p_c(\mu) < 1$. (This is a major open conjecture for quasi-transitive graphs.)

Theorem 8.11 (Critical Percolation)**.** *Let μ be an extremal unimodular non-amenable probability measure on $\mathcal{G}_*$ with $\overline{\deg}(\mu) < \infty$. There is $\mathbf{P}_{p_c(\mu)}$-a.s. no infinite cluster.*

This can be interpreted for finite graphs as follows. Suppose that G_n are finite connected graphs with bounded average degree whose random weak limit is extremal and non-amenable with critical value p_c. Consider Bernoulli(p_c) percolation on G_n. Let $\alpha_n(\ell)$ be the random variable giving the proportion of vertices of G_n that belong to simple open paths of length at least ℓ. Then $\lim_{\ell\to\infty}\lim_{n\to\infty}\alpha_n(\ell) = 0$ in probability.

The following theorem is proved similarly to Benjamini and Schramm (2001a), as extended by Lyons with Peres (2008), and by using Example 9.6.

Theorem 8.12 (Planar Percolation)**.** *Let $\mu \in \mathcal{U}$ be extremal, non-amenable, and carried by plane graphs with one end and bounded degree. Then $0 < p_c(\mu) < p_u(\mu) < 1$ and Bernoulli($p_u(\mu)$) percolation on μ has a unique infinite cluster a.s.*

The following extends Theorems 3.1, 3.2, 3.5, 3.6, and 3.10 of Benjamini, Lyons, and Schramm (1999) and is proved similarly. If $\mathbf{P}$ is a percolation on μ with $\mathbf{P} \circ \psi^{-1} = \mu$ and with $\Xi_0 \subseteq \Xi$ defining the open subgraphs, we call $\mathbf{P}'$ a **subpercolation** on $\mathbf{P}$ that **gives** subgraphs in $\mathcal{A}'$ **with positive probability** if there is a Borel function $\psi' : \Xi \to \Xi$ such that $\mathbf{P} = \mathbf{P}' \circ \psi'^{-1}$ and there is a Borel subset $\Xi_1 \subseteq \Xi$ such that if $G(\Xi_1)$ denotes the Ξ_1-open subnetwork of G, then $\mathbf{P}'\left[\psi\Big(\psi'\big(G(\Xi_1)\big)\big(\Xi_0\big)\Big) \in \mathcal{A}'\right] > 0$.

Theorem 8.13 (Non-Amenable Subgraphs)**.** *Let μ be a unimodular probability measure on $\mathcal{G}_*$ with finite expected degree and $\mathbf{P}$ be a percolation on μ with open subgraph ω.*

(i) If $h > 0$ and $\mathbf{E}[\deg_\omega o \mid o \in \omega] \ge \alpha(\mu) + 2h$, then there is a subpercolation $\mathbf{P}'$ on $\mathbf{P}$ that gives a non-empty subgraph ω' with $\iota_{\mathsf{E}}(\omega') \ge h$ with positive $\mathbf{P}'$-probability.

(ii) If ω is a forest a.s., $h > 0$ and $\mathbf{E}[\deg_\omega o \mid o \in \omega] \ge 2 + 2h$, then there is a subpercolation $\mathbf{P}'$ on $\mathbf{P}$ that gives a non-empty subgraph with $\iota_{\mathsf{E}} \ge h$ with positive probability.

(iii) If μ is non-amenable and ω has exactly one infinite cluster $\mathbf{P}$-a.s., then there is a subpercolation $\mathbf{P}'$ on $\mathbf{P}$ that gives a non-empty subgraph ω' with $\iota_{\mathsf{E}}(\omega') > 0$ $\mathbf{P}'$-a.s.

(iv) If ω has components with at least three ends $\mathbf{P}$-a.s., then there is a subpercolation $\mathbf{P}'$ on $\mathbf{P}$ that gives a non-empty forest $\mathfrak{F}$ with $\iota_{\mathsf{E}}(\mathfrak{F}) > 0$ $\mathbf{P}'$-a.s.

(v) If μ is concentrated on subgraphs with spectral radius less than 1 and ω has exactly one infinite cluster $\mathbf{P}$-a.s., then there is a subpercolation $\mathbf{P}'$ on $\mathbf{P}$ that gives a non-empty forest $\mathfrak{F}$ with $\iota_{\mathsf{E}}(\mathfrak{F}) > 0$ $\mathbf{P}'$-a.s.

The following extends a result of Häggström (1995) and is proved similarly to Corollary 6.3 of BLPS (2001).

Proposition 8.14 (Amenability and Boundary Conditions)**.** *Let μ be an amenable unimodular probability measure on $\mathcal{G}_*$. Then $\mathsf{FUSF}(\mu) = \mathsf{WUSF}(\mu)$ and $\mathsf{FMSF}(\mu) = \mathsf{WMSF}(\mu)$.*

We may now strengthen Proposition 8.7, despite the fact that not every graph is necessarily non-amenable μ-a.s. It extends Theorem 4.3 of Benjamini, Lyons, and Schramm (1999) and is proved similarly, using Theorem 8.13(iii) with $\mathbf{P} := \mu$.

