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The non-linear stochastic wave equation in high dimensions

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Abstract

We propose an extension of Walsh's classical martingale measure stochastic integral that makes it possible to integrate a general class of Schwartz distributions, which contains the fundamental solution of the wave equation, even in dimensions greater than 3. This leads to a square-integrable random-field solution to the non-linear stochastic wave equation in any dimension, in the case of a driving noise that is white in time and correlated in space. In the particular case of an affine multiplicative noise, we obtain estimates on p-th moments of the solution ($p \geqslant 1$), and we show that the solution is Hölder continuous. The Hölder exponent that we obtain is optimal .

Key words: Martingale measures, stochastic integration, stochastic wave equation, stochastic partial differential equations, moment formulae, Hölder continuity.

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1 Introduction

In this paper, we are interested in random field solutions to the stochastic wave equation

$$\frac{\partial^2}{\partial t^2}u(t,x) - \Delta u(t,x) = \alpha(u(t,x))\dot{F}(t,x) + \beta(u(t,x)), \qquad t > 0,\ x \in \mathbb{R}^d, \tag{1.1}$$

with vanishing initial conditions. In this equation, $d \geqslant 1$, Δ denotes the Laplacian on $\mathbb{R}^d$, the functions $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous and $\dot{F}$ is a spatially homogeneous Gaussian noise that is white in time. Informally, the covariance functional of $\dot{F}$ is given by

$$\mathbb{E}[\dot{F}(t,x)\dot{F}(s,y)] = \delta(t-s)f(x-y), \qquad s,t \geqslant 0,\ x,y \in \mathbb{R}^d,$$

where δ denotes the Dirac delta function and $f : \mathbb{R}^d \to \mathbb{R}_+$ is continuous on $\mathbb{R}^d \setminus \{0\}$ and even.

We recall that a random field solution to (1.1) is a family of random variables $(u(t,x),\ t \in \mathbb{R}_+,\ x \in \mathbb{R}^d)$ such that $(t,x) \mapsto u(t,x)$ from $\mathbb{R}_+ \times \mathbb{R}^d$ into $L^2(\Omega)$ is continuous and solves an integral form of (1.1): see Section 4. Having a random field solution is interesting if, for instance, one wants to study the probability density function of the random variable $u(t,x)$ for each (t,x), as in [12]. A different notion is the notion of function-valued solution, which is a process $t \to u(t)$ with values in a space such as $L^2(\Omega, L^2_{\mathrm{loc}}(\mathbb{R}^d, dx))$ (see for instance [7], [4]). In some cases, such as [6], a random field solution can be obtained from a function-valued solution by establishing (Hölder) continuity properties of $(t,x) \mapsto u(t,x)$, but such results are not available for the stochastic wave equation in dimensions $d \geqslant 4$. In other cases (see [3]), the two notions are genuinely distinct (since the latter would correspond to $(t,x) \mapsto u(t,x)$ from $\mathbb{R}_+ \times \mathbb{R}^d$ into $L^2(\Omega)$ is merely measurable), and one type of solution may exist but not the other. We recall that function-valued solutions to (1.1) have been obtained in all dimensions [14] and that random field solutions have only been shown to exist when $d \in \{1,2,3\}$ (see [1]).

In spatial dimension 1, a solution to the non-linear wave equation driven by space-time white noise was given in [24], using Walsh's martingale measure stochastic integral. In dimensions 2 or higher, there is no function-valued solution with space-time white noise as a random input: some spatial correlation is needed in this case. In spatial dimension 2, a necessary and sufficient condition on the spatial correlation for existence of a random field solution was given in [2]. Study of the probability law of the solution is carried out in [12].

In spatial dimension $d = 3$, existence of a random field solution to (1.1) is given in [1]. Since the fundamental solution in this dimension is not a function, this required an extension of Walsh's martingale measure stochastic integral to integrands that are (Schwartz) distributions. This extension has nice properties when the integrand is a non-negative measure, as is the case for the fundamental solution of the wave equation when $d = 3$. The solution constructed in [1] had moments of all orders but no spatial sample path regularity was established. Absolute continuity and smoothness of the probability law was studied in [16] and [17] (see also the recent paper [13]). Hölder continuity of the solution was only recently established in [6], and sharp exponents were also obtained.

In spatial dimension $d \geqslant 4$, random field solutions were only known to exist in the case of the linear wave equation ($\alpha \equiv 1$, $\beta \equiv 0$). The methods used in dimension 3 do not apply to higher dimensions, because for $d \geqslant 4$, the fundamental solution of the wave equation is not a measure, but a Schwartz distribution that is a derivative of some order of a measure (see Section 5). It was therefore not even clear that the solution to (1.1) should be Hölder continuous, even though this is known to be the case for the linear equation (see [20]), under natural assumptions on the covariance function f.

In this paper, we first extend (in Section 3) the construction of the stochastic integral given in [1], so as to be able to define

$$\int_0^t \int_{\mathbb{R}^d} S(s,x)Z(s,x)M(ds,dx)$$

in the case where $M(ds,dx)$ is the martingale measure associated with the Gaussian noise $\dot{F}$, $Z(s,x)$ is an L^2-valued random field with spatially homogeneous covariance, and S is a Schwartz distribution, that is not necessarily non-negative (as it was in [1]). Among other technical conditions, S must satisfy the following condition, that also appears in [14]:

$$\int_0^t ds \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\,|\mathcal{F}S(s)(\xi+\eta)|^2 < \infty,$$

where μ is the spectral measure of $\dot{F}$ (that is, $\mathcal{F}\mu = f$, where $\mathcal{F}$ denotes the Fourier transform). With this stochastic integral, we can establish (in Section 4) existence of a random field solution of a wide class of stochastic partial differential equations (s.p.d.e.'s), that contains (1.1) as a special case, in all spatial dimensions d (see Section 5).

However, for $d \geqslant 4$, we do not know in general if this solution has moments of all orders. We recall that higher order moments, and, in particular, estimates on high order moments of increments of a process, are needed for instance to apply Kolmogorov's continuity theorem and obtain Hölder continuity of sample paths of the solution.

In Section 6, we consider the special case where α is an affine function and $\beta \equiv 0$. This is analogous to the hyperbolic Anderson problem considered in [5] for $d \leqslant 3$. In this case, we show that the solution to (1.1) has moments of all orders, by using a series representation of the solution in terms of iterated stochastic integrals of the type defined in Section 3.

Finally, in Section 7, we use the results of Section 6 to establish Hölder continuity of the solution to (1.1) (Propositions 7.1 and 7.2) for α affine and $\beta \equiv 0$. In the case where the covariance function is a Riesz kernel, we obtain the optimal Hölder exponent, which turns out to be the same as that obtained in [6] for dimension 3.

2 Framework

In this section, we recall the framework in which the stochastic integral is defined. We consider a Gaussian noise $\dot{F}$, white in time and correlated in space. Its covariance function is informally given by

$$\mathbb{E}[\dot{F}(t,x)\dot{F}(s,y)] = \delta(t-s)f(x-y), \qquad s,t \geqslant 0,\ x,y \in \mathbb{R}^d,$$

where δ stands for the Dirac delta function and $f : \mathbb{R}^d \to \mathbb{R}_+$ is continuous on $\mathbb{R}^d \setminus \{0\}$ and even. Formally, let $\mathcal{D}(\mathbb{R}^{d+1})$ be the space of C^∞-functions with compact support and let $F = \{F(\varphi),\ \varphi \in \mathcal{D}(\mathbb{R}^{d+1})\}$ be an $L^2(\Omega,\mathcal{F},\mathbb{P})$-valued mean zero Gaussian process with covariance functional

$$\mathbb{E}[F(\varphi)F(\psi)] = \int_0^\infty dt \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, \varphi(t,x)f(x-y)\psi(t,y).$$

Since f is a covariance, there exists a non-negative tempered measure μ whose Fourier transform is f. That is, for all $\phi \in \mathcal{S}(\mathbb{R}^d)$, the Schwartz space of C^∞-functions with rapid decrease, we have

$$\int_{\mathbb{R}^d} f(x)\phi(x)dx = \int_{\mathbb{R}^d} \mathcal{F}\phi(\xi)\mu(d\xi).$$

As f is the Fourier transform of a tempered measure, it satisfies an integrability condition of the form

$$\int_{\mathbb{R}^d} \frac{f(x)}{1+|x|^p}\, dx < \infty, \tag{2.1}$$

for some $p < \infty$ (see [21, Theorem XIII, p.251]).

Following [2], we extend this process to a worthy martingale measure $M = (M_t(B),\ t \geqslant 0,\ B \in \mathcal{B}_b(\mathbb{R}^d))$, where $\mathcal{B}_b(\mathbb{R}^d)$ denotes the bounded Borel subsets of $\mathbb{R}$, in such a way that for all $\varphi \in \mathcal{S}(\mathbb{R}^{d+1})$,

$$F(\varphi) = \int_0^\infty \int_{\mathbb{R}^d} \varphi(t,x) M(dt,dx),$$

where the stochastic integral is Walsh's stochastic integral with respect to the martingale measure M (see [24]). The covariation and dominating measure Q and K of M are given by

$$\begin{aligned} Q([0,t]\times A\times B) &= K([0,t]\times A\times B) \\ &= \langle M(A), M(B)\rangle_t = t\int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, \mathbf{1}_A(x) f(x-y) \mathbf{1}_B(y). \end{aligned}$$

We consider the filtration $\mathcal{F}_t$ given by $\mathcal{F}_t = \mathcal{F}_t^0 \vee \mathcal{N}$, where

$$\mathcal{F}_t^0 = \sigma(M_s(B),\ s \leqslant t,\ B \in \mathcal{B}_b(\mathbb{R}^d))$$

and $\mathcal{N}$ is the σ-field generated by the $\mathbb{P}$-null sets.

Fix $T > 0$. The stochastic integral of predictable functions $g : \mathbb{R}_+ \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ such that $\|g\|_+ < \infty$, where

$$\|g\|_+^2 = \mathbb{E}\left[\int_0^T ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, |g(s,x,\cdot)|\, f(x-y)\, |g(s,y,\cdot)|\right],$$

is defined by Walsh (see [24]). The set of such functions is denoted by $\mathcal{P}_+$. Dalang [1] then introduced the norm $\|\cdot\|_0$ defined by

$$\|g\|_0^2 = \mathbb{E}\left[\int_0^T ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, g(s,x,\cdot) f(x-y) g(s,y,\cdot)\right]. \tag{2.2}$$

Recall that a function g is called *elementary* if it is of the form

$$g(s,x,\omega) = \mathbf{1}_{]a,b]}(s)\mathbf{1}_A(x)X(\omega), \tag{2.3}$$

where $0 \leqslant a < b \leqslant T$, $A \in \mathcal{B}_b(\mathbb{R}^d)$, and X is a bounded $\mathcal{F}_a$-measurable random variable. Now let $\mathcal{E}$ be the set of *simple functions*, i.e., the set of all finite linear combinations of elementary functions. Since the set of predictable functions such that $\|g\|_0 < \infty$ is not complete, let $\mathcal{P}_0$

denote the completion of the set of simple predictable functions with respect to $\|\cdot\|_0$. Clearly, $\mathcal{P}_+ \subset \mathcal{P}_0$. Both $\mathcal{P}_0$ and $\mathcal{P}_+$ can be identified with subspaces of $\overline{\mathcal{P}}$, where

$$\begin{aligned}\overline{\mathcal{P}} \quad := \quad & \Big\{t \mapsto S(t) \text{ from } [0,T]\times\Omega \to \mathcal{S}'(\mathbb{R}^d) \text{ predictable, such that } \mathcal{F}S(t) \text{ is a.s.}\\ & \text{a function and } \|S\|_0 < \infty\Big\},\end{aligned}$$

where

$$\|S\|_0^2 = \mathbb{E}\left[\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi)\,|\mathcal{F}S(t)(\xi)|^2\right]. \tag{2.4}$$

For $S(t) \in \mathcal{S}(\mathbb{R}^d)$, elementary properties of convolution and Fourier transform show that (2.2) and (2.4) are equal. When $d \geqslant 4$, the fundamental solution of the wave equation provides an example of an element of $\mathcal{P}_0$ that is not in $\mathcal{P}_+$ (see Section 5).

Consider a predictable process $(Z(t,x),\ 0 \leqslant t \leqslant T,\ x \in \mathbb{R}^d)$, such that

$$\sup_{0\leqslant t\leqslant T}\sup_{x\in\mathbb{R}^d} \mathbb{E}[Z(t,x)^2] < \infty.$$

Let M^Z be the martingale measure defined by

$$M_t^Z(B) = \int_0^t \int_B Z(s,y)M(ds,dy), \qquad 0 \leqslant t \leqslant T,\ B \in \mathcal{B}_b(\mathbb{R}^d),$$

in which we again use Walsh's stochastic integral [24]. We would like to give a meaning to the stochastic integral of a large class of $S \in \overline{\mathcal{P}}$ with respect to the martingale measure M^Z. Following the same idea as before, we will consider the norms $\|\cdot\|_{+,Z}$ and $\|\cdot\|_{0,Z}$ defined by

$$\|g\|_{+,Z}^2 = \mathbb{E}\left[\int_0^T ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\,|g(s,x,\cdot)Z(s,x)f(x-y)Z(s,y)g(s,y,\cdot)|\right]$$

and

$$\|g\|_{0,Z}^2 = \mathbb{E}\left[\int_0^T ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, g(s,x,\cdot)Z(s,x)f(x-y)Z(s,y)g(s,y,\cdot)\right]. \tag{2.5}$$

Let $\mathcal{P}_{+,Z}$ be the set of predictable functions g such that $\|g\|_{+,Z} < \infty$. The space $\mathcal{P}_{0,Z}$ is defined, similarly to $\mathcal{P}_0$, as the completion of the set of simple predictable functions, but taking completion with respect to $\|\cdot\|_{0,Z}$ instead of $\|\cdot\|_0$.

For $g \in \mathcal{E}$, as in (2.3), the stochastic integral $g\cdot M^Z = ((g\cdot M^Z)_t, 0 \leqslant t \leqslant T)$ is the square-integrable martingale

$$(g\cdot M^Z)_t = M_{t\wedge b}^Z(A) - M_{t\wedge a}^Z(A) = \int_0^t \int_{\mathbb{R}^d} g(s,y,\cdot)Z(s,y)M(ds,dy).$$

Notice that the map $g \mapsto g\cdot M^Z$, from $(\mathcal{E}, \|\cdot\|_{0,Z})$ into the Hilbert space $\mathcal{M}$ of continuous square-integrable $(\mathcal{F}_t)$-martingales $X = (X_t, 0 \leqslant t \leqslant T)$ equipped with the norm $\|X\| = \mathbb{E}[X_T^2]^{\frac{1}{2}}$, is an isometry. Therefore, this isometry can be extended to an isometry $S \mapsto S\cdot M^Z$ from $(\mathcal{P}_{0,Z}, \|\cdot\|_{0,Z})$ into $\mathcal{M}$. The square-integrable martingale $S\cdot M^Z = ((S\cdot M^Z)_t, 0 \leqslant t \leqslant T)$ is the *stochastic integral process of S with respect to M^Z*. We use the notation

$$\int_0^t \int_{\mathbb{R}^d} S(s,y)Z(s,y)M(ds,dy)$$

for $(S\cdot M^Z)_t$.

The main issue is to identify elements of $\mathcal{P}_{0,Z}$. We address this question in the next section.

3 Stochastic Integration

In this section, we extend Dalang's result concerning the class of Schwartz distributions for which the stochastic integral with respect to the martingale measure M^Z can be defined, by deriving a new inequality for this integral. In particular, contrary to [1, Theorem 2], the result presented here does not require that the Schwartz distribution be non-negative.

In Theorem 3.1 below, we show that the non-negativity assumption can be removed provided the spectral measure satisfies the condition (3.6) below, which already appears in [14] and [4]. As in [1, Theorem 3], an additional assumption similar to [1, (33), p.12] is needed (hypothesis (H2) below). This hypothesis can be replaced by an integrability condition (hypothesis (H1) below).

Suppose Z is a process such that $\sup_{0\leqslant s\leqslant T}\mathbb{E}[Z(s,0)^2] < +\infty$ and with spatially homogeneous covariance, that is $z \mapsto \mathbb{E}[Z(t,x)Z(t,x+z)]$ does not depend on x. Following [1, Theorem 3], set $f^Z(s,x) = f(x)g_s(x)$, where $g_s(x) = \mathbb{E}[Z(s,0)Z(s,x)]$.

For s fixed, the function g_s is non-negative definite, since it is a covariance function. Hence, there exists a non-negative tempered measure ν_s^Z such that $g_s = \mathcal{F}\nu_s^Z$. Note that $\nu_s^Z(\mathbb{R}^d) = g_s(0) = \mathbb{E}[Z(s,0)^2]$. Using the convolution property of the Fourier transform, we have

$$f^Z(s,\cdot) = f\cdot g_s = \mathcal{F}\mu\cdot\mathcal{F}\nu_s^Z = \mathcal{F}(\mu*\nu_s^Z),$$

where $*$ denotes convolution. Looking back to the definition of $\|\cdot\|_{0,Z}$, we obtain, for a deterministic $\varphi\in\mathcal{P}_{0,Z}$ with $\varphi(t,\cdot)\in\mathcal{S}(\mathbb{R}^d)$ for all $0\leqslant t\leqslant T$ (see [1, p.10]),

$$\begin{aligned}\|\varphi\|_{0,Z}^2 &= \int_0^T ds\int_{\mathbb{R}^d}dx\int_{\mathbb{R}^d}dy\,\varphi(s,x)f(x-y)g_s(x-y)\varphi(s,y)\\ &= \int_0^T ds\int_{\mathbb{R}^d}(\mu*\nu_s^Z)(d\xi)\,|\mathcal{F}\varphi(s,\cdot)(\xi)|^2\\ &= \int_0^T ds\int_{\mathbb{R}^d}\nu_s^Z(d\eta)\int_{\mathbb{R}^d}\mu(d\xi)\,|\mathcal{F}\varphi(s,\cdot)(\xi+\eta)|^2. \qquad (3.1)\end{aligned}$$

In particular,

$$\begin{aligned}\|\varphi\|_{0,Z}^2 &\leqslant \int_0^T ds\,\nu_s^Z(\mathbb{R}^d)\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\mu(d\xi)\,|\mathcal{F}\varphi(s,\cdot)(\xi+\eta)|^2\\ &\leqslant C\int_0^T ds\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\mu(d\xi)\,|\mathcal{F}\varphi(s,\cdot)(\xi+\eta)|^2, \qquad (3.2)\end{aligned}$$

where $C = \sup_{0\leqslant s\leqslant T}\mathbb{E}[Z(s,0)^2] < \infty$ by assumption. Taking (3.1) as the definition of $\|\cdot\|_{0,Z}$, we can extend this norm to the set $\overline{\mathcal{P}}_Z$, where

$$\begin{aligned}\overline{\mathcal{P}}_Z := \Big\{ & t\mapsto S(t) \text{ from } [0,T]\to\mathcal{S}'(\mathbb{R}^d) \text{ deterministic, such that } \mathcal{F}S(t) \text{ is}\\ & \text{a function and } \|S\|_{0,Z}<\infty\Big\}.\end{aligned}$$

The spaces $\mathcal{P}_{+,Z}$ and $\mathcal{P}_{0,Z}$ will now be considered as subspaces of $\overline{\mathcal{P}}_Z$. Let $S\in\overline{\mathcal{P}}_Z$. We will need the following two hypotheses to state the next theorem. Let $B(0,1)$ denote the open ball in $\mathbb{R}^d$ that is centered at 0 with radius 1.

