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Asymptotic evolution of acyclic random mappings

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Abstract

An acyclic mapping from an n element set into itself is a mapping φ such that if $\varphi^k(x) = x$ for some k and x, then $\varphi(x) = x$. Equivalently, $\varphi^\ell = \varphi^{\ell+1} = \dots$ for ℓ sufficiently large. We investigate the behavior as $n \to \infty$ of a sequence of a Markov chain on the collection of such mappings. At each step of the chain, a point in the n element set is chosen uniformly at random and the current mapping is modified by replacing the current image of that point by a new one chosen independently and uniformly at random, conditional on the resulting mapping being again acyclic. We can represent an acyclic mapping as a directed graph (such a graph will be a collection of rooted trees) and think of these directed graphs as metric spaces with some extra structure. Informal calculations indicate that the metric space valued process associated with the Markov chain should, after an appropriate time and "space" rescaling, converge as $n \to \infty$ to a real tree ($\mathbb{R}$-tree) valued Markov process that is reversible with respect to a measure induced naturally by the standard reflected Brownian bridge. Although we don't prove such a limit theorem, we use Dirichlet form methods to construct a Markov process that is Hunt with respect to a suitable Gromov-Hausdorff-like metric and evolves according to the dynamics suggested by the heuristic arguments. This process is similar to one that appears in earlier work by Evans and Winter as a similarly informal limit of a Markov chain related to the subtree prune and regraft tree (SPR) rearrangements from phylogenetics .

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1 Introduction

A mapping φ from the set $[n] := \{1, 2, \ldots, n\}$ into itself may be represented as a directed graph with vertex set $[n]$ and directed edges of the form $(i, \varphi(i))$, $i \in [n]$. The resulting directed graph has the feature that every vertex has out-degree 1 (with self-loops – corresponding to fixed points – allowed), and any such graph corresponds to a unique mapping. For example, the mapping $\varphi : [18] \to [18]$ in Table 1 corresponds to the directed graph in Figure 1.

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\varphi(i)$	10	3	18	10	9	2	8	4	3	7	9	2	1	9	15	1	1	9

Table 1: A mapping from [18] into itself.

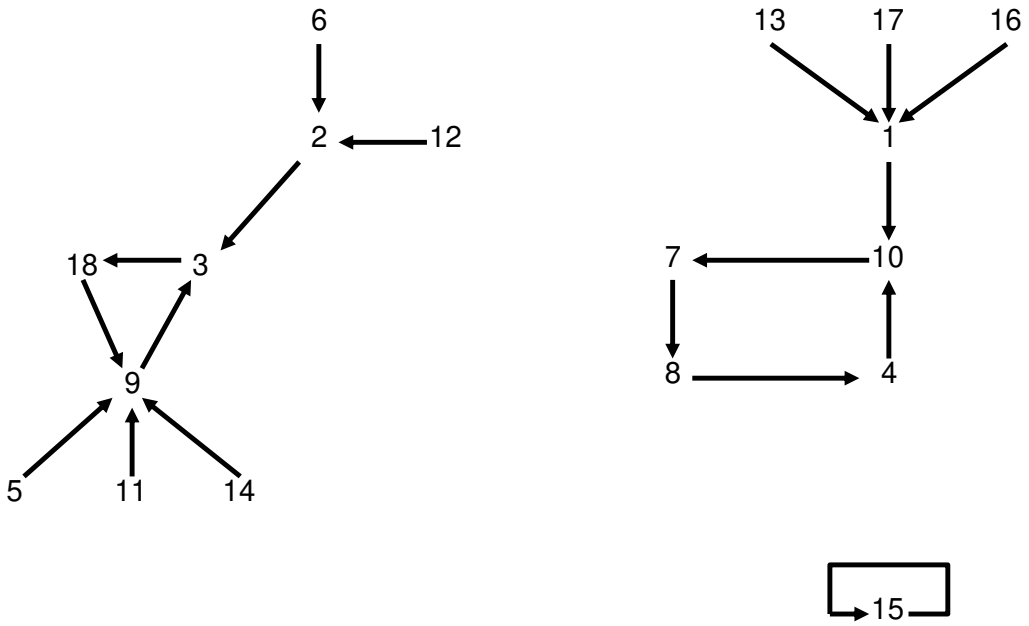

Figure 1: The directed graph corresponding to the mapping in Table 1.

The directed graph may be decomposed into a number of connected components. Each of these components consists of a single directed cycle (possibly a self-loop) plus trees rooted at each vertex on the directed cycle (such a tree may be a trivial tree consisting of only the root,

meaning that the only pre-image of that point is its predecessor on the directed cycle). We call such rooted trees the *tree components* of the graph.

Aldous and Pitman (AP94) describe a procedure for associating a mapping of $[n]$ into itself with a *lattice reflected bridge path* of length $2n$, that is, with a function $b : \{0, 1, \ldots, 2n\} \to \{0, 1, 2, \ldots\}$ such that $b(0) = b(2n) = 0$ and $|b(k+1) - b(k)| = 1$ for $0 \leq k < 2n$. The exact details of the procedure aren't important for us. However, we note that a tree component with ℓ vertices corresponds to a lattice positive excursion path from 0 with 2ℓ steps. Such a segment of path records the distance from the root plus 1 in a depth-first-search of the tree component. For example, the tree component of size 5 consisting of the vertices $\{1, 10, 13, 16, 17\}$ in Figure 1 corresponds to the excursion shown in Figure 2 after a suitable translation of the time axis. In

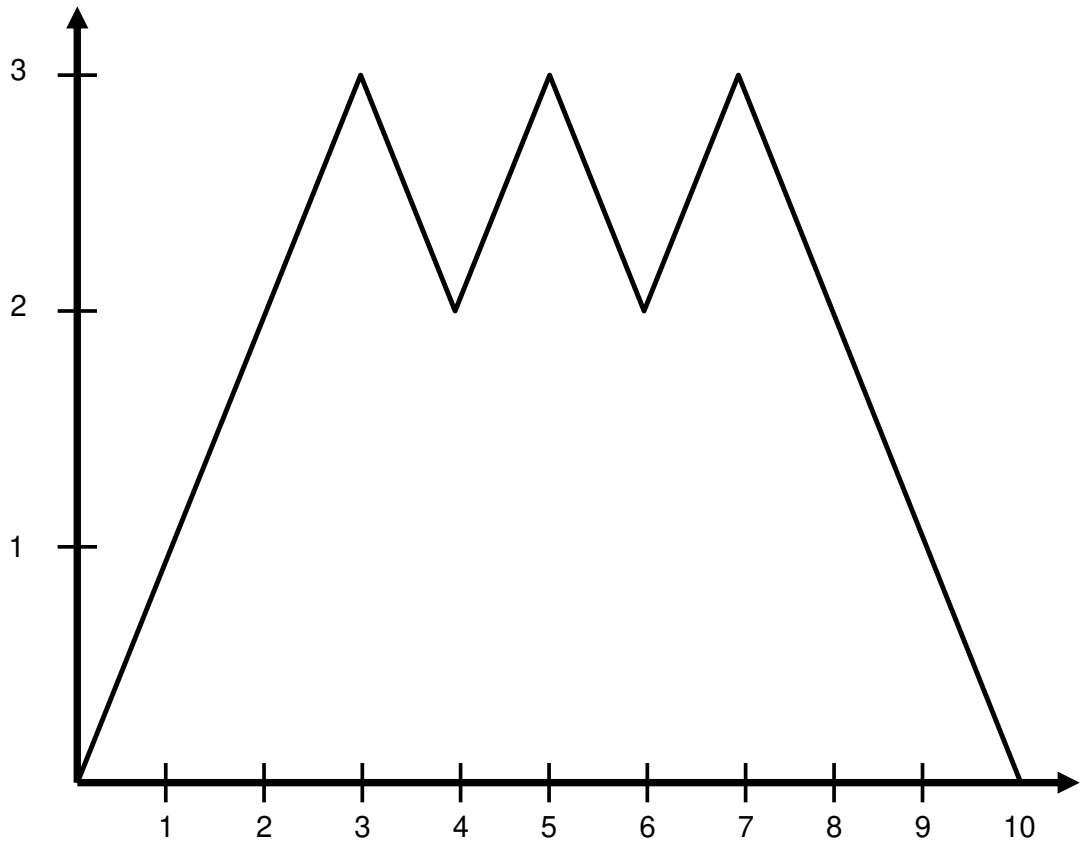

Figure 2: The excursion corresponding to the tree component rooted at vertex 10 in Figure 2 with the start of the excursion shifted to time 0.

particular, a tree component that consists of just one point (which is necessarily a point on a directed cycle) corresponds to an excursion of the form $b(k-1) = 0$, $b(k) = 1$, and $b(k+1) = 0$.

Of course, the mapping cannot be recovered from just the lattice reflected bridge path. For one thing, some extra marking of distinguished points of the zero set of the lattice path is required to split the lattice path up into sub-paths corresponding to components of the directed graph.

Once this is done, the mapping is uniquely specified by the lattice path up to a relabeling of the vertices: that is, if two mappings φ and ψ correspond to the same lattice reflected bridge path, then $\psi = \pi \circ \varphi \circ \pi^{-1}$ for some permutation π of $[n]$. Conversely, if $\psi = \pi \circ \varphi \circ \pi^{-1}$ for some permutation π of $[n]$, then the lattice path corresponding to ψ may be obtained from the lattice path corresponding to φ by composition with a bijective map of $\{0, 1, \ldots, 2n\}$ that preserves lengths of excursions above all levels. That is, if the lattice path corresponding to φ has k excursions above some level h, then the same is true of the lattice path corresponding to ψ.

Suppose now that a mapping of $[n]$ into itself is chosen uniformly at random from the n^n possibilities. This is equivalent to choosing the image of each point of $[n]$ independently and uniformly at random from $[n]$. The corresponding lattice reflected bridge path is not uniformly distributed. However, it is shown in (AP94) that if the lattice reflected bridge path is turned into a continuous time process by holding it constant between integer time points, time is rescaled by $2n$, and space is rescaled by $n^{\frac{1}{2}}$ to produce a function from $[0, 1]$ into $\mathbb{R}_+$, then this stochastic process with càdlàg sample paths converges in distribution to twice a standard reflected Brownian bridge (that is, twice the Brownian bridge reflected at 0 that goes from position 0 at time 0 to position 0 again at time 1). In particular, the proportion of vertices that lie on directed cycles converges to the proportion of time the standard reflected Brownian bridge spends at 0, which is, of course, 0, so that asymptotically almost all vertices are not roots of tree components. The asymptotics of the cyclic vertices jointly with the tree vertices are described in (AP94) using the local time at 0 of the reflected bridge and that paper also describes an auxiliary "marking" procedure for describing the joint asymptotics of the the component sizes. Some later results in this same vein may be found in (AMP05; AP02; DS97; DG99; GL00; DG04; Pit02).

