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Large time asymptotics of growth models on space-like paths I: PushASEP

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Abstract

We consider a new interacting particle system on the one-dimensional lattice that interpolates between TASEP and Toom's model: A particle cannot jump to the right if the neighboring site is occupied, and when jumping to the left it simply pushes all the neighbors that block its way.
We prove that for flat and step initial conditions, the large time fluctuations of the height function of the associated growth model along any space-like path are described by the Airy_1 and Airy_2 processes. This includes fluctuations of the height profile for a fixed time and fluctuations of a tagged particle's trajectory as special cases.

Key words: stochastic growth, KPZ, determinantal processes, Airy processes.

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1 Introduction

We consider a model of interacting particle systems, which is a generalization of the TASEP (totally asymmetric simple exclusion process) and the Toom model. Besides the extension of some universality results to a new model, the main feature of this paper is the extension of the range of analysis to any “space-like” paths in space-time, whose extreme cases are fixed time and fixed particle (tagged particle problem), see below for details.

Consider the system of N particles $x_1 > \cdots > x_N$ in $\mathbb{Z}$ that undergoes the following continuous time Markovian evolution: Each particle has two exponential clocks – one is responsible for its jumps to the left while the other one is responsible for its jumps to the right. All $2N$ clocks are independent, and the rates of all left clocks are equal to L while the rates of all right clocks are equal to R. When the ith left clock rings, the ith particle jumps to the nearest vacant site on its left. When the ith right clock rings, the ith particle jumps to the right by one provided that the site x_i+1 is empty; otherwise it stays put. The main goal of the paper is to study the asymptotic properties of this system when the number of particles and the evolution time become large.

If $L = 0$ then the dynamics is known under the name of Totally Asymmetric Simple Exclusion Process (TASEP), and if $R = 0$ the dynamics is a special case of Toom’s model studied in [9] (see references therein too). Both systems belong to the Kardar-Parisi-Zhang (KPZ) universality class of growth models in $1 + 1$ dimensions.

Particle’s jump to the nearest vacant spot on its left can be also viewed as the particle pushing all its left neighbors by one if they prevent it from jumping to the left. This point of view is often beneficial because it remains meaningful for infinite systems, and also the order of particles is not being changed. Because of this pushing effect we call our system the Pushing Asymmetric Simple Exclusion Process or PushASEP.

Observe that for a N-particle PushASEP with particles $x_1(t) > \cdots > x_N(t)$, the evolution of $(x_1, \ldots, x_M)$ for any $M \leq N$ is the M-particle PushASEP not influenced by the presence of the remaining $N - M$ particles. This "triangularity property" seems to be a key feature of our model that allows our analysis to go through.

Our results split in two groups – algebraic and analytic.

Algebraically, we derive a determinantal formula for the distribution of the N-particle PushASEP with an arbitrary fixed initial condition, and we also represent this distribution as a gap probability for a (possibly, signed) determinantal point process (see [17; 12; 21; 16; 22] for information on determinantal processes). The result is obtained in greater generality with jump rates L and R being both time and particle-dependent (Proposition 3.1). The first part (the determinantal formula, see Proposition 2.1) is a generalization of similar results due to [20; 19; 2] obtained by the Bethe Ansatz techniques. Also, a closely related result have been obtained very recently in [10] using a version of the Robinson-Schensted-Knuth correspondence.

Analytically, we use the above-mentioned determinantal process to study the large time behavior of the infinite-particle PushASEP with two initial conditions:

1. Flat initial condition with particles occupying all even integers.

2. Step initial condition with particles occupying all negative integers.

It is not obvious that the infinite-particle PushASEP started from these initial configurations is correctly defined, and some work needs to be done to prove the existence of the Markovian dynamics.

However, we take a simpler path here and consider our infinite-particle system as a limit of growing finite-particle systems. It turns out that for the above initial conditions, the distribution of any finite number of particles at any finitely many time moments stabilizes as the total number of particles in the system becomes large enough. It is this limiting distribution that we analyze.

We are able to control the asymptotic behavior of the joint distribution of $x_{n_1}(t_1), \dots, x_{n_k}(t_k)$ with $x_{n_1}(0) \geq \cdots \geq x_{n_k}(0)$ and $t_1 \geq \cdots \geq t_k$. It is the second main novel feature of the present paper (the first one being the model itself) that we can handle joint distributions of different particles at different time moments. As special cases we find distributions of several particles at a given time moment and distribution of one particle at several time moments (a.k.a. the tagged particle).

In the growth model formulation of PushASEP (that we do not give here; it can be easily reconstructed from the growth models for TASEP and Toom's model described in [9] and references therein), this corresponds to joint distributions of values of the height function at a finite number of space-time points that lie on a space-like path; for that reason we use the term 'space-like path' below. The two extreme space-like paths were described above – they correspond to $t_1 = \cdots = t_k$ and $n_1 = \cdots = n_k$.

The algebraic techniques of handling space-like paths are used in the subsequent paper [6] to analyze two different models, namely the polynuclear growth (PNG) model on a flat substrate and TASEP in discrete time with parallel update.

Our main result states that large time fluctuations of the particle positions along any space-like path have exponents $1/3$ and $2/3$, and that the limiting process is the Airy_1 process for the flat initial condition and the Airy_2 process for the step initial condition (see the review [11] and Section 2.4 below for the definition of these processes).

