Electronic Journal of Probability

Vol. 14 (2009), Paper no. , pages 1727–1744.

Journal URL
http://www.math.washington.edu/~ejpecp/

Soliton dynamics for the Korteweg-de Vries equation with multiplicative homogeneous noise

Anne de Bouard*
Centre de Mathématiques Appliquées
UMR 7641, CNRS et Ecole Polytechnique,
Route de Saclay,
91128 PALAISEAU CEDEX, FRANCE
debouard@cmapx.polytechnique.fr

Arnaud Debussche
IRMAR, ENS Cachan Bretagne, CNRS, UEB
Av. Robert Schuman,
F-35170 BRUZ, FRANCE
arnaud.debussche@bretagne.ens-cachan.fr

Abstract

We consider a randomly perturbed Korteweg-de Vries equation. The perturbation is a random potential depending both on space and time, with a white noise behavior in time, and a regular, but stationary behavior in space. We investigate the dynamics of the soliton of the KdV equation in the presence of this random perturbation, assuming that the amplitude of the perturbation is small. We estimate precisely the exit time of the perturbed solution from a neighborhood of the modulated soliton, and we obtain the modulation equations for the soliton parameters. We moreover prove a central limit theorem for the dispersive part of the solution, and investigate the asymptotic behavior in time of the limit process.

Key words: Korteweg-de Vries equation, stochastic partial differential equations, white noise, central limit theorem, solitary waves.

AMS 2000 Subject Classification: Primary 35Q53, 60H15, 76B25.

Submitted to EJP on March 31, 2009, final version accepted July 16, 2009.

*partially supported by the French ANR under ANR-07-BLAN-0250

1 Introduction

Our aim is to describe the dynamics of a soliton solution of the Korteweg-de Vries equation in the presence of a random potential, depending both on space and time and which is white in time. After the first paper [25] showing “superdiffusion" of the soliton of the KdV equation in the presence of an external force which is a white noise in time (see also [1; 16]), the interest in such questions of soliton dynamics in the presence of either deterministic or random perturbations has recently increased in the mathematical community. In [15], e.g. the question is investigated with the help of inverse scattering methods, for different types of time-white noise perturbations, still for the KdV equation, while in [11; 12], the case of a soliton of the NLS equation is studied, with the presence of a slowly varying deterministic external potential. Random potential perturbations for NLS equations have also been considered in [14; 9]. The diffusion of solitons of the KdV equation in the presence of additive noise was numerically investigated in [22; 23]. Also, in [5], we studied the soliton dynamics for a KdV equation with an additive space-time noise. Our aim here is to reproduce the analysis of [5] in the case of a random potential, which is stationary in space : the solution of the stochastic equation starting from a soliton at initial time will then stay close to a modulated soliton up to times small compared to ε^{-2} where ε is the amplitude of the random perturbation (see below). In the present case, where the noise is multiplicative (the random potential) we are then able to analyze more precisely the modulation equations for the soliton parameters and the linearized equation for the remaining (dispersive) part of the solution, and especially its asymptotic behavior in time.

We consider a stochastic KdV equation which may be written in Itô form as

$$du + (\partial_x^3 u + \frac{1}{2}\partial_x(u^2))dt = \varepsilon u dW \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, u is a random process defined on $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, W is a Wiener process on $L^2(\mathbb{R})$ whose covariance operator $\phi\phi^*$ is such that ϕ is a convolution operator on $L^2(\mathbb{R})$ defined by

$$\phi f(x) = \int_{\mathbb{R}} k(x-y)f(y)dy, \text{ for } f \in L^2(\mathbb{R}).$$

The convolution kernel k satisfies

$$\|k\|_1 := \int_{\mathbb{R}} (k^2 + (k')^2)dx < +\infty. \tag{1.2}$$

Considering a complete orthonormal system $(e_i)_{i\in\mathbb{N}}$ in $L^2(\mathbb{R})$, we may alternatively write W as

$$W(t,x) = \sum_{i\in\mathbb{N}} \beta_i(t)\phi e_i(x), \tag{1.3}$$

$(\beta_i)_{i\in\mathbb{N}}$ being an independent family of real valued Brownian motions. The correlation function of the process W is then given by

$$\mathbb{E}(W(t,x)W(s,y)) = c(x-y)(s\wedge t), \quad x,y \in \mathbb{R}, \quad s,t > 0,$$

where

$$c(z) = \int_{\mathbb{R}} k(z+u)k(u)du.$$

The existence and uniqueness of solutions for stochastic KdV equations of the type (1.1) but with an additive noise have been studied in [4; 7; 8]. The multiplicative case with homogeneous noise as described above was considered in [6]: assuming, together with the above condition, that k is an integrable function of $x \in \mathbb{R}$ allowed us to prove the global existence and uniqueness of solutions to equation (1.1) in the energy space $H^1(\mathbb{R})$, that is in the space where both the mass

$$m(u) = \frac{1}{2}\int_{\mathbb{R}} u^2(x)dx \tag{1.4}$$

and the energy

$$H(u) = \frac{1}{2}\int_{\mathbb{R}} (\partial_x u)^2 dx - \frac{1}{6}\int_{\mathbb{R}} u^3 dx \tag{1.5}$$

are well defined. Note that m and H are conserved for the equation without noise, that is

$$\partial_t u + \partial_x^3 u + \frac{1}{2}\partial_x(u^2) = 0. \tag{1.6}$$

Under the above conditions on k, it was then proved in [6] that for any given initial data $u_0 \in H^1(\mathbb{R})$, there is a unique solution u of (1.1) with paths a.s. continuous for $t \in \mathbb{R}^+$ with values in $H^1(\mathbb{R})$.

Our aim in this article is to analyze the qualitative influence of a noise on a soliton solution of the deterministic equation. More precisely, we study the qualitative behavior of solutions of (1.1) in the limit ε tends to zero, assuming that the initial state of the solution is a soliton of equation (1.6). We recall indeed that equation (1.6) possesses a two-parameter family of solitary waves (or soliton) solutions, propagating with a constant velocity $c > 0$, with the expression $u_{c,x_0}(t,x) = \varphi_c(x - ct + x_0)$, $x_0 \in \mathbb{R}$, where

$$\varphi_c(x) = \frac{3c}{2\cosh^2(\sqrt{c}\frac{x}{2})} \tag{1.7}$$

satisfies the equation

$$\varphi_c'' - c\varphi_c + \frac{1}{2}\varphi_c^2 = 0. \tag{1.8}$$

We do not recall here the well-known results concerning the stability of the soliton solutions u_{c,x_0} in equation (1.6), but we refer to [2; 3; 18] or [21] for a review of the stability questions using PDE methods, or to [13] and [24] for a review of the stability of the solitons with the help of the inverse scattering transform.

Let us consider as in [5] the solution $u^\varepsilon(t,x)$ of equation (1.1) which is such that $u^\varepsilon(0,x) = \varphi_{c_0}(x)$ where $c_0 > 0$ is fixed. Then, in Section 2, we show, as we did in [5] for the additive equation that up to times $C\varepsilon^{-2}$, where C is a constant, we may write the solution u^ε as

$$u^\varepsilon(t,x) = \varphi_{c^\varepsilon(t)}(x - x^\varepsilon(t)) + \varepsilon\eta^\varepsilon(t, x - x^\varepsilon(t)) \tag{1.9}$$

where the modulation parameters $c^\varepsilon(t)$ and $x^\varepsilon(t)$ satisfy a system of stochastic differential equations and the remaining term $\varepsilon\eta^\varepsilon$ is small in $H^1(\mathbb{R})$. We then prove in Section 3 that the process η^ε converges as ε goes to zero, in quadratic mean, to a centered Gaussian process η which satisfies an additively driven linear equation, with a conservative deterministic part; we also investigate the behavior of the process η as t goes to infinity and prove that η is in some sense a Ornstein-Uhlenbeck process, with a unique Gaussian invariant measure. In addition, the parameters $x^\varepsilon(t)$ and $c^\varepsilon(t)$ may be developed up to order one in ε and we get

$$\begin{cases} dx^\varepsilon = c_0 dt + \varepsilon B_1 dt + \varepsilon dB_2 + o(\varepsilon) \\ dc^\varepsilon = \varepsilon dB_1 + o(\varepsilon), \end{cases}$$

where B_1 and B_2 are correlated real valued Brownian motions; keeping only the order one terms in those modulation parameters, we then obtain a diffuson result on the modulated soliton similar to the result obtained by Wadati in [25], but with a different time exponent (see Section 4).

