Electronic Journal of Probability

Vol. 9 (2004), Paper no. 4, pages 57–97.

Journal URL
http://www.math.washington.edu/~ejpecp/

The Genealogy of Self-similar Fragmentations with Negative Index as a Continuum Random Tree

Bénédicte Haas
LPMA, Université Pierre et Marie Curie
175 rue du Chevaleret, F-75013 Paris, France
haas@ccr.jussieu.fr

Grégory Miermont
DMA, École Normale Supérieure, & LPMA, Université Pierre et Marie Curie
45, rue d'Ulm, 75230 Paris Cedex 05, France
miermont@dma.ens.fr

Abstract: We encode a certain class of stochastic fragmentation processes, namely self-similar fragmentation processes with a negative index of self-similarity, into a metric family tree which belongs to the family of Continuum Random Trees of Aldous. When the splitting times of the fragmentation are dense near 0, the tree can in turn be encoded into a continuous height function, just as the Brownian Continuum Random Tree is encoded in a normalized Brownian excursion. Under mild hypotheses, we then compute the Hausdorff dimensions of these trees, and the maximal Hölder exponents of the height functions.

Keywords and phrases: Self-similar fragmentation, continuum random tree, Hausdorff dimension, Hölder regularity.

A.M.S. subject classification (2000): 60G18, 60J25, 60G09.

Submitted to EJP on November 11, 2003. Final version accepted on February 9, 2004.

1 Introduction

Self-similar fragmentation processes describe the evolution of an object that falls apart, so that different fragments keep on collapsing independently with a rate that depends on their sizes to a certain power, called the *index* of the self-similar fragmentation. A genealogy is naturally associated with such fragmentation processes, by saying that the common ancestor of two fragments is the block that included these fragments for the last time, before a dislocation had definitely separated them. With an appropriate coding of the fragments, one guesses that there should be a natural way to define a genealogy tree, rooted at the initial fragment, associated with any such fragmentation. It would be natural to put a metric on this tree, e.g. by letting the distance from a fragment to the root of the tree be the time at which the fragment disappears.

Conversely, it turns out that trees have played a key role in models involving self-similar fragmentations, notably, Aldous and Pitman [3] have introduced a way to log the so-called Brownian *Continuum Random Tree* (CRT) [2] that is related to the standard additive coalescent. Bertoin [7] has shown that a fragmentation that is somehow dual to the Aldous-Pitman fragmentation can be obtained as follows. Let $\mathcal{T}_B$ be the Brownian CRT, which is considered as an "infinite tree with edge-lengths" (formal definitions are given below). Let $\mathcal{T}_t^1, \mathcal{T}_t^2, \ldots$ be the distinct tree components of the forest obtained by removing all the vertices of $\mathcal{T}$ that are at distance less than t from the root, and arranged by decreasing order of "size". Then the sequence $F_B(t)$ of these sizes defines as t varies a self-similar fragmentation. A moment of thought points out that the notion of genealogy defined above precisely coincides with the tree we have fragmented in this way, since a split occurs precisely at branchpoints of the tree. Fragmentations of CRT's that are different from the Brownian one and that follow the same kind of construction have been studied in [23].

The goal of this paper is to show that any self-similar fragmentation process with negative index can be obtained by a similar construction as above, for a certain instance of CRT. We are interested in negative indices, because in most interesting cases when the self-similarity index is non-negative, all fragments have an "infinite lifetime", meaning that the pieces of the fragmentation remain macroscopic at all times. In this case, the family tree defined above will be unbounded and without endpoints, hence looking completely different from the Brownian CRT. By contrast, as soon as the self-similarity index is negative, a loss of mass occurs, that makes the fragments disappear in finite time (see [8]). In this case, the metric family tree will be a bounded object, and in fact, a CRT. To state our results, we first give a rigorous definition of the involved objects. Call

$$S = \left\{ \mathbf{s} = (s_1, s_2, \ldots) : s_1 \geq s_2 \geq \ldots \geq 0; \sum_{i \geq 1} s_i \leq 1 \right\},$$

and endow it with the topology of pointwise convergence.

Definition 1. *A Markovian S-valued process $(F(t), t \geq 0)$ starting at $(1, 0, \ldots)$ is a ranked self-similar fragmentation with index $\alpha \in \mathbb{R}$ if it is continuous in probability and satisfies the following fragmentation property. For every $t, t' \geq 0$, given $F(t) = (x_1, x_2, \ldots)$, $F(t + t')$ has the same law as the decreasing rearrangement of the sequences $x_1 F^{(1)}(x_1^\alpha t'), x_2 F^{(2)}(x_2^\alpha t'), \ldots$, where the $F^{(i)}$'s are independent copies of F.*

By a result of Bertoin [7] and Berestycki [4], the laws of such fragmentation processes are characterized by a 3-tuple (α, c, ν), where α is the index, $c \geq 0$ is an "erosion" constant, and ν is a σ-finite measure on S that integrates $\mathbf{s} \mapsto 1 - s_1$ such that $\nu(\{(1, 0, 0 \ldots)\}) = 0$. Informally, c measures the rate at which fragments melt continuously (a phenomenon we will not be much interested in here), while ν measures instantaneous breaks of fragments: a piece with size x breaks into fragments with masses $x\mathbf{s}$ at rate $x^\alpha \nu(\mathrm{d}\mathbf{s})$. Notice that some mass can be lost within a sudden break: this happens as soon as $\nu(\sum_i s_i < 1) \neq 0$, but we will not be interested in this phenomenon here either. The loss of mass phenomenon stated above is completely different from erosion or sudden loss of mass: it is due to the fact that small fragments tend to decay faster when $\alpha < 0$.

On the other hand, let us define the notion of CRT. An $\mathbb{R}$-tree (with the terminology of Dress and Terhalle [13]; it is called a *continuum tree set* in Aldous [2]) is a complete metric space (T, d), whose elements are called *vertices*, which satisfies the following two properties:

- For $v, w \in T$, there exists a unique geodesic $[[v, w]]$ going from v to w, i.e. there exists a unique isomorphism $\varphi_{v,w} : [0, d(v, w)] \to T$ with $\varphi_{v,w}(0) = v$ and $\varphi_{v,w}(d(v, w)) = w$, and its image is called $[[v, w]]$.
- For any $v, w \in T$, the only non-self-intersecting path going from v to w is $[[v, w]]$, i.e. for any continuous injective function $s \mapsto v_s$ from $[0, 1]$ to T with $v_0 = v$ and $v_1 = w$, $\{v_s : s \in [0, 1]\} = [[v, w]]$.

We will furthermore consider $\mathbb{R}$-trees that are *rooted*, that is, one vertex is distinguished as being the root, and we call it $\varnothing$. A *leaf* is a vertex which does not belong to $[[\varnothing, w[[:= \varphi_{\varnothing,w}([0, d(\varnothing, w)))$ for any vertex w. Call $\mathcal{L}(T)$ the set of leaves of T, and $\mathcal{S}(T) = T \setminus \mathcal{L}(T)$ its skeleton. An $\mathbb{R}$ -tree is *leaf-dense* if T is the closure of $\mathcal{L}(T)$. We also call *height* of a vertex v the quantity $\mathrm{ht}(v) = d(\varnothing, v)$. Last, for T an $\mathbb{R}$-tree and $a > 0$, we let $a \otimes T$ be the $\mathbb{R}$-tree in which all distances are multiplied by a.

Definition 2. *A continuum tree is a pair (T, μ) where T is an $\mathbb{R}$-tree and μ is a probability measure on T, called the mass measure, which is non-atomic and satisfies $\mu(\mathcal{L}(T)) = 1$ and such that for every non-leaf vertex w, $\mu\{v \in T : [[\varnothing, v]] \cap [[\varnothing, w]] = [[\varnothing, w]]\} > 0$. The set of vertices just defined is called the fringe subtree rooted at w. A CRT is a random variable $\omega \mapsto (T(\omega), \mu(\omega))$ on a probability space $(\Omega, \mathcal{F}, P)$ whose values are continuum trees.*

Notice that the definition of a continuum tree implies that the $\mathbb{R}$-tree T satisfies certain extra properties, for example, its set of leaves must be uncountable and have no isolated point. Also, the definition of a CRT is a little inaccurate as we did not endow the space of $\mathbb{R}$-trees with a σ-field. This problem is in fact circumvented by the fact that CRTs are in fact entirely described by the sequence of their *marginals*, that is, of the subtrees spanned by the root and k leaves chosen with law μ given μ, and these subtrees, which are interpreted as finite *trees with edge-lengths*, are random variables (see Sect. 2.2). The reader should keep in mind that by the "law" of a CRT we mean the sequence of these marginals. Another point of view is taken in [15], where the space of $\mathbb{R}$-trees is endowed with a metric.

For (T, μ) a continuum tree, and for every $t \geq 0$, let $T_1(t), T_2(t), \ldots$ be the tree components of $\{v \in T : \mathrm{ht}(v) > t\}$, ranked by decreasing order of μ-mass. A continuum random tree (T, μ)

is said to be *self-similar* with index $\alpha < 0$ if for every $t \geq 0$, conditionally on $(\mu(T_i(t)), i \geq 1)$, $(T_i(t), i \geq 1)$ has the same law as $(\mu(T_i(t))^{-\alpha} \otimes T^{(i)}, i \geq 1)$ where the $T^{(i)}$'s are independent copies of T.

Our first result is

Theorem 1. *Let F be a ranked self-similar fragmentation process with characteristic 3-tuple (α, c, ν), with $\alpha < 0$. Suppose also that F is not constant, that $c = 0$ and $\nu(\sum_i s_i < 1) = 0$. Then there exists an α-self-similar CRT $(\mathcal{T}_F, \mu_F)$ such that, writing $F'(t)$ for the decreasing sequence of masses of connected components of the open set $\{v \in \mathcal{T}_F : \mathrm{ht}(v) > t\}$, the process $(F'(t), t \geq 0)$ has the same law as F. The tree $\mathcal{T}_F$ is leaf-dense if and only if ν has infinite total mass.*

The next statement is a kind of converse to this theorem.

Proposition 1. *Let $(\mathcal{T}, \mu)$ be a self-similar CRT with index $\alpha < 0$. Then the process $F(t) = ((\mu(\mathcal{T}_i(t), i \geq 1), t \geq 0)$ is a ranked self-similar fragmentation with index α, it has no erosion and its dislocation measure ν satisfies $\nu(\sum_i s_i < 1) = 0$. Moreover, $\mathcal{T}_F$ and $\mathcal{T}$ have the same law.*

These results are proved in Sect. 2. There probably exists some notion of continuum random tree extending the former which would include fragmentations with erosion or with sudden loss of mass, but we do not pursue this here.

The next result, to be proved in Sect. 3, deals with the Hausdorff dimension of the set of leaves of the CRT $\mathcal{T}_F$.

Theorem 2. *Let F be a ranked self-similar fragmentation with characteristics (α, c, ν) satisfying the hypotheses of Theorem 1. Writing $\dim_{\mathcal{H}}$ for Hausdorff dimension, one has*

$$\dim_{\mathcal{H}} (\mathcal{L}(\mathcal{T}_F)) = \frac{1}{|\alpha|} \quad a.s. \tag{1}$$

as soon as $\int_S \left(s_1^{-1} - 1\right) \nu(\mathrm{d}\mathbf{s}) < \infty$.

Some comments about this formula. First, notice that under the extra integrability assumption on ν, the dimension of the whole tree is $\dim_{\mathcal{H}}(\mathcal{T}_F) = (1/|\alpha|) \vee 1$ because the skeleton $\mathcal{S}(\mathcal{T}_F)$ has dimension 1 as a countable union of segments. The value -1 is therefore critical for α, since the above formula shows that the dimension of $\mathcal{T}_F$ as to be 1 as soon as $\alpha \leq -1$. It was shown in a previous work by Bertoin [8] that when $\alpha < -1$, for every fixed t the number of fragments at time t is a.s. finite, so that -1 is indeed the threshold under which fragments decay extremely fast. One should then picture the CRT $\mathcal{T}_F$ as a "dead tree" looking like a handful of thin sticks connected to each other, while when $|\alpha| < 1$ the tree looks more like a dense "bush". Last, the integrability assumption in the theorem seems to be reasonably mild; its heuristic meaning is that when a fragmentation occurs, the largest resulting fragment is not too small. In particular, it is always satisfied in the case of fragmentations for which $\nu(s_{N+1} > 0) = 0$, since then $s_1 > 1/N$ for ν-a.e. $\mathbf{s}$. Yet, we point out that when $\int_S \left(s_1^{-1} - 1\right) \nu(\mathrm{d}\mathbf{s}) = \infty$, one anyway obtains the following bounds for the Hausdorff dimension of $\mathcal{L}(\mathcal{T}_F)$:

$$\frac{\varrho}{|\alpha|} \leq \dim_{\mathcal{H}} (\mathcal{L}(\mathcal{T}_F)) \leq \frac{1}{|\alpha|} \quad \text{a.s.}$$

where

$$\varrho := \sup\left\{ p \leq 1 : \int_S \left(s_1^{-p} - 1\right) \nu(\mathrm{d}\mathbf{s}) < \infty \right\}. \tag{2}$$

We do not know whether the condition $\int_S (s_1^{-1} - 1)\nu(\mathrm{d}\mathbf{s}) < \infty$ is necessary for (1), as we are not aware of any self-similar fragmentation with index α such that the associated CRT has leaf-dimension strictly less than $1/|\alpha|$.

It is worth noting that these results allow as a special case to compute the Hausdorff dimension of the so-called *stable trees* of Duquesne and Le Gall [14], which were used to construct fragmentations in the manner of Theorem 1 in [23]. The dimension of the stable tree (as well as finer results of Hausdorff measures on more general Lévy trees) has been obtained independently in [15]. The stable tree is a CRT whose law depends on parameter $\beta \in (1, 2]$, and it satisfies the required self-similarity property of Proposition 1 with index $1/\beta - 1$. We check that the associated dislocation measure satisfies the integrability condition of Theorem 2 in Sect. 3.5, so that

Corollary 1. *Fix $\beta \in (1, 2]$. The β-stable tree has Hausdorff dimension $\beta/(\beta - 1)$.*

An interesting process associated with a given continuum tree (T, μ) is the so-called *cumulative height profile* $\bar{W}_T(h) = \mu\{v \in T : \mathrm{ht}(v) \leq h\}$, which is non-decreasing and bounded by 1 on $\mathbb{R}_+$. It may happen that the Stieltjes measure $\mathrm{d}\bar{W}_T(h)$ is absolutely continuous with respect to Lebesgue measure, in which case its density $(W_T(h), h \geq 0)$ is called the *height profile*, or *width process* of the tree. In our setting, for any fragmentation F satisfying the hypotheses of Theorem 1, the cumulative height profile has the following interpretation: one has $(\bar{W}_{\mathcal{T}_F}(h), h \geq 0)$ has the same law as $(M_F(h), h \geq 0)$, where $M_F(h) = 1 - \sum_{i \geq 1} F_i(h)$ is the total mass lost by the fragmentation at time h. Detailed conditions for existence (or non-existence) of the width profile $\mathrm{d}M_F(h)/\mathrm{d}h$ have been given in [19]. It was also proved there that under some mild assumptions $\dim_{\mathcal{H}}(\mathrm{d}M_F) \geq 1 \wedge A/|\alpha|$ a.s., where A is a ν-dependent parameter introduced in (10) below, and

$$\dim_{\mathcal{H}}(\mathrm{d}M_F) := \inf\{\dim_{\mathcal{H}}(E) \,:\, \mathrm{d}M_F(E) = 1\}.$$

The upper bound we obtain for $\dim_{\mathcal{H}}(\mathcal{L}(\mathcal{T}_F))$ allows us to complete this result:

Corollary 2. *Let F be a ranked self-similar fragmentation with same hypotheses as in Theorem 1. Then $\dim_{\mathcal{H}}(\mathrm{d}M_F) \leq 1 \wedge 1/|\alpha|$ a.s.*

Notice that this result re-implies the fact from [19] that the height profile does not exist as soon as $|\alpha| \geq 1$.

The last motivation of this paper (Sect. 4) is about relations between CRTs and their so-called encoding *height processes*. The fragmentation F_B of [7], as well as the fragmentations from [23], were defined out of certain random functions $(H_s, 0 \leq s \leq 1)$. Let us describe briefly the construction of F_B. Let B^{exc} be the standard Brownian excursion with duration 1, and consider the open set $\{s \in [0, 1] : 2B^{\mathrm{exc}}_s > t\}$. Write $F(t)$ for the decreasing sequence of the lengths of its interval components. Then F has the same law as the fragmentation F'_B defined out of the Brownian CRT in the same way as in Theorem 1. This is immediate from the description of Le Gall [22] and Aldous [2] of the Brownian tree as being encoded in the Brownian excursion. To be concise, define a pseudo-metric on $[0, 1]$ by letting $\overline{d}(s, s') = 2B^{\mathrm{exc}}_s + 2B^{\mathrm{exc}}_{s'} - 4\inf_{u \in [s, s']} B^{\mathrm{exc}}_u$, with the convention that $[s, s'] = [s', s]$ if $s' < s$. We can define a true metric space by taking the quotient with respect to the equivalence relation $s \equiv s' \iff \overline{d}(s, s') = 0$. Call $(\mathcal{T}_B, d)$ this metric space. Write μ_B for the measure induced on $\mathcal{T}_B$ by Lebesgue measure on $[0, 1]$. Then $(\mathcal{T}_B, \mu_B)$ is the Brownian CRT, and the equality in law of the fragmentations F_B and F'_B follows immediately from the definition of the mass measure. Our next result generalizes this construction.

Theorem 3. *Let F be a ranked self-similar fragmentation with same hypotheses as in Theorem 1, and suppose ν has infinite total mass. Then there exists a continuous random function $(H_F(s), 0 \leq s \leq 1)$, called the height function, such that $H_F(0) = H_F(1)$, $H_F(s) > 0$ for every $s \in (0, 1)$, and such that F has the same law as the fragmentation F' defined by: $F'(t)$ is the decreasing rearrangement of the lengths of the interval components of the open set $I_F(t) = \{s \in (0, 1) : H_F(s) > t\}$.*

An interesting point in this construction is also that it shows that a large class of self-similar fragmentation with negative index has a natural *interval representation*, given by $(I_F(t), t \geq 0)$. Bertoin [7, Lemma 6] had already constructed such an interval representation, I'_F say, but ours is different qualitatively. We will see in the sequel that our representation is intuitively obtained by putting the intervals obtained from the dislocation of a largest interval in exchangeable random order, while Bertoin's method is to put these same intervals from left to right by size-biased random order. In particular, For example, Bertoin's interval fragmentation I'_F cannot be written in the form $I'_F(t) = \{s \in (0, 1) : H(s) > t\}$ for any continuous process H.

In parallel to the computation of the Hausdorff dimension of the CRTs built above, we are able to estimate Hölder coefficients for the height processes of these CRTs. Our result is

Theorem 4. *Suppose $\nu(S) = \infty$, and set*

$$\begin{aligned} \vartheta_{\text{low}} &:= \quad \sup \left\{ b > 0 : \lim_{x \downarrow 0} x^b \nu(s_1 < 1 - x) = \infty \right\}, \\ \vartheta_{\text{up}} &:= \quad \inf \left\{ b > 0 : \lim_{x \downarrow 0} x^b \nu(s_1 < 1 - x) = 0 \right\}. \end{aligned}$$

Then the height process H_F is a.s. Hölder-continuous of order γ for every $\gamma < \vartheta_{\text{low}} \wedge |\alpha|$, and, provided that $\int_S (s_1^{-1} - 1)\nu(\mathrm{d}\mathbf{s}) < \infty$, a.s. not Hölder-continuous of order γ for every $\gamma > \vartheta_{\text{up}} \wedge |\alpha|$.

Again we point out that one actually obtains an upper bound for the maximal Hölder coefficient even when $\int_S (s_1^{-1} - 1)\nu(\mathrm{d}\mathbf{s}) = \infty$: with ϱ defined by (2), a.s. H_F cannot be Hölder-continuous of order γ for any $\gamma > \vartheta_{\text{up}} \wedge |\alpha|/\varrho$.

Note that $\vartheta_{\text{low}}, \vartheta_{\text{up}}$ depend only on the characteristics of the fragmentation process, and more precisely, on the behavior of ν when s_1 is close to 1. By contrast, our Hausdorff dimension result for the tree depended on a hypothesis on the behavior of ν when s_1 is near 0. Remark also that ϑ_{up} may be strictly smaller than 1. Therefore, the Hausdorff dimension of $\mathcal{T}_F$ is in general not equal to the inverse of the maximal Hölder coefficient of the height process, as one could have expected. However, this turns out to be true in the case of the stable tree, as will be checked in Section 4.4:

Corollary 3. *The height process of the stable tree with index $\beta \in (1, 2]$ is a.s. Hölder-continuous of any order $\gamma < 1 - 1/\beta$, but a.s. not of order $\gamma > 1 - 1/\beta$.*

When $\beta = 2$, this just states that the Brownian excursion is Hölder-continuous of any order $< 1/2$, a result that is well-known for Brownian motion and which readily transfers to the normalized Brownian excursion (e.g. by rescaling the first excursion of Brownian motion whose duration is greater than 1). The general result had been obtained in [14] by completely different methods.

Last, we mention that most of our results extend to a more general class of fragmentations in which a fragment with mass x splits to give fragments with masses $x\mathbf{s}$, $\mathbf{s} \in S$, at rate $\varsigma(x)\nu(\mathrm{d}\mathbf{s})$ for some non-negative continuous function ς on $(0, 1]$ (see [18] for a rigorous definition). The proofs of the above theorems easily adapt to give the following results: when $\liminf_{x \to 0} x^{-b}\varsigma(x) > 0$ for some $b < 0$, the fragmentation can be encoded as above into a CRT and, provided that ν is infinite, into a height function. The set of leaves of the CRT then has a Hausdorff dimension smaller than $1/|b|$ and the height function is γ-Hölder continuous for every $\gamma < \vartheta_{\mathrm{low}} \wedge |b|$. If moreover $\limsup_{x \to 0} x^{-a}\varsigma(x) < \infty$ for some $a < 0$ and $\int_S \left(s_1^{-1} - 1\right) \nu(\mathrm{d}\mathbf{s}) < \infty$, the Hausdorff dimension is larger than $1/|a|$ and the height function cannot have a Hölder coefficient $\gamma > \vartheta_{\sup} \wedge |a|$.

2 The CRT $\mathcal{T}_F$

Building the CRT $\mathcal{T}_F$ associated with a ranked fragmentation F will be done by determining its "marginals", i.e. the subtrees spanned by a finite but arbitrary number of randomly chosen leaves. To this purpose, it will be useful to use partition-valued fragmentations, which we first define, as well as a certain family of trees with edge-lengths.

