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Semiclassical analysis and a new result for Poisson - Lévy excursion measures

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Abstract

The Poisson-Lévy excursion measure for the diffusion process with small noise satisfying the Itô equation $dX^\varepsilon = b(X^\varepsilon(t))dt + \sqrt{\varepsilon}\, dB(t)$ is studied and the asymptotic behaviour in ε is investigated. The leading order term is obtained exactly and it is shown that at an equilibrium point there are only two possible forms for this term - Lévy or Hawkes – Truman. We also compute the next to leading order.

Key words: excursion measures; asymptotic expansions.

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1 Introduction

Consider a one-dimensional diffusion process defined by

$$dX(t) = b(X(t))\,dt + dB(t), \qquad X(0) = a,$$

where b is a Lipschitz-continuous function and $B(t)$ is a standard Brownian motion. The generator, G, of the above diffusion is

$$G = \frac{1}{2}\frac{d^2}{dx^2} + b(x)\frac{d}{dx},$$

and the putative invariant density is

$$\rho_0(x) = \exp\left(2\int^x b(u)\,du\right).$$

If $\rho_0 \in L^1(\mathbb{R}, dx)$ the boundary $\{-\infty, \infty\}$ is inaccessible. We assume this in what follows. The transition density

$$p_t(x,y) = \mathbb{P}(X(t) \in dy | X(0) = x)/dy,$$

satisfies

$$\begin{aligned}
\frac{\partial p_t(x,y)}{\partial t} &= \frac{\partial}{\partial y}\left(\frac{1}{2}\frac{\partial p_t(x,y)}{\partial y} - b(y)p_t(x,y)\right), \\
&= \left(G_y^* p_t\right)(x,y), \\
\lim_{t\downarrow 0} p_t(x,y) &= \delta_x(y),
\end{aligned}$$

G_y^* being the L^2 adjoint of G_y, and δ being the Dirac delta function. The density of the diffusion

$$\rho^t(y) = \int \rho_0(x)p_t(x,y)\,dx$$

therefore satisfies

$$\frac{\partial \rho^t}{\partial t} = \left(G_y^* \rho^t\right)(y).$$

Evidently,

$$\left(G_y^* \rho_0\right)(y) = 0, \qquad \frac{\partial \rho_0(y)}{\partial t} = 0,$$

so ρ_0 is the invariant density.

Crucial in what follows is the operator identity for any well-behaved f

$$Gf = -\left(\rho_0^{-1/2} H \rho_0^{1/2}\right) f,$$

or

$$G = -\left(\rho_0^{-1/2} H \rho_0^{1/2}\right)$$

and

$$G^* = -\left(\rho_0^{1/2} H \rho_0^{-1/2}\right),$$

where H is the one dimensional Schrödinger operator with potential
$V = \frac{1}{2}(b^2 + b')$,

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + V(x).$$

This follows because $\left(H\rho_0^{1/2}\right) \equiv 0$, i.e. $\rho_0^{1/2}$ is the ground state of H. For convenience, we will assume $V \in C^2(\mathbb{R})$, V bounded below together with V'', V polynomially bounded with derivatives.

2 Excursion Theory

The map $s \mapsto X(s)$ is continuous and so $\{s > 0 : X(s) \gtrless a\}$ is an open subset of $\mathbb{R}$. Therefore, $\{s > 0 : X(s) \gtrless a\}$ can be decomposed into a countable union of open intervals – $\substack{\text{upward}\\ \text{downward}}$ excursion intervals. Define

$$L^{\pm}(t) = \mathrm{Leb}\{s \in [0,t] : X(s) \gtrless a\},$$

and the local time at a

$$L^a(t) = \lim_{h \downarrow 0} h^{-1}\mathrm{Leb}\{s \in [0,t] : X(s) \in (a - h/2, a + h/2)\}.$$

$L^a(t)$ has inverse $\gamma^a(t)$, the time required to wait until L^a equals t. It can be seen that $\gamma^a(t)$ is a stopping time with $X(\gamma^a(t)) = a$. Moreover, as is intuitively obvious,

Jumps in $\gamma^a(t)$ = Excursions of X from a up to L^a equals t.

Example 1 Lévy [1954]

Lévy proved that for $b \equiv 0$, for each $\lambda > 0$,

$$\mathbb{E}_a \exp(-\lambda\gamma^a(t)) = \exp\left\{-t\int_0^\infty (1 - e^{-\lambda s})\, d\nu_a(s)\right\},$$

with Poisson-Lévy excursion measure

$$\nu_a[s,\infty) = \left(\frac{2}{\pi}\right)^{1/2} s^{-1/2}, s > 0. \tag{2.1}$$

Equating powers of λ in the above, we conclude that

$\sharp(s,t)$= Number of excursions of duration exceeding s up to L^a equals t

is Poisson with

$$\mathbb{P}(\sharp(s,t) = N) = \exp\left(-t\nu_a[s,\infty)\right)\left(t\nu_a[s,\infty)\right)^N / N!,$$

for $N = 0, 1, 2, \ldots$, and so the expected number of excursions of duration exceeding s per unit local time at a is $\nu_a[s,\infty)$, the Poisson-Lévy excursion measure.

Example 2 Hawkes and Truman [1991]

For the Ornstein-Uhlenbeck process $b(x) = -kx$, where k is a positive constant, the Hamiltonian is just

$$H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + k^2x^2 - k\right)$$

and $\rho_0(x) = C\exp\left(-kx^2\right)$. This leads to

$$\mathbb{E}_0 \exp(-\lambda\gamma^a(t)) = \exp\left\{-t\int_0^\infty (1-e^{-\lambda s})\, d\nu_0(s)\right\}, \lambda > 0,$$

with

$$\nu_0[s,\infty) = \frac{2k^{1/2}}{\pi^{1/2}}\left(e^{2ks}-1\right)^{-1/2}. \tag{2.2}$$

We discuss generalisations of the above to upward and downward excursions. Note that $\substack{\text{upward}\\\text{downward}}$ excursions can only be affected by values of $b(x)$ for $x \gtrless a$. Therefore it is natural to define the symmetrised potential

$$V^+_{\text{symm}} = \begin{cases} V(x), & x > a, \\ V(2a-x), & x < a. \end{cases}$$

with V^-_{symm} being defined in a similar manner. In an analogous manner we also define

$$H^\pm = -\frac{1}{2}\frac{d^2}{dx^2} + V^\pm_{\text{symm}}(x).$$

We now have the result due to Truman and Williams [1991]

Proposition 1. *Modulo the above assumptions*

$$\mathbb{E}_a \exp\left(-\lambda L^\pm\left(\gamma^a(t)\right)\right) = \exp\left\{-t\int_0^\infty (1-e^{-\lambda s})\, d\nu_a^\pm(s)\right\}$$

with

$$\nu_a^\pm[t,\infty) = \int_{y\gtrless a} dy\, \frac{\rho_0^{1/2}(y)}{\rho_0^{1/2}(a)}\, \frac{\partial}{\partial x}\bigg|_{x=a} \exp\left(-tH^\pm\right)(x,y).$$

Remarks

1. $L^\pm\left(\gamma^a(t)\right)$ are independent with $L^+\left(\gamma^a(t)\right) + L^-\left(\gamma^a(t)\right) = \gamma^a(t)$.

2. Jumps in $L^\pm\left(\gamma^a(t)\right) = \substack{\text{upward}\\\text{downward}}$ excursions from a up to L^a equals t.

3. $\nu_a^\pm[s,\infty)$ is the expected number of $\substack{\text{upward}\\\text{downward}}$ excursions of duration exceeding s per unit local time at a.

Proof. (Outline) The proof uses the result of Lévy [1954]

$$\mathbb{E}_a \exp(-\lambda\gamma^a(t)) = \exp\left(-t/\tilde{p}_\lambda(a,a)\right),$$

where $\tilde{p}_\lambda(x,y) = \int_0^\infty e^{-\lambda s} p_s(x,y)\, ds$ and $p_s(\cdot,\cdot)$ is the transition density.
We can deduce that

$$\tilde{p}_\lambda^{-1}(a,a) = \lambda \int_{-\infty}^{\infty} \frac{\rho_0(x)}{\rho_0(a)} \mathbb{E}e^{-\lambda\tau_x(a)}\, dx,$$

where $\tau_x(a) = \inf\{s > 0 : X(s) = a | X(0) = x\}$. Here the point is that for any point a intermediate to x and y

$$p_t(x,y) = \int_0^\infty \mathbb{P}(\tau_x(a) \in du) p_{t-u}(a,y).$$

Since the right hand side is a convolutional product, taking Laplace transforms and letting $y \to a$ gives

$$\mathbb{E}e^{-\lambda\tau_x(a)} = \tilde{p}_\lambda(x,a)/\tilde{p}_\lambda(a,a).$$

Now multiply both sides by $\rho_0(x)$ and integrate with respect to x (using the fact that ρ_0 is the invariant density) to get the desired result for $\tilde{p}_\lambda^{-1}$. Some elementary computation then leads to the result in Proposition 1.