Note that in [20], the existence of a random traveling wave is proved for a KPP equation with noise, while in [19], the asymptotic of the random traveling wave velocity is investigated as the amplitude of the noise tends to zero. Our results are rather weak compared to the results of [20] and [19]. In particular, we do not prove the existence of a random traveling wave, and we do not really believe in its existence. We only investigate the evolution of the random parameters of a soliton with a deterministic profile, in such a way that its fits the solution of the stochastic equation as long as possible. On the other hand, a key ingredient in the studies of random traveling waves for stochastic KPP equations is the comparison theorem (maximum principle) which does not hold in our case.

In all what follows, $(.,.)$ will denote the inner product in $L^2(\mathbb{R})$,

$$(u,v) = \int_{\mathbb{R}} u(x)v(x)dx$$

and we denote by $\mathcal{T}_{x_0}$ the translation operator defined for $\varphi \in C(\mathbb{R})$ by $(\mathcal{T}_{x_0}\varphi)(x) = \varphi(x+x_0)$. Note that since the process W is stationary in space, for any $x_0 \in \mathbb{R}$ the process $\mathcal{T}_{x_0}W$ is still a Wiener process with covariance $\phi\phi^*$. Indeed by (1.3),

$$\mathcal{T}_{x_0}W(t,x) = \sum_{k\in\mathbb{N}}(\phi e_k)(x+x_0)\beta_k(t) = \sum_{k\in\mathbb{N}}(\phi\tilde{e}_k)(x)\beta_k(t),$$

with $\tilde{e}_k(x) = \mathcal{T}_{x_0}e_k$.

2 Modulation and estimate on the exit time

In this section, we will prove a decomposition of the form (1.9) for the solution u^ε of equation (1.1) with $u^\varepsilon(0,x) = \varphi_{c_0}(x)$, with a small remaining term $\varepsilon\eta^\varepsilon$ in $H^1(\mathbb{R})$. We will also estimate the time τ^ε up to which such a decomposition holds. Note that the above requirements are not sufficient to ensure the uniqueness of the parameters $x^\varepsilon(t)$ and $c^\varepsilon(t)$ in the decomposition (1.9). The uniqueness is obtained by choosing some specific orthogonality conditions for the remaining term. In the next theorem, we choose these orthogonality conditions (see (2.1) and (2.2) below) in such a way that we can use the deterministic theory of orbital stability in order to estimate the exit time τ^ε. In section 4, we will slightly change the modulation parameters (hence also the orthogonality conditions) in order to be able to use, on the remaining term, the asymptotic stability theory developed in [21].

Theorem 2.1. *Assume that the kernel k of the noise satisfies (1.2) together with $k \in L^1(\mathbb{R})$ and let c_0 be fixed. For $\varepsilon > 0$, let $u^\varepsilon(t,x)$, as defined above, be the solution of (1.1) with $u(0,x) = \varphi_{c_0}(x)$. Then* *there exists $\alpha_0 > 0$ such that, for each α, $0 < \alpha \leq \alpha_0$, there is a stopping time $\tau_\alpha^\varepsilon > 0$ a.s. and there are semi-martingale processes $c^\varepsilon(t)$ and $x^\varepsilon(t)$, defined a.s. for $t \leq \tau_\alpha^\varepsilon$, with values respectively in $\mathbb{R}^{+*}$ and $\mathbb{R}$, so that if we set*

$$\varepsilon\eta^\varepsilon(t) = u^\varepsilon(t,.+x^\varepsilon(t)) - \varphi_{c^\varepsilon(t)},$$

then a.s. for $t \leq \tau_\alpha^\varepsilon$,

$$\int_{\mathbb{R}} \eta^\varepsilon(t,x)\varphi_{c_0}(x)dx = (\eta^\varepsilon, \varphi_{c_0}) = 0 \tag{2.1}$$

and

$$\int_{\mathbb{R}} \eta^\varepsilon(t,x)\partial_x\varphi_{c_0}(x)dx = (\eta^\varepsilon, \partial_x\varphi_{c_0}) = 0 \tag{2.2}$$

and for all $t \leq \tau_\alpha^\varepsilon$,

$$\|\varepsilon\eta^\varepsilon(t)\|_1 \leq \alpha$$

and

$$|c^\varepsilon(t) - c_0| \leq \alpha.$$

In addition, for α_0 sufficiently small, and any $\alpha \leq \alpha_0$, there is a constant $C > 0$, depending only on α and c_0, such that for any $T > 0$, there is an $\varepsilon_0 > 0$, with, for each $\varepsilon < \varepsilon_0$,

$$\mathbb{P}(\tau_\alpha^\varepsilon \leq T) \leq \exp\left(-\frac{C(\alpha, c_0)}{\varepsilon^2 T\|k\|_{H^1}^2}\right). \tag{2.3}$$

Remark 2.2. *It was noticed heuristically in [5], and proved in [10] that in the case of additive noise, the use of the modulation parameters $x^\varepsilon(t)$ and $c^\varepsilon(t)$ was necessary in order to get the estimate (2.3). Indeed, it was proved in [10] that if we denote*

$$\tilde{\tau}_\alpha^{\varepsilon,n} = \inf\{t > 0, \|u^{\varepsilon,n}(t,.) - \varphi_{c_0}\|_1 > \alpha\},$$

where $u^{\varepsilon,n}$ is here the solution of equation (1.1), but with an additive noise that becomes stationary in space as n goes to infinity (see [10] for a precise statement) then there exists a constant $C(\alpha, c_0)$ which depends on α and c_0 but not on T such that

$$\underline{\lim}_{n\to\infty}\underline{\lim}_{\varepsilon\to 0}\varepsilon^2 \log \mathbb{P}\left(\tilde{\tau}_\alpha^{n,\varepsilon} \leq T\right) \geq -\frac{C(\alpha, c_0)}{T^3}. \tag{2.4}$$

It is not clear that (2.4) is still true in the present multiplicative case, because the proof involves a controllability problem with a potential which – up to now – is open.

Proof of Theorem 2.1 The proof follows closely the proof of Theorem 2.1 in [5] and we refer to [5] for more details. The parameters $x^\varepsilon(t)$ and $c^\varepsilon(t)$ are obtained thanks to the use of the implicit function Theorem, as deterministic C^2 functions of $u^\varepsilon(t)$. These are then local semi-martingales defined as long as $|c^\varepsilon(t) - c_0| < \alpha$ and $\|u^\varepsilon(t,.+x^\varepsilon(t)) - \varphi_{c_0}\|_1 < \alpha$, and setting

$$\varepsilon\eta^\varepsilon(t) = u^\varepsilon(t,.+x^\varepsilon(t)) - \varphi_{c^\varepsilon(t)},$$

one has for each $\varepsilon > 0$, almost surely,

$$(\eta^\varepsilon, \varphi_{c_0}) = (\eta^\varepsilon, \partial_x\varphi_{c_0}) = 0. \tag{2.5}$$

In order to estimate the exit time

$$\tau_\alpha^\varepsilon = \inf\{t \geq 0,\ |c^\varepsilon(t) - c_0| > \alpha \text{ or } \|\varepsilon\eta^\varepsilon(t)\|_1 > \alpha\},$$

we make use, as in [5], of the functional defined for $u \in H^1(\mathbb{R})$,

$$Q_{c_0}(u) := H(u) + c_0 m(u) \tag{2.6}$$

where H and m are defined respectively in (1.4) and (1.5). Note that φ_{c_0} is a critical point of Q_{c_0}. We denote by L_{c_0} the linearized operator around φ_{c_0}, that is

$$L_{c_0} = -\partial_x^2 + c_0 - 2\varphi_{c_0}. \tag{2.7}$$

The next lemma, which is proved with the use of the Itô Formula, using the same regularization procedure as in [4], gives the evolution of H and m for the solution u^ε of (1.1) with $u^\varepsilon(0) = \varphi_{c_0}$:

Lemma 2.3. *For any stopping time $\tau < +\infty$ a.s, one has*

$$m(u^\varepsilon(\tau)) = m(\varphi_{c_0}) - \varepsilon \int_0^\tau ((u^\varepsilon)(s), dW(s)) + \varepsilon^2 |k|_{L^2}^2 \int_0^\tau m(u^\varepsilon(s))ds$$

and

$$\begin{aligned} H(u^\varepsilon(\tau)) &= H(\varphi_{c_0}) + \varepsilon \int_0^\tau (\partial_x u^\varepsilon, \partial_x(u^\varepsilon dW(s))) - \frac{\varepsilon}{2}\int_0^\tau ((u^\varepsilon)^3, dW(s)) && (2.8)\\ &+\frac{\varepsilon^2}{2}\int_0^\tau \left\{|k|_{L^2}^2 |\partial_x u^\varepsilon|_{L^2}^2 + |k'|_{L^2}^2 |u^\varepsilon|_{L^2}^2\right\} ds && (2.9)\\ &-\frac{\varepsilon^2}{2}\sum_k \int_0^\tau \int_{\mathbb{R}} (u^\varepsilon)^3 |\phi e_k|^2 dx ds. && (2.10)\end{aligned}$$