(H1) For all $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $\varphi \geqslant 0$, $\mathrm{supp}(\varphi) \subset B(0,1)$, and $\int_{\mathbb{R}^d} \varphi(x)dx = 1$, and for all $0 \leqslant a \leqslant b \leqslant T$, we have

$$\int_a^b (S(t) * \varphi)(\cdot)\, dt \,\in\, \mathcal{S}(\mathbb{R}^d), \tag{3.3}$$

and

$$\int_{\mathbb{R}^d} dx \int_0^T ds\, |(S(s) * \varphi)(x)| < \infty. \tag{3.4}$$

(H2) The function $\mathcal{F}S(t)$ is such that

$$\lim_{h \downarrow 0} \int_0^T ds \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi) \sup_{s<r<s+h} |\mathcal{F}S(r)(\xi+\eta) - \mathcal{F}S(s)(\xi+\eta)|^2 = 0. \tag{3.5}$$

This hypothesis is analogous to [1, (33), p.12]. We let $\mathcal{S}'_r(\mathbb{R}^d)$ denote the space of Schwartz distributions with rapid decrease (see [21, p.244]). We recall that for $S \in \mathcal{S}'_r(\mathbb{R}^d)$, $\mathcal{F}S$ is a function (see [21, Chapter VII, Thm. XV, p.268]).

Theorem 3.1. *Let $(Z(t,x),\ 0 \leqslant t \leqslant T,\ x \in \mathbb{R}^d)$ be a predictable process with spatially homogeneous covariance such that $\sup_{0\leqslant t\leqslant T} \sup_{x\in\mathbb{R}^d} \mathbb{E}[Z(t,x)^2] < \infty$. Let $t \mapsto S(t)$ be a deterministic function with values in the space $\mathcal{S}'_r(\mathbb{R}^d)$. Suppose that $(s,\xi) \mapsto \mathcal{F}S(s)(\xi)$ is measurable and*

$$\int_0^T ds \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}S(s)(\xi+\eta)|^2 < \infty. \tag{3.6}$$

Suppose in addition that either hypothesis (H1) or (H2) is satisfied. Then $S \in \mathcal{P}_{0,Z}$. In particular, the stochastic integral $(S \cdot M^Z)_t$ is well defined as a real-valued square-integrable martingale $((S \cdot M^Z)_t,\ 0 \leqslant t \leqslant T)$ and

$$\begin{aligned} \mathbb{E}[(S\cdot M^Z)_t^2] &= \int_0^t ds \int_{\mathbb{R}^d} \nu_s^Z(d\eta) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}S(s)(\xi+\eta)|^2 \\ &\leqslant \left(\sup_{0\leqslant s\leqslant T} \sup_{x\in\mathbb{R}^d} \mathbb{E}[Z(s,x)^2] \right) \int_0^t ds \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}S(s)(\xi+\eta)|^2. \end{aligned} \tag{3.7}$$

Proof. We are now going to show that $S \in \mathcal{P}_{0,Z}$ and that its stochastic integral with respect to M^Z is well defined. We follow the approach of [1, proof of Theorem 3].

Take $\psi \in \mathcal{D}(\mathbb{R}^d)$ such that $\psi \geqslant 0$, $\mathrm{supp}(\psi) \subset B(0,1)$, $\int_{\mathbb{R}^d} \psi(x)dx = 1$. For all $n \geqslant 1$, take $\psi_n(x) = n^d \psi(nx)$. Then $\psi_n \to \delta_0$ in $\mathcal{S}'(\mathbb{R}^d)$ as $n \to \infty$. Moreover, $\mathcal{F}\psi_n(\xi) = \mathcal{F}\psi(\frac{\xi}{n})$ and $|\mathcal{F}\psi_n(\xi)| \leqslant 1$, for all $\xi \in \mathbb{R}^d$. Define $S_n(t) = (\psi_n * S)(t)$. As $S(t)$ is of rapid decrease, we have $S_n(t) \in \mathcal{S}(\mathbb{R}^d)$ (see [21], Chap. VII, §5, p.245).

Suppose that $S_n \in \mathcal{P}_{0,Z}$ for all n. Then

$$\begin{aligned} \|S_n - S\|_{0,Z}^2 &= \int_0^T ds \int_{\mathbb{R}^d} \nu_s^Z(d\eta) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}(S_n(s) - S(s))(\xi+\eta)|^2 \\ &= \int_0^T ds \int_{\mathbb{R}^d} \nu_s^Z(d\eta) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\psi_n(\xi+\eta) - 1|^2 |\mathcal{F}S(s)(\xi+\eta)|^2. \end{aligned} \tag{3.8}$$

The expression $|\mathcal{F}\psi_n(\xi+\eta)-1|^2$ is bounded by 4 and goes to 0 as $n\to\infty$ for every ξ and η. By (3.6), the Dominated Convergence Theorem shows that $\|S_n - S\|_{0,Z}\to 0$ as $n\to\infty$. As $\mathcal{P}_{0,Z}$ is complete, if $S_n\in\mathcal{P}_{0,Z}$ for all n, then $S\in\mathcal{P}_{0,Z}$.

To complete the proof, it remains to show that $S_n\in\mathcal{P}_{0,Z}$ for all n.

First consider assumption (H2). In this case, the proof that $S_n\in\mathcal{P}_{0,Z}$ is based on the same approximation as in [1]. For n fixed, we can write $S_n(t,x)$ because $S_n(t)\in\mathcal{S}(\mathbb{R}^d)$ for all $0\leqslant t\leqslant T$. The idea is to approximate S_n by a sequence of elements of $\mathcal{P}_{+,Z}$. For all $m\geqslant 1$, set

$$S_{n,m}(t,x)=\sum_{k=0}^{2^m-1} S_n(t_m^{k+1},x)\mathbf{1}_{[t_m^k,t_m^{k+1}[}(t), \tag{3.9}$$

where $t_m^k = kT2^{-m}$. Then $S_{n,m}(t,\cdot)\in\mathcal{S}(\mathbb{R}^d)$. We now show that $S_{n,m}\in\mathcal{P}_{+,Z}$. Being a deterministic function, $S_{n,m}$ is predictable. Moreover, using the definition of $\|\cdot\|_{+,Z}$ and the fact that $|g_s(x)|\leqslant C$ for all s and x, we have

$$\begin{aligned}
\|S_{n,m}\|_{+,Z}^2 &= \int_0^T ds\int_{\mathbb{R}^d}dx\int_{\mathbb{R}^d}dy\,|S_{n,m}(s,x)|\,f(x-y)\,|g_s(x-y)||S_{n,m}(s,y)|\\
&= \sum_{k=0}^{2^m-1}\int_{t_m^k}^{t_m^{k+1}} ds\int_{\mathbb{R}^d}dx\int_{\mathbb{R}^d}dy\,|S_n(t_m^{k+1},x)|\,f(x-y)\,|g_s(x-y)|\,|S_n(t_m^{k+1},y)|\\
&\leqslant C\sum_{k=0}^{2^m-1}\int_{t_m^k}^{t_m^{k+1}} ds\int_{\mathbb{R}^d}dz\,f(z)(|S_n(t_m^{k+1},\cdot)|*|\tilde{S}_n(t_m^{k+1},\cdot)|)(z),
\end{aligned}$$

where $\tilde{S}_n(t_m^{k+1},x)=S_n(t_m^{k+1},-x)$. By Leibnitz' formula (see [22], Ex. 26.4, p.283), the function $z\mapsto(|S_n(t_m^{k+1},\cdot)|*|\tilde{S}_n(t_m^{k+1},\cdot)|)(z)$ decreases faster than any polynomial in $|z|^{-1}$. Therefore, by (2.1), the preceding expression is finite and $\|S_{n,m}\|_{+,Z}<\infty$, and $S_{n,m}\in\mathcal{P}_{+,Z}\subset\mathcal{P}_{0,Z}$.

The sequence of elements of $\mathcal{P}_{+,Z}$ that we have constructed converges in $\|\cdot\|_{0,Z}$ to S_n. Indeed,

$$\begin{aligned}
\|S_{n,m}-S_n\|_{0,Z}^2 &= \int_0^T ds\int_{\mathbb{R}^d}\nu_s^Z(d\eta)\int_{\mathbb{R}^d}\mu(d\xi)\,|\mathcal{F}(S_{n,m}(s,\cdot)-S_n(s,\cdot))(\xi+\eta)|^2\\
&\leqslant \int_0^T ds\int_{\mathbb{R}^d}\nu_s^Z(d\eta)\int_{\mathbb{R}^d}\mu(d\xi)\sup_{s<r<s+T2^{-m}}|\mathcal{F}(S_n(r,\cdot)-S_n(s,\cdot))(\xi+\eta)|^2,
\end{aligned}$$

which goes to 0 as $m\to\infty$ by (H2). Therefore, $S_{n,m}\to S_n$ as $m\to\infty$ and $S_n\in\mathcal{P}_{0,Z}$. This concludes the proof under assumption (H2).

Now, we are going to consider assumption (H1) and check that $S_n\in\mathcal{P}_{0,Z}$ under this condition. We will take the same discretization of time to approximate S_n, but we will use the mean value over the time interval instead of the value at the right extremity. That is, we are going to consider

$$S_{n,m}(t,x)=\sum_{k=0}^{2^m-1} a_{n,m}^k(x)\mathbf{1}_{[t_m^k,t_m^{k+1}[}(t), \tag{3.10}$$

where $t_m^k = kT2^{-m}$ and

$$a_{n,m}^k(x)=\frac{2^m}{T}\int_{t_m^k}^{t_m^{k+1}} S_n(s,x)\,ds. \tag{3.11}$$

By (3.3) in assumption (H1), $a^k_{n,m} \in \mathcal{S}(\mathbb{R}^d)$ for all n, m and k. Moreover, using Fubini's theorem, which applies by (3.4) since $\int_{\mathbb{R}^d} dx \int_a^b ds\, |S_n(s,x)| < \infty$ for all $0 \leqslant a < b \leqslant T$, we have

$$\begin{aligned}
\mathcal{F}a^k_{n,m}(\xi) &= \frac{2^m}{T} \int_{\mathbb{R}^d} dx \int_{t^k_m}^{t^{k+1}_m} ds\, e^{-i\langle \xi, x\rangle} S_n(s,x) \\
&= \frac{2^m}{T} \int_{t^k_m}^{t^{k+1}_m} ds\, \mathcal{F}S_n(s,\cdot)(\xi).
\end{aligned}$$

We now show that $S_{n,m} \in \mathcal{P}_{+,Z}$. We only need to show that $a^k_{n,m}(x)\mathbf{1}_{[t^k_m, t^{k+1}_m[}(t) \in \mathcal{P}_{+,Z}$ for all $k = 1, \dots, 2^m - 1$. We have

$$\|a^k_{n,m}(\cdot)\mathbf{1}_{[t^k_m, t^{k+1}_m[}(\cdot)\|_{+,Z} \leqslant C\frac{2^m}{T} \int_{\mathbb{R}^d} dz\, f(z)(|a^k_{n,m}(\cdot)| * |\widetilde{a^k_{n,m}}(\cdot)|)(z),$$

where $\widetilde{a^k_{n,m}}(x) = a^k_{n,m}(-x)$. Since $a^k_{n,m} \in \mathcal{S}(\mathbb{R}^d)$, a similar argument as above, using Leibnitz' formula, shows that this expression is finite. Hence $S_{n,m} \in \mathcal{P}_{+,Z} \subset \mathcal{P}_{0,Z}$.

It remains to show that $S_{n,m} \to S_n$ as $m \to \infty$. Indeed,

$$\begin{aligned}
&\|S_{n,m} - S_n\|^2_{0,Z} \\
&= \int_0^T ds \int_{\mathbb{R}^d} \nu^Z_s(d\eta) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}(S_{n,m}(s,\cdot) - S_n(s,\cdot))(\xi+\eta)|^2 \\
&= \sum_{k=0}^{2^m-1} \int_{t^k_m}^{t^{k+1}_m} ds \int_{\mathbb{R}^d} \nu^Z_s(d\eta) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}a^k_{n,m}(\xi+\eta) - \mathcal{F}S_n(s,\cdot)(\xi+\eta)|^2 \\
&= \sum_{k=0}^{2^m-1} \int_{t^k_m}^{t^{k+1}_m} ds \int_{\mathbb{R}^d} \nu^Z_s(d\eta) \int_{\mathbb{R}^d} \mu(d\xi) \Bigg| \frac{2^m}{T} \int_{t^k_m}^{t^{k+1}_m} \mathcal{F}S_n(u,\cdot)(\xi+\eta) du \\
&\qquad - \mathcal{F}S_n(s,\cdot)(\xi+\eta) \Bigg|^2 .
\end{aligned} \tag{3.12}$$

We are going to show that the preceding expression goes to 0 as $m \to \infty$ using the martingale L^2-convergence theorem (see [9, thm 4.5, p.252]). Take $\Omega = \mathbb{R}^d \times \mathbb{R}^d \times [0,T]$, endowed with the σ-field $\mathcal{F} = \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}([0,T])$ of Borel subsets and the measure $\mu(d\xi) \times \nu^Z_s(d\eta) \times ds$. We also consider the filtration $(\mathcal{H}_m = \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{G}_m)_{m \geqslant 0}$, where $\mathcal{G}_m = \sigma([t^k_m, t^{k+1}_m[,\ k = 0, \dots, 2^m - 1)$. For n fixed, we consider the function $X : \Omega \to \mathbb{R}$ given by $X(\xi, \eta, s) = \mathcal{F}S_n(s,\cdot)(\xi+\eta)$. This function is in $L^2(\Omega, \mathcal{F}, \mu(d\xi) \times \nu^Z_s(d\eta) \times ds)$. Indeed,

$$\begin{aligned}
&\int_0^T ds \int_{\mathbb{R}^d} \nu^Z_s(d\eta) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}S_n(s,\cdot)(\xi+\eta)|^2 \\
&\qquad\qquad \leqslant C \int_0^T ds \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}S(s,\cdot)(\xi+\eta)|^2,
\end{aligned}$$

which is finite by assumption (3.6). Then, setting

$$X_m = \mathbb{E}_{\mu(d\xi)\times\nu^Z_s(d\eta)\times ds}[X|\mathcal{H}_m] = \sum_{k=0}^{2^m-1} \left(\frac{2^m}{T} \int_{t^k_m}^{t^{k+1}_m} \mathcal{F}S_n(u,\cdot)(\xi+\eta) du \right) \mathbf{1}_{[t^k_m, t^{k+1}_m[}(s),$$

we have that $(X_m)_{m\geqslant 0}$ is a martingale. Moreover,

$$\sup_m \mathbb{E}_{\mu(d\xi)\times\nu_s^Z(d\eta)\times ds}[X_m^2] \leqslant \mathbb{E}_{\mu(d\xi)\times\nu_s^Z(d\eta)\times ds}[X^2] < \infty.$$

The martingale L^2-convergence theorem then shows that (3.12) goes to 0 as $m\to\infty$ and hence that $S_n \in \mathcal{P}_{0,Z}$.

Now, by the isometry property of the stochastic integral between $\mathcal{P}_{0,Z}$ and the set $\mathcal{M}^2$ of square-integrable martingales, $(S\cdot M^Z)_t$ is well-defined and

$$\mathbb{E}[(S\cdot M^Z)_T^2] = \|S\|_{0,Z}^2 = \int_0^T ds \int_{\mathbb{R}^d} \nu_s^Z(d\eta) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}S(s,\cdot)(\xi+\eta)|^2.$$

The bound in the second part of (3.7) is obtained as in (3.2). The result is proved. ■

Remark 3.2. As can be seen by inspecting the proof, Theorem 3.1 is still valid if we replace (H2) by the following assumptions :

- $t\mapsto \mathcal{F}S(t)(\xi)$ is continuous in t for all $\xi\in\mathbb{R}^d$;
- there exists a function $t\mapsto k(t)$ with values in the space $\mathcal{S}'_r(\mathbb{R}^d)$ such that, for all $0\leqslant t\leqslant T$ and $h\in[0,\varepsilon]$,

$$|\mathcal{F}S(t+h)(\xi) - \mathcal{F}S(t)(\xi)| \leqslant |\mathcal{F}k(t)(\xi)|,$$

  and

$$\int_0^T ds \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}k(s)(\xi+\eta)|^2 < +\infty.$$

Remark 3.3. There are two limitations to our construction of the stochastic integral in Theorem 3.1. The first concerns stationarity of the covariance of Z. Under certain conditions (which, in the case where S is the fundamental solution of the wave equation, only hold for $d\leqslant 3$), Nualart and Quer-Sardanyons [13] have removed this assumption. The second concerns positivity of the covariance function f. A weaker condition appears in [14], where function-valued solutions are studied.

