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LOCAL SUB-GAUSSIAN ESTIMATES ON GRAPHS: THE STRONGLY RECURRENT CASE

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Abstract This paper proves upper and lower off-diagonal, sub-Gaussian transition probabilities estimates for strongly recurrent random walks under sufficient and necessary conditions. Several equivalent conditions are given showing their particular role influence on the connection between the sub-Gaussian estimates, parabolic and elliptic Harnack inequality.

Keywords Random walks, potential theory, Harnack inequality, reversible Markov chains

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1 Introduction

1.1 The origin of the problem

In a recent paper ([16]) a complete characterization was given of polynomially growing (strongly) transient graphs (with volume growth $V(x,R)\simeq R^{\alpha}$) possessing sub-Gaussian behavior with mean exit time $E(x,R)\simeq R^{\beta}$ ($\alpha>\beta\geq 2$). In this setting the classical Gaussian estimates are replaced with the so called sub-Gaussian estimates which have the form

$$p_n(x,y)\leq Cn^{-\frac{\alpha}{\beta}}\exp-\left(\frac{d^{\beta}(x,y)}{Cn}\right)^{\frac{1}{\beta-1}} \qquad (UE_{\alpha,\beta})$$

$$p_n(x,y)+p_{n+1}(x,y)\geq cn^{-\frac{\alpha}{\beta}}\exp-\left(\frac{d^{\beta}(x,y)}{cn}\right)^{\frac{1}{\beta-1}} \qquad (LE_{a,\beta})$$

for $n\geq d(x,y)$ if and only if the volume growth is polynomial and the Green function decays polynomially as well. The $\beta>2$ case has the sub-Gaussian name to reflect the sub-diffusive character of the diffusion process.

The aim of this paper is to prove the strongly recurrent counterpart ($\alpha<\beta$) of the result ([16] where $\alpha>\beta$) . In fact this paper proves more. It shows a local (or as it is sometimes, a called relative) version assuming volume doubling instead of polynomial growth. This setting brings two new difficulties. One is the local formalism, the other is that due to the recurrence there is no *global* Green function (contrary to the transient case of [16]) and all the analysis is based on the *local* Green function, the Green function of the process killed on exiting from a finite set. This technique was developed in [25], [26] and in [27].

1.2 Basic objects

Let Γ be an infinite connected graph and $\mu_{x,y}$ the weight function on the connected vertices $x\sim y$, $x,y\in\Gamma$, inducing a measure μ on Γ. The measure $\mu(x)$ is defined for an $x\in\Gamma$ by

$$\mu(x)=\sum_{y:y\sim x}\mu_{x,y}$$

and for $A\subset\Gamma$

$$\mu(A)=\sum_{x\in A}\mu(x).$$

The graph is equipped with the usual (shortest path length) graph distance $d(x,y)$ and open metric balls defined for $x\in\Gamma$, $R>0$ as $B(x,R)=\{y\in\Gamma:d(x,y)<R\}$ and its $\mu-$measure is $V(x,R)$. The surface of the ball (which does not belong to it) is $S(x,R)=\{y\in\Gamma:d(x,y)=R\}$.

Definition 1.1 *The graph has* volume doubling *property if there is a constant $C_V>0$ such that for all $x\in\Gamma$ and $R>0$*

$$V(x,2R)\leq C_V V(x,R) \qquad (D)$$

It is clear that volume doubling implies $V(x,R) \leq CR^{\alpha}$ with

$$\alpha = \limsup \frac{\log V(x,R)}{\log R} \leq \log_2 C_V.$$

.

The random walk is defined by the weights via the one-step transition probabilities

$$P(x,y) = \frac{\mu_{x,y}}{\mu(x)},$$

$$\mathbb{P}(X_{n+1} = y | X_n = x) = P(x,y)$$

and

$$P_n(x,y) = \mathbb{P}(X_{n+1} = y | X_0 = x)$$

while the transition probability kernel is

$$p_n(x,y) = \frac{1}{\mu(y)} P_n(x,y).$$

Definition 1.2 *The transition probability kernel satisfies the local sub-Gaussian estimates if there are $c, C > 0$ such that for all $x, y \in \Gamma$ and $n \in \mathbb{N}$*

$$p_n(x,y) \leq \frac{C}{V(x, n^{\frac{1}{\beta}})} \exp -\left(\frac{d(x,y)^{\beta}}{Cn}\right)^{\frac{1}{\beta-1}}, \tag{UE_β}$$

$$\widetilde{p}_n(x,y) \geq \frac{c}{V(x, n^{\frac{1}{\beta}})} \exp -\left(\frac{d(x,y)^{\beta}}{cn}\right)^{\frac{1}{\beta-1}}, \tag{LE_β}$$

where $\widetilde{p_n} = p_n + p_{n+1}$.

The β- parabolic Harnack inequality can be introduced in the following way (c.f. [17] and [3]). Let $\mathcal{C} = \{C_1, C_2, C_3, C_4, \eta\}$ the profile of the parabolic Harnack inequality if $0 < C_1 < C_2 < C_3 < C_4 \leq 1, \eta < 1$ are constants.

Definition 1.3 *A weighted graph satisfies ($\beta-$parabolic or simply) parabolic Harnack inequality if for any given profile $\mathcal{C}$ there is a constant $C_H(\mathcal{C}) > 0$ for which the following is true. Assume that u is the solution of the equation*

$$u_{n+1}(x) = Pu_n(x)$$

on

$$\mathcal{U} = [k, k + R^{\beta}] \times B(x,R)$$

for $k, R \in \mathbb{N}$, then on the smaller cylinders defined by

$$\mathcal{U}^{-} = [k + C_1 R^{\beta}, k + C_2 R^{\beta}] \times B(x, \eta R)$$
$$\mathcal{U}^{+} = [k + C_3 R^{\beta}, k + C_4 R^{\beta}] \times B(x, \eta R)$$

and taking $(n_-, x_-) \in \mathcal{U}^-, (n_+, x_+) \in \mathcal{U}^+, d(x_-, x_+) \leq n_+ - n_-$ the inequality

$$u(n_-, x_-) \leq C_H \widetilde{u}(n_+, x_+) \tag{PH_β}$$

holds, where $\widetilde{u}_n = u_n + u_{n+1}$.

It is standard that if the (classical) parabolic Harnack inequality holds for a given profile, then it holds for any other profile as well, provided the volume doubling condition holds. It is clear that the same holds for the $\beta-$parabolic Harnack inequality.

The elliptic Harnack inequality is direct consequence of the β-parabolic one as it is true in the classical case.

Definition 1.4 *The graph satisfies the elliptic Harnack inequality if there is a $C>0$ such that for all $x\in\Gamma,R>1$ and $v>0$ harmonic function on $B(x,2R)$ which means that*

$$Pv=v \text{ on } B(x,2R)$$

the following inequality holds

$$\max_{B(x,R)} v\leq C\min_{B(x,R)} v. \tag{H}$$

The notation $a_\xi\simeq b_\xi$ will be used in the whole sequel if there is a $C>1$ such that $1/Ca_\xi\leq b_\xi Ca_\xi$ for all possible ξ.

Definition 1.5 *The exit time from a set A is defined as $T_A=\min\{k:X_k\in A,X_{k+1}\notin A\}$. Its expected value denoted by $E_x(A)=\mathbb{E}(T_A|X_0=x)$. Denote $T=T_R=T_{x,R}=T_{B(x,R)}$. and the mean exit time by $E(x,R)=\mathbb{E}(T_{x,R}|X_0=x)$.*

Definition 1.6 *The graph has polynomial exit time if there is a $\beta>0$ such that for all $x\in\Gamma$ and $R>0$*

$$E(x,R)\simeq R^\beta. \tag{E_β}$$

1.3 The result in brief

The main result presents a strongly recurrent counterpart ($\alpha<\beta$) of the result of [16] (where $\alpha>\beta$) and goes beyond it on one hand giving local version of the sub-Gaussian estimate and on the other hand providing a set of equivalent conditions to it (given later in Section 2 as well as the definition of strong recurrence.).

Theorem 1.1 *For strongly recurrent graphs with the property that for all $x,y\in\Gamma,x\sim y$*

$$\frac{\mu_{x,y}}{\mu(x)}\geq p_0>0 \tag{p_0}$$

the following statements are equivalent

1. *Γ satisfies $(D),(E_\beta)$ and (H)*
2. *Γ satisfies $(UE_\beta),(LE_\beta)$*
3. *Γ satisfies (PH_β)*

Remark 1.1 *We shall see that the implications* 2. $\Longrightarrow$ 3. $\Longrightarrow$ 1. *hold for all random walks on weighted graphs. The details will be given in Section 2.*

Additionally it is proved that for the same graphs (PH_β) implies the β-Poincaré inequality which is defined below.

Definition 1.7 *The generalized Poincaré inequality in our setting is the following. For for all function f on V, $x \in \Gamma, R > 0$*

$$\sum_{y\in B(x,R)} \mu(y)\left(f(y) - f_B\right)^2 \le CR^\beta \sum_{y,z\in B(x,R+1)} \mu_{y,z}(f(y) - f(z))^2 \qquad (P_\beta)$$

where

$$f_B = \frac{1}{V(x,R)} \sum_{y\in B(x,R)} \mu(y) f(y)$$

To our best knowledge the results of Theorem 1.1 is new for $\beta = 2$ as well. It is a generalization of several works having the Gaussian estimates ($\beta = 2$) ([29], [9], [17] and their bibliography).

Results on sub-diffusive behavior are well-known in the fractal settings but only in the presence of strong local symmetry and global self-similarity (c.f. [1] and its bibliography)

We recall a new result from [17, Theorem 5.2] which is in some respect generalization of [12] [13],[24],[23] and [11].

Theorem 1.2 *The following statements are equivalent for Dirichlet spaces equipped with a metric exhibiting certain properties*

1. volume doubling and (P_2)
2. (UE_2) and (PH_2) for $h_t(x,y)$
3. (PH_2)

In fact [17] provides new and simple proof of this which involves scale-invariant local Sobolev inequality eliminating the difficult part of the Moser's parabolic iterative method. A similar result for graphs with the classical method was given by [9].

These findings are partly extended in [17, Section 5.] to the sub-Gaussian case, (non-classical case as it is called there), showing that on Dirichlet spaces with proper metric

$$(UE_\beta) \text{ and } (LE_\beta) \Longrightarrow (PH_\beta) \text{ and } (D)$$

which is exactly 2. $\Longrightarrow$ 3. in Theorem 1.1 in the context of the paper [17]. Let us point out that Theorem 1.1 uses the usual shortest path metric without further assumption.

