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Tail estimates for the Brownian excursion area and other Brownian areas

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Abstract

Brownian areas are considered in this paper: the Brownian excursion area, the Brownian bridge area, the Brownian motion area, the Brownian meander area, the Brownian double meander area, the positive part of Brownian bridge area, the positive part of Brownian motion area. We are interested in the asymptotics of the right tail of their density function. Inverting a double Laplace transform, we can derive, in a mechanical way, all terms of an asymptotic expansion. We illustrate our technique with the computation of the first four terms. We also obtain asymptotics for the right tail of the distribution function and for the moments. Our main tool is the two-dimensional saddle point method.

Key words: Brownian areas, asymptotics for density functions right tail, double Laplace transform, two-dimensional saddle point method.

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1 Introduction

Let $B_{\mathrm{ex}}(t)$, $t \in [0,1]$, be a (normalized) Brownian excursion, and let $\mathcal{B}_{\mathrm{ex}} := \int_0^1 B_{\mathrm{ex}}(t)\,\mathrm{d}t$ be its area (integral). This random variable has been studied by several authors, including Louchard [14; 15], Takács [22], and Flajolet and Louchard [8]; see also the survey by Janson [10] with many further references. One reason for the interest in this variable is that it appears as a limit in many different problems, see [22] and, in particular, [8] for examples and applications.

It is known that $\mathcal{B}_{\mathrm{ex}}$ has a density function f_{ex}, which was given explicitly by Takács [22] as a convergent series involving the zeros a_j of the Airy function and the confluent hypergeometric function U:

$$f_{\mathrm{ex}}(x) = \frac{2\sqrt{6}}{x^2}\sum_{j=1}^{\infty} v_j^{2/3} e^{-v_j} U\Bigl(-\frac{5}{6},\frac{4}{3};v_j\Bigr) \quad \text{with} \quad v_j = 2|a_j|^3/27x^2. \tag{1.1}$$

(The existence and continuity of f_{ex} follows also from Theorem 3.1 below.) A related expansion for the distribution function $\mathbb{P}(\mathcal{B}_{\mathrm{ex}} \le x)$ was found by Darling [4]. (We must use the Vervaat construction [29] to relate his result to $\mathcal{B}_{\mathrm{ex}}$; see [10] and the discussion surrounding eq. (11) in Majumdar and Comtet [16].) The series expansion (1.1) for f_{ex} and Darling's result readily yield asymptotics of the left tail of the distribution, i.e., of $f_{\mathrm{ex}}(x)$ and $\mathbb{P}(\mathcal{B}_{\mathrm{ex}} \le x)$ as $x \to 0$, see Louchard [15] and Flajolet and Louchard [8] (with typos corrected in [10]); it is easily seen that with a_j ordered with $|a_1| < |a_2| < \dots$, only the first term in the sum is significant for small x because of the factor e^{-v_j}.

The main purpose of this paper is to give corresponding asymptotics for the right tail of the distribution of $\mathcal{B}_{\mathrm{ex}}$, i.e., for the density function $f_{\mathrm{ex}}(x)$ and the tail probabilities $\mathbb{P}(\mathcal{B}_{\mathrm{ex}} > x)$ for large x. This is important for large deviation properties in the above-mentioned applications. For large x, the expansion (1.1) is not very useful since many v_j are small, and to find asymptotics has been a long-standing mathematical question with only some weak results obtained so far, see (1.4) below. We have the following very precise result.

Theorem 1.1. *For the Brownian excursion area, as $x \to \infty$,*

$$f_{\mathrm{ex}}(x) \sim \frac{72\sqrt{6}}{\sqrt{\pi}} x^2 e^{-6x^2} \tag{1.2}$$

and

$$\mathbb{P}(\mathcal{B}_{\mathrm{ex}} > x) \sim \frac{6\sqrt{6}}{\sqrt{\pi}} x e^{-6x^2}. \tag{1.3}$$

More precisely, there exist asymptotic expansions in powers of x^{-2}, to arbitrary order N, as $x \to \infty$,

$$f_{\mathrm{ex}}(x) = \frac{72\sqrt{6}}{\sqrt{\pi}} x^2 e^{-6x^2}\left(1 - \frac{1}{9}\,x^{-2} - \frac{5}{1296}\,x^{-4} - \frac{25}{46656}\,x^{-6} + \dots + O\left(x^{-2N}\right)\right),$$

$$\mathbb{P}(\mathcal{B}_{\mathrm{ex}} > x) = \frac{6\sqrt{6}}{\sqrt{\pi}} x e^{-6x^2}\left(1 - \frac{1}{36}\,x^{-2} - \frac{1}{648}\,x^{-4} - \frac{7}{46656}\,x^{-6} + \dots + O\left(x^{-2N}\right)\right).$$

Unlike the left tail, it seems difficult to obtain such results from Takács's formula (1.1) for f_{ex}, and we will instead use a method by Tolmatz [26; 27; 28] that he used to obtain corresponding asymptotics for three other Brownian areas, viz., the integral $\mathcal{B}_{\mathrm{br}} := \int_0^1 |B_{\mathrm{br}}(t)|\,\mathrm{d}t$ of the absolute value of a Brownian bridge $B_{\mathrm{br}}(t)$, the integral $\mathcal{B}_{\mathrm{bm}} := \int_0^1 |B(t)|\,\mathrm{d}t$ of the absolute value of a Brownian motion $B(t)$ over $[0,1]$, and the integral $\mathcal{B}_{\mathrm{br}+} := \int_0^1 B_{\mathrm{br}}(t)_+\,\mathrm{d}t$ of the positive part of a Brownian bridge. (An application of $\mathcal{B}_{\mathrm{br}}$ can be found in Shepp [21].)

The (much weaker) fact that $-\ln \mathbb{P}(\mathcal{B}_{\mathrm{ex}} > x) \sim -6x^2$, i.e., that

$$\mathbb{P}(\mathcal{B}_{\mathrm{ex}} > x) = \exp\bigl(-6x^2 + o(x^2)\bigr), \tag{1.4}$$

was shown by Csörgő, Shi and Yor [3] as a consequence of the asymptotics of the moments $\mathbb{E}\,\mathcal{B}_{\mathrm{ex}}^n$ found by Takács [22], see Section 9. It seems difficult to obtain more precise tail asymptotics from the moment asymptotics. It is, however, easy to go in the opposite direction and obtain moment asymptotics from the tail asymptotics above, as was done by Tolmatz [27; 28] for $\mathcal{B}_{\mathrm{bm}}$, $\mathcal{B}_{\mathrm{br}}$ and $\mathcal{B}_{\mathrm{br}+}$; see again Section 9. In particular, this made it possible to guess the asymptotic formula (1.2) before we could prove it, by matching the resulting moment asymptotics with the known result by Takács [22].

An alternative way to obtain (1.4) is by large deviation theory, which easily gives (1.4) and explains the constant 6 as the result of an optimization problem, see Fill and Janson [7]. This method applies to the other Brownian areas in this paper too, and explains the different constants in the exponents below, but, again, it seems difficult to obtain more precise results by this approach.

Besides the Brownian excursion area and the three areas studied by Tolmatz, his method applies also to three further Brownian areas: the integrals $\mathcal{B}_{\mathrm{me}} := \int_0^1 |B_{\mathrm{me}}(t)|\,\mathrm{d}t$, $\mathcal{B}_{\mathrm{dm}} := \int_0^1 |B_{\mathrm{dm}}(t)|\,\mathrm{d}t$ and $\mathcal{B}_{\mathrm{bm}+} := \int_0^1 B(t)_+\,\mathrm{d}t$ of a Brownian meander $B_{\mathrm{me}}(t)$, a Brownian double meander $B_{\mathrm{dm}}(t)$, and the positive part of a Brownian motion over $[0,1]$. We define here the Brownian double meander by $B_{\mathrm{dm}}(t) := B(t) - \min_{0\le u\le 1} B(u)$; this is a non-negative continuous stochastic process on $[0,1]$ that a.s. is 0 at a unique point $\tau \in [0,1]$, and it can be regarded as two Brownian meanders on the intervals $[0,\tau]$ and $[\tau,1]$ joined back to back (with the first one reversed), see Majumdar and Comtet [17] and Janson [10]; the other processes considered here are well-known, see for example Revuz and Yor [20].

We find it illuminating to study all seven Brownian areas together, and we will therefore formulate our proof in a general form that applies to all seven areas. As a result we obtain also the following results, where we for completeness repeat Tolmatz's results. (We also extend them, since Tolmatz [26; 27; 28] gives only the leading terms, but he points out that higher order terms can be obtained in the same way.) We give the four first terms in the asymptotic expansions; they can (in principle, at least) be continued to any desired number of terms by the methods in Section 6; only even powers x^{-2k} appear in the expansions.

Theorem 1.2 (Tolmatz [26])**.** *For the Brownian bridge area, as $x \to \infty$,*

$$f_{\mathrm{br}}(x) = \frac{2\sqrt{6}}{\sqrt{\pi}}\, e^{-6x^2}\left(1 + \frac{1}{18}\,x^{-2} + \frac{1}{432}\,x^{-4} + \frac{25}{46656}\,x^{-6} + O\left(x^{-8}\right)\right),$$

$$\mathbb{P}(\mathcal{B}_{\mathrm{br}} > x) = \frac{1}{\sqrt{6\pi}}\, x^{-1} e^{-6x^2}\left(1 - \frac{1}{36}\,x^{-2} + \frac{1}{108}\,x^{-4} - \frac{155}{46656}\,x^{-6} + O\left(x^{-8}\right)\right).$$

Theorem 1.3 (Tolmatz [27])**.** *For the Brownian motion area, as $x \to \infty$,*

$$f_{\mathrm{bm}}(x) = \frac{\sqrt{6}}{\sqrt{\pi}}\, e^{-3x^2/2} \left(1 + \frac{1}{18}\, x^{-2} - \frac{1}{162}\, x^{-4} + \frac{49}{5832}\, x^{-6} + O\left(x^{-8}\right)\right),$$
$$\mathbb{P}(\mathcal{B}_{\mathrm{bm}} > x) = \frac{\sqrt{2}}{\sqrt{3\pi}}\, x^{-1} e^{-3x^2/2} \left(1 - \frac{5}{18}\, x^{-2} + \frac{22}{81}\, x^{-4} - \frac{2591}{5832}\, x^{-6} + O\left(x^{-8}\right)\right).$$

Theorem 1.4. *For the Brownian meander area, as $x \to \infty$,*

$$f_{\mathrm{me}}(x) = 3\sqrt{3}\, x e^{-3x^2/2} \left(1 - \frac{1}{18}\, x^{-2} - \frac{1}{162}\, x^{-4} + \frac{5}{5832}\, x^{-6} + O\left(x^{-8}\right)\right),$$
$$\mathbb{P}(\mathcal{B}_{\mathrm{me}} > x) = \sqrt{3}\, e^{-3x^2/2} \left(1 - \frac{1}{18}\, x^{-2} + \frac{5}{162}\, x^{-4} - \frac{235}{5832}\, x^{-6} + O\left(x^{-8}\right)\right).$$

Theorem 1.5. *For the Brownian double meander area, as $x \to \infty$,*

$$f_{\mathrm{dm}}(x) = \frac{2\sqrt{6}}{\sqrt{\pi}}\, e^{-3x^2/2} \left(1 + \frac{1}{6}\, x^{-2} + \frac{1}{18}\, x^{-4} + \frac{29}{648}\, x^{-6} + O\left(x^{-8}\right)\right),$$
$$\mathbb{P}(\mathcal{B}_{\mathrm{dm}} > x) = \frac{2\sqrt{2}}{\sqrt{3\pi}}\, x^{-1} e^{-3x^2/2} \left(1 - \frac{1}{6}\, x^{-2} + \frac{2}{9}\, x^{-4} - \frac{211}{648}\, x^{-6} + O\left(x^{-8}\right)\right).$$

Theorem 1.6 (Tolmatz [28])**.** *For the positive part of Brownian bridge area, as $x \to \infty$,*

$$f_{\mathrm{br}+}(x) = \frac{\sqrt{6}}{\sqrt{\pi}}\, e^{-6x^2} \left(1 + \frac{1}{36}\, x^{-2} - \frac{7}{5184}\, x^{-4} + \frac{17}{46656}\, x^{-6} + O\left(x^{-8}\right)\right),$$
$$\mathbb{P}(\mathcal{B}_{\mathrm{br}+} > x) = \frac{1}{2\sqrt{6\pi}}\, x^{-1} e^{-6x^2} \left(1 - \frac{1}{18}\, x^{-2} + \frac{65}{5184}\, x^{-4} - \frac{907}{186624}\, x^{-6} + O\left(x^{-8}\right)\right).$$

Theorem 1.7. *For the positive part of Brownian motion area, as $x \to \infty$,*

$$f_{\mathrm{bm}+}(x) = \frac{\sqrt{3}}{\sqrt{2\pi}}\, e^{-3x^2/2} \left(1 + \frac{1}{36}\, x^{-2} - \frac{5}{648}\, x^{-4} + \frac{109}{15552}\, x^{-6} + O\left(x^{-8}\right)\right),$$
$$\mathbb{P}(\mathcal{B}_{\mathrm{bm}+} > x) = \frac{1}{\sqrt{6\pi}}\, x^{-1} e^{-3x^2/2} \left(1 - \frac{11}{36}\, x^{-2} + \frac{193}{648}\, x^{-4} - \frac{2537}{5184}\, x^{-6} + O\left(x^{-8}\right)\right).$$

It is not surprising that the tails are roughly Gaussian, with a decay like e^{-cx^2} for some constants c. Note that the constant in the exponent is 6 for the Brownian bridge and excursion, which are tied to 0 at both endpoints, and only 3/2 for the Brownian motion, meander and double meander, which are tied to 0 only at one point. It is intuitively clear that the probability of a very large value is smaller in the former cases. There are also differences in factors of x between $\mathcal{B}_{\mathrm{br}}$ and $\mathcal{B}_{\mathrm{ex}}$, and between $\mathcal{B}_{\mathrm{bm}}$ and $\mathcal{B}_{\mathrm{me}}$, where the process conditioned to be positive has somewhat higher probabilities of large areas. These differences are in the expected direction, but we see no intuitive reason for the powers in the theorems. We have even less explanations for the constant factors in the estimates.

As said above, the weaker result (1.4) for the Brownian excursion was obtained from moment asymptotics by Csörgő, Shi and Yor [3], and the corresponding result for the double meander, $\mathbb{P}(\mathcal{B}_{\mathrm{dm}} > x) = \exp\bigl(-3x^2/2 + o(x^2)\bigr)$, was obtained in the same way by Majumdar and Comtet [16, 17].

Remark 1.8. If we define $\mathcal{B}_{\mathrm{br}-} := \int_0^1 B_{\mathrm{br}}(t)_- \,\mathrm{d}t$, we have $\mathcal{B}_{\mathrm{br}} = \mathcal{B}_{\mathrm{br}+} + \mathcal{B}_{\mathrm{br}-}$; further, $\mathcal{B}_{\mathrm{br}-} \stackrel{\mathrm{d}}{=} \mathcal{B}_{\mathrm{br}+}$ by symmetry. Hence, for any x,

$$\begin{aligned}\mathbb{P}(\mathcal{B}_{\mathrm{br}} > x) &\ge \mathbb{P}(\mathcal{B}_{\mathrm{br}+} > x \text{ or } \mathcal{B}_{\mathrm{br}-} > x)\\ &= \mathbb{P}(\mathcal{B}_{\mathrm{br}+} > x) + \mathbb{P}(\mathcal{B}_{\mathrm{br}-} > x) - \mathbb{P}(\mathcal{B}_{\mathrm{br}+} > x \text{ and } \mathcal{B}_{\mathrm{br}-} > x)\\ &\ge 2\mathbb{P}(\mathcal{B}_{\mathrm{br}+} > x) - 2\mathbb{P}(\mathcal{B}_{\mathrm{br}} > 2x).\end{aligned}$$

By Theorems 1.2 and 1.6, the ratio between the two sides is $1 + \frac{1}{36}x^{-2} + O(x^{-4})$; hence, these inequalities are tight for large x. This shows, in a very precise way, the intuitive fact that the most probable way to obtain a large value of $\mathcal{B}_{\mathrm{br}}$ is with one of $\mathcal{B}_{\mathrm{br}+}$ and $\mathcal{B}_{\mathrm{br}-}$ large and the other close to 0.