Integration with respect to Lebesgue measure

In addition to the stochastic integral defined above, we will have to define the integral of the product of a Schwartz distribution and a spatially homogeneous process with respect to Lebesgue measure. More precisely, we have to give a precise definition to the process informally given by

$$t\mapsto \int_0^t ds \int_{\mathbb{R}^d} dy\, S(s,y)Z(s,y),$$

where $t\mapsto S(t)$ is a deterministic function with values is the space of Schwartz distributions with rapid decrease and Z is a stochastic process, both satisfying the assumptions of Theorem 3.1.

In addition, suppose first that $S \in L^2([0,T], L^1(\mathbb{R}^d))$. By Hölder's inequality, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T ds \int_{\mathbb{R}^d} dx\, |S(s,x)||Z(s,x)|\right)^2\right] \\
&\leqslant\ C\mathbb{E}\left[\int_0^T ds \left(\int_{\mathbb{R}^d} dx\, |S(s,x)||Z(s,x)|\right)^2\right] \\
&\leqslant\ C\int_0^T ds \int_{\mathbb{R}^d} dx\, |S(s,x)| \int_{\mathbb{R}^d} dy\, |S(s,y)|\, \mathbb{E}[|Z(s,x)||Z(s,y)|] \\
&\leqslant\ C\int_0^T ds \int_{\mathbb{R}^d} dx\, |S(s,x)| \int_{\mathbb{R}^d} dy\, |S(s,y)| < \infty, \qquad (3.13)
\end{aligned}$$

by the assumptions on Z. Hence $\int_0^T ds \int_{\mathbb{R}^d} dx\, |S(s,x)||Z(s,x)| < \infty$ a.s. and the process

$$\int_0^t ds \int_{\mathbb{R}^d} dx\, S(s,x)Z(s,x), \qquad t \geqslant 0,$$

is a.s. well-defined as a Lebesgue-integral. Moreover,

$$\begin{aligned}
0 &\leqslant \mathbb{E}\left[\left(\int_0^T ds \int_{\mathbb{R}^d} dx\, S(s,x)Z(s,x)\right)^2\right] \\
&= \int_0^T ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, S(s,x)S(s,y)\, \mathbb{E}[Z(s,x)Z(s,y)] \\
&= \int_0^T ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, S(s,x)S(s,y) g_s(x-y) \\
&= \int_0^T ds \int_{\mathbb{R}^d} \nu_s^Z(d\eta)\, |\mathcal{F}S(s)(\eta)|^2, \qquad (3.14)
\end{aligned}$$

where ν_s^Z is the measure such that $\mathcal{F}\nu_s^Z = g_s$. Let us define a norm $\|\cdot\|_{1,Z}$ on the space $\overline{\mathcal{P}}_Z$ by

$$\|S\|_{1,Z}^2 = \int_0^T ds \int_{\mathbb{R}^d} \nu_s^Z(d\eta)\, |\mathcal{F}S(s)(\eta)|^2. \qquad (3.15)$$

This norm is similar to $\|\cdot\|_{0,Z}$, but with $\mu(d\xi) = \delta_0(d\xi)$. In order to establish the next proposition, we will need the following assumption.

(H2*) The function $\mathcal{F}S(s)$ is such that

$$\lim_{h\downarrow 0} \int_0^T ds \sup_{\eta\in\mathbb{R}^d} \sup_{s<r<s+h} |\mathcal{F}S(r)(\eta) - \mathcal{F}S(s)(\eta)|^2 = 0. \qquad (3.16)$$

This hypothesis is analogous to (H2) but with $\mu(d\xi) = \delta_0(d\xi)$.

Proposition 3.4. *Let $(Z(t,x),\ 0 \leqslant t \leqslant T,\ x \in \mathbb{R}^d)$ be a stochastic process satisfying the assumptions of Theorem 3.1. Let $t \mapsto S(t)$ be a deterministic function with values in the space $\mathcal{S}'_r(\mathbb{R}^d)$. Suppose that $(s,\xi) \mapsto \mathcal{F}S(s)(\xi)$ is measurable and*

$$\int_0^T ds \sup_{\eta\in\mathbb{R}^d} |\mathcal{F}S(s)(\eta)|^2 < \infty. \qquad (3.17)$$

Suppose in addition that either hypothesis (H1) or (H2) is satisfied. Then*

$$\mathbb{E}\left[\left(\int_0^T ds \int_{\mathbb{R}^d} dx\, S(s,x)Z(s,x)\right)^2\right]$$
$$= \qquad \|S\|_{1,Z}^2 \quad \leqslant \quad C\left(\sup_{0\leqslant s\leqslant T}\sup_{x\in\mathbb{R}^d}\mathbb{E}[Z(s,x)^2]\right)\int_0^T ds \sup_{\eta\in\mathbb{R}^d}|\mathcal{F}S(s)(\eta)|^2.$$

In particular, the process $\left(\int_0^t ds \int_{\mathbb{R}^d} dx\, S(s,x)Z(s,x),\ 0\leqslant t\leqslant T\right)$ *is well defined and takes values in* $L^2(\Omega)$.

Proof. We will consider $(S_n)_{n\in\mathbb{N}}$ and $(S_{n,m})_{n,m\in\mathbb{N}}$ to be the same approximating sequences of S as in the proof of Theorem 3.1. Recall that the sequence $(S_{n,m})$ depends on which of (H1) or (H2*) is satisfied. If (H1) is satisfied, then (3.10), (3.11) and (H1) show that $S_{n,m}\in L^2([0,T],L^1(\mathbb{R}^d))$. If (H2*) is satisfied, then (3.9) and the fact that $S_n\in\mathcal{S}(\mathbb{R}^d)$ shows that $S_{n,m}\in L^2([0,T],L^1(\mathbb{R}^d))$. Hence, by (3.13), the process $t\mapsto\int_0^t ds\int_{\mathbb{R}^d} dx\, S_{n,m}(s,x)Z(s,x)$ is well-defined.

Moreover, by arguments analogous to those used in the proof of Theorem 3.1, where we just consider $\mu(d\xi)=\delta_0(d\xi)$, replace (3.6) by (3.17) and (H2) by (H2*), we can show that

$$\|S_{n,m}-S_n\|_{1,Z}\to 0, \qquad \text{as } m\to\infty,$$

in both cases. As a consequence, the sequence

$$\left(\int_0^T ds\int_{\mathbb{R}} dx\, S_{n,m}(s,x)Z(s,x)\right)_{m\in\mathbb{N}}$$

is Cauchy in $L^2(\Omega)$ by (3.14) and hence converges. We set the limit of this sequence as the definition of $\int_0^T ds\int_{\mathbb{R}^d} dx\, S_n(s,x)Z(s,x)$ for any $n\in\mathbb{N}$. Note that (3.14) is still valid for S_n.

Using the same argument as in the proof of Theorem 3.1 again, we now can show that

$$\|S_n-S\|_{1,Z}\to 0, \qquad \text{as } n\to\infty.$$

Hence, by a Cauchy sequence argument similar to the one above, we can define the random variable $\int_0^T ds\int_{\mathbb{R}^d} dx\, S(s,x)Z(s,x)$ as the limit in $L^2(\Omega)$ of $\int_0^T ds\int_{\mathbb{R}^d} dx\, S_n(s,x)Z(s,x)$. Moreover, (3.14) remains true. ■

Remark 3.5. Assumption (3.17) appears in [6] to give estimates concerning an integral of the same type as in Proposition 3.4. In this reference, $S\geqslant 0$ and the process Z is considered to be in $L^2(\mathbb{R}^d)$, which is not the case here.

4 Application to SPDE's

In this section, we apply the preceding results on stochastic integration to construct random field solutions of non-linear stochastic partial differential equations. We will be interested in equations of the form

$$Lu(t,x)=\alpha(u(t,x))\dot F(t,x)+\beta(u(t,x)), \tag{4.1}$$

with vanishing initial conditions, where L is a second order partial differential operator with constant coefficients, $\dot{F}$ is the noise described in Section 2 and α, β are real-valued functions. Let Γ be the fundamental solution of equation $Lu(t,x) = 0$. In [1], Dalang shows that (4.1) admits a unique solution $(u(t,x),\ 0 \leqslant t \leqslant T,\ x \in \mathbb{R}^d)$ when Γ is a non-negative Schwartz distribution with rapid decrease. Moreover, this solution is in $L^p(\Omega)$ for all $p \geqslant 1$. Using the extension of the stochastic integral presented in Section 3, we are going to show that there is still a random-field solution when Γ is a (not necessarily non-negative) Schwartz distribution with rapid decrease. However, this solution will only be in $L^2(\Omega)$. We will see in Section 6 that this solution is in $L^p(\Omega)$ for any $p \geqslant 1$ in the case where α is an affine function and $\beta \equiv 0$. The question of uniqueness is considered in Theorem 4.8.

By a random-field solution of (4.1), we mean a jointly measurable process $(u(t,x),\ t \geqslant 0,\ x \in \mathbb{R}^d)$ such that $(t,x) \mapsto u(t,x)$ from $\mathbb{R}_+ \times \mathbb{R}^d$ into $L^2(\Omega)$ is continuous and satisfies the assumptions needed for the right-hand side of (4.3) below to be well defined, namely $(u(t,x))$ is a predictable process such that

$$\sup_{0 \leqslant t \leqslant T} \sup_{x \in \mathbb{R}^d} \mathbb{E}[u(t,x)^2] < \infty, \tag{4.2}$$

and such that, for $t \in [0,T]$, $\alpha(u(t,\cdot))$ and $\beta(u(t,\cdot))$ have stationary covariance and such that for all $0 \leqslant t \leqslant T$ and $x \in \mathbb{R}^d$, a.s.,

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y)\alpha(u(s,y))M(ds,dy) + \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y)\beta(u(s,y))ds\,dy. \tag{4.3}$$

In this equation, the first (stochastic) integral is defined in Theorem 3.1 and the second (deterministic) integral is defined in Proposition 3.4.

We recall the following integration result, which will be used in the proof of Lemma 4.6.

Proposition 4.1. *Let $\mathcal{B}$ be a Banach space with norm $\|\cdot\|_{\mathcal{B}}$. Let $f : \mathbb{R} \to \mathcal{B}$ be a function such that $f \in L^2(\mathbb{R}, \mathcal{B})$, i.e.*

$$\int_{\mathbb{R}} \|f(s)\|^2_{\mathcal{B}}\, ds < +\infty.$$

Then

$$\lim_{|h| \to 0} \int_{\mathbb{R}} \|f(s+h) - f(s)\|^2_{\mathcal{B}}\, ds = 0.$$

Proof. For a proof in the case where $f \in L^1(\mathbb{R}, \mathcal{B})$, see [11, Chap.XIII, Theorem 1.2, p.165]. Using the fact that simple functions are dense in $L^2(\mathbb{R}, \mathcal{B})$ (see [8, Corollary III.3.8, p.125]), the proof in the case where $f \in L^2(\mathbb{R}, \mathcal{B})$ is analogous. ■

Theorem 4.2. *Suppose that the fundamental solution Γ of equation $Lu = 0$ is a deterministic space-time Schwartz distribution of the form $\Gamma(t)dt$, where $\Gamma(t) \in \mathcal{S}'_r(\mathbb{R}^d)$, such that $(s,\xi) \mapsto \mathcal{F}\Gamma(s)(\xi)$ is measurable,*

$$\int_0^T ds \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(s)(\xi + \eta)|^2 < \infty \tag{4.4}$$

and

$$\int_0^T ds \sup_{\eta \in \mathbb{R}^d} |\mathcal{F}\Gamma(s)(\eta)|^2 < \infty. \tag{4.5}$$

Suppose in addition that either hypothesis (H1), or hypotheses (H2) and (H2), are satisfied with S replaced by Γ. Then equation (4.1), with α and β Lipschitz functions, admits a random-field solution $(u(t,x),\, 0 \leqslant t \leqslant T, x \in \mathbb{R}^d)$.*

Remark 4.3. The main example, that we will treat in the following section, is the case where $L = \frac{\partial^2}{\partial t^2} - \Delta$ is the wave operator and $d \geqslant 4$.

Proof. We are going to use a Picard iteration scheme. Suppose that α and β have Lipschitz constant K, so that $|\alpha(u)| \leqslant K(1+|u|)$ and $|\beta(u)| \leqslant K(1+|u|)$. For $n \geqslant 0$, set

$$\begin{cases} u_0(t,x) \equiv 0, \\ Z_n(t,x) = \alpha(u_n(t,x)), \\ W_n(t,x) = \beta(u_n(t,x)), \\ u_{n+1}(t,x) = \displaystyle\int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y) Z_n(s,y) M(ds,dy) \\ \qquad\qquad\qquad + \displaystyle\int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y) W_n(s,y) ds\, dy. \end{cases} \tag{4.6}$$

Now suppose by induction that, for all $T > 0$,

$$\sup_{0 \leqslant t \leqslant T} \sup_{x \in \mathbb{R}^d} \mathbb{E}[u_n(t,x)^2] < \infty. \tag{4.7}$$

Suppose also that $u_n(t,x)$ is $\mathcal{F}_t$-measurable for all x and t, and that $(t,x) \mapsto u_n(t,x)$ is L^2-continuous. These conditions are clearly satisfied for $n = 0$. The L^2-continuity ensures that $(t,x;\omega) \mapsto u_n(t,x;\omega)$ has a jointly measurable version and that the conditions of [2, Prop.2] are satisfied. Moreover, Lemma 4.5 below shows that Z_n and W_n satisfy the assumptions needed for the stochastic integral and the integral with respect to Lebesgue-measure to be well-defined. Therefore, $u_{n+1}(t,x)$ is well defined in (4.6), and is L^2-continuous by Lemma 4.6. We now show that u_{n+1} satisfies (4.7). By (4.6),

$$\begin{aligned} \mathbb{E}[u_{n+1}(t,x)^2] \;\leqslant\; & 2\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y) Z_n(s,y) M(ds,dy)\right)^2\right] \\ & + 2\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y) W_n(s,y) ds\, dy\right)^2\right]. \end{aligned}$$

Using the linear growth of α, (4.7) and the fact that $\Gamma(s,\cdot) \in \mathcal{P}_{0,Z_n}$, (4.4) and Theorem 3.1 imply that

$$\sup_{0 \leqslant t \leqslant T} \sup_{x \in \mathbb{R}^d} \|\Gamma(t-\cdot, x-\cdot)\|^2_{0,Z_n} < +\infty.$$

Further, the linear growth of β, (4.5) and Proposition 3.4 imply that

$$\sup_{0 \leqslant t \leqslant T} \sup_{x \in \mathbb{R}^d} \|\Gamma(t-\cdot, x-\cdot)\|^2_{1,W_n} < +\infty.$$

It follows that the sequence $(u_n(t,x))_{n \geqslant 0}$ is well-defined. It remains to show that it converges in $L^2(\Omega)$. For this, we are going to use the generalization of Gronwall's lemma presented in [1, Lemma 15]. We have

$$\mathbb{E}[|u_{n+1}(t,x) - u_n(t,x)|^2] \leqslant 2A_n(t,x) + 2B_n(t,x),$$

where

$$A_n(t,x) = \mathbb{E}\left[\left|\int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y)(Z_n(s,y)-Z_{n-1}(s,y))M(ds,dy)\right|^2\right]$$

and

$$B_n(t,x) = \mathbb{E}\left[\left|\int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y)(W_n(s,y)-W_{n-1}(s,y))ds\,dy\right|^2\right].$$

First consider $A_n(t,x)$. Set $Y_n = Z_n - Z_{n-1}$. By the Lipschitz property of α, the process Y_n satisfies the assumptions of Theorem 3.1 on Z by Lemma 4.5 below. Hence, by Theorem 3.1,

$$\begin{aligned}
A_n(t,x) &= C\int_0^t ds \int_{\mathbb{R}^d} \nu_s^{Y_n}(d\eta) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(t-s,x-\cdot)(\xi+\eta)|^2 \\
&\leqslant C\int_0^t ds\, \nu_s^{Y_n}(\mathbb{R}^d) \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(t-s,x-\cdot)(\xi+\eta)|^2 \\
&\leqslant C\int_0^t ds \left(\sup_{z\in\mathbb{R}^d} \mathbb{E}[Y_n(s,z)^2]\right) \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(t-s,x-\cdot)(\xi+\eta)|^2.
\end{aligned}$$

Then set $M_n(t) = \sup_{x\in\mathbb{R}^d} \mathbb{E}[|u_{n+1}(t,x)-u_n(t,x)|^2]$ and

$$J_1(s) = \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(s,\cdot)(\xi+\eta)|^2.$$

The Lipschitz property of α implies that

$$\begin{aligned}
\sup_{z\in\mathbb{R}^d} \mathbb{E}[Y_n(s,z)^2] &= \sup_{z\in\mathbb{R}^d} \mathbb{E}[(Z_n(s,z)-Z_{n-1}(s,z))^2] \\
&\leqslant \sup_{z\in\mathbb{R}^d} K^2\, \mathbb{E}[(u_n(s,z)-u_{n-1}(s,z))^2] \\
&\leqslant K^2 M_{n-1}(s),
\end{aligned}$$

and we deduce that

$$A_n(t,x) \leqslant C\int_0^t ds\, M_{n-1}(s) J_1(t-s). \tag{4.8}$$

Now consider $B_n(t,x)$. Set $V_n = W_n - W_{n-1}$. The process V_n satisfies the assumptions of Theorem 3.1 on Z by Lemma 4.5 below. Hence, by Proposition 3.4,

$$\begin{aligned}
B_n(t,x) &\leqslant C\int_0^t ds \int_{\mathbb{R}^d} \nu_s^{V_n}(d\eta)\, |\mathcal{F}\Gamma(t-s,x-\cdot)(\eta)|^2 \\
&\leqslant C\int_0^t ds\, \nu_s^{V_n}(\mathbb{R}^d) \sup_{\eta\in\mathbb{R}^d} |\mathcal{F}\Gamma(t-s,x-\cdot)(\eta)|^2 \\
&\leqslant C\int_0^t ds \left(\sup_{z\in\mathbb{R}^d} \mathbb{E}[V_n(s,z)^2]\right) \sup_{\eta\in\mathbb{R}^d} |\mathcal{F}\Gamma(t-s,x-\cdot)(\eta)|^2.
\end{aligned}$$

Then set

$$J_2(s) = \sup_{\eta\in\mathbb{R}^d} |\mathcal{F}\Gamma(s,\cdot)(\eta)|^2.$$

The Lipschitz property of β implies that

$$\begin{aligned}\sup_{z\in\mathbb{R}^d}\mathbb{E}[V_n(s,z)^2] &\leqslant \sup_{z\in\mathbb{R}^d}\mathbb{E}[(W_n(s,z)-W_{n-1}(s,z))^2]\\ &\leqslant \sup_{z\in\mathbb{R}^d}K^2\,\mathbb{E}[(u_n(s,z)-u_{n-1}(s,z))^2]\\ &\leqslant K^2M_{n-1}(s),\end{aligned}$$

and we deduce that

$$B_n(t,x)\leqslant C\int_0^t ds\,M_{n-1}(s)J_2(t-s). \tag{4.9}$$

Then, setting $J(s)=J_1(s)+J_2(s)$ and putting together (4.8) and (4.9), we obtain

$$M_n(t)\leqslant \sup_{x\in\mathbb{R}^d}(A_n(t,x)+B_n(t,x))\leqslant C\int_0^t ds\,M_{n-1}(s)J(t-s).$$

Lemma 15 in [1] implies that $(u_n(t,x))_{n\geqslant 0}$ converges uniformly in L^2, say to $u(t,x)$. As a consequence of [1, Lemma 15], u_n satisfies (4.2) for any $n\geqslant 0$. Hence, u also satisfies (4.2) as the L^2-limit of the sequence $(u_n)_{n\geqslant 0}$. As u_n is continuous in L^2 by Lemma 4.6 below, u is also continuous in L^2. Therefore, u admits a jointly measurable version, which, by Lemma 4.5 below has the property that $\alpha(u(t,\cdot))$ and $\beta(u(t,\cdot))$ have stationary covariance functions. The process u satisfies (4.3) by passing to the limit in (4.6). ■

The following definition and lemmas were used in the proof of Theorem 4.2 and will be used in Theorem 4.8.