Our paper is confined to graphs, but from the definitions, results and proof it will be clear that they generalize in measure metric spaces and in several cases the handling of continuous space and time would be even easier.

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2 Preliminaries

2.1 Basic Definitions

In this section we give the necessary definitions and formulate the main result in detail.

Condition 1 *During the whole paper for all* $x \sim y$

$$P(x,y) = \frac{\mu_{x,y}}{\mu(x)} \geq p_0 > 0 \tag{p_0}$$

is a standing assumption.

The analysis of the random walk needs some basic elements of potential theory([10]). For any finite subgraph, say for a ball $A = B(w,R)$, $w \in \Gamma$, $R > 0$ the definition of the resistance (on the subgraph induced on A) $\rho(B,C) = \rho^A(B,C)$ between two sets $B, C \subset A$ is a well defined quantity if $\mu_{x.y}^{-1}$ is the resistance associated to the edge $x \sim y$. Thanks to the monotonicity principle (c.f. [10]) this can be extended to the infinite graph, but we do not need it here. For the sake of short notation we shall introduce for $x \in \Gamma, R > r \geq 1$

$$\rho(x,R) = \rho(\{x\}, S(x,R))$$

and

$$\rho(x,r,R) = \rho(B(x,r), S(x,R))$$

for the resistance of the annulus.

Definition 2.1 *We say that the random walk (or the graph) is strongly recurrent if there is a* $c_\rho > 0, M \geq 2$ *such that for all* $x \in \Gamma$, $R \geq 1$

$$\rho(x, MR) \geq (1 + c_\rho)\rho(x,R). \tag{SR}$$

Remark 2.1 *It is evident that from (SR) it follows that there is a* $\delta > 0$ *and* $c > 0$ *for which* $\rho(x,R) > cR^\delta$ ($\delta = \log_2(1+c_\rho)$)*. It is well known that a random walk is recurrent if* $\rho(x,R) \to \infty$ *(c.f.[21], [10]), which means that strongly recurrent walks are recurrent.*

The weakly recurrent case (i.e. the random walk is recurrent but (SR) is not true) is not dealt with in the present paper. In this case, a similar result is expected along very similar arguments, but the appearance of slowly varying functions brings in extra technical difficulties.

Definition 2.2 *For $A \subset \Gamma$, $P^A = P^A(y,z) = P(y,z)|_{A\times A}$ is a sub-stochastic matrix, the restriction of P to the set A. It's iterates are denoted by P_k^A and it defines also a random walk, killed at the exiting from the ball.*

$$G^A(y,z) = \sum_{k=0}^{\infty} P_k^A(y,z),$$
$$g^A(y,z) = \frac{1}{\mu(z)} G^A(y,z)$$

is the local Green function (and Green kernel respectively). The notation $P^R = P^{x,R} = P^{B(x,R)}(y,z)$ will be used for $A = B(x,R)$ and for the corresponding Green function by G^R.

Remark 2.2 *It is well-known that (c.f. [25])*

$$G^R(x,x) = \mu(x)\rho(x,R)$$

as special case of

$$G^A(x,x) = \mu(x)\rho(x,\partial A)$$

where we have used the notation ∂A for the boundary of $A : \partial A = \{z \in \Gamma\backslash A : \exists y \in A$ and $y \sim z\}$

Definition 2.3 *We introduce the maximal recurrent resistance of a set $A \subset \Gamma$ with respect to the internal Dirichlet problem*

$$\overline{\rho}(A) = \max_{y\in A} \rho(y,\partial A)$$

which is by the above remark

$$\overline{\rho}(A) = \max_{y\in A} G^A(y,y)/\mu(y).$$

Definition 2.4 *We say that the graph has regular (relative to the volume) resistance growth if there is a $\mu > 0$ such that for all $x \in \Gamma, R > 0$*

$$\rho(x,R) \simeq \frac{R^\mu}{V(x,R)}. \qquad (\rho_\mu)$$

Definition 2.5 *The annulus resistance growth rate is defined similarly. It holds if there is a $C > 0, \mu > 0, M \geq 2$ such that for all $x \in \Gamma, R > 0$*

$$\rho(x,R,MR) \simeq \frac{R^\mu}{V(x,R)} \qquad (\rho_A,\mu)$$

The Laplace operator of finite sets is $\Delta_A = I - P^A = (I-P)|_{A\times A}$or particularly for balls is $I - P^{B(x,R)} = (I-P)|_{B(x,R)\times B(x,R)}$. The smallest eigenvalue is denoted in general by $\lambda(A)$ and for $A = B(x,R)$ by $\lambda = \lambda(x,R) = \lambda(B(x,R))$. For variational definition and properties see [8].

Definition 2.6 *We shall say that the graph has regular eigenvalue property if there is a $\nu > 0$ such that for all $x \in \Gamma, R > 0$*

$$\lambda(x,R) \simeq R^{-\nu}. \qquad (\lambda_\nu)$$

2.2 Statement of the results

The main result is the following

Theorem 2.1 *For a strongly recurrent weighted graph (Γ, μ) if (p_0) holds then the following statements are equivalent*

1. (Γ, μ) *satisfies* (D),(H) *and* $\left\{\begin{array}{c} (E_\beta) \text{ or} \\ (\rho_\beta) \text{ or} \\ (\rho_{A,\beta}) \text{ or} \\ (\lambda_\beta) \end{array}\right.$
2. (Γ, μ) *satisfies* $(UE_\beta), (LE_\beta)$
3. (Γ, μ) *satisfies* (PH_β)

In fact we show more in the course of the proof, namely.

Theorem 2.2 *For all weighted graph (Γ, μ) with (p_0) then each of the statements below imply the next one.*

1. (Γ, μ) *satisfies* $(UE_\beta), (LE_\beta)$
2. (Γ, μ) *satisfies* (PH_β)
3. (Γ, μ) *satisfies* (D),(H) *and* $(\rho_{A,\beta})$

The proof of Theorem 2.1 follows the pattern shown below.

$$\begin{array}{cc}
\underline{(p_0)+(D)+(E_\beta)+(H)} & \\
 & \overset{Proposition 3.1}{\Downarrow} \\
\Downarrow & (\overline{E}) \\
\overset{Theorem\ 4.1}{\Downarrow} & \overset{Theorem\ 3.1}{\Downarrow} \\
((E_\beta) \Longleftrightarrow (\rho_\beta) \Longleftrightarrow (\rho_{A,\beta}) \Longleftrightarrow (\lambda_\beta)) & (\Psi) \\
\hline
\end{array}$$

$$\begin{array}{ccc}
(D)+(\overline{E}) & \underbrace{(D)+(E_\beta)+(\overline{E})} & \\
\overset{Theorem\ 5.1}{\Downarrow} & \overset{Theorem\ 5.1}{\Downarrow} & +(D)+(H)+(\Psi) \\
(DLE) & (DUE)+(PUE) & \\
\hline
\end{array}$$

$$\begin{array}{cc}
\underbrace{(DUE)+(DLE)+(H)}+(D) & \underbrace{(PUE)+(\Psi)} \\
\overset{Proposition\ 6.3,6.4}{\Downarrow} & \overset{Theorem\ 6.1}{\Downarrow} \\
\underbrace{(NLE)+(D)} & (UE_\beta) \\
\overset{Proposition 6.6}{\Downarrow} & \\
(LE_\beta) &
\end{array}$$

The idea, that in statement 1. of Theorem 2.1, the conditions regarding time, resistance and eigenvalue might be equivalent is due to A. Grigor'yan, as well as the suggestion that the $R^\beta-$parabolic Harnack inequality could be inserted as a third equivalent statement.

The proof of the lower estimate is basically the same as it was given in [16]. The proof of the upper estimate and the equivalence of the conditions need several steps and new arguments. Corollary 4.6 and Theorem 4.1, collect some scaling relations. Theorem 5.1 uses the $\lambda-$resolvent technique (c.f. [5], [27]) while Theorem 6.1 is a generalization of [13].

During the whole paper several constants should be handled. To make their role transparent we introduce some convention. For important constants like C_V we introduce a separate notation, for unimportant small (< 1) constants we will use c and big (> 1) constants will be denoted by C. The by-product constants of the calculation will be absorbed into one.

3 The exit time

Let us introduce the notation

$$\overline{E}(R) = \overline{E}(x,R) = \max_{w\in B(x,R)} E(T_{B(x,R)}|X_0 = w).$$

Definition 3.1 *The graph satisfies the center-point condition if there is a $\overline{C} > 0$ such that*

$$\overline{E}(x,R) \leq \overline{C}E(x,R) \tag{$\overline{E}$}$$

for all $x \in \Gamma$ and $R > 0$.

Proposition 3.1 *For all graphs (E_β) implies $(\overline{E})$ and*

$$\overline{E}(x,R) \simeq R^\beta. \tag{$\overline{E}_\beta$}$$

Proof. It is clear that $B(x,R) \subset B(y,2R)$ for all $y \in B(x,R)$, consequently for $\overline{y}$ where the maximum of $\mathbb{E}_{(.)}(T_{B(x,R)})$ is attained

$$\overline{E}(x,R) = \mathbb{E}_{\overline{y}}(T_{B(x,R)}) \leq E(\overline{y},2R) \leq CR^\beta$$

while by definition

$$\overline{E}(x,R) \geq E(x,R) \geq cR^\beta.$$

■

The next Lemma has an important role in the estimate of the exit time and in the estimate of the $\lambda-$resolvent introduced later.

Lemma 3.1 *For all $A \subset \Gamma$, $x \in A$, and $t \geq 0$, we have*

$$\mathbb{P}_x(T_A < t) \leq 1 - \frac{E_x(A)}{\overline{E}(A)} + \frac{t}{2\overline{E}(A)}. \tag{3.1}$$

Proof. Denote $n = \lfloor t \rfloor$ and observe that

$$T_A \leq t + \mathbf{1}_{\{T_A > t\}} T_A \circ \theta_n$$

where θ_n is the time shift operator. Since $\{T_A > t\} = \{T_A > n\}$, we obtain, by the strong Markov property,

$$\mathbb{E}_x(T_A) \leq t + \mathbb{E}_x\left(\mathbf{1}_{\{T_A > t\}} \mathbb{E}_{X_n}(T_A)\right) \leq t + \mathbb{P}_x(T_A > t) \overline{E}(A).$$

Applying the definition $E_x(A) = \mathbb{E}_x(T_A)$, we obtain (3.1). ■

The following Theorem is taken from [16], see also [27],[28].

