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Competing Particle Systems Evolving by I.I.D. Increments

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Abstract

We consider competing particle systems in $\mathbb{R}^d$, i.e. random locally finite upper bounded configurations of points in $\mathbb{R}^d$ evolving in discrete time steps. In each step i.i.d. increments are added to the particles independently of the initial configuration and the previous steps. Ruzmaikina and Aizenman characterized quasi-stationary measures of such an evolution, i.e. point processes for which the joint distribution of the gaps between the particles is invariant under the evolution, in case $d = 1$ and restricting to increments having a density and an everywhere finite moment generating function. We prove corresponding versions of their theorem in dimension $d = 1$ for heavy-tailed increments in the domain of attraction of a stable law and in dimension $d \geq 1$ for lattice type increments with an everywhere finite moment generating function. In all cases we only assume that under the initial configuration no two particles are located at the same point. In addition, we analyze the attractivity of quasi-stationary Poisson point processes in the space of all Poisson point processes with almost surely infinite, locally finite and upper bounded configurations.

Key words: Competing particle systems, Evolutions of point processes, Poisson processes, Large deviations, Spin glasses.

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1 Introduction

Recently, evolutions of point processes on the real line by discrete time steps were successfully analyzed for quasi-stationary states, i.e. demanding the stationarity of the distances between the points rather than the positions of the points, see e.g. [2], [14]. In particular, the processes for which the joint distribution of the gaps stays invariant under the evolution were determined in the cases that Gaussian or i.i.d. increments having a density and an everywhere finite moment generating function are added to the particles. In the i.i.d. case Ruzmaikina and Aizenman proved that these quasi-stationary point processes are of a particularly simple form, given by superpositions of Poisson point processes with exponential densities. In the context of spin glass models their result says that quasi-stationary states in the free energy model starting with infinitely many pure states and adding a spin variable in each time step are given by superpositions of random energy model states introduced in [13]. The connection between the cavity method in the theory of spin glasses and quasi-stationary measures of evolutions of points on the real line is explained in full extent in [1] and [2]. For an introduction to spin glass models see for instance [11], [15].

The crucial assumption in [14] is that the distribution of the increments possesses a density and has an everywhere finite moment-generating function. In particular, the increments are in the domain of attraction of a normal law. Although this is the case in the context of the Sherrington-Kirkpatrick model of spin glasses, it is of interest to determine the quasi-stationary states for more general increments. Here we treat increments in the domain of attraction of a stable law and multidimensional evolutions with increments having exponential moments, thus being in the domain of attraction of a multidimensional normal law. The results for lattice type and heavy-tailed increments in dimension $d = 1$ may be as well applicable in the context of non-Gaussian spin glass models. The resulting quasi-stationary measures are superpositions of Poisson point processes, whose intensities vary with the type of the increments considered. In addition to the Ruzmaikina-Aizenman type quasi-stationary states we find completely new quasi-stationary states in the case of lattice type increments with either exponential moments or heavy tails. We also prove attractivity of the quasi-stationary Poisson point processes in the space of all Poisson point processes in $\mathbb{R}^d$ with almost surely infinite, locally finite and upper bounded configurations.

To determine the quasi-stationary measures in the case of increments with heavy tails we observe that the Poissonization Theorem of [14] can be generalized to apply in our context. Hence, we are able to write each quasi-stationary measure as a weak limit of superpositions of Poisson point processes. Subsequently, we present a direct argument in which we evaluate the limit in the Generalized Poissonization Theorem in order to conclude that it is itself a superposition of Poisson point processes. In the case of increments in the domain of attraction of a normal law we follow the approach of [14]. In our case we use a version of the Bahadur-Rao Theorem which gives the sharp asymptotics of large deviations for infinite rectangles in $\mathbb{R}^d$ and is an analog of the results in [10] for smooth domains in $\mathbb{R}^d$. This allows us to perform a compactness argument similiar to the one in [14] allowing us to pass to the limit in the Poissonization Theorem through a subsequence. One of the main obstacles hereby is the lack of a natural total order on $\mathbb{R}^d$. After proving that in both cases the quasi-stationary measures are given by superpositions of Poisson point processes we show that the intensities of the latter are solutions of Choquet-Deny type equations. This is done by extending the steepness relation on tail distribution functions to the multidimensional setting and generalizing the monotonicity argument in [14]. In our more general setting we find new intensities in addition to the ones in [14]. To prove the attractivity of certain quasi-stationary Poisson point processes in the space of all Poisson point processes with almost surely infinite, locally finite and upper bounded configurations we analyze the corresponding evolution of intensity measures and exploit the fact that the weak convergence of intensity measures implies the weak convergence of the Poisson point processes.

To define the evolution in $\mathbb{R}^d$ in full generality we consider the partial order $\geq$ on $\mathbb{R}^d$ where $a \geq b$ when $a_j \geq b_j$, $1 \leq j \leq d$. Let $l \subset \mathbb{R}^d$ be any line in $\mathbb{R}^d$ for which $\geq$ is a total order and which contains infinitely many lattice points in the case that the increments are of lattice type. Moreover, let $p : \mathbb{R}^d \to l$ be the affine map which assigns to every point x the closest point y on l with $x \geq y$. Finally, we set $a \succeq b$ if $p(a) > p(b)$ or if $a = b$ and define $a \succeq b$ in an arbitrary, but deterministic and measurable way for which $a \geq b$ implies $a \succeq b$ if $p(a) = p(b)$, $a \neq b$ (one can use induction on d to prove that this is possible). Note that $\succeq$ is a total order on $\mathbb{R}^d$ in agreement with the partial order $\geq$ and its level sets are infinite rectangles up to a modification of the boundary. We consider competing particle systems with a random μ-distributed starting configuration $(x_n)_{n\geq 1}$ ordered by $\succeq$ and evolving by i.i.d. increments $(\pi_n)_{n\geq 1}$ of distribution π, i.e. each step of the evolution is described by the mapping

$$(x_n)_{n\geq 1} \mapsto \big((x_n + \pi_n)_{n\geq 1}\big)_{\downarrow}$$

where $\downarrow$ denotes the sequence rearranged in non-ascending order $\succeq$.

From now on all considered evolutions will satisfy one of the following two assumptions: assumption 1.1 in the case of a one-dimensional evolution with heavy-tailed increments belonging to the domain of attraction of a stable law and assumption 1.2 in the case of increments being in the domain of attraction of a (possibly multidimensional) normal law.

Assumption 1.1. *$d = 1$ and there exist sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ of real numbers such that*

$$\frac{S_n - a_n}{b_n} \equiv \frac{\sum_{i=1}^n \pi_i - a_n}{b_n}$$

converges in distribution to an α-stable law with $\alpha \in (0, 2)$. Further, the initial distribution μ is simple, i.e.:

$$\mu\Big(\bigcup_{i\neq j}\{x_i = x_j\}\Big) = 0, \tag{I.1}$$

and both the evolution with increments distributed according to π and the one with increments distributed according to the corresponding α-stable law make sense (which means that the particle configuration can be reordered with probability 1 after each step of the evolution). Finally, without loss of generality $\mathbb{E}[\pi_n] = 0$ and the π_n are not almost surely equal to 0.

An example of a robust condition on μ and π which assures that the evolution makes sense is the following. Denote by λ the intensity measure of μ, i.e. define

$$\lambda(A) \equiv \mathbb{E}_\mu\Big[\sum_{n\geq 1} 1_{\{x_n \in A\}}\Big] \tag{I.2}$$

for Borel sets $A \subset \mathbb{R}$. If $\lambda * \pi$ is finite on all intervals of the type $[x, \infty)$, then the particle configuration can be reordered with probability 1, since it can be checked by a direct computation that $\lambda * \pi$ is the intensity measure of the point process resulting from μ after one step of the evolution.

Assumption 1.2. *The sequence of i.i.d. $\mathbb{R}^d$-valued random variables $(\pi_n)_{n\geq 1}$ which describes the increments of the evolution satisfies*

$$\forall \zeta \in \mathbb{R}^d, n \in \mathbb{N}: \quad \exp(\Lambda(\zeta)) \equiv \mathbb{E}[\exp(\zeta \cdot \pi_n)] < \infty \tag{I.3}$$

and each component of the π_n is of positive variance. For $d > 1$ assume further that the π_n have a density or take values in a lattice $A\mathbb{Z}^d + b$ for a fixed real $d \times d$ matrix A and a vector $b \in \mathbb{R}^d$. Moreover, the initial measure μ on particle configurations is simple and such that

$$\forall 1 \leq i \leq d \ \ \exists \zeta_i > 0: \quad \sum_{n\geq 1} \exp(\zeta_i x_n^i) < \infty \tag{I.4}$$

μ-a.s. where x_n^i is the i-th coordinate of x_n. Finally, without loss of generality $\mathbb{E}[\pi_n] = 0$.

In the case $d = 1$ assumption 1.2 allows us to deal with the evolutions considered in [14], as well as various lattice-type evolutions of interest, e.g. π_n being Bernoulli $\{-1, 1\}$-valued or following a signed Poisson distribution. Moreover, assumption 1.2 ensures that starting with a locally finite, upper bounded configuration

$$x_1 \succeq x_2 \succeq x_3 \ldots$$

we get a configuration of the same type after each step of the evolution (apply the remark in section 1.2 of [2] to each of the d coordinate processes). We will denote the space of such configurations by Ω and equip it with the σ-algebra $\widetilde{\mathscr{B}}$ which is generated by the shift invariant functions measurable with respect to the σ-algebra $\mathscr{B}$ generated by occupancy numbers of finite boxes. For the sake of full generality we include the case of configurations with finitely many particles by allowing the x_n to take the value $(-\infty, \ldots, -\infty)$. Our main result is the following:

Theorem 1.3. *Let μ be a quasi-stationary measure under an evolution satisfying assumption 1.1 or assumption 1.2. Then*

(a) μ is a superposition of Poisson point processes.

