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Asymptotic Distributions and Berry-Esseen Bounds for Sums of Record Values

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Abstract. Let $\{U_n, n \geq 1\}$ be independent uniformly distributed random variables, and $\{Y_n, n \geq 1\}$ be independent and identically distributed non-negative random variables with finite third moments. Denote $S_n = \sum_{i=1}^n Y_i$ and assume that $(U_1, \cdots, U_n)$ and S_{n+1} are independent for every fixed n. In this paper we obtain Berry-Esseen bounds for $\sum_{i=1}^n \psi(U_i S_{n+1})$, where ψ is a non-negative function. As an application, we give Berry-Esseen bounds and asymptotic distributions for sums of record values.

Key Words and Phrases: Central limit theorem, Berry-Esseen bounds, sum of records, insurance risk

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1 Introduction

Our interest in asymptotic distributions in this paper grew out of problems on sums of record values in insurance. Assume that the claim sizes $\{X_i, i \geq 1\}$ are independent and identically distributed (i.i.d.) positive random variables with a common distribution function F. X_k is called a record value if

$$X_k > \max_{1 \leq i \leq k-1} X_i.$$

By convention, X_1 is a record value and denoted by $X^{(1)}$. Define

$$L_1 = 1,\ L_n = \min\{k > L_{n-1}, X_k > X_{L_{n-1}}\} \text{ for } n \geq 2. \tag{1.1}$$

$\{L_n, n \geq 1\}$ is called the "record occurrence time" sequence of $\{X_n,\ n \geq 1\}$. Let

$$X^{(n)} = X_{L_n}. \tag{1.2}$$

Then $\{X^{(n)}, n \geq 1\}$ is the record sequence of $\{X_n, n \geq 1\}$. According to insurance theory, what leads to bankruptcy of an insurance company is usually those large claims that come suddenly. Thus, studying the laws of large claims is of significant importance for insurance industry. The limiting properties of the record values have been extensively studied in literature. Tata (1969) obtained a necessary and sufficient condition for the existence of non-degenerate limiting distribution for the standardized $X^{(n)}$. de Haan and Resnick (1973) found almost sure limit points for record values $X^{(n)}$. Arnold and Villasenor (1998) recently studied the limit laws for sums of record values

$$T_n = \sum_{i=1}^{n} X^{(i)}. \tag{1.3}$$

When X_i has an exponential distribution, Arnold and Villasenor proved that a standardized T_n is asymptotic normal. We refer to Embrechts, Klüppelberg and Mikosch (1997), Mikosch and Nagaev (1998), Su and Hu (2002) for recent developments in this area.

A key observation in those proofs is that (see Tata (1969) and Resnick (1973)) if F is an exponential distribution with mean one, then $\{X^{(n)}, n \geq 1\}$ and $\{G_n := \sum_{i=1}^{n} E_i, n \geq 1\}$ have the same distribution, where E_i are independent exponentially distributed random variables with mean one. For a general random variable X with a continuous distribution function F, it is well-known that $F(X)$ is uniformly distributed on $(0, 1)$ and $R(X) := -\ln(1 - F(X))$ has an exponential distribution with mean 1. Write

$$F^{-1}(x) = \inf\{y :\ F(y) \geq x\},\ \psi(x) = \inf\{y :\ R(y) \geq x\} \text{ for } x \geq 0.$$

Then $\psi(x) = F^{-1}(1 - e^{-x})$, and $\{X^{(n)}, n \geq 1\}$ and $\{\psi(R(X^{(n)})), n \geq 1\}$ have the same distribution. Thus, we have

$$\{X^{(n)}, n \geq 1\} \overset{d.}{=} \{\psi(G_n), n \geq 1\}$$

and in particular,

$$T_n = \sum_{i=1}^{n} X^{(i)} \overset{d.}{=} \sum_{i=1}^{n} \psi(G_i) = \sum_{i=1}^{n} \psi\Big(\frac{G_i}{G_{n+1}} G_{n+1}\Big).$$

Let $\{U_i, i \ge 1\}$ be i.i.d uniformly distributed random variables on $(0,1)$ independent of $\{E_i, i \ge 1\}$, and let $U_{n,1} \le U_{n,2} \le \cdots \le U_{n,n}$ be the order statistics of $\{U_i, 1 \le i \le n\}$. It is known that

$$\{U_{n,i}, 1 \le i \le n\} \stackrel{d.}{=} \{G_i/G_{n+1}, 1 \le i \le n\}$$

and that $\{G_i/G_{n+1}, 1 \le i \le n\}$ and G_{n+1} are independent. Therefore

$$T_n \stackrel{d.}{=} \sum_{i=1}^{n} \psi(U_{n,i}G_{n+1}) = \sum_{i=1}^{n} \psi(U_i G_{n+1}). \tag{1.4}$$

The main purpose of this paper is to study the asymptotic normality and Berry-Esseen bounds for the standardized T_n and solve an open problem posed by Arnold and Villasenor (1998).

This paper is organized as follows. The main results are stated in the next section, an application to the sum of record values is discussed in Section 3, and proofs of main results are given in Section 4. Throughout this paper, $f(x) \sim g(x)$ denotes $\lim_{x\to\infty} f(x)/g(x) = 1$, $f(x) \asymp g(x)$ means $0 < \liminf_{x\to\infty} f(x)/g(x) \le \limsup_{x\to\infty} f(x)/g(x) < \infty$.