2.1 Exchangeable partitions and partition-valued self-similar fragmentations

Let $\mathcal{P}_\infty$ be the set of (unordered) partitions of $\mathbb{N} = \{1, 2, \ldots\}$ and $[n] = \{1, 2, \ldots, n\}$. For $i, j \in \mathbb{N}$, we write $i \overset{\pi}{\sim} j$ if i and j are in the same block of π. We adopt the following ordering convention: for $\pi \in \mathcal{P}_\infty$, we let $(\pi_1, \pi_2, \ldots)$ be the blocks of π, so that π_i is the block containing i provided that i is the smallest integer of the block and $\pi_i = \varnothing$ otherwise. We let $\mathbb{O} = \{\{1\}, \{2\}, \ldots\}$ be the partition of $\mathbb{N}$ into singletons. If $B \subset \mathbb{N}$ and $\pi \in \mathcal{P}_\infty$ we let $\pi \cap B$ (or $\pi|_B$) be the restriction of π to B, i.e. the partition of B whose collection of blocks is $\{\pi_i \cap B, i \geq 1\}$. If $\pi \in \mathcal{P}_\infty$ and $B \in \pi$ is a block of π, we let

$$|B| = \lim_{n \to \infty} \frac{\#(B \cap [n])}{n}$$

be the asymptotic frequency of the block B, whenever it exists. A random variable π with values in $\mathcal{P}_\infty$ is called *exchangeable* if its law is invariant under the natural action of permutations of $\mathbb{N}$ on $\mathcal{P}_\infty$. By a theorem of Kingman [20, 1], all the blocks of such random partitions admit asymptotic frequencies a.s. For π whose blocks have asymptotic frequencies, we let $|\pi| \in S$ be the decreasing sequence of these frequencies. Kingman's theorem more precisely says that the law of any exchangeable random partition π is a (random) "paintbox process", a term we now explain. Take $\mathbf{s} \in S$ (the paintbox) and consider a sequence $U_1, U_2, \ldots$ of i.i.d. variables in $\mathbb{N} \cup \{0\}$ (the colors) with $P(U_1 = j) = s_j$ for $j \geq 1$ and $P(U_1 = 0) = 1 - \sum_k s_k$. Define a partition π on $\mathbb{N}$ by saying that $i \neq j$ are in the same block if and only if $U_i = U_j \neq 0$ (i.e. i and j have the same color, where 0 is considered as colorless). Call $\rho_{\mathbf{s}}(\mathrm{d}\pi)$ its law, the $\mathbf{s}$-*paintbox* law. Kingman's theorem says that the law of any random partition is a mixing of paintboxes, i.e. it has the form $\int_{\mathbf{s} \in S} m(\mathrm{d}\mathbf{s}) \rho_{\mathbf{s}}(\mathrm{d}\pi)$ for some probability measure m on S. A useful consequence is that the block of an exchangeable partition π containing 1, or some prescribed integer i, is a *size-biased pick* from the blocks of π, i.e. the probability it equals a non-singleton block π_j conditionally on $(|\pi_j|, j \geq 1)$ equals $|\pi_j|$. Similarly,

Lemma 1. *Let π be an exchangeable random partition which is a.s. different from the trivial partition $\mathbb{O}$, and B an infinite subset of $\mathbb{N}$. For any $i \in \mathbb{N}$, let*

$$\widetilde{i} = \inf\{j \geq i : j \in B \text{ and } \{j\} \notin \pi\},$$

then $\widetilde{i} < \infty$ a.s. and the block $\widetilde{\pi}$ of π containing $\widetilde{i}$ is a size-biased pick among the non-singleton blocks of π, i.e. if we denote these by $\pi'_1, \pi'_2, \ldots$,

$$P(\widetilde{\pi} = \pi'_k | (|\pi'_j|, j \geq 1)) = |\pi'_k| / \sum_j |\pi'_j|.$$

For any sequence of partitions $(\pi^{(i)}, i \geq 1)$, define $\pi = \bigcap_{i \geq 1} \pi^{(i)}$ by

$$k \overset{\pi}{\sim} j \iff k \overset{\pi^{(i)}}{\sim} j \quad \forall i \geq 1.$$

Lemma 2. *Let $(\pi^{(i)}, i \geq 1)$ be a sequence of independent exchangeable partitions and set $\pi := \bigcap_{i \geq 1} \pi^{(i)}$. Then, a.s. for every $j \in \mathbb{N}$,*

$$|\pi_j| = \prod_{i \geq 1} \left| \pi^{(i)}_{k(i,j)} \right|,$$

where $(k(i,j), j \geq 1)$ is defined so that $\pi_j = \bigcap_{i \geq 1} \pi^{(i)}_{k(i,j)}$.

Proof. First notice that $k(i,j) \leq j$ for all $i \geq 1$ a.s. This is clear when $\pi_j \neq \varnothing$, since $j \in \pi_j$ and then $j \in \pi^{(i)}_{k(i,j)}$. When $\pi_j = \varnothing$, $j \in \pi_m$ for some $m < j$ and then m and j belong to the same block of $\pi^{(i)}$ for all $i \geq 1$. Thus $k(i,j) \leq m < j$. Using then the paintbox construction of exchangeable partitions explained above and the independence of the $\pi^{(i)}$'s, we see that the r.v. $\prod_{i \geq 1} \mathbb{1}_{\{m \in \pi^{(i)}_{k(i,j)}\}}$, $m \geq j+1$, are iid conditionally on $(|\pi^{(i)}_{k(i,j)}|, i \geq 1)$ with a mean equal to $\prod_{i \geq 1} |\pi^{(i)}_{k(i,j)}|$. The law of large numbers therefore gives

$$\prod_{i \geq 1} \left| \pi^{(i)}_{k(i,j)} \right| = \lim_{n \to \infty} \frac{1}{n} \sum_{j+1 \leq m \leq n} \prod_{i \geq 1} \mathbb{1}_{\left\{m \in \pi^{(i)}_{k(i,j)}\right\}} \text{ a.s.}$$

On the other hand, the random variables $\prod_{i\geq 1}\mathbb{1}_{\{m\in\pi^{(i)}_{k(i,j)}\}}=\mathbb{1}_{\{m\in\pi_j\}}$, $m\geq j+1$, are i.i.d. conditionally on $|\pi_j|$ with mean $|\pi_j|$ and then the limit above converges a.s. to $|\pi_j|$, again by the law of large numbers. □

We now turn our attention to partition-valued fragmentations.

Definition 3. *Let $(\Pi(t),t\geq 0)$ be a Markovian $\mathcal{P}_\infty$-valued process with $\Pi(0)=\{\mathbb{N},\varnothing,\varnothing,\dots\}$ that is continuous in probability and exchangeable as a process (meaning that the law of Π is invariant under the action of permutations). Call it a partition-valued self-similar fragmentation with index $\alpha\in\mathbb{R}$ if moreover $\Pi(t)$ admits asymptotic frequencies for all t, a.s., if the process $(|\Pi(t)|,t\geq 0)$ is continuous in probability, and if the following fragmentation property is satisfied. For $t,t'\geq 0$, given $\Pi(t)=(\pi_1,\pi_2,\dots)$, the sequence $\Pi(t+t')$ has the same law as the partition with blocks $\pi_1\cap\Pi^{(1)}(|\pi_1|^\alpha t'),\pi_2\cap\Pi^{(2)}(|\pi_2|^\alpha t'),\dots$, where $(\Pi^{(i)},i\geq 1)$ are independent copies of Π.*

Bertoin [7] has shown that any such fragmentation is also characterized by the same 3-tuple (α,c,ν) as above, meaning that the laws of partition-valued and ranked self-similar fragmentations are in a one-to-one correspondence. In fact, for every (α,c,ν), one can construct a version of the partition-valued fragmentation Π with parameters (α,c,ν), and then $(|\Pi(t)|,t\geq 0)$ is the ranked fragmentation with parameters (α,c,ν). Let us build this version now. It is done following [6, 7] by a Poissonian construction. Recall the notation $\rho_{\mathbf{s}}(\mathrm{d}\pi)$, and define $\kappa_\nu(\mathrm{d}\pi)=\int_S\nu(\mathrm{d}\mathbf{s})\rho_{\mathbf{s}}(\mathrm{d}\pi)$. Let $\#$ be the counting measure on $\mathbb{N}$ and let (Δ_t,k_t) be a $\mathcal{P}_\infty\times\mathbb{N}$-valued Poisson point process with intensity $\kappa_\nu\otimes\#$. We may construct a process $(\Pi^0(t),t\geq 0)$ by letting $\Pi^0(0)$ be the trivial partition $(\mathbb{N},\varnothing,\varnothing,\dots)$, and saying that Π^0 jumps only at times t when an atom (Δ_t,k_t) occurs. When this is the case, Π^0 jumps from the state $\Pi^0(t-)$ to the following partition $\Pi^0(t)$: replace the block $\Pi^0_{k_t}(t-)$ by $\Pi^0_{k_t}(t-)\cap\Delta_t$, and leave the other blocks unchanged. Such a construction can be made rigorous by considering restrictions of partitions to the first n integers and by a consistency argument. Then Π^0 has the law of the fragmentation with parameters $(0,0,\nu)$.

Out of this "homogeneous" fragmentation, we construct the $(\alpha,0,\nu)$-fragmentation by introducing a time-change. Call $\lambda_i(t)$ the asymptotic frequency of the block of $\Pi^0(t)$ that contains i, and write

$$T_i(t)=\inf\left\{u\geq 0:\int_0^u\lambda_i(r)^{-\alpha}\mathrm{d}r>t\right\}. \tag{3}$$

Last, for every $t\geq 0$ we let $\Pi(t)$ be the random partition such that i,j are in the same block of $\Pi(t)$ if and only if they are in the same block of $\Pi^0(T_i(t))$, or equivalently of $\Pi^0(T_j(t))$. Then $(\Pi(t),t\geq 0)$ is the wanted version. Let $(\mathcal{G}(t),t\geq 0)$ be the natural filtration generated by Π completed up to P-null sets. According to [7], the fragmentation property holds actually for $\mathcal{G}$-stopping times and we shall refer to it as the *strong fragmentation property*. In the homogeneous case, we will rather call $\mathcal{G}^0$ the natural filtration.

When $\alpha<0$, the loss of mass in the ranked fragmentations shows up at the level of partitions by the fact that a positive fraction of the blocks of $\Pi(t)$ are singletons for some $t>0$. This last property of self-similar fragmentations with negative index allows us to build a collection of trees with edge-lengths.

2.2 Trees with edge-lengths

A tree is a finite connected graph with no cycles. It is *rooted* when a particular vertex (the root) is distinguished from the others, in this case the edges are by convention oriented, pointing from the root, and we define the out-degree of a vertex v as being the number of edges that point outward from v. A *leaf* in a rooted tree is a vertex with out-degree 0. For $k \geq 1$, let $\mathbf{T}_k$ be the set of rooted trees with exactly k labeled leaves (the names of the labels may change according to what we see fit), the other vertices (except the root) begin unlabeled , and such that the root is the only vertex that has out-degree 1. If $\mathbf{t} \in \mathbf{T}_k$, we let $E(\mathbf{t})$ be the set of its edges.

A *tree with edge-lengths* is a pair $\vartheta = (\mathbf{t}, \mathbf{e})$ for $\mathbf{t} \in \bigcup_{k \geq 1} \mathbf{T}_k$ and $\mathbf{e} = (e_i, i \in E(t)) \in (\mathbb{R}_+ \setminus \{0\})^{E(\mathbf{t})}$. Call $\mathbf{t}$ the *skeleton* of ϑ. Such a tree is naturally equipped with a distance $d(v, w)$ on the set of its vertices, by adding the lengths of edges that appear in the unique path connecting v and w in the skeleton (which we still denote by $[[v, w]]$). The height of a vertex is its distance to the root. We let $\mathbb{T}_k$ be the set of trees with edge-lengths whose skeleton is in $\mathbf{T}_k$. For $\vartheta \in \mathbb{T}_k$, let e_{root} be the length of the unique edge connected to the root, and for $e < e_{\mathrm{root}}$ write $\vartheta - e$ for the tree with edge-lengths that has same skeleton and same edge-lengths as ϑ, but for the edge pointing outward from the root which is assigned length $e_{\mathrm{root}} - e$.

We also define an operation MERGE as follows. Let $n \geq 2$ and take $\vartheta_1, \vartheta_2, \ldots, \vartheta_n$ respectively in $\mathbb{T}_{k_1}, \mathbb{T}_{k_2}, \ldots, \mathbb{T}_{k_n}$, with leaves $(L_i^1, 1 \leq i \leq k_1), (L_i^2, 1 \leq i \leq k_2), \ldots, (L_i^n, 1 \leq i \leq k_n)$ respectively. Let also $e > 0$. The tree with edge-lengths $\mathtt{MERGE}((\vartheta_1, \ldots, \vartheta_n); e) \in \mathbb{T}_{\sum_i k_i}$ is defined by merging together the roots of $\vartheta_1, \ldots, \vartheta_n$ into a single vertex $\bullet$, and by drawing a new edge root $\to \bullet$ with length e.

Last, for $\vartheta \in \mathbb{T}_k$ and i vertices $v_1, \ldots, v_i$, define the subtree spanned by the root and $v_1, \ldots, v_i$ as follows. For every $p \neq q$, let $b(v_p, v_q)$ be the branchpoint of v_p and v_q, that is, the highest point in the tree that belongs to $[[\mathrm{root}, v_p]] \cap [[\mathrm{root}, v_q]]$. The spanned tree is the tree with edge-lengths whose vertices are the root, the vertices $v_1, \ldots, v_i$ and the branchpoints $b(v_p, v_q)$, $1 \leq p \neq q \leq i$, and whose edge-lengths are given by the respective distances between this subset of vertices of the original tree.

2.3 Building the CRT

Now for $B \subset \mathbb{N}$ finite, define $\mathcal{R}(B)$, a random variable with values in $\mathbb{T}_{\#B}$, whose leaf-labels are of the form L_i for $i \in \mathbb{N}$, as follows. Let $D_i = \inf\{t \geq 0 : \{i\} \in \Pi(t)\}$ be the first time when $\{i\}$ "disappears", i.e. is isolated in a singleton of $\Pi(t)$. For B a finite subset of $\mathbb{N}$ with at least two elements, let $D_B = \inf\{t \geq 0 : \#(B \cap \Pi(t)) \neq 1\}$ be the first time when the restriction of $\Pi(t)$ to B is non-trivial, i.e. has more than one block. By convention, $D_{\{i\}} = D_i$. For every $i \geq 1$, define $\mathcal{R}(\{i\})$ as a single edge root $\to L_i$, and assign this edge the length D_i. For B with $\#B \geq 2$, let $B_1, \ldots, B_i$ be the non-empty blocks of $B \cap \Pi(D_B)$, arranged in increasing order of least element, and define a tree $\mathcal{R}(B)$ recursively by

$$\mathcal{R}(B) = \mathtt{MERGE}((\mathcal{R}(B_1) - D_B, \ldots, \mathcal{R}(B_i) - D_B); D_B).$$

Last, define $\mathcal{R}(k) = \mathcal{R}([k])$. Notice that by definition of the distance, the distance between L_i and L_j in $\mathcal{R}(k)$ for any $k \geq i \vee j$ equals $D_i + D_j - 2D_{\{i,j\}}$.

We now state the key lemma that allows us to describe the CRT out of the family $(\mathcal{R}(k), k \geq 1)$ which is the candidate for the marginals of $\mathcal{T}_F$. By Aldous [2], it suffices to check two properties, called *consistency* and *leaf-tightness*. Notice that in [2], only binary trees (in which branchpoint have out-degree 2) are considered, but as noticed therein, this translates to our setting with minor changes.

Lemma 3. *(i) The family $(\mathcal{R}(k), k \geq 1)$ is consistent in the sense that for every k and $j \leq k$, $\mathcal{R}(j)$ has the same law as the subtree of $\mathcal{R}(k)$ spanned by the root and j distinct leaves $L_1^k, \ldots, L_j^k$ taken uniformly at random from the leaves $L_1, \ldots, L_k$ of $\mathcal{R}(k)$, independently of $\mathcal{R}(k)$.*

(ii) The family $(\mathcal{R}(k), k \geq 1)$ is leaf-tight, that is, with the above notations,

$$\min_{2 \leq j \leq k} d(L_1^k, L_j^k) \xrightarrow{p} 0.$$

Proof. The consistency property is an immediate consequence of the fact that the process Π is exchangeable. Taking j leaves uniformly out of the k ones of $\mathcal{R}(k)$ is just the same as if we had chosen exactly the leaves $L_1, L_2, \ldots, L_j$, which give rise to the tree $\mathcal{R}(j)$, and this is (i).

For (ii), first notice that we may suppose by exchangeability that $L_1^k = L_1$. The only point is then to show that the minimal distance of this leaf to the leaves $L_2, \ldots, L_k$ tends to 0 in probability as $k \to \infty$. Fix $\eta > 0$ and for $\varepsilon > 0$ write $t_\varepsilon^1 = \inf\{t \geq 0 : |\Pi_1(t)| < \varepsilon\}$, where $\Pi_1(t)$ is the block of $\Pi(t)$ containing 1. Then t_ε^1 is a stopping time with respect to the natural filtration $(\mathcal{F}_t, t \geq 0)$ associated with Π and $t_\varepsilon^1 \uparrow D_1$ as $\varepsilon \downarrow 0$. By the strong Markov property and exchangeability, one has that if $K(\varepsilon) = \inf\{k > 1 : k \in \Pi_1(t_\varepsilon^1)\}$, then $P(D_1 + D_{K(\varepsilon)} - 2t_\varepsilon^1 < \eta) = E[P_{\Pi(t_\varepsilon^1)}(D_1 + D_{K(\varepsilon)} < \eta)]$ where P_π is the law of the fragmentation Π started at π (the law of Π under P_π is the same as that of the family of partitions $\left(\left\{\text{blocks of } \pi_1 \cap \Pi^{(1)}(|\pi_1|^\alpha t), \pi_2 \cap \Pi^{(2)}(|\pi_2|^\alpha t), \ldots\right\}, t \geq 0\right)$ where the $\Pi^{(i)}$'s, $i \geq 1$, are independent copies of Π under $P_{\{\mathbb{N}, \varnothing, \varnothing, \ldots\}}$). By the self-similar fragmentation property and exchangeability this is greater than $P(D_1 + D_2 < \varepsilon^\alpha \eta)$, which in turn is greater than $P(2\tau < \varepsilon^\alpha \eta)$ where τ is the first time where $\Pi(t)$ becomes the partition into singletons, which by [8] is finite a.s. This last probability thus goes to 1 as $\varepsilon \downarrow 0$. Taking $\varepsilon = \varepsilon(n) \downarrow 0$ quickly enough as $n \to \infty$ and applying the Borel-Cantelli lemma, we a.s. obtain a sequence $K(\varepsilon(n))$ such that $d(L_1, L_{K(n)}) \leq D_1 + D_{K(\varepsilon(n))} - 2t_{\varepsilon(n)} < \eta$. Hence the result. $\square$

For a rooted $\mathbb{R}$-tree T and k vertices $v_1, \ldots, v_k$, we define exactly as for marked trees the subtree spanned by the root and $v_1, \ldots, v_k$, as an element of $\mathbb{T}_k$. A consequence of [2, Theorem 3] is then:

Lemma 4. *There exists a CRT $(\mathcal{T}_\Pi, \mu_\Pi)$ such that if $Z_1, \ldots, Z_k$ is a sample of k leaves picked independently according to μ_Π conditionally on μ_Π, the subtree of $\mathcal{T}_\Pi$ spanned by the root and $Z_1, \ldots, Z_k$ has the same law as $\mathcal{R}(k)$.*

In the sequel, sequences like $(Z_1, Z_2, \ldots)$ will be called *exchangeable sequences with directing measure μ_Π*.

Proof of Theorem 1. We have to check that the tree $\mathcal{T}_\Pi$ of the preceding lemma gives rise to a fragmentation process with the same law as $F = |\Pi|$. By construction, we have that for every $t \geq 0$ the partition $\Pi(t)$ is such that i and j are in the same block of $\Pi(t)$ if and only if L_i and L_j are in the same connected component of $\{v \in \mathcal{T}_\Pi : \mathrm{ht}(v) > t\}$. Hence, the law of large numbers implies that if $F'(t)$ is the decreasing sequence of the μ-masses of these connected components, then $F'(t) = F(t)$ a.s. for every t. Hence, F' is a version of F, so we can set $\mathcal{T}_F = \mathcal{T}_\Pi$. That $\mathcal{T}_F$ is α-self-similar is an immediate consequence of the fragmentation and self-similar properties of F.

We now turn to the last statement of Theorem 1. With the notation of Lemma 4 we will show that the path $[[\varnothing, Z_1]]$ is almost-surely in the closure of the set of leaves of $\mathcal{T}_F$ if and only if $\nu(S) = \infty$. Then it must hold by exchangeability that so do the paths $[[\varnothing, Z_i]]$ for every $i \geq 1$, and this is sufficient because the definition of the CRTs implies that $\mathcal{S}(\mathcal{T}_F) = \bigcup_{i\geq 1}[[\varnothing, Z_i[[$, see [2, Lemma 6] (the fact that $\mathcal{T}_F$ is a.s. compact will be proved below). To this end, it suffices to show that for any $a \in (0,1)$, the point aZ_1 of $[[\varnothing, Z_1]]$ that is at a proportion a from $\varnothing$ (the point $\varphi_{\varnothing,Z_1}(ad(\varnothing, Z_1))$ with the above notations) can be approached closely by leaves, that is, for $\eta > 0$ there exists $j > 1$ such that $d(aZ_1, Z_j) < \eta$. It thus suffices to check that for any $\delta > 0$

$$P(\exists 2 \leq j \leq k : |D_{\{1,j\}} - aD_1| < \delta \text{ and } D_j - D_{\{1,j\}} < \delta) \underset{k\to\infty}{\to} 1, \tag{4}$$

with the above notations derived from Π (this is a slight variation of [2, (iii) a). Theorem 15]).

Suppose that $\nu(S) = \infty$. Then for every rational $r > 0$ such that $|\Pi_1(r)| \neq 0$ and for every $\delta > 0$, the block containing 1 undergoes a fragmentation in the time-interval $(r, r + \delta/2)$. This is obvious from the Poisson construction of the self-similar fragmentation Π given above, because ν is an infinite measure so there is an infinite number of atoms of (Δ_t, k_t) with $k_t = 1$ in any time-interval with positive length. Therefore, there exists an infinite number of elements of $\Pi_1(r)$ that are isolated in singletons of $\Pi(r + \delta)$, e.g. because of Lemma 5 below which asserts that only a finite number of the blocks of $\Pi(r + \delta/2)$ "survive" at time $r + \delta$, i.e. is not completely reduced to singletons. Thus, an infinite number of elements of $\Pi_1(r)$ correspond to leaves of some $\mathcal{R}(k)$ for k large enough. By taking r close to aD_1 we thus have the result.