Theorem 3.1 *Assume that the graph (Γ, μ) possesses the property (E_β), then there are $c_\Psi, C > 0$ such that for all $x \in \Gamma$, $R \geq 1$ and $n \geq 1$, we have*

$$\Psi(x, R) = \mathbb{P}_x(T_{x,R} \leq n) \leq C \exp\left(-c_\Psi \left(\frac{R^\beta}{n}\right)^{\frac{1}{\beta-1}}\right). \tag{Ψ}$$

4 Some potential theory

Before we start the potential analysis we ought to recall some properties of the measure and volume.

Proposition 4.1 *If (p_0) holds then, for all $x \in \Gamma$ and $R > 0$ and for some $C = C(p_0)$,*

$$V(x, R) \leq C^R \mu(x). \tag{4.2}$$

Remark 4.1 Inequality (4.2) implies that, for a bounded range of R, $V(x, R) \simeq \mu(x)$.

Proof. Let $x \sim y$. Since $P(x, y) = \frac{\mu_{xy}}{\mu(x)}$ and $\mu_{xy} \leq \mu(y)$, the hypothesis (p_0) implies $p_0\mu(x) \leq \mu(y)$. Similarly, $p_0\mu(y) \leq \mu(x)$. Iterating these inequalities, we obtain, for arbitrary x and y,

$$p_0^{d(x,y)} \mu(y) \leq \mu(x). \tag{4.3}$$

Another consequence of (p_0) is that any point x has at most p_0^{-1} neighbors. Therefore, any ball $B(x, R)$ has at most C^R vertices inside. By (4.3) the measure of $y \in B(x, R)$ is at most $p_0^{-R}\mu(x)$, whence (4.2) follows. ■

The volume doubling has a well-known consequence, the so-called covering principle, which is the following

Proposition 4.2 *If (p_0) and (D) hold then there is a fixed K such that for all $x \in \Gamma$, $R > 0$, $B(x, R)$ can be covered with at most K balls of radius $R/2$.*

Proof. The proof is elementary and well-known, hence it is omitted. The only point which needs some attention is that for $R < 2$ condition (p_0) has to be used. ■

We need some consequences of (D). The volume function V acts on $\Gamma \times \mathbb{N}$ and has some further remarkable properties ([8, Lemma 2.2]).

Lemma 4.1 *There is a* $C > 0$, $K > 0$ *such that for all* $x \in \Gamma, R \geq S > 0, y \in B(x,R)$

$$\frac{V(x,R)}{V(y,S)} \leq C\left(\frac{R}{S}\right)^{\alpha} \tag{V_1}$$

where $\alpha = \log_2 C_V$ *and*

$$2V(x,R) \leq V(x,KR). \tag{V_2}$$

Definition 4.1 *The graph has property* (HG) *if the local Green functions displays regular behavior in the following sense. There is a constant* $L = L(A_0, A_1, A_2, A_3) > 0$ *integer such that for all* $x \in \Gamma$, $R > 1$,

$$\max_{w \in B(x,A_2R)\setminus B(x,A_1R)} \max_{y \in B(x,A_0R)} \max_{z \in B(x,A_0R)} \frac{G^{A_3R}(y,w)}{G^{A_3R}(z,w)} < L. \tag{HG}$$

The analysis of the local Green function starts with the following Lemma which has been proved in [16, Lemma 9.2].

Lemma 4.2 *Let* $B_0 \subset B_1 \subset B_2 \subset B_3$ *be a sequence of finite sets in* Γ *such that* $\overline{B_i} \subset B_{i+1}$, $i = 0, 1, 2$. *Denote* $A = \overline{B_2} \setminus B_1$, $B = B_0$ *and* $U = B_3$. *Then, for any non-negative harmonic function* u *in* B_2,

$$\max_B u \leq H \inf_B u \tag{4.4}$$

where

$$H := \max_{x \in B} \max_{y \in B} \max_{z \in A} \frac{G^U(y,z)}{G^U(x,z)}. \tag{4.5}$$

Proof. The following potential-theoretic argument is borrowed from [6]. Denote for an $X \subset \Gamma$ $\overline{X} = X \cup \partial X$. Given a non-negative harmonic function u in B_2, denote by S_u the following class of superharmonic functions:

$$S_u = \left\{v : v \geq 0 \text{ in } \overline{U},\ \Delta v \leq 0 \text{ in } U, \text{ and } v \geq u \text{ in } \overline{B_1}\right\}.$$

Define the function w on $\overline{U}$ by

$$w(x) = \min\left\{v(x) : v \in S_u\right\}. \tag{4.6}$$

Clearly, $w \in S_u$. Since the function u itself is also in S_u, we have $w \leq u$ in $\overline{U}$. On the other hand, by definition of S_u, $w \geq u$ in $\overline{B_1}$, whence we see that $u = w$ in $\overline{B_1}$. In particular, it suffices to prove (4.4) for w instead of u.

Let us show that $w \in c_0(U)$. Indeed, let $v(x) = E_x(U)$. Let us recall that the function $E_x(U)$ solves the following boundary value problem in U:

$$\begin{cases} \Delta u = 1 & \text{in } U, \\ u = 0 & \text{outside } U. \end{cases} \tag{4.7}$$

Using this and the strong minimum principle, v is superharmonic and strictly positive in U. Hence, for a large enough constant C, we have $Cv \geq u$ in $\overline{B_1}$ whence $Cv \in S_u$ and $w \leq Cv$. Since $v = 0$ in $\overline{U} \setminus U$, this implies $w = 0$ in $\overline{U} \setminus U$ and $w \in c_0(U)$.

Denote $f := \Delta w$. Since $w \in c_0(U)$, we have, for any $x \in U$,

$$w(x) = \sum_{z \in U} G^U(x,z) f(z). \tag{4.8}$$

Next we will prove that $f = 0$ outside A so that the summation in (4.8) can be restricted to $z \in A$. Given that much, we obtain, for all $x, y \in B$,

$$\frac{w(y)}{w(x)} = \frac{\sum_{z \in A} G^U(y,z) f(z)}{\sum_{z \in A} G^U(x,z) f(z)} \leq H,$$

whence (4.4) follows.

We are left to verify that w is harmonic in B_1 and outside $\overline{B_1}$. Indeed, if $x \in B_1$ then

$$\Delta w(x) = \Delta u(x) = 0,$$

because $w = u$ in $\overline{U_1}$. Let $\Delta w(x) \neq 0$ for some $x \in U \setminus \overline{B_1}$. Since w is superharmonic, we have $\Delta w(x) < 0$ and

$$w(x) > Pw(x) = \sum_{y \sim x} P(x,y) w(y).$$

Consider the function w' which is equal to w everywhere in $\overline{U}$ except for the point x, and w' at x is defined to satisfy

$$w'(x) = \sum_{y \sim x} P(x,y) w'(y).$$

Clearly, $w'(x) < w(x)$, and w' is superharmonic in U. Since $w' = w = u$ in $\overline{B_1}$, we have $w' \in S_u$. Hence, by the definition (4.6) of w, $w \leq w'$ in $\overline{U}$ which contradicts $w(x) > w'(x)$. ■

Corollary 4.1 *If (p_0) is true then (HG) and (H) are equivalent.*

Proof. The proof of $(HG) \Longrightarrow (H)$ is just an application of the above lemma setting $B_0 = R(x, A_0 R), B_1 = B(x, A_1 R), B_2 = B(x, A_2 R)$, $B_3 = B(x, A_3 R)$. The opposite direction follows by finitely many repetition of (H) using the balls covering $B(x, A_2 R) \backslash B(x, A_1 R)$ provided by the covering principle. ■

Proposition 4.3 *If (SR) and (H) holds then there is a $c > 0$ such that for all $x \in \Gamma, R > 0$*

$$\rho(x, R, 2R) \geq c\rho(x, 2R). \tag{$\rho_A > \rho$}$$

Proof. Denote $A = B(x, MR)$ and let us define the super-level sets of G^A as $H_y = (z \in B(x, MR) : G^A(x,z) > G^A(x,y)\}$ and Γ_y the potential level of y using the linear interpolation on the edges ([26, Section 4.]). For any $y \in S(x,R)$

$$\rho(x, MR) = \rho(x, \Gamma_y) + \rho(\Gamma_y, S_{x,MR}).$$

Let us choose $w \in S(x,R)$ which maximize $\rho(\Gamma_y, S_{x,MR})$. From the maximum principle and the choice of w it follows that $\rho(x, \Gamma_y)$ is minimized and

$$\rho(x, \Gamma_w) \leq \rho(x, R)$$

on the other hand (c.f.[25]) $\rho(\Gamma_w, S(x,MR)) = \frac{1}{\mu(x)} G^A(w,x)$, and using (HG) it follows that

$$\rho(\Gamma_w, S(x,MR)) \le \frac{L}{\mu(x)} \min_{y\in S(x,R)} G^A(y,x) \le L\rho(x,R,MR)$$

which provides

$$\rho(x,MR) \le \rho(x,R) + L\rho(x,R,MR) \le \frac{1}{1+c_\rho}\rho(x,MR) + L\rho(x,R,MR)$$

where the last inequality is a consequence of (SR). Finally it follows that

$$\rho(x,MR) \le \frac{1+c_\rho}{c_\rho} L\rho(x,R,MR).$$

■

Remark 4.2 *The converse of this proposition is straightforward. If for all $x\in\Gamma, R>1$*

$$\rho(x,R,MR) \ge c\rho(x,MR)$$

then the random walk is strongly recurrent. This follows from the shorting (c.f. [25]) of $S(x,R)$ which gives the inequality

$$\rho(x,MR) \ge \rho(x,R) + \rho(x,R,MR)$$

and using the condition

$$\rho(x,MR) \ge \rho(x,R) + c\rho(x,MR)$$

follows (SR).

Corollary 4.2 *If (SR) and (H) hold then*

$$\rho(x,MR) \ge \rho(x,R,MR) \ge c\rho(x,MR)$$

and consequently

$$\rho(x,R,MR) \simeq \rho(x,MR).$$

Hence $(\rho_\mu) \Longleftrightarrow (\rho_{A,\mu})$ holds under statement 1. in Theorem 2.1.

Corollary 4.3 *If $(p_0), (D)$ and (ρ_β) hold then*

$$(SR) \Longleftrightarrow \alpha<\beta \Longleftrightarrow \rho(x,R) \ge cR^\delta$$

where $c>0, \delta>0$ independent of x and R.

We included this corollary for sake of completeness in order to connect our definition of strong recurrence with the usual one. The proof is easy, we give it in brief.