A mapping φ from $[n]$ into itself is *acyclic* if the only directed cycles in the corresponding directed graph are self-loops. That is, each $x \in [n]$ is either a fixed point of φ (so that x is a vertex on a self-loop) or $\varphi^k(x) \neq x$ for any k. Equivalently, $\varphi^\ell = \varphi^{\ell+1} = \ldots$ for ℓ sufficiently large. For such a mapping, each graph component consists of single tree component with a self-loop attached to the root, and no auxiliary marking procedure is necessary to recover the mapping up to a permutation from the corresponding lattice reflected bridge path. It is not hard to show that if we turn the lattice reflected bridge path for a uniformly chosen acyclic random mapping into a continuous time process indexed by $[0, 1]$ as above, then the resulting process also converges to twice a standard reflected Brownian bridge – as one would expect from the observation that the cyclic vertices are asymptotically negligible for a uniformly chosen random mapping,

In this paper we are interested in the asymptotic behavior as $n \to \infty$ of a simple Markov chain that randomly evolves an acyclic mapping from $[n]$ into itself. At each step of the chain, a point of $[n]$ is chosen uniformly at random and the image of this point is re-set to a new image chosen independently and uniformly at random from $[n]$, conditional on the resulting mapping being acyclic. It is clear for each n that this chain is reversible with respect to the uniform distribution on the set of acyclic mappings from $[n]$ into itself and that the chain converges to this distribution at large times.

In terms of the corresponding directed graphs, the chain evolves as follows. A directed edge is first chosen uniformly at random and deleted. The deleted edge is then replaced by another directed edge with the same initial vertex but a uniformly chosen final vertex, conditional on the resulting graph having no cycles other than self-loops. Note that the effect of such a step is the following.

- If the deleted edge is a self-loop, its deletion turns the graph component that contained the edge into a rooted subtree. Otherwise, the deletion of the directed edge splits the graph component that contained it into two pieces, one of which contains a self-loop and the other of which is a rooted subtree.
- In either case, the addition of the new directed edge either attaches the root of the subtree to itself by a self-loop, producing an extra graph component, or the new directed edge attaches the root to a vertex chosen uniformly outside the subtree (possibly to a vertex outside the subtree but within the same former graph component). All such possibilities are equally likely.

The effect on the corresponding lattice bridge path is to remove an excursion above some level, insert a suitable time-space translation of it at some time point in the lattice bridge path outside the excursion, and then close up the gap left by the removal (more precisely, this transformation may need to be followed by a bijective map of $\{0, 1, \ldots, 2n\}$ that preserves lengths of excursions above all levels because of the way that the labeling of vertices in the directed graph is used to construct the corresponding lattice bridge path).

In order to understand the asymptotic behavior of this sequence of chains as $n \to \infty$, we need to embed the state space of each chain into a common state space that will also be the state space of the limit process.

To begin with, we erase all of the self-loops in the directed graph corresponding to an acyclic mapping. This produces a forest of subtrees rooted at vertices that were formerly on self-loops. We then connect the roots of these subtrees by directed edges to a single adjoined point to produce a tree rooted at the adjoined point. Keeping in mind the rescaling identified by Aldous and Pitman, we think of this tree as a one-dimensional cell complex by regarding each edge as a segment of length $n^{-\frac{1}{2}}$. We thus have a metric space with a distinguished base point (the root). This pointed metric space is an instance of a *rooted compact real tree ($\mathbb{R}$-tree)*: see Section 2 for the precise definition of a $\mathbb{R}$-tree – for the moment, all that is important for explaining our results is that a $\mathbb{R}$-tree is a metric space that is, in some sense, "tree-like". We regard two rooted compact $\mathbb{R}$-trees as being equal if one can be mapped into the other by an isometry that preserves the root. If two mappings φ and ψ are related by a relabeling $\psi = \pi \circ \varphi \circ \pi^{-1}$ for some permutation π of $[n]$, then they correspond to the same rooted compact $\mathbb{R}$-tree.

Before we continue with the motivation of our results, we need to indicate how the rooted compact $\mathbb{R}$-tree associated with a mapping from $[n]$ into itself may be constructed directly from the corresponding lattice reflected bridge path. We begin by introducing some general notation that is useful later.

Definition 1.1. Write $C(\mathbb{R}_+, \mathbb{R}_+)$ for the space of continuous functions from $\mathbb{R}_+$ into $\mathbb{R}_+$. For $f \in C(\mathbb{R}_+, \mathbb{R}_+)$, put

$$\zeta(f) := \inf\{s > 0 : f(t) = 0 \text{ for all } t > s\}$$

with the usual convention that $\inf \emptyset = \infty$. The set of *positive bridge paths* is the set $\Omega_+ \subset C(\mathbb{R}_+, \mathbb{R}_+)$ given by

$$\Omega_+ := \left\{ f \in C(\mathbb{R}_+, \mathbb{R}_+) : \begin{array}{c} f(0) = 0,\ 0 < \zeta(f) < \infty, \\ f(t) \geq 0 \text{ for } 0 < t < \zeta(f). \end{array} \right\}$$

For $\ell > 0$, set $\Omega_+^\ell := \{f \in \Omega_+ : \zeta(f) = \ell\}$.

We associate each $f \in \Omega_+^1$ with a compact metric space as follows. Define an equivalence relation $\sim_f$ on $[0,1]$ by letting

$$u_1 \sim_f u_2, \qquad \text{iff} \qquad f(u_1) = \inf_{u \in [u_1 \wedge u_2, u_1 \vee u_2]} f(u) = f(u_2).$$

Consider the pseudo-metric d_{T_f} on $[0,1]$ defined by

$$d_{T_f}(u_1, u_2) := f(u_1) - 2 \inf_{u \in [u_1 \wedge u_2, u_1 \vee u_2]} f(u) + f(u_2).$$

This pseudo-metric becomes a true metric on the quotient space $T_f := [0,1]/_{\sim_f}$. The resulting metric space is compact and is an instance of a rooted compact $\mathbb{R}$-tree if we define the root to be the image of 0 under the quotient map.

Suppose that the function $f \in \Omega_+^1$ is obtained by first linearly interpolating the lattice reflected bridge path associated with an acyclic mapping φ of $[n]$ into itself to produce a function in Ω_+^{2n} and then rescaling time by $2n$ and space by $n^{\frac{1}{2}}$. The corresponding pointed metric space T_f is the rooted compact $\mathbb{R}$-tree associated with φ that we described above.

Any metric space of the form T_f for $f \in \Omega_+^1$ has two natural Borel measure on it. Firstly, there is the "uniform" probability measure ν_{T_f} given by the push-forward of Lebesgue measure on $[0,1]$ by the quotient map. We call this measure the *weight* on T_f. Secondly, there is the natural *length* measure μ_{T_f}, which is the one-dimensional Hausdorff measure associated with the metric d_{T_f} restricted to points of T_f that are not "leaves" (see Section 2 for a more precise definition). When f is associated with a map of $[n]$ into itself as above, then μ_{T_f} is just the "Lebesgue measure" on the cell complex T_f that assigns mass $n^{-\frac{1}{2}}$ to each edge of T_f (recall that we have rescaled so that each edge has length $n^{-\frac{1}{2}}$).

Now, if we speed up time by a factor of $n^{\frac{1}{2}}$ in our Markov chain for evolving mappings of $[n]$ into itself and look at the corresponding rooted compact $\mathbb{R}$-tree-valued process, then it is reasonable at the heuristic level that we should obtain in the limit as $n \to \infty$ a continuous time Markov process with the following informal description. The state space of the limit process is the space consisting of rooted compact $\mathbb{R}$-trees T equipped with a probability measure ν_T: we call such objects weighted rooted compact $\mathbb{R}$-trees. We note that, as in the special case of $\mathbb{R}$-trees of the form T_f for $f \in \Omega_+^1$, an arbitrary compact $\mathbb{R}$-tree has a canonical length measure μ_T given by the restriction of the one-dimensional Hausdorff measure associated with the metric to the set of points that aren't leaves. The process evolves away from its state at time 0 by choosing a point (t, v) at rate $dt \otimes \mu_T(dv)$ in time and on the current tree T, and at time t the subtree above v (that is, the subtree of points on the other side of v from the root) is re-attached at a point w chosen according to ν_T (conditional on w being outside the subtree).

In general, the measure μ_T may have infinite total mass. For example, if $f \in \Omega_+^1$ is chosen according to the distribution of standard reflected Brownian bridge, so that T_{2f} is the rooted compact $\mathbb{R}$-tree that arises from a limit as $n \to \infty$ of uniform acyclic random mappings of $[n]$ into itself, then $\mu_{T_{2f}}$ almost surely has infinite total mass. Consequently, the above specification of the dynamics of the limit process does not make rigorous sense for general weighted rooted compact $\mathbb{R}$-trees.

The aim of this paper is to use Dirichlet form methods to construct a suitably well-behaved Markov process with evolution dynamics that conform to the heuristic description. Our main

result in this direction is stated precisely as Theorem 7.5 at the end of the paper after we have developed the necessary background and notation to describe the result and the requisite technical machinery to prove it.

We stress that we do not obtain a convergence result. The process we construct has no obvious Feller-like properties and it is not clear how to define its dynamics for all starting points (as opposed to almost all starting points with respect to the symmetrizing measure, which is all the Dirichlet form approach provides). Consequently, it is not clear how standard techniques such as martingale problem methods might be used to establish convergence.

The process we construct is somewhat similar to the process constructed in (EW06) as the heuristic limit of a sequence of natural chains based on the *subtree prune and regraft (SPR)* tree rearrangement transformations from phylogenetics. Both processes involve the relocation of a subtree whose root is chosen according to the length measure on the current tree. However, the state space of the process in (EW06) consists of weighted unrooted compact $\mathbb{R}$-trees, whereas we work with weighted rooted compact $\mathbb{R}$-trees and the root plays a crucial role in defining the dynamics. The symmetrizing measures are, as a consequence, rather different: the measure in (EW06) is the distribution of the Brownian continuum random tree, which is the $\mathbb{R}$-tree "inside" twice a standard Brownian excursion, whereas our symmetrizing measure is the distribution of the $\mathbb{R}$-tree "inside" twice a standard reflected Brownian bridge. However, many of the steps in the construction are quite similar so we omit several arguments and simply refer to the analogous ones in (EW06).

We note that Markov processes with reflected bridge paths as their state space and continuous sample paths have been studied in (Zam03; Zam02; Zam01). These processes are reversible with respect to the distribution of a Bessel bridge of some index.

2 Weighted $\mathbb{R}$-trees

Definition 2.1. A metric space (T, d) is a *real tree* ($\mathbb{R}$-tree) if it satisfies the following axioms.

Axiom 0: The space (T, d) is complete.

Axiom 1: For all $x, y \in T$ there exists a unique isometric embedding $\phi_{x,y} : [0, d(x, y)] \to T$ such that $\phi_{x,y}(0) = x$ and $\phi_{x,y}(d(x, y)) = y$.

Axiom 2: For every injective continuous map $\psi : [0, 1] \to T$ one has $\psi([0, 1]) = \phi_{\psi(0),\psi(1)}([0, d(\psi(0), \psi(1))])$.

Axiom 1 says simply that there is a unique "unit speed" path between any two points x and y. We write $[x, y]$ for the image of this path and call it the *segment* with *endpoints* x and y. Axiom 2 implies that the image of any injective path connecting two points x and y coincides with the segment $[x, y]$, and so such a path may be re-parameterized to become the unit speed path. Thus, while Axiom 1 is satisfied by many other spaces such as $\mathbb{R}^d$ with the usual metric, Axiom 2 captures the essence of "treeness" and is only satisfied by $\mathbb{R}^d$ when $d = 1$. See (Dre84; DT96; DMT95; DMT96; Ter97; Chi01) for background on $\mathbb{R}$-trees. In particular, (Chi01) shows that a number of other definitions are equivalent to the one above. Much of this content is synthesized and combined with other material on probability on $\mathbb{R}$-trees in (Eva07). Also, some probabilistic aspects of $\mathbb{R}$-trees are reviewed in (LG06).