3 The Poisson-Lévy Excursion Measure for Small Noise

We will now consider the $\substack{\text{upward} \\ \text{downward}}$ excursions from the equilibrium point 0 for the one-dimensional time-homogeneous diffusion process with small noise, $X^\varepsilon(t)$, where

$$dX^\varepsilon(t) = b(X^\varepsilon(t))\, dt + \sqrt{\varepsilon}\, dB(t).$$

Introducing the small noise term into the Truman-Williams Law seen in the previous section, we get:

Proposition 2. *The expected number of* $\substack{\textit{upward} \\ \textit{downward}}$ *excursions from* 0 *of duration exceeding* s*, per unit local time at* 0 *is given by*

$$\nu_0^\pm[s,\infty) = \pm \int_{y \gtrless 0} \frac{\rho_0^{\frac{1}{2}}(y)}{\rho_0^{\frac{1}{2}}(0)}\ \varepsilon \left.\frac{\partial}{\partial x}\right|_{x=0} \exp\left(-\frac{sH^\pm}{\varepsilon}\right)(x,y)dy,$$

where ρ_0 *is the invariant density and* $H^\pm$ *is the symmetrized Hamiltonian for* $V = \frac{1}{2}(b^2 + \varepsilon b')$.

One should note the form of V, in particular the presence of ε as a multiplier of b'. This rather specific dependence originates from the Shrödinger operator mentioned earlier. Consequently, we are unable to resort to the usual methods for resolving such a dependence.

We now give a result due to Davies and Truman Davies and Truman [1982].

Proposition 3. *Let $X_{min}(\cdot)$ be the minimising path for the classical action $A(z) = 2^{-1}\left(\int_0^t \dot{z}^2(s)\,ds + \int_0^t b^2(z(s))\,ds\right)$ with $z(0)=x, z(t)=y$. Set $A(X_{min}) = A(x,y,t)$. Then for the self-adjoint quantum mechanical Hamiltonian $H(\varepsilon) = \left[-\frac{\varepsilon^2}{2}\triangle + V_\varepsilon\right]$, where $V_\varepsilon = \frac{1}{2}(b^2+\varepsilon b') \in C^\infty(R)$ and is convex (where $V_\varepsilon = V_0 + \varepsilon V_1 \in C^4$, bounded below with $V_0'' \geq -|\beta|$), then for each finite time $t \geq 0$ (for $t \leq \pi/|\beta|^{\frac{1}{2}}$).*

$$\begin{aligned}
&\exp\left(-\frac{tH(\varepsilon)}{\varepsilon}\right)(x,y)\\
&= (2\pi\varepsilon)^{-\frac{1}{2}}\exp\left(-\frac{A(x,y,t)}{\varepsilon}\right)\left\{\left|\frac{\partial^2 A(x,y,t)}{\partial x\partial y}\right|^{\frac{1}{2}}\left(1+\varepsilon K + O(\varepsilon^2)\right)\right\}.
\end{aligned}$$

K is a rather complicated expression with many terms involving sums and products of b (and its derivatives), V (and its derivatives) and the Feynman-Green function $G(\tau,\sigma)$ of the Sturm-Liouville differential operator $\frac{d^2}{d\sigma^2} - V''(X_{min}(\tau))$ with zero boundary conditions i.e. $G(0,\tau) = G(t,\tau) = 0$, and discontinuity of derivative across $\tau=\sigma$ of 1.

For a proof of this result see Davies and Truman [1982].

Henceforth, for simplicity we assume that $b^2(x)$ is an even function of x so that $\nu_0^+ = \nu_0^-$.

Theorem 1. *Using the notation and assumptions of Proposition 3, the leading term of the Poisson-Lévy excursion measure, for excursions away from the position of stable equilibrium 0, where $b(0)=0$, and $b'(0)\leq 0$ is given by*

$$\begin{aligned}
\nu_0^+[t,\infty) &\sim (2\pi\varepsilon)^{-\frac{1}{2}}\int_0^\infty dy\,\exp\left\{-\frac{y^2}{2\varepsilon}\left(\frac{\partial^2 A(0,0,t)}{\partial y^2} - b'(0)\right) - \frac{A(0,0,t)}{\varepsilon}\right\}\\
&\times \left\{\left|\frac{\partial^2 A}{\partial x\partial y}\right|_{x=0}^{\frac{1}{2}}\left(-\frac{\partial A}{\partial x}\right)_{x=0}\exp\left(\frac{1}{2}\int_0^t \left|b'(X_{min}(s))\right|\,ds\right)\right\},
\end{aligned}$$

with the action $A(x,y,t) = \frac{1}{2}\int_0^t \dot{z}^2(s)\,ds + \int_0^t V\big(z(s)\big)\,ds$, where $V = \frac{1}{2}b^2$.

Proof. As usual, the classical path $X_{min}(t) = X(x,y,t)$ satisfies, correct to first order in ε

$$\ddot{X}_{min} \sim V_0'(X_{min}) + \varepsilon\, V_1'(X_{min}).$$

For $V_0 = \frac{1}{2}b^2$ (assumed to be convex with $V_0(0)=0$, $V'(0)=0$, and $V''(0)>0$, for example $V_0(x) = \frac{1}{2}x^2$, $b(x)=-x$) , and $V_1 = \frac{1}{2}b'$, then to leading order $\ddot{X}_{min} = V_0'(X_{min})$.

The contribution to the action $A(x,y,t) = \frac{1}{2}\int_0^t \dot{z}^2\,ds + \int_0^t V\big(z(s)\big)\,ds$ from V_1 is to leading order

$$\begin{aligned}
\varepsilon\int_0^t V_1(X_{min}(s))\,ds &= \frac{\varepsilon}{2}\int_0^t b'(X_{min}(s))\,ds\\
&= -\frac{\varepsilon}{2}\int_0^t \left|b'(X_{min}(s))\right|\,ds,
\end{aligned}$$

introducing $|\cdot|$ for convenience. Therefore, from Proposition 3,

$$\exp\left(-\frac{tH(\varepsilon)}{\varepsilon}\right)(x,y) \sim (2\pi\varepsilon)^{-\frac{1}{2}} \exp\left(-\frac{A(x,y,t)}{\varepsilon}\right)\left|\frac{\partial^2 A(x,y,t)}{\partial x \partial y}\right|^{\frac{1}{2}},$$

and so, the contribution to term $\exp\left(-\frac{A(x,y,t)}{\varepsilon}\right)$ from V_1 is

$$\exp\left(\frac{1}{2}\int_0^t \left|b'(X_{min}(s))\right| ds\right), \qquad \text{the Zero Point Energy term.}$$

Therefore, we have using Proposition 2 the leading order term in the Poisson-Lévy excursion measure, for upward excursions from stable equilibrium point 0 given by

$$\begin{aligned}
&\nu_0^+[t,\infty) \\
&\sim (2\pi\varepsilon)^{-\frac{1}{2}} \int_0^\infty dy \exp\left(\frac{1}{\varepsilon}\int_0^y b(u)du\right) \\
&\times \varepsilon \left.\frac{\partial}{\partial x}\right|_{x=0}\left[\exp\left(\frac{1}{2}\int_0^t \left|b'(X_{min}(s))\right| ds\right)\exp\left(-\frac{A(x,y,t)}{\varepsilon}\right)\left|\frac{\partial^2 A}{\partial x \partial y}\right|^{\frac{1}{2}}\right].
\end{aligned}$$

Hence, for small ε, the leading order term is

$$\begin{aligned}
\nu_0^+[t,\infty) \quad \sim \quad & (2\pi\varepsilon)^{-\frac{1}{2}} \int_0^\infty dy \exp\left(\frac{1}{\varepsilon}\left(\int_0^y b(u)du - A(0,y,t)\right)\right) \\
\times \quad & \left[\left|\frac{\partial^2 A}{\partial x \partial y}\right|^{\frac{1}{2}}\left(-\frac{\partial A}{\partial x}\right)\left(\exp\left(\frac{1}{2}\int_0^t \left|b'(X_{min}(s))\right| ds\right)\right)\right]_{x=0} + O(\varepsilon).
\end{aligned} \tag{3.1}$$

Comparing this to the Laplace Integral

$$I(\varepsilon) = \int_a^b e^{-\frac{\phi(x)}{\varepsilon}}\,\theta(x)\,dx,$$

where the main contribution comes from the asymptotic behaviour at points $x_i \in [a,b]$ with $\phi'(x_i) = 0$, we can see that the main contribution to the integral in equation 3.1 comes from those $y(t,0)$ satisfying

$$b(y) = \frac{\partial A(0,y,t)}{\partial y}.$$

If we expand $\phi(y) = \int_0^y b(u)du - A(0,y,t)$ in a Taylor series about $y(t,x) = 0$ we get

$$\begin{aligned}
\phi(y) &= \phi(0) + (y - y(t,x))\phi'(0) + \frac{1}{2}(y - y(t,x))^2\,\phi''(0) + \cdots \\
&= \phi(0) + \frac{1}{2}(y - y(t,x))^2\,\phi''(0) + \cdots
\end{aligned}$$