Theorem 8.15. *Positive Speed on Non-Amenable Graphs[] If $\mu \in \mathcal{U}$ is non-amenable and concentrated on graphs with bounded degree, then the speed of simple random walk is positive μ-a.s.*

9 Examples

We present here a variety of interesting examples of unimodular measures.

Example 9.1. (Renewal Processes) Given a stationary (delayed) renewal process on $\mathbb{Z}$, let μ be the law of $(\mathbb{Z}, 0)$ with the graph $\mathbb{Z}$, some fixed mark at renewals, and some other fixed mark elsewhere. Then $\mu \in \mathcal{U}$.

Example 9.2. (Half-Plane) Fix $d \geq 3$ and let T be the d-regular tree. Let μ_1 be the random weak limit of balls of growing radii in T. Note that μ_1 is carried by trees with only one end. Let μ_2 be concentrated on the fixed graph $(\mathbb{Z}, 0)$. Now let $\mu := \mu_1 \boxtimes \mu_2$. This is a unimodular version of the half-plane $\mathbb{N} \times \mathbb{Z}$.

Example 9.3. For a rooted network (G, o), its **universal cover** is the rooted tree $(T, o) = T(G, o)$ formed as follows. The vertices of T are the finite paths in G that start at o and do not backtrack. Two such vertices are joined by an edge in T if one is an extension of the other by exactly one edge in G. The path with no edges consisting of just the vertex o in G is the root o of T. There is a natural rooted graph homomorphism $\pi : (T, o) \to (G, o)$ (the cover map) that maps paths to their last point. Marks on T are defined by lifting the marks on G via π. It is clear that if $\mu \in \mathcal{U}$ and ν is the law of $T(G, o)$ when (G, o) has the law μ, then $\nu \in \mathcal{U}$ and $\overline{\deg}(\mu) = \overline{\deg}(\nu)$.

Example 9.4. Let $\mathbf{P}$ be a unimodular percolation on μ that labels edges either open or closed. Let ν be the law of the open cluster of the root when the network is chosen according to $\mathbf{P}$ conditional on the root belonging to an infinite open cluster. Then ν is unimodular, as a direct verification of the definition shows. When μ is concentrated on a fixed unimodular graph, this fact has been widely used in the study of percolation.

Example 9.5. (Tilings) Let X be a Euclidean space or hyperbolic space (of constant curvature). Write Γ for its isometry group. There is a Mass-Transport Principle for X that says the following; see Benjamini and Schramm (2001a) for a proof. Let ρ be a positive Borel measure on $X \times X$ that is invariant under the diagonal action of Γ. Then there is a constant c such that for all Borel $A \subset X$ of volume $|A| > 0$, we have $\rho(A \times X) = \rho(X \times A) = c|A|$. Suppose that P is a (countable) point process in X whose law is Γ-invariant. For example, Poisson point processes are Γ-invariant. One often considers graphs G that are functions $G = \beta(P)$, where β commutes with the action of Γ. For a few recent examples, see Benjamini and Schramm (2001a), Holroyd and Peres (2003), or Timár (2004). For instance, the 1-skeleton of the Voronoi tessellation corresponding to P is such a graph. In general, we call such measures on graphs **Γ-equivariant factors** of P. They are necessarily Γ-invariant.

Another way that invariant measures on graphs embedded in X occur is through (aperiodic) tilings of X. Again, one can take the 1-skeleton. An important tool for studying aperiodic tilings is a limit measure obtained from translates of a given tiling (in the Euclidean case), or, more generally, invariant measures on tilings with special properties; see, e.g., Robinson (1996), Radin (1997), Solomyak (1997), Radin (1999), or Bowen and Radin (2003) for some examples.

Let ν be any Γ-invariant probability measure on graphs embedded in X. Fix a Borel set $A \subset X$ of positive finite volume. If $v(A) := \int |\mathsf{V}(G) \cap A| \, d\nu(G) < \infty$, then define μ as follows. Choose G with the law ν biased by $|\mathsf{V}(G) \cap A|$. Then choose the root o of G uniformly among all vertices that belong to A. The law of the resulting graph (G, o) is μ. We claim that μ is unimodular and does not depend on A. In fact, μ is the same as the Palm measure of (G, o), except that μ is a measure on isomorphism classes of graphs that does not involve any geometric embedding. To prove our claims, we first write μ in symbols:

$$\mu(\mathcal{A}) := v(A)^{-1} \int \sum_{o \in A} \mathbf{1}_{\{(G,o) \in \mathcal{A}\}} \, d\nu(G)$$

for Borel $\mathcal{A} \subset \mathcal{G}_*$. Let $f : \mathcal{G}_{**} \to [0, \infty]$ be Borel. Define

$$\rho(B \times C) := \int \sum_{x \in \mathsf{V}(G) \cap B} \sum_{y \in \mathsf{V}(G) \cap C} f(G, x, y) \, d\nu(G)$$

for Borel $B, C \subseteq X$. Since ν is invariant, ρ is diagonally invariant. Therefore,