On the other hand, if $\nu(S) < \infty$, it follows from the Poisson construction that the state $(1, 0, \ldots)$ is a holding state, so the first fragmentation occurs at a positive time, so the root cannot be approached by leaves. $\square$

Remark. We have seen that we may actually build simultaneously the trees $(\mathcal{R}(k), k \geq 1)$ on the same probability space as a measurable functional of the process $(\Pi(t), t \geq 0)$. This yields, by redoing the "special construction" of Aldous [2], a *stick-breaking construction* of the tree $\mathcal{T}_F$, by now considering the trees $\mathcal{R}(k)$ as $\mathbb{R}$-trees obtained as finite unions of segments rather than trees with edge-lengths (one can check that it is possible to switch between the two notions). The mass measure is then defined as the limit of the empirical measure on the leaves $L_1, \ldots, L_n$. The special CRT thus constructed is a subset of ℓ^1 in [2], but we consider it as universal, i.e. up to isomorphism. The tree $\mathcal{R}(k+1)$ is then obtained from $\mathcal{R}(k)$ by branching a new segment with length $D_{k+1} - \max_{B\subset[k], B\neq\varnothing} D_{B\cup\{k+1\}}$, and $\mathcal{T}_F$ can be reinterpreted as the completion of the metric space $\bigcup_{k\geq 1}\mathcal{R}(k)$. On the other hand, call $L_1, L_2, \ldots$ as before the leaves of $\bigcup_{k\geq 1}\mathcal{R}(k)$, L_k being the leaf corresponding to the k-th branch. One of the subtleties of the special construction of [2] is that $L_1, L_2, \ldots$ is not itself an exchangeable sample with the mass measure as directing law. However, considering such a sample $Z_1, Z_2, \ldots$, we may construct a random partition $\Pi'(t)$ for every t by letting $i \sim^{\Pi'(t)} j$ if and only if Z_i and Z_j are in the same connected component of the forest $\{v \in \mathcal{T}_F : \mathrm{ht}(v) > t\}$. Then easily $\Pi'(t)$ is again a partition-valued self-similar fragmentation, and in fact $|\Pi'(t)| = F(t)$ a.s. for every t so Π' has same law as Π (Π' can be interpreted as a "relabeling" of the blocks of Π). As a conclusion, up to this relabeling, we may and will assimilate $\mathcal{T}_F$ as the completion of the increasing union of the trees $\mathcal{R}(k)$, while $L_1, L_2, \dots$ will be considered as an exchangeable sequence with directing law μ_F.

Proof of Proposition 1. The fact that the process F defined out of a CRT $(\mathcal{T}, \mu)$ with the stated properties is a S-valued self-similar fragmentation with index α is straightforward and left to the reader. The treatment of the erosion and sudden loss of mass is a little more subtle. Let $Z_1, Z_2, \dots$ be an exchangeable sample directed by the measure μ, and for every $t \geq 0$ define a random partition $\Pi(t)$ by saying that i and j are in the same block of $\Pi(t)$ if Z_i and Z_j fall in the same tree component of $\{v \in \mathcal{T} : \mathrm{ht}(v) > t\}$. By the arguments above, Π defines a self-similar partition-valued fragmentation such that $|\Pi(t)| = F(t)$ a.s. for every t. Notice that if we show that the erosion coefficient $c = 0$ and that no sudden loss of mass occur, it will immediately follow that $\mathcal{T}$ has the same law as $\mathcal{T}_F$.

Now suppose that $\nu(\sum_i s_i < 1) \neq 0$. Then (e.g. by the Poisson construction of fragmentations described above) there exist a.s. two distinct integers i and j and a time D such that i and j are in the same block of $\Pi(D-)$ but $\{i\} \in \Pi(D)$ and $\{j\} \in \Pi(D)$. This implies that $Z_i = Z_j$, so μ has a.s. an atom and $(\mathcal{T}, \mu)$ cannot be a CRT. On the other hand, suppose that the erosion coefficient $c > 0$. Again from the Poisson construction, we see that there a.s. exists a time D such that $\{1\} \notin \Pi(D-)$ but $\{1\} \in \Pi(D)$, and nevertheless $\Pi(D) \cap \Pi_1(D-)$ is not the trivial partition $\mathbb{O}$. Taking j in a non-trivial block of this last partition and denoting its death time by D', we obtain that the distance from Z_1 to Z_j is $D' - D$, while the height of Z_1 is D and that of Z_j is D'. This implies that Z_1 is a.s. not in the set of leaves of $\mathcal{T}$, again contradicting the definition of a CRT. $\square$

3 Hausdorff dimension of $\mathcal{T}_F$

Let (M, d) be a compact metric space. For $\mathcal{E} \subseteq M$, the Hausdorff dimension of $\mathcal{E}$ is the real number

$$\dim_{\mathcal{H}}(\mathcal{E}) := \inf\{\gamma > 0 : m_\gamma(\mathcal{E}) = 0\} = \sup\{\gamma > 0 : m_\gamma(\mathcal{E}) = \infty\}, \tag{5}$$

where

$$m_\gamma(\mathcal{E}) := \sup_{\varepsilon > 0} \inf \sum_i \Delta(E_i)^\gamma, \tag{6}$$

the infimum being taken over all collections $(E_i, i \geq 1)$ of subsets of $\mathcal{E}$ with diameter $\Delta(E_i) \leq \varepsilon$, whose union covers $\mathcal{E}$. This dimension is meant to measure the "fractal size" of the considered set. For background on this subject, we mention [16] (in the case $M = \mathbb{R}^n$, but the generalization to general metric spaces of the results we will need is straightforward).

The goal of this Section is to prove Theorem 2 and more generally that

$$\frac{\varrho}{|\alpha|} \leq \dim_{\mathcal{H}}(\mathcal{L}(\mathcal{T}_F)) \leq \frac{1}{|\alpha|} \text{ a.s.}$$

where ϱ is the ν-dependent parameter defined by (2). The proof is divided in the two usual upper and lower bound parts. In Section 3.1, we first prove that $\mathcal{T}_F$ is indeed compact and that

$\dim_{\mathcal{H}}(\mathcal{L}(\mathcal{T}_F)) \leq 1/|\alpha|$ a.s., which is true without any extra integrability assumption on ν. We then show that this upper bound yields $\dim_{\mathcal{H}}(\mathrm{d}M_F) \leq 1 \wedge 1/|\alpha|$ a.s. (Corollary 2). Sections 3.2 to 3.4 are devoted to the lower bound $\dim_{\mathcal{H}}(\mathcal{L}(\mathcal{T}_F)) \geq \varrho/|\alpha|$ a.s. This is obtained by using appropriate subtrees of $\mathcal{T}_F$ (we will see that the most naive way to apply Frostman's energy method with the mass measure μ_F fails in general). That Theorem 2 applies to stable trees is proved in Sect. 3.5.

3.1 Upper bound

We begin by stating the expected

Lemma 5. *The tree $\mathcal{T}_F$ is a.s. compact.*

Proof. For $t \geq 0$ and $\varepsilon > 0$, denote by N_t^ε the number of blocks of $\Pi(t)$ not reduced to singletons that are not entirely reduced to dust at time $t+\varepsilon$. We first prove that N_t^ε is a.s. finite. Let $(\Pi_i(t), i \geq 1)$ be the blocks of $\Pi(t)$, and $(|\Pi_i(t)|, i \geq 1)$, their respective asymptotic frequencies. For integers i such that $|\Pi_i(t)| > 0$, that is $\Pi_i(t) \neq \varnothing$ and $\Pi_i(t)$ is not reduced to a singleton, let $\tau_i := \inf\{s > t : \Pi_i(t) \cap \Pi(s) = \mathbb{O}\}$ be the first time at which the block $\Pi_i(t)$ is entirely reduced to dust. Applying the fragmentation property at time t, we may write τ_i as $\tau_i = t + |\Pi_i(t)|^{|\alpha|} \widetilde{\tau}_i$ where $\widetilde{\tau}_i$ is a r.v. independent of $\mathcal{G}(t)$ that has same distribution as $\tau = \inf\{t \geq 0 : \Pi(t) = \mathbb{O}\}$, the first time at which the fragmentation is entirely reduced to dust. Now, fix $\varepsilon > 0$. The number of blocks of $\Pi(t)$ that are not entirely reduced to dust at time $t+\varepsilon$, which could be *a priori* infinite, is then given by

$$N_t^\varepsilon = \sum_{i:|\Pi_i(t)|>0} \mathbb{1}_{\left\{|\Pi_i(t)|^{|\alpha|}\widetilde{\tau}_i > \varepsilon\right\}}.$$

From Proposition 15 in [18], we know that there exist two constants C_1, C_2 such that $P(\tau > t) \leq C_1 e^{-C_2 t}$ for all $t \geq 0$. Consequently, for all $\delta > 0$,

$$\begin{aligned} E\left[N_t^\varepsilon \mid \mathcal{G}(t)\right] &\leq C_1 \sum_{i:|\Pi_i(t)|>0} e^{-C_2 \varepsilon |\Pi_i(t)|^\alpha} \\ &\leq C(\delta)\varepsilon^{-\delta} \sum_i |\Pi_i(t)|^{|\alpha|\delta}, \end{aligned} \tag{7}$$

where $C(\delta) = \sup_{x\in\mathbb{R}^+}\left\{C_1 x^\delta e^{-C_2 x}\right\} < \infty$. Since $\sum_i |\Pi_i(t)| \leq 1$ a.s, this shows by taking $\delta = 1/|\alpha|$ that $N_t^\varepsilon < \infty$ a.s.

Let us now construct a covering of supp (μ) with balls of radius 5ε. Recall that we may suppose that the tree $\mathcal{T}_F$ is constructed together with an exchangeable leaf sample $(L_1, L_2, \ldots)$ directed by μ_F. For each $l \in \mathbb{N} \cup \{0\}$, we introduce the set

$$B_l^\varepsilon = \{k \in \mathbb{N} : \{k\} \notin \Pi(l\varepsilon), \{k\} \in \Pi((l+1)\varepsilon)\},$$

some of which may be empty when $\nu(S) < \infty$, since the tree is not leaf-dense. For $l \geq 1$, the number of blocks of the partition $B_l^\varepsilon \cap \Pi((l-1)\varepsilon)$ of B_l^ε is less than or equal to $N_{(l-1)\varepsilon}^\varepsilon$ and so is a.s. finite. Since the fragmentation is entirely reduced to dust at time $\tau < \infty$ a.s., $N^{\varepsilon}_{l\varepsilon}$ is equal to zero for $l \geq \tau/\varepsilon$ and then, defining

$$N_{\varepsilon} := \sum_{l=0}^{[\tau/\varepsilon]} N^{\varepsilon}_{l\varepsilon}$$

we have $N_{\varepsilon} < \infty$ a.s. ($[\tau/\varepsilon]$ denotes here the largest integer smaller than τ/ε). Now, consider a finite random sequence of pairwise distinct integers $\sigma(1), ..., \sigma(N_{\varepsilon})$ such that for each $1 \leq l \leq [\tau/\varepsilon]$ and each non-empty block of $B^{\varepsilon}_l \cap \Pi((l-1)\,\varepsilon)$, there is a $\sigma(i), 1 \leq i \leq N_{\varepsilon}$, in this block. Then each leaf L_j belongs then to a ball of center $L_{\sigma(i)}$, for an integer $1 \leq i \leq N_{\varepsilon}$, and of radius 4ε. Indeed, fix $j \geq 1$. It is clear that the sequence $(B^{\varepsilon}_l)_{l \in \mathbb{N}\cup\{0\}}$ forms a partition of $\mathbb{N}$. Thus, there exists a unique block B^{ε}_l containing j and in this block we consider the integer $\sigma(i)$ that belongs to the same block as j in the partition $B^{\varepsilon}_l \cap \Pi(((l-1) \vee 0)\varepsilon)$. By definition (see Section 2.3), the distance between the leaves L_j and $L_{\sigma(i)}$ is $d(L_j, L_{\sigma(i)}) = D_j + D_{\sigma(i)} - 2D_{\{j,\sigma(i)\}}$. By construction, j and $\sigma(i)$ belong to the same block of $\Pi(((l-1) \vee 0)\,\varepsilon)$ and both die before $(l+1)\,\varepsilon$. In other words, $\max(D_j, D_{\sigma(i)}) \leq (l+1)\,\varepsilon$ and $D_{\{j,\sigma(i)\}} \geq ((l-1) \vee 0)\,\varepsilon$, which implies that $d(L_j, L_{\sigma(i)}) \leq 4\varepsilon$. Therefore, we have covered the set of leaves $\{L_j, j \geq 1\}$ by at most N_{ε} balls of radius 4ε. Since the sequence $(L_j)_{j\geq 1}$ is dense in supp (μ), this induces by taking balls with radius 5ε instead of 4ε a covering of supp (μ) by N_{ε} balls of radius 5ε. This holds for all $\varepsilon > 0$ so supp (μ) is a.s. compact. The compactness of $\mathcal{T}_F$ follows. $\square$

Let us now prove the upper bound for $\dim_{\mathcal{H}}(\mathcal{L}(\mathcal{T}_F))$. The difficulty for finding a "good" covering of the set $\mathcal{L}(\mathcal{T}_F)$ is that as soon as ν is infinite, this set is dense in $\mathcal{T}_F$, and thus one cannot hope to find its dimension by the plain box-counting method, because the skeleton $\mathcal{S}(\mathcal{T}_F)$ has a.s. Hausdorff dimension 1 as a countable union of segments. However, we stress that the covering with balls of radius 5ε of the previous lemma is a good covering of the *whole tree*, because the box-counting method leads to the right bound $\dim_{\mathcal{H}}(\mathcal{T}_F) \leq (1/|\alpha|) \vee 1$, and this is sufficient when $|\alpha| < 1$. When $|\alpha| \geq 1$ though, we may lose the details of the structure of $\mathcal{L}(\mathcal{T}_F)$. We will thus try to find a sharp "cutset" for the tree, motivated by the computation of the dimension of leaves of discrete infinite trees.

Proof of Theorem 2: upper bound. For every $i \in \mathbb{N}$ and $t \geq 0$ let $\Pi_{(i)}(t)$ be the block of $\Pi(t)$ containing i and for $\varepsilon > 0$ let

$$t^{\varepsilon}_i = \inf\{t \geq 0 : |\Pi_{(i)}(t)| < \varepsilon\}.$$

Define a partition Π^{ε} by $i \sim^{\Pi^{\varepsilon}} j$ if and only if $\Pi_{(i)}(t^{\varepsilon}_i) = \Pi_{(j)}(t^{\varepsilon}_j)$. One easily checks that this random partition is exchangeable, moreover it has a.s. no singleton. Indeed, notice that for any i, $\Pi_{(i)}(t^{\varepsilon}_i)$ is the block of $\Pi(t^{\varepsilon}_i)$ that contains i, and this block cannot be a singleton because the process $(|\Pi_{(i)}(t)|, t \geq 0)$ reaches 0 continuously. Therefore, $\Pi^{(\varepsilon)}$ admits asymptotic frequencies a.s., and these frequencies sum to 1. Then let

$$\tau^{\varepsilon}_{(i)} = \sup_{j \in \Pi_{(i)}(t^{\varepsilon}_i)} \inf\{t \geq t^{\varepsilon}_i : |\Pi_{(j)}(t)| = 0\} - t^{\varepsilon}_i$$

be the time after t^{ε}_i when the fragment containing i vanishes entirely (notice that $\tau^{\varepsilon}_{(i)} = \tau^{\varepsilon}_{(j)}$ whenever $i \sim^{\Pi^{\varepsilon}} j$). We also let b^{ε}_i be the unique vertex of $[[\varnothing, L_i]]$ at distance t^{ε}_i from the root, notice that again $b^{\varepsilon}_i = b^{\varepsilon}_j$ whenever $i \sim^{\Pi^{\varepsilon}} j$.

We claim that

$$\mathcal{L}(\mathcal{T}_F) \subseteq \bigcup_{i\in\mathbb{N}} \overline{B}(b_i^\varepsilon, \tau_{(i)}^\varepsilon),$$

where $\overline{B}(v,r)$ is the closed ball centered at v with radius r in $\mathcal{T}_F$. Indeed, for $L \in \mathcal{L}(\mathcal{T}_F)$, let b_L be the vertex of $[[\varnothing, L]]$ with minimal height such that $\mu_F(\mathcal{T}_{b_L}) < \varepsilon$, where $\mathcal{T}_{b_L}$ is the fringe subtree of $\mathcal{T}_F$ rooted at b_L. Since $b_L \in \mathcal{S}(\mathcal{T}_F)$, $\mu_F(\mathcal{T}_{b_L}) > 0$ and there exist infinitely many i's with $L_i \in \mathcal{T}_{b_L}$. But then, it is immediate that for any such i, $t_i^\varepsilon = \mathrm{ht}(b_L) = \mathrm{ht}(b_i^\varepsilon)$. Since $(L_i, i \geq 1)$ is dense in $\mathcal{L}(\mathcal{T}_F)$, and since for every j with $L_j \in \mathcal{T}_{b_i^\varepsilon}$ one has $d(b_i^\varepsilon, L_j) \leq \tau_{(i)}^\varepsilon$ by definition, it follows that $L \in \overline{B}(b_i^\varepsilon, \tau_{(i)}^\varepsilon)$. Therefore, $(\overline{B}(b_i^\varepsilon, \tau_{(i)}^\varepsilon), i \geq 1)$ is a covering of $\mathcal{L}(\mathcal{T}_F)$.

The next claim is that this covering is fine as $\varepsilon \downarrow 0$, namely

$$\sup_{i\in\mathbb{N}} \tau_{(i)}^\varepsilon \underset{\varepsilon\downarrow 0}{\to} 0 \qquad \text{a.s.}$$

Indeed, if it were not the case, we would find $\eta > 0$ and $i_n, n \geq 0$, such that $\tau_{(i_n)}^{1/2^n} \geq \eta$ and $d(b_{i_n}^{1/2^n}, L_{i_n}) \geq \eta/2$ for every n. Since $\mathcal{T}_F$ is compact, we may extract a subsequence such that $L_{i_n} \to v$ for some $v \in \mathcal{T}_F$. Now, since $\mu_F(\mathcal{T}_{b_{i_n}^{1/2^n}}) \leq 2^{-n}$, it follows that we may find a vertex $b \in [[\varnothing, v]]$ at distance at least $\eta/4$ from v, such that $\mu_F(\mathcal{T}_b) = 0$, and this does not happen a.s.

To conclude, let $\tau_i^\varepsilon = \tau_{(i)}^\varepsilon \mathbb{1}_{\{\Pi_{(i)}(t_i^\varepsilon) = \Pi_i(t_i^\varepsilon)\}}$ (we just choose one i representing each class of Π^ε above). By the self-similarity property applied at the $(\mathcal{G}(t), t \geq 0)$-stopping time t_i^ε, τ_i^ε has the same law as $|\Pi_i(t_i^\varepsilon)|^{|\alpha|}\tau$, where τ has same law as $\inf\{t \geq 0 : |\Pi(t)| = (0,0,\ldots)\}$ and is taken independent of $|\Pi_i(t_i^\varepsilon)|$. Therefore,

$$E\left[\sum_{i\geq 1} (\tau_i^\varepsilon)^{1/|\alpha|}\right] = E[\tau^{1/|\alpha|}] E\left[\sum_{i\geq 1} |\Pi_i(t_i^\varepsilon)|\right] = E[\tau^{1/|\alpha|}] < \infty. \tag{8}$$

The fact that $E[\tau^{1/|\alpha|}]$ is finite comes from the fact that τ has exponential moments. Because our covering is a fine covering as $\varepsilon \downarrow 0$, it finally follows that (with the above notations)

$$m_{1/|\alpha|}(\mathcal{L}(\mathcal{T}_F)) \leq \liminf_{\varepsilon\downarrow 0} \sum_{i:\Pi_{(i)}(t_i^\varepsilon) = \Pi_i(t_i^\varepsilon)} (\tau_i^\varepsilon)^{1/|\alpha|} \qquad \text{a.s.},$$

which is a.s. finite by (8) and Fatou's Lemma. □

Proof of Corollary 2. By Theorem 1, the measure $\mathrm{d}M_F$ has same law as $\mathrm{d}\overline{W}_{\mathcal{T}_F}$, the Stieltjes measure associated with the cumulative height profile $\overline{W}_{\mathcal{T}_F}(t) = \mu_F\{v \in \mathcal{T}_F : \mathrm{ht}(v) \leq t\}, t \geq 0$. To bound from above the Hausdorff dimension of $\mathrm{d}\overline{W}_{\mathcal{T}_F}$, note that

$$\mathrm{d}\overline{W}_{\mathcal{T}_F}(\mathrm{ht}(\mathcal{L}(\mathcal{T}_F))) = \int_{\mathcal{T}_F} \mathbb{1}_{\{\mathrm{ht}(v)\in\mathrm{ht}(\mathcal{L}(\mathcal{T}_F))\}}\mu_F(\mathrm{d}v) = 1$$

since $\mu_F(\mathcal{L}(\mathcal{T}_F)) = 1$. By definition of $\dim_{\mathcal{H}}\left(\mathrm{d}\overline{W}_{\mathcal{T}_F}\right)$, it is thus sufficient to show that $\dim_{\mathcal{H}}(\mathrm{ht}(\mathcal{L}(\mathcal{T}_F))) \leq 1/|\alpha|$ a.s. To do so, remark that ht is Lipschitz and that this property easily leads to

$$\dim_{\mathcal{H}}(\mathrm{ht}(\mathcal{L}(\mathcal{T}_F))) \leq \dim_{\mathcal{H}}(\mathcal{L}(\mathcal{T}_F)).$$

The conclusion hence follows from the majoration $\dim_{\mathcal{H}}(\mathcal{L}(\mathcal{T}_F))) \leq 1/|\alpha|$ proved above. □

3.2 A first lower bound

Recall that Frostman's energy method to prove that $\dim_{\mathcal{H}}(\mathcal{E}) \geq \gamma$ where $\mathcal{E}$ is a subset of a metric space (M, d) is to find a nonzero positive measure $\eta(\mathrm{d}x)$ on $\mathcal{E}$ such that $\int_{\mathcal{E}}\int_{\mathcal{E}} \frac{\eta(\mathrm{d}x)\eta(\mathrm{d}y)}{d(x,y)^\gamma} < \infty$. A naive approach for finding a lower bound of the Hausdorff dimension of $\mathcal{T}_F$ is thus to apply this method by taking $\eta = \mu_F$ and $\mathcal{E} = \mathcal{L}(\mathcal{T}_F)$. The result states as follows.