(b) The intensity measures λ of the latter are exactly those solutions of the Choquet-Deny equations $\lambda * \pi_a = \lambda$ with translates π_a of π, going over all $a \in \mathbb{R}^d$ which have no point masses and for which the corresponding Poisson point process is supported on upper bounded configurations.

(c) In case that $d = 1$ and supp π contains a non-trivial interval the intensity measures are given by $d\lambda = se^{-sx}dx$ with $s > 0$. In case that $d = 1$ and supp $\pi \subset p\mathbb{Z} + r$ the intensity measures are either of the form

$$\lambda(A) = \int_{\mathbb{R}_+ \times [0,p)} \sum_{x \in (\mathbb{Z}p + y) \cap An} e^{-\frac{sx}{n}} \, d\alpha(s, y), \quad n \in \mathbb{N}$$

or

$$\lambda(A) = \int_{\mathbb{R}_+ \times \mathbb{R}_+ \times [0,p) \times [0,w)} \sum_{k,l \in \mathbb{Z}:\, kp + lw + y \in A} e^{-s_1 k - s_2 l} \, d\beta(s_1, s_2, y), \quad w \in \mathbb{R}_+ : \ \mathbb{Z}w \cap \mathbb{Z}p = \{0\}$$

where α, β are positive Radon measures such that $\alpha(\mathbb{R}_+, dy)$, $\beta(\mathbb{R}_+, \mathbb{R}_+, dy)$ have no pure point components and we have identified $[0,p) \times [0,w)$ with a system of representatives of cosets of $\mathbb{Z}p \oplus \mathbb{Z}w$ in $\mathbb{R}$ in a canonical way.

In addition, we prove

Proposition 1.4. *Let the π_n be not almost surely constant and such that $\mathbb{E}[\pi_n] = 0$. Let further N be a Poisson point process in $\mathbb{R}^d$ with intensity measure $\lambda_\infty + \varrho$ where λ_∞ and ϱ are positive locally finite measures on $\mathbb{R}^d$ satisfying*

$$\begin{aligned} \lambda_\infty * \pi = \lambda_\infty, \\ \exists c \in \mathbb{R}^d: \ \lambda_\infty(c + (\mathbb{R}_-)^d) = \infty, \quad \lambda_\infty(\mathbb{R}^d - (c + (\mathbb{R}_-)^d)) < \infty, \\ \varrho(\mathbb{R}^d - (\mathbb{R}_-)^d) < \infty, \quad \forall a < b, y \in (\mathbb{R}_+)^d: \ \varrho((a - \gamma y, b - \gamma y)) \to_{\gamma \to \infty} 0 \end{aligned}$$

and

$$(a - \gamma y, b - \gamma y) = (a_1 - \gamma y_1, b_1 - \gamma y_1) \times \cdots \times (a_d - \gamma y_d, b_d - \gamma y_d).$$

Then the joint distribution of the gaps of N after n evolutions converges for n tending to infinity to the corresponding quantity for a Poisson point process with intensity measure λ_∞.

Remark. The quasi-stationary Poisson point processes with exponential intensities $d\lambda = re^{-r\tau}dr$, $\tau > 0$ found in [14] are quasi-stationary also in the case that π is a lattice type distribution. They can be recovered from Theorem 1.3 (c) as the special case

$$n = 1, \quad d\alpha(s, y) = d\delta_\tau(s)\tau e^{-\tau y}\, dy.$$

This is due to the computation

$$\lambda([r, \infty)) = \int_0^p \sum_{z \in \mathbb{Z} \cap \left[\frac{r-y}{p}, \infty\right)} e^{-z\tau p} \tau e^{-\tau y}\, dy = e^{-r\tau}$$

where one has to observe that the sum is a geometric series and the integrand takes only two different values.

A crucial step in the proof of Theorem 1.3 consists of writing quasi-stationary measures as weak limits of superpositions of Poisson point processes which is called Poissonization in [14]. More precisely, we use the following generalization of the Poissonization Theorem of [14]:

Theorem 1.5 (Generalized Poissonization Theorem)**.** *Let $d = 1$ and μ be a quasi-stationary measure of an evolution satisfying assumption 1.1 or 1.2, $\{F_N\}_{N \geq 1}$ be the family of functions defined by*

$$F_N(x) = \sum_{m \geq 1} \mathbb{P}_\pi(x_m + \pi_1 + \cdots + \pi_N \geq x)$$

where $(x_n)_{n \geq 1}$ is a fixed starting configuration of the particles. Then for any non-negative continuous function with compact support $f \in C_c^+(\mathbb{R})$ it holds

$$\widetilde{G}_\mu(f) = \lim_{N \to \infty} \int d\mu\, \widehat{G}_{F_N}(f). \tag{I.5}$$

Here, $\widetilde{G}_\mu$ denotes the modified probability generating functional of μ given by

$$\widetilde{G}_\mu = \mathbb{E}\Big[\exp\Big(-\sum_n f(x_1 - x_n)\Big)\Big] \tag{I.6}$$

and $\widehat{G}_{F_N}$ denotes the modified probability generating functional of the Poisson point process on $\mathbb{R}$ with intensity measure λ_N uniquely determined by

$$\lambda_N([a,b)) = F_N(a) - F_N(b).$$

To prove Theorem 1.5 it suffices to observe that the proof of the Poissonization Theorem in [14] can be adapted to our context by applying the spreading property in the form of Lemma 11.4.I of [5] which we state now for the sake of completeness:

Lemma 1.6 (Spreading property)**.** *Let $(Y_n)_{n\geq 1}$ be a sequence of non-constant i.i.d. $\mathbb{R}^d$-valued random variables. Then for any bounded Borel set $A \subset \mathbb{R}^d$ it holds*

$$\sup_{x\in\mathbb{R}^d} \mathbb{P}(Y_1 + \cdots + Y_N \in x + A) \to_{N\to\infty} 0. \tag{I.7}$$

The paper is organized as follows. We prove part (a) of Theorem 1.3 under assumption 1.1 in section 2 and under assumption 1.2 in the one-dimensional case in section 3. Having at this point the fact that each quasi-stationary measure of the one-dimensional evolution is a superposition of Poisson point processes we determine in section 4 the intensity measures of the latter, thus proving parts (b) and (c) of Theorem 1.3. Section 5 gives the proof of Theorem 1.3 for multidimensional evolutions satisfying assumption 1.2 by extending the arguments of the preceding sections to the multidimensional case. Finally, in section 6 we analyze the evolution in the space of Poisson point processes with almost surely infinite, locally finite and upper bounded configurations in order to prove Proposition 1.4.

2 Quasi-stationary measures of the evolution with heavy-tailed increments

In this section as well as sections 3 and 4 we restrict to the case $d = 1$ for the sake of a simpler notation and prove Theorem 1.3 in the one-dimensional setting. Subsequently, we show in section 5 how our arguments extend to the case $d > 1$. In this section we present the proof of Theorem 1.3 (a) for an evolution satisfying assumption 1.1. We explain the main part of the proof first and defer the technical issue of approximating the distribution of the increments by an α-stable law to the end of the proof. The proof uses the Generalized Poissonization Theorem (Theorem 1.5) in deducing that every quasi-stationary μ satisfying assumption 1.1 is a superposition of Poisson point processes.

Proof of Theorem 1.3 (a) under assumption 1.1. 1) Let $L(N)$ be a slowly varying function such that $\frac{S_N}{L(N)N^{\frac{1}{\alpha}}}$ converges to an α-stable law. Since we are only interested in the joint distribution of the gaps between the particles, we may assume that the particle configuration

$$x_1 \geq x_2 \geq x_3 \geq \dots$$

starts at $x_1 = 0$. We will shift it subsequently to the left by numbers c_N depending on the initial configuration $(x_n)_{n\geq 1}$ and tending monotonously to infinity for $N \to \infty$. The resulting configuration of particles will be denoted by

$$x_1(N) \geq x_2(N) \geq x_3(N) \geq \dots.$$

We note that

$$F_N(x) = \sum_{n\geq 1} \mathbb{P}_\pi(x_n + S_N \geq x) = \sum_{n\geq 1} \mathbb{P}_\pi\left(\frac{S_N}{L(N)N^{\frac{1}{\alpha}}} \geq \frac{x - x_n}{L(N)N^{\frac{1}{\alpha}}}\right).$$

Since a shift of the particle configuration by c_N does not affect the value of $\widetilde{G}_\mu$ and $(c_N)_{N\geq 1}$ can be chosen to converge to infinity fast enough, the functions F_N can be replaced by

$$\sum_{n\geq 1} \mathbb{P}_\pi\left(\frac{S_N}{L(N)N^{\frac{1}{\alpha}}} \geq \frac{x - x_n(N)}{L(N)N^{\frac{1}{\alpha}}}\right) \cdot 1_{\{x\geq -e_N\}} \approx CL(N)^\alpha N \sum_{n\geq 1} \frac{1}{(x - x_n(N))^\alpha} \cdot 1_{\{x\geq -e_N\}}$$

in the statement of the Generalized Poissonization Theorem where $C = C(\alpha)$ is a constant and $(e_N)_{N\geq 1}$ is an increasing sequence in $\mathbb{R}_+$ depending on the initial configuration $(x_n)_{n\geq 1}$, converging to infinity and satisfying $e_N \leq \frac{c_N}{2}$. The approximation by an α-stable law used here is justified in steps 2 to 4. We remark at this point that the right-hand side is finite due to the assumption 1.1, since it corresponds to the evolution with α-stable increments. Next, note that

$$\widehat{G}_{F_N}(f) = \int_{\mathbb{R}} F_N(dx) \exp(-F_N(x)) \exp\left(-\int_{-\infty}^{x} e^{-f(x-y)} F_N(dy)\right)$$

which follows by conditioning the Poisson point process on its leader and was shown in [14]. Here, the integrals are taken with respect to the infinite positive measures induced by the corresponding non-increasing functions. Hence, again referring to steps 2 to 4 for the justification of the approximation by an α-stable law we may conclude

$$\widehat{G}_{F_N}(f) \approx CL(N)^\alpha N \int_{-e_N}^{\infty} d\sum_{n\geq 1} \frac{1}{(x - x_n(N))^\alpha} \exp\left(-CL(N)^\alpha N \sum_{n\geq 1} \frac{1}{(x - x_n(N))^\alpha}\right)$$
$$\times \exp\left(-CL(N)^\alpha N \int_{-e_N}^{x} e^{-f(x-y)} d\sum_{n\geq 1} \frac{1}{(y - x_n(N))^\alpha}\right).$$