Consider $\nu > 0$ such that $(Q''_{c_0}(\varphi_{c_0})v, v) \geq \nu \|v\|_1^2$ for any $v \in H^1$ satisfying $(v, \varphi_{c_0}) = (v, \partial_x \varphi_{c_0}) = 0$. The existence of such a constant is a classical result (see [2] or [3]). Then it is easy to show (see [5]) that there is a constant $C(\alpha_0) > 0$ such that for any $t < \tau_\alpha^\varepsilon$,

$$Q_{c_0}(u^\varepsilon(t, . + x^\varepsilon(t))) - Q_{c_0}(\varphi_{c^\varepsilon(t)}) \geq \frac{\nu}{4}\|\varepsilon\eta^\varepsilon(t)\|_1^2 - C|c^\varepsilon(t) - c_0|^2. \tag{2.11}$$

Now, if $\tau = \tau_\alpha^\varepsilon \wedge t$, then by (2.11), the translation invariance of Q_{c_0}, and Lemma 2.3

$$\begin{aligned} \|\varepsilon\eta^\varepsilon(\tau)\|_1^2 \ \leq\ & \frac{4}{\nu}\left[Q_{c_0}(\varphi_{c_0}) - Q_{c_0}(\varphi_{c^\varepsilon(\tau)})\right] + \varepsilon\int_0^\tau (\partial_x u^\varepsilon(s), \partial_x(u^\varepsilon dW(s)))\\ &-\frac{\varepsilon}{2}\int_0^\tau ((u^\varepsilon)^3(s), dW(s)) + \frac{\varepsilon^2}{2}\int_0^\tau (|k|_{L^2}^2 |\partial_x u^\varepsilon|_{L^2}^2 + |k'|_{L^2}^2 |u^\varepsilon|_{L^2}^2) ds\\ &-\frac{\varepsilon^2}{2}\sum_k \int_0^\tau \int_{\mathbb{R}} (u^\varepsilon)^3(s) |\phi e_k|^2 dx ds - c_0\varepsilon\int_0^\tau ((u^\varepsilon)^2, dW(s))\\ &+c_0\varepsilon^2 |k|_{L^2}^2 \int_0^\tau m(u^\varepsilon(s))ds + C|c^\varepsilon(\tau) - c_0|^2. \end{aligned} \tag{2.12}$$

The term $|c^\varepsilon(\tau)-c_0|$ is then estimated thanks to the orthogonality condition $(\eta^\varepsilon,\varphi_{c_0})=0$ and the evolution of $m(u^\varepsilon(\tau))$ given in Lemma 2.3; one obtains, for some constants $\mu>0$ and $C>0$, depending only on c_0 and α_0 (with $\alpha\le\alpha_0$)

$$\begin{aligned}\mu|c^\varepsilon(\tau)-c_0| &\le \left||\varphi_{c_0}|^2_{L^2}-|\varphi_{c^\varepsilon(\tau)}|^2_{L^2}\right| \\ &\le |\varepsilon\eta^\varepsilon(\tau)|^2_{L^2}+C\alpha|c^\varepsilon(\tau)-c_0|+2\varepsilon\left|\int_0^\tau((u^\varepsilon)^2,dW(s))\right| \\ &\quad +2\varepsilon^2|k|^2_{L^2}\int_0^\tau |u^\varepsilon(s)|^2_{L^2}ds.\end{aligned}$$

Hence, choosing α_0 sufficiently small one gets

$$\begin{aligned}|c^\varepsilon(\tau)-c_0|^2 &\le C\Big[|\varepsilon\eta^\varepsilon(\tau)|^4_{L^2}+4\varepsilon^2\Big|\int_0^\tau((u^\varepsilon)^2,dW(s))\Big|^2 \\ &\quad +4\varepsilon^4|k|^4_{L^2}\Big(\int_0^\tau|u^\varepsilon(s)|^2_{L^2}ds\Big)^2\Big]\end{aligned}\tag{2.13}$$

which, once inserted into (2.12) leads to

$$\begin{aligned}\|\varepsilon\eta^\varepsilon(\tau)\|_1^2 &\le C\Big[|\varepsilon\eta^\varepsilon(\tau)|^4_{L^2}+\varepsilon\Big|\int_0^\tau(\partial_x u^\varepsilon,\partial_x(u^\varepsilon dW(s)))\Big| \\ &\quad +\varepsilon\Big|\int_0^\tau((u^\varepsilon)^3,dW(s))\Big|+c_0\varepsilon\Big|\int_0^\tau((u^\varepsilon)^2,dW(s))\Big| \\ &\quad +4\varepsilon^2\Big|\int_0^\tau((u^\varepsilon)^2,dW(s))\Big|^2+\varepsilon^2\|k\|_1^2\int_0^\tau\|u^\varepsilon(s)\|_1^2ds \\ &\quad +\varepsilon^2|k|^2_{L^2}\int_0^\tau\|u^\varepsilon(s)\|_1^3ds+\varepsilon^4|k|^4_{L^2}\Big(\int_0^\tau|u^\varepsilon(s)|^2_{L^2}ds\Big)^2\Big].\end{aligned}$$

With this estimate in hand, together with (2.13), the conclusion of Theorem 2.1 follows with the same arguments as in the proof of Proposition 3.1 in [10]. These arguments rely on classical exponential tail estimates for stochastic integrals, after noticing that $\|u^\varepsilon(s)\|_1\le C$, a.s. for $s\in[0,\tau^\varepsilon_\alpha\wedge T]$ and $\alpha\le\alpha_0$, so that the quadratic variation of each of the integrals involved in the above estimates are bounded above by CT. □

3 A central limit theorem

This section is devoted to the proof of the next theorem:

Theorem 3.1. *Under the assumptions of Theorem 2.1, let $\alpha<\alpha_0$ be fixed, and let $c^\varepsilon(t)$, $x^\varepsilon(t)$ and τ^ε_α satisfy the conclusion of Theorem 2.1. Then setting*

$$\varepsilon\eta^\varepsilon(t)=u^\varepsilon(t,.+x^\varepsilon(t))-\varphi_{c^\varepsilon(t)},$$

the process $(\eta^\varepsilon(t))_{t\in[0,T]}$ converges as ε goes to zero to a Gaussian process $\eta(t)$ satisfying a linear equation with additive noise. More precisely, for any $T>0$,

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big(\sup_{t\in[0,T\wedge\tau_\alpha^\varepsilon]}|\eta^\varepsilon(t)-\eta(t)|_{L^2}^2\Big)=0.$$

The conclusion of Theorem 3.1 will be obtained in two steps. The first step consists in estimating the modulation parameters obtained in Theorem 2.1, in terms of η^ε, using the equations for those parameters; then the convergence of η^ε as ε tends to zero is proved. In the next section, a slight change in the modulation parameters will be performed, in order to precise the asymptotic behavior (in time) of the limit process η. It will indeed be proved that part of it is actually an Ornstein-Uhlenbeck process.

From now on, we assume that α is fixed and sufficiently small, so that the conclusion of Theorem 2.1 holds, and we denote τ_α^ε by τ^ε.

3.1 Modulation equations

Since we know that the modulation parameters $x^\varepsilon(t)$ and $c^\varepsilon(t)$ are semi-martingale processes adapted to the filtration generated by $(W(t))_{t\geq 0}$, we may a priori write the stochastic evolution equations for those parameters in the form

$$\begin{cases} dx^\varepsilon = c^\varepsilon dt + \varepsilon y^\varepsilon dt + \varepsilon(z^\varepsilon, dW)\\ dc^\varepsilon = \varepsilon a^\varepsilon dt + \varepsilon(b^\varepsilon, dW)\end{cases} \tag{3.1}$$

where y^ε and a^ε are real valued adapted processes with a.s. locally integrable paths on $[0,\tau^\varepsilon)$, and $b^\varepsilon, z^\varepsilon$ are predictable processes with paths a.s. in $L^2_{loc}(0,\tau^\varepsilon;L^2(\mathbb{R}))$. We then proceed as in [5] : the Itô-Wentzell Formula applied to $u^\varepsilon(t,x+x^\varepsilon(t))$, together with equation (1.1) for u^ε and the first equation of (3.1) for x^ε give a stochastic evolution equation for $u^\varepsilon(t,x+x^\varepsilon)$. Note that one may use the Itô-Wentzell formula given in [17]. Indeed, it is easily checked that the process u^ε (stopped at τ^ε) satisfies the assumptions of Theorem 1.1 in [17]. The process $x^\varepsilon(t)$ does not readily satisfy the required assumptions, since the processes y^ε and $(z^\varepsilon,\phi e_k)$ are not bounded on $\Omega\times\mathbb{R}^+$, even when they are stopped at τ^ε. However, one may use a cut-off for y^ε and $(z^\varepsilon,\phi e_k)$, apply the formula of [17] and then remove the cut-off, noticing that each term in the equation converges a.s., at least as a distribution. On the other hand, the standard Itô Formula together with the second equation of (3.1) for c^ε give an equation for the evolution of $\varphi_{c^\varepsilon(t)}$. Replacing then $\varphi_{c^\varepsilon(t)}+\varepsilon\eta^\varepsilon(t,x)$ for $u^\varepsilon(t,x+x^\varepsilon(t))$ in the first equation leads to the following stochastic equation for the evolution of $\eta^\varepsilon(t)$:

$$\begin{aligned} d\eta^\varepsilon \;=\;& \partial_x L_{c_0}\eta^\varepsilon dt + (y^\varepsilon\partial_x\varphi_{c^\varepsilon} - a^\varepsilon\partial_c\varphi_{c^\varepsilon})dt - \partial_x((\varphi_{c^\varepsilon}-\varphi_{c_0})\eta^\varepsilon)dt\\ &+(c^\varepsilon-c_0+\varepsilon y^\varepsilon)\partial_x\eta^\varepsilon dt - \tfrac{\varepsilon}{2}\partial_x((\eta^\varepsilon)^2)dt + \varphi_{c^\varepsilon}\mathcal{T}_{x^\varepsilon}dW\\ &+\partial_x\varphi_{c^\varepsilon}(z^\varepsilon,dW) - \partial_c\varphi_{c^\varepsilon}(b^\varepsilon,dW) + \varepsilon\eta^\varepsilon\mathcal{T}_{x^\varepsilon}dW + \varepsilon\partial_x\eta^\varepsilon(z^\varepsilon,dW)\\ &+\tfrac{\varepsilon}{2}\partial_x^2\varphi_{c^\varepsilon}|\phi^* z^\varepsilon|_{L^2}^2 dt - \tfrac{\varepsilon}{2}\partial_c^2\varphi_{c^\varepsilon}|\phi^* b^\varepsilon|_{L^2}^2 dt + \varepsilon\sum_{l\in\mathbb{N}}\partial_x(\varphi_{c^\varepsilon}\mathcal{T}_{x^\varepsilon}\phi e_l)(z^\varepsilon,\phi e_l)dt\\ &+\tfrac{1}{2}\varepsilon^2\partial_x^2\eta^\varepsilon|\phi^* z^\varepsilon|_{L^2}^2 dt + \varepsilon^2\sum_{l\in\mathbb{N}}\partial_x(\eta^\varepsilon\mathcal{T}_{x^\varepsilon}\phi e_l)(z^\varepsilon,\phi e_l)dt \end{aligned} \tag{3.2}$$

where L_{c_0} is defined in (2.7). Now, taking the L^2- inner product of equation (3.2) with φ_{c_0}, on the one hand, and with $\partial_x \varphi_{c_0}$ on the other hand, then using the orthogonality conditions (2.5) and the fact that $L_{c_0}\partial_x\varphi_{c_0} = 0$, and finally identifying the drift parts and the martingale parts of each of the resulting equations lead to the same kind of system that we previously obtained in [5]; namely, setting

$$Y^\varepsilon(t) = \begin{pmatrix} y^\varepsilon(t) \\ a^\varepsilon(t) \end{pmatrix} \quad \text{and} \quad Z_l^\varepsilon(t) = \begin{pmatrix} (z^\varepsilon, \phi e_l) \\ (b^\varepsilon, \phi e_l) \end{pmatrix}$$

then one gets for the drift parts

$$A^\varepsilon(t) Y^\varepsilon(t) = G^\varepsilon(t) \tag{3.3}$$

where

$$A^\varepsilon(t) = \begin{pmatrix} (\partial_x\varphi_{c^\varepsilon} + \varepsilon\partial_x\eta^\varepsilon, \partial_x\varphi_{c_0}) & -(\partial_c\varphi_{c^\varepsilon}, \partial_x\varphi_{c_0}) \\ -(\partial_x\varphi_{c^\varepsilon}, \varphi_{c_0}) & (\partial_c\varphi_{c^\varepsilon}, \varphi_{c_0}) \end{pmatrix} \tag{3.4}$$

and

$$G^\varepsilon(t) = \begin{pmatrix} G_1^\varepsilon(t) \\ G_2^\varepsilon(t) \end{pmatrix},$$

with

$$\begin{aligned} G_1^\varepsilon(t) \;\; = \;\; & (\eta^\varepsilon, L_{c_0}\partial_x^2\varphi_{c_0}) + (c^\varepsilon - c_0)(\eta^\varepsilon, \partial_x^2\varphi_{c_0}) + \tfrac{\varepsilon}{2}(\partial_x(\eta^\varepsilon)^2, \partial_x\varphi_{c_0}) \\ & + (\partial_x((\varphi_{c^\varepsilon} - \varphi_{c_0})\eta^\varepsilon), \partial_x\varphi_{c_0}) - \tfrac{\varepsilon}{2}(\partial_x^2\varphi_{c^\varepsilon}, \partial_x\varphi_{c_0})|\phi^* z^\varepsilon|_{L^2}^2 \\ & + \tfrac{\varepsilon}{2}(\partial_c^2\varphi_{c^\varepsilon}, \partial_x\varphi_{c_0})|\phi^* b^\varepsilon|_{L^2}^2 - \varepsilon\sum_{l\in\mathbb{N}}(z^\varepsilon, \phi e_l)(\partial_x(\varphi_{c^\varepsilon}\mathcal{T}_{x^\varepsilon}\phi e_l), \partial_x\varphi_{c_0}) \\ & + \tfrac{1}{2}\varepsilon^2(\eta^\varepsilon, \partial_x^3\varphi_{c_0})|\phi^* z^\varepsilon|_{L^2}^2 - \varepsilon^2\sum_{l\in\mathbb{N}}(\partial_x(\eta^\varepsilon\mathcal{T}_{x^\varepsilon}\phi e_l), \partial_x\varphi_{c_0})(z^\varepsilon, \phi e_l) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} G_2^\varepsilon(t) \;\; = \;\; & -\tfrac{\varepsilon}{2}(\partial_x(\eta^\varepsilon)^2, \varphi_{c_0}) - (\partial_x((\varphi_{c^\varepsilon} - \varphi_{c_0})\eta^\varepsilon), \varphi_{c_0}) + \tfrac{\varepsilon}{2}(\partial_x^2\varphi_{c^\varepsilon}, \varphi_{c_0})|\phi^* z^\varepsilon|_{L^2}^2 \\ & - \tfrac{\varepsilon}{2}(\partial_c^2\varphi_{c^\varepsilon}, \varphi_{c_0})|\phi^* b^\varepsilon|_{L^2}^2 + \varepsilon\sum(z^\varepsilon, \phi e_l)(\partial_x(\varphi_{c^\varepsilon}\mathcal{T}_{x^\varepsilon}\phi e_l), \varphi_{c_0}) \\ & + \tfrac{\varepsilon^2}{2}(\eta^\varepsilon, \partial_x^2\varphi_{c_0})|\phi^* z^\varepsilon|_{L^2}^2 + \varepsilon^2\sum_{l\in\mathbb{N}}(\partial_x(\eta^\varepsilon\mathcal{T}_{x^\varepsilon}\phi e_l), \varphi_{c_0})(z^\varepsilon, \phi e_l); \end{aligned} \tag{3.6}$$

note that $A^\varepsilon(t) = A_0 + O(|c^\varepsilon - c_0| + \|\varepsilon\eta^\varepsilon\|_1)$, a.s. for $t \le \tau^\varepsilon$ with

$$A_0 = \begin{pmatrix} |\partial_x\varphi_{c_0}|_{L^2}^2 & 0 \\ 0 & (\varphi_{c_0}, \partial_c\varphi_{c_0}) \end{pmatrix}$$

and $O(|c^\varepsilon - c_0| + \|\eta^\varepsilon\|_1)$ is uniform in ε, t and ω as long as $t \le \tau^\varepsilon$. Concerning the martingale parts, one gets the equation

$$A^\varepsilon(t) Z_l^\varepsilon(t) = F_l^\varepsilon(t), \quad \forall l \in \mathbb{N} \tag{3.7}$$

with

$$F^\varepsilon(t) = \begin{pmatrix} -((\varphi_{c^\varepsilon} + \varepsilon\eta^\varepsilon)\mathcal{T}_{x^\varepsilon}\phi e_l, \partial_x\varphi_{c_0}) \\ ((\varphi_{c^\varepsilon} + \varepsilon\eta^\varepsilon)\mathcal{T}_{x^\varepsilon}\phi e_l, \varphi_{c_0}). \end{pmatrix} \tag{3.8}$$

Proposition 3.2. *Under the above assumptions, there is a constant $\alpha_1 > 0$, such that if $\alpha \leq \alpha_1$, then*

$$|\phi^* z^\varepsilon(t)|_{L^2} + |\phi^* b^\varepsilon|_{L^2} \leq C_1 |k|_{L^2}, \quad \text{a.s. for } t \leq \tau^\varepsilon \tag{3.9}$$

and

$$|a^\varepsilon(t)| + |y^\varepsilon(t)| \leq C_2 |\eta^\varepsilon(t)|_{L^2} + \varepsilon C_3, \quad \text{a.s. for } t \leq \tau^\varepsilon \tag{3.10}$$

for some constants C_1, C_2, C_3, depending only on α and c_0, and for any $\varepsilon \leq \varepsilon_0$.