Lemma 6. *For any fragmentation process F satisfying the hypotheses of Theorem 1, one has*

$$\dim_{\mathcal{H}}(\mathcal{L}(\mathcal{T}_F)) \geq \frac{A}{|\alpha|} \wedge \left(1 + \frac{\underline{p}}{|\alpha|}\right),$$

where

$$\underline{p} := -\inf\left\{q : \int_S \left(1 - \sum_{i\geq 1} s_i^{q+1}\right) \nu(\mathrm{d}\mathbf{s}) > -\infty\right\} \in [0,1], \tag{9}$$

and

$$A := \sup\left\{a \leq 1 : \int_S \sum_{1\leq i<j} s_i^{1-a} s_j \nu(\mathrm{d}\mathbf{s}) < \infty\right\} \in [0,1]. \tag{10}$$

Proof. By Lemma 4 (recall that $(\mathcal{T}_\Pi, \mu_\Pi) = (\mathcal{T}_F, \mu_F)$ by Theorem 1) we have

$$\int_{\mathcal{T}_F}\int_{\mathcal{T}_F} \frac{\mu_F(\mathrm{d}x)\mu_F(\mathrm{d}y)}{d(x,y)^\gamma} \overset{a.s.}{=} E\left[\frac{1}{d(L_1,L_2)^\gamma} \,|\mathcal{T}_F, \mu_F\right]$$

so that

$$E\left[\int_{\mathcal{T}_F}\int_{\mathcal{T}_F} \frac{\mu_F(\mathrm{d}x)\mu_F(\mathrm{d}y)}{d(x,y)^\gamma}\right] = E\left[\frac{1}{d(L_1,L_2)^\gamma}\right]$$

and by definition, $d(L_1, L_2) = D_1 + D_2 - 2D_{\{1,2\}}$. Applying the strong fragmentation property at the stopping time $D_{\{1,2\}}$, we can rewrite D_1 and D_2 as

$$D_1 = D_{\{1,2\}} + \lambda_1^{|\alpha|}(D_{\{1,2\}})\widetilde{D}_1 \qquad D_2 = D_{\{1,2\}} + \lambda_2^{|\alpha|}(D_{\{1,2\}})\widetilde{D}_2$$

where $\lambda_1(D_{\{1,2\}})$ (resp. $\lambda_2(D_{\{1,2\}})$) is the asymptotic frequency of the block containing 1 (resp. 2) at time $D_{\{1,2\}}$ and $\widetilde{D}_1$ and $\widetilde{D}_2$ are independent with the same law as D_1 and independent of $\mathcal{G}\left(D_{\{1,2\}}\right)$. Therefore,

$$d(L_1, L_2) = \lambda_1^{|\alpha|}(D_{\{1,2\}})\widetilde{D}_1 + \lambda_2^{|\alpha|}(D_{\{1,2\}})\widetilde{D}_2,$$

and

$$E\left[\frac{1}{d(L_1,L_2)^\gamma}\right] \leq 2E\left[\lambda_1^{\alpha\gamma}(D_{\{1,2\}}); \lambda_1(D_{\{1,2\}}) \geq \lambda_2(D_{\{1,2\}})\right] E\left[D_1^{-\gamma}\right]. \tag{11}$$

By [19, Lemma 2] the first expectation in the right-hand side of inequality (11) is finite as soon as $|\alpha|\gamma < A$, while by [18, Sect. 4.2.1] the second expectation is finite as soon as $\gamma < 1 + \underline{p}/|\alpha|$. That $\dim_{\mathcal{H}}(\mathcal{L}(\mathcal{T}_F)) \overset{a.s.}{\geq} \left((A/|\alpha|) \wedge \left(1 + \underline{p}/|\alpha|\right)\right)$ follows. $\square$

Let us now make a comment about this bound. For dislocation measures such that $\nu(s_{N+1} > 0) = 0$ for some $N \geq 1$, the constant A equals 1 since for all $a < 1$,

$$\int_S \sum_{i<j} s_i^{1-a} s_j \nu(\mathrm{d}\mathbf{s}) \leq \int_S (N-1) \sum_{2 \leq j \leq N} s_j \nu(\mathrm{d}\mathbf{s}) \leq (N-1) \int_S (1-s_1)\, \nu(\mathrm{d}\mathbf{s}) < \infty.$$

In such cases, if moreover $\underline{p} = 1$, the "naive" lower bound of Lemma 6 is thus equal to $1/|\alpha|$. A typical setting in which this holds is when $\nu(S) < \infty$ and $\nu(s_{N+1} > 0) = 0$ and therefore, for such dislocation measures the "naive" lower bound is also the best possible.

3.3 A subtree of $\mathcal{T}_F$ and a reduced fragmentation

In the general case, in order to improve this lower bound, we will thus try to transform the problem on F into a problem on an auxiliary fragmentation that satisfies the hypotheses above. The idea is as follows: fix an integer N and $0 < \varepsilon < 1$. Consider the subtree $\mathcal{T}_F^{N,\varepsilon} \subset \mathcal{T}_F$ constructed from $\mathcal{T}_F$ by keeping, at each branchpoint, the N largest fringe subtrees rooted at this branchpoint (that is the subtrees with the largest masses) and discarding the others in order to yield a tree in which branchpoints have out-degree at most N. Also, we remove the accumulation of fragmentation times by discarding all the fringe subtrees rooted at the branchpoints but the largest one, as soon as the proportion of its mass compared to the others is larger than $1-\varepsilon$. Then there exists a probability $\mu_F^{N,\varepsilon}$ such that $(\mathcal{T}_F^{N,\varepsilon}, \mu_F^{N,\varepsilon})$ is a CRT, to which we will apply the energy method.

Let us make the definition precise. Define $\mathcal{L}^{N,\varepsilon} \subset \mathcal{L}(\mathcal{T}_F)$ to be the set of leaves L such that for every branchpoint $b \in [[\varnothing, L]]$, $L \in \mathcal{F}_b^{N,\varepsilon}$ with $\mathcal{F}_b^{N,\varepsilon}$ defined by

$$\begin{cases} \mathcal{F}_b^{N,\varepsilon} = \mathcal{T}_b^1 \cup \ldots \cup \mathcal{T}_b^N & \text{if } \mu_F(\mathcal{T}_b^1)/\mu_F\left(\bigcup_{i\geq 1} \mathcal{T}_b^i\right) \leq 1-\varepsilon \\ \mathcal{F}_b^{N,\varepsilon} = \mathcal{T}_b^1 & \text{if } \mu_F(\mathcal{T}_b^1)/\mu_F\left(\bigcup_{i\geq 1} \mathcal{T}_b^i\right) > 1-\varepsilon \end{cases}, \tag{12}$$

where $\mathcal{T}_b^1, \mathcal{T}_b^2 \ldots$ are the connected components of the fringe subtree of $\mathcal{T}_F$ rooted at b, from whom b has been removed (the connected components of $\{v \in \mathcal{T}_F : \mathrm{ht}(v) > b\}$) and ranked in decreasing order of μ_F-mass. Then let $\mathcal{T}_F^{N,\varepsilon} \subset \mathcal{T}_F$ be the subtree of $\mathcal{T}_F$ spanned by the root and the leaves of $\mathcal{L}^{N,\varepsilon}$, i.e.

$$\mathcal{T}_F^{N,\varepsilon} = \{v \in \mathcal{T}_F : \exists L \in \mathcal{L}^{N,\varepsilon}, v \in [[\varnothing, L]]\}.$$

The set $\mathcal{T}_F^{N,\varepsilon} \subset \mathcal{T}_F$ is plainly connected and closed in $\mathcal{T}_F$, thus an $\mathbb{R}$-tree.

Now let us try to give a sense to "taking at random a leaf in $\mathcal{T}_F^{N,\varepsilon}$". In the case of $\mathcal{T}_F$, it was easy because, from the partition-valued fragmentation Π, it sufficed to look at the fragment containing 1 (or some prescribed integer). Here, it is not difficult to show (as we will see later) that the corresponding leaf L_1 a.s. never belongs to $\mathcal{T}_F^{N,\varepsilon}$ when the dislocation measure ν charges the set $\{s_1 > 1-\varepsilon\} \cup \{s_{N+1} > 0\}$. Therefore, we will have to use several random leaves of $\mathcal{T}_F$. For any leaf $L \in \mathcal{L}(\mathcal{T}_F) \setminus \mathcal{L}(\mathcal{T}_F^{N,\varepsilon})$ let $b(L)$ be the highest vertex v of $[[\varnothing, L]]$ such that $v \in \mathcal{T}_F^{N,\varepsilon}$. Call it the branchpoint of L and $\mathcal{T}_F^{N,\varepsilon}$.

Now take at random a leaf Z_1 of $\mathcal{T}_F$ with law μ_F conditionally on μ_F, and define recursively a sequence $(Z_n, n \geq 1)$ with values in $\mathcal{T}_F$ as follows. Let Z_{n+1} be independent of $Z_1, \ldots, Z_n$

conditionally on $(\mathcal{T}_F, \mu_F, b(Z_n))$, and take it with conditional law

$$P(Z_{n+1} \in \cdot | \mathcal{T}_F, \mu_F, b(Z_n)) = \mu_F(\cdot \cap \mathcal{F}^{N,\varepsilon}_{b(Z_n)}) / \mu_F(\mathcal{F}^{N,\varepsilon}_{b(Z_n)}).$$

Lemma 7. *Almost surely, the sequence $(Z_n, n \geq 1)$ converges to a random leaf $Z^{N,\varepsilon} \in \mathcal{L}(\mathcal{T}_F^{N,\varepsilon})$. If $\mu_F^{N,\varepsilon}$ denotes the conditional law of $Z^{N,\varepsilon}$ given $(\mathcal{T}_F, \mu_F)$, then $(\mathcal{T}_F^{N,\varepsilon}, \mu_F^{N,\varepsilon})$ is a CRT, provided ε is small enough.*

To prove this and for later use we first reconnect this discussion to partition-valued fragmentations. Recall from Sect. 2.1 the construction of the homogeneous fragmentation Π^0 with characteristics $(0, 0, \nu)$ out of a $\mathcal{P}_\infty \times \mathbb{N}$-valued Poisson point process $((\Delta_t, k_t), t \geq 0)$ with intensity $\kappa_\nu \otimes \#$. For any partition $\pi \in \mathcal{P}_\infty$ that admits asymptotic frequencies whose ranked sequence is $\mathbf{s}$, write $\pi_i^\downarrow$ for the block of π with asymptotic frequency s_i (with some convention for ties, e.g. taking the order of least element). We define a function $\mathtt{GRIND}^{N,\varepsilon} : \mathcal{P}_\infty \to \mathcal{P}_\infty$ that reduces the smallest blocks of the partition to singletons as follows. If π does not admit asymptotic frequencies, let $\mathtt{GRIND}^{N,\varepsilon}(\pi) = \pi$, else let

$$\mathtt{GRIND}^{N,\varepsilon}(\pi) = \begin{cases} \left(\pi_1^\downarrow, ..., \pi_N^\downarrow, \text{singletons}\right) & \text{if } s_1 \leq 1 - \varepsilon \\ \left(\pi_1^\downarrow, \text{singletons}\right) & \text{if } s_1 > 1 - \varepsilon. \end{cases}$$

Now for each $t \geq 0$ write $\Delta_t^{N,\varepsilon} = \mathtt{GRIND}^{N,\varepsilon}(\Delta_t)$, so $((\Delta_t^{N,\varepsilon}, k_t), t \geq 0)$ is a $\mathcal{P}_\infty \times \mathbb{N}$-valued Poisson point process with intensity measure $\kappa_{\nu^{N,\varepsilon}} \otimes \#$, where $\nu^{N,\varepsilon}$ is the image of ν by the function

$$\mathbf{s} \in S \mapsto \begin{cases} (s_1, ..., s_N, 0, ...) \text{ if } s_1 \leq 1 - \varepsilon \\ (s_1, 0, ...) \text{ if } s_1 > 1 - \varepsilon. \end{cases}$$

From this Poisson point process we construct first a version $\Pi^{0,N,\varepsilon}$ of the $\left(0, 0, \nu^{N,\varepsilon}\right)$ fragmentation, as explained in Section 2.1. For every time $t \geq 0$, the partition $\Pi^{0,N,\varepsilon}(t)$ is finer than $\Pi^0(t)$ and the blocks of $\Pi^{0,N,\varepsilon}(t)$ non-reduced to singleton are blocks of $\Pi^0(t)$. Next, using the time change (3), we construct from $\Pi^{0,N,\varepsilon}$ a version of the $\left(\alpha, 0, \nu^{N,\varepsilon}\right)$ fragmentation, that we denote by $\Pi^{N,\varepsilon}$.

Note that for dislocation measures ν such that $\nu^{N,\varepsilon}\left(\sum s_i < 1\right) = 0$, Theorem 2 is already proved, by the previous subsection. For the rest of this subsection and next subsection, we shall thus focus on dislocation measures ν such that $\nu^{N,\varepsilon}\left(\sum s_i < 1\right) > 0$. In that case, in $\Pi^{0,N,\varepsilon}$ (unlike for Π^0) each integer i is eventually isolated in a singleton a.s. within a sudden break and this is why a μ_F-sampled leaf on $\mathcal{T}_F$ cannot be in $\mathcal{T}_F^{N,\varepsilon}$, in other words, μ_F and $\mu_F^{N,\varepsilon}$ are a.s. singular. Recall that we may build $\mathcal{T}_F$ together with an exchangeable μ_F-sample of leaves $L_1, L_2, \ldots$ on the same probability space as Π (or Π^0). We are going to use a sub-family of $(L_1, L_2, \ldots)$ to build a sequence with the same law as $(Z_n, n \geq 1)$ built above. Let $i_1 = 1$ and

$$i_{n+1} = \inf\{i > i_n : L_{i_{n+1}} \in \mathcal{F}^{N,\varepsilon}_{b(L_{i_n})}\}.$$

It is easy to see that $(L_{i_n}, n \geq 1)$ has the same law as $(Z_n, n \geq 1)$. From this, we build a decreasing family of blocks $B^{0,N,\varepsilon}(t) \in \Pi^0(t)$, $t \geq 0$, by letting $B^{0,N,\varepsilon}(t)$ be the unique block of $\Pi^0(t)$ that contains all but a finite number of elements of $\{i_1, i_2, \ldots\}$.

Here is a useful alternative description of $B^{0,N,\varepsilon}(t)$. Let $D_i^{0,N,\varepsilon}$ be the death time of i for the fragmentation $\Pi^{0,N,\varepsilon}$ that is

$$D_i^{0,N,\varepsilon} = \inf\{t \geq 0 : \{i\} \in \Pi^{0,N,\varepsilon}(t)\}.$$

By exchangeability the $D_i^{0,N,\varepsilon}$'s are identically distributed and $D_1^{0,N,\varepsilon} = \inf\{t \geq 0 : k_t = 1 \text{ and } \{1\} \in \Delta_t^{N,\varepsilon}\}$ so it has an exponential law with parameter $\int_S (1 - \sum_i s_i)\nu^{N,\varepsilon}(\mathrm{d}\mathbf{s})$. Then notice that $B^{0,N,\varepsilon}(t)$ is the block admitting i_n as least element when $D_{i_n}^{0,N,\varepsilon} \leq t < D_{i_{n+1}}^{0,N,\varepsilon}$. Indeed, by construction we have

$$i_{n+1} = \inf\{i \in B^{0,N,\varepsilon}(D_{i_n}^{0,N,\varepsilon}-) : \{i\} \notin \Pi^{0,N,\varepsilon}(D_{i_n}^{0,N,\varepsilon})\}.$$

Moreover, the asymptotic frequency $\lambda_1^{0,N,\varepsilon}(t)$ of $B^{0,N,\varepsilon}(t)$ exists for every t and equals the μ_F-mass of the tree component of $\{v \in \mathcal{T}_F : \mathrm{ht}(v) > t\}$ containing L_{i_n} for $D_{i_n}^{0,N,\varepsilon} \leq t < D_{i_{n+1}}^{0,N,\varepsilon}$.

Notice that at time $D_{i_n}^{0,N,\varepsilon}$, either one non-singleton block coming from $B^{0,N,\varepsilon}(D_{i_n}^{0,N,\varepsilon}-)$, or up to N non-singleton blocks may appear; by Lemma 1, $B^{0,N,\varepsilon}(D_{i_n}^{0,N,\varepsilon})$ is then obtained by taking at random one of these blocks with probability proportional to its size.

Proof of Lemma 7. For $t \geq 0$ let $\lambda^{0,N,\varepsilon}(t) = |B^{0,N,\varepsilon}(t)|$ and

$$T^{0,N,\varepsilon}(t) := \inf\left\{u \geq 0 : \int_0^u \left(\lambda^{0,N,\varepsilon}(r)\right)^{-\alpha} \mathrm{d}r > t\right\} \tag{13}$$

and write $B^{N,\varepsilon}(t) := B^{0,N,\varepsilon}(T^{0,N,\varepsilon}(t))$, for $T^{0,N,\varepsilon}(t) < \infty$ and $B^{N,\varepsilon}(t) = \varnothing$ otherwise, so for all $t \geq 0$, $B^{N,\varepsilon}(t) \in \Pi^{N,\varepsilon}(t)$. Let also $D_{i_n}^{N,\varepsilon} := T^{0,N,\varepsilon}(D_{i_n}^{0,N,\varepsilon})$ be the death time of i_n in the fragmentation $\Pi^{N,\varepsilon}$. It is easy to see that $b_n = b(L_{i_n})$ is the branchpoint of the paths $[[\varnothing, L_{i_n}]]$ and $[[\varnothing, L_{i_{n+1}}]]$, so the path $[[\varnothing, b_n]]$ has length $D_{i_n}^{N,\varepsilon}$. The "edges" $[[b_n, b_{n+1}]]$, $n \in \mathbb{N}$, have respective lengths $D_{i_{n+1}}^{N,\varepsilon} - D_{i_n}^{N,\varepsilon}$, $n \in \mathbb{N}$. Since the sequence of death times $(D_{i_n}^{N,\varepsilon}, n \geq 1)$ is increasing and bounded by τ (the first time at which Π is entirely reduced to singletons), the sequence $(b_n, n \geq 1)$ is Cauchy, so it converges by completeness of $\mathcal{T}_F$. Now it is easy to show that $D_{i_n}^{0,N,\varepsilon} \to \infty$ as $n \to \infty$ a.s., so $\lambda^{0,N,\varepsilon}(t) \to 0$ as $t \to \infty$ a.s. (see also the next lemma). Therefore, the fragmentation property implies $d(L_{i_n}, b_n) \to 0$ a.s. so L_{i_n} is also Cauchy, with the same limit, and the limit has to be a leaf which we denote $L^{N,\varepsilon}$ (of course it has same distribution as the $Z^{N,\varepsilon}$ of the lemma's statement). The fact that $L^{N,\varepsilon} \in \mathcal{T}_F^{N,\varepsilon}$ a.s. is obtained by checking (12), which is true since it is verified for each branchpoint $b \in [[\varnothing, b_n]]$ for every $n \geq 1$ by construction.

We now sketch the proof that $(\mathcal{T}_F^{N,\varepsilon}, \mu_F^{N,\varepsilon})$ is indeed a CRT, leaving details to the reader. We need to show non-atomicity of $\mu_F^{N,\varepsilon}$, but it is clear that when performing the recursive construction of $Z^{N,\varepsilon}$ twice with independent variables, $(Z_n, n \geq 1)$ and $(Z'_n, n \geq 1)$ say, there exists a.s. some n such that Z_n and Z'_n end up in two different fringe subtrees rooted at some of the branchpoints b_n, provided that ε is small enough so that $\nu(1 - s_1 \geq \varepsilon) \neq 0$ (see also below the explicit construction of two independently $\mu_F^{N,\varepsilon}$-sampled leaves). On the other hand, all of the subtrees of $\mathcal{T}_F$ rooted at the branchpoints of $\mathcal{T}_F^{N,\varepsilon}$ have positive μ_F-mass, so they will end up being visited by the intermediate leaves used to construct a $\mu_F^{N,\varepsilon}$-i.i.d. sample, so the condition $\mu_F^{N,\varepsilon}(\{v \in \mathcal{T}_F^{N,\varepsilon} : [[\varnothing, v]] \cap [[\varnothing, w]] = [[\varnothing, w]]\}) > 0$ for every $w \in \mathcal{S}(\mathcal{T}_F^{N,\varepsilon})$ is satisfied. $\square$

It will also be useful to sample two leaves $(L_1^{N,\varepsilon}, L_2^{N,\varepsilon})$ that are independent with same distribution $\mu_F^{N,\varepsilon}$ conditionally on $\mu_F^{N,\varepsilon}$ out of the exchangeable family $L_1, L_2, \ldots$. A natural way to do this is to use the family $(L_1, L_3, L_5, \ldots)$ to sample the first leaf in the same way as above, and to use the family $(L_2, L_4, \ldots)$ to sample the other one. That is, let $j_1^1 = 1, j_1^2 = 2$ and define recursively $(j_n^1, j_n^2, n \geq 1)$ by letting

$$\left\{\begin{array}{l} j_{n+1}^1 = \inf\{j \in 2\mathbb{N}+1, j > j_n^1 : L_j \in \mathcal{F}^{N,\varepsilon}_{b(L_{j_n^1})}\} \\ j_{n+1}^2 = \inf\{j \in 2\mathbb{N}, j > j_{n+1}^1 : L_j \in \mathcal{F}^{N,\varepsilon}_{b(L_{j_n^2})}\} \end{array}\right. .$$

It is easy to check that $(L_{j_n^1}, n \geq 1)$ and $(L_{j_n^2}, n \geq 1)$ are two independent sequences distributed as $(Z_1, Z_2, \ldots)$ of Lemma 7. Therefore, these sequences a.s. converge to limits $L_1^{N,\varepsilon}, L_2^{N,\varepsilon}$, and these are independent with law $\mu_F^{N,\varepsilon}$ conditionally on $\mu_F^{N,\varepsilon}$. We let $\mathcal{D}_k = \mathrm{ht}(L_k^{N,\varepsilon})$, $k = 1, 2$.

Similarly as above, for every $t \geq 0$ we let $B_k^{0,N,\varepsilon}(t)$, $k = 1, 2$ (resp. $B_k^{N,\varepsilon}(t)$) be the block of $\Pi^0(t)$ (resp. $\Pi(t)$) that contains all but the first few elements of $\{j_1^k, j_2^k, \ldots\}$, and we call $\lambda_k^{0,N,\varepsilon}(t)$ (resp. $\lambda_k^{N,\varepsilon}(t)$) its asymptotic frequency. Last, let $\mathcal{D}^0_{\{1,2\}} = \inf\{t \geq 0 : B_1^{0,N,\varepsilon}(t) \cap B_2^{0,N,\varepsilon}(t) = \varnothing\}$ (and define similarly $\mathcal{D}_{\{1,2\}}$). Notice that for $t < \mathcal{D}^0_{\{1,2\}}$, we have $B_1^{0,N,\varepsilon}(t) = B_2^{0,N,\varepsilon}(t)$, and by construction the two least elements of the blocks $(2\mathbb{N}+1) \cap B_1^{0,N,\varepsilon}(t)$ and $(2\mathbb{N}) \cap B_1^{0,N,\varepsilon}(t)$ are of the form j_n^1, j_m^2 for some n, m. On the other hand, for $t \geq \mathcal{D}^0_{\{1,2\}}$, we have $B_1^{0,N,\varepsilon}(t) \cap B_2^{0,N\varepsilon}(t) = \varnothing$, and again the least elements of $(2\mathbb{N}+1) \cap B_1^{0,N,\varepsilon}(t)$ and $(2\mathbb{N}) \cap B_2^{0,N\varepsilon}(t)$ are of the form j_n^1, j_m^2 for some n, m. In any case, we let $j^1(t) = j_n^1, j^2(t) = j_m^2$ for these n, m.