Setting $K(N) \equiv CL(N)^\alpha N$, the Generalized Poissonization Theorem yields:

$$\widetilde{G}_\mu(f) = \lim_{N\to\infty} \int d\mu\, K(N) \int_{-e_N}^{\infty} d\sum_{n\geq 1} \frac{1}{(x - x_n(N))^\alpha} \exp\left(-K(N) \sum_{n\geq 1} \frac{1}{(x - x_n(N))^\alpha}\right)$$
$$\times \exp\left(-K(N) \int_{-e_N}^{x} e^{-f(x-y)} d\sum_{n\geq 1} \frac{1}{(y - x_n(N))^\alpha}\right).$$

Recalling that $x_n(N)$ was defined as $x_n - c_N$ we may rewrite the inner integral as

$$\int_{-e_N}^{\infty} K(N) d\sum_{n\geq 1} \frac{1}{(x + c_N - x_n)^\alpha} \exp\left(-K(N) \sum_{n\geq 1} \frac{1}{(x + c_N - x_n)^\alpha}\right)$$
$$\times \exp\left(-K(N) \int_{-e_N}^{x} e^{-f(x-y)} d\sum_{n\geq 1} \frac{1}{(y + c_N - x_n)^\alpha}\right).$$

Next, we enlarge the shift parameters c_N, if necessary, to have

$$H(x) \equiv \lim_{N\to\infty} K(N) \sum_{n\geq 1} \frac{1}{(x+c_N-x_n)^\alpha} 1_{\{x\geq -e_N\}} < \infty$$

such that for every $M \geq 1$ the convergence is monotone on $[-e_M, \infty)$ for $N \geq M$. Note that this is possible, because the sum on the right-hand side is finite for the original choice of $(c_N)_{N\geq 1}$ due to assumption 1.1 and, moreover, the sequence $(c_N)_{N\geq 1}$ can be adjusted separately for each starting configuration $(x_n)_{n\geq 1}$. For the sake of shorter notation we introduce positive measures α_N, α on $\mathbb{R}$ defined by

$$\alpha(dx) = H(dx), \quad \alpha_N(dx) = K(N)\, 1_{\{x\geq -e_N\}}\, d\sum_{n\geq 1} \frac{1}{(x+c_N-x_n)^\alpha}.$$

Now, we would like to interchange the limit $N \to \infty$ with the μ-integral on the right-hand side of the equation for $\widetilde{G}_\mu(f)$. To this end, we remark that the Dominated Convergence Theorem may be applied, since the integrands are dominated by

$$\int_{\mathbb{R}} \alpha_N(dx) = K(N) \sum_{n\geq 1} \frac{1}{\left(-e_N + c_N - x_n\right)^\alpha}$$

and the right-hand side can be made uniformly bounded in N by enlarging the c_N, if necessary. By interchanging the limit with the μ-integral we deduce

$$\widetilde{G}_\mu(f) = \int d\mu \lim_{N\to\infty} \int_{\mathbb{R}} \alpha_N(dx) \exp\left(-\alpha_N([x,\infty))\right) \exp\left(-\int_{-\infty}^{x} \alpha_N(dy)\, e^{-f(x-y)}\right).$$

But α_N and α were defined in such a way that α is the weak limit of the α_N. Thus, the Poisson point processes with the intensity measures α_N converge weakly to the Poisson point process with the intensity measure α (see Theorem 11.1.VII in [5] for more details). In particular, their modified probability generating functionals converge. Thus, we may pass to the limit and deduce

$$\widetilde{G}_\mu(f) = \int d\mu \int_{\mathbb{R}} \alpha(dx) \exp(-\alpha([x,\infty))) \exp\left(-\int_{-\infty}^{x} \alpha(dy)\, e^{-f(x-y)}\right).$$

In other words, μ is a superposition of Poisson point processes with intensities $\alpha(dx)$ mixed according to μ itself. This proves that each quasi-stationary measure of the evolution is a superposition of Poisson point processes given that the distribution of the increments can be approximated by an α-stable law in a suitable sense.

2) With the notation $\theta_{n,N}(x) \equiv \frac{x-x_n(N)}{L(N)N^{\frac{1}{\alpha}}}$ we need to justify that we are allowed to replace the expression

$$\sum_{n\geq 1} \mathbb{P}_\pi \left(\frac{S_N}{L(N)N^{\frac{1}{\alpha}}} \geq \theta_{n,N}(x) \right)$$

appearing on the right-hand side of the statement of the Generalized Poissonization Theorem by

$$CL(N)^\alpha N \sum_{n\geq 1} \frac{1}{(x-x_n(N))^\alpha}$$

which plays the same role on the right-hand side of the corresponding Generalized Poissonization Theorem for increments following an α-stable law. To this end, by the second remark on page 260 of [9] which characterizes domains of attraction of α-stable laws we can find constants $d_N \in [0,1]$, functions $\varepsilon_N : \mathbb{R} \to \mathbb{R}_+$ and slowly varying functions $s_N : \mathbb{R} \to \mathbb{R}_+$ such that

$$\mathbb{P}_\pi \left(\frac{S_N}{L(N)N^{\frac{1}{\alpha}}} \geq \theta_{n,N}(x) \right) = (d_N + \varepsilon_N(\theta_{n,N}(x)))L(N)^\alpha N s_N(\theta_{n,N}(x)) \frac{1}{(x - x_n(N))^\alpha}$$

and $\varepsilon_N(y) \to_{y\to\infty} 0$.

3) Suppose first that $\inf_N d_N > 0$. Choosing the shift parameters c_N introduced in step 1 to be large enough, we can achieve

$$\varepsilon_N \equiv \sup_n \varepsilon_N(\theta_{n,N}(x)) \to_{N\to\infty} 0,$$

because the functions ε_N vanish at infinity. It follows

$$\begin{aligned}\left| \sum_{n\geq 1} \mathbb{P}_\pi \left(\frac{S_N}{L(N)N^{\frac{1}{\alpha}}} \geq \theta_{n,N}(x) \right) - \sum_{n\geq 1} d_N L(N)^\alpha N s_N(\theta_{n,N}(x)) \frac{1}{(x - x_n(N))^\alpha} \right| \\ \leq \sum_{n\geq 1} \Big(\varepsilon_N(\theta_{n,N}(x)) \Big) L(N)^\alpha N s_N(\theta_{n,N}(x)) \frac{1}{(x - x_n(N))^\alpha}\end{aligned}$$

$$= \sum_{n\geq 1} \frac{\varepsilon_N(\theta_{n,N}(x))}{d_N + \varepsilon_N(\theta_{n,N}(x))} \mathbb{P}_\pi \left(\frac{S_N}{L(N)N^{\frac{1}{\alpha}}} \geq \theta_{n,N}(x) \right) \leq \frac{\varepsilon_N}{d_N + \varepsilon_N} \sum_{n\geq 1} \mathbb{P}_\pi \left(\frac{S_N}{L(N)N^{\frac{1}{\alpha}}} \geq \theta_{n,N}(x) \right)$$

by taking the absolute value inside the sum and using the monotonicity of $x \mapsto \frac{x}{d_N + x}$. Under the assumption $\inf_N d_N > 0$ we have

$$\frac{\varepsilon_N}{d_N + \varepsilon_N} \leq \frac{\varepsilon_N}{\inf_N d_N + \varepsilon_N} \to_{N\to\infty} 0.$$

We conclude

$$\lim_{N\to\infty} \int d\mu \, \widehat{G}_{F_N}(f) = \lim_{N\to\infty} \int d\mu \, \widehat{G}_{\widetilde{F}_N}(f)$$

for any test function $f \in C_c^+(\mathbb{R})$ and functions $\widetilde{F}_N$ defined by

$$\widetilde{F}_N(x) = \sum_{n\geq 1} d_N L(N)^\alpha N s_N(\theta_{n,N}(x)) \frac{1}{(x - x_n(N))^\alpha}$$

using the approximation of $\widehat{G}_F$ by functionals continuous in F presented in the proof of Theorem 6.1 in [14]. This and the fact that the $\widetilde{F}_N$'s differ from the corresponding expressions in step 1 only by the constants d_N and the slowly varying functions s_N, which both can be dominated by an appropriate choice of the sequence $(c_N)_{N\geq 1}$, justify the approximation by an α-stable law in the case $\inf_N d_N > 0$. We observe that this reasoning goes through also under the weaker assumption of $\liminf_{N\to\infty} d_N > 0$.

