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Self-repelling random walk with directed edges on $\mathbb{Z}$*

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Abstract

We consider a variant of self-repelling random walk on the integer lattice $\mathbb{Z}$ where the self-repellence is defined in terms of the local time on *oriented* edges. The long-time asymptotic scaling of this walk is surprisingly different from the asymptotics of the similar process with self-repellence defined in terms of local time on unoriented edges, examined in [10]. We prove limit theorems for the local time process and for the position of the random walker. The main ingredient is a Ray – Knight-type of approach. At the end of the paper, we also present some computer simulations which show the strange scaling behaviour of the walk considered.

Key words: random walks with long memory, self-repelling, one dimension, oriented edges, local time, Ray – Knight-theory, coupling.

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*Dedicated to József Fritz on the occasion of his 65$^{\text{th}}$ birthday

1 Introduction

The *true self-avoiding random walk* on $\mathbb{Z}$ is a nearest neighbour random walk, which is locally pushed in the direction of the negative gradient of its own local time (i.e. occupation time measure). For precise formulation and historical background, see [1], [8], [7], [10], the survey papers [12], [13], and/or further references cited there. In [10], the edge version of the problem was considered, where the walk is pushed by the negative gradient of the local time spent on *unoriented* edges. There, precise asymptotic limit theorems were proved for the local time process and position of the random walker at late times, under space scaling proportional to the 2/3-rd power of time. For a survey of these and related results, see [12], [13], [9]. Similar results for the site version have been obtained recently, [14]. In the present paper, we consider a similar problem but with the walk being pushed by the local differences of occupation time measures on *oriented* rather than unoriented edges. The behaviour is phenomenologically surprisingly different from the unoriented case: we prove limit theorems under square-root-of-time (rather than time-to-the-$\frac{2}{3}$) space-scaling but the limit laws are not the usual diffusive ones. Our model belongs to the wider class of *self-interacting random walks* which attracted attention in recent times, see e.g. [5], [3], [2], [15], [4] for a few other examples. In all these cases long memory of the random walk or diffusion is induced by a self-interaction mechanism defined locally in a natural way in terms of the local time (or occupation time) process. The main challenge is to understand the asymptotic scaling limit (at late times) of the process.

Let w be a weight function which is non-decreasing and non-constant:

$$w : \mathbb{Z} \to \mathbb{R}_+, \qquad w(z+1) \geq w(z), \qquad \lim_{z\to\infty} \big(w(z) - w(-z)\big) > 0. \tag{1}$$

We will consider a nearest neighbour random walk $X(n)$, $n \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$, on the integer lattice $\mathbb{Z}$, starting from $X(0) = 0$, which is governed by its local time process through the function w in the following way. Denote by $\ell^{\pm}(n,k)$, $(n,k) \in \mathbb{Z}_+ \times \mathbb{Z}$, the local time (that is: its occupation time measure) on oriented edges:

$$\ell^{\pm}(n,k) := \#\{0 \leq j \leq n-1 : X(j) = k,\ \ X(j+1) = k \pm 1\},$$

where $\#\{\dots\}$ denotes cardinality of the set. Note that

$$\ell^{+}(n,k) - \ell^{-}(n,k+1) = \begin{cases} +1 & \text{if} \quad 0 \leq k < X(n), \\ -1 & \text{if} \quad X(n) \leq k < 0, \\ \ \ 0 & \text{otherwise.} \end{cases} \tag{2}$$

We will also use the notation

$$\ell(n,k) := \ell^{+}(n,k) + \ell^{-}(n,k+1) \tag{3}$$

for the local time spent on the unoriented edge $\langle k, k+1 \rangle$.

Our random walk is governed by the evolution rules

$$\mathbf{P}\big(X(n+1) = X(n) \pm 1 \mid \mathscr{F}_n\big) = \frac{\omega_n^{\pm}}{\omega_n^{+} + \omega_n^{-}} \tag{4}$$

with

$$\omega_n^{\pm} = w(\mp(\ell^{+}(n,X(n)) - \ell^{-}(n,X(n)))),$$

and

$$\ell^{\pm}(n+1,k) = \ell^{\pm}(n,x) + 1\!\!1\{X(n) = k, \;\; X(n+1) = k \pm 1\}. \tag{5}$$

That is: at each step, the walk prefers to choose that oriented edge pointing away from the actually occupied site which had been less visited in the past. In this way balancing or smoothing out the roughness of the occupation time measure. We prove limit theorems for the local time process and for the position of the random walker at large times under *diffusive scaling*, that is: essentially for $n^{-1/2}\ell(n, \lfloor n^{1/2}x \rfloor)$ and $n^{-1/2}X(n)$, but with limit laws strikingly different from usual diffusions. See Theorem 1 and 2 for precise statement.

The paper is further organized as follows. In Section 2, we formulate the main results. In Section 3, we prove Theorem 1 about the convergence in sup-norm and in probability of the local time process stopped at inverse local times. As a consequence, we also prove convergence in probability of the inverse local times to *deterministic values*. In Section 4, we convert the limit theorems for the inverse local times to local limit theorems for the position of the random walker at independent random stopping times of geometric distribution with large expectation. Finally, in Section 5, we present some numerical simulations of the position and local time processes with particular choices of the weight function $w(k) = \exp(\beta k)$.

2 The main results

As in [10], the key to the proof is a Ray–Knight-approach. Let

$$T^{\pm}_{j,r} := \min\{n \geq 0 : \ell^{\pm}(n,j) \geq r\}, \qquad j \in \mathbb{Z}, \quad r \in \mathbb{Z}_+$$

be the so called inverse local times and

$$\Lambda^{\pm}_{j,r}(k) := \ell(T^{\pm}_{j,r}, k) = \ell^{+}(T^{\pm}_{j,r}, k) + \ell^{-}(T^{\pm}_{j,r}, k+1), \qquad j,k \in \mathbb{Z}, \quad r \in \mathbb{Z}_+ \tag{6}$$

the local time sequence (on unoriented edges) of the walk stopped at the inverse local times. We denote by $\lambda^{\pm}_{j,r}$ and $\rho^{\pm}_{j,r}$ the leftmost, respectively, the rightmost edges visited by the walk before the stopping time $T^{\pm}_{j,r}$:

$$\lambda^{\pm}_{j,r} := \inf\{k \in \mathbb{Z} : \Lambda^{\pm}_{j,r}(k) > 0\},$$

$$\rho^{\pm}_{j,r} := \sup\{k \in \mathbb{Z} : \Lambda^{\pm}_{j,r}(k) > 0\}.$$

The next proposition states that the random walk is recurrent in the sense that it visits infinitely often every site and edge of $\mathbb{Z}$.