We define the *η-trimming*, $R_\eta(T)$ of a compact $\mathbb{R}$-tree (T, d) for $\eta > 0$ to be the set of points $x \in T$ such that x belongs to a segment $[y, z]$ with $d(x, y) = d(x, z) = \eta$ – see Figure 3. The *skeleton* of (T, d) is the set $T^o := \bigcup_{\eta>0} R_\eta(T)$. Thus $x \in T^o$ if $x \in]y, z[$ for some y, z. The *leaf set* of (T, d) is the set $T \backslash T^o$. The *length measure* on T is the σ-finite measure μ_T on the Borel σ-field $\mathcal{B}(T)$ given by the trace onto T^o of the one-dimensional Hausdorff measure associated with d. Equivalently, μ_T is the unique measure concentrated on T^o such that $\mu_T([x, y]) = d(x, y)$ for all $x, y \in T$ (see Section 2.4 of (EPW06) or Section 2 of (EW06)).

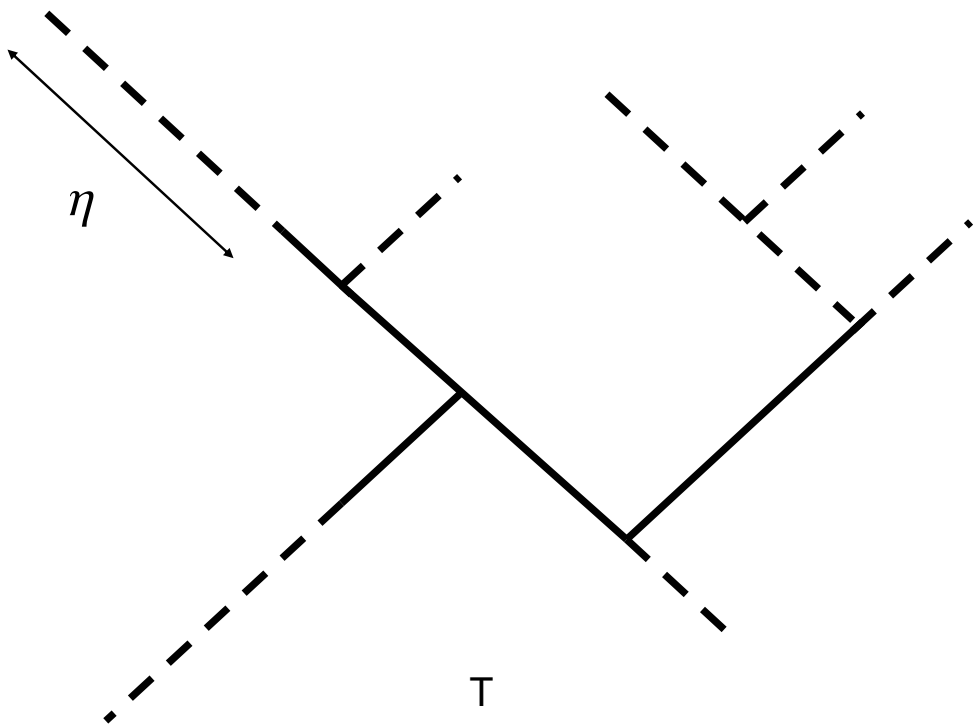

Figure 3: A $\mathbb{R}$-tree T and its η-trimming $R_\eta(T)$. The $\mathbb{R}$-tree T consists of both the solid and dashed segments, whereas the $\mathbb{R}$-tree $R_\eta(T)$ consists of just the solid segments.

In the following, we are interested in compact $\mathbb{R}$-trees (T, d) equipped with a distinguished base point $\rho \in T$ (called the *root*) and a probability measure ν on the Borel σ-field $\mathcal{B}(T)$ (called the *weight*). We call such objects *weighted rooted compact* $\mathbb{R}$*-trees*. We say that two weighted rooted compact $\mathbb{R}$-trees (X, d_X, ρ_X, ν_X) and (Y, d_Y, ρ_Y, ν_Y) are *weighted rooted isometric* if there exists a bijective isometry Φ between the metric spaces (X, d_X) and (Y, d_Y) such that $\Phi(\rho_X) = \rho_Y$ and the *push-forward* of ν_X by Φ is ν_Y, that is,

$$\nu_Y = \Phi_* \nu_X := \nu_X \circ \Phi^{-1}.$$

The property of being weighted rooted isometric is an equivalence relation. We write $\mathbf{T}^{\mathrm{wr}}$ for the collection of equivalence classes of weighted rooted compact $\mathbb{R}$-trees.

In order to define a metric on $\mathbf{T}^{\mathrm{wr}}$, we first recall the definition of the Prohorov distance between two probability measures (see, for example, (EK86)). Given two probability measures α and β on a metric space (X, d) with the corresponding collection of closed sets denoted by $\mathcal{C}$, the Prohorov distance between them is

$$d_{\mathrm{P}}(\alpha, \beta) := \inf\{\varepsilon > 0 : \alpha(C) \leq \beta(C^{\varepsilon}) + \varepsilon \text{ for all } C \in \mathcal{C}\},$$

where $C^{\varepsilon} := \{x \in X : \inf_{y \in C} d(x, y) < \varepsilon\}$. The Prohorov distance is a metric on the collection of probability measures on X.

We are now in a position to define the weighted rooted Gromov-Hausdorff distance between the two weighted rooted compact $\mathbb{R}$-trees (X, d_X, ρ_X, ν_X) and (Y, d_Y, ρ_Y, ν_Y).

For $\varepsilon > 0$, let $F^{\varepsilon}_{X,Y}$ denote the set of Borel maps $f : X \to Y$ such that $f(\rho_X) = \rho_Y$ and

$$\sup\{|d_X(x', x'') - d_Y(f(x'), f(x''))| : x', x'' \in X\} \leq \varepsilon,$$

and define $F^{\varepsilon}_{Y,X}$ similarly. Put

$$\begin{aligned}&\Delta_{\mathrm{GH}^{\mathrm{wr}}}(X, Y)\\ &:= \inf\left\{\varepsilon > 0 : \begin{array}{c}\text{exist } f \in F^{\varepsilon}_{X,Y}, g \in F^{\varepsilon}_{Y,X} \text{ such that}\\ d_{\mathrm{P}}(f_*\nu_X, \nu_Y) \leq \varepsilon,\, d_{\mathrm{P}}(\nu_X, g_*\nu_Y) \leq \varepsilon\end{array}\right\}.\end{aligned}$$

Note that the set on the right hand side is non-empty because X and Y are compact, and hence bounded in their respective metrics. Note also that $\Delta_{\mathrm{GH}^{\mathrm{wr}}}(X, Y)$ only depends on the weighted rooted isometry classes of X and Y.

It turns out that the function $\Delta_{\mathrm{GH}^{\mathrm{wr}}}$ satisfies all the properties of a metric except the triangle inequality. To rectify this, put

$$d_{\mathrm{GH}^{\mathrm{wr}}}(X, Y) := \inf\left\{\sum_{i=1}^{n-1} \Delta_{\mathrm{GH}^{\mathrm{wr}}}(Z_i, Z_{i+1})^{\frac{1}{4}}\right\},$$

where the infimum is taken over all finite sequences of weighted rooted compact $\mathbb{R}$-trees $Z_1, \ldots Z_n$ with $Z_1 = X$ and $Z_n = Y$ (the exponent $\frac{1}{4}$ is not particularly important, any sufficiently small number would suffice). Note again that $d_{\mathrm{GH}^{\mathrm{wr}}}(X, Y)$ only depends on the weighted rooted isometry classes of X and Y.

From now on, we think of $\Delta_{\mathrm{GH}^{\mathrm{wr}}}$ and $d_{\mathrm{GH}^{\mathrm{wr}}}$ as being defined on $\mathbf{T}^{\mathrm{wr}} \times \mathbf{T}^{\mathrm{wr}}$. Parts (i) and (ii) of the following result are analogous to Lemma 2.3 of (EW06), part (iv) is analogous to Proposition 2.4 of (EW06), part (v) is a re-statement of Lemma 2.6 of (EW06), and part (vi) is analogous to Theorem 2.5 of (EW06). The results in (EW06) are for $\mathbb{R}$-trees with weights but without roots, but the addition of roots does not present any new difficulties (*cf.* the passage from $\mathbb{R}$-trees without weights or roots to $\mathbb{R}$-trees without weights but with roots in Section 2.3 of (EPW06)). The space $\mathbf{T}$ in part (iv) is the collection of isometry classes of compact $\mathbb{R}$-trees (without weights or roots) and we refer the reader to Section 2.1 of (EPW06) for the definition of the associated Gromov-Hausdorff distance d_{GH}.

Proposition 2.2. *(i) The map $\Delta_{\mathrm{GH}^{\mathrm{wr}}}$ has the properties:*

(a) $\Delta_{\mathrm{GH}^{\mathrm{wr}}}(X,Y) = 0$ if and only if $X = Y$,

(b) $\Delta_{\mathrm{GH}^{\mathrm{wr}}}(X,Y) = \Delta_{\mathrm{GH}^{\mathrm{wr}}}(Y,X)$.

(ii) The map $d_{\mathrm{GH}^{\mathrm{wr}}}$ is a metric on $\mathbf{T}^{\mathrm{wr}}$.

(iii) For all $X, Y \in \mathbf{T}^{\mathrm{wr}}$,

$$\frac{1}{2}\Delta_{\mathrm{GH}^{\mathrm{wr}}}(X,Y)^{\frac{1}{4}} \leq d_{\mathrm{GH}^{\mathrm{wr}}}(X,Y) \leq \Delta_{\mathrm{GH}^{\mathrm{wr}}}(X,Y)^{\frac{1}{4}}.$$

(iv) A subset D of $(\mathbf{T}^{\mathrm{wr}}, d_{\mathrm{GH}^{\mathrm{wr}}})$ is relatively compact if and only if the subset $E := \{(T,d) : (T,d,\rho,\nu) \in D\}$ of $(\mathbf{T}, d_{\mathrm{GH}})$ is relatively compact.

(v) A subset E of $(\mathbf{T}, d_{\mathrm{GH}})$ is relatively compact if and only if

$$\sup\{\mu_T(R_\eta(T)) : T \in E\} < \infty$$

for all $\eta > 0$.

(vi) The metric space $(\mathbf{T}^{\mathrm{wr}}, d_{\mathrm{GH}^{\mathrm{wr}}})$ is complete and separable.

We note that an extensive study of spaces of metric spaces equipped with measures is given in (Stu06a; Stu06b), and the theory of weak convergence for random variables taking values in such spaces is developed in (GPW06).

3 Trees and continuous paths

Definition 3.1. The space of *positive excursion paths* is the set $\Omega_{++} \subset \Omega_+ \subset C(\mathbb{R}_+, \mathbb{R}_+)$ given by

$$\Omega_{++} := \left\{ f \in C(\mathbb{R}_+, \mathbb{R}_+) : \begin{array}{c} f(0) = 0,\ 0 < \zeta(f) < \infty, \\ f(t) > 0 \text{ for } 0 < t < \zeta(f). \end{array} \right\}$$

For $\ell > 0$, set $\Omega_{++}^{\ell} := \{f \in \Omega_{++} : \zeta(f) = \ell\}$.

The following result is a slight generalization of Lemma 3.1 in (EW06). The latter result was for the special case of $\mathbb{R}$-trees constructed from positive excursion paths rather than general positive bridge paths. The proof goes through unchanged.