Proof. The proof is exactly the same as the proof of Corollary 4.3 in [5], once noticed that, a.s. for $t \leq \tau^\varepsilon$,

$$\begin{aligned}
\sum_{l\in\mathbb{N}} |F_l^\varepsilon(t)|^2 &\leq C \sum_{l\in\mathbb{N}} |(\varphi_{c^\varepsilon} + \varepsilon\eta^\varepsilon)\mathcal{T}_{x^\varepsilon}\phi e_l|_{L^2}^2 \\
&\leq C \sum_l \int_{\mathbb{R}} (\varphi_{c^\varepsilon} + \varepsilon\eta^\varepsilon)^2(x)[(\mathcal{T}_{x^\varepsilon}k) * e_l]^2(x)dx \\
&\leq \int_{\mathbb{R}} (\varphi_{c^\varepsilon} + \varepsilon\eta^\varepsilon)^2(x) \sum_l (\mathcal{T}_{x^\varepsilon}k(x - .), e_l)^2 dx \\
&\leq C \int_{\mathbb{R}} (\varphi_{c^\varepsilon} + \varepsilon\eta^\varepsilon)^2(x) |\mathcal{T}_{x^\varepsilon}k(x - .)|_{L^2}^2 dx \\
&\leq C|k|_{L^2}^2 |\varphi_{c^\varepsilon} + \varepsilon\eta^\varepsilon|_{L^2}^2 \leq C|k|_{L^2}^2
\end{aligned}$$

where we have used the Parseval equality in the fourth line. □

3.2 Convergence of η^ε

Let us first assume that η^ε has a limit as ε goes to zero, and take formally the limit as ε goes to zero in the preceding equations. Then, as was noticed above,

$$\lim_{\varepsilon\to 0} A^\varepsilon = A_0 = \begin{pmatrix} |\partial_x \varphi_{c_0}|_{L^2}^2 & 0 \\ 0 & (\varphi_{c_0}, \partial_c \varphi_{c_0}) \end{pmatrix}$$

hence

$$\lim_{\varepsilon\to 0} \phi^* z^\varepsilon = -\frac{1}{|\partial_x \varphi_{c_0}|_{L^2}^2} (\mathcal{T}_{c_0 t}\phi)^* (\varphi_{c_0} \partial_x \varphi_{c_0}) := z \tag{3.11}$$

$$\lim_{\varepsilon\to 0} \phi^* b^\varepsilon = \frac{1}{(\varphi_{c_0}, \partial_c \varphi_{c_0})} (\mathcal{T}_{c_0 t}\phi^*)(\varphi_{c_0}^2) := b \tag{3.12}$$

$$\lim_{\varepsilon\to 0} y^\varepsilon = \frac{1}{|\partial_x \varphi_{c_0}|_{L^2}^2} (\eta, L_{c_0} \partial_x^2 \varphi_{c_0}) := y \tag{3.13}$$

and

$$\lim_{\varepsilon\to 0} a^\varepsilon = 0. \tag{3.14}$$

Moreover, formally, η satisfies the equation

$$\begin{array}{rcl} d\eta & = & \partial_x L_{c_0}\eta dt + \frac{1}{|\partial_x\varphi_{c_0}|^2_{L^2}}(\eta, L_{c_0}\partial_x^2\varphi_{c_0})\partial_x\varphi_{c_0}dt \\ & & +\varphi_{c_0}\mathcal{T}_{c_0t}dW - \frac{1}{2|\partial_x\varphi_{c_0}|^2_{L^2}}(\partial_x(\varphi_{c_0}^2),\mathcal{T}_{c_0t}dW)\partial_x\varphi_{c_0} \\ & & -\frac{1}{(\varphi_{c_0},\partial_c\varphi_{c_0})}(\varphi_{c_0}^2,\mathcal{T}_{c_0t}dW)\partial_c\varphi_{c_0}. \end{array} \tag{3.15}$$

It is easy to show that (3.15) has a unique adapted solution η with paths a.s. in $C(\mathbb{R}^+, H^1)$ satisfying $\eta(0)=0$. Moreover using the fact that $(\partial_c\varphi_{c_0}, \partial_x\varphi_{c_0})=0$, one easily gets from the above equation that $(\eta,\varphi_{c_0})=(\eta,\partial_x\varphi_{c_0})=0$, $\forall t>0$.

Next, we make use of the following lemmas, whose proofs are obtained in the same way as the corresponding Lemmas in [5].

Lemma 3.3. *Let η be the solution of (3.15) with $\eta(0)=0$. Then, for any $T>0$, there is a constant C depending only on c_0, T and $\|k\|_1$ such that*

$$\mathbb{E}\left(\|\eta(t)\|_1^4\right) \le C,\ \forall t\le T.$$

Lemma 3.4. *Let η^ε be the solution of (3.2), defined for $t\in[0,\tau^\varepsilon[$, obtained thanks to the modulation procedure of Section 2. Then, for any $T>0$,*

$$\mathbb{E}\Big(\sup_{t\le\tau^\varepsilon\wedge T}|\eta^\varepsilon(t)|^4_{L^2}\Big) \le C(T,\alpha,c_0,\|k\|_1).$$

The above lemmas show that

$$\forall T>0,\ \forall q\ge 2,\ \lim_{\varepsilon\to 0}\mathbb{E}\Big(\sup_{t\le T\wedge\tau^\varepsilon}|c^\varepsilon(t)-c_0|^q\Big)=0. \tag{3.16}$$

Indeed, the expression of $c^\varepsilon(t)-c_0$ given by (3.1) together with (3.9) and (3.10) imply easily

$$\mathbb{E}\Big(\sup_{t\le T\wedge\tau^\varepsilon}|c^\varepsilon(t)-c_0|^2\Big)\le C\varepsilon^2[1+\mathbb{E}\int_0^{T\wedge\tau^\varepsilon}|\eta^\varepsilon(s)|^2_{L^2}ds]$$

with $C=C(\alpha,c_0,T,\|k\|_1)$. Then, (3.16) is deduced form Lemma 3.4 for $q=2$, and follows for all other values of q from the uniform boundedness of $|c^\varepsilon(t)-c_0|$ on $[0,T\wedge\tau^\varepsilon]$. Note that an immediate consequence of (3.16) is the fact that

$$\forall T>0,\ \forall q\ge 2,\ \lim_{\varepsilon\to 0}\mathbb{E}\Big(\sup_{t\le T\wedge\tau^\varepsilon}\|\varphi_{c^\varepsilon(t)}-\varphi_{c_0}\|_2^2\Big)=0. \tag{3.17}$$

We will finally need the next lemma.

Lemma 3.5. *For any $T>0$, and any $q\ge 1$,*

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big(\sup_{t\le T\wedge\tau^\varepsilon}\Big(\sum_{l\in\mathbb{N}}|Z_l^\varepsilon(t)-Z_l(t)|^2\Big)^q\Big)=0$$

where we have set for $l\in\mathbb{N}$

$$Z_l(t)=\begin{pmatrix}(z,\phi e_l)\\(b,\phi e_l)\end{pmatrix},$$

z and b being given by (3.11) and (3.12), respectively.