Proposition 1. *Let $j \in \mathbb{Z}$ and $r \in \mathbb{Z}_+$ be fixed. We have*

$$\max\left\{ T^{\pm}_{j,r} \;\; , \;\; \rho^{\pm}_{j,r} - \lambda^{\pm}_{j,r} \;\; , \;\; \sup_k \Lambda^{\pm}_{j,r}(k) \right\} < \infty$$

almost surely.

Actually, we will see it from the proofs of our theorems that the quantities in Proposition 1 are finite, and much stronger results are true for them, so we do not give a separate proof of this statement.

2.1 Limit theorem for the local time process

The main result concerning the local time process stopped at inverse local times is the following:

Theorem 1. *Let $x \in \mathbb{R}$ and $h \in \mathbb{R}_+$ be fixed. Then*

$$A^{-1}\lambda^{\pm}_{\lfloor Ax\rfloor,\lfloor Ah\rfloor} \xrightarrow{\mathbf{P}} -|x|-2h, \tag{7}$$

$$A^{-1}\rho^{\pm}_{\lfloor Ax\rfloor,\lfloor Ah\rfloor} \xrightarrow{\mathbf{P}} \quad |x|+2h, \tag{8}$$

and

$$\sup_{y\in\mathbb{R}}\left|A^{-1}\Lambda^{\pm}_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(\lfloor Ay\rfloor)-\big(|x|-|y|+2h\big)_+\right| \xrightarrow{\mathbf{P}} 0 \tag{9}$$

as $A \to \infty$.

Note that

$$T^{\pm}_{j,r}=\sum_{k=\lambda^{\pm}_{j,r}}^{\rho^{\pm}_{j,r}}\Lambda^{\pm}_{j,r}(k).$$

Hence, it follows immediately from Theorem 1 that

Corollary 1. *With the notations of Theorem 1,*

$$A^{-2}T^{\pm}_{\lfloor Ax\rfloor,\lfloor Ah\rfloor} \xrightarrow{\mathbf{P}} (|x|+2h)^2 \tag{10}$$

as $A \to \infty$.

Theorem 1 and Corollary 1 will be proved in Section 3.

Remark: Note that the local time process and the inverse local times converge in probability to deterministic objects rather than converging weakly in distribution to genuinely random variables. This makes the present case somewhat similar to the weakly reinforced random walks studied in [11].

2.2 Limit theorem for the position of the walker

According to the arguments in [10], [12], [13], from the limit theorems

$$A^{-1/\nu}T^{\pm}_{\lfloor Ax\rfloor,\lfloor A^{(1-\nu)/\nu}h\rfloor} \Rightarrow \mathscr{T}_{x,h}$$

valid for any $(x,h)\in\mathbb{R}\times\mathbb{R}_+$, one can essentially derive the limit theorem for the one-dimensional marginals of the position process:

$$A^{-\nu}X(\lfloor At\rfloor)\Rightarrow \mathscr{X}(t).$$

Indeed, the summation arguments, given in detail in the papers quoted above, indicate that

$$\varphi(t,x):=2\frac{\partial}{\partial t}\int_0^\infty \mathbf{P}\left(\mathscr{T}_{x,h}<t\right)\mathrm{d}h$$

is the good candidate for the density of the distribution of $\mathscr{X}(t)$, with respect to Lebesgue-measure. The scaling relation

$$A^{1/\nu}\varphi(At, A^{1/\nu}x) = \varphi(t,x) \tag{11}$$

clearly holds. In some cases (see e.g. [10]) it is not trivial to check that $x \mapsto \varphi(t,x)$ is a bona fide probability density of total mass 1. (However, a Fatou-argument easily shows that its total mass is not more than 1.) But in our present case, this fact drops out from explicit formulas. Indeed, the weak limits (10) hold, which, by straightforward computation, imply

$$\varphi(t,x) = \frac{1}{2\sqrt{t}}1\!\!1\{|x| \le \sqrt{t}\}.$$

Actually, in order to prove limit theorem for the position of the random walker, some smoothening in time is needed, which is realized through the Laplace-transform. Let

$$\hat{\varphi}(s,x) := s\int_0^\infty e^{-st}\varphi(t,x)\,\mathrm{d}t = \sqrt{s\pi}(1 - F(\sqrt{2s}|x|))$$

where

$$F(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,\mathrm{d}y$$

is the standard normal distribution function.

We prove the following *local limit theorem* for the position of the random walker stopped at an independent geometrically distributed stopping time of large expectation:

Theorem 2. *Let $s \in \mathbb{R}_+$ be fixed and $\theta_{s/A}$ a random variable with geometric distribution*

$$\mathbf{P}\left(\theta_{s/A} = n\right) = (1 - e^{-s/A})e^{-sn/A} \tag{12}$$

which is independent of the random walk $X(n)$. Then, for almost all $x \in \mathbb{R}$,

$$A^{1/2}\mathbf{P}\left(X(\theta_{s/A}) = \lfloor A^{1/2}x\rfloor\right) \to \hat{\varphi}(s,x)$$

as $A \to \infty$.

From the above local limit theorem, the integral limit theorem follows immediately:

$$\lim_{A\to\infty}\mathbf{P}\left(A^{-1/2}X(\theta_{s/A}) < x\right) = \int_{-\infty}^{x}\hat{\varphi}(s,y)\,\mathrm{d}y.$$

From (7) and (8), the tightness of the distributions $(A^{-1/2}X(\lfloor At\rfloor))_{A\ge1}$ follows easily. Theorem 2 yields that if the random walk $X(\cdot)$ has any scaling limit, then

$$A^{-1/2}X(\lfloor At\rfloor) \Longrightarrow \mathrm{UNI}(-\sqrt{t}, \sqrt{t}) \tag{13}$$

as $A \to \infty$ holds where $\mathrm{UNI}(-\sqrt{t}, \sqrt{t})$ stands for the uniform distribution on the interval $(-\sqrt{t}, \sqrt{t})$.

The proof of Theorem 2 is presented in Section 4.