$$\begin{aligned} \int \sum_{x \in \mathsf{V}(G)} f(G, o, x) \, d\mu(G, o) &= v(A)^{-1} \rho(A \times X) = v(A)^{-1} \rho(X \times A) \\ &= \int \sum_{x \in \mathsf{V}(G)} f(G, x, o) \, d\mu(G, o) \,, \end{aligned}$$

which means that μ satisfies the Mass-Transport Principle, i.e., is unimodular. Furthermore, if we take $f(G, x, y) := \mathbf{1}_{\{x=y\}}$, then we see that $\rho(A \times X) = v(A)$, so that there is a constant c such that $v(A) = c|A|$. Likewise, if $f(G, x, y) := \mathbf{1}_{\{x=y, (G,x) \in \mathcal{A}\}}$, then we see that for each $\mathcal{A}$, there is another constant $c_{\mathcal{A}}$ such that $v(A)\mu(\mathcal{A}) = c_{\mathcal{A}}|A|$. It follows that μ does not depend on A.

Example 9.6. (Planar Duals) Let μ be a unimodular probability measure on plane graphs all of whose faces have finitely many sides. We are assuming that to each graph, there is a measurably associated plane embedding. Thus, each graph G has a plane dual $G^\dagger$ with respect to its embedding. In fact, to be technically accurate in what follows, we replace the embedding by an assignment (possibly random) of marks to the edges that indicate the cyclic order in which they appear around a vertex in a fixed orientation of the plane. (For example, if a vertex x has d edges incident to it, then one can let the d edge marks associated to x be $\{1, 2, \ldots, d\}$ in cyclic order, with the one marked 1 chosen at random, independently of marks elsewhere.) Then the plane dual graph is defined entirely with respect to the resulting network in an automorphism-equivariant way, needing no reference to the plane.

Provided a certain finiteness condition is satisfied, there is a natural unimodular probability measure on the dual graphs, constructed as follows. For a face f, let $\deg f$ denote the number of sides of f. For a vertex x, let $F(x) := \sum_{f \sim x} 1/\deg f$. Assume that $Z := \int F(o) \, d\mu(G, o) < \infty$. To create a unimodular probability measure $\mu^\dagger$ on the duals, first choose (G, o) with law μ biased by $F(o)/Z$. Then choose a face f_0 incident to o with probability proportional to $1/\deg f_0$. The law of the resulting rooted graph $(G^\dagger, f_0)$ is $\mu^\dagger$:

$$\mu^\dagger(\mathcal{A}) := Z^{-1} \int \sum_{f_0 \sim o} \frac{1}{\deg f_0} \mathbf{1}_{\{(G^\dagger, f_0) \in \mathcal{A}\}} \, d\mu(G, o)$$

for Borel $\mathcal{A} \subseteq \mathcal{G}_*$. To prove that $\mu^\dagger$ is indeed unimodular, let $k : \mathcal{G}_{**} \to [0, \infty]$ be Borel. Then

$$\begin{aligned}
Z \int \sum_{f \in \mathsf{V}(G^\dagger)} k(G^\dagger, f_0, f)\, d\mu^\dagger(G^\dagger, f_0) &= \int \sum_{f_0 \sim o} \frac{1}{\deg f_0} \sum_{f \in \mathsf{V}(G^\dagger)} k(G^\dagger, f_0, f)\, d\mu(G, o) \\
&= \int \sum_{f_0 \sim o} \sum_{f \in \mathsf{V}(G^\dagger)} \frac{1}{\deg f_0} k(G^\dagger, f_0, f) \sum_{x \sim f} \frac{1}{\deg f}\, d\mu(G, o) \\
&= \int \sum_{x \in \mathsf{V}(G)} \sum_{f_0 \sim o} \sum_{f \sim x} \frac{1}{\deg f_0} \frac{1}{\deg f} k(G^\dagger, f_0, f)\, d\mu(G, o) \\
&= \int \sum_{x \in \mathsf{V}(G)} \sum_{f_0 \sim x} \sum_{f \sim o} \frac{1}{\deg f_0} \frac{1}{\deg f} k(G^\dagger, f_0, f)\, d\mu(G, o) \\
&\qquad\qquad [\text{by the Mass-Transport Principle for } \mu] \\
&= \int \sum_{x \in \mathsf{V}(G)} \sum_{f \sim x} \sum_{f_0 \sim o} \frac{1}{\deg f_0} \frac{1}{\deg f} k(G^\dagger, f, f_0)\, d\mu(G, o) \\
&= Z \int \sum_{f \in \mathsf{V}(G^\dagger)} k(G^\dagger, f, f_0)\, d\mu^\dagger(G^\dagger, f_0)\,.
\end{aligned}$$

Thus, $\mu^\dagger$ satisfies the Mass-Transport Principle, so is unimodular. A similar argument shows that $(\mu^\dagger)^\dagger = \mu$.