2 Main results

Instead of focusing only on the sum of record values, we shall consider asymptotic distributions and Berry-Esseen bounds for a general randomly weighted sum.

Let $\{U_n, n \ge 1\}$ be independent uniformly distributed random variables, and $\{Y_n, n \ge 1\}$ be i.i.d. non-negative random variables with $EY_i = 1$ and $\mathrm{Var}(Y_i) = c_0^2$. Denote $S_n = \sum_{i=1}^n Y_i$ and assume that $(U_1, \cdots, U_n)$ and S_{n+1} are independent for every fixed n. Let $\psi: [0,\infty) \to [0,\infty)$ be a non-decreasing continuous function. Put

$$W_n = \sum_{i=1}^{n} \psi(U_i S_{n+1})$$

and define

$$\begin{aligned} \alpha(x) &= \int_0^1 \psi(ux)du, \\ \beta^2(x) &= \int_0^1 (\psi(ux) - \alpha(x))^2 du = \int_0^1 \psi^2(ux)du - \alpha^2(x), \\ \gamma(x) &= \int_0^1 \psi^3(ux)du. \end{aligned}$$

Let

$$\sigma_n^2 = \beta^2(n+1) + n(n+1)c_0^2[\alpha'(n+1)]^2 ,$$

where $\alpha'(x) = \frac{d}{dx}\alpha(x)$.

We first give a central limit theorem for a standardized W_n.

THEOREM **2.1** *Assume that the following conditions are satisfied:*

$(A1)\ \forall\ a>1,\ \lim_{n\to\infty}\sup_{|x-n|\le a\sqrt{n}}\frac{\gamma(x)}{n^{1/2}\beta^3(x)}=0,$

$(A2)\ \forall\ a>1,\ \lim_{n\to\infty}\frac{1}{n\sigma_n^2}\sup_{|x-n|\le a\sqrt{n}}\frac{\psi^4(x)}{\beta^2(x)}=0,$

$(A3)\ \forall\ a>1,\ \lim_{n\to\infty}\frac{\psi(n)+\int_{-a\sqrt{n}}^{a\sqrt{n}}|\psi(x+n)-\psi(n)|dx}{\sqrt{n}\sigma_n}=0.$

Then we have

$$\frac{W_n-n\alpha(n+1)}{\sqrt{n}\sigma_n}\xrightarrow{d.}N(0,1) \tag{2.1}$$

Next theorem provides a Berry-Esseen bound.

THEOREM **2.2** *Assume that $E(Y_i^3)<\infty$ and that ψ is differentiable. Then*

$$\begin{aligned}\sup_x\Big|P\Big(\frac{W_n-n\alpha(n+1)}{\sqrt{n}\sigma_n}\le x\Big)-\Phi(x)\Big| \\ \le Cn^{-1/2}\Big(c_0^{-3}EY_1^3+R_{n+1,1}+R_{n+1,2}+c_0^{-2}EY_1^3R_{n+1,3}\Big)\end{aligned} \tag{2.2}$$

for $n\ge(2c_0)^3$, where C is an absolute constant,

$$\begin{aligned}R_{n,1}&=\sup_{|x-n|\le c_0n^{2/3}}\frac{\gamma(x)}{\beta^3(x)},\ R_{n,2}=\frac{1}{n\alpha'(n)}\sup_{|x-n|\le c_0n^{2/3}}\frac{\psi^2(x)}{\beta(x)},\\ R_{n,3}&=\frac{1}{n\alpha'(n)}\sup_{|x-n|\le c_0n^{2/3}}(\psi(x)+x\psi'(x)).\end{aligned}$$

The results given in Theorems 2.1 and 2.2 are especially appealing when ψ is a regularly varying function of order θ. A Borel measurable function $l(x)$ on $(0,\infty)$ is said to be slowly varying (at ∞) if

$$\forall\ t>0,\ \ \lim_{x\to\infty}\frac{l(tx)}{l(x)}=1.$$

$\psi(x)$ is a regularly varying function of order θ if $\psi(x)=x^\theta l(x)$, where $l(x)$ is slowly varying. It is known that for $\theta>-1$ (see, for example, [2])

$$\lim_{x\to\infty}\frac{\int_0^\infty\psi(u)du}{x\psi(x)}=\frac{1}{1+\theta}. \tag{2.3}$$

In particular, as $x\to\infty$ for $\theta>0$

$$\alpha(x)\ \sim\ \frac{\psi(x)}{1+\theta}, \tag{2.4}$$

$$\alpha'(x) \sim \frac{\theta}{1+\theta}\frac{\psi(x)}{x}, \tag{2.5}$$

$$\beta(x) \sim \frac{\theta}{(1+\theta)\sqrt{1+2\theta}}\psi(x), \tag{2.6}$$

$$\gamma(x) \leq \frac{1}{1+3\theta}\psi^3(x), \tag{2.7}$$

$$\sigma_n^2 \sim \frac{\theta^2}{(1+\theta)^2}(\frac{1}{1+2\theta}+c_0^2)\psi^2(n). \tag{2.8}$$

Thus, we have

THEOREM **2.3** *Let $\psi(x) = x^\theta l(x)$, where $\theta > 0$ and $l(x)$ is slowly varying. Then* (2.1) *holds. In addition if $E(Y_i^3) < \infty$, and $|\psi'(x)| \leq c_1 x^{\theta-1} l(x)$ for $x > 1$, then*

$$\sup_x |P\Big(\frac{W_n - n\alpha(n+1)}{\sqrt{n}\sigma_n} \leq x\Big) - \Phi(x)| \leq An^{-1/2}, \tag{2.9}$$

where A is a constant depending only on c_0, c_1 and EY_1^3.