Proof. The implication $(SR) \Longrightarrow \rho(x,R) \ge cR^\delta$ is evident. Assume $\rho(x,R)\ge cR^\delta$. Using (ρ_β) one gets

$$\frac{R^\beta}{V(x,R)} \ge cR^\delta$$

which gives

$$V(x,R) \leq CR^{\beta-\delta}$$

and $a < \beta$, applying limpsup on both sides. Finally again from $(\rho_\beta)$and $\alpha < \beta$

$$\rho(x,MR) \geq c\frac{(MR)^\beta}{V(x,MR)} \overset{(D)}{\geq} cM^{\beta-\alpha}\frac{R^\beta}{V(x,R)} \geq cM^{\beta-\alpha}\rho(x,R)$$

and $M = \left(\frac{1+c_\rho}{c}\right)^{\frac{1}{\beta-\alpha}}$ provides (SR). ■

Corollary 4.4 *If (SR) and (H) holds then there is a $C > 1$ such that for all $x \in \Gamma, R > 0$*

$$G^{x,MR}(x,x) \leq C \min_{y\in B(x,R)} G^{x,MR}(y,x). \tag{CG}$$

Proof. Let us use Proposition 4.3.

$$\begin{aligned}\frac{1}{\mu(x)}G^{x,MR}(x,x) &= \rho(x,MR) \leq C\rho(x,R,MR) \\ &\leq \max_{y\in B(x,MR)\setminus B(x,R)} \frac{C}{\mu(x)}G^{x,MR}(y,x)\end{aligned}$$

where the last inequality follows from the maximum principle. The potential level of the vertex w maximizing $G^{x,MR}(.,x)$ runs inside of $B(x,R)$ and $w \in S(x,R)$. Here we assume that $R \geq 3$ and apply (HG) with $A_0 = 1/3, A_1 = 1/2, A_2 = 1, A_3 = M$.

$$\begin{aligned}\max_{y\in B(x,MR)\setminus B(x,R)} \frac{C}{\mu(x)}G^{x,MR}(y,x) &= \max_{y\in S(x,R)} \frac{C}{\mu(y)}G^{x,MR}(x,y) \overset{(HG)}{\leq} \\ \min_{y\in S(x,R)} \frac{CL}{\mu(y)}G^{x,MR}(x,y) &= \min_{y\in B(x,R)} \frac{C}{\mu(x)}G^{x,MR}(y,x).\end{aligned}$$

For $R \leq 2$ we use (p_0) adjusting the constant C. ■

The next proposition[1] is an easy adaptation of [25].

Proposition 4.4 *For strongly recurrent walks if (D) and (CG) hold then*

$$\rho V \simeq E.$$

More precisely there is a constant $c > 0$ such that for all $x \in \Gamma, R > 0$

$$cV(x,R)\rho(x,R) \leq E(x,R) \leq V(x,R)\rho(x,R). \tag{4.9}$$

In addition (ρ_β) holds if and only if (E_β) holds.

[1] Special thanks are due to T. Delmotte pointing out that condition of strong recurrence was missing but essential for the lower estimate in the Proposition and later in the sequel.

Proof. The upper estimate is trivial

$$E(x,R) = \sum_{y\in B(x,R)} G^R(x,y) = \sum_{y\in B(x,R)} \frac{\mu(y)}{\mu(x)} G^R(y,x)$$
$$= \sum_{y\in B(x,R)} \frac{\mu(y)}{\mu(x)} P(T_x < T_R) G^R(x,x) = \frac{1}{\mu(x)} G^R(x,x) V(x,R).$$

The lower estimate is almost as simple as the upper one.

$$E(x,MR) = \sum_{y\in B(x,MR)} G^{MR}(x,y) \geq \sum_{y\in B(x,R)} \frac{\mu(y)}{\mu(x)} G^{MR}(y,x)$$

at this point one can use (CG) to get

$$\sum_{y\in B(x,R)} \frac{\mu(y)}{\mu(x)} G^{MR}(y,x) \geq \frac{1}{C} \sum_{y\in B(x,R)} \frac{\mu(y)}{\mu(x)} G^{MR}(x,x)$$
$$= \frac{1}{C} \rho(x,MR) V(x,MR)$$

from which the statement follows for all $R = M^i$. For intermediate values of R the statement follows using $R > M^i$ trivial lover estimate and decrease of the leading constant as well as for $R < M$ using (p_0). ∎

The first eigenvalue of the Laplace operator $I - P^A$ for a set $A \subset \Gamma$ is one of the key objects in the study of random walks (c.f. [8]). Since it turned out that the other important tools are the resistance properties, it is worth finding a connection between them. Such connection was already established in [26] and [27]. Now we present some elementary observations which will be used in the rest of the proofs, and are interesting on their own.

Lemma 4.3 *For all random walks on* (Γ, μ) *and for all* $A \subset \Gamma$

$$\lambda^{-1}(A) \leq \overline{E}(A) \tag{$\lambda\overline{E}$}$$

Proof. Assume that $f \geq 0$ is the eigenfunction corresponding to $\lambda = \lambda(A)$, the smallest eigenvalue of the Laplace operator $\Delta_A = I - P^A$ on A and let f be normalized so that $\max_{y\in A} f(y) = f(x) = 1$. It is clear that

$$E(T_A) = \sum_{y\in A} G^A(x,y)$$

while $\Delta_A^{-1} = G^A$ consequently

$$\frac{1}{\lambda} = \frac{1}{\lambda} f(x) = G^A f(x) \leq \sum_{y\in A} G^A(x,y) = E_x(T_A)$$

which gives the statement. ∎

Lemma 4.4 *For all random walks on* (Γ,μ) *it is obvious that*

$$E_x(T_A)\le\rho(x,\partial A)\mu(A) \tag{4.10}$$

and

$$\overline{E}(A)\le\overline{\rho}(A)\mu(A). \tag{4.11}$$

Proof.

$$\begin{aligned}E_x(T_A)&=\sum_{y\in A}G^A(x,y)=\sum_{y\in A}G^A(y,x)\frac{\mu(y)}{\mu(x)}\\&\le\frac{G^A(x,x)}{\mu(x)}\sum_{y\in A}P_y(T_A>T_x)\mu(y)\le\rho(x,\partial A)\mu(A).\end{aligned}$$

The second statement follows from the first one taking maximum for $x\in A$ on both sides. ■

Proposition 4.5 *(c.f. [27],[28])For all random walks on* (Γ,μ) *and for* $A\subset B\subset\Gamma$ *finite sets*

$$\lambda(B)\le\frac{\rho(A,B)\mu(B\backslash intA)}{\mathbb{E}(T_{a,B})^2}$$

where $T_{a,B}$ *denotes the exit time from* B *on the modified graph* Γ_a*, where* A *shrunk into a single vertex* a *which has all the edges to vertices* $B\backslash A$ *which connects* A *and* $B\backslash A$*. (All the rest of the graph remains the same as in* Γ*.)*

Proof. We repeat here the proof of the cited works briefly. Consider the smallest eigenvalue of the Laplacian of B.

$$\lambda(B)=\inf\frac{\left((I-P)^Bf,f\right)}{\|f\|_2^2}\le\frac{\left((I-P)^Bv,v\right)}{\|v\|_2^2}$$

if $v(z)$ is the harmonic function on $B\backslash\{a\}$, $v(a)=R(a,B)$, $v(z)=0$ if $z\in\Gamma\backslash B$. It is easy to see that

$$\left((I-P)^Bv,v\right)=R(A,B)$$

while using the Cauchy-Schwarz inequality

$$\|v\|_2^2\ge\frac{\mathbb{E}(T_{a,B})^2}{\mu(B\backslash A)}.$$

■

Corollary 4.5 *(c.f. [27],[28])For all random walks on weighted graphs and* $R\ge2$

$$\lambda(x,2R)\le\frac{\rho(x,R,2R)V(x,2R)}{E(\underline{w},R/2)^2}$$

where $\underline{w}\in S(x,3/2R)$ *minimizes* $E(w,R/2)$*.*

Proof. Apply Proposition 4.5 with $A = B(x,R)$, $B = B(x,2R)$ and observe that the walk should cross $S(x,3/2R)$ before exit from B and restarting from this crossing point we get the estimate

$$\mathbb{E}(T_{a,B}) \geq \min_{w \in S(x,3/2R)} E(w,R/2)$$

which provides the statement. ∎

Proposition 4.6 *For all recurrent random walks and for all $A \subset B \subset \Gamma$*

$$\lambda(B)\rho(A,B)\mu(A) < 1 \tag{4.12}$$

particularly for $B = B(x,2R)$, $A = B(x,R)$, $x \in \Gamma, R \geq 1$

$$\lambda(x,2R)\rho(x,R,2R)V(x,R) \leq 1, \tag{4.13}$$

furthermore assuming (D)

$$\lambda(x,2R)\rho(x,R,2R)V(2R) \leq C, \tag{4.14}$$

and for $B = B(x,R)$, $A = \{x\}$ if $(D),(SR)$ and (H) hold then

$$\lambda(x,R)\rho(y,R)V(x,R) \leq C. \tag{4.15}$$

Proof. The idea of the proof is based on [15] and [26]. Consider $u(y)$ harmonic function on B defined by the boundary values $u(x) = 1$ on $x \in A, u(y) = 0$ for $y \in \Gamma \backslash B$. This is the capacity potential for the pair A, B. It is clear that $1 \geq u \geq 0$ by the maximum principle. From the variational definition of λ it follows that

$$\lambda(B) \leq \frac{((I - P^A)u,u)}{(u,u)} \leq \frac{1}{\rho(A,B)\mu(A)}$$

where we have used the Ohm law, which says that the unit potential drops from 1 to 0 between ∂A to B results $I_{eff} = 1/R_{eff} = 1/\rho(A,B)$, incoming current through ∂A and the outgoing "negative" current through ∂B. It is clear that (4.13) is just a particular case of (4.12), (4.14) follows from (4.13) using (D) finally, (4.15) can be seen applying Corollary (4.2). ∎

The above results have an important consequence. It is useful to state it separately.

Corollary 4.6 *If $(p_0),(SR)$ and (H) holds then for all $x \in \Gamma, R \geq 1$*

$$E \simeq \overline{E} \simeq \lambda^{-1} \simeq \rho V \simeq \rho_A V \simeq \overline{\rho} V \tag{4.16}$$

where the arguments (x,R) are suppressed and $\rho_A = \rho(x,R,2R)$.