Hence,

$$
\begin{aligned}
\nu_0^+[s,\infty) \quad &\sim \quad (2\pi\varepsilon)^{-\frac{1}{2}} \int_0^\infty dy \, \exp\left\{ \frac{1}{\varepsilon}\Bigg(- A(0,0,s) \right.\\
&+ \quad \left. \frac{1}{2}(y - y(s,x))^2 \frac{\partial^2}{\partial y^2} \left[\int_0^y b(u)\, du - A(0,y,s) \right] \Bigg)_{y=y(s,x)=0} \right\} \\
&\times \quad \left\{ \left(\exp\left(\frac{1}{2} \int_0^t \left| b'(X_{min}(s)) \right| \, ds \right) \left| \frac{\partial^2 A}{\partial x \partial y} \right|^{\frac{1}{2}}_{x=0} \left(- \frac{\partial A}{\partial x} \right)_{x=0} \right) \right\},
\end{aligned}
$$

giving,

$$
\begin{aligned}
&\nu_0^+[s,\infty) \\
&\sim (2\pi\varepsilon)^{-\frac{1}{2}} \int_0^\infty dy \, \exp\left\{ \frac{-A(0,0,s)}{\varepsilon} - \frac{1}{2\varepsilon} \left[\frac{\partial^2 A(0,y,s)}{\partial y^2} - b'(y) \right]_{y=0} y^2 \right\} \\
&\times \left\{ \left(\exp\left(\frac{1}{2} \int_0^t \left| b'(X_{min}(s)) \right| \, ds \right) \left(\frac{\partial^2 A}{\partial x \partial y} \right)^{\frac{1}{2}}_{x=0} \left(- \frac{\partial A}{\partial x} \right)_{x=0} \right) \right\},
\end{aligned}
$$

and so the result follows. □

4 Poisson-Lévy Excursion Measure – leading order behaviour

We have seen in the previous section that in order to calculate the Poisson-Lévy excursion measure $\nu_0^+[t,\infty)$ for a general process $X(t) = X[x,y,t]$, we require expressions for the following derivatives of the action $A(x,y,t)$.

$$
\begin{aligned}
\left.\frac{\partial A(x,y,t)}{\partial x}\right|_{x=0} \quad & \text{where} \quad && p_0 = -\frac{\partial A}{\partial x}, \\
\left.\left|\frac{\partial^2 A}{\partial x \partial y}\right|^{\frac{1}{2}}\right|_{x=0} \quad & \text{where} \quad && \frac{\partial X(t)}{\partial p_0} = \left(\frac{\partial^2 A}{\partial x \partial y}\right)^{-1} \quad \text{(the Van Vleck identity)}, \\
\left.\frac{\partial^2 A}{\partial y^2}\right|_{x=0} \quad & \text{where} \quad && b(y) = \frac{\partial A}{\partial y}(x,y,t).
\end{aligned}
$$

These expressions are evaluated in the following propositions :-

Proposition 4. *For $p(0) = b(0) = 0$, $|b'(0)| \neq 0$,*

$$
-\frac{\partial A}{\partial x}(0,y,t) \quad \sim \quad \frac{|b'(0)|\, y}{\sinh |b'(0)|\, t}, \quad \textit{as} \quad y \to 0.
$$

Proof. Observe that $p_x(y,t) = -\frac{\partial A(x,y,t)}{\partial x}$ = initial momentum at x needed to reach y in time t, satisfies

$$
t = \int_x^y \frac{du}{\left(p_0^2(y,t) + b^2(u) - b^2(x)\right)^{\frac{1}{2}}}.
$$

Therefore, $p_0(y,t)$ satisfies

$$t = \int_0^y \frac{du}{\left(p_0^2(y,t) + b^2(u)\right)^{\frac{1}{2}}}. \tag{4.1}$$

Changing integration variable $u = y\,v$,

$$t = \int_0^1 \frac{dv}{\left(\frac{p_0^2(y,t)}{y^2} + \frac{b^2(yv)}{y^2}\right)^{\frac{1}{2}}},$$

since to first order $V = \frac{1}{2}b^2$ and $V(0) = 0$ by assumption,

$$\frac{p_0(y) - p_0(0)}{y} \to p_0'(0), \quad \text{as} \quad y \to 0.$$

Therefore,

$$t = \int_0^1 \frac{dv}{\left(p_0'^2(0,t) + v^2\, b'^2(0)\right)^{\frac{1}{2}}}, \qquad \text{as } y \to 0.$$

Letting $v = \left|\frac{p_0'(0)}{b'(0)}\right| \sinh w$ we get

$$t = \frac{1}{|b'(0)|}\, \sinh^{-1}\left|\frac{b'(0)}{p'(0)}\right|, \tag{4.2}$$

giving, for $b'(0) \neq 0$,

$$|p_0'(0,t)| = \frac{|b'(0)|}{\sinh|b'(0)|\,t}. \tag{4.3}$$

$\square$

Note that for $b'(0) = 0$, we get $p_0'(0,t) = \pm 1/t$ and so equation 4.2 has the correct limiting behaviour.

Proposition 5. *For $|b'(0)| \neq 0$,*

$$\frac{\partial^2 A}{\partial x \partial y}(0,y,t) \sim \left(\frac{\sinh|b'(0)|t}{|b'(0)|}\right)^{-1}, \quad \textit{as} \quad y \to 0.$$

Proof. Using the fact that $p_0(y) = -\partial A/\partial x$ and equation 4.1 we quickly get

$$\frac{-1}{p_0'(y)} = p_0(y)\left(p_0(y)^2 + b(y)^2\right)^{\frac{1}{2}} \int_0^y \frac{du}{(p_0(y)^2 + b(u)^2)^{\frac{3}{2}}}.$$

Again, changing the variable of integration $u = yv$, we get

$$\frac{-1}{p_0'(y)} = \frac{p_0(y)}{y}\left(\left(\frac{p_0(y)}{y}\right)^2 + \left(\frac{b(y)}{y}\right)^2\right)^{\frac{1}{2}} \int_0^1 \frac{dv}{\left(\left(\frac{p_0(y)}{y}\right)^2 + \left(\frac{b(yv)}{y}\right)^2\right)^{\frac{3}{2}}}.$$

Now, following the previous argument, as $y \to 0$,

$$\frac{-1}{p_0'(y)} \to p_0'(0,t)\left(p_0'(0,t)^2 + b'(0)^2\right)^{\frac{1}{2}} \int_0^1 \frac{dv}{(p_0'(0)^2 + b'(0)^2\, v^2)^{\frac{3}{2}}}.$$

Letting $v = \left|\frac{p_0'(0)}{b'(0)}\right| \sinh w$ in the above equation gives

$$\begin{aligned} \frac{-1}{p_0'(y)} &\to \frac{\left(p_0'(0)^2 + b'(0)^2\right)^{\frac{1}{2}}}{|b'(0)|\, p_0'(0)^2} \int_0^{\sinh^{-1}\left|\frac{|b'(0)|}{p_0'(0)}\right|} \frac{1}{\cosh^2 w}\, dw \\ &= \frac{\sinh |b'(0)|t}{|b'(0)|}, \end{aligned}$$

using

$$|p_0'(0)| = \frac{|b'(0)|}{\sinh |b'(0)|t}.$$

□

Proposition 6. *For $b'(0) \le 0$, we have*

$$b'(0) - \frac{\partial^2 A(0,0,t)}{\partial y^2} = -|b'(0)|(1 + \coth |b'(0)|\, t).$$

($\frac{\partial A}{\partial y}$ is the momentum at y given that y is reached from x in time t.)

Proof. From

$$\frac{\partial^2 A}{\partial y^2} = \frac{\partial}{\partial y} p(y) = \frac{\partial\left(p_0^2(y,t) + b^2(y)\right)^{\frac{1}{2}}}{\partial y},$$

$$\begin{aligned} b'(y) - \frac{\partial^2 A}{\partial y^2} &= b'(y) - \frac{\frac{\partial p_0(y)}{\partial y} + \frac{b(y)}{p_0(y)} b'(y)}{\left(1 + \left(\frac{b(y)}{p_0(y)}\right)^2\right)^{\frac{1}{2}}} \\ &\to b'(0) - \frac{p_0'(0) + \frac{b'(0)}{p_0'(0)} b'(0)}{\left(1 + \left(\frac{b'(0)}{p_0'(0)}\right)^2\right)^{\frac{1}{2}}} \quad \text{as} \quad y \to 0, \\ &= b'(0) - \frac{\frac{|b'(0)|}{\sinh |b'(0)|t} + b'(0)^2 \frac{\sinh |b'(0)|t}{|b'(0)|}}{\left(1 + \sinh^2 |b'(0)|t\right)^{\frac{1}{2}}} \\ &= b'(0) - |b'(0)| \frac{\cosh |b'(0)|t}{\sinh |b'(0)|t} \end{aligned}$$

Therefore

$$\left\{b'(y) - \frac{\partial^2 A(0,y,t)}{\partial y^2}\right\}\Bigg|_{y=0} \longrightarrow -|b'(0)|\left(1+\coth|b'(0)|t\right),$$

and result follows. $\square$

We now come to our main result for excursions from an equilibrium point 0.

Theorem 2. *For the diffusion X^ε with small noise satisfying*

$$dX^\varepsilon(t) = b\big(X^\varepsilon(t)\big)\,dt + \sqrt{\varepsilon}\,dB(t),$$

denote the Poisson-Lévy measure for excursions from 0 by $\nu_0^\pm$.