Proof. Here again, it is sufficient to consider the case $q = 1$. We recall that Z_l^ε satisfies equation (3.7). First, it is clear that

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big(\sup_{t\le T\wedge\tau^\varepsilon}\|(A^\varepsilon(t))^{-1}-(A_0(t))^{-1}\|^{2q}\Big)=0,\ \forall q\ge 1.$$

On the other hand, in view of (3.8), denoting $F_l^0(t)$ the formal limit of $F_l^\varepsilon(t)$, one has

$$\begin{aligned}&\mathbb{E}\Big(\sup_{t\le T\wedge\tau^\varepsilon}\sum_l|F_l^\varepsilon(t)-F_l^0(t)|^2\Big)\\&\le C\mathbb{E}\Big(\sup_{t\le T\wedge\tau^\varepsilon}\sum_l|\partial_x\varphi_{c_0}(\mathscr{T}_{x^\varepsilon}\phi-\mathscr{T}_{c_0t}\phi)e_l|_{L^2}^2\Big)+C\mathbb{E}\Big(\sup_{t\le T\wedge\tau^\varepsilon}\|\varphi_{c^\varepsilon(t)}-\varphi_{c_0}\|_1^2\Big)\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\Big(\sup_{t\le T\wedge\tau^\varepsilon}\sum_l|\partial_x\varphi_{c_0}(\mathscr{T}_{x^\varepsilon}\phi-\mathscr{T}_{c_0t}\phi)e_l|_{L^2}^2\Big)\\&\le\|\varphi_{c_0}\|_1^2\mathbb{E}\Big(\sup_{t\le T\wedge\tau^\varepsilon}|k(.+x^\varepsilon(t)-c_0t)-k|_{L^2}^2\Big).\end{aligned}$$

Then, the Itô Formula applied to the function

$$\mathscr{K}^\varepsilon(t,x)=(k(x+x^\varepsilon(t)-c_0t)-k(x))^2$$

using equation (3.1) for $dx^\varepsilon(t)$, together with (3.9), (3.10), and (3.16) lead to the conclusion of Lemma 3.5. □

Now, in order to prove that

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big(\sup_{t\le T\wedge\tau^\varepsilon}|\eta^\varepsilon(t)-\eta(t)|_{L^2}^2\Big)=0,\tag{3.18}$$

where η is the solution of (3.15) with $\eta(0)=0$, it suffices to set $v^\varepsilon=\eta^\varepsilon-\eta$, to deduce from (3.15) and (3.2) the equation for dv^ε and to apply the Itô Formula to get the evolution of $|v^\varepsilon|_{L^2}^2$. We do not give the details of those tedious, but easy computations. Finally, the use of the following estimates :

$$\begin{aligned}&\varepsilon|(v^\varepsilon,\partial_x((\eta^\varepsilon)^2))|=\varepsilon|(\partial_x\eta,(\eta^\varepsilon)^2)|\le\varepsilon\|\eta\|_1|\eta^\varepsilon|_{L^4}^2\\&\le C\varepsilon\|\eta\|_1|\eta^\varepsilon|_{L^2}^{3/2}|\partial_x\eta^\varepsilon|_{L^2}^{1/2}\le C\sqrt{\varepsilon}\|\eta\|_1|\eta^\varepsilon|_{L^2}^{3/2}\end{aligned}$$

on the one hand, and

$$|y^\varepsilon-y|+|a^\varepsilon|\le C(|v^\varepsilon|_{L^2}+|c^\varepsilon-c_0||\eta^\varepsilon|_{L^2}+\varepsilon|\eta^\varepsilon|_{L^2}^2+|\eta^\varepsilon|_{L^2}\|\varphi_{c^\varepsilon}-\varphi_{c_0}\|_1+\varepsilon)$$

which is obtained as in the proof of Lemma 3.5 on the other hand, together with Lemma 3.3 to 3.5 allow to get the conclusion, that is the convergence to 0 of

$$\mathbb{E}\Big(\sup_{t\in[0,T\wedge\tau^\varepsilon]}|\eta^\varepsilon(t)-\eta(t)|_{L^2}^2\Big)$$

as ε goes to zero. □

4 Complements on the limit equation

The aim of this section is to investigate the asymptotic behavior in time of the process η. However, in the present form, the process η does not converge in law as t goes to infinity; this is due to the fact that the preceding modulation does not exactly correspond to the projection of the solution u^ε on the (two-dimensional) center manifold, in which case the remaining term would belong to the stable manifold around the soliton trajectory. We now show that by slightly changing the modulation parameters, we can get a new decomposition of the solution u^ε which is defined on the same time interval as before, but which fits with the preceding requirements. For that purpose, we first need to recall a few facts from [21].

Consider the operator $\partial_x L_{c_0}$ arising in the linearized evolution equation in the soliton frame (see (2.7)). Then $\partial_x L_{c_0}$ has a generalized nullspace spanned by he functions $\partial_x \varphi_{c_0}$ and $\partial_c \varphi_{c_0}$. More precisely, the following equalities hold, and are obtained by differentiating the soliton equation (1.8) with respect to x (resp. with respect to c) :

$$\partial_x L_{c_0} \partial_x \varphi_{c_0} = 0 \quad \text{and} \quad \partial_x L_{c_0} \partial_c \varphi_{c_0} = -\partial_x \varphi_{c_0}.$$

Note however that although L_{c_0} is a self-adjoint operator, $\partial_x L_{c_0}$ is not skewadjoint, since ∂_x and L_{c_0} do not commute. Hence, in order to define the spectral projection on the “center manifold", one has to introduce (see [21]) the functions

$$\tilde{g}_1(x) = -\theta_1 \int_{-\infty}^{x} \partial_c \varphi_{c_0}(y)dy + \theta_2 \varphi_{c_0} \quad \text{and} \quad \tilde{g}_2(x) = \theta_1 \varphi_{c_0}$$

which belong to the generalized nullspace of $-L_{c_0}\partial_x$, where θ_1 and θ_2 are constants chosen in order that

$$(\tilde{g}_1, \partial_x \varphi_{c_0}) = 1,\ (\tilde{g}_1, \partial_c \varphi_{c_0}) = 0,\ (\tilde{g}_2, \partial_x \varphi_{c_0}) = 0,\ (\tilde{g}_2, \partial_c \varphi_{c_0}) = 1$$

(note that $\theta_1 = (\varphi_{c_0}, \partial_c \varphi_{c_0})$).

Once again, this is not completely sufficient to define the spectral projection on the center manifold, because the function $\tilde{g}_1$ is not in $L^2(\mathbb{R})$ (indeed, it is easily seen from expression (1.7) for φ_c that $\int_{-\infty}^{+\infty} \partial_c \varphi_c(y)dy \neq 0$. The idea of [21] (which is classical in front propagation problems) was then to introduce spaces of functions with an exponential decay as x goes to $+\infty$, noticing that both $\partial_x \varphi_{c_0}$ and $\partial_c \varphi_{c_0}$ decay like $\exp(-\sqrt{c_0}x)$ as x goes to $+\infty$.

Hence, setting for $0 < a < \sqrt{c_0}$

$$f_1^a(x) = e^{ax}\partial_x \varphi_{c_0},\ f_2^a(x) = e^{ax}\partial_c \varphi_{c_0},\ g_1^a(x) = e^{-ax}\tilde{g}_1(x),\ g_2^a(x) = e^{-ax}\tilde{g}_2(x),$$

then all the functions f_i^a and g_j^a are in $L^2(\mathbb{R})$ and $(f_i^a, g_j^a) = \delta_{ij}$. Moreover (see Section 2.2 in [21]), if $0 < a < \sqrt{c_0/3}$, then the operator A_a defined for $a > 0$ by $A_a = e^{ax}\partial_x L_{c_0} e^{-ax}$ has a well defined generalized nullspace spanned by f_1^a, f_2^a and the spectral projection on this nullspace is given by

$$Pw = \sum_{k=1}^{2} (w, g_k^a) f_k^a, \quad w \in H^1(\mathbb{R}).$$

In addition, if $Q = I - P$, then Q is the spectral projection on the stable manifold of A_a, and under the condition $0 < a < \sqrt{c_0/3}$, there are constants $C > 0$ and $b > 0$ such that

$$\|e^{A_a t} Q w\|_1 \leq C e^{-bt} \|w\|_1,\ \forall t > 0,\ \forall w \in H^1, \tag{4.1}$$

where $e^{A_a t}$ is the C^0-semi-group generated by A_a (see Theorem 4.2 in [21]).

With these definitions in hand for the spectral projection, we are now able to state the main result of this section.

Proposition 4.1. *Let x^ε, c^ε and η^ε be as in Theorem 3.1. Then there is a predictable process $\lambda(t)$ defined on $\Omega \times \mathbb{R}^+$ such that if we set for $t \in [0, \tau^\varepsilon)$,*

$$\tilde{x}^\varepsilon(t) = x^\varepsilon(t) - \varepsilon\lambda(t)$$

and

$$\varepsilon\tilde{\eta}^\varepsilon(t,x) = u^\varepsilon(t, x + \tilde{x}^\varepsilon(t)) - \varphi_{c^\varepsilon(t)}(x)$$

then for all $T > 0$,

$$\lim_{\varepsilon\to 0} \mathbb{E}\Big(\sup_{t\in[0,T\wedge\tau^\varepsilon)} |\tilde{\eta}^\varepsilon(t) - \tilde{\eta}(t)|_{L^2}^2 \Big) = 0$$

where the Gaussian process $\tilde{\eta}$ is a solution of the additive linear equation

$$d\tilde{\eta} = \partial_x L_{c_0}\tilde{\eta}\, dt + \tilde{Q}\varphi_{c_0} d\tilde{W}, \tag{4.2}$$

with $\tilde{\eta}(0) = 0$, where $\tilde{W}$ is the Wiener process with covariance $\phi\phi^*$ given by $\tilde{W} = \mathcal{T}_{c_0 t}W$, and $\tilde{Q}$ is a projection operator. Moreover, for $0 < a < \sqrt{c_0/3}$, the process $w(t,x) = e^{ax}\tilde{\eta}(t,x)$ is a well defined H^1 valued process, of Ornstein-Uhlenbeck type, which converges in law to an H^1-valued Gaussian random variable as t goes to infinity.