3 Proof of Theorem 1

The proof is organized as follows. We introduce independent auxiliary Markov-chains associated to the vertices of $\mathbb{Z}$ in such a way that the value of the local time at the edges can be expressed with a sum of such Markov-chains. It turns out that the auxiliary Markov-chains converge exponentially fast to their common unique stationary distribution. It allows us to couple the local time process of the self-repelling random walk with the sum of i.i.d. random variables. The coupling yields that the law of large numbers for i.i.d. variables can be applied for the behaviour of the local time, with high probability. The coupling argument breaks down when the local time approaches 0. We show in Subsection 3.4, how to handle this case.

Let

$$L_{j,r}(k) := \ell^+(T_{j,r}^+, k). \tag{14}$$

Mind that due to (2), (3) and (6)

$$\left|\Lambda_{j,r}^+(k) - 2L_{j,r}(k)\right| \le 1. \tag{15}$$

We give the proof of (7), (8) and

$$\sup_{y\in\mathbb{R}} \left| A^{-1} L_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(\lfloor Ay\rfloor) - \left(\frac{|x|-|y|}{2} + h\right)_+ \right| \stackrel{\mathbf{P}}{\longrightarrow} 0$$

as $A \to \infty$, which, due to (15), implies (9) for Λ^+. The case of Λ^- can be done similarly. Without loss of generality, we can suppose that $x \le 0$.

3.1 Auxiliary Markov-chains

First we define the $\mathbb{Z}$-valued Markov-chain $l \mapsto \xi(l)$ with the following transition probabilities:

$$\mathbf{P}\left(\xi(l+1) = x+1 \mid \xi(l) = x\right) = \frac{w(-x)}{w(x)+w(-x)} =: p(x), \tag{16}$$

$$\mathbf{P}\left(\xi(l+1) = x-1 \mid \xi(l) = x\right) = \frac{w(x)}{w(x)+w(-x)} =: q(x). \tag{17}$$

Let $\tau_\pm(m)$, $m = 0, 1, 2, \dots$ be the stopping times of consecutive upwards/downwards steps of ξ:

$$\tau_\pm(0) := 0, \qquad \tau_\pm(m+1) := \min\left\{l > \tau_\pm(m) : \xi(l) = \xi(l-1) \pm 1\right\}.$$

Then, clearly, the processes

$$\eta_+(m) := -\xi(\tau_+(m)), \qquad \eta_-(m) := +\xi(\tau_-(m))$$

are themselves Markov-chains on $\mathbb{Z}$. Due to the $\pm$ symmetry of the process ξ, the Markov-chains η_+ and η_- have the same law. In the present subsection, we simply denote them by η neglecting the subscripts $\pm$. The transition probabilities of this process are

$$P(x,y) := \mathbf{P}\left(\eta(m+1) = y \mid \eta(m) = x\right) = \begin{cases} \prod_{z=x}^{y} p(z) q(y+1) & \text{if} \quad y \ge x-1, \\ 0 & \text{if} \quad y < x-1. \end{cases} \tag{18}$$

In the following lemma, we collect the technical ingredients of the forthcoming proof of our limit theorems. We identify the stationary measure of the Markov-chain η, state exponential tightness of the distributions of $\big(\eta(m)\,\big|\,\eta(0)=0\big)$ uniformly in m and exponentially fast convergence to stationarity.

Lemma 1. *(i) The unique stationary measure of the Markov-chain η is*

$$\rho(x) = Z^{-1}\prod_{z=1}^{\lfloor |2x+1|/2\rfloor}\frac{w(-z)}{w(z)} \quad \textit{with} \quad Z := 2\sum_{x=0}^{\infty}\prod_{z=1}^{x}\frac{w(-z)}{w(z)}. \tag{19}$$

(ii) There exist constants $C<\infty$, $\beta>0$ such that for all $m\in\mathbb{N}$ and $y\in\mathbb{Z}$

$$P^m(0,y)\le Ce^{-\beta|y|}. \tag{20}$$

(iii) There exist constants $C<\infty$ and $\beta>0$ such that for all $m\ge 0$

$$\sum_{y\in\mathbb{Z}}\Big|P^m(0,y)-\rho(y)\Big| < Ce^{-\beta m}. \tag{21}$$

Remark on notation: We shall use the generic notation

$$\texttt{something}\le Ce^{-\beta Y}$$

for exponentially strong bounds. The constants $C<\infty$ and $\beta>0$ will vary at different occurrences and they may (and will) depend on various fixed parameters but of course not on quantities appearing in the expression Y. There will be no cause for confusion.

Note that for any choice of the weight function w

$$\sum_{x=-\infty}^{+\infty} x\rho(x) = -\frac{1}{2}. \tag{22}$$

Proof of Lemma 1. The following proof is reminiscent of the proof of Lemmas 1 and 2 from [10]. It is somewhat streamlined and weaker conditions are assumed.

(i) The irreducibility of the Markov-chain η is straightforward. One can easily rewrite (18), using (19), as

$$P(x,y) = \begin{cases} \frac{1}{\rho(x)}\Big(p(x)\prod_{z=x+1}^{y+1}q(z)\Big)\rho(y) & \text{if} \quad y\ge x-1,\\ 0 & \text{if} \quad y<x-1.\end{cases}$$

It yields that ρ is indeed the stationary distribution for η, because

$$\sum_{x\in\mathbb{Z}}\rho(x)P(x,y) = \left(\sum_{x\le y+1}p(x)\prod_{z=x+1}^{y+1}q(z)\right)\rho(y)=\rho(y)$$

where the last equality holds, because $\lim_{z\to-\infty}\prod_{u=z}^{y+1}q(u)=0$.

(ii) The stationarity of ρ implies that

$$P^n(0,y) \le \frac{\rho(y)}{\rho(0)} = \prod_{z=1}^{\lfloor |2y+1|/2 \rfloor} \frac{w(-z)}{w(z)} \le Ce^{-\beta|y|}. \tag{23}$$

The exponential bound follows from (1). As a consequence, we get finite expectations in the forthcoming steps of the proofs below.

(iii) Define the stopping times

$$\theta_+ = \min\{n \ge 0 : \eta(n) \ge 0\},$$

$$\theta_0 = \min\{n \ge 0 : \eta(n) = 0\}.$$

From Theorem 6.14 and Example 5.5(a) of [6], we can conclude the exponential convergence (21), if for some $\gamma > 0$

$$\mathbf{E}\left(\exp(\gamma\theta_0) \mid \eta(0) = 0\right) < \infty \tag{24}$$

holds.