Assuming b is continuous, b having right and left derivatives at 0, with $b(0^\pm) \lessgtr 0$ and $b(0)=0$, then if $V_0^\pm = \frac{1}{2}b^2$ satisfies

$$V_0^{\pm\prime\prime} \geq -|\beta_\pm|, \quad \textit{for} \quad t \leq \frac{\pi}{|\beta_\pm|^{\frac{1}{2}}},$$

$$\nu_0^\pm[s,\infty) \sim \left(\frac{\varepsilon\, k_\pm}{\pi}\right)^{\frac{1}{2}} \left(e^{2k_\pm t} - 1\right)^{-\frac{1}{2}}$$

with $k_\pm = |b'(0^\pm)|$. When $|b'(0^\pm)| = 0$ the limiting behaviour is correct and yields

$$\nu_0^\pm[t,\infty) \sim \left(\frac{\varepsilon}{2\pi}\right)^{\frac{1}{2}} t^{-\frac{1}{2}}.$$

Proof. Using Theorem 1, and the expressions obtained in Propositions 4, 5 and 6, for the derivatives of the action as $y \to 0$, we get as the leading order term for excursions from the stable equilibrium position 0, (dropping $\pm$ again for convenience),

$$\begin{aligned}
&\nu_0^+[s,\infty) \\
&\sim (2\pi\varepsilon)^{-\frac{1}{2}}\, e^{\frac{s|b'(0)|}{2}} \left|\frac{\sinh|b'(0)|t}{|b'(0)|}\right|^{-\frac{1}{2}} \left(\frac{|b'(0)|}{\sinh|b'(0)|t}\right) \\
&\times \int_0^\infty dy\; y\; \exp\left(-\frac{y^2}{2\varepsilon}|b'(0)|(1+\coth|b'(0)|t)\right) \\
&= (2\pi\varepsilon)^{-\frac{1}{2}} e^{\frac{t|b'(0)|}{2}} |b'(0)|^{\frac{1}{2}} (\sinh|b'(0)|t)^{-\frac{1}{2}} \left(\frac{\varepsilon}{\cosh|b'(0)|t + \sinh|b'(0)|t}\right) \\
&= (2\pi)^{-\frac{1}{2}}\,\varepsilon^{\frac{1}{2}}\,|b'(0)|^{\frac{1}{2}}\, e^{-\frac{t|b'(0)|}{2}} \left(\frac{1}{2}(e^{t|b'(0)|} - e^{-t|b'(0)|})\right)^{-\frac{1}{2}}.
\end{aligned}$$

$\square$

These results correspond to the Poisson-Lévy excursion measures for the examples seen earlier.

5 Poisson-Lévy Excursion Measure – higher order behaviour

In calculating higher order terms in the Poisson-Lévy excursion measure $\nu_0^+[s,\infty)$, we obtain the surprising result that the next order term is identically zero. We now write the leading term as $\varepsilon^{\frac{1}{2}}\,\nu_{\frac{1}{2}}^+$.

Once again we must emphasise the particular dependence of V_ε on ε and how this requires us to follow a rather complicated route in determining the higher order dependencies on ε. This arises due to our study originating from stochastic mechanics where the Schrödinger equation and operator hold sway.

Theorem 3. *For the diffusion process with small noise, assuming $b(x) \le 0$ for all x, the Poisson-Lévy excursion measure is given by*

$$\nu_\varepsilon^+ \cong \varepsilon^{\frac{1}{2}}\,\nu_{\frac{1}{2}}^+ + O(\varepsilon^{\frac{3}{2}})$$

i. e. the second order term is identically zero.

Proof - First part. For the derivation of the next order term of the Poisson- Levy excursion measure $\nu_0^+[s,\infty)$ about $x=0$, we must include the second order term in the expression for kernel $\exp\left(-\frac{tH(\varepsilon)}{\varepsilon}\right)(x,y)$ given in Proposition 3. Hence, from Propositions 2 and 3

$$\begin{aligned}\nu_0^+[s,\infty) \quad &\sim \quad (2\pi\varepsilon)^{-\frac{1}{2}} \int_{y<0} \frac{\rho_0^{\frac{1}{2}}(y)}{\rho_0^{\frac{1}{2}}(0)}\,\varepsilon\,\frac{\partial}{\partial x}\bigg|_{x=0} \Bigg[\exp\left(-\frac{A(x,y,t)}{\varepsilon}\right) \\ &\times \quad \exp\left(\frac{1}{2}\int_0^t |b'(X_{min}(s))|\,ds\right)\left|\frac{\partial^2 A}{\partial x \partial y}\right|^{\frac{1}{2}} [1+\varepsilon K]\Bigg]\,dy,\end{aligned}$$

where K, recall, is a very complicated expression involving the Feynman-Green function.

Therefore, up to order ε, we have assuming $\frac{\partial^2 A}{\partial x \partial y} \neq 0$

$$
\begin{aligned}
&\nu_0^+[s,\infty) \\
&= (2\pi\varepsilon)^{-\frac{1}{2}} \int_0^\infty dy \, \exp\left(\frac{1}{\varepsilon}\int_0^y b(u)du\right) \\
&\times \varepsilon \left\{ \frac{1}{\varepsilon}\left[\left(-\frac{\partial A}{\partial x}\right)_{x=0} \exp\left(-\frac{A(0,y,t)}{\varepsilon}\right) \exp\left(\frac{1}{2}\int_0^t |b'(X_{min}(s))|\, ds\right) \left|\frac{\partial^2 A}{\partial x \partial y}\right|^{\frac{1}{2}}_{x=0} \right] \right. \\
&+ \varepsilon^0 \left[\frac{1}{2}\left|\frac{\partial^2 A}{\partial x \partial y}\right|^{-\frac{1}{2}}_{x=0} \frac{\partial}{\partial x}\bigg|_{x=0} \left(\frac{\partial^2 A}{\partial x \partial y}\right) \exp\left(-\frac{A(0,y,t)}{\varepsilon}\right) \exp\left(\frac{1}{2}\int_0^t |b'(X_{min}(s))|\right) \, ds \right. \\
&- \frac{1}{2}\frac{\partial}{\partial x}\bigg|_{x=0} \int_0^t b'(X_{min}(s))\, ds \left|\frac{\partial^2 A}{\partial x \partial y}\right|^{\frac{1}{2}}_{x=0} \exp\left(-\frac{A(0,y,t)}{\varepsilon}\right) \exp\left(\frac{1}{2}\int_0^t |b'(X_{min}(s))|\, ds\right) \\
&\left.\left. + \left|\frac{\partial^2 A}{\partial x \partial y}\right|^{\frac{1}{2}}_{x=0} \exp\left(-\frac{A(0,y,t)}{\varepsilon}\right) \exp\left(\frac{1}{2}\int_0^t |b'(X_{min}(s))|\, ds\right) \left(-\frac{\partial A}{\partial x}\right)_{x=0} K \right] \right\} \\
&+ O(\varepsilon^2). \qquad (5.1)
\end{aligned}
$$

We now use the result Olver [1974].

Proposition 7.

$$
\begin{aligned}
&\int_0^\infty \exp\left(-\frac{f(y)}{\varepsilon}\right) g(y)\, dy \\
&= \int_0^\infty \exp\left(-\frac{f(y)}{\varepsilon}\right) [g_0(y) + \varepsilon\, g_1(y) + \frac{\varepsilon^2}{2!}\, g_2(y) + \cdots]\, dy \qquad (5.2) \\
&\sim \quad \Gamma(1)\, \varepsilon \, \frac{g_0'}{2}\, \frac{1}{\frac{f''}{2}} + \Gamma(\frac{3}{2})\, \varepsilon^{\frac{3}{2}} \left(\frac{g_0''}{4} - 3\frac{\frac{f'''}{6}}{4\frac{f''}{2}} g_0'\right) \frac{1}{\left(\frac{f''}{2}\right)^{\frac{3}{2}}} + \Gamma(\frac{1}{2})\, \varepsilon^{\frac{3}{2}}\, \frac{g_1}{2}\, \frac{1}{\left(\frac{f''}{2}\right)^{\frac{1}{2}}} + \cdots . \qquad (5.3)
\end{aligned}
$$

Proof. For a proof of this standard result on asymptotic approximations see Olver [1974] . ☐

Comparing equation 5.2 with our expression for $\nu_0^+[s,\infty)$ to second order in equation 5.1, we get

$$
g_0(y) = (2\pi\varepsilon)^{-\frac{1}{2}} \left\{ \left|\frac{\partial^2 A(x,y,t)}{\partial x \partial y}\right|^{\frac{1}{2}} \exp\left(\frac{1}{2}\int_0^t |b'(X(s))|\, ds\right) \left(-\frac{\partial A(x,y,t)}{\partial x}\right) \right\}_{x=0},
$$

and

$$
\begin{aligned}
g_1(y) &= (2\pi\varepsilon)^{-\frac{1}{2}} \left|\frac{\partial^2 A(0,y,t)}{\partial x \partial y}\right|^{\frac{1}{2}} \exp\left(\frac{1}{2}\int_0^t |b'(X(s))| ds\right) \\
&\times \left\{ \frac{1}{2}\frac{\partial}{\partial x}\int_0^t |b'(X(s))|\, ds + \frac{1}{2}\left|\frac{\partial^2 A(x,y,t)}{\partial x \partial y}\right|^{-1} \frac{\partial}{\partial x}\left(\frac{\partial^2 A(x,y,t)}{\partial x \partial y}\right) - K\frac{\partial A(x,y,t)}{\partial x} \right\}_{x=0}.
\end{aligned}
$$