4) Now, let $\liminf_{N\to\infty} d_N = 0$. We may even assume $\lim_{N\to\infty} d_N = 0$, since we may pass to the limit in the Generalized Poissonization Theorem through any subsequence. Since $\frac{1}{d_N}\widetilde{F}_N$ is a multiple of the expected number of particles on $[x,\infty)$ after N steps in the evolution with α-stable increments, the measure induced by $\frac{1}{d_N}\widetilde{F}_N$ is not only locally finite, but also finite on intervals of the type $[x,\infty)$. Hence, by enlarging the shift parameters c_N to make $\frac{1}{d_N}\widetilde{F}_N(x)$ bounded uniformly in N for each x, we can achieve that the measures induced by $\widetilde{F}_N$ converge weakly to the zero measure on $\mathbb{R}$ for N tending to infinity. In addition, we have the estimate

$$F_N(x) = \widetilde{F}_N(x) + \sum_{n\geq 1} \left(\varepsilon_N(\theta_{n,N}(x))\right) L(N)^\alpha N s_N(\theta_{n,N}(x)) \frac{1}{(x - x_n(N))^\alpha} \leq \widetilde{F}_N(x) + \frac{\varepsilon_N}{d_N}\widetilde{F}_N(x).$$

The rightmost expression converges to 0 for $N \to \infty$ which shows that the approximation by an α-stable law may be applied with $C = 0$. This follows again by the same approximation of $\widehat{G}_F$ as in the proof of Theorem 6.1 in [14]. We observe that this case corresponds to the quasi-stationary measure in which the configuration with no particles occurs with probability 1. $\square$

3 Quasi-stationary measures of the evolution with increments in the domain of attraction of a normal law

In this section we show that a quasi-stationary measure μ of an evolution satisfying assumption 1.2 is a superposition of Poisson point processes. The main difference to the proof of Theorem 6.1 in [14] is that we apply a multidimensional version of the Bahadur-Rao Theorem which applies to any distribution π as in assumption 1.2. This leads to the replacement of Laplace transforms by modified Laplace transforms and of normalizing shifts of the whole configuration by particle dependent shifts due to the fact that the Bahadur-Rao Theorem gives only information on probabilities of large deviations for lattice points in case that π is a lattice type distribution. The version of the Bahadur-Rao Theorem we use is an analog of the results in [10] where we replace smooth domains by infinite rectangles.

Lemma 3.1 (Multidimensional Bahadur-Rao Theorem). *Let $(\pi_n)_{n\geq 1}$ be as in assumption 1.2 and set $S_N \equiv \sum_{n=1}^N \pi_n$. Then in case that $d = 1$ and the π_n are non-lattice or $d > 1$ and the π_n have a density we have for all $x \in \mathbb{R}^d$ and $\mathbb{R}^d \ni q \geq 0$:*

$$\frac{\mathbb{P}(S_N \geq x + qN)}{\mathbb{P}(S_N \geq qN)} \sim \exp(-\eta(\nu(q)) \cdot x) \tag{III.8}$$

uniformly in q where $\sim$ means that the quotient of the two expressions tends to 1, $\eta = \eta(q)$ is the unique solution of

$$\gamma(q) = \eta \cdot q - \Lambda(\eta), \tag{III.9}$$

Λ is the logarithmic moment generating function, γ is the Fenchel-Legendre transform and $\nu(q)$ is the minimizer of γ over the set $\{y \in \mathbb{R}^d | y \geq q\}$. In case that the π_n are lattice with values in $A\mathbb{Z}^d + b$, equation (I.7) holds for all $x \in A\mathbb{Z}^d$ and all lattice points $q \geq 0$ again uniformly in q where A is a real $d \times d$ matrix and $b \in \mathbb{R}^d$.

Proof. 1) In the case $d = 1$ both asymptotics and their uniformity follow directly from Lemma 2.2.5 and the proof of the one-dimensional Bahadur-Rao Theorem in [6].

2) From now on let $d > 1$ and set

$$\Gamma \equiv \{y \in \mathbb{R}^d | y \geq q\}, \quad \Gamma_N \equiv \left\{y \in \mathbb{R}^d | y \geq q + \frac{x}{N}\right\},$$
$$\Gamma \wedge \Gamma_N \equiv \left\{y \in \mathbb{R}^d | y \geq \min\left(q + \frac{x}{N}, q\right)\right\}$$

where $\geq$ and min are meant componentwise. With an abuse of notation let $\mathbb{P}$ be the distribution of the π_n on $\mathbb{R}^d$ and following [10] define the q-centered conjugate by

$$d\mathbb{P}(y;\eta) = \exp(-\Lambda(\eta) + \eta \cdot (y+q))\, d\mathbb{P}(y+q).$$

Next, choose ν to be the minimizer of γ over Γ and let ν_N be the corresponding minimizer over $\Gamma \wedge \Gamma_N$. The representation formula for large deviations of [10] implies

$$\frac{\mathbb{P}(S_N \geq x + qN)}{\mathbb{P}(S_N \geq qN)} = \frac{\int_{\sqrt{N}(\Gamma_N - \nu_N)} e^{-\sqrt{N}\eta(\nu_N)\cdot y}\, d\mathbb{P}^{*N}(\sqrt{N}y;\eta(\nu_N))}{\int_{\sqrt{N}(\Gamma - \nu_N)} e^{-\sqrt{N}\eta(\nu_N)\cdot y}\, d\mathbb{P}^{*N}(\sqrt{N}y;\eta(\nu_N))}.$$

Note further that since $\eta(\nu_N)$ solves $\nabla\gamma(\nu_N) = \eta(\nu_N)$ and ν_N is the boundary point of $\Gamma \wedge \Gamma_N$ where the level set of γ touches $\Gamma \wedge \Gamma_N$, it follows that $\eta(\nu_N)$ is the inward normal to $\Gamma \wedge \Gamma_N$ in ν_N in case that $\nu_N \neq \min\left(q + \frac{x}{N}, q\right)$ and a vector pointing inward $\Gamma \wedge \Gamma_N$ otherwise. Hence, in both cases the integrands in the numerator and denominator are bounded by 1, because $\Gamma, \Gamma_N \subset \Gamma \wedge \Gamma_N$ by definition. Next, let V be the covariance matrix of $\mathbb{P}(.\,;\eta(\nu))$ and $\varphi_{0,V}$ be the Gaussian density with mean 0 and covariance V. Applying the expansion in Lemma 1.1 of [10] and its analog for the lattice case in section 2.6 of the same paper and using the boundedness of the integrands we deduce

$$\frac{\mathbb{P}(S_N \geq x + qN)}{\mathbb{P}(S_N \geq qN)} \sim \frac{\int_{\sqrt{N}(\Gamma_N - \nu_N)} e^{-\sqrt{N}\eta(\nu_N)\cdot y}\varphi_{0,V}(y)\, dy}{\int_{\sqrt{N}(\Gamma - \nu_N)} e^{-\sqrt{N}\eta(\nu_N)\cdot y}\varphi_{0,V}(y)\, dy}$$
$$\sim \frac{\int_{N(\Gamma_N - \nu_N)} e^{-\eta(\nu_N)\cdot u}\, du}{\int_{N(\Gamma - \nu_N)} e^{-\eta(\nu_N)\cdot u}\, du} = \frac{e^{-N\eta(\nu_N)\cdot\left(q + \frac{x}{N} - \nu_N\right)}}{e^{-N\eta(\nu_N)\cdot(q - \nu_N)}} = e^{-\eta(\nu_N)\cdot x} \to_{N\to\infty} e^{-\eta(\nu)\cdot x}$$

which proves the theorem. □

Next, we define modified Laplace transforms.

Definition 3.2. *Let $\mathbb{M}$ be the space of finite measures on $(0,\infty)$ and $\mathscr{M}$ be the Borel σ-algebra on $\mathbb{M}$ for the weak topology. Moreover, denote by*

$$R_\varrho(x) \equiv \int_0^\infty e^{-ux}\varrho(du) \tag{III.10}$$

the Laplace transform of a measure $\varrho \in \mathbb{M}$ and by $\widetilde{R}_\varrho$ its modified Laplace transform given by

$$\widetilde{R}_\varrho(x) \equiv R_\varrho([x]) \tag{III.11}$$

and where $[x] = x$ in the non-lattice case and $[x]$ is the closest number to x in $p\mathbb{Z}$ not less than x in the lattice case with supp $\pi \subset p\mathbb{Z} + r$.

Now we are ready to prove that for $d = 1$ each quasi-stationary measure μ of an evolution satisfying assumption 1.2 is a superposition of Poisson point processes. This corresponds to the first part of Theorem 1.3 for evolutions satisfying the assumption 1.2.