Another important construction comes from combining the primal and dual graphs into a new plane graph by adding a vertex where each edge crosses its dual. That is, if G is a plane graph and $G^\dagger$ its dual, then every edge $e \in \mathsf{E}(G)$ intersects $e^\dagger \in \mathsf{E}(G^\dagger)$ in one point, v_e. (These are the only intersections of G and $G^\dagger$.) For $e \in \mathsf{E}(G)$, write $\widehat{e}$ for the pair of edges that result from the subdivision of e by v_e, and likewise for $\widehat{e^\dagger}$. This defines a new graph $\widehat{G}$, whose vertices are $\mathsf{V}(G) \cup \mathsf{V}(G^\dagger) \cup \left\{v_e\,;\ e \in \mathsf{E}(G)\right\}$ and whose edges are $\bigcup_{e \in \mathsf{E}(G)} (\widehat{e} \cup \widehat{e^\dagger})$. If $\overline{\deg}(\mu) < \infty$, then we may define a unimodular probability measure $\widehat{\mu}$ on the graphs $\widehat{G}$ from μ as follows. Let $\widehat{Z} := (3/2)\overline{\deg}(\mu) + Z \le (5/2)\overline{\deg}(\mu) < \infty$. For $x \in \mathsf{V}(G)$, let $\widehat{N}(x)$ be the set consisting of x itself plus the vertices of $\widehat{G}$ that correspond to edges or faces of G that are incident to x. For $w \in \mathsf{V}(\widehat{G})$, define

$$\delta(w) := |\{x \in \mathsf{V}(G)\,;\ w \in \widehat{N}(x)\}|^{-1} = \begin{cases} 1 & \text{if } w \in \mathsf{V}(G), \\ 1/2 & \text{if } w = v_e \text{ for some } e \in \mathsf{E}(G), \\ 1/\deg w & \text{if } w \in \mathsf{V}(G^\dagger). \end{cases}$$

Define

$$\widehat{\mu}(\mathcal{A}) := \widehat{Z}^{-1} \int \sum_{w_0 \in \widehat{N}(o)} \delta(w_0) \mathbf{1}_{\{(\widehat{G}, w_0) \in \mathcal{A}\}}\, d\mu(G, o)\,.$$

Note that $\widehat{\mu}$ is a probability measure. To prove that $\widehat{\mu}$ is unimodular, let $k : \mathcal{G}_{**} \to [0, \infty]$ be

Borel. Then

$$
\begin{aligned}
\widehat{Z} \int \sum_{w \in \mathsf{V}(\widehat{G})} k(\widehat{G}, w_0, w)\, d\widehat{\mu}(\widehat{G}, w_0) &= \int \sum_{w_0 \in \widehat{N}(o)} \delta(w_0) \sum_{w \in \mathsf{V}(\widehat{G})} k(\widehat{G}, w_0, w)\, d\mu(G, o) \\
&= \int \sum_{w_0 \in \widehat{N}(o)} \sum_{w \in \mathsf{V}(\widehat{G})} \delta(w_0) k(\widehat{G}, w_0, w) \sum_{x\,;\, w \in \widehat{N}(x)} \delta(w)\, d\mu(G, o) \\
&= \int \sum_{x \in \mathsf{V}(G)} \sum_{w_0 \in \widehat{N}(o)} \sum_{w \in \widehat{N}(x)} \delta(w_0)\delta(w) k(\widehat{G}, w_0, w)\, d\mu(G, o) \\
&= \int \sum_{x \in \mathsf{V}(G)} \sum_{w_0 \in \widehat{N}(x)} \sum_{w \in \widehat{N}(o)} \delta(w_0)\delta(w) k(\widehat{G}, w_0, w)\, d\mu(G, o) \\
&\qquad\qquad\qquad [\text{by the Mass-Transport Principle for } \mu] \\
&= \int \sum_{x \in \mathsf{V}(G)} \sum_{w \in \widehat{N}(x)} \sum_{w_0 \in \widehat{N}(o)} \delta(w_0)\delta(w) k(\widehat{G}, w, w_0)\, d\mu(G, o) \\
&= \widehat{Z} \int \sum_{w \in \mathsf{V}(\widehat{G})} k(\widehat{G}, w, w_0)\, d\widehat{\mu}(\widehat{G}, w_0)\,.
\end{aligned}
$$

Thus, $\widehat{\mu}$ satisfies the Mass-Transport Principle, so is unimodular.

Example 9.7. (Poisson Weighted Infinite Tree) Our definition (Section 2) of the metric on the space $\mathcal{G}_*$ of rooted graphs refers to "balls of radius r" in which distance is graph distance, i.e., edges implicitly have length 1. Aldous and Steele (2004) work in the setting of graphs whose edges have positive real lengths, so that distance becomes minimum path length. This setting permits one to consider graphs which may have infinite degree, but which are still "locally finite" in the sense that only finitely vertices fall within any finite radius ball. Of course, edge lengths are a special (symmetric) case of edge marks. An important example is the following. Consider a regular rooted tree T of infinite degree. Fix a continuous increasing function Λ on $[0, \infty)$ with $\Lambda(0) = 0$ and $\lim_{t\to\infty} \Lambda(t) = \infty$. Order the children of each vertex of T via a bijection with $\mathbb{Z}^+$. For each vertex x, consider an independent Poisson process on $\mathbb{R}^+$ with mean function Λ. Define the length of the edge joining x to its nth child to be the nth point of the Poisson process associated to x. This is a unimodular random network in the extended sense of Aldous and Steele (2004). It can be derived by taking the random weak limit of the complete graph on n vertices whose edge lengths are independent with cdf $t \mapsto 1 - e^{-\Lambda(t)/n}$ $(t \ge 0)$ and then deleting all edges in the limit whose length is ∞. (We are working here with the mark space $[0, \infty]$.) See Aldous (1992).