Since we know for $x = 0$

$$\frac{\partial A}{\partial x}\bigg|_{x=0} = -p_0(y), \text{ so } -\frac{\partial A}{\partial x}\bigg|_{x=0} = p_0(y) \ (>0), \text{ and } \frac{\partial^2 A}{\partial x \partial y}\bigg|_{x=0} = -\frac{\partial p_0(y)}{\partial y},$$

giving

$$\left|\frac{\partial^2 A}{\partial x \partial y}\right| = \frac{\partial p_0(y)}{\partial y} \quad (>0), \qquad \frac{\partial}{\partial x}\left|\frac{\partial^2 A}{\partial x \partial y}\right| = \frac{\partial^2 p_0(y)}{\partial x \partial y},$$

we can write the expressions for $g_0(y)$ and $g_1(y)$ as

$$g_0(y) = (2\pi\varepsilon)^{-\frac{1}{2}} \left\{ \left(\frac{\partial p_0(y)}{\partial y}\right)^{\frac{1}{2}} \exp\left(\frac{1}{2}\int_0^t |b'(X(s))|\, ds\right) (p_0(y)) \right\}_{x=0},$$

and

$$\begin{aligned} g_1(y) \;&=\; (2\pi\varepsilon)^{-\frac{1}{2}} \left(\frac{\partial p_0(y)}{\partial y}\right)^{\frac{1}{2}} \exp\left(\frac{1}{2}\int_0^t |b'(X(s))| ds\right) \\ &\times \left\{ \frac{1}{2}\frac{\partial}{\partial x}\int_0^t |b'(X(s))|\, ds + \frac{1}{2}\left(\frac{\partial p_0(y)}{\partial y}\right)^{-1} \left(\frac{\partial^2 p_0(y)}{\partial x \partial y}\right) - K p_0(y) \right\}_{x=0}. \end{aligned}$$

If we now expand each term in the expression for $g_0(y)$ in a Taylor series we get

$$\begin{aligned} g_0(y) = (2\pi\varepsilon)^{-\frac{1}{2}} \bigg\{ & \left(p_0'(0) + y p_0''(0) + \cdots\right)^{\frac{1}{2}} \exp\bigg(\frac{1}{2}\int_0^t |b'(X(s))|\, ds\bigg|_{y=0} \\ & + \frac{1}{2}\, y\, \frac{\partial}{\partial y}\int_0^t |b'(X(s))|\, ds\bigg|_{y=0} + \cdots \bigg) (p_0(0) + y p_0'(0) + \cdots) \bigg\}. \end{aligned}$$

Therefore, we can now see that in order to obtain the first order expressions for $g_0(y)$ and $g_1(y)$, the following terms need to be evaluated :

$$\left(\frac{\partial p_0(y)}{\partial y}\right)^{-1} \qquad \frac{\partial^2 p_0(y)}{\partial y^2} \qquad \frac{\partial^2 p_0(y)}{\partial x \partial y}$$

$$\int_0^t |b'(X(s))|\, ds \qquad \frac{\partial}{\partial x}\int_0^t |b'(X(s))|\, ds \qquad \frac{\partial}{\partial y}\int_0^t |b'(X(s))|\, ds.$$

Let us be very thankful that an evaluation of K is not needed in this rather complicated computation.

Each of these terms is evaluated in the following Propositions. Recall that we have already seen in equation 4.3

$$|p_0'(0)| = \frac{\partial p_0(y)}{\partial y} = \frac{|b'(0)|}{\sinh |b'(0)| t}.$$

□

Proposition 8. *For* $p_0(0) = b(0) = 0$ *, as* $y \to 0$*,*

$$\frac{\partial^2 p_0(y)}{\partial y^2} = \frac{b''(0)\left(\cosh|b'(0)|t - 1\right)^2}{\sinh^3|b'(0)|t}. \tag{5.4}$$

Proof. In order to calculate $\frac{\partial^2 p_0(y)}{\partial y^2}$ we return to the identity,

$$t = \int_0^y \frac{du}{\left(p_0^2(y) + b^2(u)\right)^{\frac{1}{2}}}.$$

Differentiating the equation above w.r.t. y, and then using the change of variable $u = yv$, gives dropping some inessential modulus signs for ease of presentation

$$\begin{aligned} \frac{1}{p_0'(y)} &= p_0(y)\left(p_0^2(y) + b^2(y)\right)^{\frac{1}{2}} \int_0^y \frac{du}{\left(p_0^2(y) + b^2(u)\right)^{\frac{3}{2}}} \\ &= \frac{p_0(y)}{y}\left(\left(\frac{p_0(y)}{y}\right)^2 + \left(\frac{b(y)}{y}\right)^2\right)^{\frac{1}{2}} \int_0^1 \frac{dv}{\left(\left(\frac{p_0(y)}{y}\right)^2 + \left(\frac{b(yv)}{yv}\right)^2 v^2\right)^{\frac{3}{2}}} \\ &\to p_0'(0)\ \left(p_0'(0)^2 + b'(0)^2\right)^{\frac{1}{2}} \int_0^1 \frac{dv}{\left(p_0'(0)^2 + b'(0)^2\, v^2\right)^{\frac{3}{2}}} \quad \text{as} \quad y \to 0, \end{aligned} \tag{5.5}$$

using the same argument as seen in Proposition 4

Expanding the r.h.s. of equation 5.5 in a Taylor Series, using for simplicity the notation $p = p_0(0)$ and $b = b(0)$, gives

$$\begin{aligned} &\left(p' + \frac{y}{2}p'' + \cdots\right)\left(\left(p' + \frac{y}{2}p'' + \cdots\right)^2 + \left(b' + \frac{y}{2}b'' + \cdots\right)^2\right)^{\frac{1}{2}} \\ &\qquad \times \int_0^1 \frac{dv}{\left(\left(p' + \frac{y}{2}p'' + \cdots\right)^2 + \left(b' + \frac{yv}{2}b'' + \cdots\right)^2 v^2\right)^{\frac{3}{2}}} \\ &\to \left(p' + \frac{y}{2}p'' + \cdots\right)\left(p'^2 + b'^2\right)^{\frac{1}{2}}\left(1 + y\frac{p'p'' + b'b''}{p'^2 + b'^2} + \cdots\right)^{\frac{1}{2}} \\ &\qquad \times \int_0^1 dv\ \left(p'^2 + b'^2 v^2\right)^{-\frac{3}{2}}\left(1 + y\frac{p'p'' + b'b''v^3}{p'^2 + b'^2 v^2} + \cdots\right)^{-\frac{3}{2}} \\ &= \left(p' + \frac{y}{2}p'' + \cdots\right)\left(\left(p'^2 + b'^2\right)^{\frac{1}{2}} + \frac{y}{2}\left(\frac{p'p'' + b'b''}{\left(p'^2 + b'^2\right)^{\frac{1}{2}}}\right) + \cdots\right) \\ &\qquad \times \int_0^1 dv\ \left(\left(p'^2 + b'^2 v^2\right)^{-\frac{3}{2}} - \frac{3}{2}y\left(\frac{p'p'' + b'b''v^3}{\left(p'^2 + b'^2 v^2\right)^{\frac{5}{2}}}\right) + \cdots\right) \\ &= \text{zero order term} + f.y + \text{higher order terms}(y^2 \cdots). \end{aligned}$$

The zero order term is

$$p'\,(p'^2 + b'^2)^{\frac{1}{2}} \int_0^1 dv\, \left(p'^2 + b'^2\, v^2\right)^{-\frac{3}{2}}.$$

Now the integral term in the equation above can be written as

$$\frac{1}{b'^3} \int_0^1 \frac{dv}{\left(\frac{p'^2}{b'^2} + v^2\right)^{\frac{3}{2}}}.$$

Using the change of variable $v = \left|\frac{p'}{b'}\right| \sinh w$ in the equation above gives

$$\begin{aligned}
&\frac{1}{b'^3} \int_0^{\sinh^{-1}\left|\frac{b'}{p'}\right|} dw\, \frac{\left|\frac{p'}{b'}\right| \cosh w}{\left(\frac{p'^2}{b'^2} + \frac{p'^2}{b'^2}\sinh^2 w\right)^{\frac{3}{2}}} \\
&= \frac{1}{b'\,p'^2} \int_0^{\sinh^{-1}\left|\frac{b'}{p'}\right|} dw\, \frac{1}{\cosh^2 w} = \frac{1}{|p'|^3} \cdot \frac{1}{\cosh|b'(0)|t.}
\end{aligned}$$

Therefore, the zero order term is (because of equation 4.3)

$$p'\,(p'^2 + b'^2)^{\frac{1}{2}} \cdot \frac{1}{p'^3} \cdot \frac{1}{\cosh|b'(0)|\,t} = \frac{1}{|p'|}.$$