The following decomposition is true, because the Markov-chain η can jump at most one step to the left.

$$\begin{aligned} \mathbf{E}\left(\exp(\gamma\theta_0) \mid \eta(0) = 0\right) &= e^\gamma \sum_{y\ge 0} P(0,y)\mathbf{E}\left(\exp(\gamma\theta_0) \mid \eta(0) = y\right) \\ &+ e^\gamma P(0,-1) \sum_{y \ge 0} \mathbf{E}\left(\exp(\gamma\theta_+)1\!\!1\{\eta(\theta_+) = y\} \mid \eta(0) = -1\right)\mathbf{E}\left(\exp(\gamma\theta_0) \mid \eta(0) = y\right). \end{aligned} \tag{25}$$

One can easily check that, given $\eta(0) = -1$, the random variables θ_+ and $\eta(\theta_+)$ are independent, and $\mathbf{P}\left(\eta(\theta_+) = y \mid \eta(0) = -1\right) = p(0)p(1)\dots p(y)q(y+1) = P(0,y)$, by definition. Hence for $y \ge 0$

$$\mathbf{E}\left(\exp(\gamma\theta_+)1\!\!1\{\eta(\theta_+) = y\} \mid \eta(0) = -1\right) = \frac{P(0,y)}{1 - P(0,-1)}\mathbf{E}\left(\exp(\gamma\theta_+) \mid \eta(0) = -1\right). \tag{26}$$

Combining (25) and (26) gives us

$$\begin{aligned} &\mathbf{E}\left(\exp(\gamma\theta_0) \mid \eta(0) = 0\right) \\ &= e^\gamma \sum_{y \ge 0} P(0,y)\mathbf{E}\left(\exp(\gamma\theta_0) \mid \eta(0) = y\right)\left(1 + \frac{P(0,-1)}{1 - P(0,-1)}\mathbf{E}\left(\exp(\gamma\theta_+) \mid \eta(0) = -1\right)\right). \end{aligned} \tag{27}$$

So, in order to get the result, we need to prove that for properly chosen $\gamma > 0$

$$\mathbf{E}\left(\exp(\gamma\theta_+) \mid \eta(0) = -1\right) < \infty \tag{28}$$

and

$$\mathbf{E}\left(\exp(\gamma\theta_0) \mid \eta(0) = y\right) \le Ce^{\frac{\beta}{2}y} \qquad \text{for } y \in \mathbb{Z}_+ \tag{29}$$

where β is the constant in (20).

In order to make the argument shorter, we make the assumption

$$w(-1) < w(+1),$$

or, equivalently,

$$p(1) = \frac{w(-1)}{w(+1)+w(-1)} < \frac{1}{2} < \frac{w(+1)}{w(+1)+w(-1)} = q(1).$$

The proof can be easily extended for the weaker assumption (1), but the argument is somewhat longer.

First, we prove (28). Let $x < 0$ and $x-1 \le y < 0$. Then the following stochastic domination holds:

$$\sum_{z \ge y} P(x,z) = \prod_{z=x}^{y} p(z) \ge p(-1)^{y-x+1} = q(1)^{y-x+1}. \tag{30}$$

Let $\zeta(r)$, $r = 1,2,\dots$ be i.i.d. random variables with geometric law:

$$\mathbf{P}\left(\zeta = z\right) = q(1)^{z+1} p(1), \quad z = -1,0,1,2,\dots,$$

and

$$\widetilde{\theta} := \min\{t \ge 0 : \sum_{s=1}^{t} \zeta(s) \ge 1\}.$$

Note that $\mathbf{E}\left(\zeta\right) > 0$. From the stochastic domination (30), it follows that for any $t \ge 0$

$$\mathbf{P}\left(\theta_+ > t \mid \eta(0) = -1\right) \le \mathbf{P}\left(\widetilde{\theta} > t\right),$$

and hence

$$\mathbf{E}\left(\exp(\gamma\theta_+) \mid \eta(0) = -1\right) \le \mathbf{E}\left(\exp(\gamma\widetilde{\theta})\right) < \infty$$

for sufficiently small $\gamma > 0$.

Now, we turn to (29). Let now $0 \le x-1 \le y$. In this case, the following stochastic domination is true:

$$\sum_{z \ge y} P(x,z) = \prod_{z=x}^{y} p(z) \le p(1)^{y-x+1}. \tag{31}$$

Let now $\zeta(r)$, $r = 1,2,\dots$ be i.i.d. random variables with geometric law:

$$\mathbf{P}\left(\zeta = z\right) = p(1)^{z+1} q(1), \quad z = -1,0,1,2,\dots,$$

and for $y \ge 0$

$$\widetilde{\theta}_y := \min\{t \ge 0 : \sum_{s=1}^{t} \zeta(s) \le -y\}.$$

Note that now $\mathbf{E}\left(\zeta\right) < 0$. From the stochastic domination (31), it follows now that with $y \ge 0$, for any $t \ge 0$

$$\mathbf{P}\left(\theta_0 > t \mid \eta(0) = y\right) \le \mathbf{P}\left(\widetilde{\theta}_y > t\right),$$

and hence

$$\mathbf{E}\left(\exp(\gamma\theta) \mid \eta(0) = y\right) \le \mathbf{E}\left(\exp(\gamma\widetilde{\theta}_y)\right) \le Ce^{\frac{\beta}{2}y},$$

for sufficiently small $\gamma > 0$. $\square$

3.2 The basic construction

For $j \in \mathbb{Z}$, denote the inverse local times (times of jumps leaving site $j \in \mathbb{Z}$)

$$\gamma_j(l) := \min\left\{n : \ell^+(n,j) + \ell^-(n,j) \geq l\right\}, \tag{32}$$

and

$$\xi_j(l) := \ell^+(\gamma_j(l), j) - \ell^-(\gamma_j(l), j), \tag{33}$$

$$\tau_{j,\pm}(0) := 0, \qquad \tau_{j,\pm}(m+1) := \min\left\{l > \tau_{j,\pm}(m) : \xi_j(l) = \xi_j(l-1) \pm 1\right\}, \tag{34}$$

$$\eta_{j,+}(m) := -\xi_j(\tau_{j,+}(m)), \qquad \eta_{j,-}(m) := +\xi_j(\tau_{j,-}(m)). \tag{35}$$

The following proposition is the key to the Ray – Knight-approach.