Definition 4.4 ("S" property)**.** For $z\in\mathbb{R}^d$, write $z+B=\{z+y: y\in B\}$, $M_s^{(z)}(B)=M_s(z+B)$ and $Z^{(z)}(s,x)=Z(s,x+z)$. We say that the process $(Z(s,x),\ s\geqslant 0,\ x\in\mathbb{R}^d)$ has the "S" property if, for all $z\in\mathbb{R}^d$, the finite dimensional distributions of

$$\Big((Z^{(z)}(s,x),\ s\geqslant 0,\ x\in\mathbb{R}^d)),(M_s^{(z)}(B),\ s\geqslant 0,\ B\in\mathcal{B}_b(\mathbb{R}^d))\Big)$$

do not depend on z.

Lemma 4.5. *For $n\geqslant 1$, the process $(u_n(s,x),u_{n-1}(s,x),\ 0\leqslant s\leqslant T,\ x\in\mathbb{R}^d)$ admits the "S" property.*

Proof. It follows from the definition of the martingale measure M and the fact that u_0 is constant that the finite dimensional distributions of $(u_0^{(z)}(s,x),M_s^{(z)}(B),\ s\geqslant 0,\ x\in\mathbb{R}^d,\ B\in\mathcal{B}_b(\mathbb{R}^d))$ do not depend on z. Now, we can write

$$u_1(t,x)=\int_0^t\int_{\mathbb{R}^d}\Gamma(t-s,-y)\alpha(0)M^{(x)}(ds,dy)+\int_0^t\int_{\mathbb{R}^d}\Gamma(t-s,-y)\beta(0)ds\,dy,$$

so $u_1(t,x)$ is an abstract function Φ of $M^{(x)}$. As the function Φ does not depend on x, we have $u_1^{(z)}(t,x)=\Phi(M^{(x+z)})$. Then, for $(s_1,\dots,s_k)$, $(t_1,\dots,t_j)\in\mathbb{R}_+^k,\mathbb{R}_+^j$, $(x_1,\dots,x_k)\in(\mathbb{R}^d)^k$, $B_1,\dots,B_j\in\mathcal{B}_b(\mathbb{R}^d)$, the joint distribution of

$$\Big(u_1^{(z)}(s_1,x_1),\dots,u_1^{(z)}(s_k,x_k),M_{t_1}^{(z)}(B_1),\dots,M_{t_j}^{(z)}(B_j)\Big)$$

is an abstract function of the distribution of

$$\left(M_\cdot^{(z+x_1)}(\cdot),\dots,M_\cdot^{(z+x_k)}(\cdot),M_{t_1}^{(z)}(B_1),\dots,M_{t_j}^{(z)}(B_j)\right),$$

which, as mentioned above, does not depend on z. Hence, the conclusion holds for $n=1$, because u_0 is constant. Now suppose that the conclusion holds for some $n\geqslant 1$ and show that it holds for $n+1$. We can write

$$\begin{aligned}u_{n+1}(t,x) &= \int_0^t\int_{\mathbb{R}^d}\Gamma(t-s,-y)\alpha(u_n^{(x)}(s,y))M^{(x)}(ds,dy)\\ &\qquad +\int_0^t\int_{\mathbb{R}^d}\Gamma(t-s,-y)\beta(u_n^{(x)}(s,y))ds\,dy,\end{aligned}$$

so $u_{n+1}(t,x)$ is an abstract function Ψ of $u_n^{(x)}$ and $M^{(x)}$: $u_{n+1}(t,x)=\Psi(u_n^{(x)},M^{(x)})$. The function Ψ does not depend on x and we have $u_{n+1}^{(z)}(t,x)=\Psi(u_n^{(x+z)},M^{(x+z)})$.

Hence, for every choice of $(s_1,\dots,s_k)\in\mathbb{R}_+^k$, $(t_1,\dots,t_j)\in\mathbb{R}_+^j$, $(r_1,\dots,r_\ell)\in\mathbb{R}_+^\ell$, and $(x_1,\dots,x_k)\in(\mathbb{R}^d)^k$, $(y_1,\dots,y_j)\in(\mathbb{R}^d)^j$, the joint distribution of

$$\left(u_{n+1}^{(z)}(s_1,x_1),\dots,u_{n+1}^{(z)}(s_k,x_k),u_n^{(z)}(t_1,y_1),\dots,u_n^{(z)}(t_j,y_j),M_{r_1}^{(z)}(B_1),\dots,M_{r_\ell}^{(z)}(B_\ell)\right)$$

is an abstract function of the distribution of

$$\left(u_n^{(z+x_1)}(\cdot,\cdot),\dots,u_n^{(z+x_k)}(\cdot,\cdot),u_n^{(z)}(\cdot,\cdot),M_\cdot^{(z+x_1)}(\cdot),\dots,M_\cdot^{(z+x_k)}(\cdot),M_{r_1}^{(z)}(B_1),\dots,M_{r_\ell}^{(z)}(B_\ell)\right),$$

which does not depend on z by the induction hypothesis. ∎

Lemma 4.6. *For all $n\geqslant 0$, the process $(u_n(t,x),\ t\geqslant 0,\ x\in\mathbb{R}^d)$ defined in (4.6) is continuous in $L^2(\Omega)$.*

Proof. For $n=0$, the result is trivial. We are going to show by induction that if $(u_n(t,x),\ t\geqslant 0,\ x\in\mathbb{R}^d)$ is continuous in L^2, then $(u_{n+1}(t,x),\ t\geqslant 0,\ x\in\mathbb{R}^d)$ is too.

We begin with time increments. We have

$$\mathbb{E}[(u_{n+1}(t,x)-u_{n+1}(t+h,x))^2]\leqslant 2A_n(t,x,h)+2B_n(t,x,h),$$

where

$$\begin{aligned}A_n(t,x,h) &= \mathbb{E}\left[\left(\int_0^{t+h}\int_{\mathbb{R}^d}\Gamma(t+h-s,x-y)Z_n(s,y)M(ds,dy)\right.\right.\\ &\qquad\left.\left.-\int_0^t\int_{\mathbb{R}^d}\Gamma(t-s,x-y)Z_n(s,y)M(ds,dy)\right)^2\right]\end{aligned}$$

and

$$\begin{aligned}B_n(t,x,h) &= \mathbb{E}\left[\left(\int_0^{t+h}\int_{\mathbb{R}^d}\Gamma(t+h-s,x-y)W_n(s,y)ds\,dy\right.\right.\\ &\qquad\left.\left.-\int_0^t\int_{\mathbb{R}^d}\Gamma(t-s,x-y)W_n(s,y)ds\,dy\right)^2\right].\end{aligned}$$

First of all, $A_n(t,x,h) \leqslant X_1 + X_2$, where

$$\begin{aligned}
X_1 &= \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} (\Gamma(t+h-s,x-y) - \Gamma(t-s,x-y)) Z_n(s,y) M(ds,dy)\right)^2\right],\\
X_2 &= \mathbb{E}\left[\left(\int_t^{t+h} \int_{\mathbb{R}^d} \Gamma(t+h-s,x-y) Z_n(s,y) M(ds,dy)\right)^2\right].
\end{aligned}$$

The term X_2 goes to 0 as $h \to 0$ because, by (3.7),

$$\begin{aligned}
0 \leqslant X_2 &\leqslant \sup_{0\leqslant s\leqslant T} \mathbb{E}[Z_n(s,0)^2] \int_t^{t+h} ds \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(t+h-s,x-\cdot)(\xi+\eta)|^2\\
&= \sup_{0\leqslant s\leqslant T} \mathbb{E}[Z_n(s,0)^2] \int_0^{h} ds \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(s,x-\cdot)(\xi+\eta)|^2\\
&\underset{h\to 0}{\longrightarrow} 0,
\end{aligned}$$

by the Dominated Convergence Theorem and (4.4). Concerning X_1, we have

$$\begin{aligned}
X_1 &= \int_0^t ds \int_{\mathbb{R}^d} \nu_s^{Z_n}(d\eta) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(t+h-s)(\xi+\eta) - \mathcal{F}\Gamma(t-s)(\xi+\eta)|^2\\
&= \int_0^t ds \int_{\mathbb{R}^d} \nu_{t-s}^{Z_n}(d\eta) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(s+h)(\xi+\eta) - \mathcal{F}\Gamma(s)(\xi+\eta)|^2\\
&\leqslant C \int_0^t ds \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(s+h)(\xi+\eta) - \mathcal{F}\Gamma(s)(\xi+\eta)|^2
\end{aligned}$$

This integral goes to 0 as $h \to 0$, either by (4.4) and Proposition 4.1 with $\mathcal{B} = L^\infty(\mathbb{R}^d, L^2_\mu(\mathbb{R}^d))$ and $f(s;\eta,\xi) = \mathcal{F}\Gamma(s)(\xi+\eta)\mathbf{1}_{[0,T]}(s)$, or by assumption (H2).
Secondly, $B_n(t,x,h) \leqslant Y_1 + Y_2$, where

$$\begin{aligned}
Y_1 &= \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} (\Gamma(t+h-s,x-y) - \Gamma(t-s,x-y)) W_n(s,y) ds\, dy\right)^2\right],\\
Y_2 &= \mathbb{E}\left[\left(\int_t^{t+h} \int_{\mathbb{R}^d} \Gamma(t+h-s,x-y) W_n(s,y) ds\, dy\right)^2\right].
\end{aligned}$$

The term Y_2 goes to 0 as $h \to 0$ because, by Proposition 3.4,

$$\begin{aligned}
0 \leqslant Y_2 &\leqslant \sup_{0\leqslant s\leqslant T} \mathbb{E}[W_n(s,0)^2] \int_t^{t+h} ds \sup_{\eta\in\mathbb{R}^d} |\mathcal{F}\Gamma(t+h-s,x-\cdot)(\eta)|^2\\
&= \sup_{0\leqslant s\leqslant T} \mathbb{E}[W_n(s,0)^2] \int_0^{h} ds \sup_{\eta\in\mathbb{R}^d} |\mathcal{F}\Gamma(s,x-\cdot)(\eta)|^2\\
&\underset{h\to 0}{\longrightarrow} 0,
\end{aligned}$$

by the Dominated Convergence Theorem. Concerning Y_1, we have

$$\begin{aligned} Y_1 &= \int_0^t ds \int_{\mathbb{R}^d} \nu_s^{W_n}(d\eta)\, |\mathcal{F}\Gamma(t+h-s)(\eta) - \mathcal{F}\Gamma(t-s)(\eta)|^2 \\ &= \int_0^t ds \int_{\mathbb{R}^d} \nu_{t-s}^{W_n}(d\eta)\, |\mathcal{F}\Gamma(s+h)(\eta) - \mathcal{F}\Gamma(s)(\eta)|^2 \\ &\leqslant C \int_0^t ds \sup_{\eta \in \mathbb{R}^d} |\mathcal{F}\Gamma(s+h)(\eta) - \mathcal{F}\Gamma(s)(\eta)|^2 \end{aligned}$$

This integral goes to 0 as $h \to 0$ either by (4.5) and Proposition 4.1 with $\mathcal{B} = L^\infty(\mathbb{R}^d)$ and $f(s;\eta) = \mathcal{F}\Gamma(s)(\eta)\mathbf{1}_{[0,T]}(s)$, or by assumption (H2*). This establishes the L^2-continuity in time. Turning to spatial increments, we have

$$\mathbb{E}[(u_{n+1}(t,x+z) - u_{n+1}(t,x))^2] \leqslant 2C_n(t,x,z) + 2D_n(t,x,z),$$

where

$$\begin{aligned} C_n(t,x,z) &= \mathbb{E}\Bigg[\bigg(\int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x+z-y) Z_n(s,y) M(ds,dy) \\ &\qquad - \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y) Z_n(s,y) M(ds,dy)\bigg)^2\Bigg] \end{aligned}$$

and

$$\begin{aligned} D_n(t,x,z) &= \mathbb{E}\Bigg[\bigg(\int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x+z-y) W_n(s,y) ds\, dy \\ &\qquad - \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y) W_n(s,y) ds\, dy\bigg)^2\Bigg]. \end{aligned}$$

First consider C_n. We have

$$\begin{aligned} &C_n(t,x,z) \\ &= \int_0^t ds \int_{\mathbb{R}^d} \nu_s^{Z_n}(d\eta) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(t-s,x+z-\cdot)(\xi+\eta) - \mathcal{F}\Gamma(t-s,x-\cdot)(\xi+\eta)|^2 \\ &= \int_0^t ds \int_{\mathbb{R}^d} \nu_s^{Z_n}(d\eta) \int_{\mathbb{R}^d} \mu(d\xi)\, |1 - e^{-i\langle \xi+\eta, z\rangle}|^2\, |\mathcal{F}\Gamma(t-s,\cdot)(\xi+\eta)|^2. \end{aligned}$$

Clearly, $|1 - e^{-i\langle \xi+\eta, z\rangle}|^2 \leqslant 4$ and the integrand converges to 0 as $\|z\| \to 0$. Therefore, for n fixed, by the Dominated Convergence Theorem, $C_n(t,x,z) \to 0$ as $\|z\| \to 0$.

Moreover, considering D_n, we have

$$\begin{aligned} D_n(t,x,z) &= \int_0^t ds \int_{\mathbb{R}^d} \nu_s^{W_n}(d\eta)\, |\mathcal{F}\Gamma(t-s,x+z-\cdot)(\eta) - \mathcal{F}\Gamma(t-s,x-\cdot)(\eta)|^2 \\ &= \int_0^t ds \int_{\mathbb{R}^d} \nu_s^{W_n}(d\eta)\, |1 - e^{-i\langle \eta, z\rangle}|^2\, |\mathcal{F}\Gamma(t-s,\cdot)(\eta)|^2. \end{aligned}$$

Clearly, $|1 - e^{-i\langle \eta, z\rangle}|^2 \leqslant 4$ and the integrand converges to 0 as $\|z\| \to 0$. Therefore, for n fixed, by the Dominated Convergence Theorem, $D_n(t, x, z) \to 0$ as $\|z\| \to 0$. This establishes the L^2-continuity in the spatial variable. ■

Remark 4.7. The induction assumption on the L^2-continuity of u_n is stronger than needed to show the L^2-continuity of u_{n+1}. In order that the stochastic integral process $\Gamma(t - \cdot, x - \cdot) \cdot M^Z$ be L^2-continuous, it suffices that the process Z satisfy the assumptions of Theorem 3.1.

We can now state the following theorem, which ensures uniqueness of the solution constructed in Theorem 4.2 within a more specific class of processes.

Theorem 4.8. *Under the assumptions of Theorem 4.2, let $u(t, x)$ be the solution of equation (4.3) constructed in the proof of Theorem 4.2. Let $(v(t, x),\ t \in [0, T],\ x \in \mathbb{R}^d)$ be a jointly measurable, predictable processes such that $\sup_{0 \leqslant t \leqslant T} \sup_{x \in \mathbb{R}^d} \mathbb{E}[v(t, x)^2] < \infty$, that satisfies property "S" and (4.3). Then, for all $0 \leqslant t \leqslant T$ and $x \in \mathbb{R}^d$, $v(t, x) = u(t, x)$ a.s.*

Proof. We are going to show that $\mathbb{E}[(u(t, x) - v(t, x))^2] = 0$. In the case where Γ is a non-negative distribution, we consider the sequence $(u_n)_{n \in \mathbb{N}}$ used to construct u, defined in (4.6). The approximating sequence $(\Gamma_m)_{m \geqslant 0}$ built in [1, Theorem 2] to define the stochastic integral is a positive function. Hence the stochastic integral below is a Walsh stochastic integral and using the Lipschitz property of α, we have (in the case $\beta \equiv 0$):

$$\begin{aligned}
&\mathbb{E}[(u_{n+1}(t, x) - v(t, x))^2] \\
&\quad = \lim_{m \to \infty} \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} \Gamma_m(t - s, x - y)(\alpha(u_n(s, y)) - \alpha(v(s, y)))M(ds, dy)\right)^2\right] \\
&\quad = \lim_{m \to \infty} \mathbb{E}\Bigg[\int_0^t ds \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz\, \Gamma_m(t - s, x - y)(\alpha(u_n(s, y)) - \alpha(v(s, y)))f(x - y) \\
&\qquad\qquad \times (\alpha(u_n(s, z)) - \alpha(v(s, z)))\Gamma_m(t - s, x - z)\Bigg] \\
&\quad \leqslant \lim_{m \to \infty} \int_0^t ds \sup_{y \in \mathbb{R}^d} \mathbb{E}[(u_n(s, y) - v(s, y))^2] \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}\Gamma_m(t - s, x - \cdot)(\xi)|^2.
\end{aligned}$$

Using a Gronwall-type argument ([1, Lemma 15]), uniqueness follows.