Another special case is $\psi(x) = \exp(c\,x^\tau l(x))$.

THEOREM **2.4** *Assume that $\psi(x) = \exp(cx^\tau l(x))$, where $c > 0$, $0 < \tau < 1/2$ and $l(x)$ is slowly varying at ∞. Assume that there exist $0 < c_1 \leq c_2 < \infty$ and $x_0 > 1$ such that*

$$c_1 x^{\tau-1} l(x) \leq (x^\tau l(x))' \leq c_2 x^{\tau-1} l(x) \quad \text{for } x \geq x_0. \tag{2.10}$$

Then

$$\sup_x |P\Big(\frac{W_n - n\alpha(n+1)}{\sqrt{n}\sigma_n} \leq x\Big) - \Phi(x)| \leq An^{-1/2+\tau} l(n), \tag{2.11}$$

where A is a constant depending only on c_0, c_1, c_2, x_0, τ and EY_1^3.

REMARK **2.1** *Hu, Su and Wang (2002) recently proved that if $\tau > 1/2$, there is no standardized W_n that has non-degenerate limiting distribution, which in turn indicates that assuming $\tau < 1/2$ in Theorem 2.4 is necessary.*

3 An application to the sum of record values

As we have seen in Section 1, the sum of record values T_n is a special case of W_n with Y_i i.i.d. exponentially distributed random variables. So, Theorems 2.1, 2.2 and 2.3 hold with $c_0 = 1$. When $\psi(x) = x$ or $\psi(x) = \ln x$, Arnold and Villasenor (1998) proved the asymptotic normality for T_n and also proved that the limiting distribution is not normal if $\psi(x) = 1 - \exp(-x/\beta)$. They then conjectured that the central limit theorem holds if $\psi(x) \sim x^\theta \ln^v(x)$, $\theta \geq 0, v \geq 0, \theta + v > 0$. Theorem 2.3 already implies that their conjecture is true if $\theta > 0$. Next theorem confirms that it is also true if $\theta = 0$ and $v > 0$.

THEOREM **3.1** *Let* $\psi(x) = \ln^v(1+x)$, $v > 0$. *Then*

$$\sup_x |P\Big(\frac{T_n - n\alpha(n+1)}{\sqrt{n}\sigma_n} \le x\Big) - \Phi(x)| \le An^{-1/2}\ln^3(1+n), \tag{3.1}$$

where $\alpha(n+1)$ *and* σ_n *are defined in Section 2, and* A *is a constant depending only on* v.

Proof. Obviously, for $v > 0$ and any positive integer p, it holds

$$\int_0^1 \psi^p(ux)du = \int_0^1 (\ln(1+ux))^{pv}du = \frac{1}{x}\int_1^{1+x}(\ln t)^{pv}dt\ .$$

Letting

$$c_p = \int_1^e (\ln t)^{pv}dt\ , \qquad p = 1, 2,$$

we have

$$\alpha(x) = \frac{c_1}{x} + \frac{1}{x}\int_e^{1+x} \ln^v t\ dt \sim (\ln x)^{v-1} \tag{3.2}$$

and

$$\begin{aligned} \alpha'(x) &= -\frac{c_1}{x^2} + \Big(\frac{1}{x}\int_e^{1+x}\ln^v t\ dt\Big)' \\ &= -\frac{c_1}{x^2} + \frac{\ln^v(x+1)}{x} - \frac{1}{x^2}\int_e^{1+x}\ln^v t dt \\ &= \frac{e-c_1}{x^2} - \frac{\ln^v(x+1)}{x^2} + \frac{v}{x^2}\int_e^{1+x}\ln^{v-1} t dt \\ &\sim \frac{v}{x}\ln^{v-1}(1+x)\ . \end{aligned} \tag{3.3}$$

To estimate $\beta^2(x)$, we need a more accurate estimate for $\alpha(x)$:

$$\begin{aligned} \alpha(x) &= \frac{c_1}{x} + \frac{1}{x}\int_e^{1+x}\ln^v t\ dt \\ &= \frac{1+x}{x}\ln^v(1+x) + \frac{v(v-1)}{x}\int_e^{1+x}\ln^{v-2} t dt \\ &\quad -\frac{1+x}{x}v\ln^{v-1}(1+x) + o(\ln^{v-1}(1+x))\ . \end{aligned} \tag{3.4}$$

Similarly, we have

$$\begin{aligned} \frac{1}{x}\int_1^{1+x}\ln^{2v} t dt &= \frac{c_2}{x} + \frac{1}{x}\int_e^{1+x}\ln^{2v} t\ dt \\ &= \frac{1+x}{x}\ln^{2v}(1+x) - \frac{1+x}{x}2v\ln^{2v-1}(1+x) \\ &\quad + \frac{2v(2v-1)}{x}\int_e^{1+x}\ln^{2v-2} t dt + o(\ln^{2v-2}(1+x))\ . \end{aligned} \tag{3.5}$$