Proposition 3.3. *Let $d = 1$ and μ be a quasi-stationary measure for an evolution satisfying assumption 1.2. Then there exists a measure ν on $(\mathbb{M}, \mathscr{M})$ such that for any $f \in C_c^+(\mathbb{R})$:*

$$\widetilde{G}_\mu(f) = \int_{\mathbb{M}} \nu(d\varrho)\widehat{G}_{\widetilde{R}_\varrho}(f). \tag{III.12}$$

Proof. 1) We introduce again the functions F_N defined by

$$F_N(x) = \sum_n \mathbb{P}_\pi(x_n + S_N \geq x)$$

with $S_N = \sum_{n=1}^N \pi_n$ and a starting configuration $(x_n)_{n\geq 1}$. In order to deduce Proposition 3.3 from Theorem 1.5 we want to find measures $\varrho_N \in \mathbb{M}$ such that their modified Laplace transforms $\widetilde{R}_{\varrho_N}$ are close to the functions F_N in a suitable sense. Since $\widetilde{R}_{\varrho_N}(0) = 1$, we will normalize the functions F_N such that $F_N(0)$ will be close to 1. For this purpose define numbers z_N by

$$z_N = \inf\{x \in \mathbb{R} |\, F_N(x) \leq 1\}.$$

Moreover, let $z_{n,N} = z_N$ for all n if the distribution of the π_n is non-lattice and let $z_{n,N} \geq z_N$ be closest number to z_N satisfying

$$\frac{z_{n,N} - x_n}{N} \in p\mathbb{Z} + r$$

if the distribution of the π_n is supported in $p\mathbb{Z}+r$. Lastly, define functions H_N which may be viewed as the normalized versions of the functions F_N by

$$H_N(x) = \sum_n \mathbb{P}_\pi(x_n + S_N \geq x + z_{n,N}).$$

Note that in the lattice case each function H_N is piecewise constant with jumps on a subset of $p\mathbb{Z}$. Applying Lemma 3.1 we deduce that for an appropriate $K > 0$ and all n for which $x_n \geq -KN$ it holds

$$\mathbb{P}_\pi(x_n + S_N \geq x + z_{n,N}) = \mathbb{P}_\pi(S_N \geq z_{n,N} - x_n) \exp\left(-x \cdot \eta\left(\frac{z_{n,N} - x_n}{N}\right)\right)(1 + \varepsilon_{n,N})$$

for all $x \in \mathbb{R}$ in the non-lattice case and for all $x \in p\mathbb{Z}$ in the lattice case. Moreover,

$$\sup_n |\varepsilon_{n,N}| \to_{N\to\infty} 0.$$

Hence, with high probability $H_N(x)$ can be written as

$$\int_0^\infty \varrho_N(du)e^{-ux}(1 + \varepsilon_N(u)) + \sum_{n:\, x_n < -KN} \mathbb{P}_\pi(x_n + S_N \geq x + z_{n,N})$$

where

$$\varrho_N(du) = \sum_{n\colon x_n \geq -KN} \mathbb{P}_\pi(S_N \geq z_{n,N} - x_n)\delta_{\eta\left(\frac{z_{n,N}-x_n}{N}\right)}(du),$$

because by the affine bound on $z_{n,N}$ in step 3 below we have

$$\mu(\varrho_N \in \mathbb{M}) \to_{N\to\infty} 1.$$

Choosing $\varepsilon_N(u) = \varepsilon_{n,N}$ for $u = \eta\left(\frac{z_N - x_n}{N}\right)$ and $\varepsilon_N(u) = 0$ otherwise, we have

$$\sup_{u\in\mathbb{R}} |\varepsilon_N(u)| \to_{N\to\infty} 0.$$

2) In this step we will prove that

$$\sum_{n\colon x_n < -KN} \mathbb{P}_\pi(x_n + S_N \geq x + z_{n,N})$$

tends to 0 for $N \to \infty$ and an appropriately chosen K.

In case that $\lim_{\eta\to\infty} \Lambda'(\eta) < \infty$, we conclude that the support of the distribution of the π_n is bounded from above. Hence, the expression above vanishes for a fixed large enough K and N tending to infinity (provided that $z_{n,N}$ is bounded from below by an affine function of N uniformly in n which will be proven in the next step).

It remains to consider the case $\lim_{\eta\to\infty} \Lambda'(\eta) = \infty$. As in the proof of the Bahadur-Rao Theorem in [6] we define $\psi_N(\eta) \equiv \eta\sqrt{N\Lambda''(\eta)}$ and let $\widehat{F}_N^q$ be the distribution function of $\sum_{i=1}^N \frac{\pi_i - q}{\sqrt{\Lambda''(q)}}$. In the same way as in [6] we deduce for all n with $x_n < -KN$ that

$$\mathbb{P}_\pi(x_n + S_N \geq x + z_{n,N}) =$$
$$\exp\left(-N\gamma\left(\frac{x + z_{n,N} - x_n}{N}\right)\right)\int_0^\infty \exp\left(-y\psi_N\left(\eta\left(\frac{x + z_{n,N} - x_n}{N}\right)\right)\right) d\widehat{F}_N^{\frac{x+z_{n,N}-x_n}{N}}(y).$$

Provided we have a lower bound on $z_{n,N}$ which is affine in N and uniform in n and choosing K large enough we have for large N that $\eta\left(\frac{x+z_{n,N}-x_n}{N}\right) > 0$ and hence $\psi_N\left(\eta\left(\frac{x+z_{n,N}-x_n}{N}\right)\right) > 0$, so the integral is bounded by 1. Thus, it suffices to show that for a large K

$$\sum_{n\colon x_n < -KN} \exp\left(-N\gamma\left(\frac{x + z_{n,N} - x_n}{N}\right)\right)$$

converges to 0 for $N \to \infty$. Recall the definition of ζ_1 in assumption 1.2 and assume the lower bound

$$z_{n,N} \geq AN + B$$

with A, B independent of n which will be proven in the next step. Next, choose K such that

$$\begin{aligned} \forall q \geq K + A: \quad \gamma(q) &\geq 2\zeta_1 q, \\ K &\geq -2A, \end{aligned}$$

which is possible because γ is convex with $\gamma'(q) = \eta(q) \to_{q\to\infty} \infty$. Thus, for N large enough

$$\begin{aligned}\sum_{n:\, x_n < -KN} \exp\left(-N\gamma\left(\frac{x + z_{n,N} - x_n}{N}\right)\right) &\le \sum_{n:\, x_n < -KN} \exp\left(-2\zeta_1(x + z_{n,N} - x_n)\right) \\ &\le \sum_{n:\, x_n < -KN} \exp\left(-2\zeta_1(x + AN + B - x_n)\right) \\ &\le \exp\left(-2\zeta_1(x + B)\right) \sum_{n:\, x_n < -KN} \exp\left(\zeta_1 x_n\right) \to_{N\to\infty} 0\end{aligned}$$

for μ-a.e. $(x_n)_{n\ge 1}$ and where the convergence follows from assumption 1.2.

3) We will bound $z_{n,N}$ from below by an affine function in N uniformly in n, i.e. find uniform constants A, B such that

$$z_{n,N} \ge AN + B$$

for all n, N. To this end note that by the Central Limit Theorem we have

$$F_N(x_3) \ge \mathbb{P}_\pi(x_1 + S_N \ge x_3) + \mathbb{P}_\pi(x_2 + S_N \ge x_3) + \mathbb{P}_\pi(x_3 + S_N \ge x_3) \to_{N\to\infty} \frac{3}{2},$$

hence $F_N(x_3) > 1$ for N large enough. By the definition of z_N it follows that $z_N \ge x_3$ for N large enough. Thus, we can find constants A, B such that for all N we have $z_N \ge AN + B$. The definition of $z_{n,N}$ implies immediately

$$z_{n,N} \ge z_N \ge AN + B$$

as claimed.

4) Putting the first three steps together, we conclude

$$H_N(x) = \int_0^\infty e^{-ux} \varrho_N(u)(1 + \varepsilon_N(u))du + \widetilde{\varepsilon}_N(x)$$

with $\delta_N \equiv \sup_u |\varepsilon_N(u)| \to_{N\to\infty} 0$, $\widetilde{\varepsilon}_N(x) \to_{N\to\infty} 0$ for all $x \in \mathbb{R}$ in the non-lattice case and for all $x \in p\mathbb{Z}$ in the lattice case. It follows directly that for all such x we can find positive numbers δ_N tending to 0 for N tending to infinity such that

$$|H_N(x) - R_{\varrho_N}(x)| \le \delta_N R_{\varrho_N}(x) + \widetilde{\varepsilon}_N(x).$$

Recalling that in the lattice case the functions H_N are piecewise constant having jumps only on a subset of $p\mathbb{Z}$ we may write the same inequality in terms of the functions $\widetilde{R}_{\varrho_N}$ and get

$$|H_N(x) - \widetilde{R}_{\varrho_N}(x)| \le \delta_N \widetilde{R}_{\varrho_N}(x) + \widetilde{\varepsilon}_N(x)$$

with $\delta_N \to_{N\to\infty} 0$ and $\widetilde{\varepsilon}_N(x) \to_{N\to\infty} 0$ for all $x \in \mathbb{R}$ in both cases. We will use this estimate in order to rewrite the equation

$$\widetilde{G}_\mu(f) = \lim_{N\to\infty} \int d\mu\, \widehat{G}_{F_N}(f)$$

of Theorem 1.5 in terms of the functions $\widetilde{R}_{\varrho_N}$. To this end for each N define a measurable transformation T_N of the space of configurations by

$$T_N : \Omega \to \Omega, \quad (x_n)_{n\geq 1} \mapsto (x_n + KN + z_N - z_{n,N})_{n\geq 1}$$

and let μ_N be the measure on Ω induced by μ via T_N. Then by the definition of μ_N and by the invariance of $\widehat{G}$ under the shift of particles by $KN + z_N$ we may conclude

$$\widetilde{G}_\mu(f) = \lim_{N\to\infty} \int d\mu_N \, \widehat{G}_{H_N}(f).$$

The explicit representation of the modified probability generating functional of a Poisson point process implies

$$\widetilde{G}_\mu(f) = \lim_{N\to\infty} \int d\mu_N \int_{\mathbb{R}} de^{-H_N(x)} \exp\left(- \int_{-\infty}^{x} \left(1 - e^{-f(x-y)}\right) dH_N(y) \right).$$

Taking into account the bounds

$$|e^{-x} - e^{-y}| \leq |x - y|$$

for non-negative x, y and

$$|H_N(x) - \widetilde{R}_{\varrho_N}(x)| \leq \delta_N \widetilde{R}_{\varrho_N}(x) + \widetilde{\varepsilon}_N(x)$$

it follows that

$$\begin{aligned}
\widetilde{G}_\mu(f) &= \lim_{N\to\infty} \int d\mu_N \int_{\mathbb{R}} de^{-H_N(x)} \exp\left(- \int_{-\infty}^{x} \left(1 - e^{-f(x-y)}\right) dH_N(y) \right) \\
&= \lim_{N\to\infty} \int d\mu_N \int_{\mathbb{R}} de^{-\widetilde{R}_{\varrho_N}(x)} \exp\left(- \int_{-\infty}^{x} \left(1 - e^{-f(x-y)}\right) d\widetilde{R}_{\varrho_N}(y) \right) \\
&= \lim_{N\to\infty} \int d\mu_N \, \widehat{G}_{\widetilde{R}_{\varrho_N}}(f).
\end{aligned}$$

To be precise, one can estimate $|e^{-H_N(x)} - e^{-\widetilde{R}_{\varrho_N}(x)}|$ from above to prove

$$|de^{-H_N(x)} - de^{-\widetilde{R}_{\varrho_N}(x)}| \xrightarrow{w} 0$$

and then estimate the difference between the two integrands in a similar way.