Proposition 1. *(i) The processes $l \mapsto \xi_j(l)$, $j \in \mathbb{Z}$, are independent copies of the Markov-chain $l \mapsto \xi(l)$, defined in Subsection 3.1, starting with initial conditions $\xi_j(0) = 0$.*

(ii) As a consequence: the processes $k \mapsto \eta_{j,\pm}(k)$, $j \in \mathbb{Z}$, are independent copies of the Markov-chain $m \mapsto \eta_\pm(m)$, if we consider exactly one of $\eta_{j,+}$ and $\eta_{j,-}$ for each j. The initial conditions $\eta_{j,\pm}(0) = 0$.

The statement is intuitively clear. The mathematical content of the driving rules (4) of the random walk $X(n)$ is exactly this: whenever the walk visits a site $j \in \mathbb{Z}$, the probability of jumping to the left or to the right (i.e. to site $j-1$ or to site $j+1$), conditionally on the whole past, will depend only on the difference of the number of past jumps from j to $j-1$, respectively, from j to $j+1$, and independent of what had happened at other sites. The more lengthy formal proof goes through exactly the same steps as the corresponding statement in [10]. We omit here the formal proof.

Fix now $j \in \mathbb{Z}_-$ and $r \in \mathbb{N}$. The definitions (14), (32), (33), (34) and (35) imply that

$$L_{j,r}(j) = r \tag{36}$$

$$L_{j,r}(k+1) = L_{j,r}(k) + 1 + \eta_{k+1,-}(L_{j,r}(k)+1), \qquad j \leq k < 0, \tag{37}$$

$$L_{j,r}(k+1) = L_{j,r}(k) + \eta_{k+1,-}(L_{j,r}(k)), \qquad 0 \leq k < \infty, \tag{38}$$

$$L_{j,r}(k-1) = L_{j,r}(k) + \eta_{k,+}(L_{j,r}(k)), \qquad -\infty < k \leq j, \tag{39}$$

because $\eta_{k,\pm}$-s are defined in such a way that they are the differences of the local time at two neighbouring vertices. The definition of the stopping (34) yields that the argument of $\eta_{k,\pm}$ is the value of the local time at k. Similar formulas are found for $j \in \mathbb{Z}_+$ and $r \in \mathbb{N}$.

Note that if $L_{j,r}(k_0) = 0$ for some $k_0 \geq 0$ (respectively, for some $k_0 \leq j$) then $L_{j,r}(k) = 0$ for all $k \geq k_0$ (respectively, for all $k \leq k_0$).

The idea of the further steps of proof can be summarized in terms of the above setup. With fixed $x \in \mathbb{R}_-$ and $h \in \mathbb{R}_+$, we choose $j = \lfloor Ax \rfloor$ and $r = \lfloor Ah \rfloor$ with the scaling parameter $A \to \infty$ at the end. We know from Lemma 1 that the Markov-chains $\eta_{j,\pm}$ converge exponentially fast to their stationary distribution ρ. This allows us to couple efficiently the increments $L_{\lfloor Ax \rfloor, \lfloor Ah \rfloor}(k+1) - L_{\lfloor Ax \rfloor, \lfloor Ah \rfloor}(k)$

with properly chosen i.i.d. random variables as long as the value of $L_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(k) > A^{1/2+\varepsilon}$ and to use the law of large numbers. This coupling does not apply when the value of $L_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(k) < A^{1/2+\varepsilon}$. We prove that once the value of $L_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(k)$ drops below this threshold, $L_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(k)$ hits zero (and sticks there) in $o(A)$ time, with high probability. These steps of the proof are presented in the next two subsections.

3.3 Coupling

We are in the context of the representation (36), (37), (38), (39) with $j = \lfloor Ax\rfloor$, $r = \lfloor Ah\rfloor$. Due to Lemma 1, we can realize jointly the *pairs of coupled processes*

$$\big(m \mapsto (\eta_{k,-}(m), \widetilde{\eta}_k(m))\big)_{k>j}, \qquad \big(m \mapsto (\eta_{k,+}(m), \widetilde{\eta}_k(m))\big)_{k\le j} \tag{40}$$

with the following properties.

- The pairs of coupled processes with different k-indices are independent.
- The processes $\big(m \mapsto \eta_{k,-}(m)\big)_{k>j}$ and $\big(m \mapsto \eta_{k,+}(m)\big)_{k\le j}$ are those of the previous subsection. I.e. they are independent copies of the Markov-chain $m \mapsto \eta(m)$ with initial conditions $\eta_{k,\pm}(0) = 0$.
- The processes $\big(m \mapsto \widetilde{\eta}_k(m)\big)_{k\in\mathbb{Z}}$ are independent copies of the *stationary* process $m \mapsto \eta(m)$. I.e. these processes are initialized independently with $\mathbf{P}\big(\widetilde{\eta}_k(0) = x\big) = \rho(x)$ and run independently of one another.
- The pairs of coupled processes $m \mapsto (\eta_{k,\pm}(m), \widetilde{\eta}_k(m))$ are coalescing. This means the following: we define the coalescence time

  $$\mu_k := \inf\{m \ge 0 : \eta_{k,\pm}(m) = \widetilde{\eta}_k(m)\}. \tag{41}$$

  Then, for $m \ge \mu_k$, the two processes stick together: $\eta_{k,\pm}(m) = \widetilde{\eta}(m)$. Mind that the random variables μ_k, $k \in \mathbb{Z}$ are i.i.d.
- The tail of the distribution of the coalescence times decays exponentially fast:

  $$\mathbf{P}\big(\mu_k > m\big) < Ce^{-\beta m}. \tag{42}$$

We define the processes $k \mapsto \widetilde{L}_{j,r}(k)$ similarly to the processes $k \mapsto L_{j,r}(k)$ in (36), (37), (38), (39), with the η-s replaced by the $\widetilde{\eta}$-s:

$$\begin{aligned}
&\widetilde{L}_{j,r}(j) = r \\
&\widetilde{L}_{j,r}(k+1) = \widetilde{L}_{j,r}(k) + 1 + \widetilde{\eta}_{k+1,-}(\widetilde{L}_{j,r}(k) + 1), && j \le k < 0, \\
&\widetilde{L}_{j,r}(k+1) = \widetilde{L}_{j,r}(k) + \widetilde{\eta}_{k+1,-}(\widetilde{L}_{j,r}(k)), && 0 \le k < \infty, \\
&\widetilde{L}_{j,r}(k-1) = \widetilde{L}_{j,r}(k) + \widetilde{\eta}_{k,+}(\widetilde{L}_{j,r}(k)), && -\infty < k \le j.
\end{aligned}$$