Proof. First of all, we note that the modulation equations may be written at order one in ε as

$$\begin{cases} dx^\varepsilon = c_0 dt + \varepsilon y dt + \varepsilon W_1 dt + \varepsilon dW_2 + o(\varepsilon) \\ dc^\varepsilon = \varepsilon dW_1 + o(\varepsilon) \end{cases}$$

where

$$y = |\partial_x \varphi_{c_0}|_{L^2}^{-2}(\eta, L_{c_0}\partial_x^2 \varphi_{c_0}),$$

$$W_1(t) = (\varphi_{c_0}, \partial_c \varphi_{c_0})^{-1}(\varphi_{c_0}^2, \tilde{W}(t))$$

and

$$W_2(t) = -\frac{1}{2}|\partial_x \varphi_{c_0}|_{L^2}^{-2}(\partial_x({\varphi_{c_0}}^2), \tilde{W}(t)).$$

Note that W_1 and W_2 are real valued Brownian motions, which are independent since

$$\mathbb{E}(W_1(t)W_2(s)) = -\frac{1}{2}|\partial_x\varphi_{c_0}|_{L^2}^{-2}(\varphi_{c_0}, \partial_c\varphi_{c_0})^{-1}(\phi^*(\partial_x(\varphi_{c_0}^2)), \phi^*(\varphi_{c_0}^2))(t \wedge s) = 0$$

because the operator ϕ^* commutes with spatial derivation.

Now, let η be the solution of (3.15) with $\eta(0) = 0$, and consider $w(t,x) = e^{ax}\eta(t,x)$, with $0 < a < \sqrt{c_0/3}$. Note that w satisfies the equation

$$\begin{aligned} dw \ &= \ A_a w dt + \tfrac{1}{|\partial_x\varphi_{c_0}|_{L^2}^2}(\eta, L_{c_0}\partial_x^2\varphi_{c_0})e^{ax}\partial_x\varphi_{c_0} dt \\ &\quad + e^{ax}\varphi_{c_0} d\tilde{W} - \tfrac{1}{2|\partial_x\varphi_{c_0}|_{L^2}^2}(\partial_x(\varphi_{c_0}^2), d\tilde{W})\eta^{ax}\partial_x\varphi_{c_0} \\ &\quad - \tfrac{1}{(\varphi_{c_0}, \partial_c\varphi_{c_0})}(\varphi_{c_0}^2, d\tilde{W})e^{ax}\partial_c\varphi_{c_0}. \end{aligned} \tag{4.3}$$

Since A_a generates a C^0 semi-group in $H^1(\mathbb{R})$ (see Theorem 4.2 in [21]) and since the functions $e^{ax}\partial_x\varphi_{c_0}$ and $e^{ax}\partial_c\varphi_{c_0}$ are H^1 functions (due to the decay of $\partial_x\varphi_{c_0}$ and $\partial_c\varphi_{c_0}$ as x goes to $+\infty$), it follows easily that w has path a.s. in $C(\mathbb{R}^+, H^1(\mathbb{R}))$.

Note that the orthogonality condition $(\eta, \varphi_{c_0}) = 0$ implies $(w, g_2^a) = 0$, so that $Pw = \lambda(t) f_1^a$ with $\lambda(t) = (w(t), g_1^a)$ a real valued stochastic process whose evolution is given by

$$\begin{aligned}\lambda(t) \quad = \quad & \int_0^t |\partial_x\varphi_{c_0}|_{L^2}^{-2}(\eta(s), L_{c_0}\partial_x^2\varphi_{c_0})ds - \int_0^t |\partial_x\varphi_{c_0}|_{L^2}^{-2}(\varphi_{c_0}\partial_x\varphi_{c_0}, d\tilde{W}(s)) \\ & + \int_0^t (e^{ax}\varphi_{c_0} d\tilde{W}(s), g_1^a)\end{aligned} \tag{4.4}$$

where we have used (3.15) and the fact that $A_a Pw = 0$ and $\lambda(0) = 0$. Hence, $\mathbb{E}\big(\sup_{t\in[0,T\wedge\tau^\varepsilon)} |\lambda(t)|^4\big)$ is finite by Lemma 3.3. Let us set $\tilde{x}^\varepsilon(t) = x^\varepsilon(t) - \varepsilon\lambda(t)$ for $t \in [0, \tau^\varepsilon[$. Then

$$u^\varepsilon(t, x + \tilde{x}^\varepsilon(t)) = \varphi_{c^\varepsilon(t)}(x) + \varepsilon\tilde{\eta}^\varepsilon(t, x) \tag{4.5}$$

with

$$\tilde{\eta}^\varepsilon(t, x) = \frac{1}{\varepsilon}(\varphi_{c^\varepsilon(t)}(x - \varepsilon\lambda(t)) - \varphi_{c^\varepsilon(t)}(x)) + \eta^\varepsilon(t, x - \varepsilon\lambda(t)).$$

Note that, a.s. for $t \le \tau^\varepsilon$:

$$|\varphi_{c^\varepsilon(t)}(. - \varepsilon\lambda(t)) - \varphi_{c^\varepsilon(t)} - \varepsilon\lambda(t)\partial_x\varphi_{c^\varepsilon(t)}|_{L^2} \le \varepsilon^2\lambda^2(t)C(c_0, \alpha).$$

Hence, it follows from Lemma 3.3, 3.4 and the above bound on λ that

$$\lim_{\varepsilon\to 0} \mathbb{E}\big(\sup_{t\le T\wedge\tau^\varepsilon} |\tilde{\eta}^\varepsilon(t) - \tilde{\eta}(t)|_{L^2}^2\big) = 0 \tag{4.6}$$

with $\tilde{\eta}(t) = \eta(t) - \lambda(t)\partial_x\varphi_{c_0}$. So now, with this new decomposition, we clearly have, setting $\tilde{w}(t, x) = e^{ax}\tilde{\eta}(t, x)$:

$$P\tilde{w} = 0, \quad Q\tilde{w} = Qw.$$

Also, if $w_2 = Qw$, then the equation (3.15) implies

$$dw_2 = A_a w_2 dt + Qe^{ax}\varphi_{c_0} d\tilde{W} \tag{4.7}$$

hence

$$w_2(t) = \int_0^t e^{A_a(t-\sigma)} Q[e^{ax}\varphi_{c_0} d\tilde{W}(\sigma)];$$

the trace of the covariance operator of the Gaussian process w_2 in H^1 may be easily computed and estimated thanks to (4.1) as

$$\int_0^t \sum_l \|e^{A_a\sigma} Q e^{ax}\varphi_{c_0}\phi e_l\|_1^2 d\sigma \le C\Big(\int_0^t e^{-b\sigma}d\sigma\Big) \sum_l \|e^{ax}\varphi_{c_0}\phi e_l\|_1^2 d\sigma \le C\|k\|_1^2 \|e^{ax}\varphi_{c_0}\|_1^2.$$

Moreover, this covariance operator converges as t goes to infinity and it follows that w_2 converges in law in H^1 to a Gaussian random variable. The end of the statement of Proposition 4.1 follows, setting $\tilde{Q}v = e^{-ax}Qe^{ax}v$. □