The same is true for $\mathcal{B}_{\mathrm{bm}}$ and $\mathcal{B}_{\mathrm{bm}\pm}$ by Theorems 1.2 and 1.6. It is interesting to note that for both $\mathcal{B}_{\mathrm{br}}$ and $\mathcal{B}_{\mathrm{bm}}$, the ratio $\mathbb{P}(\mathcal{B} > x)/2\mathbb{P}(\mathcal{B}_+) = 1 + \frac{1}{36}x^{-2} + O(x^{-4})$, with the first two terms equal for the two cases (the third terms differ).

Remark 1.9. Series expansions similar to (1.1) for the density functions f_{br}, f_{bm}, f_{me} and f_{dm} are known, see [23; 24; 25; 17; 10]. As for the Brownian excursion, these easily yield asymptotics as $x \to 0$ [10] but not as $x \to \infty$.

Tolmatz's method is based on inverting a double Laplace transform; this double Laplace transform has simple explicit forms (involving the Airy function) for all seven Brownian areas, see the survey [10] and the references given there. The inversion is far from trivial; a straightforward inversion leads to a double integral that is not even absolutely convergent, and not easy to estimate. Tolmatz found a clever change of contour that together with properties of the Airy function leading to near cancellations makes it possible to rewrite the integral as a double integral of a rapidly decreasing function, for which the saddle point method can be applied. (Kearney, Majumdar and Martin [13] have recently used a similar change of contour together with similar near cancellations to invert a (single) Laplace transform for another type of Brownian area.) We follow Tolmatz's approach, and state his inversion using a change of contour in a rather general form in Section 3; the proof is given in Section 8. This inversion formula is then applied to the seven Brownian areas in Sections 4–6. Moment asymptotics are derived in Section 9.

A completely different proof for the asymptotics of $\mathbb{P}(\mathcal{B}_{\mathrm{br}} > x)$ and $\mathbb{P}(\mathcal{B}_{\mathrm{bm}} > x)$ in Theorems 1.2 and 1.3 has been given by Fatalov [6] using Laplace's method in Banach spaces. This method seems to be an interesting and flexible alternative way to obtain at least first order asymptotics in many situations, and it would be interesting to extend it to cover all cases treated here.

We use $C_1, C_2, \ldots$ and $c_1, c_2, \ldots$ to denote various positive constants; explicit values could be given but are unimportant. We also write, for example, $C_1(M)$ to denote dependency on a parameter (but not on anything else).

2 Asymptotics of density and distribution functions

The relation between the asymptotics for density functions and distribution functions in Theorems 1.1–1.7 can be obtained as follows.

Suppose that X is a positive random variable with a density function f satisfying

$$f(x) \sim a x^{\alpha} e^{-bx^2}, \qquad x \to \infty, \tag{2.1}$$

for some numbers $a, b > 0$, $\alpha \in \mathbb{R}$. It is easily seen, e.g. by integration by parts, that (2.1) implies

$$\mathbb{P}(X > x) \sim \frac{a}{2b} x^{\alpha-1} e^{-bx^2}, \qquad x \to \infty. \tag{2.2}$$

Obviously, there is no implication in the opposite direction; X may even satisfy (2.2) without having a density at all. On the other hand, if it is known that (2.1) holds with some unknown constants a, b, α, then the constants can be found from the asysmptotics of $\mathbb{P}(X > x)$ by (2.2).

The argument extends to asymptotic expansions with higher order terms. If, as for the Brownian areas studied in this paper, there is an asymptotic expansion

$$f(x) = x^{\alpha} e^{-bx^2} \left(a_0 + a_2 x^{-2} + a_4 x^{-4} + \cdots + O(x^{-2N})\right), \qquad x \to \infty, \tag{2.3}$$

then repeated integrations by parts yield a corresponding expansion

$$\mathbb{P}(X > x) = x^{\alpha-1} e^{-bx^2} \left(a_0' + a_2' x^{-2} + a_4' x^{-4} + \cdots + O(x^{-2N})\right), \qquad x \to \infty, \tag{2.4}$$

where $a_0' = a_0/(2b)$, $a_2' = a_0(\alpha-1)/(2b)^2 + a_2/(2b)$, $\ldots$; in general, the expansion (2.3) is recovered by formal differentiation of (2.4), which gives a simple method to find the coefficients in (2.4).

3 A double Laplace inversion

We state the main step in (our version of) Tolmatz' method as the following inversion formula, which is based on and generalizes formulas in Tolmatz [26; 27; 28].

Fractional powers of complex numbers below are interpreted as the principal values, defined in $\mathbb{C} \setminus (-\infty, 0]$.

Theorem 3.1. *Let X be a positive random variable and let $\psi(s) := \mathbb{E}\, e^{-sX}$ be its Laplace transform. Suppose that $0 < \nu < 3/2$ and that*

$$\frac{1}{\Gamma(\nu)} \int_0^{\infty} e^{-xs} \psi(s^{3/2}) s^{\nu-1} \, \mathrm{d}s = \Psi(x), \qquad x > 0, \tag{3.1}$$

where Ψ is an analytic function in the sector $\{z \in \mathbb{C} : |\arg z| < 5\pi/6\}$ such that

$$\Psi(z) = o(|z|^{-\nu}), \qquad z \to 0 \text{ with } |\arg z| < 5\pi/6, \tag{3.2}$$

$$\Psi(z) = O(1), \qquad |z| \to \infty \text{ with } |\arg z| < 5\pi/6. \tag{3.3}$$

Let

$$\Psi^*(z) := e^{2\pi\nu\mathbf{i}/3} \Psi\big(e^{2\pi\mathbf{i}/3} z\big) - e^{-2\pi\nu\mathbf{i}/3} \Psi\big(e^{-2\pi\mathbf{i}/3} z\big). \tag{3.4}$$

Finally, assume that

$$\Psi^*(z) = O(|z|^{-6}), \qquad |z| \to \infty \text{ with } |\arg z| < \pi/6. \tag{3.5}$$

Then X is absolutely continuous with a continuous density function f given by, for $x > 0$ and every $\xi > 0$,

$$f(x) = \frac{3\Gamma(\nu)}{8\pi^2\mathbf{i}}\xi^{5/2-\nu}x^{2\nu/3-5/3} \cdot \int_{\theta=-\pi/2}^{\pi/2}\int_{r=0}^{\infty} \exp\Bigl(\xi x^{-2/3}\sec\theta e^{\mathbf{i}\theta} - e^{\mathbf{i}\theta}(\xi\sec\theta)^{3/2}r^{-3/2}\Bigr) \cdot e^{(1-2\nu/3)\mathbf{i}\theta}(\sec\theta)^{7/2-\nu}r^{\nu-5/2}\Psi^*(re^{\mathbf{i}\theta/3})\,\mathrm{d}r\,\mathrm{d}\theta. \quad (3.6)$$

Note that Ψ^* is analytic in the sector $|\arg z| < \frac{\pi}{6}$, with, by (3.2) and (3.3),

$$\Psi^*(z) = o(|z|^{-\nu}), \qquad z \to 0 \text{ with } |\arg z| < \frac{\pi}{6}, \quad (3.7)$$

$$\Psi^*(z) = O(1), \qquad |z| \to \infty \text{ with } |\arg z| < \frac{\pi}{6}. \quad (3.8)$$

However, we need, as assumed in (3.5), a more rapid decay as $|z| \to \infty$ than this.

Remark 3.2. In all our applications, Ψ is, in fact, analytic in the slit plane $\mathbb{C} \setminus (-\infty, 0]$, and (3.2) and (3.3) hold in any sector $|\arg z| \le \pi - \delta$; thus Ψ^* is analytic in $|\arg z| < \pi/3$, and (3.7) and (3.8) hold for $|\arg z| \le \pi/3 - \delta$.

Remark 3.3. To obtain Tolmatz' version of the formulas, for example [26, (30)] (correcting a typo there), take $\nu = 1/2$ and Ψ^* as in (4.5) below, and make the substitutions $x = \lambda$, $\xi = a\lambda^{2/3}$ and $r = a\rho^{-2/3}\sec\theta$.

We prove Theorem 3.1 in Section 8, but show first how it applies to the Brownian areas.

4 The function Ψ^* for Brownian areas

For the Brownian bridge area $\mathcal{B}_{\mathrm{br}}$ we have $\nu = 1/2$ and, see e.g. [10, (126)],

$$\Psi(z) = -2^{1/6}\frac{\mathrm{Ai}(2^{1/3}z)}{\mathrm{Ai}'(2^{1/3}z)}, \quad (4.1)$$

which by the formula [1, 10.4.9]

$$\mathrm{Ai}(ze^{\pm 2\pi\mathbf{i}/3}) = \tfrac{1}{2}e^{\pm\pi\mathbf{i}/3}\bigl(\mathrm{Ai}(z) \mp \mathbf{i}\mathrm{Bi}(z)\bigr) \quad (4.2)$$

and its consequence

$$\mathrm{Ai}'(ze^{\pm 2\pi\mathbf{i}/3}) = \tfrac{1}{2}e^{\mp\pi\mathbf{i}/3}\bigl(\mathrm{Ai}'(z) \mp \mathbf{i}\mathrm{Bi}'(z)\bigr) \quad (4.3)$$

together with the Wronskian [1, 10.4.10]

$$\mathrm{Ai}(z)\mathrm{Bi}'(z) - \mathrm{Ai}'(z)\mathrm{Bi}(z) = \pi^{-1} \quad (4.4)$$

by a simple calculation leads to, as shown by Tolmatz [26, Lemma 2.1], see (4.7) below,

$$\Psi^*(z) = \frac{2^{7/6}\pi^{-1}\mathbf{i}}{\mathrm{Ai}'(2^{1/3}z)^2 + \mathrm{Bi}'(2^{1/3}z)^2}. \quad (4.5)$$

It seems simpler to instead consider $\sqrt{2}\,\mathcal{B}_{\mathrm{br}}$. Note that, by the simple change of variables $s \mapsto 2^{1/3}s$ and $x \mapsto 2^{-1/3}x$ in (3.1), if (3.1) holds for some random variable X and a function Ψ, it holds for $\sqrt{2}\,X$ and $2^{-\nu/3}\Psi(2^{-1/3}z)$. We use the notations Ψ_{br} and Ψ^*_{br} for the case $X = \sqrt{2}\,\mathcal{B}_{\mathrm{br}}$ and obtain from (4.1) the simpler

$$\Psi_{\mathrm{br}}(z) = -\frac{\mathrm{Ai}(z)}{\mathrm{Ai}'(z)} \tag{4.6}$$

and thus, by (4.2), (4.3) and (4.4),

$$\begin{aligned}
\Psi^*_{\mathrm{br}}(z) &= \sum_{\pm} \pm e^{\pm\pi\mathbf{i}/3}\Psi\bigl(e^{\pm 2\pi\mathbf{i}/3}z\bigr) = \sum_{\pm} \mp e^{\pm 3\pi\mathbf{i}/3}\frac{\mathrm{Ai}(z) \mp \mathbf{i}\mathrm{Bi}(z)}{\mathrm{Ai}'(z) \mp \mathbf{i}\mathrm{Bi}'(z)} \\
&= \sum_{\pm} \pm \frac{\bigl(\mathrm{Ai}(z) \mp \mathbf{i}\mathrm{Bi}(z)\bigr)\bigl(\mathrm{Ai}'(z) \pm \mathbf{i}\mathrm{Bi}'(z)\bigr)}{\mathrm{Ai}'(z)^2 + \mathrm{Bi}'(z)^2} \\
&= \frac{2\pi^{-1}\mathbf{i}}{\mathrm{Ai}'(z)^2 + \mathrm{Bi}'(z)^2}.
\end{aligned} \tag{4.7}$$

For the Brownian excursion area $\mathcal{B}_{\mathrm{ex}}$ we have $\nu = 1/2$ and by Louchard [15], see also [10, (80)],

$$\Psi(z) = -2^{5/6}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{Ai}'(2^{1/3}z)}{\mathrm{Ai}(2^{1/3}z)}\right) = 2^{1/2}\left(2^{1/3}\frac{\mathrm{Ai}'(2^{1/3}z)}{\mathrm{Ai}(2^{1/3}z)}\right)^2 - 2^{3/2}z, \tag{4.8}$$

Again, it seems simpler to instead consider $\sqrt{2}\,\mathcal{B}_{\mathrm{ex}}$, for which we use the notation Ψ_{ex} and Ψ^*_{ex}. We have, see Louchard [15] and [10, (81)], or by (4.8) and the general relation above,

$$\Psi_{\mathrm{ex}}(z) = -2\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{Ai}'(z)}{\mathrm{Ai}(z)}\right) = 2\left(\frac{\mathrm{Ai}'(z)}{\mathrm{Ai}(z)}\right)^2 - 2z, \tag{4.9}$$

and thus by (4.2), (4.3) and (4.4)

$$\begin{aligned}
\Psi^*_{\mathrm{ex}}(z) &= \sum_{\pm} \pm e^{\pm\pi\mathbf{i}/3}\Psi_{\mathrm{ex}}\bigl(e^{\pm 2\pi\mathbf{i}/3}z\bigr) \\
&= \sum_{\pm} \pm\left(e^{\pm\pi\mathbf{i}/3}2\left(e^{\mp 2\pi\mathbf{i}/3}\frac{\mathrm{Ai}'(z) \mp \mathbf{i}\mathrm{Bi}'(z)}{\mathrm{Ai}(z) \mp \mathbf{i}\mathrm{Bi}(z)}\right)^2 - 2e^{\pm 2\pi\mathbf{i}/3}z\right) \\
&= 2\sum_{\pm} \pm e^{\mp 3\pi\mathbf{i}/3}\frac{\bigl(\mathrm{Ai}'(z) \mp \mathbf{i}\mathrm{Bi}'(z)\bigr)^2\bigl(\mathrm{Ai}(z) \pm \mathbf{i}\mathrm{Bi}(z)\bigr)^2}{\bigl(\mathrm{Ai}(z)^2 + \mathrm{Bi}(z)^2\bigr)^2} + 0 \\
&= 2\sum_{\pm} \mp \frac{\bigl(\mathrm{Ai}'(z)\mathrm{Ai}(z) + \mathrm{Bi}'(z)\mathrm{Bi}(z) \mp \mathbf{i}\pi^{-1}\bigr)^2}{\bigl(\mathrm{Ai}(z)^2 + \mathrm{Bi}(z)^2\bigr)^2} \\
&= \frac{8\pi^{-1}\mathbf{i}\bigl(\mathrm{Ai}(z)\mathrm{Ai}'(z) + \mathrm{Bi}(z)\mathrm{Bi}'(z)\bigr)}{\bigl(\mathrm{Ai}(z)^2 + \mathrm{Bi}(z)^2\bigr)^2}.
\end{aligned} \tag{4.10}$$

The Brownian motion area $\mathcal{B}_{\mathrm{bm}}$ is another case treated by Tolmatz [27]. Note that in this case $\nu = 1$. For $\sqrt{2}\,\mathcal{B}_{\mathrm{bm}}$, we have by Takács [24], see also Kac [11], Perman and Wellner [19], and [10, Section 20 and Appendix C.1], $\nu = 1$ and

$$\Psi_{\mathrm{bm}}(z) = -\frac{\mathrm{AI}(z)}{\mathrm{Ai}'(z)}, \tag{4.11}$$

where we use the notation, see [10, Appendix A],

$$\mathrm{AI}(z) := \int_z^{+\infty} \mathrm{Ai}(t)\,\mathrm{d}t = \frac{1}{3} - \int_0^z \mathrm{Ai}(t)\,\mathrm{d}t. \tag{4.12}$$

If we further define

$$\mathrm{BI}(z) := \int_0^z \mathrm{Bi}(t)\,\mathrm{d}t, \tag{4.13}$$

we have by (4.2)

$$\begin{aligned}
\mathrm{AI}(ze^{\pm 2\pi\mathbf{i}/3}) &= \frac{1}{3} - \int_0^{ze^{\pm 2\pi\mathbf{i}/3}} \mathrm{Ai}(t)\,\mathrm{d}t \\
&= \frac{1}{3} - e^{\pm 2\pi\mathbf{i}/3} \int_0^z \mathrm{Ai}(te^{\pm 2\pi\mathbf{i}/3})\,\mathrm{d}t \\
&= \frac{1}{3} - \frac{1}{2} e^{\pm 3\pi\mathbf{i}/3} \int_0^z \big(\mathrm{Ai}(t) \mp \mathbf{i}\mathrm{Bi}(t)\big)\,\mathrm{d}t \\
&= \tfrac{1}{2} - \tfrac{1}{2}\mathrm{AI}(z) \mp \tfrac{1}{2}\mathbf{i}\mathrm{BI}(z).
\end{aligned} \tag{4.14}$$