Lemma 3.2. *For each $f \in \Omega_+^1$, the metric space (T_f, d_{T_f}) is a compact $\mathbb{R}$-tree.*

We root a $\mathbb{R}$-tree (T_f, d_{T_f}) coming from a positive bridge path in $f \in \Omega_+^1$ by taking the root to be the point corresponding to $0 \in [0,1]$ under the quotient map. We equip (T_f, d_{T_f}) with the weight ν_{T_f} given by the push-forward of Lebesgue measure on $[0,1]$ by the quotient map.

For a positive bridge path $f \in \Omega^1_+$, we identify the length measure μ_{T_f} on the associated compact $\mathbb{R}$-tree (T_f, d_{T_f}) as follows (the discussion is essentially the same as that in Section 3 of (EW06) which considered $\mathbb{R}$-trees coming from positive excursion paths). For $a \geq 0$, let

$$\mathcal{G}(f,a) := \left\{ s \in [0,1] : \begin{array}{c} f(s) = a \text{ and, for some } t > s, \\ f(r) > a \text{ for all } r \in]s,t[, \\ f(t) = a. \end{array} \right\} \tag{1}$$

denote the countable set of starting points of excursions of the function f above the level a – see Figure 4. Then, the length measure μ_{T_f} is the push-forward of the measure

$$m_f := \int_0^\infty \mathrm{d}a \sum_{t \in \mathcal{G}(f,a)} \delta_t \tag{2}$$

by the quotient map, where δ_t is the unit point mass at t.

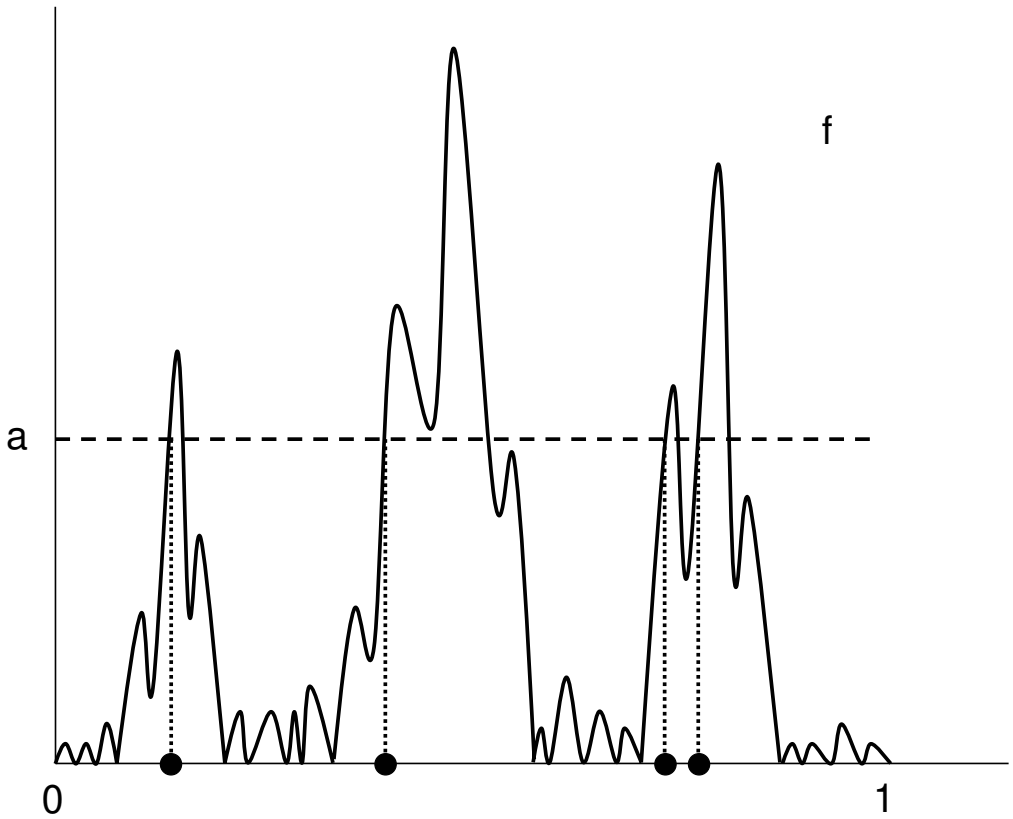

Figure 4: The set $\mathcal{G}(f,a)$ for the reflected bridge path f and the level a is indicated by the four dots.

Alternatively, write

$$\Gamma(f) := \{(s,a) : \ s \in]0,1[, \ a \in [0, f(s)[\}$$

for the region between the time axis and the graph of f, and for $(s,a) \in \Gamma(f)$ denote by

$$\underline{s}(f,s,a) := \sup\{r < s : f(r) = a\} \tag{3}$$

and

$$\bar{s}(f,s,a) := \inf\{t > s : f(t) = a\} \tag{4}$$

the start and finish of the excursion of e above level a that straddles time s. Then,

$$m_f = \int_{\Gamma(f)} \mathrm{d}s \otimes \mathrm{d}a \, \frac{1}{\bar{s}(f,s,a) - \underline{s}(f,s,a)} \delta_{\underline{s}(f,s,a)}. \tag{5}$$

4 A path transformation connecting reflected Brownian bridge and Brownian excursion

Write $\mathbb{P}_+$ for the law of the standard Brownian bridge reflected at 0 that goes from 0 at time 0 to 0 at time 1. Write $\mathbb{P}_{++}$ for the law of standard Brownian excursion. Thus, $\mathbb{P}_+$ is a probability measure on Ω^1_+ and $\mathbb{P}_{++}$ is a probability measure on Ω^1_{++}. We show in this section how various computations for $\mathbb{P}_+$ can be reduced to computations for $\mathbb{P}_{++}$ using a result of Bertoin and Pitman.

Given $f \in \Omega^\ell_+$, put

$$L(t;f) := \begin{cases} \limsup_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_{[0,t]} ds \, \mathbf{1}\{f(s) < \varepsilon\}, & \text{if } \limsup_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_{[0,\ell]} ds \, \mathbf{1}\{f(s) < \varepsilon\} \\ & < \infty, \\ 0, & \text{otherwise,} \end{cases}$$

for $0 \le t \le \ell$, and set $L(t;f) = L(\ell;f)$ for $t \ge \ell$.

Denote by $\tilde{\Omega}^\ell_+$ the subset of Ω^ℓ_+ consisting of functions f with the properties:

- the closed set $\{t \in [0,\ell] : f(t) = 0\}$ is perfect (that is, has no isolated points) and has Lebesgue measure zero;
- for $0 \le t \le \ell$,
$$L(t;f) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_{[0,t]} ds \, \mathbf{1}\{f(s) < \varepsilon\};$$
- the function $t \mapsto L(t;f)$ is continuous;
- the set of points of increase of the function $t \mapsto L(t;f)$ coincides with $\{t \in [0,\ell] : f(t) = 0\}$.

Note that if $f \in \tilde{\Omega}^\ell_+$, then $L(\cdot;f)$ is not identically 0 (indeed, $L(\cdot;f)$ has 0 as a point of increase). Of course, $\mathbb{P}_+(\tilde{\Omega}^1_+) = 1$.

For $f \in \Omega^\ell_+$, set

$$U(f) := \sup\left\{0 \le t \le \ell : L(t;f) \le \frac{1}{2} L(\ell;f)\right\}$$

and put

$$K^{\rightarrow}(t;f) := \begin{cases} L(t;f), & 0 \le t \le U(f), \\ L(\ell;f) - L(t;f), & U(f) \le t \le \ell, \\ 0, & t \ge \ell. \end{cases}$$

For $f \in \Omega_+^\ell$ and $u \in [0,\ell]$, set

$$K^{\leftarrow}(t;f,u) := \begin{cases} \min_{t \le s \le u} f(s), & 0 \le t \le u, \\ \min_{u \le s \le t} f(s), & u \le t \le \ell, \\ 0, & t \ge \ell. \end{cases}$$

The following result is elementary and we leave the proof to the reader. The construction it describes is illustrated in Figure 5.

Lemma 4.1. *Fix a function $f \in \tilde{\Omega}_+^\ell$. Set*

$$e = K^{\rightarrow}(\cdot;f) + f.$$

Then, $e \in \Omega_{++}^\ell$ and

$$f = K^{\leftarrow}(\cdot;e,U(f)).$$

The next result, which is Lemma 3.3 of (BP94), says that under $\mathbb{P}_+$ the path-valued random variable $f \mapsto K^{\rightarrow}(\cdot;f) + f$ has law $\mathbb{P}_{++}$, the random variable $f \mapsto U(f)$ is uniformly distributed on $[0,1]$, and these two random variables are independent.

Proposition 4.2. *For any Borel function $F : \Omega_+^1 \times [0,1] \to \mathbb{R}_+$,*

$$\int \mathbb{P}_+(\mathrm{d}f)\, F(K^{\rightarrow}(\cdot;f) + f, U(f)) = \int \mathbb{P}_{++}(\mathrm{d}e) \int_{[0,1]} \mathrm{d}u\ F(e,u).$$

In order to apply Proposition 4.2, we need to understand for a fixed positive bridge path $f \in \tilde{\Omega}_+^1$ how the measure m_f of (2) or (5) is related to the analogous measure for the associated positive excursion path $K^{\rightarrow}(\cdot;f) + f \in \Omega_{++}^1$.

Definition 4.3. For $e \in \Omega_{++}^1$ and $u \in [0,1]$, write

$$\Gamma^*(e,u) := \{(s,a) \in \Gamma(e) : u \notin [\underline{s}(e,s,a), \bar{s}(e,s,a)]\}$$

for the set of points in $\Gamma(e)$ such that the corresponding straddling sub-excursion does not straddle the time u – see Figure 6.

Lemma 4.4. *Fix $f \in \tilde{\Omega}_+^1$. Set $e = K^{\rightarrow}(\cdot;f) + f \in \Omega_{++}^1$, so that*

$$\Gamma^*(e,U(f)) = \{(s,a) : s \in]0,1[,\ K^{\rightarrow}(s;f) \le a < K^{\rightarrow}(s;f) + f(s)\}.$$

Define a bijection $\xi : \Gamma(f) \to \Gamma^*(e,U(f))$ by setting

$$\xi(s,a) := (s, a + K^{\rightarrow}(s;f)).$$

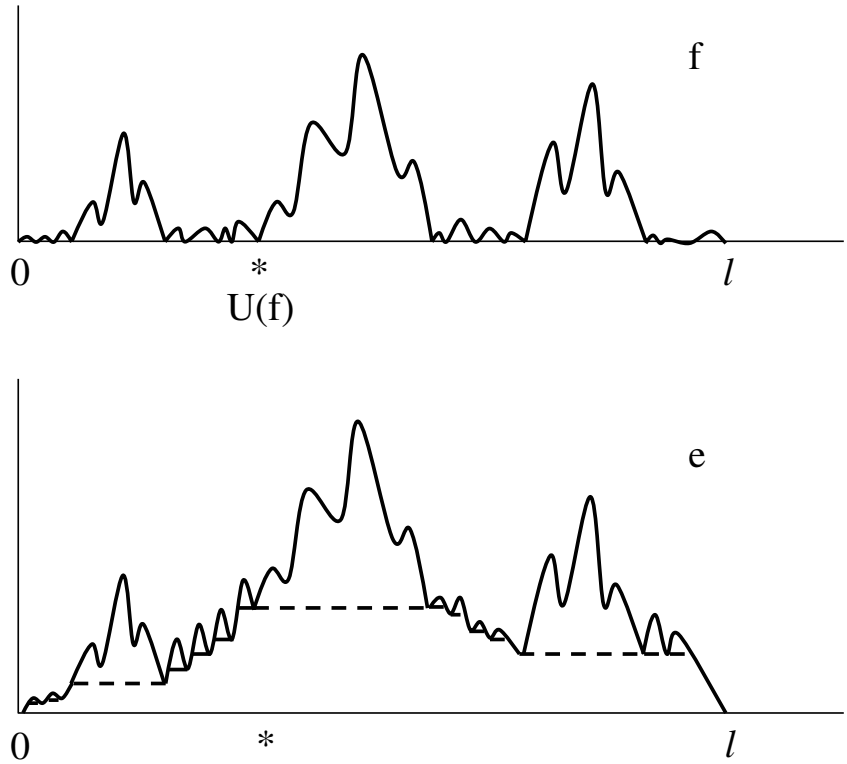

Figure 5: The mapping of Lemma 4.1 between a reflected bridge path f and an excursion paths e.