3.4 Lower bound

Since $\mu_F^{N,\varepsilon}$ is a measure on $\mathcal{L}(\mathcal{T}_F)$, we want to show that for every $a < \varrho$, the integral $\int_{\mathcal{T}_F^{N,\varepsilon}} \int_{\mathcal{T}_F^{N,\varepsilon}} \frac{\mu_F^{N,\varepsilon}(\mathrm{d}x)\mu_F^{N,\varepsilon}(\mathrm{d}y)}{d(x,y)^{a/|\alpha|}}$ is a.s. finite for suitable N and ε. So consider $a < \varrho$, and note that

$$E\left[\int_{\mathcal{T}_F^{N,\varepsilon}} \int_{\mathcal{T}_F^{N,\varepsilon}} \frac{\mu_F^{N,\varepsilon}(\mathrm{d}x)\mu_F^{N,\varepsilon}(\mathrm{d}y)}{d(x,y)^{a/|\alpha|}}\right] = E\left[\frac{1}{d(L_1^{N,\varepsilon}, L_2^{N,\varepsilon})^{a/|\alpha|}}\right],$$

where $d(L_1^{N,\varepsilon}, L_2^{N,\varepsilon}) = \mathcal{D}_1 + \mathcal{D}_2 - 2\mathcal{D}_{\{1,2\}}$, with notations above. The fragmentation property at the stopping time $\mathcal{D}_{\{1,2\}}$ leads to

$$\mathcal{D}_k = \mathcal{D}_{\{1,2\}} + \lambda_k^{N,\varepsilon}(\mathcal{D}_{\{1,2\}})^{|\alpha|}\widetilde{\mathcal{D}}_k,\ k = 1, 2,$$

where $\widetilde{\mathcal{D}}_1, \widetilde{\mathcal{D}}_2$ are independent with the same distribution as $\mathcal{D}$, the height of the leaf $L^{N,\varepsilon}$ constructed above, and independent of $\mathcal{G}(\mathcal{D}_{\{1,2\}})$. Therefore, the distance $d(L_1^{N,\varepsilon}, L_2^{N,\varepsilon})$ can be rewritten as

$$d(L_1^{N,\varepsilon}, L_2^{N,\varepsilon}) = \left(\lambda_1^{N,\varepsilon}(\mathcal{D}_{\{1,2\}})\right)^{|\alpha|}\widetilde{\mathcal{D}}_1 + \left(\lambda_2^{N,\varepsilon}(\mathcal{D}_{\{1,2\}})\right)^{|\alpha|}\widetilde{\mathcal{D}}_2$$

and

$$E\left[d(L_1^{N,\varepsilon}, L_2^{N,\varepsilon})^{-a/|\alpha|}\right] \leq 2E\left[\left(\lambda_1^{N,\varepsilon}(\mathcal{D}_{\{1,2\}})\right)^{-a}; \lambda_1^{N,\varepsilon}(\mathcal{D}_{\{1,2\}}) \geq \lambda_2^{N,\varepsilon}(\mathcal{D}_{\{1,2\}})\right] E\left[\mathcal{D}^{-a/|\alpha|}\right].$$

Therefore, that $\dim_{\mathcal{H}}(\mathcal{L}(\mathcal{T}_F)) \geq \varrho/|\alpha|$ is directly implied by the following Lemmas 8 and 10.

Lemma 8. *The quantity $E[\mathcal{D}^{-\gamma}]$ is finite for every $0 \leq \gamma \leq \varrho / |\alpha|$.*

The proof uses the following technical lemma. Recall that $\lambda^{N,\varepsilon}(t) = |B^{N,\varepsilon}(t)|$.

Lemma 9. *One can write $\lambda^{N,\varepsilon} = \exp\left(-\xi_{\rho(\cdot)}\right)$, where ξ (tacitly depending on N, ε) is a subordinator with Laplace exponent*

$$\Phi_\xi(q) = \int_S \left((1 - s_1^q)\, \mathbb{1}_{\{s_1 > 1-\varepsilon\}} + \sum_{i=1}^N (1 - s_i^q) \frac{s_i \mathbb{1}_{\{s_1 \leq 1-\varepsilon\}}}{s_1 + \ldots + s_N} \right) \nu(\mathrm{d}\mathbf{s}), \;\; q \geq 0, \tag{14}$$

and ρ is the time-change

$$\rho(t) = \inf\left\{ u \geq 0 : \int_0^u \exp(\alpha \xi_r) \mathrm{d}r > t \right\}, \;\; t \geq 0.$$

Proof. Recall the construction of the process $B^{0,N,\varepsilon}$ from Π^0, which itself was constructed from a Poisson process $(\Delta_t, k_t, t \geq 0)$. From the definition of $B^{0,N,\varepsilon}(t)$, we have

$$B^{0,N,\varepsilon}(t) = \bigcap_{0 \leq s \leq t} \bar{\Delta}_s^{N,\varepsilon},$$

where the sets $\bar{\Delta}_s^{N,\varepsilon}$ are defined as follows. For each $s \geq 0$, let $i(s)$ be the least element of the block $B^{0,N,\varepsilon}(s-)$ (so that $B^{0,N,\varepsilon}(s-) = \Pi^0_{i(s)}(s-)$), so $(i(s), s \geq 0)$ is an $(\mathcal{F}(s-), s \geq 0)$-adapted jump-hold process, and the process $\{\Delta_s : k_s = i(s), s \geq 0\}$ is a Poisson point process with intensity κ_ν. Then for each s such that $k_s = i(s)$, $\bar{\Delta}_s^{N,\varepsilon}$ consists in a certain block of Δ_s, and precisely, $\bar{\Delta}_s^{N,\varepsilon}$ is the block of Δ_s containing

$$\inf\left\{ i \in B^{0,N,\varepsilon}(s-) : \{i\} \notin \Delta_s^{N,\varepsilon} \right\},$$

the least element of $B^{0,N,\varepsilon}(s-)$ which is not isolated in a singleton of $\Delta_s^{N,\varepsilon}$ (such an integer must be of the form i_n for some n by definition). Now $B^{0,N,\varepsilon}(s-)$ is $\mathcal{F}(s-)$-measurable, hence independent of Δ_s. By Lemma 1, $\bar{\Delta}_s^{N,\varepsilon}$ is thus a size-biased pick among the non-void blocks of $\Delta_s^{N,\varepsilon}$, and by definition of the function $\mathtt{GRIND}^{N,\varepsilon}$, the process $(|\bar{\Delta}_s^{N,\varepsilon}|, s \geq 0)$ is a $[0,1]$-valued Poisson point process with intensity $\omega(s)$ characterized by

$$\int_{[0,1]} f(s) \omega(\mathrm{d}s) = \int_S \left(\mathbb{1}_{\{s_1 > 1-\varepsilon\}} f(s_1) + \mathbb{1}_{\{s_1 \leq 1-\varepsilon\}} \sum_{i=1}^N f(s_i) \frac{s_i}{s_1 + \ldots + s_N} \right) \nu(\mathrm{d}\mathbf{s}),$$

for every positive measurable function f. Then $|B^{0,N,\varepsilon}(t)| = \prod_{0 \leq s \leq t} |\bar{\Delta}_s^{N,\varepsilon}|$ a.s. for every $t \geq 0$. To see this, denote for every $k \geq 1$ by $\Delta_{s_1}^{N,\varepsilon,k}, \Delta_{s_2}^{N,\varepsilon,k}, \ldots$ the atoms $\Delta_s^{N,\varepsilon}$, $s \leq t$, such that $|\Delta_s^{N,\varepsilon}|_1 \in [1 - k^{-1}, 1 - (k+1)^{-1})$. Complete this a.s. finite sequence of partitions by partitions $\mathbf{1}$ and call $\Gamma^{(k)}$ their intersection, i.e. $\Gamma^{(k)} := \bigcap_{i \geq 1} (\Delta_{s_i}^{N,\varepsilon,k})$. By Lemma 2, $|\Gamma^{(k)}_{n_k}| \overset{a.s.}{=} \prod_{i \geq 1} |\overline{\Delta}_{s_i}^{N,\varepsilon,k}|$, where n_k is the index of the block $\bigcap_{i \geq 1} \overline{\Delta}_{s_i}^{N,\varepsilon,k}$ in the partition $\Gamma^{(k)}$. These partitions $\Gamma^{(k)}$, $k \geq 1$, are exchangeable and clearly independent. Applying again Lemma 2 gives $|\bigcap_{k \geq 1} \Gamma^{(k)}_{n_k}| \overset{a.s.}{=} \prod_{k \geq 1} \prod_{i \geq 1} |\overline{\Delta}_{s_i}^{N,\varepsilon,k}|$, which is exactly the equality mentioned above. The exponential formula for Poisson processes then shows that $(\xi_t, t \geq 0) = (-\log(\lambda^{0,N,\varepsilon}(t)), t \geq 0)$ is a subordinator with Laplace exponent Φ_ξ. The result is now obtained by noticing that (3) rewrites $\lambda^{N,\varepsilon}(t) = \lambda^{0,N,\varepsilon}(\rho(t))$ in our setting. □

Proof of Lemma 8. By the previous lemma, $\mathcal{D} = \inf\{t \geq 0 : \lambda^{N,\varepsilon}(t) = 0\}$, which equals $\int_0^\infty \exp(\alpha\xi_t)\mathrm{d}t$ by the definition of ρ. According to Theorem 25.17 in [25], if for some positive γ the quantity

$$\Phi_\xi(-\gamma) := \int_S \left(\left(1 - s_1^{-\gamma}\right) \mathbb{1}_{\{s_1 > 1-\varepsilon\}} + \sum_{i=1}^N \left(1 - s_i^{-\gamma}\right) \frac{s_i \mathbb{1}_{\{s_i > 0\}} \mathbb{1}_{\{s_1 \leq 1-\varepsilon\}}}{s_1 + \ldots + s_N} \right) \nu(\mathrm{d}\mathbf{s})$$

is finite, then $E[\exp(\gamma\xi_t)] < \infty$ for all $t \geq 0$ and it equals $\exp(-t\Phi_\xi(-\gamma))$. Notice that $\Phi_\xi(-\gamma) > -\infty$ for $\gamma < \varrho \leq 1$. Indeed for such γ's, $\int_S \left(s_1^{-\gamma} - 1\right) \mathbb{1}_{\{s_1 > 1-\varepsilon\}} \nu(\mathrm{d}\mathbf{s}) < \infty$ by definition and

$$\int_S \left(\sum_{i=1}^N \left(s_i^{1-\gamma} - s_i\right) \frac{\mathbb{1}_{\{s_1 \leq 1-\varepsilon\}}}{s_1 + \ldots + s_N} \right) \nu(\mathrm{d}\mathbf{s}) \leq N \int_S \frac{s_1^{1-\gamma} \mathbb{1}_{\{s_1 \leq 1-\varepsilon\}}}{s_1} \nu(\mathrm{d}\mathbf{s}),$$

which is finite by the definition of ϱ and since ν integrates $(1 - s_1)$. This implies in particular that ξ_t has finite expectation for every t, and it follows by [11] that $E[\mathcal{D}^{-1}] < \infty$. Then, following the proof of Proposition 2 in [9] and using again that $\Phi_\xi(-\gamma) > -\infty$ for $\gamma < \varrho$,

$$E\left[\left(\int_0^\infty \exp(\alpha\xi_t)\mathrm{d}t \right)^{-k-1} \right] = \frac{-\Phi_\xi(-|\alpha|\, k)}{k} E\left[\left(\int_0^\infty \exp(\alpha\xi_t)\mathrm{d}t \right)^{-k} \right]$$

for every integer $k < \varrho / |\alpha|$. Hence, using induction, $E[(\int_0^\infty \exp(\alpha\xi_t))^{-k-1}]$ is finite for $k = [\varrho/|\alpha|]$ if $\varrho/|\alpha| \notin \mathbb{N}$ and for $k = \varrho/|\alpha| - 1$ otherwise. In both cases, we see that $E[\mathcal{D}^{-\gamma}] < \infty$ for every $\gamma \leq \varrho/|\alpha|$. □

Lemma 10. *For any $a < \varrho$, there exists N, ε such that*

$$E\left[\left(\lambda_1^{N,\varepsilon}(\mathcal{D}_{\{1,2\}}) \right)^{-a} ; \lambda_1^{N,\varepsilon}(\mathcal{D}_{\{1,2\}}) \geq \lambda_2^{N,\varepsilon}(\mathcal{D}_{\{1,2\}}) \right] < \infty.$$

The ingredient for proving Lemma 10 is the following lemma, which uses the notations around the construction of the leaves $(L_1^{N,\varepsilon}, L_2^{N,\varepsilon})$.

Lemma 11. *With the convention $\log(0) = -\infty$, the process*

$$\sigma(t) = -\log \left| B_1^{0,N,\varepsilon}(t) \cap B_2^{0,N,\varepsilon}(t) \right| \quad , \quad t \geq 0$$

is a killed subordinator (its death time is $\mathcal{D}^0_{\{1,2\}}$) with Laplace exponent

$$\Phi_\sigma(q) = \mathtt{k}^{N,\varepsilon} + \int_S \left((1 - s_1^q)\, \mathbb{1}_{\{s_1 > 1-\varepsilon\}} + \sum_{i=1}^N (1 - s_i^q) \frac{s_i^2 \mathbb{1}_{\{s_1 \leq 1-\varepsilon\}}}{(s_1 + \ldots + s_N)^2} \right) \nu(\mathrm{d}\mathbf{s}), \ q \geq 0, \qquad (15)$$

where the killing rate $\mathtt{k}^{N,\varepsilon} := \int_S \sum_{i \neq j} s_i s_j \frac{\mathbb{1}_{\{s_1 \leq 1-\varepsilon\}}}{(s_1 + \ldots + s_N)^2} \nu(\mathrm{d}\mathbf{s}) \in (0, \infty)$. Moreover, the pair

$$(l_1^{N,\varepsilon}, l_2^{N,\varepsilon}) = \exp(\sigma(\mathcal{D}^0_{\{1,2\}}-))(\lambda_1^{0,N,\varepsilon}(\mathcal{D}^0_{\{1,2\}}), \lambda_2^{0,N,\varepsilon}(\mathcal{D}^0_{\{1,2\}}))$$

is independent of $\sigma(\mathcal{D}^0_{\{1,2\}}-)$ with law characterized by

$$E\left[f\left(l_1^{N,\varepsilon}, l_2^{N,\varepsilon}\right)\right] = \frac{1}{\mathtt{k}^{N,\varepsilon}} \int_S \sum_{1\le i\ne j\le N} f(s_i, s_j) \frac{s_i s_j \mathbb{1}_{\{s_1\le 1-\varepsilon\}} \mathbb{1}_{\{s_i>0\}} \mathbb{1}_{\{s_j>0\}}}{(s_1+...+s_N)^2} \nu(\mathrm{d}\mathbf{s})$$

for any positive measurable function f.

Proof. We again use the Poisson construction of Π^0 out of $(\Delta_t, k_t, t\ge 0)$ and follow closely the proof of Lemma 9. For every $t\ge 0$ we have

$$B_k^{0,N,\varepsilon}(t) = \bigcap_{0\le s\le t} \bar{\Delta}_s^k \quad , \quad k=1,2,$$

where $\bar{\Delta}_s^k$ is defined as follows. Let $J^k(s), k=1,2$ be the integers such that $B_k^{0,N,\varepsilon}(s-) = \Pi^0_{J^k(s)}(s-)$, so $\{\Delta_s : k_s = J^k(s), s\ge 0\}$, $k=1,2$ are two Poisson processes with same intensity κ_ν, which are equal for s in the interval $[0, \mathcal{D}^0_{\{1,2\}})$. Then for s with $k_s = J^k(s)$, let $\bar{\Delta}_s^k$ be the block of Δ_s containing $j^k(s)$. If $B_1^{0,N,\varepsilon}(s-) = B_2^{0,N,\varepsilon}(s-)$ notice that $j^1(s), j^2(s)$ are the two least integers of $(2\mathbb{N}+1)\cap B_1^{0,N,\varepsilon}(s-)$ and $(2\mathbb{N})\cap B_2^{0,N,\varepsilon}(s-)$ respectively that are not isolated as singletons of $\Delta_s^{N,\varepsilon}$, so $\bar{\Delta}_s^1 = \bar{\Delta}_s^2$ if these two integers fall in the same block of $\Delta_s^{N,\varepsilon}$. Hence by a variation of Lemma 1, $(|\bar{\Delta}_s^1\cap\bar{\Delta}_s^2|, s\ge 0)$ is a Poisson process whose intensity is the image measure of $\kappa_{\nu^{N,\varepsilon}}(\pi\mathbb{1}_{\{1\sim 2\}})$ by the map $\pi\mapsto|\pi|$, and killed at an independent exponential time (namely $\mathcal{D}^0_{\{1,2\}}$) with parameter $\kappa_{\nu^{N,\varepsilon}}(1\nsim 2)$ (here $1\sim 2$ means that 1 and 2 are in the same block of π). This implies (15).

The time $\mathcal{D}^0_{\{1,2\}}$ is the first time when the two considered integers fall into two distinct blocks of $\Delta_s^{N,\varepsilon}$. It is then easy by the Poissonian construction and the paintbox representation to check that these blocks have asymptotic frequencies $(l_1^{N,\varepsilon}, l_2^{N,\varepsilon})$ which are independent of $\sigma(\mathcal{D}^0_{\{1,2\}}-)$, and have the claimed law. □

Proof of Lemma 10. First notice, from the fact that self-similar fragmentations are time-changed homogeneous fragmentations, that

$$(\lambda_1^{N,\varepsilon}(\mathcal{D}_{\{1,2\}}), \lambda_2^{N,\varepsilon}(\mathcal{D}_{\{1,2\}})) \stackrel{d}{=} (\lambda_1^{0,N,\varepsilon}(\mathcal{D}^0_{\{1,2\}}), \lambda_2^{0,N,\varepsilon}(\mathcal{D}^0_{\{1,2\}})).$$

Thus, with the notations of Lemma 11,

$$\begin{aligned} &E\left[\left(\lambda_1^{N,\varepsilon}(\mathcal{D}_{\{1,2\}})\right)^{-a}; \lambda_1^{N,\varepsilon}(\mathcal{D}_{\{1,2\}}) \ge \lambda_2^{N,\varepsilon}(\mathcal{D}_{\{1,2\}})\right] \\ &\quad = E\left[\exp(a\sigma(\mathcal{D}^0_{\{1,2\}}-))\right] E\left[\left(l_1^{N,\varepsilon}\right)^{-a}; l_1^{N,\varepsilon}\ge l_2^{N,\varepsilon}\right]. \end{aligned}$$

First, define for every $a>0$ $\Phi_\sigma(-a)$ by replacing q by $-a$ in (15) and then remark that $\Phi_\sigma(-a) > -\infty$ when $a<\varrho$. Indeed, $\int_S \left(s_1^{-a}-1\right)\mathbb{1}_{\{s_1>1-\varepsilon\}}\nu(\mathrm{d}\mathbf{s})$ is then finite and, since $\sum_{1\le i\le N} s_i^{2-a} \le \left(\sum_{1\le i\le N} s_i\right)^{2-a}$ $(2-a\ge 1)$,

$$\sum_{1\le i\le N}\left(s_i^{2-a}-s_i^2\right)\frac{\mathbb{1}_{\{s_1\le 1-\varepsilon\}}}{(s_1+...+s_N)^2} \le \frac{\mathbb{1}_{\{s_1\le 1-\varepsilon\}}}{s_1^a}$$

which, by assumption, is integrable with respect to ν. Then, consider a subordinator $\widetilde{\sigma}$ with Laplace transform $\Phi_\sigma - \mathtt{k}^{N,\varepsilon}$ and independent of $\mathcal{D}^0_{\{1,2\}}$, such that $\sigma = \widetilde{\sigma}$ on $(0, \mathcal{D}^0_{\{1,2\}})$. As in the proof of Lemma 8, we use Theorem 25.17 of [25], which gives $E\left[\exp(a\widetilde{\sigma}(t))\right] = \exp\left(-t\left(\Phi_\sigma(-a) - \mathtt{k}^{N,\varepsilon}\right)\right)$ for all $t \geq 0$. Hence, by independence of $\widetilde{\sigma}$ and $\mathcal{D}^0_{\{1,2\}}$,

$$\begin{aligned} E\left[\exp(a\sigma(\mathcal{D}^0_{\{1,2\}}-))\right] &= E\left[\exp(a\widetilde{\sigma}(\mathcal{D}^0_{\{1,2\}}))\right] \\ &= \mathtt{k}^{N,\varepsilon} \int_0^\infty \exp(-t\mathtt{k}^{N,\varepsilon}) \exp\left(-t(\Phi_\sigma(-a) - \mathtt{k}^{N,\varepsilon})\right) \mathrm{d}t, \end{aligned}$$

which is finite if and only if $\Phi_\sigma(-a) > 0$. Recall that $\Phi_\sigma(-a)$ is equal to

$$\int_S \left(1 - s_1^{-a}\right) \mathbb{1}_{\{s_1 > 1-\varepsilon\}} \nu(\mathrm{d}\mathbf{s}) + \int_S \left(\sum_{1 \leq i \neq j \leq N} s_i s_j + \sum_{1 \leq i \leq N} \left(s_i^2 - s_i^{2-a}\right) \right) \frac{\mathbb{1}_{\{s_1 \leq 1-\varepsilon\}}}{(s_1 + \ldots + s_N)^2} \nu(\mathrm{d}\mathbf{s}). \quad (16)$$

Since

$$\sum_{1 \leq i \neq j \leq N} s_i s_j + \sum_{1 \leq i \leq N} (s_i^2 - s_i^{2-a}) = (\sum_{1 \leq i \leq N} s_i)^2 - \sum_{1 \leq i \leq N} s_i^{2-a},$$

the integrand in the second term converges to $\left(1 - \sum_i s_i^{2-a}\right) \mathbb{1}_{\{s_1 \leq 1-\varepsilon\}}$ as $N \to \infty$ and is dominated by $\left(1 + s_1^{-a}\right) \mathbb{1}_{\{s_1 \leq 1-\varepsilon\}}$. So, by dominated convergence, the second term of (16) converges to $\int_S (1 - \sum_i s_i^{2-a}) \mathbb{1}_{\{s_1 \leq 1-\varepsilon\}} \nu(\mathrm{d}\mathbf{s})$ as $N \to \infty$. This last integral converges to a strictly positive quantity as $\varepsilon \downarrow 0$, and since $\int_S \left(1 - s_1^{-a}\right) \mathbb{1}_{\{s_1 > 1-\varepsilon\}} \nu(\mathrm{d}\mathbf{s}) \to 0$ as $\varepsilon \to 0$, $\Phi_\sigma(-a)$ is strictly positive for N and $1/\varepsilon$ large enough. Hence $E[\exp(a\sigma(\mathcal{D}^0_{\{1,2\}}-))] < \infty$ for N and $1/\varepsilon$ large enough.