Consequently, using (4.11) and (4.3),

$$\begin{aligned}
\Psi^*_{\mathrm{bm}}(z) &= \sum_{\pm} \pm e^{\pm 2\pi\mathbf{i}/3} \Psi_{\mathrm{bm}}\big(e^{\pm 2\pi\mathbf{i}/3} z\big) \\
&= \sum_{\pm} \mp e^{\pm 3\pi\mathbf{i}/3} \frac{1 - \mathrm{AI}(z) \mp \mathbf{i}\mathrm{BI}(z)}{\mathrm{Ai}'(z) \mp \mathbf{i}\mathrm{Bi}'(z)} \\
&= \sum_{\pm} \pm \frac{\big(1 - \mathrm{AI}(z) \mp \mathbf{i}\mathrm{BI}(z)\big)\big(\mathrm{Ai}'(z) \pm \mathbf{i}\mathrm{Bi}'(z)\big)}{\mathrm{Ai}'(z)^2 + \mathrm{Bi}'(z)^2} \\
&= 2\mathbf{i} \frac{\mathrm{Bi}'(z) - \mathrm{AI}(z)\mathrm{Bi}'(z) - \mathrm{Ai}'(z)\mathrm{BI}(z)}{\mathrm{Ai}'(z)^2 + \mathrm{Bi}'(z)^2}.
\end{aligned} \tag{4.15}$$

For the Brownian meander, or more precisely $\sqrt{2}\,\mathcal{B}_{\mathrm{me}}$, by Takács [25], see also [10, Section 22 and Appendix C.3], (3.1) holds with $\nu = 1/2$ and

$$\Psi_{\mathrm{me}}(z) = \frac{\mathrm{AI}(z)}{\mathrm{Ai}(z)}. \tag{4.16}$$

Consequently, using (4.14) and (4.2),

$$\begin{aligned}
\Psi^*_{\mathrm{me}}(z) &= \sum_{\pm} \pm e^{\pm \pi\mathbf{i}/3} \Psi_{\mathrm{me}}\big(e^{\pm 2\pi\mathbf{i}/3} z\big) \\
&= \sum_{\pm} \pm \frac{1 - \mathrm{AI}(z) \mp \mathbf{i}\mathrm{BI}(z)}{\mathrm{Ai}(z) \mp \mathbf{i}\mathrm{Bi}(z)} \\
&= \sum_{\pm} \pm \frac{\big(1 - \mathrm{AI}(z) \mp \mathbf{i}\mathrm{BI}(z)\big)\big(\mathrm{Ai}(z) \pm \mathbf{i}\mathrm{Bi}(z)\big)}{\mathrm{Ai}(z)^2 + \mathrm{Bi}(z)^2} \\
&= 2\mathbf{i} \frac{\mathrm{Bi}(z) - \mathrm{AI}(z)\mathrm{Bi}(z) - \mathrm{Ai}(z)\mathrm{BI}(z)}{\mathrm{Ai}(z)^2 + \mathrm{Bi}(z)^2}.
\end{aligned} \tag{4.17}$$

For the Brownian double meander, or more precisely $\sqrt{2}\,\mathcal{B}_{\mathrm{dm}}$, by Majumdar and Comtet [17], see also [10, Section 23], (3.1) holds with $\nu = 1$ and

$$\Psi_{\mathrm{dm}}(z) = \left(\frac{\mathrm{AI}(z)}{\mathrm{Ai}(z)}\right)^2. \tag{4.18}$$

Consequently, using (4.14) and (4.2),

$$\begin{aligned}
\Psi^*_{\mathrm{dm}}(z) &= \sum_{\pm} \pm e^{\pm 2\pi\mathbf{i}/3}\Psi_{\mathrm{dm}}\big(e^{\pm 2\pi\mathbf{i}/3}z\big) \\
&= \sum_{\pm} \pm \left(\frac{1 - \mathrm{AI}(z) \mp \mathbf{i}\mathrm{BI}(z)}{\mathrm{Ai}(z) \mp \mathbf{i}\mathrm{Bi}(z)}\right)^2 \\
&= \sum_{\pm} \pm \frac{\big(\big(1 - \mathrm{AI}(z) \mp \mathbf{i}\mathrm{BI}(z)\big)\big(\mathrm{Ai}(z) \pm \mathbf{i}\mathrm{Bi}(z)\big)\big)^2}{\big(\mathrm{Ai}(z)^2 + \mathrm{Bi}(z)^2\big)^2} \\
&= 4\mathbf{i}\frac{\big((1 - \mathrm{AI}(z))\mathrm{Ai}(z) + \mathrm{BI}(z)\mathrm{Bi}(z)\big)\big((1 - \mathrm{AI}(z))\mathrm{Bi}(z) - \mathrm{BI}(z)\mathrm{Ai}(z)\big)}{\big(\mathrm{Ai}(z)^2 + \mathrm{Bi}(z)^2\big)^2}.
\end{aligned} \tag{4.19}$$

The positive part of a Brownian bridge is another case treated by Tolmatz [28]. For $\sqrt{2}\,\mathcal{B}_{\mathrm{br}+}$, by Perman and Wellner [19], see also Tolmatz [28] and [10, Section 22 and Appendix C.2], (3.1) holds with $\nu = 1/2$ and

$$\Psi_{\mathrm{br}+}(z) = 2\frac{\mathrm{Ai}(z)}{z^{1/2}\mathrm{Ai}(z) - \mathrm{Ai}'(z)}. \tag{4.20}$$

Consequently, by (4.2), (4.3) and (4.4),

$$\begin{aligned}
\Psi^*_{\mathrm{br}+}(z) &= \sum_{\pm} \pm e^{\pm \pi\mathbf{i}/3}\Psi_{\mathrm{br}+}\big(e^{\pm 2\pi\mathbf{i}/3}z\big) \\
&= \sum_{\pm} \pm 2e^{\pm \pi\mathbf{i}/3}\frac{\mathrm{Ai}\big(e^{\pm 2\pi\mathbf{i}/3}z\big)}{e^{\pm \pi\mathbf{i}/3}z^{1/2}\mathrm{Ai}\big(e^{\pm 2\pi\mathbf{i}/3}z\big) - \mathrm{Ai}'\big(e^{\pm 2\pi\mathbf{i}/3}z\big)} \\
&= 2\sum_{\pm} \pm \frac{\mathrm{Ai}(z) \mp \mathbf{i}\mathrm{Bi}(z)}{z^{1/2}\big(\mathrm{Ai}(z) \mp \mathbf{i}\mathrm{Bi}(z)\big) - e^{\mp 3\pi\mathbf{i}/3}\big(\mathrm{Ai}'(z) \mp \mathbf{i}\mathrm{Bi}'(z)\big)} \\
&= 2\sum_{\pm} \pm \frac{\mathrm{Ai}(z) \mp \mathbf{i}\mathrm{Bi}(z)}{\big(z^{1/2}\mathrm{Ai}(z) + \mathrm{Ai}'(z)\big) \mp \mathbf{i}\big(z^{1/2}\mathrm{Bi}(z) + \mathrm{Bi}'(z)\big)} \\
&= 2\sum_{\pm} \pm \frac{\big(\mathrm{Ai}(z) \mp \mathbf{i}\mathrm{Bi}(z)\big)\big(z^{1/2}\mathrm{Ai}(z) + \mathrm{Ai}'(z) \pm \mathbf{i}\big(z^{1/2}\mathrm{Bi}(z) + \mathrm{Bi}'(z)\big)\big)}{\big(z^{1/2}\mathrm{Ai}(z) + \mathrm{Ai}'(z)\big)^2 + \big(z^{1/2}\mathrm{Bi}(z) + \mathrm{Bi}'(z)\big)^2} \\
&= \frac{4\mathbf{i}\pi^{-1}}{\big(z^{1/2}\mathrm{Ai}(z) + \mathrm{Ai}'(z)\big)^2 + \big(z^{1/2}\mathrm{Bi}(z) + \mathrm{Bi}'(z)\big)^2}.
\end{aligned} \tag{4.21}$$

For the positive part of a Brownian motion, or more precisely $\sqrt{2}\,\mathcal{B}_{\mathrm{bm}+}$, by Perman and Wellner [19], see also [10, Section 23 and Appendix C.1], (3.1) holds with $\nu = 1$ and

$$\Psi_{\mathrm{bm}+}(z) = \frac{z^{-1/2}\mathrm{Ai}(z) + \mathrm{AI}(z)}{z^{1/2}\mathrm{Ai}(z) - \mathrm{Ai}'(z)}. \tag{4.22}$$

Note that this Ψ is singular at 0, but still satisfies (3.2). By (4.2), (4.3), (4.14) and (4.4),

$$\begin{aligned}
\Psi^*_{\mathrm{bm}+}(z) &= \sum_{\pm} \pm e^{\pm 2\pi\mathbf{i}/3}\Psi_{\mathrm{bm}+}\big(e^{\pm 2\pi\mathbf{i}/3}z\big) \\
&= \sum_{\pm} \pm e^{\pm 2\pi\mathbf{i}/3}\frac{e^{\mp\pi\mathbf{i}/3}z^{-1/2}\mathrm{Ai}\big(e^{\pm 2\pi\mathbf{i}/3}z\big) + \mathrm{AI}\big(e^{\pm 2\pi\mathbf{i}/3}z\big)}{e^{\pm\pi\mathbf{i}/3}z^{1/2}\mathrm{Ai}\big(e^{\pm 2\pi\mathbf{i}/3}z\big) - \mathrm{Ai}'\big(e^{\pm 2\pi\mathbf{i}/3}z\big)} \\
&= \sum_{\pm} \pm \frac{z^{-1/2}\big(\mathrm{Ai}(z) \mp \mathbf{i}\mathrm{Bi}(z)\big) + 1 - \mathrm{AI}(z) \mp \mathbf{i}\mathrm{BI}(z)}{z^{1/2}\big(\mathrm{Ai}(z) \mp \mathbf{i}\mathrm{Bi}(z)\big) - e^{\mp 3\pi\mathbf{i}/3}\big(\mathrm{Ai}'(z) \mp \mathbf{i}\mathrm{Bi}'(z)\big)} \\
&= \sum_{\pm} \pm \frac{z^{-1/2}\mathrm{Ai}(z) + 1 - \mathrm{AI}(z) \mp \mathbf{i}\big(z^{-1/2}\mathrm{Bi}(z) + \mathrm{BI}(z)\big)}{\big(z^{1/2}\mathrm{Ai}(z) + \mathrm{Ai}'(z)\big) \mp \mathbf{i}\big(z^{1/2}\mathrm{Bi}(z) + \mathrm{Bi}'(z)\big)} \\
&= 2\mathbf{i}\frac{\big(z^{-1/2}\mathrm{Ai}(z) + 1 - \mathrm{AI}(z)\big)\big(z^{1/2}\mathrm{Bi}(z) + \mathrm{Bi}'(z)\big)}{\big(z^{1/2}\mathrm{Ai}(z) + \mathrm{Ai}'(z)\big)^2 + \big(z^{1/2}\mathrm{Bi}(z) + \mathrm{Bi}'(z)\big)^2} \\
&\qquad - 2\mathbf{i}\frac{\big(z^{-1/2}\mathrm{Bi}(z) + \mathrm{BI}(z)\big)\big(z^{1/2}\mathrm{Ai}(z) + \mathrm{Ai}'(z)\big)}{\big(z^{1/2}\mathrm{Ai}(z) + \mathrm{Ai}'(z)\big)^2 + \big(z^{1/2}\mathrm{Bi}(z) + \mathrm{Bi}'(z)\big)^2} \\
&= 2\mathbf{i}\frac{\big(1 - \mathrm{AI}(z)\big)\big(z^{1/2}\mathrm{Bi}(z) + \mathrm{Bi}'(z)\big) - \mathrm{BI}(z)\big(z^{1/2}\mathrm{Ai}(z) + \mathrm{Ai}'(z)\big) + z^{-1/2}\pi^{-1}}{\big(z^{1/2}\mathrm{Ai}(z) + \mathrm{Ai}'(z)\big)^2 + \big(z^{1/2}\mathrm{Bi}(z) + \mathrm{Bi}'(z)\big)^2}
\end{aligned} \tag{4.23}$$

Note that the functions Ψ_{br}, Ψ_{ex}, Ψ_{bm}, Ψ_{me} and Ψ_{dm} given above in (4.6), (4.9), (4.11), (4.16), (4.18) are meromorphic, with poles only on the negative real axis, because the only zeros of Ai and Ai$'$ are on the negative real axis [1, p. 450]. The functions $\Psi_{\mathrm{br}+}$ and $\Psi_{\mathrm{bm}+}$ in (4.20) and (4.22) are analytic in the slit plane $\mathbb{C}\setminus(-\infty, 0]$, since Tolmatz [28] showed that $z^{1/2}\mathrm{Ai}(z) - \mathrm{Ai}'(z)$ has no zeros in the slit plane; see Appendix A for an alternative proof. In particular, all seven functions are analytic in the slit plane. Furthermore, all except $\Psi_{\mathrm{bm}+}$ have finite limits as $z \to 0$, and in particular they are $O(1)$ as $z \to 0$ so (3.2) holds. By (4.22), we have $\Psi_{\mathrm{bm}+}(z) \sim z^{-1/2}\mathrm{Ai}(0)/\mathrm{Ai}'(0)$ and thus $\Psi_{\mathrm{bm}+} = O(|z|^{-1/2})$ as $z \to 0$; since in this case $\nu = 1$, (3.2) holds for $\Psi_{\mathrm{bm}+}$ too.

Next we consider asymtotics as $|z| \to \infty$. The Airy functions have well-known asymptotics, see [1, 10.4.59, 10.4.61, 10.4.63, 10.4.66, 10.4.82, 10.4.84]. The leading terms are, as $|z| \to \infty$ and uniformly in the indicated sectors for any $\delta > 0$,

$$\mathrm{Ai}(z) \sim \frac{\pi^{-1/2}}{2}z^{-1/4}e^{-2z^{3/2}/3}, \qquad |\arg(z)| \le \pi - \delta, \tag{4.24}$$

$$\mathrm{Ai}'(z) \sim -\frac{\pi^{-1/2}}{2}z^{1/4}e^{-2z^{3/2}/3}, \qquad |\arg(z)| \le \pi - \delta, \tag{4.25}$$

$$\mathrm{AI}(z) \sim \frac{\pi^{-1/2}}{2}z^{-3/4}e^{-2z^{3/2}/3}, \qquad |\arg(z)| \le \pi - \delta, \tag{4.26}$$

$$\mathrm{Bi}(z) \sim \pi^{-1/2}z^{-1/4}e^{2z^{3/2}/3}, \qquad |\arg(z)| \le \pi/3 - \delta, \tag{4.27}$$

$$\mathrm{Bi}'(z) \sim \pi^{-1/2}z^{1/4}e^{2z^{3/2}/3}, \qquad |\arg(z)| \le \pi/3 - \delta, \tag{4.28}$$

$$\mathrm{BI}(z) \sim \pi^{-1/2}z^{-3/4}e^{2z^{3/2}/3}, \qquad |\arg(z)| \le \pi/3 - \delta. \tag{4.29}$$

It follows by using (4.24), (4.25) and (4.26) in (4.6), (4.9), (4.16), (4.11), (4.20), (4.22) that in all seven cases (3.3) holds; more precisely, $\Psi(z) \sim z^{-\nu}$ as $|z| \to \infty$ with $|\arg z| < \pi - \delta$. (For real

$z > 0$, this is always true, as follows from (3.1) by the change of variables $s = t/x$ and monotone (or dominated) convergence.)