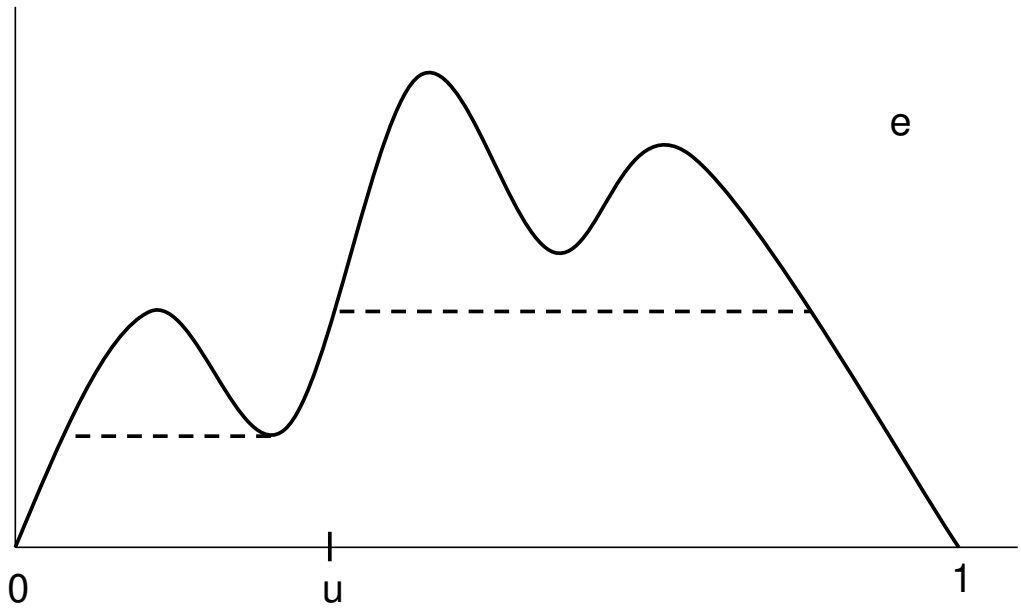

Figure 6: The set $\Gamma^*(e, u)$ of Definition 4.3 is the region above the horizontal dashed lines and below the graph of the excursion path e.

The map ξ is a measure-preserving bijection between the set $\Gamma(f)$ equipped with the measure

$$\mathrm{d}s \otimes \mathrm{d}a \, \frac{1}{\bar{s}(f,s,a) - \underline{s}(f,s,a)}$$

and the set $\Gamma^*(e, U(f))$ equipped with the measure

$$\mathrm{d}s \otimes \mathrm{d}a \, \frac{1}{\bar{s}(e,s,a) - \underline{s}(e,s,a)}.$$

Proof. Decompose the open set $\{t \in [0,1] : f(t) > 0\}$ into a countable union of intervals A_k, $k \in \mathbb{N}$. Set $B_k = \{(s,a) \in \Gamma(f) : s \in A_k\}$, $k \in \mathbb{N}$, and $C_k = \{(s,a) \in \Gamma^*(e, U(f)) : s \in A_k\}$, $k \in \mathbb{N}$. We have $\lambda([0,1] \backslash \bigcup_k A_k) = 0$, where λ is Lebesgue measure. Thus, $\lambda \otimes \lambda(\Gamma(f) \backslash \bigcup_k B_k) = 0$ and $\lambda \otimes \lambda(\Gamma^*(e, U(f)) \setminus \bigcup_k C_k) = 0$.

The function $t \mapsto L(t; f)$ is constant on each of the sets A_k, and so the same is true of the function $t \mapsto K^{\rightarrow}(t; f)$. Write c_k for this constant. The function ξ maps B_k bijectively into C_k and the restriction of ξ to B_k is the translation $(s,a) \mapsto (s, a + c_k)$.

Therefore, ξ is a measure-preserving bijection between the set $\Gamma(f)$ equipped with the measure $\mathrm{d}s \otimes \mathrm{d}a$ and the set $\Gamma^*(e, U(f))$ equipped with the measure $\mathrm{d}s \otimes \mathrm{d}a$.

It remains to note that if, for some $(s,a) \in \Gamma(f)$, we write $\xi(s,a) = (s, a')$, then we have $\underline{s}(f,s,a) = \underline{s}(e,s,a')$ and $\bar{s}(f,s,a) = \bar{s}(e,s,a')$, so that, in particular, $\bar{s}(f,s,a) - \underline{s}(f,s,a) = \bar{s}(e,s,a') - \underline{s}(e,s,a')$. □

Remark 4.5. Assume that $f \in \tilde{\Omega}^1_+$. For $a \geq 0$, recall the definition of $\mathcal{G}(f,a)$ from (1). For $u \in [0,1]$ and $e \in \Omega^1_{++}$ put

$$\mathcal{G}^*(e,a,u) := \left\{ s \in [0,1] : \begin{array}{c} e(s) = a \text{ and, for some } t > s, \\ e(r) > a \text{ for all } r \in]s,t[, \\ e(t) = a, \\ u \notin [s,t]. \end{array} \right\}$$

That is, $\mathcal{G}^*(e,a,u)$ is the countable set of starting points of excursions of e above the level a that don't straddle the time u. A consequence of Lemma 4.4 is that the measure m_f coincides with the measure

$$\int_0^\infty \mathrm{d}a \sum_{t \in \mathcal{G}^*(K^{\rightarrow}(\cdot;f)+f,a,U(f))} \delta_t.$$

As explained in the Introduction, the dynamics of the process we wish to construct involves "picking" a point v in a rooted compact $\mathbb{R}$-tree (T, d_T, ρ_T) according to the length measure μ_T and then re-rooting the subtree above v (that is, the subtree consisting of points of $x \in T$ such that $v \in [\rho_T, x[$) at a new location w. When $T = T_f$ for some $f \in \Omega^1_+$, this re-rooting of a subtree corresponds to a rearrangement of f by relocating an excursion of f above some level. We introduce the following notation to describe such rearrangements – see also Figure 7.

Definition 4.6. For $f \in \Omega_+^1$ and $(s,a) \in \Gamma(f)$, define $\hat{f}^{s,a} \in \Omega_{++}$ and $\check{f}^{s,a} \in \Omega_+$, by

$$\hat{f}^{s,a}(t) := \begin{cases} f(\underline{s}(f,s,a)+t) - a, & 0 \le t \le \bar{s}(f,s,a) - \underline{s}(f,s,a), \\ 0, & t > \bar{s}(f,s,a) - \underline{s}(f,s,a), \end{cases}$$

and

$$\check{f}^{s,a}(t) := \begin{cases} f(t), & 0 \le t \le \underline{s}(f,s,a), \\ f(t + \bar{s}(f,s,a) - \underline{s}(f,s,a)), & t > \underline{s}(f,s,a). \end{cases}$$

That is, $\hat{f}^{s,a}$ is the sub-excursion of f that straddles (s,a) shifted to start at position 0 at time 0, and $\check{f}^{s,a}$ is f with the sub-excursion that straddles (s,a) excised and the resulting gap closed up.

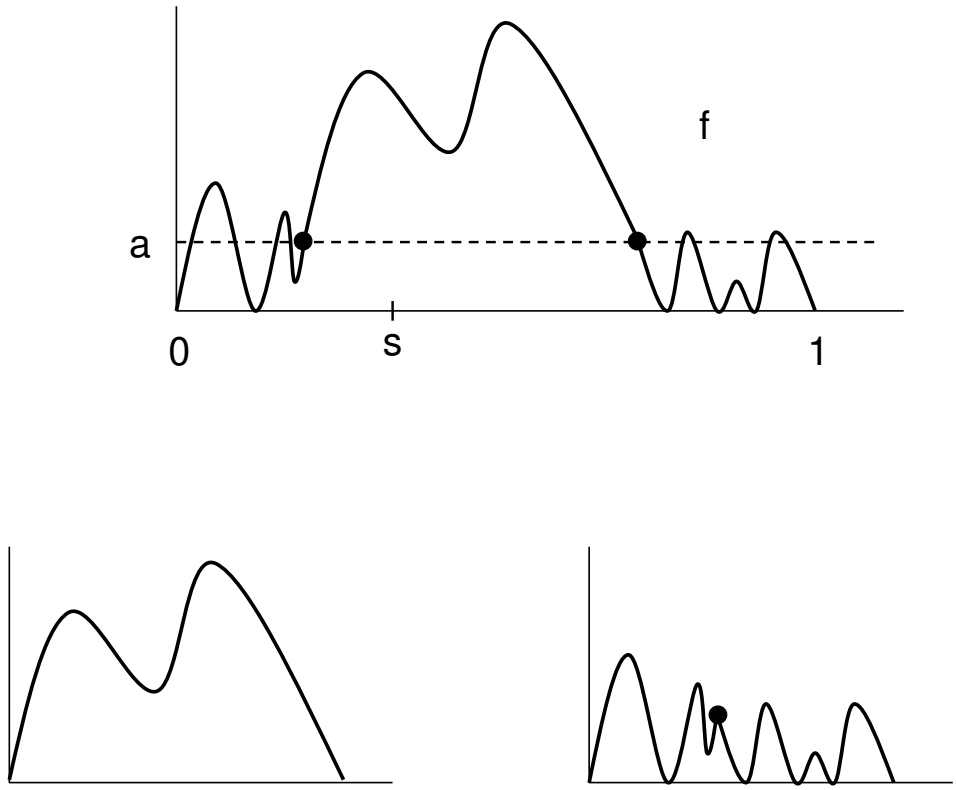

Figure 7: The decomposition described in Definition 4.6. The excursion path at the bottom left is $\hat{f}^{s,a}$ and the reflected bridge path at the bottom right is $\check{f}^{s,a}$. The two points marked by dots in the graph of the reflected bridge path f at the top correspond to the single point marked by a dot in the graph of the reflected bridge path $\check{f}^{s,a}$ at the bottom right.

Definition 4.7. For $f \in \Omega_+^1$, $u \in [0,1]$, and $(s,a) \in \Gamma^*(f,u)$, put

$$\check{U}(f,u,s,a) = \begin{cases} u, & 0 \le u < \underline{s}(f,s,a), \\ u - \bar{s}(f,s,a) + \underline{s}(f,s,a), & \bar{s}(f,s,a) < u \le 1. \end{cases}$$

By definition of $\Gamma^*(f,u)$, the point u belongs to the set

$$[0, \underline{s}(f,s,a)[\cup]\bar{s}(f,s,a), 1]$$

of length

$$\zeta(\check{f}^{s,a}) = 1 - \zeta(\hat{f}^{s,a}) = 1 - (\bar{s}(f,s,a) - \underline{s}(f,s,a)),$$

and $\check{U}(f,u,s,a)$ is where u is moved to when we close up the gap to form the interval $[0, \zeta(\check{f}^{s,a})]$.