5) Define by ν_N the probability measure on $\mathbb{M}$ induced by μ_N through the measurable mapping $\widetilde{T}_N$ given by

$$\widetilde{T}_N : \Omega \to \mathbb{M}, \quad (x_n)_{n\geq 1} \mapsto \varrho_N$$

if $\varrho_N \in \mathbb{M}$ and $\widetilde{T}_N((x_n)_{n\geq 1}) \equiv 0$ otherwise. Step 4 and the definition of ν_N imply

$$\widetilde{G}_\mu(f) = \lim_{N\to\infty} \int_{\mathbb{M}} \nu_N(d\varrho) \widehat{G}_{\widetilde{R}_\varrho}(f).$$

Our next claim is that the sequence $(\nu_N)_{N\ge1}$ is tight. To this end we will show that for each $\delta > 0$ there exists a function M such that for all N it holds

$$\nu_N(\varrho \in \mathbb{M}|\, R_\varrho(x) \le M(x)) \ge 1-\delta.$$

The compactness of $\{\varrho \in \mathbb{M}|\, R_\varrho(x) \le M(x)\}$ will then imply the claim. Note that because of the monotonicity of R_ϱ we may replace R_ϱ by $\widetilde{R}_\varrho$ in the last inequality without loss of generality. Recalling the definition of ν_N we observe that the inequality corresponds to

$$\mu_N((x_n)_{n\ge1} \in \Omega|\, \widetilde{R}_{\varrho_N}(x) \le M(x)) \ge 1-\delta.$$

The upper bound on $|H_N(x) - \widetilde{R}_{\varrho_N}(x)|$ and the definition of μ_N allows us to replace $\widetilde{R}_{\varrho_N}$ by H_N and subsequently to rewrite the inequality as

$$\mu((x_n)_{n\ge1} \in \Omega|\, \sum_n \mathbb{P}_\pi(x_n + S_N \ge x + KN + z_N) \le M(x)) \ge 1-\delta.$$

We can get an upper bound on $\sum_n \mathbb{P}_\pi(x_n+S_N \ge x+KN+z_N)$ which is uniform in N by adapting the estimates of step 2 to the present situation and enlarging the constant K if necessary. More precisely, in

$$\mathbb{P}_\pi(x_n + S_N \ge x + KN + z_N) \le \exp\left(-N\gamma\left(\frac{x+KN+z_N-x_n}{N}\right)\right)$$

we can bound $N\gamma\left(\frac{x+KN+z_N-x_n}{N}\right)$ from below by $\zeta_1(x+KN+z_N-x_n)$ and subsequently estimate z_N by $AN+B$ leading to

$$\sum_n \mathbb{P}_\pi(x_n + S_N \ge x + KN + z_N) \le \sum_n \exp\left(-\zeta_1(x+KN+AN+B-x_n)\right).$$

Finally, the right-hand side is bounded by $e^{-\zeta_1(x+B)}\sum_n e^{\zeta_1 x_n}$ which is finite μ-a.s. The claim follows now by choosing $M(x)$ satisfying

$$\mu((x_n)_{n\ge1}| e^{-\zeta_1(x+B)}\sum_n e^{\zeta_1 x_n} \le M(x)) \ge 1-\delta.$$

Strictly speaking we may have to choose a larger $M(x)$, because the estimate above is only valid for $N \ge N_0(x)$ with probability tending to 1 for $N_0(x)$ tending to infinity. Hence, we can choose $N_0(x)$ such that the probability is larger than $1-\frac{\delta}{2}$ and fix an $M(x)$ large enough for our purposes afterwards. The claim readily follows.

6) Define ν to be the limit point of a converging subsequence of $(\nu_N)_{N\ge1}$. Then the same approximation of $\widehat{G}_F$ by functionals continuous in F as in [14] implies

$$\widetilde{G}_\mu(f) = \lim_{N\to\infty}\int_{\mathbb{M}} \nu_N(d\varrho)\widehat{G}_{\widetilde{R}_\varrho}(f) = \int_{\mathbb{M}} \nu(d\varrho)\widehat{G}_{\widetilde{R}_\varrho}(f)$$

which proves the proposition. □

4 Poisson intensities

Up to this point we have shown that a quasi-stationary measure of a one-dimensional evolution satisfying the assumption 1.1 or the assumption 1.2 is given by a superposition of Poisson point processes. In this section we provide the exact shape of the Poisson intensity measures both in case that assumption 1.1 and in case that assumption 1.2 is satisfied. This is done by exploiting the properties of the steepness relation defined next.

Definition 4.1. *Let $\mathscr{F}$ be the space of positive, non-increasing, left-continuous functions F on $\mathbb{R}$ with $F(x) \to_{x\to\infty} 0$. For any two functions F, $G \in \mathscr{F}$ define G to be steeper than F if*

$$\forall u > 0: \quad G(x) = F(y) \;\Rightarrow\; G(x+u) \leq F(y+u). \tag{IV.13}$$

Remark. Note that since the measures $\varrho \in \mathbb{M}$ are supported on $(0,\infty)$, their Laplace transforms R_ϱ and also their modified Laplace transforms $\widetilde{R}_\varrho$ are elements of $\mathscr{F}$.

The main tool in the characterization of the Poisson intensities is the following result.

Lemma 4.2. *Let ϱ be a measure in $\mathbb{M}$, $F = \widetilde{R}_\varrho \in \mathscr{F}$ and λ be the unique positive measure on $\mathbb{R}$ with*

$$\forall a < b: \quad \lambda([a,b)) = F(a) - F(b). \tag{IV.14}$$

If the π_n satisfy assumption 1.2, π is the probability distribution of each of the π_n and $G \in \mathscr{F}$ is the unique function with

$$\forall a < b: \quad G(a) - G(b) = (\lambda * \pi)([a,b)), \tag{IV.15}$$

then G is steeper than F.

Proof. Let $G(a) = F(b)$ for some $a, b \in \mathbb{R}$. Without loss of generality we may assume $b = 0$, because otherwise we can replace ϱ by $\widetilde{\varrho}(du) \equiv e^{su}\varrho(du)$ yielding shifted versions $\widetilde{F}$, $\widetilde{G}$ of F, G with $\widetilde{F}(0) = F(b)$ for a suitable s. By performing the same argument as below for $\widetilde{F}$, $\widetilde{G}$ instead of F, G we can conclude that $\widetilde{G}$ is steeper than $\widetilde{F}$. Thus, G is steeper than F by the invariance of the steepness relation under shifts. Furthermore we may assume that $a \geq 0$, because otherwise we can replace π by a shifted version of itself and note that the following argument does not depend on the fact that the expectation of π is zero. We need to show $G(a+v) \leq F(v)$ for any $v > 0$. By Fubini's Theorem and integration by parts we can estimate $G(x)$ for $x \geq 0$ in the following way:

$$\begin{aligned}
G(x) = \int_{-\infty}^{\infty} \lambda([x-y,\infty))\pi(dy) = \int_{-\infty}^{\infty} R_\varrho([x-y])\pi(dy) = \int_{-\infty}^{\infty}\int_0^{\infty} e^{-u[x-y]}\varrho(du)\pi(dy) \\
= \int_0^{\infty}\int_{-\infty}^{\infty} e^{-u[x-y]}\pi(dy)\varrho(du) \leq \int_0^{\infty} e^{-u[x]}\left[\int_{-\infty}^{\infty} e^{u[y]}\pi(dy)\right]\varrho(du) \\
= \int_0^{\infty} e^{-u[x]}\left[\int_{-\infty}^{\infty} e^{uy}\pi(dy)\right]\varrho(du) = \int_0^{\infty} [x]e^{-ta[x]}\int_0^t\left[\int_{-\infty}^{\infty} e^{uy}\pi(dy)\right]\varrho(du)\,dt
\end{aligned}$$

where we have used $-[x-y] \leq -[x]+[y]$ and the fact that $[y] = y$ on the support of π. A similar but simpler calculation for F implies for $x \geq 0$ that

$$F(x) = R_\varrho([x]) = \int_0^{\infty} e^{-u[x]}\varrho(du) = \int_0^{\infty}[x]e^{-t[x]}\int_0^t \varrho(du)\,dt.$$

A calculation for $G(a+v)$ similar to that for $G(x)$ together with the calculation for $F(x)$ imply

$$\begin{aligned} G(a+v) &\leq \int_0^\infty [v]e^{-t[v]} \int_0^t \left[\int_{-\infty}^\infty e^{u(y-a)}\pi(dy)\right] \varrho(du)\,dt \\ &\leq \int_0^\infty [v]e^{-t[v]} \int_0^t \varrho(du)\,dt = F(v) \end{aligned}$$

where we have used the inequality

$$\int_0^t \left[\int_{-\infty}^\infty e^{u(y-a)}\pi(dy)\right] \varrho(du) \leq \int_0^t \varrho(du)$$

of Lemma 7.2 in [14] which relies only on the fact that π is a probability measure and not on its shape. $\square$

The last tool we need is the following classical version of the Choquet-Deny Theorem (Theorem 3 in [7]).