In the case considered here, the sequence $(\Gamma_m)_{m \geqslant 0}$ is not necessarily positive and the argument above does not apply. We need to know a priori that the processes $Z(t, x) = \alpha(u_n(t, x)) - \alpha(v(t, x))$ and $W(t, x) = \beta(u_n(t, x)) - \beta(v(t, x))$ have a spatially homogeneous covariance. This is why we consider the restricted class of processes satisfying property "S".

As $u_0 \equiv 0$, it is clear that the joint process $(u_0(t, x), v(t, x),\ t \geqslant 0,\ x \in \mathbb{R}^d)$ satisfies the "S" property. A proof analogous to that of Lemma 4.5 with u_{n-1} replaced by v shows that the process $(u_n(t, x), v(t, x),\ t \geqslant 0,\ x \in \mathbb{R}^d)$ also satisfies the "S" property. Then $\alpha(u_n(t, \cdot)) - \alpha(v(t, \cdot))$ and $\beta(u_n(t, \cdot)) - \beta(v(t, \cdot))$ have spatially homogeneous covariances. This ensures that the stochastic integrals below are well defined. We have

$$\mathbb{E}[(u_n(t, x) - v(t, x))^2] \leqslant 2A(t, x) + 2B(t, x),$$

where

$$A_n(t,x) = \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y)(\alpha(u_n(t,x)) - \alpha(v(t,x)))M(ds,dy)\right)^2\right]$$

and

$$B_n(t,x) = \mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y)(\beta(u_n(t,x)) - \beta(v(t,x)))ds\, dy\right)^2\right].$$

Clearly,

$$\begin{aligned} &A_n(t,x) \\ &\leqslant C\int_0^t ds \sup_{x\in\mathbb{R}^d} \mathbb{E}[(u_{n-1}(t,x) - v(t,x))^2] \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(t-s,\cdot)(\xi+\eta)|^2. \end{aligned} \tag{4.10}$$

Setting

$$\tilde{M}_n(t) = \sup_{x\in\mathbb{R}^d} \mathbb{E}[(u_n(t,x) - v(t,x))^2]$$

and using the notations in the proof of Theorem 4.2 we obtain, by (4.10),

$$A_n(t,x) \leqslant \int_0^t \tilde{M}_{n-1}(s) J_1(t-s) ds.$$

Moreover,

$$B_n(t,x) \quad \leqslant \quad C\int_0^t ds \sup_{x\in\mathbb{R}^d} \mathbb{E}[(u_{n-1}(t,x) - v(t,x))^2] \sup_{\eta\in\mathbb{R}^d} |\mathcal{F}\Gamma(t-s,\cdot)(\eta)|^2, \tag{4.11}$$

so

$$B_n(t,x) \leqslant \int_0^t \tilde{M}_{n-1}(s) J_2(t-s) ds.$$

Hence,

$$\tilde{M}_n(t) \leqslant \int_0^t \tilde{M}_{n-1}(s) J(t-s) ds.$$

By [1, Lemma 15], this implies that

$$\tilde{M}_n(t) \leqslant \left(\sup_{0\leqslant s\leqslant t} \sup_{x\in\mathbb{R}^d} \mathbb{E}[v(s,x)^2]\right) a_n,$$

where $(a_n)_{n\in\mathbb{N}}$ is a sequence such that $\sum_{n=0}^{\infty} a_n < \infty$. This shows that $\tilde{M}_n(t) \to 0$ as $n \to \infty$. Finally, we conclude that

$$\mathbb{E}[(u(t,x) - v(t,x))^2] \leqslant 2\mathbb{E}[(u(t,x) - u_n(t,x))^2] + 2\mathbb{E}[(u_n(t,x) - v(t,x))^2] \to 0, \tag{4.12}$$

as $n \to \infty$. This establishes the theorem. ■

5 The non-linear wave equation

As an application of Theorem 4.2, we check the different assumptions in the case of the non-linear stochastic wave equation in dimensions greater than 3. The case of dimensions 1, 2 and 3 has been treated in [1]. We are interested in the equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = \alpha(u)\dot{F} + \beta(u), \tag{5.1}$$

with vanishing initial conditions, where $t \geqslant 0$, $x \in \mathbb{R}^d$ with $d > 3$ and $\dot{F}$ is the noise presented in Section 2. In the case of the wave operator, the fundamental solution (see [10, Chap.5]) is

$$\Gamma(t) = \frac{2\pi^{\frac{d}{2}}}{\gamma(\frac{d}{2})} \mathbf{1}_{\{t>0\}} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{d-3}{2}} \frac{\sigma_t^d}{t}, \qquad \text{if } d \text{ is odd}, \tag{5.2}$$

$$\Gamma(t) = \frac{2\pi^{\frac{d}{2}}}{\gamma(\frac{d}{2})} \mathbf{1}_{\{t>0\}} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{d-2}{2}} (t^2 - |x|^2)_+^{-\frac{1}{2}}, \qquad \text{if } d \text{ is even}, \tag{5.3}$$

where σ_t^d is the Hausdorff surface measure on the d-dimensional sphere of radius t and γ is Euler's gamma function. The action of $\Gamma(t)$ on a test function is explained in (5.6) and (5.7) below. It is also well-known (see [23, §7]) that

$$\mathcal{F}\Gamma(t)(\xi) = \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|},$$

in all dimensions. Hence, there exist constants C_1 and C_2, depending on T, such that for all $s \in [0, T]$ and $\xi \in \mathbb{R}^d$,

$$\frac{C_1}{1+|\xi|^2} \leqslant \frac{\sin^2(2\pi s|\xi|)}{4\pi^2|\xi|^2} \leqslant \frac{C_2}{1+|\xi|^2}. \tag{5.4}$$

Theorem 5.1. *Let $d \geqslant 1$, and suppose that*

$$\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi+\eta|^2} < \infty. \tag{5.5}$$

Then equation (5.1), with α and β Lipschitz functions, admits a random-field solution $(u(t,x),\ 0 \leqslant t \leqslant T,\ x \in \mathbb{R}^d)$. In addition, the uniqueness statement of Theorem 4.8 holds.

Proof. We are going to check that the assumptions of Theorem 4.2 are satisfied. The estimates in (5.4) show that Γ satisfies (4.4) since (5.5) holds. This condition can be shown to be equivalent to the condition (40) of Dalang [1], namely $\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty$ since $f \geqslant 0$ (see [4, Lemma 8] and [14]). Moreover, taking the supremum over ξ in (5.4) shows that (4.5) is satisfied.

To check (H1), and in particular, (3.3) and (3.4), fix $\varphi \in \mathcal{D}(\mathbb{R}^d)$ such that $\varphi \geqslant 0$, $\operatorname{supp}\varphi \subset B(0,1)$ and $\int_{\mathbb{R}^d} \varphi(x)\,dx = 1$. From formulas (5.2) and (5.3), if d is odd, then

$$(\Gamma(t-s) * \varphi)(x) = c_d \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{\frac{d-3}{2}} \left[r^{d-2} \int_{\partial B_d(0,1)} \varphi(x+ry)\ \sigma_1^{(d)}(dy) \right]\Bigg|_{r=t-s}, \tag{5.6}$$

where $\sigma_1^{(d)}$ is the Hausdorff surface measure on $\partial B_d(0,1)$, and when d is even,

$$(\Gamma(t-s)*\varphi)(x) = c_d \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{\frac{d-2}{2}} \left[r^{d-2} \int_{B_d(0,1)} \frac{dy}{\sqrt{1-|y|^2}}\, \varphi(x+ry) \right] \Bigg|_{r=t-s}. \tag{5.7}$$

For $0 \leqslant a \leqslant b \leqslant T$ and $a \leqslant t \leqslant b$, this is a uniformly bounded C^∞-function of x, with support contained in $B(0,T+1)$, and (3.3) and (3.4) clearly hold. Indeed, $(\Gamma(t-s)*\varphi)(x)$ is always a sum of products of a positive power of r and an integral of the same form as above but with respect to the derivatives of φ, evaluated at $r = t-s$. This proves Theorem 5.1. ■

Remark 5.2. When $f(x) = \|x\|^{-\beta}$, with $0 < \beta < d$, then (5.5) holds if and only if $0 < \beta < 2$.

6 Moments of order p of the solution $(p > 2)$: the case of affine multiplicative noise

In the preceding sections, we have seen that the stochastic integral constructed in Section 3 can be used to obtain a random field solution to the non-linear stochastic wave equation in dimensions greater than 3 (Sections 4 and 5). As for the stochastic integral proposed in [1], this stochastic integral is square-integrable if the process Z used as integrand is square-integrable. This property makes it possible to show that the solution $u(t,x)$ of the non-linear stochastic wave equation is in $L^2(\Omega)$ in any dimension.

Theorem 5 in [1] states that Dalang's stochastic integral is L^p-integrable if the process Z is. We would like to extend this result to our generalization of the stochastic integral, even though the approach used in the proof of Theorem 5 in [1] fails in our case. Indeed, that approach is strongly based on Hölder's inequality which can be used when the Schwartz distribution S is non-negative.

The main interest of a result concerning L^p-integrability of the stochastic integral is to show that the solution of an s.p.d.e. admits moments of any order and to deduce Hölder-continuity properties. The first question is whether the solution of the non-linear stochastic wave equation admits moments of any order, in any dimension ? We are going to prove that this is indeed the case for a particular form of the non-linear stochastic wave equation, where α is an affine function and $\beta \equiv 0$. This will not be obtained via a result on the L^p-integrability of the stochastic integral. However, a slightly stronger assumption on the integrability of the Fourier transform of the fundamental solution of the equation is required ((6.1) below instead of (4.4)). The proof is based mainly on the specific form of the process that appears in the Picard iteration scheme when α is affine. Indeed, we will be able to use the fact that the approximating random variable $u_n(t,x)$ is an n-fold iterated stochastic integral.

Theorem 6.1. *Suppose that the fundamental solution Γ of the equation $Lu = 0$ is a space-time Schwartz distribution of the form $\Gamma(t)dt$, where $\Gamma(t) \in \mathcal{S}'(\mathbb{R}^d)$ satisfies*

$$\sup_{0\leqslant s\leqslant T} \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(s)(\xi+\eta)|^2 < \infty, \tag{6.1}$$

as well as the assumptions of Theorem 4.2. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be an affine function given by $\alpha(u) = au + b$, $a, b \in \mathbb{R}$, and let $\beta \equiv 0$. Then equation (4.1) admits a random-field solution $(u(t,x),\ 0 \leqslant t \leqslant T,\ x \in \mathbb{R}^d)$ that is unique in the sense of Theorem 4.8, given by

$$u(t,x) = \sum_{n=1}^{\infty} v_n(t,x), \tag{6.2}$$

where

$$v_1(t,x) = b \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y) M(ds, dy) \tag{6.3}$$

and v_n is defined recursively for $n \geqslant 1$ by

$$v_{n+1}(t,x) = a \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y) v_n(s,y) M(ds, dy). \tag{6.4}$$

Moreover, for all $p \geqslant 1$ and all $T > 0$, this solution satisfies,

$$\sup_{0 \leqslant t \leqslant T} \sup_{x \in \mathbb{R}^d} \mathbb{E}[|u(t,x)|^p] < \infty.$$

Proof. The existence and uniqueness are a consequence of Theorems 4.2 and 4.8. Multiplying the covariance function f by a, we can suppose, without loss of generality, that the affine function is $\alpha(u) = u + b$ ($b \in \mathbb{R}$), that is, $a = 1$. In this case, the Picard iteration scheme defining the sequence $(u_n)_{n \in \mathbb{N}}$ is given by $u_0 \equiv 0$ and

$$u_{n+1}(t,x) = \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y) u_n(s,y) M(ds, dy) + b \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y) M(ds, dy), \tag{6.5}$$

where the stochastic integrals are well defined by Theorem 3.1. Set $v_n(t,x) = u_n(t,x) - u_{n-1}(t,x)$ for all $n \geqslant 1$. Then

$$v_1(t,x) = u_1(t,x) = b \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y) M(ds, dy).$$

Hence, $u(t,x) = \lim_{m \to \infty} u_m(t,x) = \lim_{m \to \infty} \sum_{n=1}^{m} v_n(t,x) = \sum_{n=1}^{\infty} v_n(t,x)$ and (6.2) is proved.

By Theorem 3.1 and because $v_1(t,x)$ is a Gaussian random variable, $v_1(t,x)$ admits finite moments of order p for all $p \geqslant 1$. Suppose by induction that for some $n \geqslant 1$, v_n satisfies, for all $p \geqslant 1$,

$$\sup_{0 \leqslant t \leqslant T} \sup_{x \in \mathbb{R}^d} \mathbb{E}[|v_n(t,x)|^p] < \infty. \tag{6.6}$$

We are going to show that v_{n+1} also satisfies (6.6).

By its definition and (6.5), v_{n+1} satisfies the recurrence relation

$$v_{n+1}(t,x) = \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y) v_n(s,y) M(ds, dy), \tag{6.7}$$

for all $n \geqslant 1$. The stochastic integral above is defined by Theorem 3.1 using the approximating sequence $\Gamma_{m,k} \in \mathcal{P}_+$, denoted $S_{n,m}$ in the proof of Theorem 3.1 (whose definition depends on which of (H1) or (H2) is satisfied). For $s \leqslant t \leqslant T$, we set

$$M_1(s;t,x) = \int_0^s \int_{\mathbb{R}^d} \Gamma(t-\rho, x-y) M(d\rho, dy),$$
$$M_1^{(m,k)}(s;t,x) = \int_0^s \int_{\mathbb{R}^d} \Gamma_{m,k}(t-\rho, x-y) M(d\rho, dy),$$

and, for $n \geqslant 1$,

$$M_{n+1}(s;t,x) = \int_0^s \int_{\mathbb{R}^d} \Gamma(t-\rho, x-y) v_n(\rho, y) M(d\rho, dy)$$

and

$$M_{n+1}^{(m,k)}(s;t,x) = \int_0^s \int_{\mathbb{R}^d} \Gamma_{m,k}(t-\rho, x-y) v_n(\rho, y) M(d\rho, dy).$$

For all $n \geqslant 1$, set also $v_n^{(m,k)}(t,x) = M_n^{(m,k)}(t;t,x)$.

Fix an even integer p and set $q = \frac{p}{2}$. We know that $s \mapsto M_n^{(m,k)}(s;t,x)$ is a continuous martingale and so, by Burkholder's inequality (see [15, Chap. IV, Theorem 73]),

$$\mathbb{E}[|v_{n+1}^{(m,k)}(t,x)|^p] = \mathbb{E}[|M_{n+1}^{(m,k)}(t;t,x)|^p] \leqslant C\, \mathbb{E}[\langle M_{n+1}^{(m,k)}(\cdot;t,x)\rangle_t^q],$$

and by Theorem 2.5 in [24] and Hölder's inequality, the last expectation above is bounded by

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t ds \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz\, \Gamma_{m,k}(t-s, x-y) f(y-z) \Gamma_{m,k}(t-s, x-z) v_n(s,y) v_n(s,z)\right)^q\right] \\
&\leqslant t^{q-1}\mathbb{E}\left[\int_0^t ds \left(\int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz\, \Gamma_{m,k}(t-s, x-y) f(y-z) \Gamma_{m,k}(t-s, x-z) v_n(s,y) v_n(s,z)\right)^q\right] \\
&= t^{q-1} \int_0^t ds \int_{\mathbb{R}^d} dy_1 \int_{\mathbb{R}^d} dz_1\, \Gamma_{m,k}(t-s, x-y_1) f(y_1 - z_1) \Gamma_{m,k}(t-s, x-z_1) \\
&\qquad\qquad \times \cdots \times \int_{\mathbb{R}^d} dy_q \int_{\mathbb{R}^d} dz_q\, \Gamma_{m,k}(t-s, x-y_q) f(y_q - z_q) \Gamma_{m,k}(t-s, x-z_q) \\
&\qquad\qquad\qquad \times \mathbb{E}[v_n(s,y_1) v_n(s,z_1) \cdots v_n(s,y_q) v_n(s,z_q)]. \qquad (6.8)
\end{aligned}$$

The last step uses Fubini's theorem, the assumptions of which are satisfied because $\Gamma_{m,k} \in \mathcal{P}_+$ and is deterministic for all m, k, and $v_n(t,x)$ has finite moments of any order by the induction assumption. In particular, the right-hand side of (6.8) is finite.

We are going to study the expression $\mathbb{E}[v_n(s,y_1) v_n(s,z_1) \cdots v_n(s,y_q) v_n(s,z_q)]$ and come back to (6.8) later on. More generally, we consider a term of the form

$$\mathbb{E}\left[\prod_{i=1}^p M_{n_i}(s; t_i, x_i)\right],$$

where p is a fixed even integer, $s \in [0,T]$ and for all i, $1 \leqslant n_i \leqslant n$, $x_i \in \mathbb{R}$, and $t_i \in [s,T]$. In the next lemma, we provide an explicit expression for this expectation.