The coefficient of y is given by

$$\begin{aligned}
f = {} & \frac{p''}{2}\left(p'^2 + b'^2\right)^{\frac{1}{2}} \int_0^1 dv\, \left(p'^2 + b'^2 v^2\right)^{-\frac{3}{2}} + \frac{p'}{2}\frac{p'p'' + b'b''}{\left(p'^2 + b'^2\right)^{\frac{1}{2}}} \int_0^1 dv\, \left(p'^2 + b'^2 v^2\right)^{-\frac{3}{2}} \\
& - \frac{3}{2}\,p'\left(p'^2 + b'^2\right)^{\frac{1}{2}} \int_0^1 dv\, \frac{p'p'' + b'b''v^3}{(p'^2 + b'^2 v^2)^{\frac{5}{2}}}.
\end{aligned}$$

Therefore, we have that

$$\frac{\partial p}{\partial y} = \frac{1}{p'^{-1} + yf + \cdots} = \frac{p'}{1 + yp'f + \cdots} = p'(1 - yp'f - \cdots). \tag{5.6}$$

Now, by letting $|p'| = -\frac{|b'|}{a}$ giving $a = -\sinh|b'(0)|t,$ we can write

$$\begin{aligned}
p'^2 f \;=\; & \frac{p''}{2} + \frac{p'^3}{2}\,\frac{p'p'' + b'b''}{p'(1 + a^2)^{\frac{1}{2}}}\, p'^{-3}\,(1 + a^2)^{-\frac{1}{2}} \\
- \; & \frac{3}{2}\,p'^3\,p'\,(1 + a^2)^{\frac{1}{2}} \left\{ \frac{p'p''}{p'^5}\left(\frac{2}{3(1 + a^2)} + \frac{1}{3(1 + a^2)^{\frac{3}{2}}}\right)\right. \\
+ \; & \left.\frac{b'b''}{p'^5}\left(\frac{2}{3a^2} - \frac{2 + 3a^2}{3a^4(1 + a^2)^{\frac{3}{2}}}\right)\right\},
\end{aligned}$$

which simplifies to

$$
\begin{aligned}
p'^2 f &= \frac{p''}{2} + \frac{1}{2}\frac{p'' - ab''}{(1+a^2)} - \frac{3}{2}p''\left(\frac{2}{3} + \frac{1}{3(1+a^2)}\right) \\
&- \frac{3}{2}(-a)\,b''\left(\frac{2(1+a^2)^{\frac{1}{2}}}{3a^4} - \frac{2+3a^2}{3a^4(1+a^2)}\right).
\end{aligned}
$$

Hence, from equation 5.6

$$
-p'' = \frac{p''}{2} + \frac{1}{2}\frac{p'' - ab''}{(1+a^2)} - p'' - \frac{p''}{2(1+a^2)} + \frac{3ab''}{2}\left(\frac{2(1+a^2)^{\frac{1}{2}}}{3a^4} - \frac{2+3a^2}{3a^4(1+a^2)}\right),
$$

giving

$$
0 = \frac{p''}{2} - \frac{ab''}{2(1+a^2)} + \frac{b''(1+a^2)^{\frac{1}{2}}}{a^3} - \frac{1+\frac{3}{2}a^2}{a^3(1+a^2)}b''.
$$

Now since $a = -\frac{|b'|}{|p'|}$ and $|p'| = \frac{|b'(0)|}{\sinh|b'(0)|t}$, and again using the obvious notation

$$
s = \sinh|b'(0)|\,t \quad \text{and} \quad c = \cosh|b'(0)|\,t,
$$

we can write the equation above as

$$
\begin{aligned}
0 &= p'' + b''\left(-\frac{s}{c^2} + 2\frac{c}{s^3} - \frac{2+3s^2}{s^3c^2}\right) \\
&= p'' + \frac{b''}{s^3}\left(-\frac{s^4}{c^2} + 2c - \frac{2+3s^2}{c^2}\right) \\
&= p'' + \frac{b''}{s^3}\left(-\frac{(c^4 - 2c^2 + 1)}{c^2} + \frac{2c^3}{c^2} - \frac{2+3(c^2-1)}{c^2}\right) \\
&= p'' + \frac{b''}{s^3}\left(\frac{-c^4 + 2c^3 - c^2}{c^2}\right).
\end{aligned}
$$

Hence, we get the result

$$
p_0''(0) = \frac{b''(0)\left(\cosh|b'(0)|t - 1\right)^2}{\sinh^3|b'(0)|t}.
$$

□

Proposition 9. *As $y \to 0$,*

$$
\frac{\partial^2 p_0}{\partial x \partial y} \to \frac{b''(0)\left(\cosh|b'(0)|t - 1\right)^2}{\sinh^3|b'(0)|t} = p_0''(0).
$$

Proof. We begin with

$$
t = \int_x^y \frac{du}{(p^2(x,y) + b^2(u) - b^2(x))^{\frac{1}{2}}}.
$$

Differentiating both sides w.r.t. x gives,

$$
0 = -\frac{1}{|p(x,y)|} - \int_x^y du\,\frac{p\,\frac{\partial p}{\partial x} - b(x)\,\frac{\partial b(x)}{\partial x}}{(p^2(x,y) + b^2(u) - b^2(x))^{\frac{3}{2}}}.
$$

Therefore, as $x \to 0$

$$0 \;=\; \frac{1}{|p_0(y)|} \;+\; p_0(y)\frac{\partial p}{\partial x}\bigg|_{x=0} \int_0^y \frac{du}{\left(p_0^2(y)+b^2(u)\right)^{\frac{3}{2}}}.$$

Hence,

$$\frac{\partial p}{\partial x}\bigg|_{x=0} \;=\; \frac{-1}{p_0^2(y)\int_0^y \left(p_0^2(y)+b^2(u)\right)^{-3/2} du}. \tag{5.7}$$

If we consider the quotient term on the r.h.s. of equation 5.7, with a change of variable $u = yv$ and again letting $y \to 0$, we get

$$r.h.s. \;=\; \frac{-1}{p_0'^2(0)\int_0^y \left(p_0'^2(0)+b'^2(0)v^2\right)^{-3/2} dv}.$$

Expanding the denominator of equation 5.7 in a Taylor series [$p(0) = 0$, $p'(0) \neq 0$], using the same notation as in the previous Proposition

$$\begin{aligned}
&(p'+\frac{y}{2}\,p''+\cdots)^2\int_0^1 \frac{dv}{\left((p'+\frac{y}{2}\,p''+\cdots)^2+(b'+\frac{yv}{2}\,b''+\cdots)^2\,v^2\right)^{\frac{3}{2}}}\\
&=(p'^2+y\,p'\,p''+\cdots)\int_0^1 \frac{dv}{\left(p'^2+b'^2\,v^2+y(p'\,p''+v^3\,b'\,b'')+\cdots\right)^{\frac{3}{2}}}\\
&=(p'^2+y\,p'\,p''+\cdots)\int_0^1 \frac{1}{\left(p'^2+b'^2\,v^2\right)^{\frac{3}{2}}}\left\{1-\frac{3}{2}\,y\,\frac{p'\,p''+v^3\,b'\,b''}{(p'+b'^2\,v^2)}-\cdots\right\}dv.
\end{aligned}$$

Therefore,

$$\left(\frac{\partial p}{\partial x}\right)^{-1}_{x=0} = \text{zero order term} \;+\; f y+\cdots\,,$$

where f is the coefficient of the y term as shown below

$$\begin{aligned}
\left(\frac{\partial p}{\partial x}\right)^{-1}_{x=0} \;&=\; p'^2\int_0^1 \frac{dv}{(p'^2+b'^2\,v^2)^{\frac{3}{2}}}+y\left(p'\,p''\int_0^1 \frac{dv}{(p'^2+b'^2\,v^2)^{\frac{3}{2}}}\right.\\
&\left.-\;\frac{3}{2}\,p'^2\int_0^1 \frac{p'\,p''+b'\,b''\,v^3}{(p'^2+b'^2\,v^2)^{\frac{5}{2}}}\,dv\right)+\cdots
\end{aligned}$$

Inverting this equation gives

$$\begin{aligned}
\frac{\partial p}{\partial x}\bigg|_{x=0} = &\left(p'^2\int_0^1 \frac{dv}{(p'^2+b'^2\,v^2)^{\frac{3}{2}}}\right)^{-1}\left\{1-y\left(\vphantom{\frac{1}{1}}\right.\right.\\
&\left.\left.\frac{p'\,p''\int_0^1 \frac{dv}{(p'^2+b'^2\,v^2)^{\frac{3}{2}}}-\frac{3}{2}p'^2\int_0^1 \frac{p'\,p''+b'\,b''\,v^3}{(p'^2+b'^2\,v^2)^{\frac{5}{2}}}\,dv}{p'^2\int_0^1 \frac{dv}{(p'^2+b'^2\,v^2)^{\frac{3}{2}}}}\right)-\cdots\right\}.
\end{aligned}$$

Now since

$$\left.\frac{\partial p(x,y)}{\partial x}\right|_{x=0} = \left.\frac{\partial p(x,0)}{\partial x}\right|_{x=0} + y \left.\frac{\partial^2 p(x,0)}{\partial x \partial y}\right|_{x=0} + \cdots .$$