Turning to Ψ^*, we observe first that, by (3.4), in all seven cases, $\Psi^*(z)$ is analytic in $|\arg z| < 1/3$. Next, (4.24)–(4.29) show that, as $|z| \to \infty$ in a sector $|\arg(z)| \le \pi/3 - \delta$, $\mathrm{Ai}, \mathrm{Ai}', \mathrm{AI}$ decrease superexponentially while $\mathrm{Bi}, \mathrm{Bi}', \mathrm{BI}$ increase superexponentially. Hence, we can ignore all terms involving Ai. More precisely, (4.7), (4.10), (4.15), (4.17), (4.19), (4.21), (4.23) together with (4.24)–(4.29) yield the asymptotics, as $|z| \to \infty$ with (for example) $|\arg z| \le \pi/6$,

$$\Psi^*_{\mathrm{br}}(z) = \frac{2\pi^{-1}\mathbf{i}}{\mathrm{Bi}'(z)^2}\Bigl(1 + O\left(e^{-8z^{3/2}/3}\right)\Bigr), \tag{4.30}$$

$$\Psi^*_{\mathrm{ex}}(z) = 8\pi^{-1}\mathbf{i}\,\frac{\mathrm{Bi}'(z)}{\mathrm{Bi}(z)^3}\Bigl(1 + O\left(e^{-8z^{3/2}/3}\right)\Bigr), \tag{4.31}$$

$$\Psi^*_{\mathrm{bm}}(z) = \frac{2\mathbf{i}}{\mathrm{Bi}'(z)}\Bigl(1 + O\left(e^{-2z^{3/2}/3}\right)\Bigr), \tag{4.32}$$

$$\Psi^*_{\mathrm{me}}(z) = \frac{2\mathbf{i}}{\mathrm{Bi}(z)}\Bigl(1 + O\left(e^{-2z^{3/2}/3}\right)\Bigr), \tag{4.33}$$

$$\Psi^*_{\mathrm{dm}}(z) = 4\mathbf{i}\,\frac{\mathrm{BI}(z)}{\mathrm{Bi}(z)^2}\Bigl(1 + O\left(e^{-2z^{3/2}/3}\right)\Bigr), \tag{4.34}$$

$$\Psi^*_{\mathrm{br}+}(z) = \frac{4\mathbf{i}\pi^{-1}}{\left(z^{1/2}\mathrm{Bi}(z) + \mathrm{Bi}'(z)\right)^2}\Bigl(1 + O\left(e^{-4z^{3/2}/3}\right)\Bigr), \tag{4.35}$$

$$\Psi^*_{\mathrm{bm}+}(z) = \frac{2\mathbf{i}}{z^{1/2}\mathrm{Bi}(z) + \mathrm{Bi}'(z)}\Bigl(1 + O\left(e^{-2z^{3/2}/3}\right)\Bigr). \tag{4.36}$$

In all seven cases, Ψ^* decreases superexponentially in the sector; in particular, (3.5) holds. It is remarkable that in all seven cases, $\Psi(z)$ decreases slowly, as $z^{-1/2}$ or z^{-1}, but the linear combination $\Psi^*(z)$ decreases extremely rapidly in a sector around the positive real axis; there are thus almost complete cancellations between the values of $\Psi(z)$ at, say, $\arg z = \pm 2\pi\mathbf{i}/3$. These cancellations are an important part of the success of Tolmatz's method.

We have verified all the conditions of Theorem 3.1. Hence, the theorem shows that the variables have continuous density functions given by (3.6).

5 The saddle point method

We proceed to show how the tail asymptotics for the Brownian areas follow from Theorem 3.1 and the formulae in Section 4 by straightforward applications of the saddle point method. For simplicity, we give first a derivation of the leading terms. In the next section we show how the calculations can be refined to obtain the asymptotic expansions in Theorems 1.1–1.7.

We use $\Xi \in \{\mathrm{br}, \mathrm{ex}, \mathrm{bm}, \mathrm{me}, \mathrm{dm}, \mathrm{br}+, \mathrm{bm}+\}$ as a variable indicating the different Brownian areas we consider. We begin by writing (4.30)–(4.36), using (4.27) and (4.28), as

$$\Psi^*_\Xi(z) = h_\Xi(z)e^{-\gamma_\Xi z^{3/2}}, \tag{5.1}$$

where $\gamma_{\mathrm{br}} = \gamma_{\mathrm{ex}} = \gamma_{\mathrm{br}+} = 4/3$ and $\gamma_{\mathrm{bm}} = \gamma_{\mathrm{me}} = \gamma_{\mathrm{dm}} = \gamma_{\mathrm{bm}+} = 2/3$ (note that these cases differ

by having two or one points tied to 0) and, as $|z| \to \infty$ with $|\arg z| \le \pi/6$,

$$h_{\mathrm{br}}(z) \sim 2\mathbf{i} z^{-1/2} \tag{5.2}$$
$$h_{\mathrm{ex}}(z) \sim 8\mathbf{i} z \tag{5.3}$$
$$h_{\mathrm{bm}}(z) \sim 2\mathbf{i}\pi^{1/2} z^{-1/4} \tag{5.4}$$
$$h_{\mathrm{me}}(z) \sim 2\mathbf{i}\pi^{1/2} z^{1/4} \tag{5.5}$$
$$h_{\mathrm{dm}}(z) \sim 4\mathbf{i}\pi^{1/2} z^{-1/4} \tag{5.6}$$
$$h_{\mathrm{br}+}(z) \sim \mathbf{i} z^{-1/2} \tag{5.7}$$
$$h_{\mathrm{bm}+}(z) \sim \mathbf{i}\pi^{1/2} z^{-1/4}. \tag{5.8}$$

We write the right hand sides as $\mathbf{i}h^0_{\mathrm{br}}(z), \dots, \mathbf{i}h^0_{\mathrm{bm}+}(z)$, and thus these formulae can be written

$$h_\Xi(z) \sim \mathbf{i} h^0_\Xi(z), \tag{5.9}$$

where $h^0_{\mathrm{br}}(z) = 2z^{-1/2}$, $h^0_{\mathrm{ex}}(z) = 8z$, and so on.

Consider, for simplicity, first the cases $\Xi \in \{\mathrm{br}, \mathrm{ex}, \mathrm{me}, \mathrm{br}+\}$ where $\nu = 1/2$. We then rewrite (3.6) as, using f^*_Ξ for the density of $\sqrt{2}\,\mathcal{B}_\Xi$,

$$f^*_\Xi(x) = \xi^2 x^{-4/3} \int_{-\pi/2}^{\pi/2} \int_0^\infty F_0(r,\theta) e^{\varphi_0(r,\theta;x,\xi)}\, \mathrm{d}r\, \mathrm{d}\theta \tag{5.10}$$

where, with $\gamma = \gamma_\Xi$,

$$F_0(r,\theta) := \frac{3\pi^{-3/2}}{8\mathbf{i}} e^{2\mathbf{i}\theta/3} (\sec\theta)^3 r^{-2} h_\Xi(r e^{\mathbf{i}\theta/3}), \tag{5.11}$$
$$\varphi_0(r,\theta;x,\xi) := \xi x^{-2/3} \sec\theta e^{\mathbf{i}\theta} - e^{\mathbf{i}\theta}(\xi \sec\theta/r)^{3/2} - \gamma r^{3/2} e^{\mathbf{i}\theta/2}. \tag{5.12}$$

Remember that ξ is arbitrary; we choose $\xi = \rho x^{8/3}$ for a positive constant ρ that will be chosen later. Further, make the change of variables $r = x^{4/3} s^{2/3}$. Thus,

$$f^*_\Xi(x) = \rho^2 x^{8/3} \int_{\theta=-\pi/2}^{\pi/2} \int_{s=0}^\infty F_1(s,\theta;x) e^{x^2 \varphi_1(s,\theta)}\, \mathrm{d}s\, \mathrm{d}\theta \tag{5.13}$$

where

$$F_1(s,\theta;x) := \frac{1}{4\pi^{3/2}\mathbf{i}} e^{2\mathbf{i}\theta/3} (\sec\theta)^3 s^{-5/3} h_\Xi(x^{4/3} s^{2/3} e^{\mathbf{i}\theta/3}), \tag{5.14}$$
$$\varphi_1(s,\theta) := \rho\big(1 + \mathbf{i}\tan\theta\big) - \rho^{3/2} s^{-1} e^{\mathbf{i}\theta} (\sec\theta)^{3/2} - \gamma s e^{\mathbf{i}\theta/2}. \tag{5.15}$$

In particular,

$$\operatorname{Re}\varphi_1(s,\theta) = \rho - \rho^{3/2} s^{-1} (\cos\theta)^{-1/2} - \gamma s \cos(\theta/2). \tag{5.16}$$

In the cases $\Xi \in \{\mathrm{bm}, \mathrm{dm}, \mathrm{bm}+\}$ when $\nu = 1$, we obtain similarly

$$f^*_\Xi(x) = \rho^{3/2} x^{7/3} \int_{\theta=-\pi/2}^{\pi/2} \int_{s=0}^\infty F_1(s,\theta;x) e^{x^2 \varphi_1(s,\theta)}\, \mathrm{d}s\, \mathrm{d}\theta \tag{5.17}$$

where

$$F_1(s,\theta;x) := \frac{1}{4\pi^2\mathbf{i}} e^{\mathbf{i}\theta/3} (\sec\theta)^{5/2} s^{-4/3} h_\Xi(x^{4/3} s^{2/3} e^{\mathbf{i}\theta/3}) \tag{5.18}$$

and φ_1 is the same as above.

Consider first $\theta = 0$; then

$$\varphi_1(s,0) = \operatorname{Re}\varphi_1(s,0) = \rho - \rho^{3/2} s^{-1} - \gamma s, \tag{5.19}$$

which has a maximum at $s = s_0 := \rho^{3/4}\gamma^{-1/2}$. In order for $(s_0, 0)$ to be a saddle point of φ_1, we need also

$$0 = \frac{\partial\varphi_1}{\partial\theta}(s_0,0) = \mathbf{i}\rho - \mathbf{i}\rho^{3/2} s^{-1} - \tfrac{1}{2}\mathbf{i}\gamma s = \mathbf{i}\big(\rho - \tfrac{3}{2}\rho^{3/4}\gamma^{1/2}\big) \tag{5.20}$$

and thus

$$\rho = \rho_\Xi := \Big(\frac{3\gamma^{1/2}}{2}\Big)^4 = \Big(\frac{9\gamma}{4}\Big)^2 = \begin{cases} 9, & \Xi \in \{\mathrm{br}, \mathrm{ex}, \mathrm{br}+\}, \\ 9/4, & \Xi \in \{\mathrm{bm}, \mathrm{me}, \mathrm{dm}, \mathrm{bm}+\}. \end{cases} \tag{5.21}$$

With this choice of ρ, we find from (5.15) and (5.20) that the value at the saddle point is

$$\varphi_1(s_0,0) = \rho - 2\rho^{3/4}\gamma^{1/2} = -\frac{\rho}{3} = \begin{cases} -3, & \Xi \in \{\mathrm{br}, \mathrm{ex}, \mathrm{br}+\}, \\ -3/4, & \Xi \in \{\mathrm{bm}, \mathrm{me}, \mathrm{dm}, \mathrm{bm}+\}. \end{cases} \tag{5.22}$$

This yields the constant coefficient in the exponent of the asymptotics. We denote this value by $-b = -b_\Xi$, and have thus, using (5.20),

$$\rho = 3b, \qquad\qquad \rho^{3/4}\gamma^{1/2} = 2b. \tag{5.23}$$

Further, by (5.23) and (5.22)

$$s_0 = \rho^{3/4}\gamma^{-1/2} = 2b\gamma^{-1} = \begin{cases} 9/2, & \Xi \in \{\mathrm{br}, \mathrm{ex}, \mathrm{br}+\}, \\ 9/4, & \Xi \in \{\mathrm{bm}, \mathrm{me}, \mathrm{dm}, \mathrm{bm}+\}. \end{cases} \tag{5.24}$$

The significant part of the integrals in (5.13) and (5.17) comes from the square

$$Q := \big\{(s,\theta) : |s - s_0| \le \log x/x,\ |\theta| \le \log x/x\big\} \tag{5.25}$$

around the saddle point, as we will see in Lemma 5.1 below. We consider first this square.

By (5.14), (5.18) and (5.2)–(5.8), uniformly for $(s,\theta) \in Q$, as $x \to \infty$,

$$F_1(s,\theta;x) = \begin{cases} \frac{1+o(1)}{4\pi^{3/2}} s_0^{-5/3} h_\Xi^0(x^{4/3} s_0^{2/3}), & \Xi \in \{\mathrm{br}, \mathrm{ex}, \mathrm{me}, \mathrm{br}+\}, \\ \frac{1+o(1)}{4\pi^2} s_0^{-4/3} h_\Xi^0(x^{4/3} s_0^{2/3}), & \Xi \in \{\mathrm{bm}, \mathrm{dm}, \mathrm{bm}+\}. \end{cases} \tag{5.26}$$

For the exponential part, we let $s = s_0(1 + u/x)$ and $\theta = 2v/x$, and note that Q corresponds to

$$Q' := \big\{(u,v) : |u| \le (\log x)/s_0,\ |\theta| \le (\log x)/2\big\}. \tag{5.27}$$

A Taylor expansion yields, for $(u,v) \in Q'$, after straightforward computations,

$$\varphi_1(s,\theta) = -b - 2bu^2 x^{-2} + 2\mathbf{i}buvx^{-2} - bv^2 x^{-2} + O\big((|u|^3 + |v|^3)x^{-3}\big). \tag{5.28}$$

Hence,

$$\begin{aligned}\iint_Q e^{x^2\varphi_1(s,\theta)}\,\mathrm{d}s\,\mathrm{d}\theta &= 2s_0x^{-2}\iint_{Q'} e^{-bx^2-2bu^2+2\mathbf{i}buv-bv^2+o(1)}\,\mathrm{d}u\,\mathrm{d}v\\ &= 2s_0x^{-2}e^{-bx^2}\biggl(\int_0^\infty\int_0^\infty e^{-2bu^2+2\mathbf{i}buv-bv^2+o(1)}\,\mathrm{d}u\,\mathrm{d}v+o(1)\biggr)\\ &= 2s_0x^{-2}e^{-bx^2}\biggl(\pi\begin{vmatrix}2b & -\mathbf{i}b\\ -\mathbf{i}b & b\end{vmatrix}^{-1/2}+o(1)\biggr)\\ &\sim \frac{2s_0\pi}{\sqrt{3}\,b}x^{-2}e^{-bx^2}.\end{aligned}\tag{5.29}$$

Further, $\iint_Q\bigl|e^{x^2\varphi_1(s,\theta)}\bigr|\,\mathrm{d}s\,\mathrm{d}\theta$ is of the same order. Consequently, if we write

$$G_1(s,\theta;x) := F_1(s,\theta;x)e^{x^2\varphi_1(s,\theta)},$$

then (5.29) and (5.26) yield

$$\iint_Q G_1(s,\theta;x)\,\mathrm{d}s\,\mathrm{d}\theta = \begin{cases}\frac{1+o(1)}{2\sqrt{3\pi}\,b}s_0^{-2/3}h^0_\Xi(x^{4/3}s_0^{2/3})x^{-2}e^{-bx^2}, & \Xi\in\{\mathrm{br},\mathrm{ex},\mathrm{me},\mathrm{br}+\},\\ \frac{1+o(1)}{2\sqrt{3}\,\pi b}s_0^{-1/3}h^0_\Xi(x^{4/3}s_0^{2/3})x^{-2}e^{-bx^2}, & \Xi\in\{\mathrm{bm},\mathrm{dm},\mathrm{bm}+\}.\end{cases}\tag{5.30}$$

For the complement $Q^c := (0,\infty)\times(-\pi/2,\pi/2)\setminus Q$, we have the following.