From (3.4) and (3.5) it follows that

$$
\begin{aligned}
(3.6) \qquad \beta^2(x) &= \frac{1}{x}\int_1^{1+x} \ln^{2v} t dt - \alpha^2(x) \\
&\sim \frac{2v(1-v)(x+1)}{x^2} \ln^v(1+x) \int_e^{1+x} \ln^{v-2} t dt \\
&\quad + \frac{2v(2v-1)}{x} \int_e^{1+x} \ln^{2v-2} t dt - \left(\frac{v(1+x)}{x}\ln^{v-1}(1+x)\right)^2 \\
&\sim v^2 \ln^{2v-2}(1+x)\ .
\end{aligned}
$$

Thus, by the above estimates

$$
(3.7) \qquad R_{n,1} \le A\ln^3(1+n),\ R_{n,2} \le A\ln^2(1+n),\ R_{n,3} \le A\ln(1+n).
$$

This proves Theorem 3.1, by Theorem 2.2 and (3.7). ∎

4 Proofs

The proofs are based on the fact that given S_{n+1}, $\{\psi(U_i S_{n+1}), 1 \le i \le n\}$ is a sequence of i.i.d. random variables with mean $\alpha(S_{n+1})$ and variance $\beta(S_{n+1})$, and that $S_{n+1}/(n+1) \to 1$ with probability one by the law of large numbers. Let

$$
(4.1) \qquad H_n = \frac{\sum_{i=1}^n \psi(U_i S_{n+1}) - n\alpha(S_{n+1})}{\sqrt{n}\beta(S_{n+1})}.
$$

Then we have

$$
\begin{aligned}
(4.2) \qquad \frac{W_n - n\alpha(n+1)}{\sqrt{n}\sigma_n} &= \frac{\beta(S_{n+1})}{\sigma_n} H_n + \frac{\sqrt{n}}{\sigma_n}(\alpha(S_{n+1}) - \alpha(n+1)) \\
&:= H_{n,1} + H_{n,2},
\end{aligned}
$$

where

$$
\begin{aligned}
(4.3) \qquad H_{n,1} &= \frac{\beta(n+1)}{\sigma_n} H_n + \frac{c_0\alpha'(n+1)\sqrt{n(n+1)}}{\sigma_n} \frac{(S_{n+1} - (n+1))}{\sqrt{n+1}c_0}, \\
(4.4) \qquad H_{n,2} &= \frac{\beta(S_{n+1}) - \beta(n+1)}{\sigma_n} H_n + \frac{\sqrt{n}}{\sigma_n}\int_{n+1}^{S_{n+1}} (\alpha'(t) - \alpha'(n+1))dt.
\end{aligned}
$$

We need some facts on $\alpha'(x)$, $\alpha''(x)$ and $\beta'(x)$. The proof will be given in the Appendix.

LEMMA **4.1** *We have*

$$
\begin{aligned}
(4.5) \qquad \alpha'(x) &= \frac{1}{x^2}\int_0^x (\psi(x) - \psi(u))du, \\
(4.6) \qquad |\alpha'(x)| &\le \frac{\psi(x)}{x}; \\
(4.7) \qquad |\beta'(x)| &\le \frac{\psi^2(x)}{x\beta(x)}, \\
(4.8) \qquad |\alpha'(x) - \alpha'(y)| &\le \frac{2(y-x)\psi(x)}{xy} + \frac{\psi(y) - \psi(x)}{y} \ \textit{for } y \ge x > 0,
\end{aligned}
$$

and

$$(4.9) \qquad |\alpha''(x)| \le \frac{2\psi(x)}{x^2} + \frac{\psi'(x)}{x}$$

if $\psi(x)$ is differentiable.

Proof of Theorem 2.1. By (4.2), it suffices to show that

$$(4.10) \qquad H_{n,1} \xrightarrow{d.} N(0,1)$$

and

$$(4.11) \qquad H_{n,2} \to 0 \text{ in probability.}$$

In view of conditions (A1) - (A3), there exists a sequence of a_n such that $a_n \to \infty$, $a_n \le 1/2\sqrt{n}$ and

$$(A1)^* \quad \lim_{n\to\infty} \sup_{|x-n|\le a_n\sqrt{n}} \frac{\gamma(x)}{n^{1/2}\beta^3(x)} = 0,$$

$$(A2)^* \quad \lim_{n\to\infty} \frac{1}{n\sigma_n^2} \sup_{|x-n|\le a_n\sqrt{n}} \frac{\psi^4(x)}{\beta^2(x)} = 0,$$

$$(A3)^* \quad \lim_{n\to\infty} \frac{a_n^2\psi(n) + \int_{-a_n\sqrt{n}}^{a_n\sqrt{n}} |\psi(x+n) - \psi(n)|dx}{\sqrt{n}\sigma_n} = 0.$$