Comparing y-terms yields

$$\left.\frac{\partial^2 p(x,0)}{\partial x \partial y}\right|_{x=0} = \frac{p'\,p'' \int_0^1 \frac{dv}{\left(p'^2+b'^2\,v^2\right)^{\frac{3}{2}}} - \frac{3}{2} p'^2 \int_0^1 \frac{p'\,p''+b'\,b''\,v^3}{\left(p'^2+b'^2\,v^2\right)^{\frac{5}{2}}}\,dv}{\left(p'^2 \int_0^1 \frac{dv}{\left(p'^2+b'^2\,v^2\right)^{\frac{3}{2}}}\right)^2}.$$

Letting $a = -\left|\frac{b'}{p'}\right|$,

$$\begin{aligned}\left.\frac{\partial^2 p}{\partial x \partial y}\right|_{x=0} &= \frac{\frac{p''}{p'^2} \int_0^1 \frac{dv}{\left(1+a^2\,v^2\right)^{\frac{3}{2}}} - \frac{3}{2}\frac{1}{p'^3} \int_0^1 \frac{p'\,p''+b'\,b''\,v^3}{\left(1+a^2\,v^2\right)^{\frac{5}{2}}}\,dv}{\left(\frac{1}{p'} \int_0^1 \frac{dv}{\left(1+a^2 v^2\right)^{\frac{3}{2}}}\right)^2} \\ &= -\frac{1}{2}\,\frac{p''}{p'}\,\frac{1}{(1+a^2)} - \frac{1}{2}\,\frac{b'\,b''}{p'^2}\,\frac{(1+a^2)^{\frac{1}{2}}}{a^2} + \frac{1}{2}\,\frac{b'\,b''}{p'^2}\,\frac{1}{a^2(1+a^2)}.\end{aligned}$$

Now, substituting for p' and p'' gives the result. ∎

Remark. A by product of the above is

$$\frac{\partial p_0(y)}{\partial x} \to -p_0' \cosh|b'(0)|\,t.$$

Proposition 10. *As $y \to 0$,*

$$\int_0^t |b'(X_{min}(s))|\,ds \to t|b'(0)|$$

with X_{min} satisfying

$$\ddot{X}_{min}(s) = b(X_{min}(s))b'(X_{min}(s)) \quad \text{and} \quad X_{min}(0) = x, X_{min}(t) = y.$$

Proof. For $u = X_{min}(s)$, $du = \dot{X}_{min}(s)\,ds$, and considering

$$\begin{aligned}\int_0^t |b'(X_{min}(s))|\,ds &= s\,|b'(X_{min}(s))|\Big|_0^t + \int_0^t s\,b''(X_{min}(s))\,\dot{X}_{min}(s)\,ds \\ &= t\,|b'(y)| + \int_x^y s(u)\,b''(u)\,du \\ &= t\,|b'(y)| + \int_x^y du \int_x^u dv \frac{b''(u)}{(p^2(x,y)+b^2(v)-b^2(x))^{\frac{1}{2}}}\end{aligned} \tag{5.8}$$

since,

$$t(u) = \int_x^y \frac{du}{(p^2(x,y)+b^2(u)-b^2(x))^{\frac{1}{2}}}.$$

Therefore, the integral term on the r.h.s. of equation 5.8 $\to 0$ as $y \to 0$, and so the result follows (recall $y > x > 0$). ∎

Proposition 11.

$$\left.\frac{\partial}{\partial x}\right|_{x=0} \int_0^t |b'(X_{min}(s))|\, ds \quad \to \quad -\, b''(0)\, p_0'(0) \left.\frac{\partial p}{\partial x}\right|_{x=0} \int_0^1 du \int_u^1 \frac{dv}{(p_0'^2(0) + b'^2(0) v^2)^{\frac{3}{2}}}$$
$$= \quad \frac{b''(0)}{|b'(0)|} \frac{\cosh |b'(0)|t - 1}{\sinh |b'(0)|t}, \quad \textit{as} \quad y \to 0. \tag{5.9}$$

Proof. Using equations 5.7 and 5.8, a calculation along the lines of the proof of Theorem 4, using integration by parts, yields

$$\left.\frac{\partial}{\partial x}\right|_{x=0} \int_0^t |b'(X_{min}(s))|\, ds \;=\; \int_0^y du \left\{ p_0(y) \left.\frac{\partial p}{\partial x}\right|_{x=0} \int_0^u \frac{dv}{\left(p_0^2(y) + b^2(v)\right)^{\frac{3}{2}}} \right\} b''(u)$$
$$\to -b''(0)\, p_0'(0) \left.\frac{\partial p}{\partial x}\right|_{y=x=0} \int_0^1 du \int_u^1 \frac{dv}{\left(p_0'^2(0) + b'^2(0)\, v^2\right)^{\frac{3}{2}}} \quad \text{as} \quad y \to 0$$

and result follows. □

Proposition 12.

$$\frac{\partial}{\partial y} \int_0^t |b'(X(s))|\, ds \quad \to \quad b''(0)(p_0'(0))^2 \int_0^1 du \int_0^u \frac{dv}{(p_0'^2(0) + b'^2(0) v^2)^{\frac{3}{2}}}$$
$$= \quad \frac{b''(0)}{|b'(0)|} \frac{\cosh |b'(0)|t - 1}{\sinh |b'(0)|t}, \quad \textit{as} \quad y \to 0. \tag{5.10}$$

Proof. The proof of this is similar to that of Proposition 11. □

Proof - Second part. Therefore, by substituting equations 4.3, 5.4, 5.7, 5.9, 5.10 and Proposition 9 into the expressions obtained for $g_0(y)$ and $g_1(y)$, and using Proposition 7, we can complete the proof of Theorem 3.

$$g_0(y) \sim (2\pi\varepsilon)^{-\frac{1}{2}} \left(p' + p'' y + \cdots\right)^{\frac{1}{2}} \exp\left(\frac{1}{2}|b'(X(0))|t\right.$$
$$\left. + \frac{1}{2}\, y \frac{b''(0)}{|b'(0)|} \frac{\cosh |b'(0)|t - 1}{\sinh |b'(0)|t} + \cdots \right) \left(p(0) + y p'(0) + \frac{y^2}{2} p''(0) + \cdots \right), \quad \text{as} \quad y \to 0$$
$$= (2\pi\varepsilon)^{-\frac{1}{2}} (p_0')^{\frac{3}{2}} \exp\left(\frac{t|b'(0)|}{2}\right) \left(1 + \frac{p''}{p'} y + \cdots \right)^{\frac{1}{2}}$$
$$\times \left(y + \frac{p''}{p'} \frac{y^2}{2} - \frac{b''(0)}{2|b'(0)|} \frac{\cosh |b'(0)|t - 1}{\sinh |b'(0)|t} y^2 - \frac{b''(0) p''}{4|b'(0)| p'} \frac{\cosh |b'(0)|t - 1}{\sinh |b'(0)|t} y^3 + \cdots \right)$$
$$= (2\pi\varepsilon)^{-\frac{1}{2}} (p')^{\frac{3}{2}} \exp\left(\frac{t|b'(0)|}{2}\right) \left[y + \frac{1}{2} y^2 \left(2\frac{p''}{p'} - \frac{b''(0)}{|b'(0)|} \frac{\cosh |b'(0)|t - 1}{\sinh |b'(0)|t} \right) + \cdots \right.$$

Similarly,

$$g_1(y) \quad \sim \quad (2\pi\varepsilon)^{-\frac{1}{2}} (p_0')^{\frac{1}{2}} \exp\left(\frac{t|b'(0)|}{2}\right)$$
$$\times \quad \left(-\frac{1}{2} \frac{b''(0)}{|b'(0)|} \frac{\cosh |b'(0)|t - 1}{\sinh |b'(0)|t} + \frac{1}{2} \frac{b''(0)}{|b'(0)|} \frac{(\cosh |b'(0)|t - 1)^2}{(\sinh |b'(0)|t)^2} + y(\cdots) + \cdots \right).$$

Therefore,

$$g_0'(0) = (2\pi\varepsilon)^{-\frac{1}{2}} (p')^{\frac{3}{2}} \exp\left(\frac{t|b'(0)|}{2}\right),$$

and

$$g_0''(0) = (2\pi\varepsilon)^{-\frac{1}{2}} (p')^{\frac{3}{2}} \exp\left(\frac{t|b'(0)|}{2}\right)\left(2\frac{p''}{p'} - \frac{b''(0)}{|b'(0)|}\frac{\cosh|b'(0)|t - 1}{\sinh|b'(0)|t}\right).$$

Now, in our case, the expression for $f(y)$ in Proposition 7 is

$$f(y) = A(0,y,t) - \int_0^y b(u)\,du,$$

so

$$f(0) = A(0,0,t) - \int_0^0 b(u)\,du = 0.$$

Also, since $p(t) = \sqrt{p_0^2(y) + b^2(y)}$,

$$f'(y) = \frac{\partial A(0,y,t)}{\partial y} - \frac{\partial}{\partial y}\int_0^y b(u)du = p_0(t) - b(y) = \sqrt{p_0^2(y) + b^2(y)} - b(y),$$

so

$$f'(0) = \sqrt{p_0^2(0) + b^2(0)} - b(0) = 0,$$

since $p_0(0) = 0$, (the momentum required to go from $x = 0$ to $y = 0$).