Lemma 5.1. *For every $N<\infty$, for large x,*

$$\iint_{Q^c}|G_1(s,\theta;x)|\,\mathrm{d}s\,\mathrm{d}\theta = O\Bigl(x^{-N}e^{-bx^2}\Bigr).$$

We postpone the proof and find from (5.13), (5.17) and (5.30), using (5.23),

$$f^*(x)\sim\begin{cases}\frac{\sqrt{3}\rho}{2\sqrt{\pi}}s_0^{-2/3}h^0_\Xi(x^{4/3}s_0^{2/3})x^{2/3}e^{-bx^2}, & \Xi\in\{\mathrm{br},\mathrm{ex},\mathrm{me},\mathrm{br}+\},\\ \frac{\sqrt{3\rho}}{2\pi}s_0^{-1/3}h^0_\Xi(x^{4/3}s_0^{2/3})x^{1/3}e^{-bx^2}, & \Xi\in\{\mathrm{bm},\mathrm{dm},\mathrm{bm}+\}.\end{cases}$$

Substituting the functions h^0_Ξ implicit in (5.2)–(5.8) and the values of ρ, b and s_0 given in

(5.21)–(5.24), we finally find

$$f^*_{\mathrm{br}}(z) \sim \frac{\sqrt{3}\,\rho}{\sqrt{\pi}} s_0^{-1} e^{-bx^2} = \frac{2\sqrt{3}}{\sqrt{\pi}} e^{-3x^2}, \tag{5.31}$$

$$f^*_{\mathrm{ex}}(z) \sim \frac{4\sqrt{3}\,\rho}{\sqrt{\pi}} x^2 e^{-bx^2} = \frac{36\sqrt{3}}{\sqrt{\pi}} x^2 e^{-3x^2}, \tag{5.32}$$

$$f^*_{\mathrm{bm}}(z) \sim \frac{\sqrt{3\rho}}{\sqrt{\pi}} s_0^{-1/2} e^{-bx^2} = \frac{\sqrt{3}}{\sqrt{\pi}} e^{-3x^2/4}, \tag{5.33}$$

$$f^*_{\mathrm{me}}(z) \sim \sqrt{3}\,\rho s_0^{-1/2} x e^{-bx^2} = \frac{3\sqrt{3}}{2} x e^{-3x^2/4}, \tag{5.34}$$

$$f^*_{\mathrm{dm}}(z) \sim 2\frac{\sqrt{3\rho}}{\sqrt{\pi}} s_0^{-1/2} e^{-bx^2} = \frac{2\sqrt{3}}{\sqrt{\pi}} e^{-3x^2/4}, \tag{5.35}$$

$$f^*_{\mathrm{br}+}(z) \sim \frac{\sqrt{3}\,\rho}{2\sqrt{\pi}} s_0^{-1} e^{-bx^2} = \frac{\sqrt{3}}{\sqrt{\pi}} e^{-3x^2}, \tag{5.36}$$

$$f^*_{\mathrm{bm}+}(z) \sim \frac{\sqrt{3\rho}}{2\sqrt{\pi}} s_0^{-1/2} e^{-bx^2} = \frac{\sqrt{3}}{2\sqrt{\pi}} e^{-3x^2/4}. \tag{5.37}$$

Recall that these are the densities of $\sqrt{2}\,\mathcal{B}_\Xi$. The density of $\mathcal{B}_\Xi$ is $f_\Xi(x) = \sqrt{2}\, f^*_\Xi(\sqrt{2}\,x)$, and we obtain the leading term of the asymptotics in Theorems 1.1–1.7. The leading terms of the asymptotics for $\mathbb{P}(\mathcal{B}_\Xi > x)$ follow by integration by parts, as discussed in Section 2.

It remains to prove Lemma 5.1. We begin by observing that by (3.7), (5.1) and (5.2)–(5.8),

$$|h(z)| = O\bigl(|z| + |z|^{-1}\bigr), \qquad |\arg z| < \pi/6. \tag{5.38}$$

Hence (5.14) and (5.18) show that, with some margin,

$$|F_1(s,\theta;x)| \le C_1\bigl(x^2 s^{-1} + x^{-2} s^{-3}\bigr)(\cos\theta)^{-3}. \tag{5.39}$$

and thus by (5.16), for $x \ge 1$,

$$|G_1(s,\theta;x)| \le C_2 x^2 (\cos\theta)^{-3}\bigl(s^{-1} + s^{-3}\bigr) e^{\rho x^2 - x^2 A(\theta) s^{-1} - x^2 B(\theta) s}, \tag{5.40}$$

where $A(\theta) = \rho^{3/2}(\cos\theta)^{-1/2}$ and $B(\theta) := \gamma\cos(\theta/2)$. We integrate over s, using the following lemma.

Lemma 5.2. *Let $M \ge 0$.*

(i) *If A and B are positive numbers and $AB \ge 1$, then*

$$\int_0^\infty s^{-M-1} e^{-As^{-1} - Bs}\,\mathrm{d}s \le C_3(M)(B/A)^{M/2} e^{-2\sqrt{AB}}. \tag{5.41}$$

(ii) *If further $0 < \delta < 1$, then*

$$\int_{\left|s - \sqrt{A/B}\right| > \delta\sqrt{A/B},\, s>0} s^{-M-1} e^{-As^{-1} - Bs}\,\mathrm{d}s \le C_4(M)(B/A)^{M/2} e^{-(2+\delta^2/2)\sqrt{AB}}. \tag{5.42}$$

Proof. (i): The change of variables $s = \sqrt{A/B}\,t$ followed by $t \mapsto t^{-1}$ for $t > 1$ yields

$$\begin{aligned}\int_0^\infty s^{-M} e^{-As^{-1}-Bs}\,\frac{\mathrm{d}s}{s} &= (B/A)^{M/2}\int_0^\infty t^{-M} e^{-\sqrt{AB}(t^{-1}+t)}\,\frac{\mathrm{d}t}{t} \\ &= (B/A)^{M/2}\int_0^1 \big(t^{-M}+t^{M}\big) e^{-\sqrt{AB}(t^{-1}+t)}\,\frac{\mathrm{d}t}{t}.\end{aligned} \tag{5.43}$$

For $t \in (\frac{1}{6}, 1)$ we write $t = 1-u$ and use $(1-u)^{-1} + 1 - u \geq 2 + u^2$; hence the integral over $(\frac{1}{6}, 1)$ is bounded by

$$C_5(M)\int_0^\infty e^{-\sqrt{AB}(2+u^2)}\,\mathrm{d}u \leq C_6(M) e^{-2\sqrt{AB}}$$

For $t \in (0, \frac{1}{6})$ we use

$$t^{-M-1} e^{-\sqrt{AB}\,t^{-1}/2} \leq C_7(M)(AB)^{-(M+1)/2} \leq C_7(M);$$

hence the integral over $(0, \frac{1}{6})$ is bounded by

$$C_7(M)\int_0^{1/6} e^{-\sqrt{AB}\,t^{-1}/2}\,\mathrm{d}t \leq C_7(M) e^{-3\sqrt{AB}}. \tag{5.44}$$

(ii): Arguing as in (5.43), we see that the integral is bounded by

$$(B/A)^{M/2}\int_0^{1/(1+\delta)} 2t^{-M} e^{-\sqrt{AB}(t^{-1}+t)}\,\frac{\mathrm{d}t}{t}.$$

The integral over $(0, 1/6)$ is bounded by (5.44), and the integral over $(1/6, 1/(1+\delta))$ by

$$C_8(M) e^{-\sqrt{AB}\big(1+\delta+1/(1+\delta)\big)} \leq C_8(M) e^{-\sqrt{AB}(2+\delta^2/2)}. \qquad \square$$

Proof of Lemma 5.1. Returning to (5.40), we have $B(\theta)/A(\theta) \leq \gamma/\rho^{2/3}$ and

$$A(\theta)B(\theta) = \rho^{2/3}\gamma(\cos\theta)^{-1/2}\cos(\theta/2). \tag{5.45}$$

Noting that $\rho^{2/3}\gamma = (2b)^2$ by (5.23) and

$$\frac{\cos(\theta/2)^2}{\cos\theta} = \frac{\cos\theta + 1}{2\cos\theta} = \frac{1}{2} + \frac{1}{2\cos\theta} \geq 1 + c_1\theta^2, \qquad |\theta| < \pi/2,$$

we see that

$$\sqrt{A(\theta)B(\theta)} \geq 2b + c_2\theta^2. \tag{5.46}$$

Hence Lemma 5.2 applies with $A = x^2 A(\theta)$ and $B = x^2 B(\theta)$ when $x^2 \geq 1/(2b)$ and shows, using (5.40), that for every θ with $|\theta| < \pi/2$,

$$\begin{aligned}\int_0^\infty |G_1(s,\theta;x)|\,\mathrm{d}s &\leq C_9 x^2 (\cos\theta)^{-3} e^{\rho x^2 - 4bx^2 - 2c_2 x^2\theta^2} \\ &= C_9 x^2 (\cos\theta)^{-3} e^{-bx^2 - c_3 x^2\theta^2}.\end{aligned} \tag{5.47}$$

For $|\theta|$ close to $\pi/2$, we use instead of (5.46)

$$\sqrt{A(\theta)B(\theta)} \geq c_4(\cos\theta)^{-1/4}, \tag{5.48}$$

another consequence of (5.45). Hence, (5.40) and Lemma 5.2(i) show that if $\varepsilon > 0$ is small enough, and $|\theta| > \pi/2 - \varepsilon$, then

$$\begin{aligned}\int_0^\infty |G_1(s,\theta;x)|\,\mathrm{d}s &\leq C_{10}x^2(\cos\theta)^{-3}e^{\rho x^2 - c_4x^2(\cos\theta)^{-1/4}} \\ &\leq C_{11}e^{-bx^2-c_4x^2/2}.\end{aligned} \tag{5.49}$$

Moreover, (5.45) implies that if $|\theta| \leq 1$, say, then

$$\sqrt{A(\theta)/B(\theta)} = s_0 + O(\theta^2).$$

Hence, if $|\theta| \leq (\log x)/x$ and $|s - s_0| > (\log x)/x$, then, for large x,

$$\left|s - \sqrt{A(\theta)/B(\theta)}\right| > \frac{\log x}{2x} > c_5\frac{\log x}{x}\sqrt{A(\theta)/B(\theta)},$$

and Lemma 5.2(ii) implies, using (5.46), that if $|\theta| \leq (\log x)/x$, then

$$\begin{aligned}\int_{|s-s_0|>\log x/x,\, s>0} |G_1(s,\theta;x)|\,\mathrm{d}s &\leq C_{12}x^2e^{\rho x^2 - 4bx^2 - c_6(\log x)^2} \\ &\leq C_{13}e^{-bx^2-c_7(\log x)^2}.\end{aligned} \tag{5.50}$$

The lemma follows by using (5.50) for $|\theta| \leq (\log x)/x$, (5.49) for $|\theta| > \pi/2 - \varepsilon$, and (5.47) for the remaining θ, and integrating over θ. □

6 Higher order terms

The asymptotics for $f_\Xi(x)$ obtained above can be refined to full asymptotic expansions by standard methods and straightforward, but tedious, calculations. With possible future extensions in view, we find it instructive to present two versions of this; the first is more straightforward brute force, while the second (in the next section) performs a change of variables leading to simpler integrals.

First, the asymptotics (4.27) and (4.28) can be refined into well-known asymptotic series [1, 10.4.63,10.4.66 (with a typo in early printings)]; we write these as

$$\mathrm{Bi}(z) = \pi^{-1/2}z^{-1/4}e^{2z^{3/2}/3}\beta_0(z), \tag{6.1}$$

$$\mathrm{Bi}'(z) = \pi^{-1/2}z^{1/4}e^{2z^{3/2}/3}\beta_1(z), \tag{6.2}$$

with

$$\beta_0(z) = 1 + \frac{5}{48}z^{-3/2} + \frac{385}{4608}z^{-3} + \cdots + O\bigl(z^{-3N/2}\bigr), \qquad |\arg(z)| \leq \pi/3 - \delta, \tag{6.3}$$

$$\beta_1(z) = 1 - \frac{7}{48}z^{-3/2} - \frac{455}{4608}z^{-3} + \cdots + O\bigl(z^{-3N/2}\bigr), \qquad |\arg(z)| \leq \pi/3 - \delta, \tag{6.4}$$

where the expansions can be continued to any desired power N of $z^{3/2}$. Similarly, (4.29) can be refined to an asymptotic series

$$\mathrm{BI}(z) = \pi^{-1/2} z^{-3/4} e^{2z^{3/2}/3} \beta_{-1}(z), \tag{6.5}$$

with

$$\beta_{-1}(z) = 1 + \frac{41}{48} z^{-3/2} + \frac{9241}{4608} z^{-3} + \cdots + O\big(z^{-3N/2}\big), \qquad |\arg(z)| \le \pi/3 - \delta; \tag{6.6}$$

this is easily verified by writing (4.13) as $\mathrm{BI}(z) = \mathrm{BI}(1) + \int_1^z t^{-1}\mathrm{Bi}''(t)\,\mathrm{d}t$ followed by repeated integrations by parts, as in corresponding argument for $\mathrm{AI}(z)$ in [10, Appendix A]. The coefficients in (6.6) are easily found noting that a formal differentiation of (6.5) yields (6.1). (They are the numbers denoted β_k in [10].)

Hence, by (4.30)–(4.36), (5.9) can be refined to, for $|\arg z| \le \pi/6$,

$$h_\Xi(z) = \mathbf{i} h^0_\Xi(z)\Big(h^1_\Xi(z) + O\left(e^{-2z^{3/2}/3}\right)\Big), \tag{6.7}$$

where

$$h^1_{\mathrm{br}}(z) := \beta_1(z)^{-2} = 1 + \frac{7}{24} z^{-3/2} + \dots, \tag{6.8}$$

$$h^1_{\mathrm{ex}}(z) := \beta_1(z)\beta_0(z)^{-3} = 1 - \frac{11}{24} z^{-3/2} + \dots, \tag{6.9}$$

$$h^1_{\mathrm{bm}}(z) := \beta_1(z)^{-1} = 1 + \frac{7}{48} z^{-3/2} + \dots, \tag{6.10}$$

$$h^1_{\mathrm{me}}(z) := \beta_0(z)^{-1} = 1 - \frac{5}{48} z^{-3/2} + \dots, \tag{6.11}$$

$$h^1_{\mathrm{dm}}(z) := \beta_{-1}\beta_0(z)^{-2} = 1 + \frac{31}{48} z^{-3/2} + \dots, \tag{6.12}$$

$$h^1_{\mathrm{br}+}(z) := \big((\beta_0(z) + \beta_1(z))/2\big)^{-2} = 1 + \frac{1}{24} z^{-3/2} + \dots, \tag{6.13}$$

$$h^1_{\mathrm{bm}+}(z) := \big((\beta_0(z) + \beta_1(z))/2\big)^{-1} = 1 + \frac{1}{48} z^{-3/2} + \dots \tag{6.14}$$

By (6.3) and (6.4), h^1_Ξ has an asymptotic series expansion

$$h^1_\Xi(z) = 1 + d^\Xi_1 z^{-3/2} + d^\Xi_2 z^{-3} + \cdots + O\big(z^{-3N/2}\big), \qquad |\arg(z)| \le \pi/3 - \delta, \tag{6.15}$$

for some readily computed coefficients d^Ξ_k; moreover, we can clearly ignore the O term in (6.7).

Next, by Lemma 5.1, it suffices to consider $(s,\theta) \in Q$ in (5.13) and (5.17). We use (6.7) and (6.15) in (5.14) and (5.18) and obtain, for example, for $\Xi = \mathrm{ex}$,

$$F_1(s,\theta;x) = \frac{2x^{4/3}}{\pi^{3/2}}\Big(e^{i\theta}(\sec\theta)^3 s^{-1} - \frac{11}{24} e^{i\theta/2}(\sec\theta)^3 s^{-2} x^{-2} + \cdots + O(x^{-2N})\Big). \tag{6.16}$$

We substitute $s = s_0(1 + u/x)$ and $\theta = 2v/x$ as above and obtain by Taylor expansions a series in x^{-1} (with a prefactor $x^{4/3}$) where the coefficients are polynomials in u and v.

Similarly, the Taylor expansion (5.28) can be continued; we write the remainder term as $R(u,v;x)$ and have

$$R(u,v;x) = r_3(u,v)x^{-3} + \cdots + O(x^{-2N-2}), \tag{6.17}$$

for some polynomials $r_k(u,v)$. Another Taylor expansion then yields

$$e^{x^2R(u,v;x)} = r_1^*(u,v)x^{-1} + \cdots + O(x^{-2N}), \tag{6.18}$$

for some polynomials $r_k^*(u,v)$. We multiply this, the expansion of F_1 and the main term $\exp(-bx^2 - 2bu^2 + 2\mathbf{i}buv - bv^2)$, and integrate over Q; we may extend the integration domain to $\mathbb{R}^2$ with a negligible error. This yields an asymptotic expansion for $f_\Xi^*(x)$, and thus for $f_\Xi(x)$, where the leading term found above is multiplied by a series in x^{-1}, up to any desired power. Furthermore, it is easily seen that all coefficients for odd powers of x^{-1} vanish, since they are given by the integrals of an odd functions of u and v; hence this is really an asymptotic series in x^{-2}.