Lemma 6.2. *Let p be a fixed even integer, $(n_i)_{i=1}^p$ be a sequence of integers such that $1 \leqslant n_i \leqslant n$ for all i, let $s \in [0,T]$, $(t_i)_{i=1}^p \subset [s,T]$ and $(x_i)_{i=1}^p \subset \mathbb{R}^d$. Suppose moreover that n is such that for all $m \leqslant n$ and all $q \geqslant 1$,*

$$\sup_{0\leqslant s\leqslant t\leqslant T} \sup_{x\in\mathbb{R}^d} \mathbb{E}[|M_m(s;t,x)|^q] < \infty.$$

If the sequence (n_i) is such that each term in this sequence appears an even number of times, then

$$\begin{aligned}&\mathbb{E}\left[\prod_{i=1}^p M_{n_i}(s;t_i,x_i)\right] \\ &\overset{S}{=} \int_0^s d\rho_1 \cdots \int_0^{\rho_{N-1}} d\rho_N \prod_{j=1}^N \int_{\mathbb{R}^d} \mu(d\xi_j)\overline{\mathcal{F}\Gamma(\sigma_j-\rho_j)(\xi_j+\eta_j)}\mathcal{F}\Gamma(\sigma_j'-\rho_j)(\xi_j+\eta_j') \\ &\quad\times \left(\prod_{k=1}^p e^{i\langle x_k,\delta_k\rangle}\right),\end{aligned} \tag{6.9}$$

where

(a) $\overset{S}{=}$ means "is a sum of terms of the form" (a bound on the number of terms is given in Lemma 6.4 below);

(b) $N = \frac{1}{2}\sum_{i=1}^p n_i$;

(c) σ_j and σ_j' are linear combinations of $\rho_1,\dots,\rho_N,t_1,\dots,t_p$ $(j=1,\dots,N)$;

(d) η_j and η_j' are linear combinations of $\xi_1,\dots,\xi_{j-1}$ $(j=1,\dots,N)$;

(e) δ_k is a linear combination of $\xi_1,\dots,\xi_N$ $(k=1,\dots,p)$.

(f) In (c)-(e), the linear combinations only admit 0, $+1$ and -1 as coefficients.

Remark 6.3. (a) We will see in the proof of Lemma 6.2 that if the elements of the sequence (n_i) do not appear an even number of times, then the expectation vanishes.

(b) It is possible to give an exact expression for the linear combinations in (c)-(e). The exact expression is not needed to prove Theorem 6.1.

Proof. We want to calculate $\mathbb{E}[\prod_{i=1}^p M_{n_i}(s;t_i,x_i)]$. We say that we are interested in the expectation with respect to a *configuration* $(n_i)_{i=1}^p$. The *order* of this configuration (n_i) is defined to be the number $N = \frac{1}{2}\sum_{i=1}^p n_i$.

The proof of the lemma will be based on Itô's formula (see [18, Theorem 3.3, p.147]), by induction on the order of the configuration considered. Suppose first that we have a configuration of order $N=1$. The only case for which the expectation does not vanish is $p=2$, $n_1=n_2=1$ in which the term 1 appears an even number of times. In this case, by [24, Theorem 2.5] and properties of the Fourier transform,

$$\begin{aligned}&\mathbb{E}[M_1^{(m,k)}(s;t_1,x_1)M_1^{(m,k)}(s;t_2,x_2)] \\ &\quad= \int_0^s d\rho_1 \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz\, \Gamma_{m,k}(t_1-\rho_1,x_1-y)f(y-z)\Gamma_{m,k}(t_2-\rho_1,x_2-z) \\ &\quad= \int_0^s d\rho_1 \int_{\mathbb{R}^d} \mu(d\xi)\, \overline{\mathcal{F}\Gamma_{m,k}(t_1-\rho_1)(\xi_1)}\mathcal{F}\Gamma_{m,k}(t_2-\rho_1)(\xi_1)e^{i\langle \xi_1,x_1-x_2\rangle}.\end{aligned}$$

Taking limits as k, then m tend to infinity, we obtain

$$\mathbb{E}[M_1(s;t_1,x_1)M_1(s;t_2,x_2)] = \int_0^s d\rho_1 \int_{\mathbb{R}^d} \mu(d\xi_1)\, \overline{\mathcal{F}\Gamma(t_1-\rho_1)(\xi_1)} \mathcal{F}\Gamma(t_2-\rho_1)(\xi_1) e^{i\langle \xi_1, x_1-x_2\rangle}.$$

This expression satisfies (6.9) with $N=1$, $\sigma_1=t_1$, $\sigma_1'=t_2$, $\eta_1=\eta_1'=0$, $\delta_1=\xi_1$, $\delta_2=-\xi_1$.

Now suppose that (6.9) is true for all configurations of order not greater than N and consider a configuration $(n_i)_{i=1}^p$ of order $N+1$. For all $i=1,\dots,p$, the process $s\mapsto M_{n_i}(s;t_i,x_i)$ is a continuous martingale. We want to find the expectation of $h(M_{n_1},\dots,M_{n_p})$, where $h(x_1,\dots,x_p)=x_1\cdots x_p$. To evalute this expectation, we first use Itô's formula with the function h and the processes $M_{n_i}^{(m_i,k_i)}$ $(i=1,\dots,p)$. We obtain

$$\begin{aligned}
&\mathbb{E}\left[\prod_{i=1}^p M_{n_i}^{(m_i,k_i)}(s;t_i,x_i)\right] \\
&\quad = \sum_{i=1}^p \mathbb{E}\left[\int_0^s \prod_{\substack{j=1\\ j\neq i}}^p M_{n_j}^{(m_j,k_j)}(\rho;t_j,x_j)\, dM_{n_i}^{(m_i,k_i)}(\rho;t_i,x_i)\right] \qquad (6.10)\\
&\qquad +\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^p \mathbb{E}\left[\int_0^s \prod_{\substack{\ell=1\\ \ell\neq i,j}}^p M_{n_\ell}^{(m_\ell,k_\ell)}(\rho;t_\ell,x_\ell)\, d\left\langle M_{n_i}^{(m_i,k_i)}(\cdot;t_i,x_i); M_{n_j}^{(m_j,k_j)}(\cdot;t_j,x_j)\right\rangle_\rho\right].
\end{aligned}$$

As the processes $M_{n_i}^{(m_i,k_i)}$ admit finite moments for all $i=1,\dots,p$, the process in the expectation in the first sum of the right-hand side of (6.10) is a martingale that vanishes at time zero. Hence, this expectation is zero. In the second sum on the right-hand side of (6.10), all terms are similar. For the sake of simplicity, we will only consider here the term for $i=1$, $j=2$: the right-hand side of (6.9) is a sum of terms similar to this one. In the case where $n_1\neq n_2$, the cross-variation is zero. Indeed, the two processes are multiple stochastic integrals of different orders and hence do not belong to the same Wiener chaos. Otherwise, using [24, Theorem 2.5] and Fubini's theorem (which is valid because $M_{n_i}^{(m_i,k_i)}$ has finite moments of any order for all i and $\Gamma_{m,k}\in\mathcal{P}_+$), we have

$$\begin{aligned}
&\mathbb{E}\left[\prod_{i=1}^p M_{n_i}^{(m_i,k_i)}(s;t_i,x_i)\right] \qquad (6.11)\\
&\overset{S}{=} \int_0^s d\rho \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz\, \Gamma_{m_1,k_1}(t_1-\rho,x_1-y) f(y-z) \Gamma_{m_2,k_2}(t_2-\rho,x_2-z)\\
&\qquad \times \mathbb{E}\left[M_{n_1-1}(\rho;\rho,y) M_{n_2-1}(\rho;\rho,z) \prod_{j=3}^p M_{n_j}^{(m_j,k_j)}(\rho;t_j,x_j)\right].
\end{aligned}$$

(We set $M_0\equiv 1$ when $n_1=n_2=1$.) Because $M_{n_j}^{(m_j,k_j)}$ have finite moments of any order and $M_{n_j}^{(m_j,k_j)}\to M_{n_j}$ in $L^2(\Omega)$ by the definition of the stochastic integral (see the proof of Theorem 3.1), we know that $M_{n_j}^{(m_j,k_j)}\to M_{n_j}$ in $L^p(\Omega)$. As $\Gamma_{m,k}\in\mathcal{P}_+$, taking limits as $k_3,\dots,k_p$ tend to

$+\infty$ and then as $m_3, \dots, m_p$ tend to $+\infty$, we obtain

$$\mathbb{E}\left[M_{n_1}^{(m_1,k_1)}(s;t_1,x_1)M_{n_2}^{(m_2,k_2)}(s;t_2,x_2)\prod_{i=3}^{p}M_{n_i}(s;t_i,x_i)\right] \tag{6.12}$$
$$\overset{S}{=} \int_0^s d\rho \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz\, \Gamma_{m_1,k_1}(t_1-\rho, x_1-y) f(y-z) \Gamma_{m_2,k_2}(t_2-\rho, x_2-z)$$
$$\times\, \mathbb{E}\left[M_{n_1-1}(\rho;\rho,y)M_{n_2-1}(\rho;\rho,z)\prod_{j=3}^{p}M_{n_j}(\rho;t_j,x_j)\right].$$

At this point in the proof, we can see why the terms of (n_i) have to appear an even number of times. Indeed, if we consider $n_1 \neq n_2$, we have seen that the expectation is zero. When $n_1 = n_2$, the product in the expectation on the right-hand side of (6.12) is of order N. Hence, we can use the induction assumption to express it as in (6.9). By the induction assumption, if the terms of (n_i) do not appear an even number of times, the expectation on the right-hand side of (6.12) vanishes and hence the one on the left-hand side does too. If these terms do appear an even number of times, then setting $t_1 = s = \rho$, $t_2 = \rho$, $x_1 = y$, $x_2 = z$ in (6.9) and substituting into (6.12), we obtain

$$\mathbb{E}\left[M_{n_1}^{(m_1,k_1)}(s;t_1,x_1)M_{n_2}^{(m_2,k_2)}(s;t_2,x_2)\prod_{i=3}^{p}M_{n_i}(s,t_i,x_i)\right] \tag{6.13}$$
$$\overset{S}{=} \int_0^s d\rho \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz\, \Gamma_{m_1,k_1}(t_1-\rho, x_1-y) f(y-z) \Gamma_{m_2,k_2}(t_2-\rho, x_2-z)$$
$$\times \int_0^{\rho} d\rho_1 \cdots \int_0^{\rho_{N-1}} d\rho_N \prod_{j=1}^{N} \int_{\mathbb{R}^d} \mu(d\xi_j) \overline{\mathcal{F}\Gamma(\sigma_j-\rho_j)(\xi_j+\eta_j)}\mathcal{F}\Gamma(\sigma'_j-\rho_j)(\xi_j+\eta'_j)$$
$$\times \left(e^{i\langle y,\delta_1\rangle}\cdot e^{i\langle z,\delta_2\rangle}\cdot \prod_{k=3}^{p} e^{i\langle x_k,\delta_k\rangle}\right),$$

where

(i) σ_j and σ'_j are linear combinations of $\rho_1, \dots, \rho_N$, ρ, $t_3, \dots, t_p$ $(j = 1, \dots, N)$;

(ii) η_j and η'_j are linear combinations of $\xi_1, \dots, \xi_{j-1}$ $(j = 1, \dots, N)$;

(iii) δ_k is a linear combination of $\xi_1, \dots, \xi_N$ $(k = 1, \dots, p)$.

Since the modulus of the exponentials is 1, by (ii), (6.1) and because $\Gamma_{m,k} \in \mathcal{P}_+$, we see that the right-hand side of (6.13) is finite. So, by Fubini's theorem, we permute the integrals in dy

and dz first with the $d\rho_i$-integrals, then with the $\mu(d\xi_j)$-integrals, to obtain

$$\begin{aligned}
&\mathbb{E}\left[M_{n_1}^{(m_1,k_1)}(s;t_1,x_1)M_{n_2}^{(m_2,k_2)}(s;t_2,x_2)\prod_{i=3}^{p}M_{n_i}(s;t_i,x_i)\right]\\
&\overset{S}{=}\int_0^s d\rho\int_0^{\rho}d\rho_1\cdots\int_0^{\rho_{N-1}}d\rho_N\prod_{j=1}^{N}\int_{\mathbb{R}^d}\mu(d\xi_j)\overline{\mathcal{F}\Gamma(\sigma_j-\rho_j)(\xi_j+\eta_j)}\mathcal{F}\Gamma(\sigma_j'-\rho_j)(\xi_j+\eta_j')\\
&\quad\times\left(\prod_{k=3}^{p}e^{i\langle x_k,\delta_k\rangle}\right)\int_{\mathbb{R}^d}dy\int_{\mathbb{R}^d}dz\,\Gamma_{m_1,k_1}(t_1-\rho,x_1-y)e^{i\langle y,\delta_1\rangle}f(y-z)\\
&\qquad\qquad\times\Gamma_{m_2,k_2}(t_2-\rho,x_2-z)e^{i\langle z,\delta_2\rangle}.
\end{aligned}$$

Rewriting the last two integrals with the Fourier transforms, we have

$$\begin{aligned}
&\mathbb{E}\left[M_{n_1}^{(m_1,k_1)}(s;t_1,x_1)M_{n_2}^{(m_2,k_2)}(s;t_2,x_2)\prod_{i=3}^{p}M_{n_i}(s;t_i,x_i)\right]\\
&\overset{S}{=}\int_0^s d\rho\int_0^{\rho}d\rho_1\cdots\int_0^{\rho_{N-1}}d\rho_N\prod_{j=1}^{N}\int_{\mathbb{R}^d}\mu(d\xi_j)\overline{\mathcal{F}\Gamma(\sigma_j-\rho_j)(\xi_j+\eta_j)}\mathcal{F}\Gamma(\sigma_j'-\rho_j)(\xi_j+\eta_j')\\
&\quad\times\left(\prod_{k=3}^{p}e^{i\langle x_k,\delta_k\rangle}\right)\int_{\mathbb{R}^d}\mu(d\xi)\,\overline{\mathcal{F}\Gamma_{m_1,k_1}(t_1-\rho)(\xi+\delta_1)}\mathcal{F}\Gamma_{m_2,k_2}(t_2-\rho)(\xi+\delta_2)\\
&\qquad\qquad\times e^{i\langle x_1,\xi+\delta_1\rangle}\cdot e^{i\langle x_2,\xi+\delta_2\rangle}.
\end{aligned}\tag{6.14}$$

Setting $\xi_{n+1}=\xi$, $\sigma_{N+1}=t_1$, $\sigma_{N+1}'=t_2$, $\eta_{N+1}=\delta_1$, $\eta_{N+1}'=\delta_2$, $\tilde{\delta}_1=\xi+\delta_1$, $\tilde{\delta}_2=\xi+\delta_2$, the assumptions needed on these linear combinations are satisfied and (6.14) is of the desired form. It remains to take limits as k_1,k_2 and then m_1,m_2 tend to infinity.

The left-hand side has the desired limit because M_{n_i} has finite moments of any order and $\lim_{m_i\to\infty}\lim_{k_i\to\infty}M_{n_i}^{(m_i,k_i)}(s;t_i,x_i)=M_{n_i}(s;t_i,x_i)$ in $L^2(\Omega,\mathcal{F},\mathbb{P})$, $i=1,2$. For the right-hand side, first consider the limit with respect to k_1 and k_2. To show convergence, we consider the left-hand side of (6.14) as the inner product of $\mathcal{F}\Gamma_{m_1,k_1}(t_1-\rho)(\xi+\delta_1)$ and $\mathcal{F}\Gamma_{m_2,k_2}(t_2-\rho)(\xi+\delta_2)$ in the L^2-space with respect to the measure

$$ds\times\cdots\times d\rho_N\times\left(\times_{j=1}^{N}\overline{\mathcal{F}\Gamma(\sigma_j-\rho_j)(\xi_j+\eta_j)}\mathcal{F}\Gamma(\sigma_j'-\rho_j)(\xi_j+\eta_j')\,\mu(d\xi_j)\right)\times\mu(d\xi).\tag{6.15}$$

Note that the exponentials are of modulus one and hence do not play any role in the convergence. Therefore, it is sufficient to consider $i=1$ and to show that

$$\begin{aligned}
&\int_0^s d\rho\int_0^{\rho}d\rho_1\cdots\int_0^{\rho_{N-1}}d\rho_N\prod_{j=1}^{N}\int_{\mathbb{R}^d}\mu(d\xi_j)\overline{\mathcal{F}\Gamma(\sigma_j-\rho_j)(\xi_j+\eta_j)}\mathcal{F}\Gamma(\sigma_j'-\rho_j)(\xi_j+\eta_j')\\
&\quad\times\left(\prod_{k=3}^{p}e^{i\langle x_k,\delta_k\rangle}\right)\int_{\mathbb{R}^d}\mu(d\xi)\,|\mathcal{F}\Gamma_{m,k}(t_1-\rho)(\xi+\delta_1)-\mathcal{F}\Gamma_m(t_1-\rho)(\xi+\delta_1)|^2
\end{aligned}$$

goes to 0 as k tends to infinity. This limit has to be treated differently according to which assumption (H1) or (H2) in Theorem 3.1 is satisfied.

In the case where assumption (H1) is satisfied, the proof of convergence is based on the martingale convergence theorem in a way analogous to the approach used in the proof of Theorem 3.1 with the measure $ds \times \nu_s(d\eta) \times \mu(d\xi)$ replaced by the one in (6.15). Assumption (6.1) allows to bound the $\mu(d\xi_j)$-integrals $(1 \leqslant j \leqslant N)$ when we check the L^2-boundedness of $\mathcal{F}\Gamma_m(t_1 - \rho)(\xi + \delta_1)$.

In the case where (H2) is satisfied, we bound the $\mu(d\xi_j)$-integrals by (6.1) again, compute the time-integrals (except the one with respect to ρ) and finally the continuity assumption (H2) shows the desired convergence.

Finally, the limit with respect to m_1 and m_2 is treated as in the proof of Theorem 3.1 by the Dominated Convergence Theorem. Lemma 6.2 is proved. ■

Proof of Theorem 6.1 (continued)

We use (6.9) with $n_i = n$, $t_i = s$ for all $i = 1, \ldots, p$, to express the expectation in (6.8). Using the same idea as in the proof of Lemma 6.2, we can permute the integrals to obtain

$$\begin{aligned}
&\mathbb{E}[|v_{n+1}^{(m,k)}(t,x)|^p] \\
&\overset{S}{\leqslant} \; t^{q-1} \int_0^t ds \int_0^s d\rho_1 \cdots \int_0^{\rho_{N-1}} d\rho_N \prod_{j=1}^{N} \int_{\mathbb{R}^d} \mu(d\xi_j) \overline{\mathcal{F}\Gamma(\sigma_j - \rho_j)(\xi_j - \eta_j)} \mathcal{F}\Gamma(\sigma_j' - \rho_j)(\xi_j - \eta_j') \\
&\qquad \times \prod_{\ell=1}^{q} \int_{\mathbb{R}^d} \mu(d\beta_\ell) \overline{\mathcal{F}\Gamma_{m,k}(t-s)(\beta_\ell - \gamma_\ell)} \mathcal{F}\Gamma_{m,k}(t-s)(\beta_\ell - \gamma_\ell') e^{i\langle x, \delta\rangle}, \qquad (6.16)
\end{aligned}$$

where $\overset{S}{\leqslant}$ means "is bounded by a sum of terms of the form" and $N = nq$ is the order of the particular configuration considered in that case. The variables $\sigma_j, \sigma_j', \eta_j, \eta_j'$ $(j = 1, \ldots, N)$ satisfy the same assumptions as in Lemma 6.2, the variables $\gamma_\ell, \gamma_\ell'$ $(\ell = 1, \ldots, q)$ are linear combinations of $\xi_1, \ldots, \xi_N$ and δ is a linear combination of $\xi_1, \ldots, \xi_N, \beta_1, \ldots, \beta_q$. When using (6.9) in (6.8), exponentials of the form $e^{i\langle y_j, \delta_j\rangle}$ and $e^{i\langle z_j, \tilde{\delta}_j\rangle}$ appear. When writing the y_ℓ, z_ℓ-integrals as a $\mu(d\beta_\ell)$-integral, these exponentials become shifts. This explains why the variables $\gamma_\ell, \gamma_\ell'$ $(\ell = 1, \ldots, q)$ and δ appear.