Now, $f''(y) = \frac{\partial}{\partial y}p(0,y,t) - b'(y)$, so differentiating $p(t) = \sqrt{p_0^2(y) + b^2(y)}$ twice with respect to y, and using the fact that $b(0) = 0$ and $p(t) \to 0$ as $y \to 0$, we get

$$\left.\left\{\left(\frac{\partial p(t)}{\partial y}\right)^2 + p(t)\,\frac{\partial^2 p(t)}{\partial y^2}\right\}\right|_{x=0} = p_0'(y)^2 + p_0(y)\,p_0''(y) + b'(y)^2 + b(y)\,b''(y). \tag{5.11}$$

As $y \to 0$,

$$\left(\frac{\partial p_0(t)}{\partial y}\right)^2 \to p_0'(0)^2 + b'(0)^2,$$

since $p_0(0) = b(0) = 0$, and $p_0(t)\,\frac{\partial^2 p_0(t)}{\partial y^2} \to 0$, as $y \to 0$. Hence,

$$\begin{aligned}
\left.\left(\frac{\partial p_0(t)}{\partial y}\right)^2\right|_{x=0} &= p_0'(0)^2 + b'(0)^2 \\
&= \frac{(b'(0))^2}{(\sinh|b'(0)|t)^2} + b'(0)^2 \\
&= (b'(0))^2\left(\frac{1 + \sinh^2|b'(0)|t}{\sinh^2|b'(0)|t}\right) \\
&= |b'(0)|^2\coth^2|b'(0)|t.
\end{aligned}$$

Therefore,

$$f''(y) \to b'(0)\,\coth|b'(0)|t - b'(0) = |b'(0)|\,(\coth|b'(0)|t + 1),$$

since we are dealing with $b(x) < 0$.

Similarly,

$$f'''(y) = \frac{\partial^2 p}{\partial y^2}(0, y, t) - b''(y).$$

Differentiating equation 5.11 again w.r.t. y gives,

$$\begin{aligned} \left\{ \frac{\partial^3 p(t)}{\partial y^3}\, p(t) + 3\, \frac{\partial p(t)}{\partial y}\, \frac{\partial^2 p(t)}{\partial y^2} \right\} \Bigg|_{x=0} \quad &= \quad 3\, p_0'(y)\, p_0''(y) \;+\; p_0(y)\, p'''(y) \\ &+ \quad 3\, b'(y)\, b'''(y) \;+ b(y)\, b'''(y). \end{aligned}$$

And again, since $p_0(0) = b(0) = 0$, and $p_0(t) \to 0$ as $y \to 0$, we get

$$\left\{ \frac{\partial p_0(t)}{\partial y}\, \frac{\partial^2 p_0(t)}{\partial y^2} \right\} \Bigg|_{y=0} = p_0'(0)\, p_0''(0) + b'(0)\, b''(0),$$

giving, after a little calculation

$$\left\{ \frac{\partial^2 p_0(t)}{\partial y^2} \right\} \Bigg|_{y=0} \quad = \quad \frac{p_0'(0)\, p_0''(0) + |b'(0)|\, b''(0)}{|b'(0)|\; \coth |b'(0)|\, t}.$$

Therefore, as $y \to 0$,

$$\begin{aligned} f'''(y) \quad &\to \quad \frac{p_0' p_0'' + |b'(0)| b''(0)}{|b'(0)| \coth |b'(0)| t} - b''(0) \\ &= \quad b''(0) \left(\frac{(\cosh |b'(0)| t - 1)^2}{\cosh |b'(0)| t (\sinh |b'(0)| t)^3} - \frac{\sinh |b'(0)| t}{\cosh |b'(0)| t} - 1 \right). \end{aligned}$$

Finally, we conclude the proof of Theorem 3 by substituting into equation 5.3 to get

$$\begin{aligned} \nu_0^+[t, \infty) = \int_0^\infty dy\, \exp\left(\frac{1}{\varepsilon} \int_0^y b(u)\, du - A(0, y, t) \right) (g_0(y) + \varepsilon g_1(y) + \cdots)\, dy \\ \sim \quad & \frac{\varepsilon}{2}\, (2\pi\varepsilon)^{-\frac{1}{2}}\, (p')^{\frac{3}{2}}\, \exp\left(\frac{t\, |b'(0)|}{2} \right) \frac{1}{\frac{f''}{2}} \\ + \quad & \sqrt{\frac{\pi}{2 f''}}\, \varepsilon^{\frac{3}{2}} \left\{ \frac{(2\pi\varepsilon)^{-\frac{1}{2}}}{4}\, (p')_0^{\frac{3}{2}}\, \exp\left(\frac{t\, |b'(0)|}{2} \right) \left(\frac{2 p_0''}{p_0'} - \frac{b''(0)}{|b'(0)|} \frac{\cosh |b'(0)| t - 1}{\sinh |b'(0)| t} \right) \right. \\ - \quad & \left. \frac{3}{4}\, \frac{\frac{f'''}{6}}{\frac{f''}{2}}\, (2\pi\varepsilon)^{-\frac{1}{2}}\, (p')_0^{\frac{3}{2}}\, \exp\left(\frac{t\, |b'(0)|}{2} \right) \right\} \frac{1}{\frac{f''}{2}} \\ + \quad & \sqrt{\frac{\pi}{2 f''}}\, \varepsilon^{\frac{3}{2}}\, (2\pi\varepsilon)^{-\frac{1}{2}}\, (p')_0^{\frac{1}{2}}\, \exp\left(-\frac{t\, |b'(0)|}{2} \right) \\ \times \quad & \left(-\frac{1}{2} \frac{b''(0)}{|b'(0)|} \frac{\cosh |b'(0)| t - 1}{\sinh |b'(0)| t} + \frac{1}{2} \frac{b''(0)}{|b'(0)|} \frac{(\cosh |b'(0)| t - 1)^2}{(\sinh |b'(0)| t)^2} \right). \end{aligned}$$

Substituting for p' and p'' eventually gives, using an obvious shorthand notation

$$\begin{aligned}
\nu_0^+[t,\infty) &\sim \varepsilon(2\pi\varepsilon)^{-\frac{1}{2}}\,(p')^{\frac{3}{2}}\,\exp\left(\frac{t|b'(0)|}{2}\right)\frac{1}{f''}\\
&+ \varepsilon^{\frac{3}{2}}(2\pi\varepsilon)^{-\frac{1}{2}}\,(p')^{\frac{3}{2}}\,\exp\left(\frac{t|b'(0)|}{2}\right)\left(\frac{\pi}{2f''}\right)^{\frac{1}{2}}\left\{\left[\frac{b''(0)(c-1)^2}{2p's^3}-\frac{b''(0)(c-1)}{4|b'(0)|s}\right.\right.\\
&- \left.\left.\frac{b''}{4}\left(\frac{(c-1)^2-s^4-cs^3}{b'cs^2(c+s)}\right)\right]\frac{2s}{b'(c+s)}-\frac{b''(c-1)}{2p'b's}+\frac{b''(c-1)^2}{2p'b's^2}\right\}\\
&= \varepsilon\,(2\pi\varepsilon)^{-\frac{1}{2}}\,(p')^{\frac{3}{2}}\,\exp\left(\frac{t|b'(0)|}{2}\right)\frac{1}{f''}+\varepsilon^{\frac{3}{2}}\,(2\pi\varepsilon)^{-\frac{1}{2}}\,(p')^{\frac{1}{2}}\,\exp\left(\frac{t|b'(0)|}{2}\right)\\
&\times \sqrt{\frac{\pi}{2f''}}\left(-\frac{b''(0)}{2|b'(0)|}\right)\left[\frac{c-1}{s}-\frac{(c-1)^2}{s^2}+\frac{1}{1+\frac{c}{s}}\left(\frac{c-1}{s^2}-\frac{2(c-1)^2}{s^3}\right)\right.\\
&+ \left.\frac{1}{s\left(1+\frac{c}{s}\right)^2}\left(\frac{(c-1)^2}{cs^3}-\frac{s}{c}-1\right)\right]\\
&= \varepsilon\,(2\pi\varepsilon)^{-\frac{1}{2}}\,(p')^{\frac{3}{2}}\,\exp\left(\frac{t|b'(0)|}{2}\right)\frac{1}{f''}\\
&+ \frac{\varepsilon}{2}\,(p')^{\frac{1}{2}}\,\exp\left(\frac{t|b'(0)|}{2}\right)\frac{1}{(f'')^{\frac{1}{2}}}\left(-\frac{b''(0)}{2|b'(0)|}\right)\Big[\cdots\cdots\Big].
\end{aligned}$$

A tedious calculation shows that the terms within the $\Big[\cdots\Big]$ cancel leaving the result,

$$\begin{aligned}
\nu_0^+[t,\infty) &= \varepsilon\,(2\pi\varepsilon)^{-\frac{1}{2}}\,(p_0')^{\frac{3}{2}}\,\exp\left(\frac{t|b'(0)|}{2}\right)\frac{1}{f''}\;+\;O(\varepsilon^{\frac{3}{2}}),\\
&= \left(\frac{\varepsilon|b'(0)|}{\pi}\right)^{\frac{1}{2}}\left(e^{2|b'(0)|t}-1\right)^{-\frac{1}{2}}+\;O(\varepsilon^{\frac{3}{2}}).
\end{aligned}$$

□

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