Note that the increments of this process are *independent* with distribution

$$\mathbf{P}\left(\widetilde{L}_{j,r}(k+1)-\widetilde{L}_{j,r}(k)=z\right)=\rho(z-1), \qquad j\le k<0,$$

$$\mathbf{P}\left(\widetilde{L}_{j,r}(k+1)-\widetilde{L}_{j,r}(k)=z\right)=\rho(z), \qquad 0\le k<\infty,$$

$$\mathbf{P}\left(\widetilde{L}_{j,r}(k-1)-\widetilde{L}_{j,r}(k)=z\right)=\rho(z), \qquad -\infty<k\le j.$$

Hence, from (22), it follows that for any $K<\infty$

$$\sup_{|y|\le K}\left|A^{-1}\widetilde{L}_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(\lfloor Ay\rfloor)-\left((|x|-|y|)/2+h\right)\right| \xrightarrow{\mathbf{P}} 0. \tag{43}$$

Actually, by Doob's inequality, the following large deviation estimate holds: for any $x\in\mathbb{R}$, $h\in\mathbb{R}_+$ and $K<\infty$ fixed

$$\mathbf{P}\Big(\sup_{|k|\le AK}\left|\widetilde{L}_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(k)-\left((A|x|-|k|)/2+Ah\right)\right|>A^{1/2+\varepsilon}\Big)<Ce^{-\beta A^{2\varepsilon}}. \tag{44}$$

(The constants $C<\infty$ and $\beta>0$ do depend on the fixed parameters x, h and K.) Denote now

$$\kappa^+_{j,r}:=\min\{k\ge j: L_{j,r}(k)\ne\widetilde{L}_{j,r}(k)\},$$

$$\kappa^-_{j,r}:=\max\{k\le j: L_{j,r}(k)\ne\widetilde{L}_{j,r}(k)\}.$$

Then, for $k\ge j$:

$$\begin{aligned}
&\mathbf{P}\left(\kappa^+_{j,r}\le k+1\right)-\mathbf{P}\left(\kappa^+_{j,r}\le k\right)=\\
&\quad=\mathbf{P}\left(\kappa^+_{j,r}=k+1,\ \ \widetilde{L}_{j,r}(k)\le A^{1/2+\varepsilon}\right)+\mathbf{P}\left(\kappa^+_{j,r}=k+1,\ \ \widetilde{L}_{j,r}(k)\ge A^{1/2+\varepsilon}\right)\\
&\quad\le\mathbf{P}\left(\widetilde{L}_{j,r}(k)\le A^{1/2+\varepsilon}\right)+\mathbf{P}\left(\kappa^+_{j,r}=k+1\mid\kappa^+_{j,r}>k,\ \ L_{j,r}(k)=\widetilde{L}_{j,r}(k)\ge A^{1/2+\varepsilon}\right).
\end{aligned} \tag{45}$$

Similarly, for $k\le j$:

$$\begin{aligned}
&\mathbf{P}\left(\kappa^-_{j,r}\ge k-1\right)-\mathbf{P}\left(\kappa^-_{j,r}\ge k\right)=\\
&\quad=\mathbf{P}\left(\kappa^-_{j,r}=k-1,\ \ \widetilde{L}_{j,r}(k)\le A^{1/2+\varepsilon}\right)+\mathbf{P}\left(\kappa^-_{j,r}=k-1,\ \ \widetilde{L}_{j,r}(k)\ge A^{1/2+\varepsilon}\right)\\
&\quad\le\mathbf{P}\left(\widetilde{L}_{j,r}(k)\le A^{1/2+\varepsilon}\right)+\mathbf{P}\left(\kappa^-_{j,r}=k-1\mid\kappa^-_{j,r}<k,\ \ L_{j,r}(k)=\widetilde{L}_{j,r}(k)\ge A^{1/2+\varepsilon}\right).
\end{aligned} \tag{46}$$

Now, from (44), it follows that for $|k|\le A(|x|+2h)-4A^{1/2+\varepsilon}$

$$\mathbf{P}\left(\widetilde{L}_{j,r}(k)\le A^{1/2+\varepsilon}\right)\le Ce^{-\beta A^{2\varepsilon}}. \tag{47}$$

On the other hand, from (42),

$$\mathbf{P}\left(\kappa^+_{j,r}=k+1\mid\kappa^+_{j,r}>k,\ \ L_{j,r}(k)=\widetilde{L}_{j,r}(k)\ge A^{1/2+\varepsilon}\right)\le Ce^{-\beta A^{1/2+\varepsilon}}, \tag{48}$$

$$\mathbf{P}\left(\kappa^-_{j,r}=k-1\mid\kappa^-_{j,r}<k,\ \ L_{j,r}(k)=\widetilde{L}_{j,r}(k)\ge A^{1/2+\varepsilon}\right)\le Ce^{-\beta A^{1/2+\varepsilon}} \tag{49}$$

with some constants $C < \infty$ and $\beta > 0$, which do depend on all fixed parameters and may vary from formula to formula.

Putting together (45), (47), (48), respectively, (46), (47), (49) and noting that $\mathbf{P}\left(\kappa^+_{j,r} = j\right) = 0$, we conclude that

$$\mathbf{P}\left(\min\left\{|k| : L_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(k) \neq \widetilde{L}_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(k)\right\} \leq A(|x|+2h) - 4A^{1/2+\varepsilon}\right) \leq CAe^{-\beta A^{2\varepsilon}}, \tag{50}$$

$$\mathbf{P}\left(L_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(\pm\lfloor A(|x|+2h) - 4A^{1/2+\varepsilon}\rfloor) \geq 3A^{1/2+\varepsilon}\right) \leq Ce^{-\beta A^{2\varepsilon}}. \tag{51}$$

3.4 Hitting of 0

It follows from Lemma 1 that all moments of the distributions $P^n(0,\cdot)$ converge to the corresponding moments of ρ. In particular, for any $\delta > 0$ there exists $n_\delta < \infty$, such that

$$\sum_{x\in\mathbb{Z}} P^n(0,x)x \leq -\frac{1}{2+\delta}$$

holds if $n \geq n_\delta$.