Example 9.8. (Edge Replacement) Here is a general way to create unimodular random rooted graphs from existing unimodular fixed graphs. This is an extension of the random subdivision (or stretching) introduced by Adams and Lyons (1991) and studied further in Example 2.4.4 of Kaimanovich (1998) and Chen and Peres (2004). Let FG_2 be the set of isomorphism classes of finite graphs with an ordered pair of distinct distinguished vertices. For our construction, we may start with a fixed unimodular quasi-transitive connected graph, G, or, more generally, with any unimodular probability measure μ on $\mathcal{G}_*$. In the former case, fix an orientation of the edges of G and let L be a random field on the oriented edges of G that is invariant under the automorphism group of G and takes values in FG_2 and such that $\big|\mathsf{V}\big(L(e)\big)\big|$ has finite mean for each edge e. Replace each edge e with the graph $L(e)$, where the first of the distinguished vertices of $L(e)$ is identified with the tail of e and the second of its distinguished vertices is identified with the head of e. Call the resulting random graph H. There is a unimodular probability measure that is equivalent to this measure on random graphs. Namely, let μ_i be the law of (H, o_i), where $\{o_i\}$ is a complete section of the vertex orbits of G. Given H, write

$$A(x) := 2 + \sum_{e \sim x} \Big(\big|\mathsf{V}\big(L(e)\big)\big| - 2 \Big)$$

and

$$c := \sum_i \mathbf{E}\big[A(o_i)\big] \big|\mathsf{Stab}(o_i)\big|^{-1} .$$

Choose o_i with probability $c^{-1}\mathbf{E}\big[A(o_i)\big]\big|\mathsf{Stab}(o_i)\big|^{-1}$. Given o_i, choose (H, o_i) with distribution μ_i. Given this, list the non-distinguished vertices of all $L(e)$ for e incident to o_i as $z_1, z_2, \ldots, z_{A(o_i)-2}$ and set $z_{A(o_i)-1} := z_{A(o_i)} := o_i$. Let U be a uniform integer in $\big[1, A(o_i)\big]$. Then (H, z_U) is unimodular and, clearly, has law with respect to which $\sum_i \mu_i$ is absolutely continuous.

Indeed, we state and prove this more generally. Suppose that μ is a unimodular probability measure on $\mathcal{G}_*$. Orient the edges of the rooted networks arbitrarily. Let $\psi(e)$ denote the ordered pair of the marks of the edge e (ordered by the orientation of e). Suppose $L : \Xi^2 \to \mathsf{FG}_2$ is Borel with the property that whenever $L(\xi_1, \xi_2) = (G, x, y)$, we also have $L(\xi_2, \xi_1) = (G, y, x)$. (This will ensure that the orientation of the edges will not affect the result.) If

$$\int \sum_{e \sim o} \Big[\Big|\mathsf{V}\Big(L\big(\psi(e)\big)\Big)\Big| - 2 \Big] \, d\mu(G, o) < \infty ,$$

then let μ' be the following measure. Define

$$A(G, o) := 2 + \sum_{e \sim o} \Big[\Big|\mathsf{V}\Big(L\big(\psi(e)\big)\Big)\Big| - 2 \Big] .$$

Choose (G, o) with probability distribution μ biased by $A(G, o)$ and replace each edge e by the graph $L\big(\psi(e)\big)$, where the tail and head of e are identified with the first and second distinguished vertices of $L\big(\psi(e)\big)$, respectively; call the resulting graph H. Write $A := A(G, o)$ and list the non-distinguished vertices of all $L\big(\psi(e)\big)$ for e incident to o as $z_1, z_2, \ldots, z_{A-2}$ and set $z_{A-1} := z_A := o$. Let U be a uniform integer in $[1, A]$. Finally, let μ' be the distribution of (H, z_U).