Let Z_1 and Z_2 be independent standard normal random variables independent of $\{U_i, i \ge 1\}$ and $\{Y_i, i \ge 1\}$. For given S_{n+1} satisfying $|S_{n+1} - (n+1)| \le a_n\sqrt{n+1}$, applying the Berry-Esseen (see, e.g., [8]) bound to H_n yields

$$\begin{aligned}(4.12) \qquad & \sup_x |P(H_{n,1} \le x) - P\Big(\frac{\beta(n+1)}{\sigma_n}Z_1 + \frac{c_0\alpha'(n+1)\sqrt{n(n+1)}}{\sigma_n}\frac{(S_{n+1} - \alpha(n+1))}{\sqrt{n}c_0} \le x\Big)| \\ \le\ & P(|S_{n+1} - (n+1)| > a_{n+1}\sqrt{n+1}) \\ & +10n^{-1/2}E\Big(\frac{\gamma(S_{n+1})}{\beta^3(S_{n+1})}I\{|S_{n+1} - (n+1)| \le a_{n+1}\sqrt{n+1}\}\Big) \\ \le\ & \frac{\mathrm{Var}(Y_1)}{a_{n+1}^2} + 10n^{-1/2} \sup_{|s-(n+1)|\le a_{n+1}\sqrt{n+1}} \frac{\gamma(s)}{\beta^3(s)} \\ \to\ & 0\end{aligned}$$

as $n \to \infty$, by $(A1)^*$. Now, for given Z_1, applying the central limit theorem to $(S_{n+1} - n - 1)/\sqrt{n+1}c_0$ gives

$$\begin{aligned}(4.13) \qquad \sup_x \ | \ & P\Big(\frac{\beta(n+1)}{\sigma_n}Z_1 + \frac{c_0\alpha'(n+1)\sqrt{n(n+1)}}{\sigma_n}\ \frac{(S_{n+1} - n - 1)}{\sqrt{n}c_0} \le x\Big) \\ & -P\Big(\frac{\beta(n+1)}{\sigma_n}Z_1 + \frac{c_0\alpha'(n+1)\sqrt{n(n+1)}}{\sigma_n}Z_2 \le x\Big)| \to 0\end{aligned}$$

as $n \to \infty$. It is easy to see that

$$\frac{\beta(n+1)}{\sigma_n} Z_1 + \frac{c_0 \alpha'(n+1)\sqrt{n(n+1)}}{\sigma_n} Z_2$$

has a standard normal distribution, by the definition of σ_n. This proves (4.10), by (4.12) and (4.13).

As to (4.11), observe that conditioning on S_{n+1}, H_n is a standardized sum of i.i.d. random variables and hence

$$\begin{aligned}
(4.14) \qquad & \sigma_n^{-2} E\Big\{(\beta(S_{n+1}) - \beta(n+1))^2 I\{|S_{n+1} - (n+1)| \le a_{n+1}\sqrt{n+1}\} H_n^2\Big\} \\
&= \sigma_n^{-2} E\Big\{(\beta(S_{n+1}) - \beta(n+1))^2 I\{|S_{n+1} - (n+1)| \le a_{n+1}\sqrt{n+1}\}\Big\} \\
&\le \sigma_n^{-2} \sup_{|x-(n+1)| \le a_{n+1}\sqrt{n+1}} (\beta'(x))^2 E(S_{n+1} - (n+1))^2 \\
&\le (n+1) c_0^2 \sigma_n^{-2} \sup_{|x-(n+1)| \le a_{n+1}\sqrt{n+1}} \frac{\psi^4(x)}{x^2 \beta^2(x)} \\
&\le \frac{4c_0^2}{n\sigma_n^2} \sup_{|x-(n+1)| \le a_{n+1}\sqrt{n+1}} \frac{\psi^4(x)}{\beta^2(x)} \\
&\to 0
\end{aligned}$$

as $n \to \infty$, by (A2)*.

It follows from (4.8) that

$$\begin{aligned}
& I\{|S_{n+1} - (n+1)| \le a_{n+1}\sqrt{n+1}\} \int_{n+1}^{S_{n+1}} |\alpha'(t) - \alpha'(n+1)| dt \\
&\le \int_{-a_{n+1}\sqrt{n+1}}^{a_{n+1}\sqrt{n+1}} |\alpha'(t+n+1) - \alpha'(n+1)| dt \\
&\le \frac{8\psi(n+1)}{n^2} \int_{-a_{n+1}\sqrt{n+1}}^{a_{n+1}\sqrt{n+1}} |t| dt + \frac{1}{n} \int_{-a_{n+1}\sqrt{n+1}}^{a_{n+1}\sqrt{n+1}} |\psi(t+n+1) - \psi(n+1)| dt \\
&\le \frac{8\psi(n+1) a_{n+1}^2}{n} + \frac{1}{n} \int_{-a_{n+1}\sqrt{n+1}}^{a_{n+1}\sqrt{n+1}} |\psi(t+n+1) - \psi(n+1)| dt.
\end{aligned}$$

Thus, by $(A3)^*$

$$\begin{aligned}
(4.15) \qquad & \frac{\sqrt{n}}{\sigma_n} I\{|S_{n+1} - n - 1| \le a_{n+1}\sqrt{n+1}\} \Big| \int_{n+1}^{S_{n+1}} (\alpha'(t) - \alpha'(n+1)) dt \Big| \\
&\le \frac{8\psi(n+1) a_{n+1}^2}{\sqrt{n}\sigma_n} + \frac{1}{\sqrt{n}\sigma_n} \int_{-a_{n+1}\sqrt{n+1}}^{a_{n+1}\sqrt{n+1}} |\psi(t+n+1) - \psi(n+1)| dt \\
&\to 0.
\end{aligned}$$

In view of the fact that $|S_{n+1} - (n+1)| = o(a_{n+1}\sqrt{n+1})$ in probability, (4.11) holds by (4.14) and (4.15). ∎

The proof of Theorem 2.2 is based on the following Berry-Esseen bound for non-linear statistics.