Consider now the Markov-chains defined by (38) or (39) (the two are identical in law):

$$L(k+1) = L(k) + \eta_{k+1}(L(k)), \qquad L(0) = r \in \mathbb{N},$$

where $m \mapsto \eta_k(m)$, $k = 1,2,3,\ldots$ are i.i.d. copies of the Markov-chain $m \mapsto \eta(m)$ with initial conditions $\eta_k(0) = 0$. Define the stopping times

$$\tau_x := \min\{k : L(k) \leq x\}, \qquad x = 0,1,2,\ldots.$$

Lemma 2. *For any $\delta > 0$ there exists $K_\delta < \infty$ such that for any $r \in \mathbb{N}$:*

$$\mathbf{E}\left(\tau_0 \mid L(0) = r\right) \leq (2+\delta)r + K_\delta.$$

Proof. Clearly,

$$\mathbf{E}\left(\tau_0 \mid L(0) = r\right) \leq \mathbf{E}\left(\tau_{n_\delta} \mid L(0) = r\right) + \max_{0\leq s\leq n_\delta} \mathbf{E}\left(\tau_0 \mid L(0) = s\right).$$

Now, by optional stopping,

$$\mathbf{E}\left(\tau_{n_\delta} \mid L(0) = r\right) \leq (2+\delta)r,$$

and obviously,

$$K_\delta := \max_{0\leq s\leq n_\delta} \mathbf{E}\left(\tau_0 \mid L(0) = s\right) < \infty.$$

□

In particular, choosing $\delta = 1$ and applying Markov's inequality, it follows that

$$\mathbf{P}\left(\rho^+_{\lfloor Ax\rfloor,\lfloor Ax\rfloor} > A(|x|+2h)+A^{1/2+2\varepsilon} \,\middle|\, L_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(\lfloor A(|x|+2h)-4A^{1/2+\varepsilon}\rfloor) \le 3A^{1/2+\varepsilon}\right)$$
$$\le \frac{9A^{1/2+\varepsilon}+K_1}{5A^{1/2+2\varepsilon}} < 2A^{-\varepsilon}, \tag{52}$$

and similarly,

$$\mathbf{P}\left(\lambda^+_{\lfloor Ax\rfloor,\lfloor Ax\rfloor} < -A(|x|+2h)-A^{1/2+2\varepsilon} \,\middle|\, L_{\lfloor Ax\rfloor,\lfloor Ah\rfloor}(-\lfloor A(|x|+2h)+4A^{1/2+\varepsilon}\rfloor) \le 3A^{1/2+\varepsilon}\right)$$
$$\le \frac{9A^{1/2+\varepsilon}+K_1}{5A^{1/2+2\varepsilon}} < 2A^{-\varepsilon}. \tag{53}$$

Eventually, Theorem 1 follows from (43), (50), (51), (52) and (53).

4 Proof of the theorem for the position of the random walker

First, we introduce the following notations. For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, let

$$P(n,k) := \mathbf{P}\left(X(n)=k\right)$$

be the distribution of the position of the random walker. For $s \in \mathbb{R}_+$,

$$R(s,k) := (1-e^{-s})\sum_{n=0}^{\infty} e^{-sn}P(n,k) \tag{54}$$

is the distribution of $X(\theta_s)$ where θ_s has geometric distribution (12) and it is independent of $X(n)$. Also (11) tells us that the proper definition of the rescaled distribution is

$$\varphi_A(t,x) := A^{1/2}P(\lfloor At\rfloor, \lfloor A^{1/2}x\rfloor),$$

if $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$. Let

$$\hat\varphi_A(s,x) := A^{1/2}R(A^{-1}s, \lfloor A^{1/2}x\rfloor), \tag{55}$$

which is asymptotically the Laplace-transform of φ_A as $A \to \infty$.

With these definitions, the statement of Theorem 2 is equivalent to

$$\hat\varphi_A(s,x) \to \hat\varphi(s,x),$$

which is proved below.

We will need the Laplace-transform

$$\hat\rho(s,x,h) = s\,\mathbf{E}\left(\exp\left(-s\,\mathcal{T}_{x,h}\right)\right) = se^{-s(|x|+2h)^2},$$

for which

$$\hat\varphi(s,x) = 2\int_0^\infty \hat\rho(s,|x|,h)\,\mathrm{d}h$$

holds.

Proof of Theorem 2. Fix $x \geq 0$. We can proceed in the case $x \leq 0$ similarly. We start with the identity

$$P(n,k) = \mathbf{P}\left(X_n = k\right) = \sum_{m=0}^{\infty} \Big(\mathbf{P}\left(T^{+}_{k-1,m} = n\right) + \mathbf{P}\left(T^{-}_{k+1,m} = n\right)\Big), \tag{56}$$

which is easy to check. From the definitions (54) and (55),

$$\begin{aligned} \hat{\varphi}_A(s,x) &= \frac{1-e^{-s/A}}{s/A} s \sum_{n=0}^{\infty} \frac{1}{\sqrt{A}} e^{-ns/A} P(n, \lfloor A^{1/2} x \rfloor) \\ &= \frac{1-e^{-s/A}}{s/A} s \sum_{n=0}^{\infty} \frac{1}{\sqrt{A}} \Big(\mathbf{E}\left(e^{-sA^{-1} T^{+}_{\lfloor A^{1/2}x\rfloor - 1, n}}\right) + \mathbf{E}\left(e^{-sA^{-1} T^{-}_{\lfloor A^{1/2}x\rfloor + 1, n}}\right)\Big), \end{aligned} \tag{57}$$

where we used (56) in the second equality. Let

$$\hat{\rho}^{\pm}_A(s,x,h) = s\, \mathbf{E}\left(\exp\left(-\frac{s}{A} T^{\pm}_{\lfloor A^{1/2}x\rfloor, \lfloor A^{1/2}h\rfloor}\right)\right).$$

Then (57) can be written as

$$\hat{\varphi}_A(s,x) = \frac{1-e^{-s/A}}{s/A} \int_0^{\infty} \left(\hat{\rho}^{+}_A(s, x - A^{-1/2}, h) + \hat{\rho}^{-}_A(s, x + A^{-1/2}, h)\right) \mathrm{d}h. \tag{58}$$