Now, using the Cauchy-Schwartz inequality and setting

$$I = \sup_{0 \leqslant s \leqslant T} \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(s)(\xi + \eta)|^2,$$

which is finite by (6.1), and taking limits as k and m tend to $+\infty$, we obtain

$$\begin{aligned}
\mathbb{E}[|v_{n+1}(t,x)|^p] \;&\overset{S}{\leqslant}\; t^{q-1} \int_0^t ds \int_0^s d\rho_1 \cdots \int_0^{\rho_{N-1}} d\rho_N \, I^{N+q} \\
&= \frac{t^{N+q}}{(N+1)!} I^{N+q} = \frac{t^{(n+1)q}}{(nq+1)!} I^{(n+1)q}, \qquad (6.17)
\end{aligned}$$

where $q = \frac{p}{2}$. We have obtained an expression that bounds the moment of order p of v_{n+1} as a finite sum of finite terms. In order to have a bound for this moment, it remains to estimate the number of terms in the sum. This is the goal of Lemma 6.4.

Lemma 6.4. *In the case where $n_i = n$, for all $i = 1, \ldots, p$ and $q = \frac{p}{2}$, then the number of terms in the sum implied by $\overset{S}{=}$ in (6.17) is bounded by $R = (q(p-1))^{nq}$.*

Proof. We have to estimate the number of terms appearing in the sum when we use Itô's formula. For each application of Itô's formula, we have to sum over all choices of pairs in $(n_i)_{i=1}^p$. Hence, we have at most $\frac{1}{2}p(p-1)$ choices. Moreover, Itô's formula has to be iterated at most $N = nq$ times to completely develop the expectation. Hence, the number of terms in the sum implied by $\overset{S}{=}$ is bounded by $R = (q(p-1))^{nq}$. ∎

Proof of Theorem 6.1 (continued)

We return to the proof of Theorem 6.1. Using Lemma 6.4 together with (6.17), we obtain

$$\mathbb{E}[|v_{n+1}(t,x)|^p] \leqslant (q(p-1))^{nq} \frac{t^{(n+1)q}}{(nq+1)!} I^{(n+1)q}. \tag{6.18}$$

Clearly, the series $\sum_{n=0}^{\infty} \|v_{n+1}(t,x)\|_p$ converges, where $\|\cdot\|_p$ stands for the norm in $L^p(\Omega)$. Hence,

$$\|u_n(t,x)\|_p = \|v_1(t,x) + \cdots + v_n(t,x)\|_p \leqslant \sum_{i=0}^{n-1} \|v_{i+1}(t,x)\|_p.$$

As the bound on the series does not depend on x and as $t \leqslant T$, we have

$$\sup_{n\in\mathbb{N}} \sup_{0\leqslant t\leqslant T} \sup_{x\in\mathbb{R}^d} \mathbb{E}[|u_n(t,x)|^p] < \infty, \tag{6.19}$$

for all even integers p. Jensen's inequality then shows that (6.19) is true for all $p \geqslant 1$. As the sequence $(u_n(t,x))_{n\in\mathbb{N}}$ converges in $L^2(\Omega)$ to $u(t,x)$ by Theorem 3.1, (6.19) ensures the convergence in $L^p(\Omega)$ and we have

$$\sup_{0\leqslant t\leqslant T} \sup_{x\in\mathbb{R}^d} \mathbb{E}[|u(t,x)|^p] < \infty,$$

for all $p \geqslant 1$. Theorem 6.1 is proved. ■

Remark 6.5. The fact that α is an affine function is strongly used in this proof. The key fact is that its derivative is constant and so Itô's formula can be applied iteratively. This is not the case for a general Lipschitz function α.

7 Hölder continuity

In this section, we are going to study the regularity of the solution of the non-linear wave equation (4.1) in the specific case considered in Theorem 6.1 : let $u(t,x)$ be the random field solution of the equation

$$Lu = (u+b)\dot{F}, \tag{7.1}$$

with vanishing initial conditions, where $b \in \mathbb{R}$ and the spatial dimension is $d \geqslant 1$. We will need the following hypotheses, which are analogous to those that appear in [20], in order to guarantee the regularity of the solution.

(H3) For all $T > 0$, $h \geqslant 0$, there exist constants C, $\gamma_1 \in]0,+\infty[$ such that

$$\sup_{0\leqslant s\leqslant T} \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(s+h)(\xi+\eta) - \mathcal{F}\Gamma(s)(\xi+\eta)|^2 \leqslant Ch^{2\gamma_1}.$$

(H4) For all $T > 0$, $t \in [0,T]$, there exist constants C, $\gamma_2 \in]0,+\infty[$ such that

$$\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(t)(\xi+\eta)|^2 \leqslant Ct^{2\gamma_2}.$$

(H5) For all $T > 0$ and compact sets $K \subset \mathbb{R}^d$, there exist constants C, $\gamma_3 \in]0, +\infty[$ such that for any $z \in K$,

$$\sup_{0 \leqslant s \leqslant T} \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi) \, |\mathcal{F}\Gamma(s, z - \cdot)(\xi + \eta) - \mathcal{F}\Gamma(s, \cdot)(\xi + \eta)|^2 \leqslant C|z|^{2\gamma_3}.$$

The next result concerns the regularity in time of the solution of (7.1).

Proposition 7.1. *Suppose that the fundamental solution of $Lu = 0$ satisfies the assumptions of Theorem 6.1, (H3) and (H4), and u is the solution of (7.1) given by Theorem 6.1. Then for any $x \in \mathbb{R}^d$, $t \mapsto u(t, x)$ is a.s. γ-Hölder-continuous, for any $\gamma \in]0, \gamma_1 \wedge (\gamma_2 + \frac{1}{2})[$.*

Proof. Following Theorem 6.1, the solution $u(t, x)$ to (7.1) is given recursively by (6.2)-(6.4). Hence, for any $h \geqslant 0$ and $t \in [0, T - h]$, we have

$$u(t + h, x) - u(t, x) = \sum_{n=1}^{\infty} (v_n(t + h, x) - v_n(t, x)). \tag{7.2}$$

The Gaussian process v_1 is given by (6.3). Hence,

$$v_1(t + h, x) - v_1(t, x) = A_1(t, x; h) + B_1(t, x; h),$$

where

$$A_1(t, x; h) = \int_0^t \int_{\mathbb{R}^d} (\Gamma(t + h - s, x - y) - \Gamma(t - s, x - y)) M(ds, dy) \tag{7.3}$$

and

$$B_1(t, x; h) = \int_t^{t+h} \int_{\mathbb{R}^d} \Gamma(t + h - s, x - y) M(ds, dy). \tag{7.4}$$

Fix p an even integer. By Burkholder's inequality (see [15, Chap. IV, Theorem 73]),

$$\begin{aligned}
&\mathbb{E}[|A_1(t, x; h)|^p] \\
&\leqslant C \left(\int_0^t ds \int_{\mathbb{R}^d} \nu_s(d\eta) \int_{\mathbb{R}^d} \mu(d\xi) \, |\mathcal{F}\Gamma(t + h - s)(\xi + \eta) - \mathcal{F}\Gamma(t - s)(\xi + \eta)| \right)^{\frac{p}{2}} \\
&\leqslant C \left(\int_0^t ds \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi) \, |\mathcal{F}\Gamma(t + h - s)(\xi + \eta) - \mathcal{F}\Gamma(t - s)(\xi + \eta)| \right)^{\frac{p}{2}} \\
&\leqslant Ch^{p\gamma_1}
\end{aligned} \tag{7.5}$$

by (H3). On another hand, using again Burkholder's inequality, we see that

$$\begin{aligned}
\mathbb{E}[|B(t, x; h)|^p] &\leqslant \left(\int_t^{t+h} ds \int_{\mathbb{R}^d} \nu_s(d\eta) \int_{\mathbb{R}^d} \mu(d\xi) \, |\mathcal{F}\Gamma(t + h - s)(\xi + \eta)|^2 \right)^{\frac{p}{2}} \\
&\leqslant \left(\int_t^{t+h} ds \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi) \, |\mathcal{F}\Gamma(t + h - s)(\xi + \eta)|^2 \right)^{\frac{p}{2}} \\
&\leqslant C \left(\int_t^{t+h} ds \, (t + h - s)^{2\gamma_2} \right) \\
&\leqslant Ch^{p(\gamma_2 + \frac{1}{2})},
\end{aligned} \tag{7.6}$$

by (H4). Hence, putting together (7.5) and (7.6), we see that there exists a constant C_0 such that

$$\mathbb{E}[|v_1(t+h,x) - v_1(t,x)|^p] \leqslant C_0 h^{p(\gamma_1 \wedge (\gamma_2 + \frac{1}{2}))}. \tag{7.7}$$

For $n \geqslant 2$, set $w_n(t,x;h) = v_n(t+h,x) - v_n(t,x)$, where v_n is defined by (6.4). Then

$$w_{n+1}(t,x;h) = A_n(t,x;h) + B_n(t,x;h),$$

where

$$A_n(t,x;h) = \int_0^t \int_{\mathbb{R}^d} (\Gamma(t+h-s,x-y) - \Gamma(t-s,x-y)) v_n(s,y) M(ds,dy) \tag{7.8}$$

and

$$B_n(t,x;h) = \int_t^{t+h} \int_{\mathbb{R}^d} \Gamma(t+h-s,x-y) v_n(s,y) M(ds,dy). \tag{7.9}$$

Setting $\tilde{\Gamma}(s,y) = \Gamma(t+h-s,x-y) - \Gamma(t-s,x-y)$ and letting $A_n^{(m,k)}$ be the approximation of A_n with Γ replaced by $\Gamma_{m,k}$ in (7.8), we can use the same argument as in (6.8) to see that

$$\begin{aligned} \mathbb{E}[|A_n^{(m,k)}(t,x;h)|^p] &\leqslant C \int_0^t ds \prod_{j=1}^q \int_{\mathbb{R}^d} dy_j \int_{\mathbb{R}^d} dz_j \, \tilde{\Gamma}_{m,k}(s,y_j) f(y_j - z_j) \tilde{\Gamma}_{m,k}(s,z_j) \\ &\quad \times \mathbb{E}[v_n(s,y_1) v_n(s,z_1) \cdots v_n(s,y_q) v_n(s,z_q)], \end{aligned} \tag{7.10}$$

where p is an even integer and $q = \frac{p}{2}$. Using Lemma 6.2 to express the expectation and using the same argument as used to reach (6.16), we obtain

$$\begin{aligned} &\mathbb{E}[|A_n^{(m,k)}(t,x;h)|^p] \\ &\overset{S}{\leqslant} \int_0^t ds \int_0^s d\rho_1 \cdots \int_0^{\rho_{N-1}} d\rho_N \prod_{j=1}^N \int_{\mathbb{R}^d} \mu(d\xi_j) \overline{\mathcal{F}\Gamma(\sigma_j - \rho_j)(\xi_j - \eta_j)} \mathcal{F}\Gamma(\sigma_j' - \rho_j)(\xi_j - \eta_j') \\ &\quad \times \prod_{\ell=1}^q \int_{\mathbb{R}^d} \mu(d\beta_\ell) \overline{\mathcal{F}\tilde{\Gamma}_{m,k}(s)(\beta_\ell - \gamma_\ell)} \mathcal{F}\tilde{\Gamma}_{m,k}(s)(\beta_\ell - \gamma_\ell') e^{i\langle x,\delta\rangle}, \end{aligned} \tag{7.11}$$

where $\overset{S}{\leqslant}$ means "is bounded by a sum of terms of the form", $N = nq$ and $\sigma_j, \sigma_j', \eta_j, \eta_j', \gamma_\ell, \gamma_\ell'$ and δ ($1 \leqslant j \leqslant N$, $1 \leqslant \ell \leqslant q$) satisfy the same assumptions as in (6.16). Notice that Γ appears in the first N integrals and $\tilde{\Gamma}$ in the last q integrals.

We take limits in (7.11) as k and m tend to $+\infty$. Then, using the Cauchy-Schwartz inequality, we bound the first N spatial integrals in (7.11) using (6.1), bound the other q spatial integrals by hypothesis (H3), compute the time integrals and bound the number of terms in the sum by Lemma 6.4 and, similarly to (6.18), we obtain

$$\mathbb{E}[|A_n(t,x;h)|^p] \leqslant (q(p-1))^{nq} \frac{T^{(n+1)q}}{(nq+1)!} I^{nq} h^{p\gamma_1} = C_n^{(1)} h^{p\gamma_1}, \tag{7.12}$$

where $C_n^{(1)} = (q(p-1))^{nq} \frac{T^{(n+1)q}}{(nq+1)!} I^{nq}$.

On another hand, let $B_n^{(m,k)}$ be the corresponding approximation of B_n. The same arguments as those used to obtain (6.8) show that

$$\begin{aligned}
&\mathbb{E}[|B_n^{(m,k)}(t,x;h)|^p] \\
&\quad \leqslant \; Ch^{q-1}\int_t^{t+h} ds \prod_{j=1}^{q} \int_{\mathbb{R}^d} dy_j \int_{\mathbb{R}^d} dz_j \, \Gamma_{m,k}(t+h-s,y_j) f(y_j-z_j) \Gamma_{m,k}(t+h-s,z_j) \\
&\qquad\qquad \times \mathbb{E}[v_n(s,y_1)v_n(s,z_1)\cdots v_n(s,y_q)v_n(s,z_q)]. \qquad (7.13)
\end{aligned}$$

Note that the factor h^{q-1} appears because Hölder's inequality is used on the interval $[t,t+h]$ instead of $[0,t]$. Using Lemma 6.2 and the argument used to reach (6.16), we obtain

$$\begin{aligned}
&\mathbb{E}[|B_n^{(m,k)}(t,x;h)|^p] \\
&\quad \overset{S}{\leqslant} \; Ch^{q-1}\int_t^{t+h} ds \int_0^s d\rho_1 \cdots \int_0^{\rho_{N-1}} d\rho_N \qquad (7.14) \\
&\qquad \times \prod_{j=1}^{N} \int_{\mathbb{R}^d} \mu(d\xi_j)\, \overline{\mathcal{F}\Gamma(\sigma_j-\rho_j)(\xi_j-\eta_j)} \mathcal{F}\Gamma(\sigma_j'-\rho_j)(\xi_j-\eta_j') \\
&\qquad \times \prod_{\ell=1}^{q} \int_{\mathbb{R}^d} \mu(d\beta_\ell)\, \overline{\mathcal{F}\Gamma_{m,k}(t+h-s)(\beta_\ell-\gamma_\ell)} \mathcal{F}\Gamma_{m,k}(t+h-s)(\beta_\ell-\gamma_\ell') e^{i\langle x,\delta\rangle},
\end{aligned}$$

where $\overset{S}{\leqslant}$ means "is bounded by a sum of terms of the form", $N = nq$ and $\sigma_j, \sigma_j', \eta_j, \eta_j', \gamma_\ell, \gamma_\ell'$ and δ ($1 \leqslant j \leqslant N$, $1 \leqslant \ell \leqslant q$) satisfy the same assumptions as in (6.16).

We take limits in (7.14) as k and m tend to $+\infty$. Then, using the Cauchy-Schwartz inequality, we bound the first N spatial integrals in (7.14) using (6.1), bound the other q spatial integrals by hypothesis (H4) and bound the number of terms in the sum by Lemma 6.4. Then

$$\mathbb{E}[|B_n(t,x;h)|^p] \leqslant Ch^{q-1}(q(p-1))^{nq} I^{nq} \int_t^{t+h} ds \int_0^s d\rho_1 \cdots \int_0^{\rho_{N-1}} d\rho_N \, (t+h-s)^{p\gamma_2}.$$

The n-fold integral is bounded by

$$\int_t^{t+h} ds \, \frac{s^{nq}}{(nq)!} (t+h-s)^{p\gamma_2} \leqslant \frac{T^{nq}}{(nq)!} \int_t^{t+h} ds \, (t+h-s)^{p\gamma_2} = \frac{T^{nq}}{(nq)!} h^{p\gamma_2+1},$$

Therefore,

$$\mathbb{E}[|B_n(t,x;h)|^p] \leqslant C_n^{(2)} h^{p(\gamma_2+\frac{1}{2})}, \qquad (7.15)$$

where $C_n^{(2)} = C(q(p-1))^{nq} I^{nq} \frac{T^{nq}}{(nq)!}$.

Finally, putting (7.12) and (7.15) together, we have for any $n \geqslant 2$,

$$\mathbb{E}[|w_{n+1}(t,x;h)|^p] \leqslant (C_n^{(1)} + C_n^{(2)}) h^{p(\gamma_1 \wedge (\gamma_2+\frac{1}{2}))} \qquad (7.16)$$

and, by (7.7) and (7.16),

$$\mathbb{E}[|u(t+h,x)-u(t,x)|^p] \leqslant \left(\sum_{n=1}^{\infty} (C_n^{(1)} + C_n^{(2)}) \right) h^{p(\gamma_1 \wedge (\gamma_2+\frac{1}{2}))}, \qquad (7.17)$$

for any even integer p and $h \geqslant 0$. The series $\sum_{n=1}^{\infty}(C_n^{(1)} + C_n^{(2)})$ converges, as in (6.18). Jensen's inequality establishes that (7.17) holds for an arbitrary $p \geqslant 1$, which shows γ-Hölder-continuity of $t \mapsto u(t,x)$ for any $\gamma \in]0, \gamma_1 \wedge (\gamma_2 + \frac{1}{2})[$ by Kolmogorov's continuity theorem (see [18, Theorem 2.1, p.26]). ■

The next result concerns the spatial regularity of the solution.