The following result is immediate from Lemma 4.1 and Lemma 4.4.

Corollary 4.8. *Fix $f \in \tilde{\Omega}_+^1$. Set $e = K^{\rightarrow}(\cdot; f) + f \in \Omega_{++}^1$. Then, for any Borel function $F : \Omega_+ \times \Omega_+ \to \mathbb{R}_+$,*

$$\begin{aligned} &\int_{\Gamma(f)} \mathrm{d}s \otimes \mathrm{d}a \, \frac{1}{\bar{s}(f,s,a) - \underline{s}(f,s,a)} F(\hat{f}^{s,a}, \check{f}^{s,a}) \\ &\quad = \int_{\Gamma^*(e,U(f))} \mathrm{d}s \otimes \mathrm{d}a \, \frac{1}{\bar{s}(e,s,a) - \underline{s}(e,s,a)} F(\hat{e}^{s,a}, K^{\leftarrow}(\cdot; \check{e}^{s,a}, \check{U}(e, U(f), s, a)). \end{aligned}$$

5 Standard Brownian excursion and length measure

We first recall a result (Proposition 5.2 below) that appears as Corollary 5.2 in (EW06). It says that if we pick an excursion e according to the standard excursion distribution $\mathbb{P}_{++}$ and then pick a point $(s,a) \in \Gamma(e)$ according to the σ-finite measure

$$\mathrm{d}s \otimes \mathrm{d}a \, \frac{1}{\bar{s}(e,s,a) - \underline{s}(e,s,a)}$$

so that the time point $\underline{s}(e,s,a)$ is picked according to the σ-finite measure m_e, then the following objects are independent:

(a) the length of the excursion above level a that straddles time s;

(b) the excursion obtained by taking the excursion above level a that straddles time s, turning it (by a shift of axes) into an excursion $\hat{e}^{s,a}$ above level zero starting at time zero, and then Brownian re-scaling $\hat{e}^{s,a}$ to produce an excursion of unit length;

(c) the excursion obtained by taking the excursion $\check{e}^{s,a}$ that comes from excising $\hat{e}^{s,a}$ and closing up the gap, and then Brownian re-scaling $\check{e}^{s,a}$ to produce an excursion of unit length;

(d) the starting time $\underline{s}(e,s,a)$ of the excursion above level a that straddles time s rescaled by the length of $\check{e}^{s,a}$ to give a time in the interval $[0,1]$.

Moreover,

- the length in (a) is "distributed" according to the σ-finite measure

$$\frac{1}{2\sqrt{2\pi}}\frac{\mathrm{d}r}{\sqrt{(1-r)r^3}}, \quad r \in [0,1];$$

- the unit length excursions in (b) and (c) are both distributed as standard Brownian excursions (that is, according to $\mathbb{P}_{++}$);
- the time in (d) is uniformly distributed on the interval $[0,1]$.

Definition 5.1. For $c > 0$, let $\mathcal{S}_c : \Omega_+^1 \to \Omega_+^c$ be the Brownian re-scaling map defined by

$$\mathcal{S}_c f := \sqrt{c} f(\cdot/c).$$

Proposition 5.2. *For any Borel function $F : [0,1] \times \Omega_{++} \times \Omega_{++} \to \mathbb{R}_+$,*

$$\begin{aligned}
&\int \mathbb{P}_{++}(\mathrm{d}e) \int_{\Gamma(e)} \frac{\mathrm{d}s \otimes \mathrm{d}a}{\bar{s}(e,s,a) - \underline{s}(e,s,a)} F\Big(\frac{\underline{s}(e,s,a)}{\zeta(\check{e}^{s,a})}, \hat{e}^{s,a}, \check{e}^{s,a}\Big) \\
&= \int_{[0,1]} \mathrm{d}v \, \frac{1}{2\sqrt{2\pi}} \int_{[0,1]} \frac{\mathrm{d}r}{\sqrt{(1-r)r^3}} \int \mathbb{P}_{++}(\mathrm{d}e') \otimes \mathbb{P}_{++}(\mathrm{d}e'') \, F(v, \mathcal{S}_r e', \mathcal{S}_{1-r} e'').
\end{aligned}$$

With Proposition 4.2 and Corollary 4.8 in mind, we want to obtain an analogous result with $\Gamma(e)$ replaced by $\Gamma^*(e,u)$, where u is picked uniformly from $[0,1]$.

Corollary 5.3. *For any Borel function $G : [0,1] \times [0,1] \times \Omega_{++} \times \Omega_{++} \to \mathbb{R}_+$,*

$$\begin{aligned}
&\int_{[0,1]} \mathrm{d}u \int \mathbb{P}_{++}(\mathrm{d}e) \int_{\Gamma^*(e,u)} \frac{\mathrm{d}s \otimes \mathrm{d}a}{\bar{s}(e,s,a) - \underline{s}(e,s,a)} \\
&\quad \times G\left(\frac{\breve{U}(e,u,s,a)}{\zeta(\check{e}^{s,a})}, \frac{\underline{s}(e,s,a)}{\zeta(\check{e}^{s,a})}, \hat{e}^{s,a}, \check{e}^{s,a}\right) \\
&= \int_{[0,1]} \mathrm{d}u \int_{[0,1]} \mathrm{d}v \, \frac{1}{2\sqrt{2\pi}} \int_{[0,1]} \mathrm{d}r \sqrt{\frac{1-r}{r^3}} \int \mathbb{P}_{++}(\mathrm{d}e') \otimes \mathbb{P}_{++}(\mathrm{d}e'') \\
&\quad \times G(u, v, \mathcal{S}_r e', \mathcal{S}_{1-r} e'').
\end{aligned}$$

Proof. For $v, r \in [0,1]$ and $u \in [0, (1-r)v[\cup](1-r)v + r, 1]$, put

$$\breve{U}(u,v,r) = \begin{cases} \frac{u}{1-r}, & 0 \le u < (1-r)v, \\ \frac{u-r}{1-r}, & (1-r)v + r < u \le 1. \end{cases}$$

From Proposition 5.2, we have

$$\begin{aligned}
&\int \mathbb{P}_{++}(\mathrm{d}e) \int_{\Gamma^*(e,u)} \frac{\mathrm{d}s \otimes \mathrm{d}a}{\bar{s}(e,s,a) - \underline{s}(e,s,a)} \\
&\quad \times G\left(\frac{\check{U}(e,u,s,a)}{\zeta(\check{e}^{s,a})}, \frac{\underline{s}(e,s,a)}{\zeta(\check{e}^{s,a})}, \hat{e}^{s,a}, \check{e}^{s,a} \right) \\
&= \int \mathbb{P}_{++}(\mathrm{d}e) \int_{\Gamma(e)} \frac{\mathrm{d}s \otimes \mathrm{d}a}{\bar{s}(e,s,a) - \underline{s}(e,s,a)} \mathbf{1}\{u \notin [\underline{s}(e,s,a), \bar{s}(e,s,a)]\} \\
&\quad \times G\left(\frac{\check{U}(e,u,s,a)}{\zeta(\check{e}^{s,a})}, \frac{\underline{s}(e,s,a)}{\zeta(\check{e}^{s,a})}, \hat{e}^{s,a}, \check{e}^{s,a} \right) \\
&= \int_{[0,1]} \mathrm{d}v \, \frac{1}{2\sqrt{2\pi}} \int_{[0,1]} \frac{\mathrm{d}r}{\sqrt{(1-r)r^3}} \int \mathbb{P}_{++}(\mathrm{d}e') \otimes \mathbb{P}_{++}(\mathrm{d}e'') \\
&\quad \times \mathbf{1}\{u \notin [(1-r)v, (1-r)v + r]\} \\
&\quad \times G\left(\check{U}(u,v,r), v, \mathcal{S}_r e', \mathcal{S}_{1-r} e'' \right).
\end{aligned}$$

The change of variable $w = \check{U}(u,v,r)$ gives

$$\begin{aligned}
&\int_{[0,1]} \mathrm{d}u \, \mathbf{1}\{u \notin [(1-r)v, (1-r)v + r]\} G\left(\check{U}(u,v,r), v, \mathcal{S}_r e', \mathcal{S}_{1-r} e'' \right) \\
&\quad = (1-r) \int_{[0,1]} \mathrm{d}w \, G\left(w, v, \mathcal{S}_r e', \mathcal{S}_{1-r} e'' \right),
\end{aligned}$$

and the result follows. □

6 A symmetric measure on $\Omega_+^1 \times \Omega_+^1$

Definition 6.1. Fix a function $f \in \Omega_+^1$ and suppose that $v \in G(f,a)$ is the starting point of an excursion of f above some level a. Write

$$\delta(f,v) := \inf\{t > v : f(t) = a\}$$

for the time at which the excursion finishes. Thus, $\underline{s}(f,s,a) = v$ and $\bar{s}(f,s,a) = \delta(f,v)$ for any $s \in]v, \delta(f,v)[$. Define $\tilde{e}^{f,v} \in \Omega_{++}$ by

$$\tilde{e}^{f,v} := \begin{cases} f(t+v) - f(v), & 0 \le t \le v - \delta(f,v), \\ 0, & t > v - \delta(f,v). \end{cases}$$

That is, $\tilde{e}^{f,v}$ is the result of taking the excursion starting and ending at times v and $\delta(f,v)$, respectively, and shifting the time and space axes to obtain an excursion that starts at position 0 at time 0. Given $w \in [0,1] \setminus [v, \delta(f,v)]$, denote by $f^{v,w} \in \Omega_+^1$ the path defined as follows. If $w > v$ (so that $w > \delta(f,v)$), then

$$f^{v,w}(t) := \begin{cases} f(t), & 0 \le t < v, \\ f(t - v + \delta(f,v)), & v \le t < v - \delta(f,v) + w, \\ \tilde{e}^{f,v}(t - (v - \delta(f,v) + w)) + f(w), & v - \delta(f,v) + w \le t < w, \\ f(t), & t \ge w. \end{cases}$$

If $w < v$, then

$$f^{v,w}(t) := \begin{cases} f(t), & 0 \le t < w, \\ \tilde{e}^{f,v}(t-w) + f(w), & w \le t < w - v + \delta(f,v), \\ f(t + v - \delta(f,v)), & w - v + \delta(f,v) \le t < \delta(f,v), \\ f(t), & t \ge \delta(f,v). \end{cases}$$

In other words, the excursion of f starting at time v is first moved so that it starts at w and then the resulting gap left between times v and $\delta(f,v)$ is closed up – see Figure 8.