It follows from (10) that for all $s > 0$, $x \in \mathbb{R}$ and $h > 0$,

$$\hat{\rho}^{\pm}_A(s,x,h) \to \hat{\rho}(s,x,h)$$

as $A \to \infty$. Applying Fatou's lemma in (58) yields

$$\liminf_{A\to\infty} \hat{\varphi}_A(s,x) \geq 2 \int_0^{\infty} \hat{\rho}(s,x,h)\, \mathrm{d}h = \hat{\varphi}(s,x).$$

If we use Fatou's lemma again, we get

$$1 = \int_{-\infty}^{\infty} \hat{\varphi}(s,x)\, \mathrm{d}x \leq \int_{-\infty}^{\infty} \liminf_{A\to\infty} \hat{\varphi}_A(s,x)\, \mathrm{d}x \leq \liminf_{A\to\infty} \int_{-\infty}^{\infty} \hat{\varphi}_A(s,x)\, \mathrm{d}x = 1,$$

which gives for all $s \in \mathbb{R}$ that

$$\hat{\varphi}(s,x) = \liminf_{A\to\infty} \hat{\varphi}_A(s,x) \tag{59}$$

holds for almost all $x \in \mathbb{R}$. Note that (59) is also true for any subsequence $A_k \to \infty$, which implies the assertion of Theorem 2. ☐

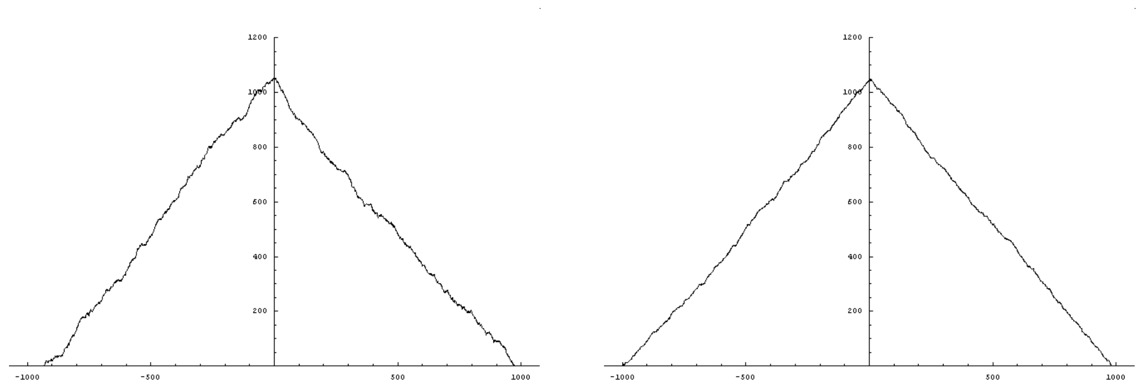

Figure 1: The local time process of the random walk with $w(k) = 2^k$ and $w(k) = 10^k$

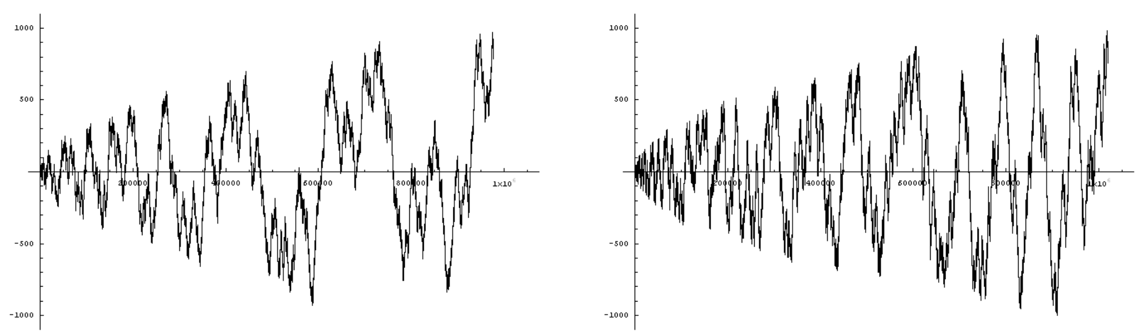

Figure 2: The trajectories of the random walk with $w(k) = 2^k$ and $w(k) = 10^k$

5 Computer simulations

We have prepared computer simulations with exponential weight functions $w(k) = 2^k$ and $w(k) = 10^k$.

Note that the limit objects in our theorems do not depend on the choice of the weight function w. Therefore, we expect that the behaviour of the local time and the trajectories is qualitatively similar, and we will find only quantitative differences.

Figure 1 shows the local time process of the random walk after approximately 10^6 steps. More precisely, we have plotted the value of $\Lambda^+_{100,800}$ with $w(k) = 2^k$ and $w(k) = 10^k$ respectively. One can see that the limits are the same in the two cases – according to Theorem 1 – but the rate of convergence does depend on the choice of w. We can conclude the empirical rule that the faster the weight function grows at infinity, the faster the convergence of the local time process is.

The difference between the trajectories of random walks generated with various weights is more conspicuous. On Figure 2, the trajectories of the walks with $w(k) = 2^k$ and $w(k) = 10^k$ are illustrated, respectively. The number of steps is random, it is about 10^6. The data comes from the same sample as that shown on Figure 1.

The first thing that we can observe on Figure 2 is that the trajectories draw a sharp upper and lower hull according to $\sqrt{t}$ and $-\sqrt{t}$, which agrees with our expectations after (13). On the other hand, the trajectories oscillate very heavily between their extreme values, especially in the case $w(k) = 10^k$, there are almost but not quite straight crossings from $\sqrt{t}$ to $-\sqrt{t}$ and back. It shows that there is no continuous scaling limit of the self-repelling random walk with directed edges.

The shape of the trajectories are slightly different in the cases $w(k) = 2^k$ and $w(k) = 10^k$. The latter has heavier oscillations, because it corresponds to a higher rate of growth of the weight function. Note that despite this difference in the oscillation, the large scale behaviour is the same on the two pictures on Figure 2. The reason for this is that if the random walk explores a new region, e.g. it exceeds its earlier maximum, then the probability of the reversal does not depend on w, since the both outgoing edges have local time 0. It can be a heuristic argument, why the upper and lower hulls $\sqrt{t}$ and $-\sqrt{t}$ are universal.

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