Lemma 4.3 (Choquet-Deny Theorem). *Let π be a probability measure on a locally compact abelian group X. Then the positive measures λ on X satisfying $\lambda * \pi = \lambda$ are given by*

$$\lambda = \int_{\mathscr{E}\times\Gamma} (f\,\omega) * \delta_x \, d\nu(f,x)$$

where Γ is a set containing exactly one representative of each coset of the subgroup Y generated by the support of π in X, the set $\mathscr{E}$ is given by $\mathscr{E} = \{f \,|\, \forall g_1, g_2 \in Y :\ f(g_1+g_2) = f(g_1)f(g_2)\}$, the measure ω is a Haar measure on Y and ν is a positive Radon measure on $\mathscr{E}\times\Gamma$.

The Choquet-Deny Theorem in this general form allows us to finish the proof of Theorem 1.3 in the one-dimensional setting which is one of the main results of the paper.

Proof of Theorem 1.3 in the one-dimensional setting. 1) We have shown above that under assumption 1.1 or 1.2 each quasi-stationary measure is a superposition of Poisson point processes. To prove that the intensity measures of the latter have the desired form, we will restrict to the case that assumption 1.2 is satisfied. The other case is completely analogous and requires only a replacement of the space over which the superposition is taken. From now on we consider an evolution satisfying assumption 1.2 and let F, G be functions constructed as in Lemma 4.2 where ϱ will vary over $\mathbb{M}$, so that F varies over the space of the modified Laplace transforms of measures in $\mathbb{M}$. Making $N+1$ steps of the evolution instead of N in the proof of Proposition 3.3 shows that

$$\widetilde{G}_\mu(f) = \int_{\mathbb{M}} \nu(d\varrho)\widehat{G}_F(f) = \int_{\mathbb{M}} \nu(d\varrho)\widehat{G}_G(f).$$

From here it follows that $\lambda * \pi$ is a translate of λ with the notation used in Lemma 4.2. This can be shown as in the proof of Theorem 8.1 in [14], applying our Lemma 4.2 instead of their Theorem 7.3.

2) Applying the Choquet-Deny Theorem in the case that *supp* π contains a non-trivial interval (and thus the same holds for *supp* π_a) we get $Y = X = \mathbb{R}$; moreover, Γ can be chosen as $\{0\}$ and ω may be chosen as the Lebesgue measure $\mathscr{L}^1$ on $\mathbb{R}$. Hence,

$$\lambda = \int_{\mathscr{E}} (f\mathscr{L}^1)\,\alpha(df)$$

for a Radon measure α on $\mathscr{E}$. Noting that λ corresponds to a function in $\mathscr{F}$, we conclude that α is supported on $f \in \mathscr{E}$ with the continuity and decay properties of functions in $\mathscr{F}$, i.e. $f \in \{e^{-sx} | s > 0\}$ as claimed.

3) In the case *supp* $\pi \subset p\mathbb{Z} + r$ for some $p > 0$, $r \geq 0$, so *supp* $\pi_a \subset p\mathbb{Z} + (r-a)$, we distinguish the cases $\mathbb{Z}p + \mathbb{Z}(r-a) = \mathbb{Z}\frac{p}{n}$ for an $n \in \mathbb{N}$ and $\mathbb{Z}p \cap \mathbb{Z}(r-a) = \{0\}$ which will correspond to the two different types of the Poisson intensity measures.

In the first case we may apply the Choquet-Deny Theorem with $Y = \mathbb{Z}p + \mathbb{Z}(r-a) = \mathbb{Z}\frac{p}{n}$, $\Gamma = [0, \frac{p}{n})$ and ω being the counting measure ω_c on Y and conclude

$$\lambda(A) = \int_{\mathscr{E}\times\left[0,\frac{p}{n}\right)} \left((f\,\omega_c) * \delta_y\right)(A)\, d\widetilde{\nu}(f,y)$$

for a positive Radon measure $\widetilde{\nu}$ on $\mathscr{E} \times \left[0, \frac{p}{n}\right)$. Noting that the only elements of $\mathscr{E}$ with the decay properties of functions in $\mathscr{F}$ are $\{e^{-sx} | s > 0\}$ restricted to $\mathbb{Z}\frac{p}{n}$, we deduce

$$\lambda(A) = \int_{\mathbb{R}_+\times\left[0,\frac{p}{n}\right)} \left((e^{-sx}\omega_c) * \delta_y\right)(A)\, d\widetilde{\alpha}(s,y)$$

for a suitable Radon measure $\widetilde{\alpha}$ on $\mathbb{R}_+ \times \left[0, \frac{p}{n}\right)$. The last equation can be rewritten as

$$\begin{aligned}\lambda(A) = \int_{\mathbb{R}_+\times\left[0,\frac{p}{n}\right)} \sum_{x\in\left(\mathbb{Z}\frac{p}{n}+y\right)\cap A} e^{-sx}\, d\widetilde{\alpha}(s,y) &= \int_{\mathbb{R}_+\times\left[0,\frac{p}{n}\right)} \sum_{x\in\left(\mathbb{Z}p+ny\right)\cap An} e^{-s\frac{x}{n}}\, d\widetilde{\alpha}(s,y)\\ &= \int_{\mathbb{R}_+\times\left[0,p\right)} \sum_{x\in\left(\mathbb{Z}p+y\right)\cap An} e^{-s\frac{x}{n}}\, d\alpha(s,y)\end{aligned}$$

with a positive Radon measure α on $\mathbb{R}_+ \times [0, p)$. Finally, a Poisson point process with intensity measure λ on $\mathbb{R}$ is supported on simple configurations iff $\alpha(\mathbb{R}_+, dy)$ has no point masses.

In the second case we apply the Choquet-Deny Theorem with $Y = \mathbb{Z}p \oplus \mathbb{Z}(r-a)$, noting that ω may be chosen as the counting measure ω_c on Y and identifying Γ with $[0,p) \times [0, r-a)$ via the Chinese Remainder Theorem. The result is

$$\lambda(A) = \int_{\mathscr{E}\times[0,p)\times[0,r-a)} \left((f\,\omega_c) * \delta_y\right) d\widetilde{\nu}(f, dy)$$

for a positive Radon measure $\widetilde{\nu}$ on $\mathscr{E} \times [0,p) \times [0,r-a)$. The set $\mathscr{E}$ of exponentials on Y with the decay properties of functions in $\mathscr{F}$ is given here by $f(kp + l(r-a)) = e^{-s_1 k - s_2 l}$ with parameters $s_1, s_2 > 0$. Hence, the last equation can be rewritten as

$$\lambda(A) = \int_{\mathbb{R}_+ \times \mathbb{R}_+ \times [0,p) \times [0,r-a)} \sum_{k,l \in \mathbb{Z}:\, kp + l(r-a) + y \in A} e^{-s_1 l - s_2 k}\, d\beta(s_1, s_2, y)$$

with a suitable positive Radon measure β on $\mathbb{R}_+ \times \mathbb{R}_+ \times [0,p) \times [0,r-a)$. The equation in the theorem follows by setting $w \equiv r - a$. Finally, note that a Poisson point process with intensity measure μ is supported on simple configurations iff $\beta(\mathbb{R}_+, \mathbb{R}_+, dy)$ has no point masses.

4) In this last step we will prove that a Poisson point process with the intensity measure λ is a simple quasi-stationary measure for our evolution (hence, superpositions of such processes are also simple and quasi-stationary). The simplicity follows from the corresponding remarks in step 3. For quasi-stationarity note that λ was constructed as the solution of a Choquet-Deny equation which by Lemma 7.4 in [14] implies

$$\mathbb{P}_F(x_1 - x_2 \geq u) = \int_{-\infty}^{\infty} e^{-F(x-u)} dF(x) = \int_{-\infty}^{\infty} e^{-G(x-u)} dG(x) = \mathbb{P}_G(x_1 - x_2 \geq u).$$

Here $\mathbb{P}_F$, $\mathbb{P}_G$ denote probabilities associated with Poisson point processes corresponding to F, G, respectively. Thus, the distribution of the first gap is invariant under the evolution. A similar calculation proves that this holds for any finite number of gaps which proves quasi-stationarity. $\square$

5 Quasi-stationary measures of the multidimensional evolution

This section contains a sketch of the proof of Theorem 1.3 under the assumption 1.2 for any dimension d in which we explain how the arguments of sections 3 and 4 generalize to the multidimensional case.

Proof of Theorem 1.3 under assumption 1.2. 1) For the extension of the Generalized Poissonization Theorem define functions F_N in the same way as for $d = 1$ with $\succeq$ instead of $\geq$ and let measures λ_N be defined analogously to the case $d = 1$ by setting $\lambda_N([a,b))$ for any finite box $[a,b)$ to be the alternating sum of values of F_N at the vertices of the box. Note that λ_N is a positive measure, because every summand $\mathbb{P}_\pi(x_n + S_N \succeq x)$ in the definition of F_N defines a probability measure on $\mathbb{R}^d$. Moreover, for test functions $f \in C_c^+(\mathbb{R}^d)$ let the modified probability generating functional $\widetilde{G}_\mu(f)$ be defined by

$$\widetilde{G}_\mu(f) = \mathbb{E}\Big[\exp\Big(-\sum_n f(x_1 - x_n)\Big)\Big]$$

where the configurations are arranged in the non-ascending order $\succeq$. Define $\widehat{G}_{F_N}$ as the corresponding quantity for the d-dimensional Poisson point process with intensity measure λ_N. Then performing the proof of the Poissonization Theorem of [14] with $\geq$ replaced by $\succeq$ and $<$ replaced by $\not\succeq$ and applying the spreading property (Lemma 1.6) to each of the components to bound

$\mathbb{P}_\pi(S_N \succeq x,\ S_N \not\succeq x + D)$ uniformly in x one deduces

$$\widetilde{G}_\mu(f) = \lim_{N\to\infty} \int d\mu\, \widehat{G}_{F_N}(f)$$

for any $f \in C_c^+(\mathbb{R}^d)$.