This is unimodular by the following calculation. Write $H(G)$ for the graph H formed as above from the network G. Let $\mathsf{V}_0(\xi_1, \xi_2)$ be the set of non-distinguished vertices of the graph $L(\xi_1, \xi_2)$. Write $z_i(G, o)$ $(1 \le i \le A(G, o) - 2)$ for the vertices of the neighborhood $B(G, o) := \bigcup_{e \sim o} \mathsf{V}_0\big(\psi(e)\big)$. Write $z_i(G, o) := o$ for $i = A(G, o) - 1, A(G, o)$. Put $c := \int A(G, o) \, d\mu(G, o)$. In order to show that μ' is unimodular, let $f : \mathcal{G}_{**} \to [0, \infty]$ be

Borel. Define $\overline{f} : \mathcal{G}_{**} \to [0, \infty]$ by

$$\overline{f}(G,x,y) := \frac{1}{c} \sum_{z \in B(G,x)} \sum_{z' \in B(G,y)} f\big(H(G), z, z'\big) + \frac{2}{c} \sum_{z \in B(G,x)} f\big(H(G), z, y\big) + \frac{2}{c} \sum_{z' \in B(G,y)} f\big(H(G), x, z'\big) + \frac{2}{c} f\big(H(G), x, y\big) .$$

Then

$$\begin{aligned}
\int \sum_{z \in \mathsf{V}(H)} & f(H,o,z)\, d\mu'(H,o) \\
&= \frac{1}{c} \int \frac{1}{A(G,o)} \sum_{i=1}^{A(G,o)} \sum_{z \in \mathsf{V}(H(G))} f\big(H(G), z_i(G,o), z\big) A(G,o)\, d\mu(G,o) \\
&= \int \sum_{x \in \mathsf{V}(G)} \overline{f}(G,o,x)\, d\mu(G,o) \\
&= \int \sum_{x \in \mathsf{V}(G)} \overline{f}(G,x,o)\, d\mu(G,o) \\
&= \int \sum_{z \in \mathsf{V}(H)} f(H,z,o)\, d\mu'(H,o) .
\end{aligned}$$

Our final example details the correspondence between random rooted graphs and graphings of equivalence relations.

Example 9.9. Let μ be a Borel probability measure on a topological space X and R be a Borel subset of X^2 that is an equivalence relation with finite or countable equivalence classes. We call the triple (X, μ, R) a **measured equivalence relation**. For $x \in X$, denote its R-equivalence class by $[x]$. We call R **measure preserving** if

$$\int_{x \in X} \sum_{y \in [x]} f(x,y)\, d\mu(x) = \int_{x \in X} \sum_{y \in [x]} f(y,x)\, d\mu(x)$$

for all Borel $f : X^2 \to [0, \infty]$. A **graphing** Φ of R is a Borel subset of X^2 such that the smallest equivalence relation containing Φ is R. A graphing Φ induces the structure of a graph on the vertex set X by defining an edge between x and y if $(x,y) \in \Phi$ or $(y,x) \in \Phi$. Denote the subgraph induced on $[x]$ and rooted at x by $\Phi(x)$. Given Borel maps $\psi : X \to \Xi$ and $\phi : X^2 \to \Xi$, we regard $\psi(x)$ as the mark at x and $\phi(x,y)$ as the mark at x of the edge from x to y. Thus, $\Phi(x)$ is a random rooted network. Its law (or, rather, the law of its rooted isomorphism class) is unimodular iff R is measure preserving.

Conversely, suppose that μ is a probability measure on $\mathcal{G}_*$. Add independent uniform marks as second coordinates to the existing marks and call the resulting measure ν. Write $\mathcal{G}_\# \subset \mathcal{G}_*$ for the set of (isomorphism classes of) rooted networks with distinct marks. Thus, ν is concentrated on $\mathcal{G}_\#$. Define $R \subset \mathcal{G}_\#^2$ to be the set of pairs of (isomorphism classes of) rooted networks that are non-rooted isomorphic. Define $\Phi \subset R$ to be the set of pairs of isomorphic rooted networks whose roots are neighbors in the unique (non-rooted) isomorphism. Then $(\mathcal{G}_{\#}, \nu, R)$ is a measured equivalence relation with graphing Φ. We have that R is measure preserving iff μ is unimodular. If we define the mark map prj that forgets the second coordinate, then ν pushes forward to μ, i.e., $\mu = \nu \circ \mathsf{prj}^{-1}$.

Thus, the theory of unimodular random rooted networks has substantial overlap with the theory of graphed measure-preserving equivalence relations. The largest difference between the two theories lies in the foci of attention: We focus on probabilistic aspects of the graphing, while the other theory focuses on ergodic aspects of the equivalence relation (and, thus, considers all graphings of a given equivalence relation). The origins of our work lie in two distinct areas: one is group-invariant percolation on graphs, while the other is asymptotic analysis of finite graphs. The origin of the study of measured equivalence relations lies in the ergodic theory of group actions. Some references for the latter work, showing relations to von Neumann algebras and logic, among other things, are Feldman and Moore (1977a,b), Feldman, Hahn, and Moore (1978), Connes, Feldman, and Weiss (1981), Zimmer (1984), Kechris and Miller (2004), Becker and Kechris (1996), Kechris (1991), Adams and Lyons (1991), Kaimanovich (1997, 1998), Paulin (1999), Gaboriau (2000, 2002), and Furman (1999a,b).

10 Finite Approximation

Although we do not present any theorems in this section, because of its potential importance, we have devoted the whole section to the question of whether finite networks are weakly dense in $\mathcal{U}$. Let us call random weak limits of finite networks **sofic**.

Question 10.1. (Finite Approximation) Is every probability measure in $\mathcal{U}$ sofic? In other words, if μ is a unimodular probability measure on $\mathcal{G}_*$, do there exist finite networks G_n such that $U(G_n) \Rightarrow \mu$?