5 A remark on the soliton diffusion

Let us go back to the stochastic evolution equations for the new modulation parameters, that we may write as

$$\left\{\begin{array}{l} d\tilde{x}^\varepsilon = c_0 dt + \varepsilon B_1 dt + \varepsilon dB_2 + o(\varepsilon) \\ dc^\varepsilon = \varepsilon dB_1 + o(\varepsilon) \end{array}\right. \tag{5.1}$$

with $B_1 = W_1$ and $B_2 = -(e^{ax}\varphi_{c_0}\tilde{W}(t), g_1^a) = -(\tilde{W}(t), \varphi_{c_0}\tilde{g}_1)$. Note that B_1 and B_2 are now correlated Brownian motions. We denote by

$$\sigma = (\sigma_{ij})_{i,j} = \mathrm{cov}(B_1, B_2).$$

If we keep only the order one terms in ε i.e. we consider the solution $(X^\varepsilon(t), C^\varepsilon(t))$ of the system of SDEs

$$\left\{\begin{array}{l} dX^\varepsilon = c_0 dt + \varepsilon B_1 dt + \varepsilon dB_2 \\ dC^\varepsilon = \varepsilon dB_1, \end{array}\right.$$

then $(X^\varepsilon(t) - c_0 t, C^\varepsilon(t) - c_0)$ is a centered Gaussian vector, and it is easy to compute its covariance matrix. Let us denote by μ_t^ε the law of $(X^\varepsilon(t) - c_0 t, C^\varepsilon(t) - c_0)$; we may compute

$$\begin{aligned} &\max_{x\in\mathbb{R}} \mathbb{E}\Big(\varphi_{C^\varepsilon(t)}(x - X^\varepsilon(t))\Big) \\ &= \max_{x\in\mathbb{R}} \iint \varphi_{c+c_0}(x - c_0 t - y)\mu_t^\varepsilon(dy, dc) \\ &= \max_{x\in\mathbb{R}} \frac{1}{(\det\Sigma)^{1/2}} \iint \varphi_{c+c_0}(x - c_0 t - y)\exp\Big(-\frac{1}{2}\Sigma^{-1}\begin{pmatrix} c \\ y \end{pmatrix}\cdot\begin{pmatrix} c \\ y \end{pmatrix}\Big) dc\, dy \end{aligned} \tag{5.2}$$

where Σ is the covariance matrix of $(X^\varepsilon(t) - c_0 t, C^\varepsilon(t) - c_0)$, given by

$$\Sigma = \varepsilon^2 \begin{pmatrix} \sigma_{11} t & \sigma_{12} t + \sigma_{11}\frac{t^2}{2} \\ \sigma_{12} t + \sigma_{11}\frac{t^2}{2} & \sigma_{22} t + \sigma_{12} t^2 + \sigma_{11}\frac{t^3}{3} \end{pmatrix}.$$

It is not difficult to see that

$$\exp\Big(-\frac{1}{2}\Sigma^{-1}\begin{pmatrix} c \\ y \end{pmatrix}\cdot\begin{pmatrix} c \\ y \end{pmatrix}\Big) \le \exp\Big(-\frac{1}{2}\frac{\varepsilon^2}{\det\Sigma}\big(\sigma_{11}\frac{t^3}{12} + (\sigma_{22} - \frac{\sigma_{12}^2}{\sigma_{11}} t)\big)c^2\Big).$$

Inserting this inequality in (5.2), using the fact that $\varphi_c(x) = c\varphi_1(\sqrt{c}x)$ and integrating in y give the bound

$$\mathbb{E}\Big(\varphi_{C^\varepsilon(t)}(x - X^\varepsilon(t))\Big) \le \frac{K}{(\det\Sigma)^{1/2}} \int_0^{+\infty} \sqrt{c + c_0}\, e^{-\frac{1}{2}\frac{\varepsilon^2}{\det\Sigma}[\sigma_{11}\frac{t^3}{12} + (\sigma_{22} - \frac{\sigma_{12}^2}{\sigma_{11}} t)]c^2} dc$$

where K is a constant, and since

$$\int_0^{+\infty} \sqrt{c}\, e^{-\frac{c^2}{2\alpha^2}} dc \le K\alpha^{3/2}$$

for another constant K, it follows

$$\max_{x\in\mathbb{R}} \mathbb{E}\Big(\varphi_{C^\varepsilon(t)}(x - X^\varepsilon(t))\Big) \le K_0 \varepsilon^{-1/2} t^{-5/4} \tag{5.3}$$

for t large enough.

This inequality has to be compared to the result of [25] where an additive equation with a white noise in time was considered. An inequality of the form (5.3) was obtained, but with a power $t^{-3/2}$ instead of $t^{-5/4}$.

References

[1] F.G. Bass, Y.S. Kivshar, V.V. Konotop, Y.A. Sinitsyn, *Dynamics of solitons under random perturbations*, Phys. Rep. **157**, (1988), p. 63-181. MR0920335

[2] T.B. Benjamin, *The stability of solitary waves*, Proc. Roy. Soc. London, **A 328**, (1972), p. 153-183. MR0338584

[3] J. L. Bona, P.E. Souganidis, W. Strauss, *Stability and instability of solitary waves of Korteweg-de Vries type*, Proc. Roy. Soc. London, **A 411** (1987), p. 395-412. MR0897729

[4] A. de Bouard, A. Debussche, *On the stochastic Korteweg-de Vries equation*, J. Funct. Anal. **154**, (1998), p. 215-251. MR1616536

[5] A. de Bouard, A. Debussche, *Random modulation of solitons for the stochastic Korteweg-de Vries equation*, Ann. IHP. Analyse Non linéaire **24**, (2007), p. 251-278. MR2310695

[6] A. de Bouard, A. Debussche, *The Korteweg-de Vries equation with multiplicative homogeneous noise*, in "Stochastic Differential Equations : Theory and Applications", P.H. Baxendale and S.V. Lototsky Ed., Interdisciplinary Math. Sciences vol. 2, World Scientific, 2007. MR2393574

[7] A. de Bouard, A. Debussche, Y. Tsutsumi, *White noise driven Korteweg-de Vries equation*, J. Funct. Anal. **169**, (1999), p. 532-558. MR1730557

[8] A. de Bouard, A. Debussche, Y. Tsutsumi, *Periodic solutions of the Korteweg-de Vries equation driven by white noise*, S.I.A.M. J. Math. Anal. **36**, (2004), p. 815-855. MR2111917

[9] A. de Bouard, R. Fukuizumi, *Modulation analysis for a stochastic NLS equation arising in Bose-Einstein condensation*, to appear in Asymptotic Analysis.

[10] A. de Bouard, E. Gautier, *Exit problems related to the persistence of solitons for the Korteweg-de Vries equation with small noise*, to appear in Discrete Contin. Dyn. Syst. Ser. A

[11] J. Fröhlich, S. Gustafson, B. L. Jonsson, and I. M. Sigal, *Solitary wave dynamics in an external potential,* Commun. Math. Phys. **250**, (2004), p. 613-642. MR2094474

[12] J. Fröhlich, S. Gustafson, B. L. Jonsson, and I. M. Sigal, *Long time motion of NLS solitary waves in a confining potential,* Ann. Henri Poincaré **7**, (2006), p. 621-660. MR2232367

[13] C.S. Gardner, J.M. Greene, M.D. Kruskal, R.M. Miura, *Method for solving the Korteweg-de Vries equation*, Phys. Rev. Lett. **19**, (1967), p. 1095-1097.

[14] J. Garnier, *Asymptotic transmission of solitons through random media*, SIAM J. Appl. Math. **58**, (1998), p. 1969-1995. MR1644311

[15] J. Garnier, *Long-time dynamics of Korteweg-de Vries solitons driven by random perturbations*, J. Stat. Phys. **105**, (2001), p. 789-833. MR1869566

[16] R. Herman, *The stochastic, damped Korteweg-de Vries equation*, J. Phys. A. **23**, (1990), p. 1063-1084. MR1049519

[17] N.V. Krylov, *On the Itô-Wentzell formula for distribution-valued processes and related topics*, preprint arXiv:0904.2752v1.

[18] Y. Martel, F. Merle, *Asymptotic stability of solitons of the subcritical gKdV equations revisited*, Nonlinearity **18**, (2005), p. 55-80. MR2109467

[19] C. Mueller, L. Mytnik, J. Quastel, *Small noise asymptotics of traveling waves*, Markov Process. Related Fields **14**, (2008), p. 333-342. MR2453698

[20] C. Mueller, R.B. Sowers, *Random travelling waves for the KPP equation with noise* J. Funct. Anal. **128**, (1995), p. 439-498. MR1319963

[21] R.L. Pego, M.I. Weinstein, *Asymptotic stability of solitary waves*, Commun. Math. Phys. **164**, (1994), p. 305-349. MR1289328

[22] J. Printems, Aspects Théoriques et numériques de l'équation de Korteweg-de Vries stochastique, thèse, Université Paris-Sud, Orsay, France, 1998.

[23] M. Scalerandi, A. Romano, C.A. Condat, *Korteweg-de Vries solitons under additive stochastic perturbations*, Phys. Review E **58**, (1998), p. 4166-4173.

[24] P.C. Schuur, Asymptotic analysis of soliton problems, Lect. Notes in Math. 1232, Berlin, New York, Springer, 1986. MR0874343

[25] M. Wadati, *Stochastic Korteweg-de Vries equations*, J. Phys. Soc. Japan **52**, (1983), p. 2642-2648. MR0722214