On the other hand, Lemma 11 implies that the finiteness of $E[(l_1^{N,\varepsilon})^{-a} \mathbb{1}_{\{l_1^{N,\varepsilon} \geq l_2^{N,\varepsilon}\}}]$ is equivalent to that of $\int_S \sum_{1 \leq i \neq j \leq N} s_i^{1-a} s_j \frac{\mathbb{1}_{\{s_1 \leq 1-\varepsilon\}}}{(s_1 + \ldots + s_N)^2} \nu(\mathrm{d}\mathbf{s})$. But this integral is finite for all integers N and every $0 < \varepsilon < 1$, since $\sum_{1 \leq i \neq j \leq N} s_i^{1-a} s_j \leq N^2 s_1^{2-a}$ and ν integrates $s_1^{-a} \mathbb{1}_{\{s_1 \leq 1-\varepsilon\}}$. Hence the result. $\square$

3.5 Dimension of the stable tree

This section is devoted to the proof of Corollary 1. Recall from [23] that the fragmentation F_- associated with the β-stable tree has index $1/\beta - 1$ (where $\beta \in (1, 2]$). In the case $\beta = 2$, the tree is the Brownian CRT and the fragmentation is binary (it is the fragmentation F_B of the Introduction), so that the integrability assumption of Theorem 2 is satisfied and then the dimension is 2. So suppose $\beta < 2$. The main result of [23] is that the dislocation measure $\nu_-(\mathrm{d}\mathbf{s})$ of F_- has the form

$$\nu_-(\mathrm{d}\mathbf{s}) = C(\beta) E\left[T_1; \frac{\Delta T_{[0,1]}}{T_1} \in \mathrm{d}\mathbf{s}\right]$$

for some constant $C(\beta)$, where $(T_x, x \geq 0)$ is a stable subordinator with index $1/\beta$ and $\Delta T_{[0,1]} = (\Delta_1, \Delta_2, \ldots)$ is the decreasing rearrangement of the sequence of jumps of T accomplished within the time-interval $[0, 1]$ (so that $\sum_i \Delta_i = T_1$). By Theorem 2, to prove Corollary 1 it thus suffices to check that $E[T_1(T_1/\Delta_1 - 1)]$ is finite. The problem is that computations involving jumps of subordinators are often quite involved; they are sometimes eased by using size-biased picked jumps, whose laws are more tractable. However, one can check that if Δ_* is a size-biased picked jump among $(\Delta_1, \Delta_2, \ldots)$, the quantity $E[T_1(T_1/\Delta_* - 1)]$ is infinite, therefore we really have to study the joint law of (T_1, Δ_1). This has been done in Perman [24], but we will re-explain all the details we need here.

Recall that the process $(T_x, x \geq 0)$ can be put in the Lévy-Itô form $T_x = \sum_{0\leq y\leq x} \Delta(y)$, where $(\Delta(y), y \geq 0)$ is a Poisson point process with intensity $cu^{-1-1/\beta}\mathrm{d}u$ (the Lévy measure of T) for some constant $c > 0$. Therefore, the law of the largest jump of T before time 1 is characterized by

$$P(\Delta_1 < v) = P\left(\sup_{0\leq y\leq 1} \Delta(y) < v\right) = \exp\left(-c\beta v^{-1/\beta}\right) \qquad v > 0,$$

and by the restriction property of Poisson processes, conditionally on $\Delta_1 = v$, one can write $T_1 = v + T_1^{(v)}$, where $(T_x^{(v)}, x \geq 0)$ is a subordinator with Lévy measure $cu^{-1-1/\beta}\mathbb{1}_{\{0\leq u\leq v\}}\mathrm{d}u$. The Laplace transform of $T_x^{(v)}$ is given by the Lévy-Khintchine formula

$$E[\exp(-\lambda T_x^{(v)})] = \exp\left(-x\int_0^v \frac{c(1-e^{-\lambda u})}{u^{1+1/\beta}}\mathrm{d}u\right) \quad \lambda, x \geq 0,$$

in particular, $T_1^{(v)}$ admits moments of all order (by differentiating in λ) and $v^{-1}T_1^{(v)}$ has the same law as $T^{(1)}_{v^{-1/\beta}}$ (by changing variables). We then obtain

$$\begin{aligned} E[T_1(T_1/\Delta_1 - 1)] &= E\left[\Delta_1\left(1 + \frac{T_1^{(\Delta_1)}}{\Delta_1}\right)\frac{T_1^{(\Delta_1)}}{\Delta_1}\right] \\ &= K_1\int_{\mathbb{R}_+} \mathrm{d}v\, v^{-1/\beta}e^{-\beta c v^{-1/\beta}} E\left[\left(1 + \frac{T_1^{(v)}}{v}\right)\frac{T_1^{(v)}}{v}\right] \\ &= K_1\int_{\mathbb{R}_+} \mathrm{d}v\, v^{-1/\beta}e^{-\beta c v^{-1/\beta}} E\left[\left(1 + T^{(1)}_{v^{-1/\beta}}\right)T^{(1)}_{v^{-1/\beta}}\right] \end{aligned}$$

where $K_1 = K(\beta) > 0$. Since $T_1^{(1)}$ has a moment of orders 1 and 2, the expectation in the integrand is dominated by some $K_2 v^{-1/\beta} + K_3 v^{-2/\beta}$. It is then easy to see that the integrand is integrable both near 0 and ∞ since $\beta < 2$. Hence $\int_S \left(s_1^{-1} - 1\right)\nu_-(\mathrm{d}\mathbf{s}) < \infty$.

4 The height function

We now turn to the proof of the results related to the height function, starting with Theorem 3. The height function we are going to build will in fact satisfy more than stated there: we will show that under the hypotheses of Theorem 3, there exists a process H_F that encodes $\mathcal{T}_F$ in the sense given in the introduction, that is, $\mathcal{T}_F$ is isometric to the quotient $((0,1), \overline{d})/\equiv$, where $\overline{d}(u,v) = H_F(u) + H_F(v) - 2\inf_{s\in[u,v]} H_F(s)$ and $u \equiv v \iff \overline{d}(u,v) = 0$. Once we have proved this, the result is obvious since $I_F(t)/\equiv$ is the set of vertices of $\mathcal{T}_F$ that are above level t.

4.1 Construction of the height function

Recall from [2] that to encode a CRT, defined as a projective limit of consistent random $\mathbb{R}-$trees $(\mathcal{R}(k), k \geq 1)$, in a continuous height process, one first needs to enrich the structure of the $\mathbb{R}$-trees with consistent *orders* on each set of children of some node. The sons of a given node of $\mathcal{R}(k)$ are thus labelled as first, second, etc... This induces a *planar* representation of the tree. This representation also induces a total order on the vertices of $\mathcal{R}(k)$, which we call $\preceq_k$, by the rule $v \preceq w$ if either v is an ancestor of w, or the branchpoint $b(v, w)$ of v and w is such that the edge leading toward v is earlier than the edge leading toward w (for the ordering on children of $b(v, w)$). In turn, the knowledge of $\mathcal{R}(k), \preceq_k$, or even of $\mathcal{R}(k)$ and the restriction of $\preceq_k$ to the leaves $L_1, \ldots, L_k$ of $\mathcal{R}(k)$, allows us to recover the planar structure of $\mathcal{R}(k)$. The family of planar trees $(\mathcal{R}(k), \preceq_k, k \geq 1)$ is said to be *consistent* if furthermore for every $1 \leq j < k$ the planar tree $(\mathcal{R}(j), \preceq_j)$ has the same law as the planar subtree of $(\mathcal{R}(k), \preceq_k)$ spanned by j leaves $L_1^1, \ldots, L_j^k$ taken independently uniformly at random among the leaves of $\mathcal{R}(k)$.

We build such a consistent family out of the consistent family of unordered trees $(\mathcal{R}(k), k \geq 1)$ as follows. Starting from the tree $\mathcal{R}(1)$, which we endow with the trivial order on its only leaf, we build recursively the total order on $\mathcal{R}(k+1)$ from the order $\preceq_k$ on $\mathcal{R}(k)$, so that the restriction of $\preceq_{k+1}$ to the leaves $L_1, \ldots, L_k$ of $\mathcal{R}(k)$ equals $\preceq_k$. Given $\mathcal{R}(k+1), \preceq_k$, let $b(L_{k+1})$ be the father of L_{k+1}. We distinguish two cases:

1. if $b(L_{k+1})$ is a vertex of $\mathcal{R}(k)$, which has r children $c_1, c_2, \ldots, c_r$ in $\mathcal{R}(k)$, choose J uniformly in $\{1, 2, \ldots, r+1\}$ and let $c_{J-1} \preceq_{k+1} L_{k+1} \preceq_{k+1} c_J$, that is, turn L_{k+1} into the j-th son of $b(L_{k+1})$ in $\mathcal{R}(k+1)$ with probability $1/(r+1)$ (here c_0 (resp. c_{r+1}) is the predecessor (resp. successor) of c_1 (resp. c_r) for $\preceq_k$; we simply ignore them if they do not exist)

2. else, $b(L_{k+1})$ must have a unique son s besides L_{k+1}. Let s' be the predecessor of s for $\preceq_k$ and s'' its successor (if any), and we let $s' \preceq_{k+1} L_{k+1} \preceq_{k+1} s$ with probability $1/2$ and $s \preceq_{k+1} L_{k+1} \preceq_{k+1} s''$ with probability $1/2$.

It is easy to see that this procedure uniquely determines the law of the total order $\preceq_{k+1}$ on $\mathcal{R}(k+1)$ given $\mathcal{R}(k+1), \preceq_k$, and hence the law of $(\mathcal{R}(k), \preceq_k, k \geq 1)$ (the important thing being that the order is total).

Lemma 12. *The family of planar trees $(\mathcal{R}(k), \preceq_k, k \geq 1)$ is consistent. Moreover, given $\mathcal{R}(k)$, the law of $\preceq_k$ can be obtained as follows: for each vertex v of $\mathcal{R}(k)$, endow the (possibly empty) set $\{c_1(v), \ldots, c_i(v)\}$ of children of v in uniform random order, this independently over different vertices.*

Proof. The second statement is obvious by induction. The first statement follows, since we already know that the family of unordered trees $(\mathcal{R}(k), k \geq 1)$ is consistent. $\square$

As a consequence, there exists a.s. a unique total order $\preceq$ on the set of leaves $\{L_1, L_2 \ldots\}$ such that the restriction $\preceq_{|[k]} = \preceq_k$. One can check that this order extends to a total order on the set $\mathcal{L}(\mathcal{T}_F)$: if L, L' are distinct leaves, we say that $L \preceq L'$ if and only if there exist two sequences $L_{\phi(k)} \preceq L_{\varphi(k)}, k \geq 1$, the first one decreasing and converging to L and the second increasing and converging to L'. In turn, this extends to a total order (which we still call $\preceq$) on the whole tree $\mathcal{T}_F$. Theorem 3 is now a direct application of [2, Theorem 15 (iii)], the only thing to check being the conditions a) and b) therein (since we already know that $\mathcal{T}_F$ is compact). Precisely, condition (iii) a) rewritten to fit our setting spells:

$$\lim_{k\to\infty} P(\exists 2 \le j \le k : |D_{\{1,j\}} - aD_1| \le \delta \text{ and } D_j - D_{\{1,j\}} < \delta \text{ and } L_j \preceq L_1) = 1.$$

This is thus a slight modification of (4), and the proof goes similarly, the difference being that we need to keep track of the order on the leaves. Precisely, consider again some rational $r < aD_1$ close to aD_1, so that $|\Pi_1(r)| \neq 0$. The proof of (4) shows that within the time-interval $[r, r+\delta]$, infinitely many integers of $\Pi_1(r)$ have been isolated into singletons. Now, by definition of $\preceq$, the probability that any of these integers j satisfies $L_j \preceq_j L_1$ is $1/2$. Therefore, infinitely many integers of $\Pi_1(r)$ give birth to a leaf L_j that satisfy the required conditions, a.s. The proof of [2, Condition (iii) b)] is exactly similar, hence proving Theorem 3.

It is worth recalling the detailed construction of the process H_F, which is taken from the proof of [2, Theorem 15] with a slight modification (we use the leaves L_i rather than a new sample $Z_i, i \ge 1$, but one checks that the proof remains valid). Given the continuum ordered tree $(\mathcal{T}_F, \mu_F, \preceq, (L_i, i \ge 1))$,

$$U_i = \lim_{n\to\infty} \frac{\#\{j \le n : L_j \preceq L_i\}}{n},$$

a limit that exists a.s. Then the family $(U_i, i \ge 1)$ is distributed as a sequence of independent sequence of uniformly distributed random variables on $(0,1)$, and since $\preceq$ is a total order, one has $U_i \le U_j$ if and only if $L_i \preceq L_j$. Next, define $H_F(U_i)$ to be the height of L_i in $\mathcal{T}_F$, and extend it by continuity on $[0,1]$ (which is a.s. possible according to [2, Theorem 15]) to obtain H_F. In fact, one can define $\tilde{H}_F(U_i) = L_i$ and extend it by continuity on $\mathcal{T}_F$, in which case $\tilde{H}_F$ is an isometry between $\mathcal{T}_F$ and $((0,1), \overline{d})/\equiv$ that maps (the equivalence class of) U_i to L_i for $i \ge 1$, and which preserves order.

Writing $I_F(t) = \{s \in (0,1) : H_F(s) > t\}$, and $|I_F(t)|$ for the decreasing sequence of the lengths of the interval components of $I_F(t)$, we know from the above that $(|I_F(t)|, t \ge 0)$ has the same law as F. More precisely,

Lemma 13. *The processes* $(|I_F(t)|, t \ge 0)$ *and* $(F(t), t \ge 0)$ *are equal.*

Proof. Let $\Pi'(t)$ be the partition of $\mathbb{N}$ such that $i \sim^{\Pi'(t)} j$ if and only if U_i and U_j fall in the same interval component of $I_F(t)$. The isometry $\tilde{H}_F$ allows us to assimilate L_i to U_i, then the interval component of $I_F(t)$ containing U_i corresponds to the tree component of $\{v \in \mathcal{T}_F : \mathrm{ht}(v) > t\}$ containing L_i, therefore U_j falls in this interval if and only if $i \sim^{\Pi(t)} j$, and $\Pi'(t) = \Pi(t)$. By the law of large numbers and the fact that $(U_j, j \ge 1)$ is distributed as a uniform i.i.d. sample, it follows that the length of the interval equals the asymptotic frequency of the block of $\Pi(t)$ containing i, a.s. for every t. One inverts the assertions "a.s." and "for every t" by a simple monotony argument, showing that if $(U_i, i \ge 1)$ is a uniform i.i.d. sample, then a.s. for every sub-interval (a,b) of $(0,1)$, the asymptotic frequency $\lim_{n\to\infty} n^{-1}\#\{i \le n : U_i \in (a,b)\} = b - a$ (use distribution functions). $\square$

We will also need the following result, which is slightly more accurate than just saying, as in the introduction, that $(I_F(t), t \ge 0)$ is an "interval representation" of F:

Lemma 14. *The process $(I_F(t), t \geq 0)$ is a self-similar interval fragmentation, meaning that it is nested ($I_F(t') \subseteq I_F(t)$ for every $0 \leq t \leq t'$), continuous in probability, and for every $t, t' \geq 0$, given $I_F(t) = \bigcup_{i \geq 1} I_i$ where I_i are pairwise disjoint intervals, $I_F(t+t')$ has the same law as $\bigcup_{i \geq 1} g_i(I_F^{(i)}(t'|I_i|^\alpha))$, where the $I_F^{(i)}, i \geq 1$ are independent copies of I_F, and g_i is the orientation-preserving affine function that maps $(0,1)$ to I_i.*

Here, the "continuity in probability" is with respect to the Hausdorff metric D on compact subsets of $[0,1]$, and it just means that $P(D(I_F^c(t_n), I_F^c(t)) > \varepsilon) \to 0$ as $n \to \infty$ for any sequence $t_n \to t$ and $\varepsilon > 0$ (here $A^c = [0,1] \setminus A$).

Proof. The fact that $I_F(t)$ is nested is trivial. Now recall that the different interval components of $I_F(t)$ encode the tree components of $\{v \in \mathcal{T}_F : \mathrm{ht}(v) > t\}$, call them $\mathcal{T}_1(t), \mathcal{T}_2(t), \ldots$. We already know that these trees are rescaled independent copies of $\mathcal{T}_F$, that is, they have the same law as $\mu_F(\mathcal{T}_i(t))^{-\alpha} \otimes \mathcal{T}^{(i)}, i \geq 1$, where $\mathcal{T}^{(i)}, i \geq 1$ are independent copies of $\mathcal{T}_F$. So let $\mathcal{T}^{(i)} = \mu_F(\mathcal{T}_i(t))^\alpha \otimes \mathcal{T}_i(t)$. Now, the orders induced by $\preceq$ on the different $\mathcal{T}^{(i)}$'s have the same law as $\preceq$ and are independent, because they only depend on the L_j's that fall in each of them. Therefore, the trees $(\mathcal{T}^{(i)}, \mu^{(i)}, \preceq^{(i)})$ are independent copies of $(\mathcal{T}_F, \mu_F, \preceq)$, where $\mu^{(i)}(\cdot) = \mu_F((\mu_F(\mathcal{T}_i(t))^{-\alpha} \otimes \cdot) \cap \mathcal{T}_i(t))/\mu_F(\mathcal{T}_i(t))$ and $\preceq^{(i)}$ is the order on $\mathcal{T}^{(i)}$ induced by the restriction of $\preceq$ to $\mathcal{T}_i(t)$. It follows by our previous considerations that their respective height processes $H^{(i)}$ are independent copies of H_F, and it is easy to check that given $I_F(t) = \bigcup_{i \geq 1} I_i$ (where I_i is the interval corresponding to $\mathcal{T}_i(t)$), the excursions of H_F above t are precisely the processes $\mu(\mathcal{T}_i(t))^{-\alpha} H^{(i)} = |I_i|^{-\alpha} H^{(i)}$. The self-similar fragmentation property follows at once, as the fact that I_F is Markov. Thanks to these properties, we may just check the continuity in probability at time 0, and it is trivial because H_F is a.s. continuous and positive on $(0,1)$. $\square$

It appears that besides these elementary properties, the process H_F is quite hard to study. In order to move one step further, we will try to give a "Poissonian construction" of H_F, in the same way as we used properties of the Poisson process construction of Π^0 to study $\mathcal{T}_F$. To begin with, we move "back to the homogeneous case" by time-changing. For every $x \in (0,1)$, let $I_x(t)$ be the interval component of $I_F(t)$ containing x, and $|I_x(t)|$ be its length ($= 0$ if $I_x(t) = \varnothing$). Then set

$$T_t^{-1}(x) = \inf\left\{u \geq 0 : \int_0^u |I_x(r)|^\alpha \mathrm{d}r > t\right\},$$

and let $I_F^0(t)$ be the open set constituted of the union of the intervals $I_x(T_t^{-1}(x)), x \in (0,1)$ (it suffices in fact to take the union of the $I_{U_i}(T_t^{-1}(U_i)), i \geq 1$). From [7] and Lemma 14, $(I_F^0(t), t \geq 0)$ is a self-similar homogeneous interval fragmentation.

4.2 A Poissonian construction

Recall that the process $(\Pi(t), t \geq 0)$ is constructed out of a homogeneous fragmentation $(\Pi^0(t), t \geq 0)$, which has been appropriately time-changed, and where $(\Pi^0(t), t \geq 0)$ has itself been constructed out of a Poisson point process $(\Delta_t, k_t, t \geq 0)$ with intensity $\kappa_\nu \otimes \#$. Further, we mark this Poisson process by considering, for each jump time t of this Poisson process, a sequence $(U_i(t), i \geq 1)$ of i.i.d. random variables that are uniform on $(0,1)$, so that these sequences are independent over different such t's. We are going to use the marks to build an order on the non-void blocks of Π^0. It is convenient first to formalize what we precisely call an *order* on a set A: it is a subset $\mathcal{O}$ of $A \times A$ satisfying:

1. $(i,i) \in \mathcal{O}$ for every $i \in A$
2. $(i,j) \in \mathcal{O}$ and $(j,i) \in \mathcal{O}$ imply $i = j$
3. $(i,j) \in \mathcal{O}$ and $(j,k) \in \mathcal{O}$ imply $(i,k) \in \mathcal{O}$.

If $B \subseteq A$, the restriction to B of the order $\mathcal{O}$ is $\mathcal{O}_{|B} = \mathcal{O} \cap (B \times B)$. We now construct a process $(\mathcal{O}(t), t \geq 0)$, with values in the set of orders of $\mathbb{N}$, as follows. Let $\mathcal{O}(0) = \{(i,i), i \in \mathbb{N}\}$ be the trivial order, and let $n \in \mathbb{N}$. Let $0 < t_1 < t_2 < \ldots < t_K$ be the times of occurrence of jumps of the Poisson process $(\Delta_t, k_t, t \geq 0)$ such that both $k_t \leq n$ and $(\Delta_t)_{|[n]}$ (the restriction of Δ_t to $[n]$) is non-trivial. Let $\mathcal{O}^n(0) = \mathcal{O}_{|[n]}(0)$, and define a process $\mathcal{O}^n(t)$ to be constant on the time-intervals $[t_{i-1}, t_i)$ (where $t_0 = 0$), where inductively, given $\mathcal{O}^n(t_{i-1}) = \mathcal{O}^n(t_i-)$, $\mathcal{O}^n(t_i)$ is defined as follows. Let $J_n(t_i) = \{j \in \Pi^0_{k_{t_i}}(t_i-) : j \leq n \text{ and } \Pi^0_j(t_i) \neq \varnothing\}$ so that $k_{t_i} \in J_n(t_i)$ as soon as $\Pi^0_{k_{t_i}}(t_i-) \neq \varnothing$. Let then

$$\mathcal{O}^n(t_i) = \mathcal{O}^n(t_i-) \cup \bigcup_{\substack{j,k \in J_n(t_i): \\ U_j(t_i) < U_k(t_i)}} \{(j,k)\} \cup \bigcup_{\substack{j:(j,k_{t_i}) \in \mathcal{O}_{|[n]}(t_i-) \\ k \in J_n(t_i)}} \{(j,k)\} \cup \bigcup_{\substack{j:(k_{t_i},j) \in \mathcal{O}_{|[n]}(t_i-) \\ k \in J_n(t_i)}} \{(k,j)\}.$$

In words, we order each set of new blocks in random order in accordance with the variables $U_m(t_i), 1 \leq m \leq n$, and these new blocks have the same relative position with other blocks as had their father, namely the block $\Pi^0_{k_{t_i}}(t_i-)$.

It is easy to see that the orders thus defined are consistent as n varies, i.e. $(\mathcal{O}^{n+1}(t))_{|[n]} = \mathcal{O}^n(t)$ for every n, t, and it easily follows that there exists a unique process $(\mathcal{O}(t), t \geq 0)$ such that $\mathcal{O}_{|[n]}(t) = \mathcal{O}^n(t)$ for every n, t (for existence, take the union over $n \in \mathbb{N}$, and unicity is trivial). The process $\mathcal{O}$ thus obtained allows us to build an interval-valued version of the fragmentation $\Pi^0(t)$, namely, for every $t \geq 0$ and $j \geq 0$ let

$$I^0_j(t) = \left(\sum_{k \neq j : (k,j) \in \mathcal{O}(t)} |\Pi^0_k(t)|, \sum_{k:(k,j) \in \mathcal{O}(t)} |\Pi^0_k(t)| \right)$$

(notice that $I^0_j(t) = \varnothing$ if $\Pi^0_j(t) = \varnothing$). Write $I^0(t) = \bigcup_{j \geq 1} I^0_j(t)$, and notice that the length $|I^0_j(t)|$ of $I^0_j(t)$ equals the asymptotic frequency of $\Pi^0_j(t)$ for every $j \geq 1, t \geq 0$.

Proposition 2. *The processes $(I^0_F(t), t \geq 0)$ and $(I^0(t), t \geq 0)$ have the same law.*

As a consequence, we have obtained a construction of an object with the same law as I^0_F with the help of a marked Poisson process in $\mathcal{P}_\infty$, and this is the one we are going to work with.