Proof. The proof is straightforward from Corollary 4.2, proposition 4.4,4.6 and Lemma4.3. ∎

Theorem 4.1 *Assume $(p_0),(SR)$ and (H) then the following statements are equivalent for all $x \in \Gamma, R \geq 1$*

$$E(x,R) \simeq R^\beta \tag{4.17}$$

follows

$$\lambda(x,R) \simeq R^{-\beta} \tag{4.18}$$

and

$$\rho(x,R) \simeq \frac{R^{\beta}}{V(x,R)}, \tag{4.19}$$

$$\rho(x,R,2R) \simeq \frac{R^{\beta}}{V(x,R)}. \tag{4.20}$$

Proof. Thanks to Corollary 4.4 (SR) and (H) implies (CG) and by Proposition 4.4 from (CG) follows (4.9) and directly (4.17) $\Longleftrightarrow$ (4.19),while (4.19) $\Longleftrightarrow$ (4.20) follows from Corollary 4.2. On the other hand (4.17) $\Longleftrightarrow$ (4.18) is a direct consequence of Proposition 3.1 and Corollary 4.6. ∎

This Theorem shows that the alternatives under the first condition in Theorem 2.1 are equivalent.

5 The diagonal estimates

The on-diagonal estimates basically were given in [26]. There the main goal was to get a Weyl type result by controlling of the spectral density via the diagonal upper (and lower) bounds of the process, killed at leaving $B(x,R)$. The result immediately extends to the transition probabilities of the original chain.

Theorem 5.1 *If* $(p_0)(D),(E_\beta)$ *and* (H) *hold then there are* $c_i, C_j > 0$ *such that for* $n, R \geq 1, x \in \Gamma$

$$P_n(x,x) \leq C_1 \frac{\mu(x)}{V(x,n^{\frac{1}{\beta}})} \tag{DUE}$$

$$P_n(x,y) \leq C_2 \frac{\mu(y)}{\left(V(x,n^{\frac{1}{\beta}})V(y,n^{\frac{1}{\beta}})\right)^{1/2}} \tag{PUE}$$

and furthermore if $n \leq c_3 R^{\beta}$ *then*

$$P_{2n}(x,x) \geq P_n^{B(x,R)}(x,x) \geq C_4 \frac{\mu(x)}{V(x,n^{\frac{1}{\beta}})}. \tag{DLE}$$

The (DLE) follows from the next simple observation

Proposition 5.1 *For all* (Γ,μ) *for* $A \subset \Gamma$, *and fixed* $w \in A$ *if*

$$\overline{E}(A) \leq C_0 E_w(A)$$

then for $n \leq \frac{1}{2} E_w(A)$

$$P_{2n}^A(w,w) \geq \frac{c\mu(w)}{\mu(A)}.$$

Proof. ¿From the condition using Lemma 3.1 it follows, that if $n \leq \frac{1}{2}E_w(A)$ then

$$P_w(T_R > n) > \frac{E_w(A) - n}{\overline{E}(A)} = \frac{1}{2C_0} = c > 0$$

$$\begin{aligned}
c^2 &\leq P_w(T_R > n)^2 \leq \left(e_w^* P_n^A \mathbf{1}\right)^2 \qquad (5.21)\\
&\leq \left(\sum_{y\in A} P_n^A(w,y)\sqrt{\frac{\mu(y)}{\mu(y)}}\right)^2 \\
&\leq \left(\sum_{y\in A}\mu(y)\right)\left(\sum_{y\in A}\frac{P_n^A(w,y)^2}{\mu(y)}\right) \\
&= \mu(A)\left(\sum_{y\in A} P_n^A(w,y)\frac{P_n^A(y,w)}{\mu(w)}\right) \\
&\leq \frac{1}{\mu(w)}\mu(A)P_{2n}^A(w,w)
\end{aligned}$$

which was to be shown. ■

Corollary 5.1 *If (p_0) and $(\overline{E})$ holds then*

$$P_{2n}^{B(x,R)}(x,x) \geq c\frac{\mu(x)}{V(x,R)} \geq c\frac{\mu(x)}{V(x,\frac{1}{\delta}n^{\frac{1}{\beta}})}$$

if $\delta < 1$ and $\delta R^\beta > n$.

Proof. The statement follows from Proposition 5.1.

Proposition 5.2 *If $(p_0), (D)$, (E_β) and (H) holds then there is a $\delta > 0$ such that for all $R > 0$ and $1 \leq n < \delta R^\beta$*

$$P_{2n}(x,x) \geq P_{2n}^{B(x,R)}(x,x) \geq c\frac{\mu(x)}{V(x,n^{\frac{1}{\beta}})}. \qquad (5.22)$$

■

Proof. We can apply Proposition 5.1 for $A = B(x,R)$, to get (5.22) with $w = x$ and having $(\overline{E})$ thanks to Proposition 3.1. ■

Definition 5.1 *Let us define the $\lambda-$resolvent and recall the local Green function as follows*

$$G_\lambda(x,x) = \sum_{k=0}^{\infty} e^{-\lambda k}P_k(x,x)$$

and

$$G^R(x,x) = \sum_{k=0}^{\infty} P_k^{B(x,R)}(x,y).$$

The starting point of the proof of the (DUE) is the following lemma (from [27]) for the $\lambda-$resolvent without any change.

Lemma 5.1 *In general if* $\lambda^{-1}=n$ *then*

$$P_{2n}(x,x)\leq c\lambda G_{\lambda}(x,x).$$

Proof. The proof is elementary. It follows from the eigenfunction decomposition that $P_{2n}^{B(x,R)}(x,x)$ is non-increasing in n (c.f. [16] or [27]). For $R>2n$ $P_{2n}^{B(x,R)}(x,x)=P_{2n}(x,x)$, hence the monotonicity holds for $P_{2n}(x,x)$ in the $2n<R$ time range. But R is chosen arbitrarily, hence P_{2n} is non-increasing and we derive

$$G_{\lambda}(x,x)=\sum_{k=0}^{\infty}e^{-\lambda k}P_{k}(x,x)\geq\sum_{k=0}^{\infty}e^{-\lambda 2k}P_{2k}(x,x)\geq\sum_{k=0}^{n-1}e^{-\lambda 2k}P_{2k}(x,x)$$

$$\geq P_{2n}(x,x)\frac{1-e^{-\lambda 2n}}{1-e^{-2\lambda}}.$$

Choosing $\lambda^{-1}=n$ follows the statement ■

Lemma 5.2 *If* $(\overline{E})$ *holds then*

$$G_{\lambda}(x,x)\leq cG^{R}(x,x).$$

Proof. The argument is taken from [26, Lemma 6.4]. Let ξ_{λ} be a geometrically distributed random variable with parameter a $e^{-\lambda}$. One can see easily that

$$\begin{aligned}G_{R}(x,x)&=G_{\lambda}(x,x)+E_{x}(I(T_{R}\geq\xi_{\lambda})G_{R}(X_{\xi_{\lambda}},x))\\ &-E_{x}(I(T_{R}<\xi_{\lambda})G_{\lambda}(X_{T_{R}},x))\end{aligned}\tag{5.23}$$

from which

$$G_{\lambda}(x,x)\leq P(T_{R}\geq\xi_{\lambda})^{-1}G_{R}(x,x).$$

Here $P(T_{R}\geq\xi_{\lambda})$ can be estimated thanks to Lemma 3.1

$$\begin{aligned}P(T_{R}\geq\xi_{\lambda})&\geq P(T_{R}>n,\xi_{\lambda}\leq n)\\ &\geq P(\xi_{\lambda}\leq n)P(T_{R}>n)\geq c'\frac{E-n}{2C\overline{E}}>c.\end{aligned}$$

if $\lambda^{-1}=n=\frac{1}{2}E(x,R)$ and $(\overline{E})$ holds. ■

Proof of Theorem 5.1. Combining the previous lemmas with $\lambda^{-1}=n=\frac{1}{2}E(x,R)$ one gets

$$P_{2n}(x,x)\leq c\lambda G_{\lambda}(x,x)\leq cE(x,R)^{-1}G^{R}(x,x).$$

Now let us recall from Remark 2.2 , that $G^{R}(x,x)=\mu(x)\rho(x,R)$ and let us use the conditions

$$\begin{aligned}G^{R}(x,x)=\mu(x)\rho(x,R)&\overset{(\lambda\rho\mu)}{\leq}\frac{C\mu(x)}{\lambda(x,R)V(x,R)}\\ &\overset{(\lambda\overline{E})}{\leq}\frac{C\mu(x)\overline{E}(x,R)}{V(x,R)}\end{aligned}$$

and by Lemma 5.1 and 5.2

$$P_{2n}(x,x) \le C\mu(x)E(x,R)^{-1}\rho(x,R)$$
$$\le \frac{C\mu(x)\overline{E}}{E(x,R)V(x,R)} \overset{(\overline{E})}{\le} \frac{C\mu(x)}{V(x,R)} \overset{(D)}{\le} \frac{C\mu(x)}{V(x,n^{\frac{1}{\beta}})}.$$

¿From this it follows that $P_{2n+1}(x,x) \le c\mu(x)\ V(x,n^{\frac{1}{\beta}})^{-1}$ and with Cauchy-Schwartz and the standard argument (c.f. [8]) one has that

$$P_n(x,y) \le \mu(y)\sqrt{\frac{P_n(x,x)}{\mu(x)}\frac{P_n(y,y)}{\mu(y)}} \tag{5.24}$$

consequently

$$P_n(x,y) \le \mu(y)\left(\frac{1}{V(x,n^{\frac{1}{\beta}})V(y,n^{\frac{1}{\beta}})}\right)^{1/2}.$$

This proves (DUE) and (PUE) and (DLE) follows from Proposition 5.2. ■

6 Off-diagonal estimates

In this section we deduce the off-diagonal estimates based on the diagonal ones.

6.1 Upper estimate

The upper estimate uses an idea of [13].