Lemma **4.2** *[Chen and Shao (2003)] Let $\{\xi_i, 1 \le i \le n\}$ be independent random variables, $g_i : R^1 \to R^1$ and $\Delta = \Delta(X_i, 1 \le i \le n) : R^n \to R^1$ are Borel measurable functions. Put $G_n = \sum_{i=1}^n g_i(X_i)$. Assume that*

$$Eg_i(X_i) = 0, \ \sum_{i=1}^n Eg_i^2(X_i) = 1.$$

Then

$$\begin{aligned} & \sup_x |P(G_n + \Delta \le x) - \Phi(x)| \\ \le\ & 6.1\Big(\sum_{i=1}^n E|g_i(X_i)|^2 I\{|g_i(X_i)| > 1\} + \sum_{i=1}^n E|g_i(X_i)|^3 I\{|g_i(X_i)| \le 1\}\Big) \\ & + E|G_n \Delta| + \sum_{i=1}^n E|g_i(X_i)(\Delta - \Delta_i)| \end{aligned} \tag{4.16}$$

for any Borel measurable functions $\Delta_i = \Delta_i(X_j, 1 \le j \le n, j \ne i)$.

Proof of Theorem 2.2. In what follows, we use C to denote an absolute constant, but its value could be different from line to line. By the Rosenthal inequality, we have

$$P(|S_{n+1} - n - 1| > c_0(n+1)^{2/3}) \le n^{-2} c_0^{-3} E|S_{n+1} - n - 1|^3 \le C n^{-1/2} c_0^{-3} EY_1^3 \tag{4.17}$$

Define

$$\bar{x} = \begin{cases} n + 1 - c_0(n+1)^{2/3} & \text{for } x < n + 1 - c_0(n+1)^{2/3} \\ x & \text{for } n + 1 - c_0(n+1)^{2/3} \le x \le n + 1 + c_0(n+1)^{2/3} \\ n + 1 + c_0(n+1)^{2/3} & \text{for } x > n + 1 + c_0(n+1)^{2/3}, \end{cases}$$

$\bar{\beta}(x) = \beta(\bar{x})$ and $\beta^*(x) = \beta(x) - \bar{\beta}(x)$. Similar to (4.2), we have

$$\begin{aligned} \frac{W_n - n\alpha(n+1)}{\sqrt{n}\sigma_n} &= \frac{\beta(\bar{S}_{n+1})}{\sigma_n} H_n + \frac{\sqrt{n}}{\sigma_n}(\alpha(S_{n+1}) - \alpha(n+1)) + \frac{\beta^*(S_{n+1})}{\sigma_n} H_n \\ &= H_{n,3} + H_{n,4}, \end{aligned} \tag{4.18}$$

where

$$\begin{aligned} H_{n,3} &= \frac{\beta(\bar{S}_{n+1})}{\sigma_n} H_n + \frac{\alpha'(n+1)\sqrt{n}}{\sigma_n}(S_{n+1} - n - 1) + \frac{\sqrt{n}}{\sigma_n} \int_{n+1}^{\bar{S}_{n+1}} (\alpha'(t) - \alpha(n+1)) dt, \\ H_{n,4} &= \frac{\beta^*(S_{n+1})}{\sigma_n} H_n + \frac{\sqrt{n}}{\sigma_n} \int_{\bar{S}_{n+1}}^{S_{n+1}} (\alpha'(t) - \alpha(n+1)) dt. \end{aligned}$$

Noting that $H_{n,4} = 0$ on $\{|S_{n+1} - (n+1)| > c_0(n+1)^{2/3}\}$, we only need to show that

$$\sup_x |P(H_{n,3} \le x) - \Phi(x)| \le C n^{-1/2} \Big(c_0^{-3} EY_1^3 + R_{n+1,1} + R_{n+1,2} + c_0^{-2} EY_1^3 R_{n+1,3} \Big) \tag{4.19}$$

by (4.17). Again, let Z_1 and Z_2 be independent standard normal random variables independent of $\{U_i, i \geq 1\}$ and $\{Y_i, i \geq 1\}$. Let

$$H_{n,3}(Z_1) = \frac{\bar{\beta}(S_{n+1})}{\sigma_n} Z_1 + \frac{\alpha'(n+1)\sqrt{n}}{\sigma_n}(S_{n+1} - n - 1) + \frac{\sqrt{n}}{\sigma_n} \int_{n+1}^{\bar{S}_{n+1}} (\alpha'(t) - \alpha(n+1)) dt.$$

Similar to (4.12)

$$\begin{aligned}
(4.20) \qquad & \sup_x |P(H_{n,3} \leq x) - P(H_{n,3}(Z_1) \leq x)| \\
&\leq P(|S_{n+1} - n - 1| > c_0 (n+1)^{2/3}) \\
&\quad + 10 n^{-1/2} E\Big(\frac{\gamma(S_{n+1})}{\beta^3(S_{n+1})} I\{|S_{n+1} - (n+1)| \leq c_0 (n+1)^{2/3}\}\Big) \\
&\leq C n^{-1/2} c_0^{-3} E Y_1^3 + 10 n^{-1/2} R_{n+1,1}.
\end{aligned}$$