Proof. Let $I^0_F(i,t)$ be the interval component of $I^0_F(t)$ containing U_i if i is the least j such that U_j falls in this component, and $I^0_F(i,t) = \varnothing$ otherwise. Let $\mathcal{O}_F(t) = \{(i,i), i \in \mathbb{N}\} \cup \{(j,k) : I^0_F(j,t) \text{ is located to the left of } I^0_F(k,t) \text{ and both are nonempty}\}$. Since the lengths of the interval components of I^0_F and I^0 are the same, the only thing we need to check is that the processes $\mathcal{O}$ and $\mathcal{O}_F$ have the same law. But then, for $j \neq k$, $(j,k) \in \mathcal{O}_F(t)$ means that the branchpoint $b(L_j, L_k)$ of L_j and L_k has height less than t, and the subtree rooted at $b(L_j, L_k)$ containing L_j has been placed before that containing L_k. Using Lemma 12, we see that given $\mathcal{T}_F, L_1, L_2, \ldots$, the subtrees rooted at any branchpoint b of $\mathcal{T}_F$ are placed in exchangeable random order independently over branchpoints. Precisely, letting $\mathcal{T}_1^b$ be the subtree containing the leaf with least label, $\mathcal{T}_2^b$ the subtree different from $\mathcal{T}_1^b$ containing the leaf with least label, and so on, the first subtrees $\mathcal{T}_1^b, \ldots, \mathcal{T}_k^b$ are placed in any of the $k!$ possible linear orders, consistently as k varies. Therefore (see e.g. [2, Lemma 10]), there exist independent uniform$(0,1)$ random variables $U_1^b, U_2^b, \ldots$ independent over b's such that $\mathcal{T}_i^b$ is on the "left" of $\mathcal{T}_j^b$ (for the order $\mathcal{O}_F$) if and only if $U_i^b \leq U_j^b$. This is exactly how we defined the order $\mathcal{O}(t)$. □

Remark. As the reader may have noticed, this construction of an interval-valued fragmentation has in fact little to do with pure manipulation of intervals, and it is actually almost entirely performed in the world of partitions. We stress that it is in fact quite hard to construct directly such an interval fragmentation out of the plain idea: "start from the interval $(0,1)$, take a Poisson process $(s(t), k_t, t \geq 0)$ with intensity $\nu(\mathrm{d}\mathbf{s}) \otimes \#$, and at a jump time of the Poisson process turn the k_t-th interval component $I_{k_t}(t-)$ of $I(t-)$ (for some labeling convention) into the open subset of $I_{k_t}(t-)$ whose components sizes are $|I_{k_t}(t-)|s_i(t), i \geq 1$, and placed in exchangeable order". Using partitions helps much more than plainly giving a natural "labeling convention" for the intervals. In the same vein, we refer to the work of Gnedin [17], which shows that exchangeable interval (composition) structures are in fact equivalent to "exchangeable partitions+order on blocks".

For every $x \in (0,1)$, write $I_x^0(t)$ for the interval component of $I_F^0(t)$ containing x, and notice that $I_x^0(t-) = \bigcap_{s \uparrow t} I_x^0(s)$ is well-defined as a decreasing intersection. For $t \geq 0$ such that $I_x^0(t) \neq I_x^0(t-)$, let $s^x(t)$ be the sequence $|I_F^0(t) \cap I_x^0(t-)|/|I_x^0(t-)|$, where $|I_F^0(t) \cap I_x^0(t-)|$ is the decreasing sequence of lengths of the interval components of $I_F^0(t) \cap I_x^0(t-)$. The useful result on the Poissonian construction is given in the following

Lemma 15. *The process $(s^x(t), t \geq 0)$ is a Poisson point process with intensity $\nu(\mathrm{d}\mathbf{s})$, and more precisely, the order of the interval components of $I_F^0(t) \cap I_x^0(t-)$ is exchangeable: there exists a sequence of i.i.d. uniform random variables $(U_i^x(t), i \geq 1)$, independent of $(\mathcal{G}^0(t-), s^x(t))$ such that the interval with length $s_i^x(t)|I_x^0(t-)|$ is located on the left of the interval with length $s_j^x(t)|I_x^0(t-)|$ if and only if $U_i^x(t) \leq U_j^x(t)$.*

Proof. Let $i(t,x) = \inf\{i \in \mathbb{N} : U_i \in I_x^0(t)\}$. Then $i(t,x)$ is an increasing jump-hold process in $\mathbb{N}$. If now $I_x^0(t) \neq I_x^0(t-)$, it means that there has been a jump of the Poisson process Δ_t, k_t at time t, so that $k_t = i(t,x)$, and then $s^x(t)$ is equal to the decreasing sequence $|\Delta_t|$ of asymptotic frequencies of Δ_t, therefore $s^x(t) = |\Delta_t|$ when $k_t = i(t-,x)$, and since $i(t-,x)$ is progressive, its jump times are stopping times so the process $(s^x(t), t \geq 0)$ is in turn a Poisson process with intensity $\nu(\mathrm{d}\mathbf{s})$. Moreover, by Proposition 2 and the construction of I^0, each time an interval splits, the corresponding blocks are put in exchangeable order, which gives the second half of the lemma. □

4.3 Proof of Theorem 4

4.3.1 Hölder-continuity of H_F

We prove here that the height process is a.s. Hölder-continuous of order γ for every $\gamma < \vartheta_{\text{low}} \wedge |\alpha|$. The proof will proceed in three steps.

First step: Reduction to the behavior of H_F near 0. By a theorem of Garsia, Rodemich and Rumsey (see e.g. [12]), the finiteness of $\int_0^1 \int_0^1 \frac{|H_F(x)-H_F(y)|^{n+n_0}}{|x-y|^{\gamma n}} \mathrm{d}x\mathrm{d}y$ leads to the $\left(\frac{\gamma n-2}{n+n_0}\right)$-Hölder-continuity of H_F, so that when the previous integral is finite for every n, the height process H_F is Hölder-continuous of order δ for every $\delta < \gamma$, whatever is n_0. To prove Theorem 4 it is thus sufficient to show that for every $\gamma < \vartheta_{\text{low}} \wedge |\alpha|$ there exists a $n_0(\gamma)$ such that

$$E\left[\int_0^1 \int_0^1 \frac{|H_F(x) - H_F(y)|^{n+n_0(\gamma)}}{|x-y|^{\gamma n}} \mathrm{d}x\mathrm{d}y\right] < \infty \text{ for every positive integer } n.$$

Now take V_1, V_2 uniform independent on $(0,1)$, independently of H_F. The expectation above then becomes $E\left[\frac{|H_F(V_1)-H_F(V_2)|^{n+n_0(\gamma)}}{|V_1-V_2|^{\gamma n}}\right]$.

Consider next I_F the interval fragmentation constructed from H_F (see Section 4.1). By Lemma 14, $H_F(V_1)$ and $H_F(V_2)$ may be rewritten as

$$H_F(V_i) = D_{\{1,2\}} + \lambda_i^{|\alpha|}(D_{\{1,2\}})\widetilde{D}_i, \ i = 1, 2,$$

where $D_{\{1,2\}}$ is the first time at which V_1 and V_2 belong to different intervals of I_F and $\widetilde{D}_1, \widetilde{D}_2$ have the same law as $H_F(V_1)$ and are independent of $\mathcal{H}(D_{\{1,2\}})$, where $\mathcal{H}(t), t \geq 0$ is the natural completed filtration associated with I_F. The r.v. $\widetilde{D}_1$ and $\widetilde{D}_2$ can actually be described more precisely. Say that at time $D_{\{1,2\}}$, V_1 belongs to an interval $\left(a_1, a_1 + \lambda_1(D_{\{1,2\}})\right)$ and V_2 to $\left(a_2, a_2 + \lambda_2(D_{\{1,2\}})\right)$. Then there exist two iid processes independent of $\mathcal{H}(D_{\{1,2\}})$ and with the same law as H_F, let us denote them $H_F^{(1)}$ and $H_F^{(2)}$, such that $\widetilde{D}_i = H_F^{(i)}\left(\frac{V_i - a_i}{\lambda_i(D_{\{1,2\}})}\right)$, $i = 1, 2$. Since $V_i \in \left(a_i, a_i + \lambda_i(D_{\{1,2\}})\right)$, the random variables $\widetilde{V}_i = (V_i - a_1)\,\lambda_i^{-1}(D_{\{1,2\}})$ are iid, with the uniform law on $(0,1)$ and independent of $H_F^{(1)}, H_F^{(2)}$ and $\mathcal{H}(D_{\{1,2\}})$. And when $V_1 > V_2$,

$$V_1 - V_2 \geq \lambda_1(D_{\{1,2\}})\widetilde{V}_1 + \lambda_2(D_{\{1,2\}})\left(1 - \widetilde{V}_2\right)$$

since a_1 is then larger than $a_2 + \lambda_2(D_{\{1,2\}})$. This gives

$$\begin{aligned} E\left[\frac{|D_1 - D_2|^{n+n_0(\gamma)}}{|V_1 - V_2|^{\gamma n}}\right] &= 2E\left[\frac{|D_1 - D_2|^{n+n_0(\gamma)}}{(V_1 - V_2)^{\gamma n}} \mathbb{1}\{V_1 > V_2\}\right] \\ &\leq 2E\left[\frac{\left(\lambda_1^{|\alpha|}(D_{\{1,2\}})\widetilde{D}_1 + \lambda_2^{|\alpha|}(D_{\{1,2\}})\widetilde{D}_2\right)^{n+n_0(\gamma)}}{\left(\lambda_1(D_{\{1,2\}})\widetilde{V}_1 + \lambda_2(D_{\{1,2\}})\left(1 - \widetilde{V}_2\right)\right)^{\gamma n}}\right] \end{aligned}$$

and this last expectation is bounded from above by

$$2^{n+n_0(\gamma)} E\left[\left(\lambda_1(D_{\{1,2\}})\right)^{(n+n_0(\gamma))|\alpha| - \gamma n}\right] \left(E\left[\frac{H_F^{n+n_0(\gamma)}(V_1)}{V_1^{\gamma n}}\right] + E\left[\frac{H_F^{n+n_0(\gamma)}(V_1)}{(1-V_1)^{\gamma n}}\right]\right).$$

The expectation involving λ_1 is bounded by 1 since $\gamma < |\alpha|$. And since V_1 is independent of H_F, the two expectations in the parenthesis are equal (reversing the order $\preceq$ and performing the construction of H_F gives a process with the same law and shows that $H_F(x) \overset{law}{=} H_F(1-x)$ for every $x \in (0,1)$) and finite as soon as

$$\sup_{x\in(0,1)} E\left[H_F(x)^{n+n_0(\gamma)}\right] x^{-\gamma n} < \infty. \tag{17}$$

The rest of the proof thus consists of finding an integer $n_0(\gamma)$ such that (17) holds for every n. To do so, we will have to observe the interval fragmentation I_F at nice stopping times depending on x, say $\mathbb{T}_x^{(\gamma)}$, and then use the strong fragmentation property at time $\mathbb{T}_x^{(\gamma)}$. This gives

$$H_F(x) = \mathbb{T}_x^{(\gamma)} + \left(S_x(\mathbb{T}_x^{(\gamma)})\right)^{|\alpha|} \overline{H}_F(P_x(\mathbb{T}_x^{(\gamma)})) \tag{18}$$

where $S_x(\mathbb{T}_x^{(\gamma)})$ is the length of the interval containing x at time $\mathbb{T}_x^{(\gamma)}$, $P_x(\mathbb{T}_x^{(\gamma)})$ the relative position of x in that interval and $\overline{H}_F$ a process with the same law as H_F and independent of $\mathcal{H}(\mathbb{T}_x^{(\gamma)})$.

Second step: Choice and properties of $\mathbb{T}_x^{(\gamma)}$. Let us first introduce some notation in order to prove the forthcoming Lemma 16. Recall that we have called I_F^0 the homogeneous interval fragmentation related to I_F by the time changes $T_t^{-1}(x)$ introduced in Section 4.1. In this homogeneous fragmentation, let

$I_x^0(t) = (a_x(t), b_x(t))$ be the interval containing x at time t

$S_x^0(t)$ the length of this interval

$P_x^0(t) = (x - a_x(t))/S_x^0(t)$ the relative position of x in $I_x(t)$.

Similarly, we define $P_x^0(t-)$ to be the relative position of x in the interval $I_x^0(t-)$, which is well-defined as an intersection of nested intervals. $S_x^0(t-)$ is the size of this interval. We will need the following inequalities in the sequel:

$$P_x^0(t) \leq x/S_x^0(t) \qquad P_x^0(t-) \leq x/S_x^0(t-).$$

Next recall the Poisson point process construction of the interval fragmentation I_F^0, and the Poisson point process $(s^x(t))_{t\geq 0}$ of Lemma 15. Set

$$\sigma(t) := -\ln\left(\prod_{s\leq t} s_1^x(t)\right) \qquad t \geq 0,$$

with the convention $s_1^x(t) = 1$ when t is not a time of occurrence of the point process. By Lemma 15, the process σ is a subordinator with intensity measure $\nu(-\ln s_1 \in x)$, which is infinite. Consider then T_x^{exit}, the first time at which x is not in the largest sub-interval of I_x^0 when I_x^0 splits, that is

$$T_x^{\text{exit}} := \inf\left\{t : S_x^0(t) < \exp(-\sigma(t))\right\}.$$

By definition, the size of the interval containing x at time $t < T_x^{\text{exit}}$ is given by $S_x^0(t) = \exp(-\sigma(t))$. We will need to consider the first time at which this size is smaller than a, for a in $(0,1)$, and so we introduce

$$T_a^\sigma := \inf\left\{t : \exp(-\sigma(t)) < a\right\}.$$

Note that $P_x^0(t) \le x\exp(\sigma(t))$ when $t < T_x^{\text{exit}}$ and that $P_x^0(T_x^{\text{exit}}-) \le x\exp(\sigma(T_x^{\text{exit}}-))$.

Finally, to obtain a nice $\mathbb{T}_x^{(\gamma)}$ as required in the preceding step, we stop the homogeneous fragmentation at time

$$T_x^{\text{exit}} \wedge T_{x^\varepsilon}^\sigma$$

for some ε to be determined (and depending on γ) and then take for $\mathbb{T}_x^{(\gamma)}$ the self-similar counterpart of this stopping time, that is $\mathbb{T}_x^{(\gamma)} = T^{-1}_{T_x^{\text{exit}} \wedge T_{x^\varepsilon}^\sigma}(x)$. More precisely, we have

Lemma 16. *For every $\gamma < \vartheta_{\text{low}} \wedge |\alpha|$, there exists a family of random stopping times $\mathbb{T}_x^{(\gamma)}, x \in (0,1)$, and an integer $N(\gamma)$ such that*

(i) for every $n \ge 0$, $\exists C_1(n) : E\left[\left(\mathbb{T}_x^{(\gamma)}\right)^n\right] \le C_1(n)x^{\gamma n} \quad \forall\ x \in (0,1)$,

(ii) $\exists C_2$ such that $E\left[\left(S_x(\mathbb{T}_x^{(\gamma)})\right)^n\right] \le C_2 x^\gamma$ for every x in $(0,1)$ and $n \ge N(\gamma)$.

Proof. Fix $\gamma < \vartheta_{\text{low}} \wedge |\alpha|$ and then $\varepsilon < 1$ such that $\gamma/(1-\varepsilon) < \vartheta_{\text{low}}$. The times $\mathbb{T}_x^{(\gamma)}, x \in (0,1)$, are constructed from this ε by

$$\mathbb{T}_x^{(\gamma)} = T^{-1}_{T_x^{\text{exit}} \wedge T_{x^\varepsilon}^\sigma}(x),$$

and it may be clear that these times are stopping times with respect to $\mathcal{H}$. A first remark is that the function $x \in (0,1) \mapsto S_x(\mathbb{T}_x^{(\gamma)})$ is bounded from above by 1 and that $x \in (0,1) \mapsto \mathbb{T}_x^{(\gamma)}$ is bounded from above by τ, the first time at which the fragmentation is entirely reduced to dust, that is, in others words, the supremum of H_F on $[0,1]$. Since τ has moments of all orders, it is thus sufficient to prove statements (i) and (ii) for $x \in (0, x_0)$ for some well chosen $x_0 > 0$. Another remark, using the definition of $T_t^{-1}(x)$, is that $\mathbb{T}_x^{(\gamma)} \le T_x^{\text{exit}} \wedge T_{x^\varepsilon}^\sigma$ and $S_x(\mathbb{T}_x^{(\gamma)}) = S_x^0\left(T_x^{\text{exit}} \wedge T_{x^\varepsilon}^\sigma\right)$, so that we just have to prove (i) and (ii) by replacing in the statement $\mathbb{T}_x^{(\gamma)}$ by $T_x^{\text{exit}} \wedge T_{x^\varepsilon}^\sigma$ and $S_x(\mathbb{T}_x^{(\gamma)})$ by $S_x^0\left(T_x^{\text{exit}} \wedge T_{x^\varepsilon}^\sigma\right)$.

We shall thus work with the homogeneous fragmentation. When I_x^0 splits to give smaller intervals, we divide these sub-intervals into three groups: the largest sub-interval, the group of sub-intervals on its left and the group of sub-intervals on its right. With the notations of Lemma 15, the lengths of the intervals belonging to the group on the left are the $s_i^x(t)S_x^0(t-)$ with i such that $U_i^x(t) < U_1^x(t)$ and similarly, the lengths of the intervals on the right are the $s_i^x(t)S_x^0(t-)$ with i such that $U_i^x(t) > U_1^x(t)$. An important point is that when $T_x^{\text{exit}} < T_{x^\varepsilon}^\sigma$, then at time T_x^{exit}, the point x belongs to the group of sub-intervals on the left resulting from the fragmentation of $I_x^0(T_x^{\text{exit}}-)$. Indeed, when $T_x^{\text{exit}} < T_{x^\varepsilon}^\sigma$, then $\exp(-\sigma(T_x^{\text{exit}})) \ge x^\varepsilon \ge x$, which becomes $s_1^x(T_x^{\text{exit}})\exp(-\sigma(T_x^{\text{exit}}-)) \ge x$. Then using that $P_x^0(T_x^{\text{exit}}-) \le x\exp(\sigma(T_x^{\text{exit}}-))$, we obtain $s_1^x(T_x^{\text{exit}}) \ge P_x^0(T_x^{\text{exit}}-)$ and thus that x does not belong to the group on the right at time T_x^{exit} (x belongs to the group on the right at a time t if and only if $P_x^0(t-) > \sum_{i:U_i^x(t) \le U_1^x(t)} s_i^x(t)$). Hence x belongs to the union of intervals on the left at time T_x^{exit} when $T_x^{\text{exit}} < T_{x^\varepsilon}^\sigma$. In other words,

$$T_x^{\text{exit}} = \inf\left\{t : \sum\nolimits_{i:U_i^x(t)<U_1^x(t)} s_i^x(t) > P_x^0(t-)\right\} \text{ when } T_x^{\text{exit}} < T_{x^\varepsilon}^\sigma.$$

The key-point, consequence of Lemma 15, is that the process $\left(\sum_{i:U_i^x(t)<U_1^x(t)} s_i^x(t)\right)_{t\ge 0}$ is a

marked Poisson point process with an intensity measure on $[0,1]$ given by

$$\mu(\mathrm{d}u) := \int_S p(\mathbf{s}, \mathrm{d}u)\nu(\mathrm{d}\mathbf{s}),\ u \in [0,1],$$

where for a fixed $\mathbf{s}$ in S, $p(\mathbf{s}, \mathrm{d}u)$ is the law of $\sum_{i:U_i<U_1} s_i$, the U_i's being uniform and independent random variables. We refer to Kingman [21] for details on marked Poisson point processes. Observing then that for any a in $(0,1/2)$ and for a fixed $\mathbf{s}$ in S

$$\mathbb{1}_{\{1-s_1>2a\}} \leq \mathbb{1}_{\left\{\sum_{i:U_i<U_1} s_i>a\right\}} + \mathbb{1}_{\left\{\sum_{i:U_i>U_1} s_i>a\right\}},$$

we obtain that $\mathbb{1}_{\{1-s_1>2a\}} \leq 2P\left(\sum_{i:U_i<U_1} s_i > a\right)$ and then the following inequality

$$\mu\left((a,1]\right) \geq \frac{1}{2}\nu\left(s_1 < 1-2a\right).$$

This, recalling the definition of ϑ_{low} and that $\gamma/(1-\varepsilon) < \vartheta_{\mathrm{low}}$, leads to the existence of a positive x_0 and a positive constant C such that

$$\mu\left(\left(x^{1-\varepsilon},1\right]\right) \geq C\left(x^{-(1-\varepsilon)}\right)^{\gamma/(1-\varepsilon)} = Cx^{-\gamma} \text{ for all } x \text{ in } (0,x_0). \tag{19}$$

Proof of (i). We again have to introduce a hitting time, that is the first time at which the Poisson point process $\left(\sum_{i:U_i^x(t)<U_1^x(t)} s_i^x(t), t \geq 0\right)$ belongs to $(x^{1-\varepsilon},1)$:

$$H_{x^{1-\varepsilon}} := \inf\left\{t : \sum_{i:U_i^x(t)<U_1^x(t)} s_i^x(t) > x^{1-\varepsilon}\right\}.$$

By the theory of Poisson point processes, this time has an exponential law with parameter $\mu\left((x^{1-\varepsilon},1]\right)$. Hence, given inequality (19), it is sufficient to show that $T_x^{\mathrm{exit}} \wedge T_{x^\varepsilon}^\sigma \leq H_{x^{1-\varepsilon}}$ to obtain (i) for x in $(0,x_0)$ and then (i) (we recall that it is already known that $\sup_{x\in[x_0,1)} x^{-\gamma n} E\left[\left(\mathbb{T}_x^{(\gamma)}\right)^n\right]$ is finite). On the one hand, since $P_x^0(t) \leq x\exp(\sigma(t))$ when $t < T_x^{\mathrm{exit}}$,

$$P_x^0(H_{x^{1-\varepsilon}}-) \leq x\exp(\sigma(H_{x^{1-\varepsilon}}-)) < x\exp(\sigma(H_{x^{1-\varepsilon}})) \text{ when } H_{x^{1-\varepsilon}} < T_x^{\mathrm{exit}}.$$

On the other hand, $H_{x^{1-\varepsilon}} < T_{x^\varepsilon}^\sigma$ yields

$$x\exp(\sigma(H_{x^{1-\varepsilon}})) \leq x^{1-\varepsilon} < \sum_{i:U_i^x(H_{x^{1-\varepsilon}})<U_1^x(H_{x^{1-\varepsilon}})} s_i^x(H_{x^{1-\varepsilon}}),$$

and combining these two remarks, we get that $H_{x^{1-\varepsilon}} < T_x^{\mathrm{exit}} \wedge T_{x^\varepsilon}^\sigma$ implies

$$P_x^0(H_{x^{1-\varepsilon}}-) < \sum_{i:U_i^x(H_{x^{1-\varepsilon}})<U_1^x(H_{x^{1-\varepsilon}})} s_i^x(H_{x^{1-\varepsilon}}).$$

Yet this is not possible, because this last relation on $H_{x^{1-\varepsilon}}$ means that, at time $H_{x^{1-\varepsilon}}$, x is not in the largest sub-interval resulting from the splitting of $I_x^0(H_{x^{1-\varepsilon}}-)$, which implies $H_{x^{1-\varepsilon}} \geq T_x^{\mathrm{exit}}$ and this does not match with $H_{x^{1-\varepsilon}} < T_x^{\mathrm{exit}} \wedge T_{x^\varepsilon}^\sigma$. Hence $T_x^{\mathrm{exit}} \wedge T_{x^\varepsilon}^\sigma \leq H_{x^{1-\varepsilon}}$ and (i) is proved.