We obtain the explicit expansions for $f_\Xi(x)$ in Theorems 1.1–1.7 by calculations with `Maple`. The asymptotics for $\mathbb{P}(\mathcal{B}_\Xi > x)$ follow by integration by parts, see Section 2.

Remark 6.1. In particular, since $h^0_{\mathrm{br}+} = \frac12 h^0_{\mathrm{br}}$ and $h^0_{\mathrm{bm}+} = \frac12 h^0_{\mathrm{bm}}$, the leading terms for br+ and bm+ differ from those of br and bm by a factor $\frac12$ as discussed in Remark 1.8. The second order terms in h^1 are different, as is seen above; more precisely, $h^1_{\mathrm{br}+} = h^1_{\mathrm{br}} - \frac14 z^{-3/2} + O(z^{-3})$ and $h^1_{\mathrm{bm}+} = h^1_{\mathrm{bm}} - \frac18 z^{-3/2} + O(z^{-3})$; it is easily seen that if we ignore terms beyond the second, this difference transfers into factors $1-(4s_0)^{-1}x^{-2}$ and $1-(8s_0)^{-1}x^{-2}$, respectively, for f_Ξ^*, which in both cases equals $1-\frac{1}{18}x^{-2}$, and thus a factor $1-\frac{1}{36}x^{-2}$ for f_Ξ, which explains the difference between the second order terms in f_{br} or f_{bm} and $2f_{\mathrm{br}+}$ or $2f_{\mathrm{bm}+}$; cf. again Remark 1.8.

7 Higher order terms, version II

Our second version of the saddle point method leads to simpler calculations (see for instance, Bleistein and Handelsman [2]). We illustrate it with $\mathcal{B}_{\mathrm{ex}}$; the other Brownian areas are treated similarly. We use again (5.13), and recall that by Lemma 5.1, it suffices to consider (s,θ) close to $(s_0,0) = (9/2,0)$.

We make first the substitution $s = \frac92(\sec\theta)^{3/2}u^{-1}$ (this is not necessary, but makes the integral more similar to Tolmatz' versions). This transforms (5.13) into

$$f^*_{\mathrm{ex}}(x) = \int_{\theta=-\pi/2}^{\pi/2}\int_{u=0}^{\infty} F_2(u,\theta;x)e^{x^2\varphi_2(u,\theta)}\,\mathrm{d}u\,\mathrm{d}\theta, \tag{7.1}$$

where, by (5.14), (5.15), (5.21), (6.7), (5.3), (6.9), for u bounded, at least,

$$\begin{aligned}
F_2(u,\theta;x) &= \frac{81x^{8/3}}{4\pi^{3/2}\mathbf{i}}e^{2\mathbf{i}\theta/3}(\sec\theta)^2\left(\tfrac92\right)^{-2/3}u^{-1/3}h_{\mathrm{ex}}\Bigl(\left(\tfrac92\right)^{2/3}x^{4/3}u^{-2/3}e^{\mathbf{i}\theta/3}\sec\theta\Bigr),\\
&= \frac{162x^4e^{\mathbf{i}\theta}}{\pi^{3/2}u(\cos\theta)^3}h^1_{\mathrm{ex}}\Bigl(\left(\tfrac92\right)^{2/3}x^{4/3}u^{-2/3}e^{\mathbf{i}\theta/3}\sec\theta\Bigr) \qquad (7.2)\\
&= \frac{162x^4e^{i\theta}}{\pi^{3/2}u(\cos\theta)^3} - \frac{33x^2e^{\mathbf{i}\theta/2}}{2\pi^{3/2}(\cos\theta)^{3/2}} + O(1), \qquad (7.3)\\
\varphi_2(u,\theta) &= 9(1+\mathbf{i}\tan\theta) - 6ue^{\mathbf{i}\theta} - \frac{6e^{-\mathbf{i}\theta}(1+\mathbf{i}\tan\theta)^{3/2}}{u}. \qquad (7.4)
\end{aligned}$$

The saddle point is now $(u,\theta)=(1,0)$, and in a neighbourhood we have, cf. (5.28), with $v=u-1$,

$$\varphi_2(u,\theta) = -3 - 6v^2 - 3\mathbf{i}v\theta - \tfrac{3}{4}\theta^2 + O\bigl(|v|^3+|\theta|^3\bigr). \tag{7.5}$$

The function φ_2 has a non-degenerate critical point at $(1,0)$, and by the Morse lemma, see e.g. Milnor [18], Lemma 2.2, we can make a complex analytic change of variables in a neighbourhood of $(1,0)$ such that in the new variables φ_2+3 becomes a diagonal quadratic form. (The Morse lemma is usually stated for real variables, but the standard proof in e.g. [18] applies to the complex case too.) The quadratic part of (7.5) is diagonalized by $(\tilde v,\theta)$ with $v=\tilde v-\mathbf{i}\theta/4$; we may thus choose the new variables $\tilde u$ and $\tilde\theta$ such that $\tilde u\sim\tilde v$ and $\tilde\theta\sim\theta$ at the critical point, and thus

$$u = 1+\tilde u - \mathbf{i}\tilde\theta/4 + O\bigl(|\tilde u|^2+|\tilde\theta|^2\bigr), \tag{7.6}$$

$$\theta = \tilde\theta + O\bigl(|\tilde u|^2+|\tilde\theta|^2\bigr), \tag{7.7}$$

$$\varphi_2(u,\theta) = -3 - 6\tilde u^2 - \tfrac{9}{8}\tilde\theta^2. \tag{7.8}$$

Note that the new coordinates are not uniquely determined; we will later use this and simplify by letting some Taylor coefficients be 0. In the new coordinates, (7.1) yields,

$$f^*_{\mathrm{ex}}(x) \sim \int_{\tilde\theta}\int_{\tilde u} F_3(\tilde u,\tilde\theta;x) e^{-3x^2-x^2(6\tilde u^2+\frac98\tilde\theta^2)} J(\tilde u,\tilde\theta)\,\mathrm{d}\tilde u\,\mathrm{d}\tilde\theta, \tag{7.9}$$

where F_3 is obtained by substituting $u=u(\tilde u,\tilde\theta)$ and $\theta=\theta(\tilde u,\tilde\theta)$ in (7.2) and $J(\tilde u,\tilde\theta)=\frac{\partial u}{\partial\tilde u}\frac{\partial\theta}{\partial\tilde\theta}-\frac{\partial u}{\partial\tilde\theta}\frac{\partial\theta}{\partial\tilde u}$ is the Jacobian. Recall that, up to a negligible error, we only have to integrate in (7.1) over a small disc, say with radius $\log x/x$; this becomes in the new coordinates a surface in $\mathbb{C}^2$ as the integration domain in (7.9). The next step is to replace this integration domain by, for example, the disc $\{(\tilde u,\tilde\theta)\in\mathbb{R}^2 : |\tilde u|^2+|\tilde\theta|^2\le(\log x/x)^2\}$, in analogy with the much more standard change of integration contour in one complex variable. To verify the change of integration domain, note that if $F(z_1,z_2)$ is any analytic function of two complex variables, then $F(z_1,z_2)\,\mathrm{d}z_1\wedge\mathrm{d}z_2$ is a closed differential form in $\mathbb{C}^2$ (regarded as a real manifold of dimension four), and thus $\int_{\partial M}F(z_1,z_2)\,\mathrm{d}z_1\wedge\mathrm{d}z_2=0$ by Stokes' theorem for any compact submanifold M with boundary ∂M. In our case, it follows that the difference between the integrals over the two domains equals an integral over boundary terms at a distance $\asymp\log x/x$ from the origin, which is negligible. (The careful reader may parametrize the two domains by suitable mappings $\psi_0,\psi_1:U\to\mathbb{C}^2$, where U is the unit disc in $\mathbb{R}^2$, and apply Stokes' theorem to the cylinder $U\times[0,1]$ and the pullback of $F(z_1,z_2)\,\mathrm{d}z_1\wedge\mathrm{d}z_2$ by the map $(w,t)\mapsto(1-t)\psi_0(w)+t\psi_1(w)$.)

We next change variable again to $w=x\tilde u$, $t=x\tilde\theta$, and obtain by (7.9)

$$f^*_{\mathrm{ex}}(x) \sim x^{-2}e^{-3x^2}\iint F_3(w/x,t/x;x)J(w/x,t/x)e^{-6w^2-\frac98t^2}\,\mathrm{d}w\,\mathrm{d}t, \tag{7.10}$$

integrating over $(w,t)\in\mathbb{R}^2$ with, say, $w^2+t^2\le(\log x)^2$. To obtain the desired asymptotics for f^*_{ex}, and thus for f_{ex}, we mechanically expand F_3 and J in Taylor series up to any desired order and compute the resulting Gaussian integrals, extending the integration domains to $\mathbb{R}^2$.

We illustrate this by giving the details for the first two terms in (1.2). We have, cf. (7.6) and (7.7), expansions

$$u = 1+\tilde u - \mathbf{i}\tilde\theta/4 + \alpha_1\tilde u^2 + \alpha_2\tilde u\tilde\theta + O(|\tilde u|^3+|\tilde\theta|^3),$$
$$\theta = \tilde\theta + \alpha_3\tilde\theta^2 + \alpha_4\tilde u\tilde\theta + O(|\tilde u|^3+|\tilde\theta|^3),$$

where we, as we may, have chosen two Taylor coefficients to be 0. To determine $\alpha_1, \dots, \alpha_4$, we substitute into $\varphi_2(u,\theta)$. We obtain from (7.4), up to terms of order three,

$$\varphi_2(u,\theta) \sim -3 - [6\tilde{u}^2 + 9\tilde{\theta}^2/8] + (6 - 12\alpha_1)\tilde{u}^3 + (-3\mathbf{i}\alpha_4 - 12\alpha_2 - 15\mathbf{i}/2)\tilde{\theta}\tilde{u}^2 + \left(-\frac{9\alpha_4}{4} - 3\mathbf{i}\alpha_3 + \frac{33}{8}\right)\tilde{\theta}^2\tilde{u} + \left(-\frac{9\alpha_3}{4} + \frac{15\mathbf{i}}{32}\right)\tilde{\theta}^3.$$

Annihilating the coefficients, cf. (7.8), leads to a linear system, the solution of which is

$$\alpha_1 = 1/2, \qquad \alpha_2 = -83\mathbf{i}/72, \qquad \alpha_3 = 5\mathbf{i}/24, \qquad \alpha_4 = 19/9.$$

This leads to the Jacobian

$$J(\tilde{u},\tilde{\theta}) = 1 + \left(-\frac{5\mathbf{i}\tilde{\theta}}{24} + \frac{28\tilde{u}}{9}\right) + O\big(\tilde{\theta}^2 + \tilde{u}^2\big).$$

Furthermore, by (7.3), with $w = x\tilde{u}$ and $t = x\tilde{\theta}$,

$$F_3(\tilde{u},\tilde{\theta};x) = F_2(u,\theta;x) \sim \frac{162x^4}{\pi^{3/2}} + \frac{162x^3}{\pi^{3/2}}\Big(\frac{5}{4}\mathbf{i}t - w\Big) + O\big(x^2(1 + w^2 + t^2)\big).$$

Integrating in (7.10) yields the leading term

$$f^*_{\mathrm{ex}}(x) \sim \frac{36\sqrt{3}}{\pi^{1/2}}\, x^2 e^{-3x^2} \tag{7.11}$$

together with correction terms of order xe^{-3x^2} that all vanish by symmetry, since they involve integrals of odd functions, plus a remainder term of order e^{-3x^2}.

The next term in the expansion of $e^{3x^2} f^*_{\mathrm{ex}}$ is thus the constant term. To find it, we try, again setting some Taylor coefficients to 0 as we may,

$$\begin{aligned} u &\sim 1 + (\tilde{u} - \mathbf{i}\tilde{\theta}/4) + \tilde{u}(\alpha_1\tilde{u} + \alpha_2\tilde{\theta}) + \tilde{u}(\beta_1\tilde{u}^2 + \beta_2\tilde{u}\tilde{\theta} + \beta_3\tilde{\theta}^2), \\ \theta &\sim \tilde{\theta} + \tilde{\theta}(\alpha_3\tilde{\theta} + \alpha_4\tilde{u}) + \tilde{\theta}(\beta_4\tilde{u}^2 + \beta_5\tilde{u}\tilde{\theta} + \beta_6\tilde{\theta}^2). \end{aligned}$$

We obtain now

$$\begin{aligned} \varphi_2(u,\theta) \sim -3 - [6\tilde{u}^2 + 9\tilde{\theta}^2/8] + (3/2 - 12\beta_1)\tilde{u}^4 + (-131\mathbf{i}/6 - 3\mathbf{i}\beta_4 - 12\beta_2)\tilde{u}^3\tilde{\theta} \\ + (-9\beta_4/4 + 2627/288 - 12\beta_3 - 3\mathbf{i}\beta_5)\tilde{u}^2\tilde{\theta}^2 + (-9\beta_5/4 + 535\mathbf{i}/96 - 3\mathbf{i}\beta_6)\tilde{u}\tilde{\theta}^3 \\ + (-1283/768 - 9\beta_6/4)\tilde{\theta}^4. \end{aligned}$$

We set for instance $\beta_4 = 0$. This gives

$$\beta_1 = 1/8, \beta_2 = -131\mathbf{i}/72, \beta_3 = 16867/10368, \beta_5 = 4493\mathbf{i}/1296, \beta_6 = -1283/1728.$$

The Jacobian becomes

$$J(\tilde{u},\tilde{\theta}) \sim 1 + \left(\frac{28\tilde{u}}{9} - \frac{5\mathbf{i}\tilde{\theta}}{24}\right) + \left(\frac{179}{72}\tilde{u}^2 + \frac{2405}{648}\mathbf{i}\tilde{u}\tilde{\theta} - \frac{379}{384}\tilde{\theta}^2\right).$$

The first term in (7.3) becomes

$$\sim \frac{162x^4}{\pi^{3/2}} + \frac{162x^4}{\pi^{3/2}}\Big(\frac{5}{4}\mathbf{i}\tilde{\theta} - \tilde{u}\Big) + \frac{x^4}{\pi^{3/2}}\Big(81\tilde{u}^2 + \frac{1143}{4}\mathbf{i}\tilde{u}\tilde{\theta} + \frac{621}{8}\tilde{\theta}^2\Big).$$

and the second is

$$\sim -\frac{33x^2}{2\pi^{3/2}}.$$

Collecting terms, the coefficient of x^2 in $F_3(w/x, t/x)J(w/x, t/x)$ equals

$$-\frac{1296w^2 - 99248\mathbf{i}wt + 2565t^2}{64\pi^{3/2}} - \frac{33}{2\pi^{3/2}}.$$

Multiplying by $\exp(-6w^2 - \frac{9}{8}t^2)$ and integrating yields the contribution

$$x^2\frac{-8\sqrt{3}}{\pi^{1/2}} \tag{7.12}$$

to the integral in (7.10). So, finally, combining (7.11) and (7.12),

$$f^*_{\mathrm{ex}}(x) \sim \frac{3^{1/2}e^{-3x^2}}{\pi^{1/2}}\left[36x^2 - 8\right],$$

which fits with the first two terms for $f_{\mathrm{ex}}(x)$ in Theorem 1.1. More terms can be found in a mechanical way.