Theorem 6.1 $(p_0) + (D) + (E_\beta) + (H) \Longrightarrow (UE_\beta)$

For the proof we generalize the inequality (c.f. [12, Proposition 5.1])

Lemma 6.1 *For all random walks and for any $L(s) \ge 0$ convex (non-concave from below) function $(s > 0)$ and $D > 0$*

$$P_n(x,y) \le (M(x,n)M(y,n))^{1/2} \exp -2L\left(d(x,y)\right)$$

where

$$M(w,n) = \sum_{z\in\Gamma} \frac{P_n(w,z)^2}{\mu(z)} \exp L\left(d(w,z)\right).$$

Proof. Let us observe first that the triangular inequality

$$d(x,y) \le d(x,z) + d(z,y)$$

implies using the Jensen inequality that

$$\begin{aligned}L(d(x,y)) &\leq L(d(x,z)+d(z,y)) \\ &\leq \frac{1}{2}\left(L\left(d(x,z)\right)+L\left(d(z,y)\right)\right).\end{aligned}$$

This means that

$$\exp\left(-2L(d(x,y))+L\left(d(x,z)\right)+L\left(d(z,y)\right)\right)\geq e>1$$

hence

$$\begin{aligned}P_n(x,y) &= \sum_{z\in\Gamma}P_n(x,z)P_n(z,y) = \sum_{z\in\Gamma}P_n(x,z)\frac{\mu(y)}{\mu(z)}P_n(y,z) \\ &\leq \mu(y)\sum_{z\in\Gamma}\frac{P_n(x,z)}{\mu(z)^{1/2}}\frac{P_n(y,z)}{\mu(z)^{1/2}}e^{\left(-L(d(x,y))+\frac{1}{2}(L(d(x,z))+L(d(z,y)))\right)} \\ &\leq \mu(y)e^{-L(d(x,y))}\left(\sum_{z\in\Gamma}\frac{p(x,z)^2}{\mu(z)}e^{\frac{1}{2}L(d(x,z))}\right)^{1/2} \\ &\times\left(\sum_{z\in\Gamma}\frac{p(y,z)^2}{\mu(z)}e^{\frac{1}{2}L(d(z,y))}\right)^{1/2}.\end{aligned}$$

■

Corollary 6.1 *For all random walks and $D>0,\beta>1$*

$$P_n(x,y)\leq\left(E_D(x,n)E_D(y,n)\right)^{1/2}\exp-\left(\frac{d(x,y)}{D\left(2n\right)^{\frac{1}{\beta}}}\right)^{\frac{\beta}{\beta-1}}$$

where

$$E_D(w,n)=\sum_{z\in\Gamma}\frac{P_n(w,z)^2}{\mu(z)}\exp\left(\frac{d(w,z)}{Dn^{\frac{1}{\beta}}}\right)^{\frac{\beta}{\beta-1}}$$

Proof. Consider the $L(s)=\left(\frac{s^\beta}{Dn}\right)^{\frac{1}{\beta-1}}$ function. L is non-concave if $\beta>1$ hence Lemma 6.1 applicable. ■

The next step towards to the proof of (UE_β) is to get an estimate of $E_D(w,n)$.

Lemma 6.2 *For all $w\in\Gamma$, $n\in\mathbb{N}$ (PUE) and (Ψ) implies*

$$E_D(w,n)\leq\frac{C}{V(w,n^{\frac{1}{\beta}})}$$

Proof. Let us assume first that $d(w,z) < n^{\frac{1}{\beta}}$ in the summa of E_D. In this case

$$\frac{P_n(w,z)^2}{\mu(z)} \exp\left(\frac{d(w,z)^\beta}{Dn}\right)^{\frac{1}{\beta-1}} \leq C \frac{P_n(w,z)^2}{\mu(z)}$$

hence

$$\begin{aligned}
&\sum_{d(w,z)<n^{\frac{1}{\beta}}} \frac{P_n(w,z)^2}{\mu(z)} \exp\left(\frac{d(w,z)^\beta}{Dn}\right)^{\frac{1}{\beta-1}} \\
&\leq C \sum_{d(w,z)<n^{\frac{1}{\beta}}} \frac{P_n(w,z)P_n(z,w)}{\mu(w)} \\
&\leq C \sum_{z\in\Gamma} \frac{P_n(w,z)P_n(z,w)}{\mu(w)} \overset{(DUE)+(D)}{\leq} \frac{C}{V(w,n^{\frac{1}{\beta}})}.
\end{aligned} \tag{6.25}$$

Let us consider the sum of "far away" vertices and denote $\delta = \left(\frac{1}{D}\right)^{\frac{1}{\beta-1}}$.

$$\begin{aligned}
&\sum_{d(w,z)\geq n^{\frac{1}{\beta}}} \frac{P_n(w,z)^2}{\mu(z)} \exp\left(\frac{d(w,z)^\beta}{Dn}\right)^{\frac{1}{\beta-1}} \\
&\leq \sum_{d(w,z)\geq n^{\frac{1}{\beta}}} P_n(w,z) \exp\delta\left(\frac{d(w,z)^\beta}{n}\right)^{\frac{1}{\beta-1}} \max_z \frac{P_n(w,z)}{\mu(z)}.
\end{aligned}$$

The max can be handled as usual using (PUE)

$$\begin{aligned}
\max_z \frac{P_n(w,z)}{\mu(z)} &\leq \frac{C}{V(w,n^{\frac{1}{\beta}})}\left(\frac{V(w,n^{\frac{1}{\beta}})}{V(z,n^{\frac{1}{\beta}})}\right)^{1/2} \leq \frac{C}{V(w,n^{\frac{1}{\beta}})}\left(\frac{V(w,d(w,z))}{V(z,n^{\frac{1}{\beta}})}\right)^{1/2} \\
&\overset{(D)}{\leq} \frac{C}{V(w,n^{\frac{1}{\beta}})}\left(\frac{d(w,z)}{n^{\frac{1}{\beta}}}\right)^{\alpha/2} \leq \frac{C}{V(w,n^{\frac{1}{\beta}})} C_\varepsilon \exp\frac{\varepsilon\alpha}{2}\frac{d(w,z)}{n^{\frac{1}{\beta}}} \\
&\leq \frac{C}{V(w,n^{\frac{1}{\beta}})} C_\varepsilon \exp\frac{\varepsilon\alpha}{2}\left(\frac{d(w,z)^{\frac{\beta}{\beta-1}}}{n^{\frac{1}{\beta-1}}}\right) \leq \frac{C}{V(w,n^{\frac{1}{\beta}})} \exp\frac{\varepsilon\alpha}{2}\left(\frac{d(w,z)^\beta}{n}\right)^{\frac{1}{\beta-1}}
\end{aligned}$$

Applying this in the sum

$$\begin{aligned}
&\sum_{d(w,z)\geq n^{\frac{1}{\beta}}} \frac{P_n(w,z)^2}{\mu(z)} \exp\delta\left(\frac{d(w,z)^\beta}{n}\right)^{\frac{1}{\beta-1}} \\
&\leq \frac{C}{V(w,n^{\frac{1}{\beta}})} \sum_{r=n^{\frac{1}{\beta}}}^{\infty} \sum_{z\in S(w,r)} P_n(w,z) \exp\left(\frac{\varepsilon\alpha}{2}+\delta\right)\left(\frac{r^\beta}{n}\right)^{\frac{1}{\beta-1}} \\
&= \frac{C}{V(w,n^{\frac{1}{\beta}})} \sum_{r=n^{\frac{1}{\beta}}}^{\infty} P_n(w,S(w,r)) \exp\frac{\varepsilon\alpha}{2}\left(\frac{r^\beta}{n}\right)^{\frac{1}{\beta-1}}
\end{aligned}$$

Here $P_n(w,S(w,r))$ can be estimated using (Ψ) to get further upper bound by

$$\frac{C}{V(w,n^{\frac{1}{\beta}})}\sum_{r=n^{\frac{1}{\beta}}}^{\infty} P_n(w,S(w,r))\exp\left(\frac{\varepsilon\alpha}{2}+\delta\right)\left(\frac{r^\beta}{n}\right)^{\frac{1}{\beta-1}}$$

$$\leq \frac{C}{V(w,n^{\frac{1}{\beta}})}\sum_{r=n^{\frac{1}{\beta}}}^{\infty} \exp\left(\frac{\varepsilon\alpha}{2}+\delta\right)\left[\left(\frac{r^\beta}{n}\right)^{\frac{1}{\beta-1}} - c_{\gneqq}\left(\frac{r^\beta}{n}\right)^{\frac{1}{\beta-1}}\right]$$

and if ε is chosen $\varepsilon=\frac{2c_{\gneqq}}{3\alpha}$ and $\delta=\frac{c_{\gneqq}}{3}$ (i.e. $D=\left(\frac{3}{c_{\gneqq}}\right)^{\beta-1}$) then finally we obtain

$$\frac{C}{V(w,n^{\frac{1}{\beta}})}\sum_{r=n^{\frac{1}{\beta}}}^{\infty} \exp\left(\frac{\varepsilon\alpha}{2}+\delta\right)\left[\left(\frac{r^\beta}{n}\right)^{\frac{1}{\beta-1}} - c_{\gneqq}\left(\frac{r^\beta}{n}\right)^{\frac{1}{\beta-1}}\right] \tag{6.26}$$

$$\leq \frac{C}{V(w,n^{\frac{1}{\beta}})}\sum_{r=n^{\frac{1}{\beta}}}^{\infty} \exp-\frac{c_{\gneqq}}{3}\left(\frac{r^\beta}{n}\right)^{\frac{1}{\beta-1}} \tag{6.27}$$

where the last sum is evidently bounded by a constant depending only on c_Ψ and β. The estimates in (6.25) and (6.27) provide the statement. ■

Proof of Theorem 6.1. Now we collect our findings. By Proposition 3.1 (E_β) implies (Ψ) while (PUE) is given by Theorem 5.1, consequently we can apply Lemma 6.2 in Lemma 6.1. The final step is standard to replace $V(y,n^{\frac{1}{\beta}})^{1/2}$ with

$$V(x,n^{\frac{1}{\beta}})^{1/2}C_\varepsilon\exp\frac{\varepsilon\alpha}{2}\left(\frac{d(x,y)^\beta}{n}\right)^{\frac{1}{\beta-1}}$$

as in the proof of Lemma 6.1 with a slight further decrease of the leading constant in the exponent. ■

Let us remark that this proof is considerably simpler than those given in [13], [8] with the aid of integral estimates and mean value inequalities while here we have the full power of (Ψ).

6.2 Lower estimate

For the lower estimate it is common to use the upper estimate of the time derivative of the heat kernel.

Definition 6.1 *For a function $u_n(x)$ on $\mathbb{N}\times V$ we define the time derivative as*

$$\partial_n u = u_{n+2}-u_n.$$

Definition 6.2 *For any set U and a function u on U, denote*

$$\operatorname*{osc}_U u := \max_U u - \min_U u.$$

The following statements are taken from [16].