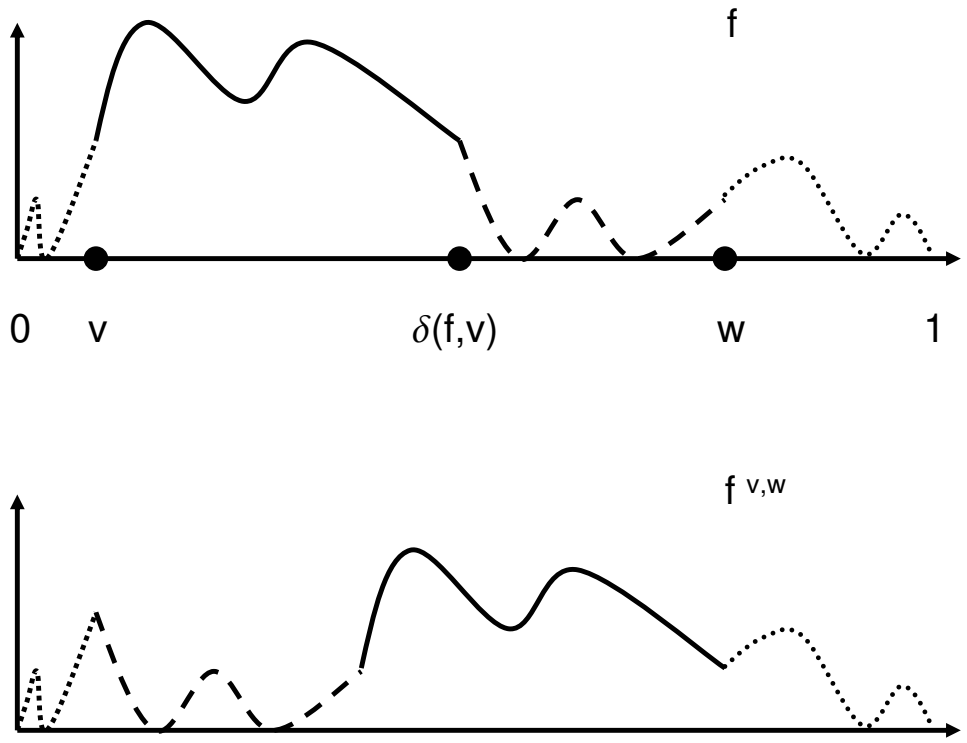

Figure 8: The transformation taking the path f to the path $f^{v,w}$ when $w > v$. The figure for $w < v$ is similar.

Definition 6.2. Define a kernel κ_+ on Ω_+^1 by

$$\kappa_+(f,B) := \int_0^\infty \mathrm{d}a \sum_{v \in \mathcal{G}(f,a)} \frac{1}{1 - (\delta(f,v) - v)} \int_{[0,1]\setminus[v,\delta(f,v)]} \mathrm{d}w \, \mathbf{1}(f^{v,w} \in B).$$

That is, a starting point v of an excursion is chosen according to the measure m_f corresponding to length measure μ_{T_f} on the $\mathbb{R}$-tree associated with f, this excursion is then relocated so that it starts at a uniformly chosen point $w \in [0,1] \setminus [v, \delta(f,v)]$, and finally the resulting gap is closed up. Define a measure $\mathbb{J}_+$ on $\Omega^1_+ \times \Omega^1_+$ by

$$\mathbb{J}_+(\mathrm{d}f', \mathrm{d}f'') := \mathbb{P}_+(\mathrm{d}f')\kappa_+(f', \mathrm{d}f'').$$

Proposition 6.3. *The measure $\mathbb{J}_+$ is symmetric.*

Proof. Given $e', e'' \in \Omega^1_{++}$, $v \in [0,1]$, and $r \in]0,1]$, define $e^\circ(\cdot; e', e'', v, r) \in \Omega^1_{++}$ by

$$\begin{aligned} &e^\circ(t; e', e'', v, r) \\ &\quad := \begin{cases} \mathcal{S}_{1-r} e''(t), & 0 \le t \le (1-r)v, \\ \mathcal{S}_{1-r} e''((1-r)v) + \mathcal{S}_r e'(t - (1-r)v), & (1-r)v \le t \le (1-r)v + r, \\ \mathcal{S}_{1-r} e''(t-r), & (1-r)v + r \le t \le 1. \end{cases} \end{aligned}$$

That is, $e^\circ(\cdot; e', e'', v, r)$ is the excursion that arises from Brownian re-scaling e' and e'' to have lengths r and $1-r$, respectively, and then inserting the re-scaled version of e' into the re-scaled version of e'' at a position that is a fraction v of the total length of the re-scaled version of e''.

Also, for $u \in [0,1]$ set

$$\tilde{U}(u, v, r) := \begin{cases} (1-r)u, & 0 \le u \le v, \\ r + (1-r)u, & v < u \le 1, \end{cases}$$

so that $\tilde{U}(u, v, r)$ belongs to the set $[0, (1-r)v[\cup](1-r)v + r, 1]$ for Lebesgue almost all $u \in [0,1]$ and the push-forward of Lebesgue measure on $[0,1]$ by the map $u \mapsto \tilde{U}(u, v, r)$ is the uniform distribution on this union of two intervals.

Define a measure $\mathbb{J}_{++}$ on $[0,1] \times [0,1] \times \Omega^1_{++} \times \Omega^1_{++}$ by

$$\begin{aligned} &\int \mathbb{J}_{++}(\mathrm{d}u^*, \mathrm{d}u^{**}, \mathrm{d}e^*, \mathrm{d}e^{**}) G(u^*, u^{**}, e^*, e^{**}) \\ &\quad := \int_{[0,1]^3} \mathrm{d}u \otimes \mathrm{d}v \otimes \mathrm{d}w \frac{1}{2\sqrt{2\pi}} \int_{[0,1]} \mathrm{d}r \sqrt{\frac{1-r}{r^3}} \int \mathbb{P}_{++}(\mathrm{d}e') \otimes \mathbb{P}_{++}(\mathrm{d}e'') \\ &\qquad \times G\left(\tilde{U}(u, v, r), \tilde{U}(u, w, r), e^\circ(\cdot; e', e'', v, r), e^\circ(\cdot; e', e'', w, r)\right) \end{aligned}$$

for any non-negative Borel function G.

Clearly, the measure $\mathbb{J}_{++}$ is preserved by pushing it forward with the map $(u^*, u^{**}, e^*, e^{**}) \mapsto (u^{**}, u^*, e^{**}, e^*)$. Also, it follows from Lemma 4.1, Proposition 4.2, Corollary 4.8 and Corollary 5.3 that the measure $\mathbb{J}_+$ is the push-forward of the measure $\mathbb{J}_{++}$ by the map

$$(u^*, u^{**}, e^*, e^{**}) \mapsto (K^{\leftarrow}(\cdot; e^*, u^*), K^{\leftarrow}(\cdot; e^{**}, u^{**})),$$

and the result follows. □

By construction, the measure $\mathbb{J}_+$ is concentrated on pairs $(f', f'') \in \Omega^1_+ \times \Omega^1_+$ such that f'' is obtained from f' by the re-location of an excursion. If we shift the starting point of this excursion in space and time to the origin to obtain an element of Ω_{++}, then the σ-finite law of this shifted excursion is

$$\begin{aligned}&\mathbb{P}_+\left[\int_0^\infty \mathrm{d}a \sum_{v\in\mathcal{G}(f,a)} \mathbf{1}\left(\tilde{e}^{f,v} \in \cdot\right)\right]\\ &\quad = \mathbb{P}_+\left[\int_{\Gamma(f)} \mathrm{d}s \otimes \mathrm{d}a \, \frac{1}{\bar{s}(f,s,a) - \underline{s}(f,s,a)} \mathbf{1}\left(\hat{f}^{s,a} \in \cdot\right)\right].\end{aligned}$$

Informally, this is the law of the excursion under $\mathbb{J}_+$, but we note while $\mathbb{J}_+$ is concentrated on pairs (f', f'') of the form $(f, f^{v,w})$ for some $v, w \in [0,1]$, the value of v and the corresponding excursion $\tilde{e}^{f,v}$ cannot be uniquely reconstructed from (f', f''). Arguing as in the proof of Proposition 6.3, this law is given by

$$\frac{1}{2\sqrt{2\pi}} \int_{[0,1]} \mathrm{d}r \sqrt{\frac{1-r}{r^3}} \mathbb{P}_{++}\left\{e \in \Omega^1_{++} : \mathcal{S}_r(e) \in \cdot\right\}.$$

We need the following properties of this law.

Proposition 6.4. *(i) For $0 < t \le 1$,*

$$\int \mathbb{P}_+(\mathrm{d}f) \int_0^\infty \mathrm{d}a \sum_{v\in\mathcal{G}(f,a)} \mathbf{1}(\zeta(\tilde{e}^{f,v}) > t) = \frac{1}{\sqrt{2\pi}} \left(\sqrt{\frac{1}{t} - 1} + \arcsin\left(\sqrt{t}\right) - \frac{\pi}{2}\right),$$

and hence

$$\int \mathbb{P}_+(\mathrm{d}f) \int_0^\infty \mathrm{d}a \sum_{v\in\mathcal{G}(f,a)} (\zeta(\tilde{e}^{f,v}))^2 = \frac{\pi^{\frac{1}{2}}}{16\sqrt{2}}.$$

(ii) For $x > 0$,

$$\int \mathbb{P}_+(\mathrm{d}f) \int_0^\infty \mathrm{d}a \sum_{v\in\mathcal{G}(f,a)} \mathbf{1}\left(\max(\tilde{e}^{f,v}) > x\right) = \sum_{n=1}^\infty \int_{2nx}^\infty \mathrm{d}z \, \exp\left(-\frac{z^2}{2}\right),$$

and hence

$$\int \mathbb{P}_+(\mathrm{d}f) \int_0^\infty \mathrm{d}a \sum_{v\in\mathcal{G}(f,a)} \left(\max(\tilde{e}^{f,v})\right)^2 = \frac{\pi^{\frac{5}{2}}}{24\sqrt{2}}.$$

Proof. (i) By the remarks prior the statement of the proposition, the quantity in the first claim is

$$\frac{1}{2\sqrt{2\pi}} \int_{[t,1]} \mathrm{d}r \sqrt{\frac{1-r}{r^3}},$$

and a straightforward integration shows that this has the stated value. The second claim follows by an equally straightforward integration by parts.

(ii) Again by the remarks prior the statement of the proposition, the quantity in the first claim is

$$\frac{1}{2\sqrt{2\pi}} \int_{[0,1]} \mathrm{d}r \sqrt{\frac{1-r}{r^3}} \mathbb{P}_{++} \left\{ e \in \Omega^1_{++} : \max(e) > \frac{x}{\sqrt{r}} \right\}.$$

From Theorem 5.2.10 in (Kni81), we have that

$$\mathbb{P}_{++} \left\{ e \in \Omega^1_{++} : \max(e) > y \right\} = 2 \sum_{n=1}^{\infty} (4n^2y^2 - 1) \exp(-2n^2y^2),$$

and an integration establishes the claim.

An integration by parts shows that the quantity in the second claim is

$$\sqrt{\frac{\pi}{32}} \sum_{n=1}^{\infty} \frac{1}{n^2} = \sqrt{\frac{\pi}{32}} \frac{\pi^2}{6},$$

as required. □

7 A Dirichlet form

Recall that any $f \in \Omega^1_+$ is associated with a $\mathbb{R}$-tree (T_f, d_{T_f}) that arises as a quotient of $[0,1]$ under an equivalence relation defined by f. Moreover, we may equip this $\mathbb{R}$-tree with the root ρ_{T_f} that is the image of 0 under the quotient and the weight ν_{T_f} that is the push-forward of Lebesgue measure on $[0,1]$ by the quotient map.