8 Proof of Theorem 3.1

Let $T \sim \Gamma(\nu)$ be a Gamma distributed random variable independent of X and let $X_T := T^{3/2}X$. Then T has the density $\Gamma(\nu)^{-1}t^{\nu-1}e^{-t}$, $t > 0$, and thus X_T has, using (3.1), the Laplace transform

$$\begin{aligned}
\psi_T(u) &:= \mathbb{E}\, e^{-uT^{3/2}X} = \mathbb{E}\,\psi\big(uT^{3/2}\big) \\
&= \Gamma(\nu)^{-1}\int_0^\infty \psi\big(ut^{3/2}\big)t^{\nu-1}e^{-t}\,\mathrm{d}t \\
&= \Gamma(\nu)^{-1}\int_0^\infty \psi\big(s^{3/2}\big)u^{-2\nu/3}s^{\nu-1}e^{-u^{-2/3}s}\,\mathrm{d}s \\
&= u^{-2\nu/3}\Psi\big(u^{-2/3}\big), \qquad u > 0.
\end{aligned} \tag{8.1}$$

By (8.1) and our assumption on Ψ, ψ_T extends to an analytic function in $\mathbb{C}\backslash(\infty, 0]$. Furthermore, X_T has a density g on $(0, \infty)$, because $T^{3/2}$ has, and it is easily verified that this density is continuous. We next use Laplace inversion for X_T. The Laplace transform ψ_T is, by (8.1), *not* absolutely integrable on vertical lines (at least not in our cases, where $\Psi(z)$ is bounded away from 0 as $z \to 0$), so we will use the following form of the Laplace inversion formula, assuming only conditional convergence of the integral.

Lemma 8.1. *Let h be a measurable function on $\mathbb{R}$. Suppose that the Laplace transform $\tilde{h}(z) := \int_{-\infty}^{\infty} h(y)e^{-zy}\,\mathrm{d}y$ exists in a strip $a < \operatorname{Re} z < b$, and that $\sigma \in (a,b)$ is a real number such that the generalized integral $\int_{\sigma-\mathbf{i}\infty}^{\sigma+\mathbf{i}\infty} e^{xz}\tilde{h}(z)\,\mathrm{d}z$ exists in the sense that the limit $\lim_{A\to\infty} s_A$ exists, where*

$$s_A := \int_{\sigma-\mathbf{i}A}^{\sigma+\mathbf{i}A} e^{xz}\tilde{h}(z)\,\mathrm{d}z.$$

If further x is a continuity point (or, more generally, a Lebesgue point) of h, then

$$\int_{\sigma-\mathbf{i}\infty}^{\sigma+\mathbf{i}\infty} e^{xz}\tilde{h}(z)\,\mathrm{d}z := \lim_{A\to\infty} s_A = 2\pi\mathbf{i}h(x).$$

Proof. By considering instead $e^{-\sigma y}h(y)$, we may suppose that $\sigma = 0$. In this case, h is integrable and $\tilde{h}(\mathbf{i}t) = \hat{h}(t)$, the Fourier transform of h, and the result is a classical result on Fourier inversion. (It is the analogue for Fourier transforms of the more well-known fact that if a Fourier series converges at a continuity (or Lebesgue) point of the function, then the limit equals the function value.) For a proof, note that if s_A converges as $A \to \infty$, then so does the Abel mean $\int_0^\infty ye^{-yA}s_A\,\mathrm{d}A$ as $y \to 0$, and this Abel mean equals $2\pi\mathbf{i}$ times the Poisson integral $\int_{-\infty}^{\infty} \pi^{-1}y(u^2+y^2)^{-1}h(x-u)\,\mathrm{d}u$, which converges to $h(x)$. □

We verify the condition of the lemma with $h = g$ and $\sigma = 1$, recalling that $\tilde{g} = \psi_T$. Thus, by (8.1),

$$s_A := \int_{1-\mathbf{i}A}^{1+\mathbf{i}A} e^{xz}\psi_T(z)\,\mathrm{d}z = \int_{1-\mathbf{i}A}^{1+\mathbf{i}A} e^{xz}z^{-2\nu/3}\Psi(z^{-2/3})\,\mathrm{d}z.$$

We may here change the integration path from the straight line segment $[1-\mathbf{i}A, 1+\mathbf{i}A]$ to the path consisting of the following seven parts:

γ_1: the line segment $[1-\mathbf{i}A, -A-\mathbf{i}A]$,

γ_2: the line segment $[-A-\mathbf{i}A, -A-\mathbf{i}0]$,

γ_3: the line segment $[-A-\mathbf{i}0, -\varepsilon-\mathbf{i}0]$,

γ_4: the circle $\{\varepsilon e^{\mathbf{i}t} : t \in [-\pi,\pi]\}$.

γ_5: the line segment $[-\varepsilon+\mathbf{i}0, -A+\mathbf{i}0]$,

γ_6: the line segment $[-A+\mathbf{i}0, -A+\mathbf{i}A]$,

γ_7: the line segment $[-A+\mathbf{i}A, 1+\mathbf{i}A]$.

(Here, γ_3 could formally be interpreted as the line segment $[-A-\mathbf{i}\eta, -\sqrt{\varepsilon^2-\eta^2}-\mathbf{i}\eta]$ for a small positive η, taking the limit of the integral as $\eta \to 0$, and similarly for the other parts with $\pm\mathbf{i}0$.) Letting $A \to \infty$, we see that we essentially change the integration path from a vertical line to a Hankel contour; however, we do this carefully since, as said above, the integral along the vertical line is not absolutely convergent.

We now first let $\varepsilon \to 0$. By (3.3),

$$\int_{\gamma_4} e^{xz} z^{-2\nu/3} \Psi(z^{-2/3})\,\mathrm{d}z = O\left(\varepsilon^{1-2\nu/3}\right) \to 0,$$

and, again by (3.3), the integrals along γ_3 and γ_5 converge to the absolutely convergent integrals

$$\int_{-A-\mathbf{i}0}^{-\mathbf{i}0} e^{xz} z^{-2\nu/3} \Psi(z^{-2/3})\,\mathrm{d}z = \int_0^A e^{-x\rho} \rho^{-2\nu/3} e^{2\pi\nu\mathbf{i}/3} \Psi\big(e^{2\pi\mathbf{i}/3}\rho^{-2/3}\big)\,\mathrm{d}\rho$$

and

$$\int_{\mathbf{i}0}^{-A+\mathbf{i}0} e^{xz} z^{-2\nu/3} \Psi(z^{-2/3})\,\mathrm{d}z = -\int_0^A e^{-x\rho} \rho^{-2\nu/3} e^{-2\pi\nu\mathbf{i}/3} \Psi\big(e^{-2\pi\mathbf{i}/3}\rho^{-2/3}\big)\,\mathrm{d}\rho,$$

which together make

$$I_A := \int_0^A e^{-x\rho} \rho^{-2\nu/3} \Psi^*\big(\rho^{-2/3}\big)\,\mathrm{d}\rho.$$

Hence, for every $A > 0$,

$$s_A = I_A + \left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_6} + \int_{\gamma_7}\right) e^{xz} z^{-2\nu/3} \Psi(z^{-2/3})\,\mathrm{d}z.$$

Now let $A \to \infty$. By (3.2),

$$\int_{\gamma_1} e^{xz} z^{-2\nu/3} \Psi(z^{-2/3})\,\mathrm{d}z = o\left(\int_{-\infty}^1 e^{xt}\,\mathrm{d}t\right) = o(1),$$

and similarly $\int_{\gamma_2} = o(1)$, $\int_{\gamma_6} = o(1)$, $\int_{\gamma_7} = o(1)$. Finally, $I_A \to I_\infty$, and Lemma 8.1 applies and yields the following.

Lemma 8.2. *For every $x > 0$, we have*

$$g(x) = \frac{1}{2\pi\mathbf{i}} \int_0^\infty e^{-x\rho} \rho^{-2\nu/3} \Psi^*(\rho^{-2/3})\,\mathrm{d}\rho, \tag{8.2}$$

where the integral is absolutely convergent by (3.7) *and* (3.8).

By the change of variables $\rho = u^{-3/2}$, (8.2) may be rewritten as

$$g(x) = \frac{3}{4\pi\mathbf{i}} \int_0^\infty e^{-xu^{-3/2}} u^{\nu-5/2} \Psi^*(u)\,\mathrm{d}u, \qquad x > 0. \tag{8.3}$$

We can here, using (3.7) and (3.8), change the integration path from the positive real axis to the line $\{re^{\mathbf{i}\varphi} : r > 0\}$, for every fixed φ with $|\varphi| < \frac{\pi}{6}$. Consequently, we further have, for $x > 0$ and $|\varphi| < \frac{\pi}{6}$,

$$g(x) = \frac{3}{4\pi\mathbf{i}} e^{(\nu-3/2)\mathbf{i}\varphi} \int_0^\infty \exp\big(-e^{-3\mathbf{i}\varphi/2} x r^{-3/2}\big) r^{\nu-5/2} \Psi^*(re^{\mathbf{i}\varphi})\,\mathrm{d}r. \tag{8.4}$$

The right hand side of (8.4) is an analytic function of x in the sector $\{x : |\arg x - 3\varphi/2| < \pi/2\}$, which contains the positive real axis; together, these thus define an analytic extension of $g(x)$

to the sector $|\arg x| < 3\pi/4$ such that (8.4) holds whenever $|\arg x| < 3\pi/4$, $|\varphi| < \frac{\pi}{6}$ and $|\arg x - 3\varphi/2| < \pi/2$.

We next find the density of X from g by another Laplace inversion. Assume first, for simplicity, that we already know that X has a continuous density f on $(0,\infty)$. Then $t^{3/2}X$ has the density $t^{-3/2}f(t^{-3/2}x)$, and thus (using $t = x^{2/3}s$), for $x > 0$,

$$\begin{aligned} g(x) &= \Gamma(\nu)^{-1}\int_0^\infty t^{\nu-1}e^{-t}t^{-3/2}f\bigl(t^{-3/2}x\bigr)\,\mathrm{d}t \\ &= \Gamma(\nu)^{-1}x^{2\nu/3-1}\int_0^\infty e^{-x^{2/3}s}s^{\nu-5/2}f\bigl(s^{-3/2}\bigr)\,\mathrm{d}s. \end{aligned} \tag{8.5}$$

Let

$$F(s) := s^{\nu-5/2}f\bigl(s^{-3/2}\bigr). \tag{8.6}$$

Then (8.5) can be written, with $x = y^{3/2}$,

$$g(y^{3/2}) = \Gamma(\nu)^{-1}y^{\nu-3/2}\int_0^\infty e^{-ys}F(s)\,\mathrm{d}s, \qquad y > 0. \tag{8.7}$$

In other words, F has the Laplace transform

$$\widetilde{F}(y) := \int_0^\infty e^{-ys}F(s)\,\mathrm{d}s = \Gamma(\nu)y^{3/2-\nu}g(y^{3/2}), \qquad y > 0. \tag{8.8}$$

Since this is finite for all $y > 0$, the Laplace transform $\widetilde{F}$ is analytic in the half-plane $\operatorname{Re} y > 0$. Hence, using our analytic exytension of g to $|\arg z| < 3\pi/4$,, (8.8) holds for all y with $\operatorname{Re} y > 0$. Consequently, by standard Laplace inversion, for every $s > 0$ and every $\xi > 0$ such that the integrals are absolutely (or even conditionally, see Lemma 8.1) convergent,

$$F(s) = \frac{1}{2\pi\mathbf{i}}\int_{\xi-\mathbf{i}\infty}^{\xi+\mathbf{i}\infty} e^{sy}\widetilde{F}(y)\,\mathrm{d}y = \frac{\Gamma(\nu)}{2\pi\mathbf{i}}\int_{\xi-\mathbf{i}\infty}^{\xi+\mathbf{i}\infty} e^{sy}y^{3/2-\nu}g(y^{3/2})\,\mathrm{d}y. \tag{8.9}$$

We have for (mainly notational) simplicity assumed that X has a density. In general, we may replace the density function f in (8.5) and (8.6) by a probability measure μ (with suitable interpretations; we identify here absolutely continuous measures and their densities as in the theory of distributions). Then F is a (positive) measure on $(0,\infty)$, and its Laplace transform is still given by (8.8). The fact, proved below, that $\widetilde{F}$ is absolutely integrable on a vertical line $\operatorname{Re} y = \xi$ implies by standard Fourier analysis that F actually is the continuous function given by (8.9), and thus the measure μ too is a continuous function; i.e., X has a continuous density f as asserted, and (8.5) and (8.6) hold.

We change variables in (8.9) to $\theta := \arg y \in (-\pi/2, \pi/2)$, that is $y = \xi(1 + \mathbf{i}\tan\theta) = \xi\sec(\theta)e^{\mathbf{i}\theta}$. We further express $g(y^{3/2})$ by (8.4) with $\varphi = \theta/3$ (which satisfies the conditions above for (8.4)); this yields, assuming absolute convergence of the double integral,

$$\begin{aligned} F(s) = \frac{3\Gamma(\nu)}{8\pi^2\mathbf{i}}\int_{\theta=-\pi/2}^{\pi/2}\int_{r=0}^{\infty}\exp\Bigl(\xi s(1+\mathbf{i}\tan\theta) - e^{\mathbf{i}\theta}(\xi\sec\theta)^{3/2}r^{-3/2}\Bigr) \\ e^{(1-2\nu/3)\mathbf{i}\theta}\xi^{5/2-\nu}(\sec\theta)^{7/2-\nu}r^{\nu-5/2}\Psi^*(re^{\mathbf{i}\theta/3})\,\mathrm{d}r\,\mathrm{d}\theta. \end{aligned} \tag{8.10}$$

To verify absolute convergence of this double integral, take absolute values inside the integral. Since

$$\operatorname{Re}\bigl(e^{\mathrm{i}\theta}(\xi\sec\theta)^{3/2}r^{-3/2}\bigr) \ge \xi^{3/2}(\sec\theta)^{1/2}r^{-3/2},$$

the resulting integral is, using (3.5) and (3.7), for fixed s and ξ bounded by

$$C_{14}(s,\xi)\int_0^\infty\int_0^\infty e^{-\xi^{3/2}(\sec\theta)^{1/2}r^{-3/2}}(\sec\theta)^{7/2-\nu}r^{\nu-5/2}\min\bigl(r^{-\nu},r^{-6}\bigr)\,\mathrm{d}r\,\mathrm{d}\theta.$$

We split this double integral into the two parts: $0<\theta<1$ and $1<\theta<\infty$. For $0<\theta<1$, $\sec\theta$ is bounded above and below, and it is easy to see that the integral is finite. For $\theta>1$, $\tan\theta<\sec\theta<2\tan\theta$, and with $t=\tan\theta$ we obtain at most

$$C_{15}\int_1^\infty\int_0^\infty e^{-\xi^{3/2}t^{1/2}r^{-3/2}}t^{3/2-\nu}r^{\nu-5/2}\min\bigl(r^{-\nu},r^{-6}\bigr)\,\mathrm{d}r\,\mathrm{d}t.$$

Substituting $t=r^3u$, we find that this is at most

$$C_{15}\int_0^\infty e^{-\xi^{3/2}u^{1/2}}u^{3/2-\nu}\,\mathrm{d}u\int_0^\infty r^{5-2\nu}\min\bigl(r^{-\nu},r^{-6}\bigr)\,\mathrm{d}r<\infty.$$

This verifies absolute convergence of the double integral in (8.10) for every $\xi>0$, which implies absolute convergence of the integrals in (8.9). Consequently, (8.9) and (8.10) are valid for every $s>0$ and $\xi>0$. We now put $s=x^{-2/3}$ in (8.10) and obtain by (8.6) the sought result (3.6).

Remark 8.3. We have chosen $\varphi=\theta/3$, which leads to (3.6) and, see Remark 3.3, the formulas by Tolmatz [26, 27, 28]. Other choices of φ are possible and lead to variations of the inversion formula (3.6). In particular, it may be noted that we may take $\varphi=0$ for, say, $|\theta|<1$; this yields a formula that, apart from a small contribution for $|\theta|>\pi/4$, involves $\Psi^*(x)$ for real x only. However, we do not find that this or any other variation of (3.6) simplifies the application of the saddle method, and we leave these versions to the interested reader.

9 Moment asymptotics

Suppose that X is a positive random variable with a density function f satisfying (2.1). Then, as $r\to\infty$, using Stirling's formula,

$$\begin{aligned}\mathbb{E}\,X^r &\sim \int_0^\infty ax^{r+\alpha}e^{-bx^2}\,\mathrm{d}x\\ &= \frac{a}{2}\int_0^\infty y^{(r+\alpha+1)/2-1}e^{-by}\,\mathrm{d}y\\ &= \frac{a}{2}b^{-(r+\alpha+1)/2}\Gamma\Bigl(\frac{r+\alpha+1}{2}\Bigr)\\ &\sim \frac{a}{2}b^{-(r+\alpha+1)/2}\Bigl(\frac{r}{2}\Bigr)^{(\alpha+1)/2}\Gamma\Bigl(\frac{r}{2}\Bigr)\\ &= a\sqrt{\pi}(2b)^{-(\alpha+1)/2}r^{\alpha/2}\Bigl(\frac{r}{2eb}\Bigr)^{r/2}.\end{aligned}\tag{9.1}$$

(It is easily seen, by an integration by parts, that the same result follows from the weaker assumption (2.2).)