Proposition 6.1 *([16, Proposition 11.2]) Assume that the elliptic Harnack inequality (H) holds on (Γ, μ). Let $u \in c_0(B(x,R))$ satisfy in $B(x,R)$ the equation $\Delta u = f$. Then for any $\varepsilon > 0$, there exists $\sigma = \sigma(\varepsilon, H) < 1$ such that for any positive $r < R$,*

$$\underset{B(x,\sigma r)}{\operatorname{osc}} u \leq 2\left(\overline{E}(x,r) + \varepsilon \overline{E}(x,R)\right) \max |f| . \tag{6.28}$$

Proposition 6.2 *([16, Proposition 12.1, 12.2]) For all $A \subset \Gamma, x, y \in \Gamma$, $n \in \mathbb{N}$*

$$\left|\partial_n p^A(x,y)\right| \leq Cn^{-1} \left(p_n^A(x,x) p_n^A(y,y)\right)^{1/2} . \tag{6.29}$$

Corollary 6.2 *If the (D) and (DUE) hold then there is a $C > 1$ such that for all $x, y \in \Gamma$, $n \in \mathbb{N}$, $d(x,y) \leq cn^{\frac{1}{\beta}}$*

$$\left|\partial_n P^{B(x,R)}(x,y)\right| \leq C \frac{\mu(y)}{nV(x, n^{\frac{1}{\beta}})} \tag{6.30}$$

¿From these propositions the next particular diagonal lower estimate follows again as in [16, Proposition 13.1].

Proposition 6.3 *Assume (p_0) then from $(DUE) + (DLE) + (H)$ follows that for all there is a $C > 1$ such that $x, y \in \Gamma, n \in \mathbb{N}, R > Cn^{\beta}$*

$$P_n^{B(x,R)}(x,y) + P_{n+1}^{B(x,R)}(x,y) \geq c \frac{\mu(y)}{V(x, n^{\frac{1}{\beta}})} \tag{PLE}$$

provided $d(x,y) \leq \delta n^{\frac{1}{\beta}}$.

Proof. Let us fix $x \in \Gamma$, $n \in \mathbb{N}$ and set

$$R = \left(\frac{n}{\varepsilon}\right)^{1/\beta}, \tag{6.31}$$

for small enough positive ε. So far we assume only that ε satisfies the restriction in (DLE) but later one more upper bound on ε will be imposed. Denote $A := B(x,R)$ and introduce the function

$$u(y) := p_n^A(x,y) + p_{n+1}^A(x,y).$$

By the hypothesis (DLE), we have $u(x) \geq cV(x, n^{\frac{1}{\beta}})^{-1}$. Let us show that

$$|u(x) - u(y)| \leq \frac{c}{2} \frac{1}{V(x, n^{\frac{1}{\beta}})}, \tag{6.32}$$

provided $d(x,y) \leq \delta n^{1/\beta}$, which would imply $u(y) \geq \frac{c}{2} V(x, n^{\frac{1}{\beta}})^{-1}$, hence proving (PLE).

The function $u(y)$ is in class $c_0(A)$ and solves the equation $\Delta u(y) = f(y)$ where

$$f(y) := p^A_{n+2}(x,y) - p^A_n(x,y).$$

The on-diagonal upper bound (DUE) implies, by Corollary 6.2,

$$\max|f| \leq \frac{C}{nV(x,n^{\frac{1}{\beta}})}. \tag{6.33}$$

By (H) and Proposition 6.1, we have, for any $0 < r < R$ and for some $\sigma = \sigma(\varepsilon^2) \in (0,1)$,

$$\operatorname*{osc}_{B(x,\sigma r)} u \leq 2\left(\overline{E}(x,r) + \varepsilon^2 \overline{E}(x,R)\right) \max|f|\,.$$

As it is derived in Proposition 3.1, $(E \leq)$ implies a similar upper bound for $\overline{E}$. Estimating $\max|f|$ by (6.33), we obtain

$$\operatorname*{osc}_{B(x,\sigma r)} u \leq C\frac{r^\beta + \varepsilon^2 R^\beta}{V(x,n^{\frac{1}{\beta}})}.$$

Choosing r to satisfy $r = \varepsilon R$ and substituting from (6.31) $n = (\varepsilon R)^\beta$, we obtain

$$\operatorname*{osc}_{B(x,\sigma r)} u \leq C\frac{\varepsilon^2 R^\beta}{nV(x,n^{\frac{1}{\beta}})} = \frac{\varepsilon C}{V(x,n^{\frac{1}{\beta}})},$$

which implies

$$\operatorname*{osc}_{B(x,\sigma r)} u \leq \frac{c}{2}\frac{1}{V(x,n^{\frac{1}{\beta}})}, \tag{6.34}$$

provided ε is small enough.

Note that

$$\sigma r = \sigma\varepsilon^{2/\beta} R = \sigma\varepsilon^{2/\beta}\left(\frac{n}{\varepsilon}\right)^{1/\beta} = \sigma\varepsilon^{1/\beta} n^{1/\beta} = \delta n^{1/\beta}$$

where $\delta := \sigma\varepsilon^{1/\beta}$. Hence, (6.34) implies

$$|u(x) - u(y)| \leq \frac{c}{2}\frac{1}{V(x,n^{\frac{1}{\beta}})},$$

provided $d(x,y) \leq \delta n^{1/\beta}$, which was to be proved. ■

Proposition 6.3 immediately implies the near diagonal lower estimate.

Proposition 6.4 *Assume (p_0) then from $(DUE) + (DLE) + (H)$ follows that for all $x, y \in \Gamma, n \in \mathbb{N}$*

$$\widetilde{P}_n(x,y) = P_n(x,y) + P_{n+1}(x,y) \geq c\frac{\mu(y)}{V(x,n^{\frac{1}{\beta}})} \tag{NLE}$$

provided $d(x,y) \leq \delta n^{\frac{1}{\beta}}$.

The next proposition embeds the above statement in our chain of proofs.

Proposition 6.5 *Assume* (p_0) *then*

$$(D) + (E_\beta) + (H) \Longrightarrow (NLE).$$

Proof. The statement follows from Theorem 5.1 and Proposition 6.4.

Proposition 6.6 *If* (p_0) *and* (D) *are true then*

$$(NLE) \Longrightarrow (LE_\beta)$$

■

Proof. Using (NLE) and the standard chaining argument the lower estimate (LE_β) can be easily seen. It is exhaustively elaborated for the regular volume growth case ([16, Proposition 13.3]) (or see [9, Theorem 3.8 lower bound] under the doubling condition.) and the proof generalizes automatically to the recent situation hence we give here just the key step of the proof. Assume that $d = d(x,y)$, $\delta n^{\frac{1}{\beta}} < d \le \varepsilon n$ and consider a sequence of vertices $o_i \in \Gamma$ for $i = 1, 2.. = k = C\left(\frac{d^\beta}{n}\right)^{\frac{1}{\beta-1}}$, $o_0 = x$, $o_k = y$ where $C = C(\varepsilon, \delta)$ is a big constant and

$$d(o_i, o_{i+1}) \le \left\lceil \frac{d}{k} \right\rceil =: r$$

and set $m = \left\lfloor \frac{n}{k} \right\rfloor - 1$. Recognize that $r \simeq \left(\frac{n}{d}\right)^{\frac{1}{\beta-1}}$ and $m \simeq \left(\frac{n}{d}\right)^{\frac{\beta}{\beta-1}}$ and apply (NLE) for $z_i \in B(o_i, r) =: B_i$, with $d(z_i, z_{i+1}) \le 3r \le \delta m^{\frac{1}{\beta}}$ (which can be ensured with the right choice of C). (NLE) holds in the form

$$\widetilde{P}_m(z_i, z_{i+1}) \ge c_l \frac{\mu(z_{i+1})}{V(z_i, n^{\frac{1}{\beta}})}$$

$(c_l < 1)$. This can be applied to get

$$\begin{aligned}
\widetilde{P}_n(x,y) &\ge \sum_{(z_1,..z_{k-1}) \in B_1 \times .. B_{k-1}} \frac{c_l \mu(z_1)}{V(x,r)} \frac{c_l \mu(z_2)}{V(o_1,r)} \cdots \frac{c_l \mu(y)}{V(o_{k-1},r)} \\
&\ge c' \frac{\mu(y) \exp\left(-k \log 1/c_l\right)}{V(x, n^{\frac{1}{\beta}})}
\end{aligned}$$

which is the (LE_β). For the trivial $d(x,y) \le \delta n^{\frac{1}{\beta}}$ and $> \varepsilon n$ cases see the arguments in [16]. ■

7 The return route of the proof

In this section the proof of the implications 2. $\Longrightarrow$ 3. $\Longrightarrow$ 1. in Theorem 2.1 are given. Particularly we shall prove $(\rho_{A,\beta})$ from the set of equivalent conditions in 1. and hence Theorem 2.2. It should be emphasized again that we do not use the recurrence assumption.

Theorem 7.1 $(p_0) + (UE_\beta) + (LE_\beta) \Longrightarrow (PH_\beta)$

This is proved in [17] based on a method of [11, Section 3] developed for the case $\beta = 2$, and can be reconstructed from the clear interpretation of [9, Section 3.3] as well.

Theorem 7.2 *If (p_0) and (PH_β) hold, then (D), $(\rho_{A,\beta})$ and the elliptic Harnack inequality are true.*

The elliptic Harnack inequality evidently follows from the $\beta-$parabolic one. The proof of the rest is via proving (DUE) and (PLE), namely for all $x \in \Gamma, R > 0, A = B(x, 2R)$

$$\widetilde{P}_n^A(x,y) = P_n^A(x,y) + P_{n+1}^A(x,y) \geq \frac{c\mu(y)}{V(x, n^{\frac{1}{\beta}})} \tag{7.35}$$

if $d(x,y) < R$ and $\frac{4}{9}R^\beta \leq n \leq \frac{5}{9}R^\beta$.

Proposition 7.1 $(p_0) + (PH_\beta) \Longrightarrow (DUE), (PLE\ as\ in\ 7.35), (NLE)$

Proof. Let us show (borrowing the idea from [9, Proposition 3.1]) the diagonal upper estimate first. Let us fix $y \in \Gamma$ and let $u_n(x) = P_n(x,y)$ the solution on $[0, R^\beta)] \times B(x, 2R)$. Let us use (PH_β) for the profile $\mathcal{C} = \{\frac{4}{9}, \frac{5}{9}, \frac{6}{9}, 1, \frac{1}{2}\}$ to $x, y, z \in \Gamma, d(x,y) \leq R$, $d(z,y) \leq R$. It provides

$$P_n(x,y) \leq C_H \widetilde{P}_{2n}(z,y) \tag{7.36}$$

which can be summed over $B(x,R)$

$$\begin{aligned} P_n(x,y) &\leq \frac{C_H}{V(x,R)} \sum_{z \in B(x,R)} \mu(z) \widetilde{P}_{2n}(z,y) \\ &= \frac{C_H \mu(y)}{V(x,R)} \sum_{z \in V(x,R)} \widetilde{P}_{2n}(y,z) \\ &\leq \frac{C\mu(y)}{V(x,R)} \leq C \frac{\mu(y)}{V(x, n^{\frac{1}{\beta}})}. \end{aligned}$$

Let us remark that the particular choice of the profile has no real importance, the only point is to ensure that $n_- = n, n_+ = 2n$ can be chosen. This will be applied repeatedly without any further comment.