2) To prove that μ is a superposition of Poisson point processes we generalize the proof of Proposition 3.3. Using again the Central Limit Theorem we can find points z_N on the line $l \subset \mathbb{R}^d$ appearing in the definition of $\succeq$ with $F_N(z_N) \leq 1$ and having coordinates which are bounded from below by an affine function of N. Define the points $z_{n,N}$ such that $z_{n,N} \geq z_N$ and $p\left(\frac{z_{n,N}-x_n}{N}\right)$ is a lattice point which is possible by the definition of p (see the definition of $\succeq$). Then each component of $z_{n,N}$ is also bounded from below by an affine function of N. Applying the d-dimensional version of the Bahadur-Rao Theorem (Lemma 3.1) with points $p\left(\frac{z_{n,N}-x_n}{N}\right)$ and introducing again measures μ_N, ν_N we can conclude

$$\widetilde{G}_\mu(f) = \lim_{N\to\infty} \int_{\mathbb{M}} \nu_N(d\varrho)\widehat{G}_{\widetilde{R}_\varrho}(f)$$

where now $\mathbb{M}$ denotes the space of finite measures on $(\mathbb{R}_+)^d$ and $\widetilde{R}_\varrho$ denotes the Laplace transform of a $\varrho \in \mathbb{M}$ with the argument modified to the closest lattice point from above if necessary. Observe that $\widetilde{R}_\varrho$ is a tail distribution function of a locally finite positive measure for $\varrho \in \mathbb{M}$, because for any $x \geq 0$ the function $e^{-u\cdot x}$ is the tail distribution function of a product of exponential distributions on $\mathbb{R}$. By exactly the same argument as in section 3 it follows that the sequence ν_N is tight and we let ν be a subsequential limit of it. By the approximation of the functional $\widetilde{G}_F$ by functionals continuous in F as done in [14] for $d = 1$ one may conclude

$$\widetilde{G}_\mu(f) = \int_{\mathbb{M}} \nu(d\varrho)\widehat{G}_{\widetilde{R}_\varrho}(f),$$

so μ is a superposition of Poisson point processes.

3) To extend section 4 to the case $d > 1$ we extend first the steepness relation to the space of tail distribution functions F of positive measures on $\mathbb{R}^d$ which satisfy $F(\lambda x) \to_{\lambda\to\infty} 0$ for each $x \in \mathbb{R}^d - (\mathbb{R}_-)^d$. We call G steeper than F if $F(x) = G(y)$ for some $x, y \in \mathbb{R}^d$ implies $F(x + a) \geq G(y + a)$ for all $a \in (\mathbb{R}_+)^d$. The same calculation as before for each of the d coordinates yields that functions become steeper if convolved with probability measures in the sense of Lemma 4.2. Finally, the same monotonicity argument as in section 4 shows that for intensity measures λ of quasi-stationary Poisson point processes $\lambda * \pi$ has to be a translate of λ. Finally, the Poisson point process with intensity measure λ has to be supported on upper bounded configurations and be simple, hence λ has no point masses. □

6 Attractivity

In this concluding section we prove attractivity of certain quasi-stationary Poisson point processes in the space of all Poisson point processes with almost surely infinite, locally finite and upper bounded configurations by analyzing the corresponding evolution of intensity measures. The latter will be assumed locally finite satisfying

$$\lambda(c+(\mathbb{R}_-)^d)=\infty, \quad \lambda(\mathbb{R}^d-(c+(\mathbb{R}_-)^d))<\infty$$

for a $c\in\mathbb{R}^d$ depending on λ which means precisely that the corresponding Poisson point processes are supported on infinite, locally finite and upper bounded configurations. As before we will denote the points of such configurations by $x_1, x_2, \dots$ in descending order $\succeq$ and call the Poisson point processes and intensity measures of this type regular. In particular, it turns out that the space of regular Poisson point processes is invariant under evolutions with i.i.d. increments. For the increments $(\pi_n)_{n\geq 1}$ we assume for this section that $\mathbb{E}[\pi_n]=0$ and the π_n are not almost surely equal to 0. Obviously, the two assumptions can be made without loss of generality since a recentering of the increments does not affect the joint distribution of the gaps of the evolved process which will be the only quantity of interest. Lemma 6.1 and Lemma 6.2 are the key to the attractivity result. The first is taken from section 11.4 of [5], so we omit the proof and the proof of the second is given below.

Lemma 6.1. *Let N be a regular Poisson point process on $\mathbb{R}^d$ with corresponding intensity measure λ and configurations $(x_n)_{n\geq 1}$. Define $\widetilde{N}$ to be the point processes with configurations*

$$(y_n)_{n\geq 1}\equiv(x_n+\pi_n)_{n\geq 1}$$

where π_n are i.i.d. random variables with distribution π which are independent of N. Then $\widetilde{N}$ is a Poisson point process with intensity measure $\lambda*\pi$.

Lemma 6.2. *Let N be a regular Poisson point process with intensity measure λ and suppose that there exists a measure λ_∞ corresponding to a regular Poisson point process and satisfying*

$$\lambda*\pi^{*n}\xrightarrow{w}_{n\to\infty}\lambda_\infty \tag{VI.16}$$

where π is the distribution of the increments of the evolution as in Lemma 6.1. Then the joint distribution of the gaps of N after n evolutions converges to the corresponding quantity for the Poisson point process with intensity measure λ_∞ for $n\to\infty$.

Proof. From the multidimensional version of the Levy Continuity Theorem it can be deduced that the convergence of the joint distribution of the gaps follows from the convergence of the corresponding modified probability generating functionals

$$\mathbb{E}\Big[\exp\Big(-\sum_n f(x_1-x_n)\Big)\Big]$$

for test functions $f\in C_c^+(\mathbb{R}^d)$, so it suffices to show the convergence of the latter. To this end let N_k be the point process resulting from N after k steps of the evolution. By Lemma 6.1 it is a Poisson point process with intensity measure $\lambda*\pi^{*k}$. By the general formula for modified probability generating functionals of Poisson point processes the modified probability generating functional of N_k is given by

$$\widehat{G}_{\lambda*\pi^{*k}}(f)\equiv\int_{\mathbb{R}^d}\left(\lambda*\pi^{*k}\right)(dx)\exp\left(-(\lambda*\pi^{*k})([x,\infty))\right)\exp\left(-\int_{-\infty}^{x}e^{-f(x-y)}(\lambda*\pi^{*k})(dy)\right).$$

Even though $\widehat{G}_\lambda$ is not a continuous functional of λ, it can be approximated arbitrarily well by continuous functionals of λ in the appropriate L^1 sense as was done in the proof of Theorem 6.1 in [14]. Hence, the weak convergence of $\lambda * \pi^{*k}$ towards λ_∞ implies

$$\widehat{G}_{\lambda * \pi^{*k}}(f) \to_{k\to\infty} \widehat{G}_{\lambda_\infty}(f)$$

for all test functions $f \in C_c^+(\mathbb{R}^d)$. The claimed convergence of joint gap distributions readily follows. $\square$

We prove now Proposition 1.4. It shows that any regular solution λ_∞ of the Choquet-Deny equation $\lambda * \pi = \lambda$ is attractive in the direction of a measure ϱ where ϱ can be infinite, but "small" enough in the sense that ϱ is regular and its tail at infinity is dominated by the tail of any multiple of the Lebesgue measure. Since any of these multiples are fixed points under the evolution in case that the π_n have a density, adding something of the order of such a multiple $\alpha\mathscr{L}^d$ to λ_∞ will result in attractiveness towards $\lambda_\infty + \alpha\mathscr{L}^d$ instead of λ_∞. Hence, Proposition 1.4 is the strongest attractivity result possible in this context.

Proof of Proposition 1.4. By Lemma 6.2 and the bilinearity of the convolution it suffices to show

$$(\varrho * \pi^{*n})((a,b)) \to_{n\to\infty} 0$$

for any $a \le b$. To this end fix a and b and define $f(y) = \varrho((a-y, b-y))$. The spreading property (Lemma 1.6) implies that for any $0 < \varepsilon < 1$ there exists a sequence $0 < a_n(\varepsilon) \to_{n\to\infty} \infty$ with

$$\pi^{*n}((-a_n(\varepsilon), a_n(\varepsilon))^d) \le \varepsilon.$$

We conclude that for any fixed ε it holds

$$\begin{aligned}(\varrho * \pi^{*n})((a,b)) = \int_{\mathbb{R}^d} f(y)\, \pi^{*n}(dy) &\le \sup_{|y|_\infty \ge a_n(\varepsilon)} f(y) + \varepsilon \sup_{|y|_\infty \le a_n(\varepsilon)} f(y) \\ &\le \sup_{|y|_\infty \ge a_n(\varepsilon)} f(y) + \varepsilon \sup_{y\in\mathbb{R}^d} f(y).\end{aligned}$$

Hence, by the assumptions on ϱ we have

$$\limsup_{n\to\infty} (\varrho * \pi^{*n})((a,b)) \le \varepsilon \sup_{y\in\mathbb{R}^d} f(y).$$

By letting ε tend to 0 we deduce that the limit $\lim_{n\to\infty}(\varrho * \pi^{*n})((a,b))$ exists and equals to 0. $\square$

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