Rewrite

$$H_{n,3}(Z_1) = \frac{c_0 \alpha'(n+1)\sqrt{n(n+1)}}{\sigma_n} \Big(G_n + \Delta + \frac{\bar{\beta}(n+1) Z_1}{c_0 \alpha'(n+1)\sqrt{n(n+1)}} \Big),$$

where

$$\begin{aligned}
G_n &= \frac{S_{n+1} - n - 1}{\sqrt{n+1} c_0}, \\
\Delta &= \frac{(\bar{\beta}(S_{n+1}) - \bar{\beta}(n+1)) Z_1}{c_0 \alpha'(n+1)\sqrt{n(n+1)}} + \frac{1}{c_0 \alpha'(n+1)\sqrt{n+1}} \int_{n+1}^{\bar{S}_{n+1}} (\alpha'(t) - \alpha'(n+1)) dt.
\end{aligned}$$

By (4.6), (4.7) and (4.9), we have

$$\begin{aligned}
(4.21) \quad E(|G_n \Delta| \mid Z_1) &\leq \frac{|Z_1| E|G_n (\bar{\beta}(S_{n+1}) - \bar{\beta}(n+1))|}{c_0 n \alpha'(n+1)} \\
&\quad + \frac{1}{c_0 \alpha'(n+1)\sqrt{n}} E\Big\{ G_n \int_{n+1}^{\bar{S}_{n+1}} |\alpha'(t) - \alpha'(n+1)| dt \Big\} \\
&\leq \frac{|Z_1|}{c_0^2 n^{3/2} \alpha'(n+1)} \sup_{|x-(n+1)| \leq (n+1)^{2/3}} \frac{\psi^2(x)}{x \beta(x)} E(S_{n+1} - n - 1)^2 \\
&\quad + \frac{1}{c_0^2 \alpha'(n+1) n} \sup_{|x-n-1| \leq (n+1)^{2/3}} \Big(\frac{2\psi(x)}{x^2} + \frac{x \psi'(x)}{x^2} \Big) E|S_{n+1} - n - 1|^3 \\
&\leq \frac{C|Z_1|}{n^{3/2} \alpha'(n+1)} \sup_{|x-(n+1)| \leq (n+1)^{2/3}} \frac{\psi^2(x)}{\beta(x)} \\
&\quad + \frac{C E Y_1^3}{c_0^2 n^{3/2} \alpha'(n+1)} \sup_{|x-n-1| \leq (n+1)^{2/3}} (\psi(x) + x\psi'(x)) \\
&\leq C |Z_1| n^{-1/2} R_{n+1,2} + C E Y_1^3 c_0^{-2} n^{-1/2} R_{n+1,3}.
\end{aligned}$$

To apply Lemma 4.2, let $g_i(Y_i) = (Y_i - 1)/(c_0\sqrt{n+1})$, $S^{(i)} = S_{n+1} - X_i$ and

$$\Delta_i = \frac{(\bar\beta(S^{(i)}) - \bar\beta(n+1))Z_1}{c_0\alpha'(n+1)\sqrt{n(n+1)}} + \frac{1}{c_0\alpha'(n+1)\sqrt{n+1}}\int_{n+1}^{S^{(i)}}(\alpha'(t) - \alpha'(n+1))dt$$

for $1 \le i \le n+1$. Following the proof of (4.21), we have

$$\begin{aligned} &E(|g_i(Y_i)(\Delta - \Delta_i)| \mid Z_1) \\ \le\ & \frac{|Z_1|E|g_i(Y_i)(\bar\beta(S_{n+1}) - \bar\beta(S^{(i)}))|}{c_0 n\alpha'(n+1)} \\ & + \frac{1}{c_0\alpha'(n+1)\sqrt{n}}E\Big\{g_i(Y_i)\int_{\bar S^{(i)}}^{\bar S_{n+1}}|\alpha'(t) - \alpha'(n+1)|dt\Big\} \\ \le\ & C|Z_1|n^{-3/2}R_{n+1,2} + CEY_1^3c_0^{-2}n^{-2}R_{n+1,3}. \end{aligned} \tag{4.22}$$

Letting

$$H_{n,3}(Z_1, Z_2) = \frac{c_0\alpha'(n+1)\sqrt{n(n+1)}}{\sigma_n}\Big(Z_2 + \frac{\bar\beta(n+1)Z_1}{c_0\alpha'(n+1)\sqrt{n(n+1)}}\Big)$$

and applying Lemma 4.2 for given Z_1 yields

$$\begin{aligned} &\sup_x |P(H_{n,3}(Z_1) \le x) - P(H_{n,3}(Z_1, Z_2) \le x)| \\ \le\ & Cn^{-1/2}c_0^{-3}EY_1^3 + Cn^{-1/2}R_{n+1,2}E|Z_1| + CEY_1^3c_0^{-2}n^{-1/2}R_{n+1,3} \\ \le\ & Cn^{-1/2}\Big(c_0^{-3}EY_1^3 + R_{n+1,2} + c_0^{-2}EY_1^3R_{n+1,3}\Big). \end{aligned} \tag{4.23}$$

On the other hand, it is easy to see that $H_{n,3}(Z_1, Z_2)$ has the standard normal distribution. This proves (4.19), as desired. ∎