To appreciate why the answer is not obvious, consider the special case of the (non-random) graph consisting of the infinite rooted 3-regular tree. It is true that there exist finite graphs G_n that approximate the infinite 3-regular tree in the sense of random weak convergence; a moment's thought shows these *cannot* be finite trees. This special case is of course known (one can use finite quotient groups of $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$, random 3-regular graphs, or expanders Lubotzky (1994)), but the constructions in this special case do not readily extend to the general case.

Another known case of sofic measures is more difficult to establish. Namely, Bowen (2003) showed that all unimodular networks on regular trees are sofic. (To deduce this from his result, one must use the fact, easily established, that networks with marks from a finite set are dense in $\mathcal{G}_*$.)

Example 10.2. The general unimodular Galton-Watson measure UGW (Example 1.1) is also sofic. To see this, consider the following random networks, sometimes called "fixed-degree distribution networks" and first studied by Molloy and Reed (1995). Given $\langle r_k \rangle$ and n vertices, give each vertex k balls with probability r_k, independently. Then pair the balls at random and place an edge for each pair between the corresponding vertices. There may be one ball left over; if so, ignore it. Let $m_0 := \sum k r_k$, which we assume is finite. In the limit, we get a tree where the root has degree k with probability r_k, each neighbor of the root, if any, has degree k with probability kr_k/m_0, etc. In fact, we get UGW for the offspring distribution $k \mapsto (k+1)r_{k+1}/m_0$. Thus, if we want the offspring distribution $\langle p_k \rangle$, we need merely start with $r_k := c^{-1}p_{k-1}/k$ for $k \geq 1$ and $r_0 := 0$, where $c := \sum_{k \geq 0} p_k/(k+1)$.

Let us compare the intuitions behind amenability and unimodularity. One can define an average of any bounded function on the vertices of, say, an amenable Cayley graph; the average will be the same for any translate of the function. This can also be regarded as an average of the function with respect to a probability measure that chooses a group element uniformly at random; however, the precise justification of this requires a measure that, though group invariant, is only finitely additive. Nevertheless, this invariant measure is approximated by uniform measures on finite sets, namely, Følner sets. By contrast, the justification that a unimodular random rooted graph provides a uniform distribution on the vertices is via the Mass-Transport Principle. The measure itself is, of course, countably additive; if it is sofic, then it, too, is approximated by uniform measures on finite sets. The two intuitions concerning uniform measures that come from amenability and from unimodularity agree insofar as every amenable quasi-transitive graph is unimodular, as shown by Soardi and Woess (1990) and Salvatori (1992).

One might think that if a sequence $\langle G_n \rangle$ of finite graphs has a fixed transitive graph G as its random weak limit, then any unimodular probability measure on networks supported by G would be a random weak limit of some choice of networks on the same sequence $\langle G_n \rangle$. This is false, however; e.g., if G is a 3-regular tree, then almost any choice of a sequence of growing 3-regular graphs has G as its random weak limit. However, there is a random independent set[3] of density $1/2$ on G whose law is automorphism invariant, while the density of independent sets in random 3-regular graphs is bounded away from $1/2$ (see Frieze and Suen (1994)). Nevertheless, if Question 10.1 has a positive answer, then it is not hard to show that there is some sequence of finite graphs that has this property of carrying arbitrary networks.

Recall that the **Cayley diagram** of a group Γ generated by a finite subset S is the network (G, o) with vertex set Γ, edge set $\big\{(x, xs)\,;\; x \in \Gamma,\; s \in S\big\}$, root o the identity element of Γ, and edge marks s at the endpoint x of (x, xs) and s^{-1} at the endpoint xs of (x, xs), as in Remark 3.3. We do not mark the vertices (or mark them all the same). Weiss (2000) defined Γ to be **sofic** if its Cayley diagram is a random weak limit of finite networks with edge marks from $S \cup S^{-1}$. It is easy to check that this property does not depend on the generating set S chosen. By embedding $S \cup S^{-1}$ into Ξ, we can use a positive answer to Question 10.1 to give every Cayley diagram as a random weak limit of *some* finite networks. Changing the marks on the finite networks to their nearest points in $S \cup S^{-1}$ gives the kind of approximating networks desired. That is, we would have that every finitely generated group is sofic. As we mentioned in the introduction, this would have plenty of consequences.

To illustrate additional consequences of a positive answer to Question 10.1, we establish the existence of various probability measures on sofic networks. The idea is that if a class of probability measures is specified by a sequence of local "closed" conditions in such a way that there is a measure in that class on any finite graph, then there is an automorphism-invariant measure in that class on any sofic quasi-transitive graph. Rather than state a general theorem to that effect, we shall illustrate the principle by two examples.

[3] This means that no two vertices of the set are adjacent.