Proposition 7.2. *Suppose that the fundamental solution of $Lu = 0$ satisfies the assumptions of Theorem 6.1 and (H5) and u is the solution of (7.1) built in Theorem 6.1. Then for any $t \in [0,T]$, $x \mapsto u(t,x)$ is a.s. γ-Hölder-continuous, for any $\gamma \in]0, \gamma_3[$.*

Proof. The proof is similar to that of Proposition 7.1. We know that $u(t,x)$ is given by (6.2)-(6.4). Hence, for any compact set $K \subset \mathbb{R}^d$ and for any $z \in K$,

$$u(t,x+z) - u(t,x) = \sum_{n=1}^{\infty}(v_n(t,x+z) - v_n(t,x)).$$

The Gaussian process v_1 is given by (6.3). Hence,

$$v_1(t,x+z) - v_1(t,x) = \int_0^t \int_{\mathbb{R}^d} (\Gamma(t-s,x+z-y) - \Gamma(t-s,x-y))M(ds,dy).$$

By Burkholder's inequality,

$$\begin{aligned}
&\mathbb{E}[|v_1(t,x+z) - v_1(t,x)|^p] \\
&\leqslant \left(\int_0^t ds \int_{\mathbb{R}^d} \nu_s(d\eta) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(t-s,x+z-\cdot)(\xi+\eta) - \mathcal{F}\Gamma(t-s,x-\cdot)(\xi+\eta)|^2\right)^{\frac{p}{2}} \\
&\leqslant \left(\int_0^t ds \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(t-s,z-\cdot)(\xi+\eta) - \mathcal{F}\Gamma(t-s,\cdot)(\xi+\eta)|^2\right)^{\frac{p}{2}} \\
&\leqslant C|z|^{p\gamma_3}, \qquad (7.18)
\end{aligned}$$

by (H5). Therefore, there exists a constant C_0 such that

$$\mathbb{E}[|v_1(t,x+z) - v_1(t,x)|^p] \leqslant C_0|z|^{p\gamma_3}. \qquad (7.19)$$

For $n \geqslant 2$, set $w_n(t,x;z) = v_n(t,x+z) - v_n(t,x)$, where v_n is defined by (6.4). Then

$$w_{n+1}(t,x;z) = \int_0^t \int_{\mathbb{R}^d} (\Gamma(t-s,x+z-y) - \Gamma(t-s,x-y))v_n(s,y)M(ds,dy). \qquad (7.20)$$

Setting $\check{\Gamma}(s,y) = \Gamma(t-s,z+y) - \Gamma(t-s,y)$ and letting $w_n^{(m,k)}$ be the approximation of w_n with Γ replaced by $\Gamma_{m,k}$ in (7.20), we can use the same argument as in (6.8) to see that

$$\begin{aligned}
\mathbb{E}[|w_{n+1}^{(m,k)}(t,x;z)|^p] \;\leqslant\; & t^{q-1}\int_0^t ds \prod_{j=1}^{q} \int_{\mathbb{R}^d} dy_j \int_{\mathbb{R}^d} dz_j \check{\Gamma}_{m,k}(s,x-y_j) f(y_j-z_j) \check{\Gamma}_{m,k}(s,x-z_j) \\
& \times \mathbb{E}[v_n(s,y_1)v_n(s,z_1)\cdots v_n(s,y_q)v_n(s,z_q)], \qquad (7.21)
\end{aligned}$$

where p is an even integer and $q = \frac{p}{2}$. Using Lemma 6.2 to express the expectation and using the same argument as used to reach (6.16), we obtain

$$
\begin{aligned}
&\mathbb{E}[|w_{n+1}^{(m,k)}(t,x;z)|^p] \\
&\overset{S}{\leqslant} \; t^{q-1}\int_0^t ds \int_0^s d\rho_1 \cdots \int_0^{\rho_{N-1}} d\rho_N \prod_{j=1}^{N} \int_{\mathbb{R}^d} \mu(d\xi_j)\overline{\mathcal{F}\Gamma(\sigma_j-\rho_j)(\xi_j-\eta_j)}\mathcal{F}\Gamma(\sigma_j'-\rho_j)(\xi_j-\eta_j') \\
&\qquad \times \prod_{l=1}^{q}\int_{\mathbb{R}^d}\mu(d\beta_l)\overline{\mathcal{F}\check{\Gamma}_{m,k}(s)(\beta_k-\gamma_k)}\mathcal{F}\check{\Gamma}_{m,k}(s)(\beta_k-\gamma_k')e^{i\langle x,\delta\rangle}, \qquad (7.22)
\end{aligned}
$$

where $\overset{S}{\leqslant}$ means "is bounded by a sum of terms of the form", $N = nq$ and $\sigma_j, \sigma_j', \eta_j, \eta_j', \gamma_k, \gamma_k'$ and δ ($1 \leqslant j \leqslant N$, $1 \leqslant k \leqslant q$) satisfy the same assumptions as in (6.16). Notice that Γ appears in the first N integrals and $\check{\Gamma}$ in the last q integrals.

We take limits in (7.22) as k and m tend to $+\infty$, then bound the first N spatial integrals in (7.22) using (6.1), bound the other q spatial integrals by hypothesis (H5), compute the time integrals and bound the number of terms in the sum by Lemma 6.4 and we finally reach

$$
\mathbb{E}[|w_{n+1}(t,x;z)|^p] \leqslant (q(p-1))^{nq}\frac{T^{(n+1)q}}{(nq+1)!}I^{nq}|z|^{p\gamma_3} = C_n^{(3)}|z|^{p\gamma_3}, \qquad (7.23)
$$

where $C_n^{(3)} = (q(p-1))^{nq}\frac{T^{(n+1)q}}{(nq+1)!}I^{nq}$. Finally, by (7.19) and (7.23), we have

$$
\mathbb{E}[|u(t,x+z)-u(t,x)|^p] \leqslant \sum_{n=1}^{\infty} C_n^{(3)}|z|^{p\gamma_3}, \qquad (7.24)
$$

for any even integer p and $z \in K$. The series $\sum_{n=1}^{\infty} C_n^{(3)}$ converges, as in (6.18). Jensen's inequality establishes (7.24) for an arbitrary $p \geqslant 1$, , which shows γ-Hölder-continuity of $x \mapsto u(t,x)$ for any $\gamma \in]0,\gamma_3[$ by Kolmogorov's continuity theorem (see [18, Theorem 2.1, p.26]). ∎

As a consequence of Propositions 7.1 and 7.2, we easily obtain the following corollary.

Corollary 7.3. *Suppose that the fundamental solution of $Lu = 0$ satisfies the assumptions of Theorem 6.1 as well as (H3) to (H5), and u is the solution of (7.1) given by Theorem 6.1. Then $(t,x) \mapsto u(t,x)$ is a.s. jointly γ-Hölder-continuous in time and space for any $\gamma \in]0, \gamma_1 \wedge (\gamma_2 + \frac{1}{2}) \wedge \gamma_3[$.*

Proof. By (7.17) and (7.24),

$$
\mathbb{E}[|u(t,x)-u(s,y)|^p] \leqslant C\left(|t-s|^{\gamma_1\wedge(\gamma_2+\frac{1}{2})} + |x-y|^{\gamma_3}\right)^p,
$$

so the conclusion follows from Kolmogorov's continuity theorem (see [18, Theorem 2.1, p.26]). ∎

Now, we are going to check that the fundamental solution of the wave equation satisfies hypotheses (H3) to (H5). This requires an integrability condition on the covariance function f (or the spectral measure μ) of $\dot{F}$: we suppose that there exists $\alpha \in]0,1[$ such that

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^\alpha} < \infty. \tag{7.25}$$

This assumption is the same as condition (40) in [1]. Since $f \geqslant 0$, it is equivalent (see [4, Lemma 8] and [14]) to the property

$$\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi+\eta|^2)^\alpha} < \infty. \tag{7.26}$$

Proposition 7.4. *Suppose (7.26) is satisfied for some $\alpha \in]0,1[$. Then the fundamental solution of the wave equation satisfies hypotheses (H3) to (H5) for any $\gamma_i \in]0, 1-\alpha]$, $i = 1,2,3$.*

Proof. Omitting the factors 2π, which do not play any role, we recall that the fundamental solution Γ of the wave equation satisfies

$$\mathcal{F}\Gamma(s)(\xi) = \frac{\sin(s|\xi|)}{|\xi|}$$

in any spatial dimension $d \geqslant 1$. Consider first hypothesis (H3). Fix Q sufficiently large. For any $s \in [0,T]$ and $h \geqslant 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(s+h)(\xi+\eta) - \mathcal{F}\Gamma(s)(\xi+\eta)|^2 \\
&\qquad = \int_{\mathbb{R}^d} \mu(d\xi)\, \frac{|\sin((s+h)|\xi+\eta|) - \sin(s|\xi+\eta|)|^2}{|\xi+\eta|^2} \\
&\qquad = \int_{|\xi+\eta|\leqslant Q} \mu(d\xi)\, \frac{|\sin((s+h)|\xi+\eta|) - \sin(s|\xi+\eta|)|^2}{|\xi+\eta|^2} \\
&\qquad\quad + \int_{|\xi+\eta|> Q} \mu(d\xi)\, \frac{|\sin((s+h)|\xi+\eta|) - \sin(s|\xi+\eta|)|^2}{|\xi+\eta|^2}.
\end{aligned}$$

Using elementary properties of trigonometric functions and the fact that $|\sin(x)| \leqslant x$ for all $x \geqslant 0$ in the first integral and using the same on the $2(1-\alpha)$ power in the second integral, the previous expression is bounded by

$$\begin{aligned}
&\int_{|\xi+\eta|\leqslant Q} \mu(d\xi)\, 4\,h^2\, \cos^2((2s+h)|\xi+\eta|) \\
&\quad + \int_{|\xi+\eta|> Q} \mu(d\xi)\, \frac{|\sin((s+h)|\xi+\eta|) - \sin(s|\xi+\eta|)|^{2\alpha}}{|\xi+\eta|^{2\alpha}} (2h|\cos((2s+h)|\xi+\eta|)|)^{2(1-\alpha)}.
\end{aligned}$$

Bounding the trigonometric functions by 1 and using properties of the domain of integration of each integral, the previous expression is not greater than

$$\begin{aligned}
&\left(\int_{|\xi+\eta|\leqslant Q} \mu(d\xi) \frac{4(1+Q^2)}{1+|\xi+\eta|^2}\right) h^2 + \left(\int_{|\xi+\eta|> Q} \mu(d\xi) \frac{4(1+\frac{1}{Q^2})^\alpha}{(1+|\xi+\eta|^2)^\alpha}\right) h^{2(1-\alpha)} \\
&\qquad \leqslant C \left(\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi+\eta|^2)^\alpha}\right) h^{2(1-\alpha)}.
\end{aligned}$$

Hence,

$$\sup_{0\leqslant s\leqslant T}\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\mu(d\xi)\,|\mathcal{F}\Gamma(s+h)(\xi+\eta)-\mathcal{F}\Gamma(s)(\xi+\eta)|^2$$
$$\leqslant C\left(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\mu(d\xi)}{(1+|\xi+\eta|^2)^\alpha}\right)h^{2(1-\alpha)},$$

and hypothesis (H3) is satisfied for any $\gamma_1\in\,]0,1-\alpha]$.

For hypothesis (H4), for any $s\in[0,T]$,

$$\begin{aligned}\int_{\mathbb{R}^d}\mu(d\xi)\,|\mathcal{F}\Gamma(s)(\xi+\eta)|^2 &= \int_{\mathbb{R}^d}\mu(d\xi)\,\frac{\sin^2(s|\xi+\eta|)}{|\xi+\eta|^2}\\ &\leqslant \int_{|\xi+\eta|<1}\mu(d\xi)\,\frac{\sin^2(s|\xi+\eta|)}{|\xi+\eta|^2}+\int_{|\xi+\eta|\geqslant 1}\mu(d\xi)\,\frac{\sin^2(s|\xi+\eta|)}{|\xi+\eta|^2}.\end{aligned}$$

Using the fact that $|\sin(x)|\leqslant x$ for all $x\geqslant 0$ in the first integral and the same on the $2(1-\alpha)$ power in the second integral, the previous expression is bounded by

$$s^2\int_{|\xi+\eta|<1}\mu(d\xi)+\int_{|\xi+\eta|\geqslant 1}\mu(d\xi)\,s^{2(1-\alpha)}\frac{|\sin(s|\xi+\eta|)|^{2\alpha}}{|\xi+\eta|^{2\alpha}}.$$

Bounding the trigonometric function by 1 and using properties of the domain of integration of each integral, the previous expression is not greater than

$$\begin{aligned}&\leqslant\quad s^2\int_{|\xi+\eta|<1}\mu(d\xi)\,\frac{2}{1+|\xi+\eta|^2}+s^{2(1-\alpha)}\int_{|\xi+\eta|\geqslant 1}\mu(d\xi)\,\frac{2^\alpha}{(1+|\xi+\eta|^2)^\alpha}\\ &\leqslant\quad C\left(\int_{\mathbb{R}^d}\frac{\mu(d\xi)}{(1+|\xi+\eta|^2)^\alpha}\right)|s|^{2(1-\alpha)}.\end{aligned}$$

Hence,

$$\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\mu(d\xi)\,|\mathcal{F}\Gamma(s)(\xi+\eta)|^2\leqslant C\left(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\mu(d\xi)}{(1+|\xi+\eta|^2)^\alpha}\right)s^{2(1-\alpha)},$$

and hypothesis (H4) is satisfied for any $\gamma_2\in\,]0,1-\alpha]$.

Finally, for hypothesis (H5), for any $x\in\mathbb{R}$ and $z\in K$, K a compact subset of $\mathbb{R}^d$,

$$\begin{aligned}&\int_{\mathbb{R}^d}\mu(d\xi)\,|\mathcal{F}\Gamma(t-s,z-\cdot)(\xi+\eta)-\mathcal{F}\Gamma(t-s,\cdot)(\xi+\eta)|^2\\ &=\int_{|\xi+\eta|<1}\mu(d\xi)\,|e^{-i\langle\xi+\eta,z\rangle}-1|^2\,\frac{\sin^2((t-s)|\xi+\eta|)}{|\xi+\eta|^2}\\ &\quad+\int_{|\xi+\eta|\geqslant 1}\mu(d\xi)\,|e^{-i\langle\xi+\eta,z\rangle}-1|^2\,\frac{\sin^2((t-s)|\xi+\eta|)}{|\xi+\eta|^2}.\end{aligned}$$

Bounding the trigonometric functions by 1, using properties of the domain of integration in the first integral and bounding the 2α power of the second factor by 2 in the second integral, the previous expression is not greater than

$$\int_{|\xi+\eta|<1} \mu(d\xi)\, |e^{-i\langle \xi+\eta, z\rangle} - 1|^2 \frac{2}{1+|\xi+\eta|^2}$$
$$+ \int_{|\xi+\eta|\geqslant 1} \mu(d\xi)\, |e^{-i\langle \xi+\eta, z\rangle} - 1|^{2(1-\alpha)} 2^{2\alpha} \frac{1}{|\xi+\eta|^2}.$$

Using the fact that $|e^{-i\langle \xi+\eta, z\rangle} - 1| \leqslant |\xi+\eta||z|$ and properties of the domain of integration of each integral, the previous expression is bounded by

$$|z|^2 \int_{|\xi+\eta|<1} \mu(d\xi) \frac{2}{1+|\xi+\eta|^2} + |z|^{2(1-\alpha)} \int_{|\xi+\eta|\geqslant 1} \mu(d\xi) \frac{4^{2\alpha}}{(1+|\xi+\eta|^2)^\alpha}$$
$$\leqslant C \left(\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi+\eta|^2)^\alpha} \right) |z|^{2(1-\alpha)}.$$

Hence,

$$\sup_{0\leqslant s\leqslant T} \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(t-s, x+z-\cdot)(\xi+\eta) - \mathcal{F}\Gamma(t-s, x-\cdot)(\xi+\eta)|^2$$
$$\leqslant C \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi+\eta|^2)^\alpha} \right) |z|^{2(1-\alpha)},$$

and hypothesis (H5) is satisfied for any $\gamma_3 \in]0, 1-\alpha]$. ■

We recall the following result for the covariance function $f(x) = \frac{1}{|x|^\beta}$, with $0 < \beta < d$. For a proof, see [20, Prop.5.3].

Proposition 7.5. *If $f(x) = \frac{1}{|x|^\beta}$, where $0 < \beta < d$, then $\mu(dx) = \frac{dx}{|x|^{d-\beta}}$ and (7.25) (hence (7.26)) is satisfied for any $\alpha \in]\frac{\beta}{2}, +\infty[$.*

Putting together Propositions 7.1-7.4, Corollary 7.3 and Proposition 7.5, we have the following.

Theorem 7.6. *If $f(x) = \frac{1}{|x|^\beta}$, with $0 < \beta < 2$, then the random-field solution $u(t,x)$ of the non-linear wave equation with spatial dimension $d > 3$ built in Theorem 6.1 is jointly γ-Hölder-continuous in time and space for any exponent $\gamma \in]0, \frac{2-\beta}{2}[$.*

Remark 7.7. (a) Note that Theorem 7.6 and its proof are still valid when the spatial dimension is less than or equal to 3. In these cases, the regularity of the solution has already been obtained for a more general class of non-linear functions α, namely Lipschitz continuous functions. For more details, see [24] for $d=1$, [12] for $d=2$ and [6] for $d=3$.
(b) The exponent $\frac{2-\beta}{2}$ in Theorem 7.6 is the optimal exponent. Indeed, $u(t,x)$ is not γ-Hölder-continuous for any exponent $\gamma > \frac{2-\beta}{2}$ as is shown in [6, Theorem 5.1]. Their proof applies to the general d-dimensional case, essentially without change.

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