The next step is to show $(PH_\beta) \Longrightarrow$ (7.35) This can be seen again from (PH_β) applied to P_n^A getting $P_n^A(z,y) \leq C_H \widetilde{P}_{2n}^A(x,y)$ for $z \in B(x,R)$ and for an other solution $\widetilde{u}$ of a parabolic equation with boundary conditions as follows. Let $u_k(w)$ defined on $[0, R^\beta] \times B(x, 2R)$ and

$$u_k(w) = \begin{cases} 1 \text{ if } 0 \leq k \leq n \\ \sum_{z \in B(x,R)} P_{k-n}^A(w,z) \text{ if } n < k \leq 2n \end{cases}$$

(PH_β) provides for $y \in B(x, R/2)$

$$\begin{aligned}
\frac{1}{C_H} = \frac{1}{C_H} u_n(x) \leq \widetilde{u}_{2n}(y) &= \sum_{z \in B(x,R)} \widetilde{P}_n^A(y,z) \\
&= \sum_{y \in B(x,R)} \widetilde{P}_n^A(z,y) \frac{\mu(z)}{\mu(y)} \\
&\leq \sum_{y \in B(x,R)} 2C_H \widetilde{P}_{2n}^A(x,y) \frac{\mu(z)}{\mu(y)} \\
&= \frac{2C_H V(x,R)}{\mu(y)} \widetilde{P}_{2n}^A(x,y).
\end{aligned}$$

This proves (7.35) for $R \geq 9$, for small $R - s$ the statement follows from (p_0). We have got (PLE) for $2n$ and it follows for $2n+1$ using (p_0) in the one step decomposition. It is clear that (NLE) follows from (7.35) imposing the condition $d(x,y) \leq n^{\frac{1}{\beta}}$ (which is stronger than the assumed $d(x,y) \leq R$, $n \leq \frac{5}{9} R^\beta$). ■

Proposition 7.2 *(p_0) and (PH_β) imply (D).*

Proof. The volume doubling is easy consequence of (DUE) and (NLE) provided by Proposition 7.1 from (PH_β). Consider $n = R^\beta, m = (2R)^\beta$ and apply the conditions to get

$$c\frac{\mu(x)}{V(x,R)} \leq \widetilde{P}_n(x,x) \overset{(PH)}{\leq} C\widetilde{P}_m(x,x) \leq C\frac{\mu(x)}{V(x,2R)} \tag{7.37}$$

■

Proposition 7.3 *If $(p_0), (D)$ and (DUE) hold then*

$$\lambda(x,R) \geq cR^{-\beta} \tag{7.38}$$

Proof. We shall choose later $0 < \varepsilon < 1$ and assume $R \leq \varepsilon n^{\frac{1}{\beta}}$ and $y, z \in B(x,R) =: A$. At the end of the proof of Theorem 5.1 we have seen that (DUE) implies upper estimate of $P_n(y,z)$ which is in our case

$$P_n^A(y,z) \leq C \frac{\mu(z)}{\left(V(y, n^{\frac{1}{\beta}}) V(z, n^{\frac{1}{\beta}}))\right)^{1/2}}.$$

For $w = y$ or z $d(x,w) \leq R$ using (D_β) one has

$$\frac{V(x, n^{\frac{1}{\beta}})}{V(w, n^{\frac{1}{\beta}})} \leq C$$

which results that

$$P_n^A(y,z) \leq C \frac{\mu(z)}{V(x, n^{\frac{1}{\beta}})}.$$

If ϕ is the eigenfunction belonging to the smallest eigenvalue λ of $I - P^{B(x,R)}$ normalized to $(\phi 1) = 1$ then

$$(1-\lambda)^n = \phi P_n^A 1 \le \sum_{y,z\in B(x,R)} \phi(y) P_n^A(y,z)$$

$$\le \sum_{z\in B(x,R)} \frac{C\mu(z)}{\min_{y\in B(x,R)} V(y, n^{\frac{1}{\beta}})} \le C\frac{V(x,R)}{V(x,n^{\frac{1}{\beta}})} \le C\frac{V(x,\varepsilon n^{\frac{1}{\beta}})}{V(x,n^{\frac{1}{\beta}})}$$

but using (V_2) for $\varepsilon = \left(\frac{1}{K}\right)^m$

$$C\frac{V(x,\varepsilon n^{\frac{1}{\beta}})}{V(x,n^{\frac{1}{\beta}})} \le C\frac{2^{-m}V(x,n^{\frac{1}{\beta}})}{V(x,n^{\frac{1}{\beta}})} < \frac{1}{2}$$

if $m = \lceil \log_2 C + 1 \rceil$. Finally, using the inequality and $1 - \xi \ge \frac{1}{2}\log\frac{1}{\xi}$ for $\xi \in [\frac{1}{2}, 1]$ for $\xi = 1 - \lambda(x,R)$ one has

$$\lambda(x,R) \ge \frac{\log 2}{2n} \ge cR^{-\beta}.$$

■

Corollary 7.1 *If* (p_0) *and* (PH_β) *hold, then there is a* $C > 0$ *such that for all* $x \in \Gamma, R > 1$

$$\rho(x,R,2R) \le C\frac{R^\beta}{V(x,2R)}.$$

Proof. The statement direct consequence of (4.13) and Proposition 7.3. ■

The lower estimate of E is quite simple in the possession of the (7.35) which is a consequence again of (PH_β) by Proposition 7.1.

Proposition 7.4 *If* (p_0) *and* (7.35) *(variant of* (PLE) *) holds, then there is a* $c > 0$ *such that for all* $x \in \Gamma, R \ge 0$

$$E(x,R) \ge cR^\beta$$

Proof. We assumed that for all $x \in \Gamma, R \ge 1$

$$\widetilde{P}_n^{B(x,4R)}(x,y) = P_n^{B(x,4R)}(x,y) + P_{n+1}^{B(x,4R)}(x,y) \ge \frac{c\mu(y)}{V(x,R)}$$

if $\frac{4}{9}R^\beta \le n \le \frac{5}{9}R^\beta$, $y \in B(x,R/2)$. By definition

$$E(x,R) = \sum_{k=0}^{\infty} \sum_{y\in B(x,R/2)} P_k^{B(x,2R)}(x,y) \ge \sum_{k=0}^{\infty} \sum_{y\in B(x,R/2)} \frac{1}{2}\widetilde{P}_k^{B(x,2R)}(x,y)$$

$$\sum_{k=\left(\frac{1}{4}R\right)^\beta}^{\frac{5}{9}R^{\beta\prime}} \sum_{y\in B(x,R/2)} \frac{1}{2}\widetilde{P}_k^{B(x,2R)}(x,y) \overset{(PLE)}{\ge} c\frac{V(x,R/2)}{V(x,\left(\frac{5}{9}\right)^{\frac{1}{\beta}}R)}R^\beta \overset{(D)}{=} cR^\beta.$$

■

Corollary 7.2 *If* (7.35)*(variant of* (PLE) *) holds then*

$$\rho(x,R,2R) \geq c\frac{R^\beta}{V(2R)}$$

Proof. Let us use Corollary 4.5

$$\rho(x,R,2R) \geq \frac{\lambda(x,2R)E(\underline{w},R/2)^2}{V(x,2R)}$$

and Proposition 7.3 and 7.4 to get immediately the statement. ∎

Proof of Theorem 7.2. Proposition 7.1 gives (D) and $(\rho_{A,\beta})$ and the elliptic Harnack inequality evidently follows from the parabolic one. This finishes the proof of 3. $\Longrightarrow$ 1. in Theorem 2.1 and hence the whole proof is complete. ∎

Proof of Theorem 2.2. As we stated during the steps 2. $\Longrightarrow$ 3. $\Longrightarrow$ 1. particularly when 1. is reached at the condition $(\rho_{A,\beta})$ neither the recurrence nor it's strong form was used, hence we have proved this Theorem as well. ∎

8 Poincaré Inequality

In this section we show that a Poincaré type inequality follows from the parabolic Harnack inequality. The opposite direction is not clear. There are some indications that it might be generally not true.

Proposition 8.1 *¿From* (PH_β) *follows that for all function* f *on* V, $x \in \Gamma, R > 0$

$$\sum_{y\in B(x,R)} \mu(y)\left(f(y)-f_B\right)^2 \leq CR^\beta \sum_{y,z\in B(x,R+1)} \mu_{y,z}(f(y)-f(z))^2 \tag{8.39}$$

where

$$f_B = \frac{1}{V(x,R)} \sum_{y\in B(x,R)} \mu(y)f(y)$$

Proof. The proof is easy adaptation of [9, Theorem 3.11]. We consider the Neumann boundary conditions on $B(x,2R)$ which has the new transition probability $P'(y,z) \geq P(y,z)$ equality holds everywhere inside and strict inequality holds at the boundary. Consider the operator

$$Qg(y) = \sum P'(y,z)g(z)$$

and $W = Q^K$ where $K = CR^\beta$. Since $Wg > 0$ for $g > 0$ is a solution of the parabolic equation on $B(x,2R)$ using the hypothesis we have

$$\begin{aligned} W(f-Wf(y))^2(y) &\geq \sum_{z\in B(y,R+1)} \frac{c\mu(z)}{V(y,2R)}(f(z)-Wf(y))^2 \\ &\geq \frac{c}{V(x,3R)} \sum_{z\in B(x,R)} \mu(y)(f(y)-f_{B(x,R)})^2 \end{aligned}$$

since $\sum_{z\in B(x,R)}\mu(y)(f(y)-\lambda)^2$is minimal at $\lambda = f_B$. We conclude to

$$\begin{aligned}\sum_{y\in B(x,R)}\mu(y)(f(y)-f_{B(x,R)})^2 &\le c\sum_{y\in B(x,R+1)}W(f-Wf(y))^2(y)\\ &\le c\left(\|f\|_2^2-\|Wf\|_2^2\right)\\ &\le cK\,\|\nabla f\|_2^2\end{aligned}$$

where $\|f\|_2^2=\sum_{y\in B(x,2R)}\mu'(y)f(y)^2$ and $\|\nabla f\|_2^2=\sum_{y,z\in B(x,2R)}\mu_{y,z}(f(y)-f(z))^2$. The last inequality is the result of the repeated application of

$$\|Wf\|_2^2\le\|f\|_2^2 \text{ and } \|f\|_2^2-\|Wf\|_2^2\le\|\nabla f\|_2^2$$

Recalling the definition of K, the result follows . ∎

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