Example 10.3. (Invariant Markov Random Fields) Consider networks with vertex marks ± 1 and no (or constant) edge marks. Given a finite graph G, $h \in \mathbb{R}$, and $\beta > 0$, the probability measure ν_G on mark maps $\psi : \mathsf{V}(G) \to \{-1, +1\}$ given by

$$\nu_G(\psi) := Z^{-1} \exp\Big\{ \sum_{x \in \mathsf{V}(G)} \beta h \psi(x) - \sum_{x \sim y} \beta \psi(x)\psi(y) \Big\} ,$$

where Z is the normalizing constant required to make these probabilities add to 1, is known as the anti-ferromagnetic Ising model at inverse temperature β and with external field h on G. Let G now be an infinite sofic transitive graph. Let $\langle G_n \rangle$ be an approximating sequence of finite graphs and let $\nu_n := \nu_{G_n}$ be the corresponding probability measures. Write $\overline{G}_n$ for the corresponding random network on G_n and, as usual, $U(\overline{G}_n)$ for the uniformly rooted network. Since $U(\overline{G}_n)$ is unimodular, so is any weak limit point, μ. By tightness, there is such a weak limit point, and it is concentrated on networks with underlying graph G. By Theorem 3.2, we may lift μ to an automorphism-invariant measure $\nu = \lambda_\mu$ on networks on G. This measure ν is a Markov random field with the required Gibbs specification, meaning that for any finite subgraph H of G, the conditional ν-distribution of $\psi{\restriction}\mathsf{V}(H)$ given $\psi{\restriction}\partial_{\mathsf{V}} H$ is equal to the conditional ν_H-distribution of the same thing. One is often interested in invariant random fields, not just any random fields with the given Gibbs specification. One can easily get a Markov random field with the required Gibbs specification by taking a limit over subgraphs of G, but this will not necessarily produce an invariant measure. In case G is amenable, one could take a limit of averages of the resulting measure to obtain an invariant measure, but this will not work in the non-amenable case. That is the point of the present construction. A variation on this is spin glasses: Here, for finite graphs G, the measure ν_G is

$$\nu_G(\psi) := Z^{-1} \exp\Big\{ \sum_{x \in \mathsf{V}(G)} \beta h \psi(x) + \sum_{x \sim y} \beta J_{x,y} \psi(x)\psi(y) \Big\} ,$$

where $J_{x,y}$ are, say, independent ± 1-valued random variables. Again, one can find an invariant spin glass (coupled to the independent interactions $J_{x,y}$) with the same parameters h and β on any transitive sofic graph by the above method.

Example 10.4. (Invariant Sandpiles) Consider now networks with vertex marks in $\mathbb{N}$ and no edge marks. Given a finite rooted graph (G, o), a mark map $\psi : \mathsf{V}(G) \to \mathbb{N}$ is called **critical** (or **stable and recurrent**) if for all $x \in \mathsf{V}(G)$, we have $\psi(x) < \deg(x)$ and for all subgraphs W of $G \setminus \{o\}$, there is some $x \in \mathsf{V}(W)$ such that $\psi(x) \ge \deg_W(x)$; we may take $\psi(o) \equiv 0$. In this context, one usually calls the root "the sink". It turns out that the set of such mark maps form a very interesting group, called the **sandpile group** or **chip-firing group** of G; see Bak, Tang, and Wiesenfeld (1988), Dhar (1999), Biggs (1997), and Meester, Redig, and Znamenski (2001). Let $\nu_{(G,o)}$ be the uniform measure on critical mark maps. Given a transitive sofic graph G and an approximating sequence $\langle G_n \rangle$ of finite graphs, let $\nu_n := \nu_{(G_n, o_n)}$ be the corresponding probability measures for any fixed choice of roots o_n. Write $(\overline{G}_n, o_n)$ for the corresponding random network on (G_n, o_n) and, as usual, $U(\overline{G}_n)$ for the uniformly rooted network. (This root is unrelated to o_n.) Since $U(\overline{G}_n)$ is unimodular, so is any weak limit point, μ. By tightness, there is such a weak limit point, and it is concentrated on networks with underlying graph G. (Probably the entire sequence $\langle U(\overline{G}_n) \rangle$ in fact converges to μ.) By Theorem 3.2, we may lift μ to an automorphism-invariant measure ν on networks on G.

This measure ν is supported by networks with only critical mark maps in the sense that for all $x \in \mathsf{V}(G)$, we have $\psi(x) < \deg(x)$ and for all subgraphs W of G, there is some $x \in \mathsf{V}(W)$ such that $\psi(x) \geq \deg_W(x)$.

However, we do not know how to answer the following question.

Question 10.5. (Invariant Coloring) Given a quasi-transitive infinite graph G and a number c that is at least the chromatic number of G, is there an $\mathrm{Aut}(G)$-invariant probability measure on proper c-colorings of the vertices of G? Let $D := \max_{x \in \mathsf{V}(G)} \deg_G x$. A positive answer for $c \geq D+1$ is due to Schramm (personal communication, 1997). If G is sofic, then we can also obtain such a measure for $c = D$ by using a well-known result of Brooks (see, e.g., Bollobás (2001), p. 148, Theorem V.3.) The question is particularly interesting when G is planar and $c = 4$. In fact, it is then also of great interest to know whether there is a quasi-transitive proper 4-coloring of G.

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