For the Brownian areas studied in this paper, Theorems 1.1–1.7 thus imply the following.

Corollary 9.1. *As $n \to \infty$,*

$$\mathbb{E}\,\mathcal{B}_{\mathrm{ex}}^n \sim 3\sqrt{2}\,n\Big(\frac{n}{12e}\Big)^{n/2}, \tag{9.2}$$

$$\mathbb{E}\,\mathcal{B}_{\mathrm{br}}^n \sim \sqrt{2}\,\Big(\frac{n}{12e}\Big)^{n/2}, \tag{9.3}$$

$$\mathbb{E}\,\mathcal{B}_{\mathrm{bm}}^n \sim \sqrt{2}\,\Big(\frac{n}{3e}\Big)^{n/2}, \tag{9.4}$$

$$\mathbb{E}\,\mathcal{B}_{\mathrm{me}}^n \sim \sqrt{3\pi}n^{1/2}\Big(\frac{n}{3e}\Big)^{n/2}, \tag{9.5}$$

$$\mathbb{E}\,\mathcal{B}_{\mathrm{dm}}^n \sim 2\sqrt{2}\,\Big(\frac{n}{3e}\Big)^{n/2}, \tag{9.6}$$

$$\mathbb{E}\,\mathcal{B}_{\mathrm{br}+}^n \sim \frac{1}{\sqrt{2}}\Big(\frac{n}{12e}\Big)^{n/2}, \tag{9.7}$$

$$\mathbb{E}\,\mathcal{B}_{\mathrm{bm}+}^n \sim \frac{1}{\sqrt{2}}\Big(\frac{n}{3e}\Big)^{n/2}. \tag{9.8}$$

Most of these results have been found earlier: (9.2) by Takács [22], (9.3) by Takács [23] and Tolmatz [26], (9.4) by Takács [24] and Tolmatz [27], (9.5) by Takács [25], (9.6) by Janson [10], (9.7) by Tolmatz [28]; Takács used recursion formulas derived by other methods, while Tolmatz used the method followed here. Note that, as remarked by Tolmatz [28], $\mathbb{E}\,\mathcal{B}_{\mathrm{br}+}^n \sim \frac{1}{2}\,\mathbb{E}\,\mathcal{B}_{\mathrm{br}}^n$ and similarly $\mathbb{E}\,\mathcal{B}_{\mathrm{bm}+}^n \sim \frac{1}{2}\,\mathbb{E}\,\mathcal{B}_{\mathrm{bm}}^n$, cf. Remark 1.8.

In the opposite direction, we do not know any way to get precise asymptotics of the form (2.1) or (2.2) from moment asymptotics, but, as observed by Csörgő, Shi and Yor [3], the much weaker estimate (1.4) and its analogue for other Brownian areas can be obtained by the following special case of results by Davies [5] and Kasahara [12]. (See [9, Theorem 4.5] for a more general version with an arbitrary power x^p instead of x^2 in the exponent.)

Proposition 9.2. *If X is a positive random variable and $b > 0$, then the following are equivalent:*

$$\begin{aligned} -\ln \mathbb{P}(X > x) &\sim bx^2, && x \to \infty,\\ \big(\mathbb{E}\,X^n\big)^{1/n} &\sim \sqrt{\frac{n}{2eb}}, && n \to \infty,\\ \ln\big(\mathbb{E}\,e^{tX}\big) &\sim \frac{1}{4b}t^2, && t \to \infty. \end{aligned}$$

Returning to (9.1), we obtain in the same way more precise asymptotics for the moments if we are given an asymptotic series for f or $\mathbb{P}(X > x)$. For simplicity, we consider only the next term, but the calculations can be extended to an asymptotic expansion with any number of terms. Thus, suppose that, as for the Brownian areas, (2.1) is sharpened to (2.3) with $N \ge 2$. Then,

also using further terms in Stirling's formula,

$$\begin{aligned}\mathbb{E}\, X^n &= \frac{a_0}{2} b^{-(n+\alpha+1)/2}\Gamma\Big(\frac{n+\alpha+1}{2}\Big) + \frac{a_2}{2} b^{-(n+\alpha-1)/2}\Gamma\Big(\frac{n+\alpha-1}{2}\Big) \\ &\qquad + O\left(b^{-n/2}\Gamma\Big(\frac{n+\alpha-3}{2}\Big)\right) \\ &= \frac{1}{2} b^{-(n+\alpha+1)/2}\Gamma\Big(\frac{n+\alpha+1}{2}\Big)\left(a_0 + a_2 b\frac{2}{n} + O(n^{-2})\right) \\ &= \sqrt{2\pi}(2b)^{-(\alpha+1)/2} n^{\alpha/2}\Big(\frac{n}{2eb}\Big)^{n/2} \\ &\qquad \cdot \left(a_0 + \left(a_0\frac{\alpha^2-1}{4} + \frac{a_0}{6} + 2a_2 b\right) n^{-1} + O(n^{-2})\right).\end{aligned}$$

In particular, for the Brownian excursion, where by Theorem 1.1 (2.3) holds with $\alpha = 2$, $b = 6$, $a_0 = 72\sqrt{6/\pi}$ and $a_2 = -8\sqrt{6/\pi}$,

$$\begin{aligned}\mathbb{E}\, \mathcal{B}_{\mathrm{ex}}^n &= \frac{1}{2\sqrt{2}}\Big(\frac{n}{12e}\Big)^{n/2} n\left(12 + \frac{9+2-16}{n} + O(n^{-2})\right) \\ &= 3\sqrt{2}\Big(\frac{n}{12e}\Big)^{n/2} n\left(1 - \frac{5}{12n} + O(n^{-2})\right).\end{aligned} \tag{9.9}$$

If we, following Takács [22], introduce K_n defined by

$$\mathbb{E}\, \mathcal{B}_{\mathrm{ex}}^n = \frac{4\sqrt{\pi}\, 2^{-n/2} n!}{\Gamma((3n-1)/2)} K_n,$$

further applications of Stirling's formula shows that (9.9) is equivalent to

$$K_n = (2\pi)^{-1/2} n^{-1/2}\Big(\frac{3n}{4e}\Big)^n \left(1 - \frac{7}{36n} + O(n^{-2})\right). \tag{9.10}$$

Again, the leading term is given by Takács [22], in the equivalent form

$$K_n \sim \frac{1}{2\pi}\Big(\frac{3}{4}\Big)^n (n-1)! \qquad \text{as } n \to \infty. \tag{9.11}$$

Takács [22] further gave the recursion formula (with $K_0 = -1/2$)

$$K_n = \frac{3n-4}{4} K_{n-1} + \sum_{j=1}^{n-1} K_j K_{n-j}, \qquad n \ge 1, \tag{9.12}$$

It is easy to obtain from (9.11) and (9.12) the refined asymptotics

$$K_n = \frac{1}{2\pi}\Big(\frac{3}{4}\Big)^n (n-1)!\left(1 - \frac{5}{18n} + O(n^{-2})\right), \tag{9.13}$$

which is equivalent to (9.10). and, by recursion, (9.13) can be extended to an asymptotic expansion of arbitrary length. (Another method to obtain an asymptotic expansion of K_n is given by Kearney, Majumdar and Martin [13].) Hence (9.9) (also with further terms) can, alternatively, be derived from (9.11) and (9.12) by straightforward calculations. However, as said above, we do not know any way to derive Theorem 1.1 from this. (Nevertheless, the calculations above serve as a check of the coefficients in Theorem 1.1.)

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A Proof that $\sqrt{z}\mathrm{Ai}(z) - \mathrm{Ai}'(z)$ has no zeros

The double Laplace transforms for the positive part areas $\mathcal{B}_{\mathrm{br}+}$ and $\mathcal{B}_{\mathrm{bm}+}$ have both the denominator $\sqrt{z}\mathrm{Ai}(z) - \mathrm{Ai}'(z)$, and it is important that this function has no zeros, whence $\Psi^*_{\mathrm{br}+}$ and $\Psi^*_{\mathrm{bm}+}$ are analytic in the slit plane $\mathbb{C} \setminus (-\infty, 0]$. This was proved by Tolmatz [28] (for the same reason), but we give here an alternative proof that does not need the careful numerical integration done by Tolmatz. (Our proof is, like Tolmatz', based on the argument principle.)

Lemma A.1 (Tolmatz [28])**.** *The function $\sqrt{z}\mathrm{Ai}(z) - \mathrm{Ai}'(z)$ is non-zero for all $z = re^{\mathbf{i}\theta}$ with $r \geq 0$ and $|\theta| \leq \pi$.*

Proof. We use the notations

$$\begin{aligned}\zeta(z) &:= \tfrac{2}{3}z^{3/2},\\ f(z) &:= \sqrt{\pi}\left(\sqrt{z}\mathrm{Ai}(z) - \mathrm{Ai}'(z)\right),\\ g(z) &:= e^{\zeta(z)}f(z).\end{aligned}$$

Note that these functions are analytic in the slit plane $\mathbb{C} \setminus (-\infty, 0]$ and extend continuously to $(-\infty, 0]$ from each side, so we can regard them as continuous functions of $re^{\mathbf{i}\theta}$ with $r \geq 0$ and $-\pi \leq r \leq \pi$, where we regard the two sides $re^{\pm\mathbf{i}\pi} = -r \pm \mathbf{i}0$ of the negative real axis as different. (The reader that dislikes this can reformulate the proof and study $z\mathrm{Ai}(z^2) - \mathrm{Ai}'(z^2)$ for $\operatorname{Re} z \geq 0$; this avoids the ambiguities of square roots.)

We will use the argument principle on $g(z)$ and the contour γ_R consisting of the interval from 0 to $-R - \mathbf{i}0$ along the lower side of the negative real axis, the circle $Re^{\mathbf{i}\theta}$ for $-\pi \leq \theta \leq \pi$ and the interval from $-R + \mathbf{i}0$ back to 0, where R is a large real number.

First, fix a small $\delta > 0$. By (4.24) and (4.25), as $|z| \to \infty$,

$$f(z) \sim z^{1/4}e^{-\zeta(z)}, \qquad |\arg z| \leq \pi - \delta. \tag{A.1}$$

Next, assume $0 < \arg z < 2\pi/3 - \delta$. Note that then $\arg(-z) = \arg(z) - \pi \in (-\pi, -\pi/3 - \delta)$ and thus

$$(-z)^{1/2} = -\mathbf{i}z^{1/2}, \qquad \zeta(-z) = e^{-(3/2)\mathbf{i}\pi}\zeta(z) = \mathbf{i}\zeta(z). \tag{A.2}$$

Furthermore, we have as $|z| \to \infty$ with $|\arg z| < 2\pi/3 - \delta$ the expansions [1, 10.4.60,10.4.62]

$$\begin{aligned}\mathrm{Ai}(-z) &= \pi^{-1/2}z^{-1/4}\Big(\sin\big(\zeta(z) + \tfrac{\pi}{4}\big)\big(1 + O(\zeta^{-2})\big) - \cos\big(\zeta(z) + \tfrac{\pi}{4}\big)\cdot O(\zeta^{-1})\Big)\\ \mathrm{Ai}'(-z) &= \pi^{-1/2}z^{1/4}\Big(-\cos\big(\zeta(z) + \tfrac{\pi}{4}\big)\big(1 + O(\zeta^{-2})\big) + \cos\big(\zeta(z) + \tfrac{\pi}{4}\big)\cdot O(\zeta^{-1})\Big)\end{aligned}$$

and thus

$$f(-z) = z^{1/4}\Big(\cos\big(\zeta(z) + \tfrac{\pi}{4}\big)\big(1 + O(\zeta^{-1})\big) - \mathbf{i}\sin\big(\zeta(z) + \tfrac{\pi}{4}\big)\big(1 + O(\zeta^{-1})\big)\Big).$$

In the range $\arg z \in (0, 2\pi/3 - \delta)$, further $\Im\zeta(z) > 0$, and thus by Euler's formulas

$$\left|\cos\bigl(\zeta(z) + \frac{\pi}{4}\bigr)\right| + \left|\sin\bigl(\zeta(z) + \frac{\pi}{4}\bigr)\right| \le \left|e^{\mathbf{i}\zeta(z)}\right| + \left|e^{-\mathbf{i}\zeta(z)}\right| \le 2e^{\Im\zeta(z)} = 2\left|e^{-\mathbf{i}\zeta(z)}\right|.$$

Hence, as $|z| \to \infty$ with $\arg z \in (0, 2\pi/3 - \delta)$, using (A.2).

$$f(-z) = z^{1/4}e^{-\mathbf{i}(\zeta(z)+\pi/4)}\bigl(1 + O(\zeta^{-1})\bigr) \sim (-z)^{1/4}e^{-\zeta(-z)}.$$

Consequently, (A.1) holds as $|z| \to \infty$ with $-\pi < \arg z < -\pi/3 - \delta$ too. Since $f(\overline{z}) = \overline{f(z)}$, it holds for $\pi/3 + \delta < \arg z < \pi$ too, and combining the three ranges, we see that as $|z| \to \infty$, for all $|\arg z| < \pi$,

$$f(z) \sim z^{1/4}e^{-\zeta(z)}, \tag{A.3}$$

and thus

$$g(z) = z^{1/4}\bigl(1 + o(1)\bigr). \tag{A.4}$$

Consider now $f(z)$ on the lower side of the negative real axis, i.e. for $z = re^{-\mathbf{i}\pi} = -r - \mathbf{i}0$, $r \ge 0$. Note that then $\mathrm{Ai}(z)$ and $\mathrm{Ai}'(z)$ are real and $z^{1/2}$ purely imaginary. Since Ai and Ai′ have no common zeros, and $\mathrm{Ai}'(0) \ne 0$, $f(-r - \mathbf{i}0) \ne 0$. Moreover, $f(0) > 0$, and as r grows from 0 to ∞, $f(-r - \mathbf{i}0)$ is real at $r = 0$ and at the zeros $-r = a_k$ of Ai, and imaginary at the zeros $-r = a_k'$ of Ai′. Consider continuous determinations of $\arg f(z)$ and $\arg g(z)$ along the neagtive real axis, starting with $\arg f(0) = \arg g(z) = 0$. It is easily seen that $\arg f(z)$ then is $-\pi/2$ for $z = a_1'$, $-\pi$ for $z = a_1$, and so on, with $\arg f(a_k - \mathbf{i}0) = -k\pi$. Furthermore, for $z = -r - \mathbf{i}0$,

$$\arg g(z) = \arg f(z) + \Im\zeta(z) = \arg f(z) + \tfrac{2}{3}\Im z^{3/2} = \arg f(z) + \tfrac{2}{3}r^{3/2}.$$

In particular, using the asymptotic formula [1, 10.4.94] for the Airy zeros a_k,

$$\begin{aligned}\arg g(a_k) &= -\pi k + \tfrac{2}{3}|a_k|^{3/2} = -\pi k + \frac{2}{3}\,\frac{3\pi(4k-1)}{8}\bigl(1 + O(k^{-2})\bigr)\\ &= -\frac{\pi}{4} + O(k^{-1}).\end{aligned} \tag{A.5}$$

Consider now a continuous determination of $\arg g(z)$ along the contour γ_R, with $R = |a_k|$ for a large k. On the part from 0 to $-R - \mathbf{i}0$, the argument decreases by $-\pi/4 + O(k^{-1})$ by (A.5), and on the half-circle from $-R - \mathbf{i}0$ to R, it increases by (A.4) by $\pi/4 + o(1)$, so the total change from 0 to R is $o(1)$, i.e., tends to 0 as $k \to \infty$. Since furthermore $g(R) > 0$, the change is a multiple of 2π, and thus exactly 0 for large k. By symmetry, the change of the argument on the remaining half of γ_R is the same, so the total change along γ_R is 0, which proves that $g(z)$ has no zero inside γ_R for $R = |a_k|$ with k large. Letting $k \to \infty$, we see that $g(z)$ has no zeros. □

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