Proof of Theorem 2.4. Let $\rho(x) = c\,x^\tau l(x)$. It is easy to see that condition (2.10) implies for $a > 0$

$$\int_0^x e^{a\rho(t)}dt \asymp \frac{x}{\rho(x)}e^{a\rho(x)} \tag{4.24}$$

as $x \to \infty$, where $g(x) \asymp h(x)$ denotes $0 < \liminf_{x\to\infty} g(x)/h(x) \le \limsup_{x\to\infty} g(x)/h(x) < \infty$. Thus, we have

$$\begin{aligned} \alpha(x) &\asymp \frac{1}{\rho(x)}e^{\rho(x)},\ \alpha'(x) \asymp \frac{1}{x}e^{\rho(x)}, \\ \beta^2(x) &\asymp \frac{1}{\rho(x)}e^{2\rho(x)},\ \gamma(x) \asymp \frac{1}{\rho(x)}e^{3\rho(x)} \end{aligned} \tag{4.25}$$

Let $d_n = n/\rho(n)$. Then

$$P(|S_{n+1} - (n+1)| > d_n) \le (d_n)^{-2}\mathrm{Var}(S_n) \le 2c_0^2\rho^2(n)/n = 2c_0^2(cn^{\tau-1/2}l(n))^2.$$

Following the proof of Theorem 2.2, we have

$$\sup_x \left|P\Big(\frac{W_n - n\alpha(n+1)}{\sqrt{n}\sigma_n} \le x\Big) - \Phi(x)\right| \le A(n^{\tau-1/2}l(n))^2 + An^{-1/2}\Big(R^*_{n,1} + R^*_{n,2} + R^*_{n,3}\Big), \tag{4.26}$$

where

$$\begin{aligned} R^*_{n,1} &= \sup_{|x-n-1|\le d_n} \frac{\gamma(x)}{\beta^3(x)},\ R^*_{n,2} = \frac{1}{n\alpha'(n)} \sup_{|x-n-1|\le d_n} \frac{\psi^2(x)}{\beta(x)}, \\ R^*_{n,3} &= \frac{1}{n\alpha'(n)} \sup_{|x-n-1|\le d_n} (\psi(x) + x\psi'(x)). \end{aligned}$$

Note that for x satisfying $|x-n-1| \le d_n$, by (2.10)

$$|\rho(x) - \rho(n)| \le A_1|x-n|\rho(n)/n \le A_2 d_n \rho(n)/n \le A_3,$$

where A_1, A_2, A_3 denote constants that do not depend on n. Hence

$$\psi(x) = \psi(n)e^{\rho(x)-\rho(n)} \asymp \psi(n),$$

which combines with (4.25) gives

$$R^*_{n,1} \le A\rho^{1/2}(n),\ R^*_{n,2} \le A\rho^{1/2}(n),\ R^*_{n,3} \le A\rho(n). \tag{4.27}$$

This proves (2.11), by (4.26) and (4.27). ∎

5 Appendix

Proof of Lemma 4.1. Rewrite

$$\alpha(x) = \frac{1}{x}\int_0^x \psi(u)du.$$

Recall that $\psi(x)$ is non-decreasing and continuous. We have

$$\begin{aligned}
\alpha'(x) &= \frac{-1}{x^2}\int_0^x \psi(u)du + \frac{\psi(x)}{x} = \frac{1}{x^2}\int_0^x (\psi(x)-\psi(u))du, \\
0 &\le \alpha'(x) \le \frac{\psi(x)}{x}, \\
(\beta^2(x))' &= -\frac{1}{x^2}\int_0^x \psi^2(u)du + \frac{\psi^2(x)}{x} - 2\alpha(x)\alpha'(x) \\
&= \frac{1}{x^2}\int_0^x (\psi^2(x)-\psi(u))du - \frac{2}{x^3}\int_0^x \psi(u)du\int_0^x (\psi(x)-\psi(v))dv, \\
|\beta'(x)| &= |\frac{(\beta^2(x))'}{2\beta(x)}| \le \frac{\psi^2(x)}{x\beta(x)}, \\
\alpha'(y)-\alpha'(x) &= \frac{1}{y^2}\int_0^y (\psi(y)-\psi(u))du - \frac{1}{x^2}\int_0^x (\psi(x)-\psi(u))du, \\
&= \frac{1}{y^2}\int_x^y (\psi(y)-\psi(u))du + \Big(\frac{1}{y^2}-\frac{1}{x^2}\Big)\int_0^x (\psi(x)-\psi(u))du + \frac{1}{y^2}\int_0^x (\psi(y)-\psi(x))du, \\
\alpha'(y)-\alpha'(x) &\le \frac{1}{y^2}\int_x^y (\psi(y)-\psi(x))du + \frac{1}{y^2}x(\psi(y)-\psi(x)) = \frac{\psi(y)-\psi(x)}{y} \quad \text{for } y > x > 0 \\
\alpha'(y)-\alpha'(x) &\ge \Big(\frac{1}{y^2}-\frac{1}{x^2}\Big)\int_0^x (\psi(x)-\psi(u))du \\
&\ge \frac{(x^2-y^2)}{x^2y^2}x\psi(x) \\
&\ge \frac{2(x-y)\psi(x)}{xy} \quad \text{for } y > x > 0
\end{aligned}$$

and if $\psi(x)$ is differentiable,

$$\begin{aligned}
\alpha''(x) &= -\frac{2}{x^3}\int_0^x (\psi(x)-\psi(u))du + \frac{1}{x^2}\int_0^x \psi'(x)du \\
&= -\frac{2}{x^3}\int_0^x (\psi(x)-\psi(u))du + \frac{\psi'(x)}{x}, \\
|\alpha''(x)| &\le \frac{2\psi(x)}{x^2} + \frac{\psi'(x)}{x}.
\end{aligned}$$

This proves Lemma 4.1. ∎

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