Proof of (ii). Take $N(\gamma) \geq \gamma/\varepsilon \vee 1$. When $T^{\sigma}_{x^{\varepsilon}} \leq T^{\text{exit}}_x$, using the definition of $T^{\sigma}_{x^{\varepsilon}}$ and the right continuity of σ, we have

$$S^0_x(T^{\text{exit}}_x \wedge T^{\sigma}_{x^{\varepsilon}}) \leq \exp(-\sigma(T^{\sigma}_{x^{\varepsilon}})) \leq x^{\varepsilon}$$

and consequently $\left(S^0_x(T^{\text{exit}}_x \wedge T^{\sigma}_{x^{\varepsilon}})\right)^{N(\gamma)} \leq x^{\gamma}$. Thus it just remains to show that

$$E\left[\left(S^0_x(T^{\text{exit}}_x \wedge T^{\sigma}_{x^{\varepsilon}})\right)^{N(\gamma)} \mathbb{1}_{\left\{T^{\text{exit}}_x < T^{\sigma}_{x^{\varepsilon}}\right\}}\right] \leq x^{\gamma} \text{ for } x < x_0.$$

When $T^{\text{exit}}_x < T^{\sigma}_{x^{\varepsilon}}$, we know - as explained at the beginning of the proof - that x belongs at time T^{exit}_x to the group of sub-intervals on the left resulting from the fragmentation of $I^0_x(T^{\text{exit}}_x -)$ and hence that $S^0_x(T^{\text{exit}}_x \wedge T^{\sigma}_{x^{\varepsilon}})^{N(\gamma)} \leq s^x_i(T^{\text{exit}}_x)$ for some i such that $U^x_i(T^{\text{exit}}_x) < U^x_1(T^{\text{exit}}_x)$. More roughly,

$$S^0_x(T^{\text{exit}}_x \wedge T^{\sigma}_{x^{\varepsilon}})^{N(\gamma)} \mathbb{1}_{\left\{T^{\text{exit}}_x < T^{\sigma}_{x^{\varepsilon}}\right\}} \leq \sum\nolimits_{i:U^x_i(T^{\text{exit}}_x) < U^x_1(T^{\text{exit}}_x)} s^x_i(T^{\text{exit}}_x) \mathbb{1}_{\left\{T^{\text{exit}}_x < T^{\sigma}_{x^{\varepsilon}}\right\}}.$$

To evaluate the expectation of this random sum, recall from the proof of (i) that $T^{\text{exit}}_x \leq H_{x^{1-\varepsilon}}$ when $T^{\text{exit}}_x < T^{\sigma}_{x^{\varepsilon}}$ and remark that either $T^{\text{exit}}_x < H_{x^{1-\varepsilon}}$ and then

$$\sum\nolimits_{i:U^x_i(T^{\text{exit}}_x) < U^x_1(T^{\text{exit}}_x)} s^x_i(T^{\text{exit}}_x) \leq x^{1-\varepsilon} \leq x^{\gamma} \quad (\gamma < \vartheta_{\text{low}}(1-\varepsilon) \leq 1-\varepsilon)$$

or $T^{\text{exit}}_x = H_{x^{1-\varepsilon}}$ and then

$$\sum\nolimits_{i:U^x_i(T^{\text{exit}}_x) < U^x_1(T^{\text{exit}}_x)} s^x_i(T^{\text{exit}}_x) = \sum\nolimits_{i:U^x_i(H_{x^{1-\varepsilon}}) < U^x_1(H_{x^{1-\varepsilon}})} s^x_i(H_{x^{1-\varepsilon}}).$$

There we conclude with the following inequality

$$\begin{aligned} E\left[\sum\nolimits_{i:U^x_i(H_{x^{1-\varepsilon}}) < U^x_1(H_{x^{1-\varepsilon}})} s^x_i(H_{x^{1-\varepsilon}})\right] &= \frac{\int_S E\left[\sum_{i:U_i < U_1} s_i \mathbb{1}_{\left\{\sum_{i:U_i<U_1} s_i > x^{1-\varepsilon}\right\}}\right] \nu(\mathrm{d}s)}{\mu\left((x^{1-\varepsilon}, 1]\right)} \\ &\leq C^{-1} x^{\gamma} \int_S (1 - s_1)\, \nu(\mathrm{d}s), \ x \in (0, x_0). \end{aligned}$$

□

Third step: Proof of (17). Fix $\gamma < \vartheta_{\text{low}} \wedge |\alpha|$ and take $\mathbb{T}^{(\gamma)}_x$ and $N(\gamma)$ as introduced in Lemma 16. Let then $n_0(\gamma)$ be an integer larger than $N(\gamma)/|\alpha|$. According to the first step, Theorem 4 is proved if (17) holds for this $n_0(\gamma)$ and every integer $n \geq 1$. To show this, it is obviously sufficient to prove that for all integers $n \geq 1$ and $m \geq 0$, there exists a finite constant $C(n,m)$ such that

$$E\left[H_F(x)^{m+n+n_0(\gamma)}\right] \leq C(n,m) x^{\gamma n} \quad \forall x \in (0,1).$$

This can be proved by induction: for $n = 1$ and every $m \geq 0$, using (18), we have

$$\begin{aligned} E\left[H_F(x)^{m+1+n_0(\gamma)}\right] &\leq 2^{m+1+n_0(\gamma)} \\ &\times \left(E\left[\left(\mathbb{T}^{(\gamma)}_x\right)^{m+1+n_0(\gamma)}\right] + E\left[\left(S_x(\mathbb{T}^{(\gamma)}_x)\right)^{|\alpha|(m+1+n_0(\gamma))} \widetilde{\tau}^{m+1+n_0(\gamma)}\right]\right) \end{aligned}$$

where $\widetilde{\tau}$ is the maximum of $\overline{H}_F$ on $(0,1)$. Recall that this maximum is independent of $S_x(\mathbb{T}_x^{(\gamma)})$ and has moments of all orders. Since moreover $|\alpha|(m+1+n_0(\gamma)) \geq N(\gamma)$, we can apply Lemma 16 to deduce the existence of a constant $C(1,m)$ such that

$$E\left[H_F(x)^{m+1+n_0(\gamma)}\right] \leq C(1,m)x^{\gamma} \text{ for } x \text{ in } (0,1).$$

Now suppose that for some fixed n and every $m \geq 0$,

$$E\left[H_F(x)^{m+n+n_0(\gamma)}\right] \leq C(n,m)x^{\gamma n} \quad \forall x \in (0,1).$$

Then,

$$\begin{aligned} E\left[\left(\overline{H}_F\left(P_x(\mathbb{T}_x^{(\gamma)})\right)\right)^{m+n+1+n_0(\gamma)} \mid \mathcal{H}\left(\mathbb{T}_x^{(\gamma)}\right)\right] &\leq C(n,m+1)\left(P_x(\mathbb{T}_x^{(\gamma)})\right)^{\gamma n} \\ &\leq C(n,m+1)\left(S_x(\mathbb{T}_x^{(\gamma)})\right)^{-\gamma n} x^{\gamma n} \end{aligned}$$

since $P_x(\mathbb{T}_x^{(\gamma)}) \leq x/S_x(\mathbb{T}_x^{(\gamma)})$. Next, by (18),

$$\begin{aligned} E\left[H_F(x)^{m+n+1+n_0(\gamma)}\right] &\leq 2^{m+n+1+n_0(\gamma)} E\left[\left(\mathbb{T}_x^{(\gamma)}\right)^{m+n+1+n_0(\gamma)}\right] \\ &+ 2^{m+n+1+n_0(\gamma)} C(n,m+1) E\left[\left(S_x(\mathbb{T}_x^{(\gamma)})\right)^{|\alpha|(m+n+1+n_0(\gamma))-\gamma n}\right] x^{\gamma n}. \end{aligned}$$

Since $\gamma < |\alpha|$, the exponent $|\alpha|(m+n+1+n_0(\gamma)) - \gamma n \geq N(\gamma)$, and hence Lemma 16 applies to give, together with the previous inequality, the existence of a finite constant $C(n+1,m)$ such that

$$E\left[H_F(x)^{m+n+1+n_0(\gamma)}\right] \leq C(n+1,m)x^{\gamma(n+1)}$$

for every x in $(0,1)$. This holds for every m and hence the induction, formula (17) and Theorem 4 are proved.

4.3.2 Maximal Hölder exponent of the height process

The aim of this subsection is to prove that a.s. H_F cannot be Hölder-continuous of order γ for any $\gamma > \vartheta_{\mathrm{up}} \wedge |\alpha|/\varrho$.

We first prove that H_F cannot be Hölder-continuous with an exponent γ larger than ϑ_{up}. To see this, consider the interval fragmentation I_F and let U be a r.v. independent of I_F and with the uniform law on $(0,1)$. By Corollary 2 in [7], there is a subordinator $(\theta(t), t \geq 0)$ with no drift and a Lévy measure given by

$$\pi_\theta(dx) = e^{-x} \sum_{i=1}^{\infty} \nu(-\log s_i \in dx), x \in (0,\infty),$$

such that the length of the interval component of I_F containing U at time t is equal to $\exp(-\theta(\rho_\theta(t)))$, $t \geq 0$, ρ_θ being the time-change

$$\rho_\theta(t) = \inf\left\{u \geq 0 : \int_0^u \exp\left(\alpha\theta(r)\right) dr > t\right\}, t \geq 0.$$

Denoting by Leb the Lebesgue measure on $(0,1)$, we then have that

$$\text{Leb}\,\{x\in(0,1):H_F(x)>t\}\geq\exp(-\theta(\rho_\theta(t))). \tag{20}$$

On the other hand, recall that H_F is anyway a.s. continuous and introduce for every $t>0$

$$x_t:=\inf\{x:H_F(x)=t\},$$

so that $x<x_t\Rightarrow H_F(x)<t$. Hence $x_t\leq$Leb$\{x\in(0,1):H_F(x)<t\}$ and this yields, together with (20),

$$x_t\leq 1-\exp(-\theta(\rho_\theta(t))\text{ a.s. for every }t\geq 0.$$

Now suppose that H_F is a.s. Hölder-continuous of order γ. The previous inequality then gives

$$t=H_F(x_t)\leq Cx_t^\gamma\leq C\left(\theta(\rho_\theta(t))\right)^\gamma \tag{21}$$

so that it is sufficient to study the behavior of $\theta(\rho_\theta(t))$ as $t\to 0$ to obtain an upper bound for γ. It is easily seen that $\rho_\theta(t)\sim t$ as $t\downarrow 0$, so we just have to focus on the behavior of $\theta(t)$ as $t\to 0$. By [5, Theorem III.4.9], for every $\delta>1$, $\lim_{t\to 0}\left(\theta(t)/t^\delta\right)=0$ as soon as $\int_0^1\overline{\pi_\theta}(t^\delta)dt<\infty$, where $\overline{\pi_\theta}(t^\delta)=\int_{t^\delta}^\infty\pi_\theta(dx)$. To see when this quantity is integrable near 0, remark first that

$$\overline{\pi_\theta}(u)=\overline{\pi_\theta}(1)+\int_u^1 e^{-x}\nu(-\log s_1\in dx)\text{ when }u<1,$$

(since $s_i\leq 1/2$ for $i\geq 2$) and second that

$$\int_u^1 e^{-x}\nu(-\log s_1\in dx)\leq\nu(s_1<e^{-u}).$$

Hence,

$$\int_0^1\overline{\pi_\theta}(t^\delta)dt\leq\overline{\pi_\theta}(1)+\int_0^1\nu(s_1<e^{-t^\delta})dt$$

and by the definition of ϑ_{up} this integral is finite as soon as $1/\delta>\vartheta_{\text{up}}$. Thus $\lim_{t\to 0}\left(\theta(t)/t^\delta\right)=0$ for every $\delta<1/\vartheta_{\text{up}}$ and this implies, recalling (21), that $\gamma\delta<1$ for every $\delta<1/\vartheta_{\text{up}}$. Which gives $\gamma\leq\vartheta_{\text{up}}$.

It remains to prove that H_F cannot be Hölder-continuous with an exponent γ larger than $|\alpha|/\varrho$. This is actually a consequence of the results we have on the minoration of $\dim_{\mathcal{H}}(\mathcal{T}_F)$. Indeed, recall the definition of the function $\tilde{H}_F:(0,1)\to\mathcal{T}_F$ introduced Section 4.1 and in particular that for $0<x<y<1$

$$d\left(\tilde{H}_F(x),\tilde{H}_F(y)\right)=H_F(x)+H_F(y)-2\inf_{z\in[x,y]}H_F(z),$$

which shows that the γ-Hölder continuity of H_F implies that of $\tilde{H}_F$. It is now well known that, since $\tilde{H}_F:(0,1)\to\mathcal{T}_F$, the γ-Hölder continuity of $\tilde{H}_F$ leads to $\dim_{\mathcal{H}}(\mathcal{T}_F)\leq\dim_{\mathcal{H}}((0,1))/\gamma=1/\gamma$. Hence H_F cannot be Hölder-continuous with an order $\gamma>1/\dim_{\mathcal{H}}(\mathcal{T}_F)$. Recall then that $\dim_{\mathcal{H}}(\mathcal{T}_F)\geq\varrho/|\alpha|$. Hence H_F cannot be Hölder-continuous with an order $\gamma>|\alpha|/\varrho$.

4.4 Height process of the stable tree

To prove Corollary 3, we will check that $\nu_-(1-s_1>x)\sim Cx^{1/\beta-1}$ for some $C>0$ as $x\downarrow 0$, where ν_- is the dislocation measure of the fragmentation F_- associated with the stable (β) tree. In view of Theorem 4 this is sufficient, since the index of self-similarity is $1/\beta-1$ and $\int_S\left(s_1^{-1}-1\right)\nu(\mathrm{d}\mathbf{s})<\infty$, as proved in Sect. 3.5. Recalling the definition of ν_- in Sect. 3.5 and the notations therein, we want to prove

$$E\left[T_1\mathbb{1}_{\{1-\Delta_1/T_1>x\}}\right]\sim Cx^{1/\beta-1}\qquad \text{as } x\downarrow 0$$

Using the above notations, the quantity on the left can be rewritten as

$$E\left[(\Delta_1+T_1^{(\Delta_1)})\mathbb{1}_{\left\{T_1^{(\Delta_1)}/(\Delta_1+T_1^{(\Delta_1)})>x\right\}}\right]=E\left[\Delta_1(1+\Delta_1^{-1}T_1^{(\Delta_1)})\mathbb{1}_{\left\{\Delta_1^{-1}T_1^{(\Delta_1)}>x(1-x)^{-1}\right\}}\right].$$

Recalling the law of Δ_1 and the fact that $v^{-1}T_1^{(v)}$ has same law as $T^{(1)}_{v^{-1/\beta}}$, this is

$$c\int_0^\infty \mathrm{d}v\, v^{-1/\beta}e^{-c\beta v^{-1/\beta}}E\left[(1+T^{(1)}_{v^{-1/\beta}})\mathbb{1}_{\left\{T^{(1)}_{v^{-1/\beta}}>x(1-x)^{-1}\right\}}\right].$$

By [25, Proposition 28.3], since $T^{(1)}$ and T share the same Lévy measure on a neighborhood of 0, $T_v^{(1)}$ admits a continuous density $q_v^{(1)}(x), x\geq 0$ for every $v>0$. We thus can rewrite the preceding quantity as

$$c\int_0^\infty\frac{\mathrm{d}v}{v^{1/\beta}}e^{-c\beta v^{-1/\beta}}\int_{x/(1-x)}^\infty(1+u)q^{(1)}_{v^{-1/\beta}}(u)\mathrm{d}u=c\beta\int_{x/(1-x)}^\infty \mathrm{d}u(1+u)\int_0^\infty\frac{\mathrm{d}w}{w^\beta}e^{-c\beta w}q_w^{(1)}(u)$$

by Fubini's theorem and the change of variables $w=v^{-1/\beta}$. The behavior of this as $x\downarrow 0$ is the same as that of $c\beta J(x)$ where $J(x)=\int_x^\infty \mathrm{d}u j(u)$, and where $j(u)=\int_0^\infty \mathrm{d}w\, w^{-\beta}e^{-c\beta w}q_w^{(1)}(u)$. Write $\mathcal{J}(x)=\int_0^x J(u)\mathrm{d}u$ for $x\geq 0$, and consider the Stieltjes-Laplace transform $\hat{\mathcal{J}}$ of $\mathcal{J}$ evaluated at $\lambda\geq 0$:

$$\begin{aligned}\hat{\mathcal{J}}(\lambda)=\int_0^\infty e^{-\lambda u}J(u)\mathrm{d}u &= \lambda^{-1}\int_0^\infty(1-e^{-\lambda u})j(u)\mathrm{d}u\\ &= \lambda^{-1}\int_0^\infty\frac{\mathrm{d}w}{w^\beta}e^{-c\beta w}\int_0^\infty \mathrm{d}u q_w^{(1)}(u)(1-e^{-\lambda u})\\ &= \lambda^{-1}\int_0^\infty\frac{\mathrm{d}w}{w^\beta}e^{-c\beta w}(1-e^{-w\Phi^{(1)}(\lambda)})\end{aligned}$$

where as above $\Phi^{(1)}(\lambda)=c\int_0^1 u^{-1-1/\beta}(1-e^{-\lambda u})\mathrm{d}u$. Integrating by parts yields

$$\begin{aligned}\hat{\mathcal{J}}(\lambda) &= \frac{\lambda^{-1}}{\beta-1}\int_0^\infty\frac{\mathrm{d}w}{w^{\beta-1}}e^{-c\beta w}((c\beta+\Phi^{(1)}(\lambda))e^{-w\Phi^{(1)}(\lambda)}-c\beta)\\ &= \lambda^{-1}\frac{\Gamma(2-\beta)}{\beta-1}((c\beta+\Phi^{(1)}(\lambda))^{\beta-1}-(c\beta)^{\beta-1})\end{aligned}$$

Changing variables in the definition of $\Phi^{(1)}$, we easily obtain that $\Phi^{(1)}(\lambda)\sim C\lambda^{1/\beta}$ as $\lambda\to\infty$ for some $C>0$, so finally we obtain that $\hat{\mathcal{J}}(\lambda)\sim C\lambda^{-1/\beta}$ as $\lambda\to\infty$ for some other $C>0$. Since

$\mathcal{J}$ is non-decreasing, Feller's version of Karamata's Tauberian theorem [10, Theorem 1.7.1'] gives $\mathcal{J}(x) \sim Cx^{1/\beta}$ as $x \downarrow 0$, and since J is monotone, the monotone convergence theorem [10, Theorem 1.7.2b] gives $J(x) \sim \beta^{-1}Cx^{1/\beta-1}$ as $x \downarrow 0$, as wanted.

Acknowledgement

Many thanks to the referee for a very careful reading of this paper.

References

[1] D. J. Aldous, *Exchangeability and related topics*, in École d'été de probabilités de Saint-Flour, XIII—1983, vol. 1117 of Lecture Notes in Math., Springer, Berlin, 1985, pp. 1–198.

[2] ——, *The continuum random tree. III*, Ann. Probab., 21 (1993), pp. 248–289.

[3] D. J. Aldous and J. Pitman, *The standard additive coalescent*, Ann. Probab., 26 (1998), pp. 1703–1726.

[4] J. Berestycki, *Ranked fragmentations*, ESAIM Probab. Statist., 6 (2002), pp. 157–175 (electronic). Available via `http://www.emath.fr/ps/`.

[5] J. Bertoin, *Lévy processes*, Cambridge University Press, Cambridge, 1996.

[6] ——, *Homogeneous fragmentation processes*, Probab. Theory Relat. Fields, 121 (2001), pp. 301–318.

[7] ——, *Self-similar fragmentations*, Ann. Inst. Henri Poincare Probab. Stat., 38 (2002), pp. 319–340.

[8] ——, *The asymptotic behavior of fragmentation processes*, J. Eur. Math. Soc. JEMS, 5 (2003), pp. 395–416.

[9] J. Bertoin and M. Yor, *On the entire moments of self-similar Markov processes and exponential functionals of Lévy processes*, Ann. Fac. Sci. Toulouse VI. Ser. Math., 11 (2002), no. 1, pp. 33–45.

[10] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular variation*, vol. 27 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1989.

[11] P. Carmona, F. Petit, and M. Yor, *On the distribution and asymptotic results for exponential functionals of Lévy processes*, in Exponential functionals and principal values related to Brownian motion, Bibl. Rev. Mat. Iberoamericana, Rev. Mat. Iberoamericana, Madrid, 1997, pp. 73–130.

[12] C. Dellacherie, B. Maisonneuve, and P. A. Meyer, *Probabilités et Potentiel, Processus de Markov (fin), Compléments de Calcul Stochastique*, Herrmann , Paris, 1992.

[13] A. W. M. Dress and W. F. Terhalle, *The real tree*, Adv. Math., 120 (1996), pp. 283–301.

[14] T. Duquesne and J.-F. Le Gall, *Random trees, Lévy processes and spatial branching processes*, Astérisque, 281 (2002), pp. vi+147.

[15] ——, *Probabilistic and fractal aspects of Lévy trees*, in preparation (2003).

[16] K. J. Falconer, *The geometry of fractal sets*, vol. 85 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 1986.

[17] A. V. Gnedin, *The representation of composition structures*, Ann. Probab., 25 (1997), pp. 1437–1450.

[18] B. Haas, *Loss of mass in deterministic and random fragmentations*, Stoch. Proc. App., 106 (2003), pp. 245-277.

[19] ——, *Regularity of formation of dust in self-similar fragmentations*, Ann. Inst. Henri Poincare Probab. Stat. (2004). To appear.

[20] J. F. C. Kingman, *The representation of partition structures*, J. London Math. Soc. (2), 18 (1978), pp. 374–380.

[21] ——, *Poisson processes*, vol. 3 of Oxford Studies in Probability, The Clarendon Press Oxford University Press, New York, 1993. Oxford Science Publications.

[22] J.-F. Le Gall, *The uniform random tree in a Brownian excursion*, Probab. Theory Relat. Fields, 96 (1993), pp. 369–383.

[23] G. Miermont, *Self-similar fragmentations derived from the stable tree I: splitting at heights*, Probab. Theory Relat. Fields, 127 (2003), pp. 423–454.

[24] M. Perman, *Order statistics for jumps of normalised subordinators*, Stochastic Process. Appl., 46 (1993), pp. 267–281.

[25] K.-i. Sato, *Lévy processes and infinitely divisible distributions*, vol. 68 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1999.