Definition 7.1. Define the probability measure $\mathbf{P}$ on $\mathbf{T}^{\mathrm{wr}}$ to be the push-forward of the probability measure $\mathbb{P}_+$ on Ω^1_+ by the map

$$f \mapsto (T_{2f}, d_{T_{2f}}, \rho_{T_{2f}}, \nu_{T_{2f}}).$$

Define the measure $\mathbf{J}$ on $\mathbf{T}^{\mathrm{wr}} \times \mathbf{T}^{\mathrm{wr}}$ to be **twice** the push-forward of $\mathbb{J}_+$ by the map

$$(f', f'') \mapsto ((T_{2f'}, d_{T_{2f'}}, \rho_{T_{2f'}}, \nu_{T_{2f'}}), (T_{2f''}, d_{T_{2f''}}, \rho_{T_{2f''}}, \nu_{T_{2f''}})).$$

Proposition 7.2. *(i) The measure* $\mathbf{J}$ *is symmetric.*

(ii) For each compact subset $K \subset \mathbf{T}^{\mathrm{wr}}$ *and open subset* U *such that* $K \subset U \subseteq \mathbf{T}^{\mathrm{wr}}$,

$$\mathbf{J}(K \times (\mathbf{T}^{\mathrm{wr}} \setminus U)) < \infty.$$

(iii) The function $\Delta_{\mathrm{GH}^{\mathrm{wr}}}$ *is square-integrable with respect to* J, *that is,*

$$\int \mathbf{J}(\mathrm{d}T', \mathrm{d}T'') \, \Delta^2_{\mathrm{GH}^{\mathrm{wr}}}(T', T'') < \infty.$$

Proof. (i) This is immediate from Proposition 6.3.

(ii) By construction, the measure $\mathbf{J}$ has the following description. Firstly, a weighted rooted compact $\mathbb{R}$-tree $T' \in \mathbf{T}^{\mathrm{wr}}$ is chosen according to $\mathbf{P}$. A point $v \in T'$ is chosen according to the length measure $\mu_{T'}$ and another point $w \in T'$ is chosen according to the renormalization of the weight $\nu_{T'}$ outside of the subtree $S^{T',v}$ of points "above" v (that is, of points x such that v belongs to the segment $[\rho_{T'}, x[$). The subtree $S^{T',v}$ is then pruned off and re-attached at w to form a new $\mathbb{R}$-tree T''. More formally, the $\mathbb{R}$-tree T'' can be identified as the set T' equipped with new metric $d_{T''}$ given by

$$d_{T''}(x,y) := \begin{cases} d(x,y), & x,y \in S^{T',v}, \\ d(x,y), & x,y \in T' \setminus S^{T',v}, \\ d(x,v) + d(w,y), & x \in S^{T',v},\ y \in T' \setminus S^{T',v}, \\ d(y,v) + d(w,x), & y \in S^{T',v}, x \in T' \setminus S^{T',v}. \end{cases}$$

With this identification, $\rho_{T''} = \rho_{T'}$ and $\nu_{T''} = \nu_{T'}$.

We claim that if, for some $\varepsilon > 0$,

$$\max_{x \in S^{T',v}} d_{T'}(v,x) \le \varepsilon$$

and

$$\nu_{T'}(S^{T',v}) \le \varepsilon,$$

then $\Delta_{\mathrm{GH}^{\mathrm{wr}}}(T', T'') \le \varepsilon$. Firstly, the map $f : T' \to T''$ defined by

$$f(x) := \begin{cases} x, & x \in T' \setminus S^{T',v}, \\ u, & x \in S^{T',v}, \end{cases}$$

is such that $f(\rho_{T'}) = \rho_{T''}$ and

$$\sup\{|d_{T'}(x,y) - d_{T''}(f(x), f(y))| : x, y \in T'\} \le \varepsilon.$$

Moreover, it is immediate that

$$d_P(f_* \nu_{T'}, \nu_{T''}) \le \varepsilon,$$

and so $f \in F^{\varepsilon}_{T',T''}$. Note also that $S^{T'',w} = S^{T',v}$ as sets,

$$\max_{x \in S^{T'',w}} d_{T''}(w,x) = \max_{x \in S^{T',v}} d_{T'}(v,x),$$

and

$$\nu_{T''}(S^{T'',w}) = \nu_{T'}(S^{T',v}),$$

and so a similar argument shows that the map $g : T'' \to T'$ defined by

$$g(x) := \begin{cases} x, & x \in T'' \setminus S^{T'',w}, \\ w, & x \in S^{T'',w} \end{cases}$$

belongs to $F^{\varepsilon}_{T'',T'}$. Thus, $\Delta_{\mathrm{GH}^{\mathrm{wr}}}(T', T'') \le \varepsilon$ as required.

Now let K and U be as in the statement of part (ii). The result is trivial if $K = \emptyset$, so we assume that $K \neq \emptyset$. Since $\mathbf{T}^{\mathrm{wr}} \setminus U$ and K are disjoint closed sets and K is compact, we have that

$$c := \inf_{T' \in K, T'' \in U} \Delta_{\mathrm{GH}^{\mathrm{wr}}}(T', T'') > 0.$$

From what we have just observed, if $0 < \varepsilon < c$, then, by Proposition 6.4,

$$\begin{aligned}
\mathbf{J}(K \times (\mathbf{T}^{\mathrm{wr}} \setminus U)) &\leq \mathbf{J}\left\{(T', T'') : \Delta_{\mathrm{GH}^{\mathrm{wr}}}(T', T'') > \varepsilon\right\} \\
&\leq \int \mathbf{P}(\mathrm{d}T') \int_{T'} \mu_{T'}(\mathrm{d}v)\, \mathbf{1}\left(\max_{x \in S^{T',v}} d_{T'}(v, x) > \varepsilon\right) \\
&\quad + \int \mathbf{P}(\mathrm{d}T') \int_{T'} \mu_{T'}(\mathrm{d}v)\, \mathbf{1}\left(\nu(S^{T',v}) > \varepsilon\right) \\
&= \int \mathbb{P}_+(\mathrm{d}f)\, 2 \int_0^\infty \mathrm{d}a \sum_{v \in \mathcal{G}(f,a)} \mathbf{1}\left(\max(\tilde{e}^{f,v}) > \varepsilon/2\right) \\
&\quad + \int \mathbb{P}_+(\mathrm{d}f)\, 2 \int_0^\infty \mathrm{d}a \sum_{v \in \mathcal{G}(f,a)} \mathbf{1}\left(\zeta\left(\tilde{e}^{f,v}\right) > \varepsilon\right) \\
&< \infty,
\end{aligned}$$

as required.

(iii) By an argument similar to that in part (ii) and Proposition 6.4, we have

$$\begin{aligned}
&\int \mathbf{J}(\mathrm{d}T', \mathrm{d}T'')\, \Delta^2_{\mathrm{GH}^{\mathrm{wr}}}(T', T'') \\
&= \int_0^\infty \mathrm{d}\varepsilon\, 2\varepsilon\, \mathbf{J}\left\{(T', T'') : \Delta_{\mathrm{GH}^{\mathrm{wr}}}(T', T'') > \varepsilon\right\} \\
&\leq \int_0^\infty \mathrm{d}\varepsilon\, 2\varepsilon \int \mathbf{P}(\mathrm{d}T') \int_{T'} \mu_{T'}(\mathrm{d}v)\, \mathbf{1}\left(\max_{x \in S^{T',v}} d_{T'}(v, x) > \varepsilon\right) \\
&\quad + \int_0^\infty \mathrm{d}\varepsilon\, 2\varepsilon \int \mathbf{P}(\mathrm{d}T') \int_{T}' \mu_{T'}(\mathrm{d}v)\, \mathbf{1}\left(\nu(S^{T',v}) > \varepsilon\right) \\
&= \int_0^\infty \mathrm{d}\varepsilon\, 2\varepsilon \int \mathbb{P}_+(\mathrm{d}f)\, 2 \int_0^\infty \mathrm{d}a \sum_{v \in \mathcal{G}(f,a)} \mathbf{1}\left(\max(\tilde{e}^{f,v}) > \varepsilon/2\right) \\
&\quad + \int_0^\infty \mathrm{d}\varepsilon\, 2\varepsilon \int \mathbb{P}_+(\mathrm{d}f)\, 2 \int_0^\infty \mathrm{d}a \sum_{v \in \mathcal{G}(f,a)} \mathbf{1}(\zeta(\tilde{e}^{f,v}) > \varepsilon) \\
&= \int \mathbb{P}_+(\mathrm{d}f)\, 8 \int_0^\infty \mathrm{d}a \sum_{v \in \mathcal{G}(f,a)} \left(\max(\tilde{e}^{f,v})\right)^2 \\
&\quad + \int \mathbb{P}_+(\mathrm{d}f)\, 2 \int_0^\infty \mathrm{d}a \sum_{v \in \mathcal{G}(f,a)} (\zeta(\tilde{e}^{f,v}))^2 \\
&< \infty,
\end{aligned}$$

as required. □

Definition 7.3. Define a bilinear form

$$\mathcal{E}(f, g) := \int \mathbf{J}(\mathrm{d}T', \mathrm{d}T'') \left(f(T'') - f(T')\right)\left(g(T'') - g(T')\right),$$

for f, g in the domain

$$\mathcal{D}^*(\mathcal{E}) := \{f \in L^2(\mathbf{T}^{\mathrm{wr}}, \mathbf{P}) : \ f \text{ is Borel and } \mathcal{E}(f,f) < \infty\}.$$

Here, as usual, $L^2(\mathbf{T}^{\mathrm{wr}}, \mathbf{P})$ is equipped with the inner product

$$(f,g)_{\mathbf{P}} := \int \mathbf{P}(\mathrm{d}x)\, f(x)g(x).$$

Definition 7.4. Let $\mathcal{L}$ denote the collection of functions $f : \mathbf{T}^{\mathrm{wr}} \to \mathbb{R}$ such that

$$\sup_{T \in \mathbf{T}^{\mathrm{wr}}} |f(T)| < \infty$$

and

$$\sup_{T', T'' \in \mathbf{T}^{\mathrm{wr}},\, T' \neq T''} \frac{|f(T') - f(T'')|}{\Delta_{\mathrm{GH}^{\mathrm{wr}}}(T', T'')} < \infty.$$

Part (i) (respectively, parts (ii) and (iii)) of the following result may be proved in the same manner as Lemma 7.1 (respectively, Lemma 7.2 and Theorem 7.3) of (EW06), with Proposition 7.2 above playing the role played in (EW06) by Lemma 6.2 of that paper. Our setting is slightly different, in that we are working with weighted rooted compact $\mathbb{R}$-trees rather than just weighted compact $\mathbb{R}$-trees, but this doesn't require any significant changes in the arguments. In particular, the analogues of Lemmas 7.5, 7.6 and 7.7 of (EW06) go through quite straightforwardly to establish the tightness property required for part (iii) to hold.

Theorem 7.5. *(i) The form $(\mathcal{E}, \mathcal{D}^*(\mathcal{E}))$ is Dirichlet (that is, it is symmetric, non-negative definite, Markovian, and closed).*

(ii) The set $\mathcal{L}$ is a vector lattice and an algebra, and $\mathcal{L} \subseteq \mathcal{D}^*(\mathcal{E})$. Hence, if $\mathcal{D}(\mathcal{E})$ denotes the closure of $\mathcal{L}$ in $\mathcal{D}^*(\mathcal{E}))$, then $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is also a Dirichlet form.

(iii) There is a recurrent $\mathbf{P}$-symmetric Hunt process $X = (X_t, \mathbb{P}^T)$ on $\mathbf{T}^{\mathrm{wr}}$ with Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

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