In the PushASEP model, we have the fluctuation exponent $1/3$ even in the case of zero drift. This is due to the asymmetry in the dynamical rules and it is consistent with the KPZ hypothesis. In fact, from KPZ we expect to have the $1/3$ exponent when $j''(\rho) \neq 0$, where $j(\rho)$ is the current of particles as a function of their density ρ, and $j''(\rho) = -2(R + L/(1-\rho)^3)$ for PushASEP.

We find it remarkable that up to scaling factors, the fluctuations are independent of the space-like path we choose (this phenomenon was also observed in [7] for the polynuclear growth model (PNG) with step initial condition). It is natural to conjecture that this type of universality holds at least as broadly as KPZ-universality does.

Interestingly enough, so far it is unknown how to study the joint distribution of $x_{n_1}(t_1)$ and $x_{n_2}(t_2)$ with $x_{n_1}(0) > x_{n_2}(0)$ and $t_1 < t_2$ (two points on a time-like path); this question remains a major open problem of the subject.

Previous results. For the TASEP and PNG models, large time fluctuation results have already been obtained in the following cases: For the step initial condition the Airy_2 process has been shown to occur in the scaling limit for fixed time [18; 14; 15], and more recently for tagged particle [13]. For TASEP, the Airy_1 process occurs for flat initial conditions in continuous time [4] and in discrete time with sequential update [3] with generalization to the initial condition of one particle every $d \geq 2$ sites[1]. Also, a transition between the Airy_2 and Airy_1 processes was obtained in [5]. These are fixed time results; the only previous result concerning general space-like paths is to be found in [7] in the

[1]Similar results for discrete time TASEP with parallel update and PNG model will follow from more general results of [6].

context of the PNG model, where the Airy$_2$ process was obtained as a limit for a directed percolation model.

Outline. The paper is organized as follows. In Section 2 we describe the model and the results. In Proposition 2.1 the transition probability of the model is given. Then, we define what we mean by space-like paths, and formulate the scaling limit results; the definitions of the Airy$_1$ and Airy$_2$ processes are recalled in Section 2.4. In Section 3 we state the general kernel for PushASEP (Proposition 3.1) and then particularize it to step and flat initial conditions (Proposition 3.4 and 3.6). In Section 4 we first prove Proposition 2.1 and then obtain the general kernel for a determinantal measure of a certain form (Theorem 4.2), which includes the one of PushASEP. Finally, the asymptotic analysis is the content of Section 5.

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2 The PushASEP model and limit results

2.1 The PushASEP

The model we consider is an extension of the well known totally asymmetric simple exclusion process (TASEP) on $\mathbb{Z}$. The allowed configuration are like in the TASEP, i.e., configurations consist of particles on $\mathbb{Z}$, with the constraint that at each site can be occupied by at most one particle (exclusion constraint). We consider a dynamics in continuous time, where particles are allowed to jump to the right and to the left as follows. A particle jumps to its right-neighbor site with some rate, provided the site is empty (TASEP dynamics). To the left, a particle jump to its left-neighbor site with some rate and, if the site is already occupied by another particle, this is pushed to its left-neighbor and so on (push dynamics).

To define precisely the jump rates, we need to introduce a few notations. Since the dynamics preserves the relative order of particles, we can associate to each particle a label. Let $x_k(t)$ be the position of particle k at time t. We choose the right-left labeling, i.e., $x_k(t) > x_{k+1}(t)$ for all $k \in I \subseteq \mathbb{Z}$, $t \geq 0$. With this labeling, we consider $v_k > 0$, $k \in I$, and some smooth positive increasing functions $a(t), b(t)$ with $a(0) = b(0) = 0$. Then, the right jump rate of particle k is $\dot{a}(t)v_k$, while its left jump rate is $\dot{b}(t)/v_k$.

In Proposition 2.1 we derive the expression of the transition probability from time $t = 0$ to time t for N particles, proven in Section 4.

Proposition 2.1. *Consider N particles with initial conditions $x_i(0) = y_i$. Denote its transition probability until time t by*

$$G(x_N, \ldots, x_1; t) = \mathbb{P}(x_i(t) = x_i, 1 \leq i \leq N | x_i(0) = y_i, 1 \leq i \leq N). \tag{2.1}$$

Then

$$G(x_N,\dots,x_1;t) \tag{2.2}$$
$$= \left(\prod_{n=1}^{N} v_n^{x_n-y_n} e^{-a(t)v_n} e^{-b(t)/v_n}\right) \det\left[F_{k,l}(x_{N+1-l}-y_{N+1-k},a(t),b(t))\right]_{1\leq k,l\leq N},$$

where

$$F_{k,l}(x,a,b) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z z^{x-1} \frac{\prod_{i=1}^{k-1}(1-v_{N+1-i}z)}{\prod_{j=1}^{l-1}(1-v_{N+1-j}z)} e^{bz} e^{a/z}, \tag{2.3}$$

where Γ_0 is any anticlockwise oriented simple loop with including only the pole at $z=0$.

2.2 Space-like paths

The computation of the joint distribution of particle positions at a given time t can be obtained from Proposition 2.1 by adapting the method used in [4] for the TASEP. However, one of the main motivation for this work is to enlarge the spectrum of the situations which can be analyzed to what we call *space-like* paths. In this context, space-like paths are sequences of particle numbers and times in the ensemble

$$\mathscr{S} = \{(n_k,t_k), k\geq 1 | (n_k,t_k) \prec (n_{k+1},t_{k+1})\}, \tag{2.4}$$

where, by definition,

$$(n_i,t_i) \prec (n_j,t_j) \text{ if } n_j \geq n_i, t_j \leq t_i, \text{ and the two couples are not identical.} \tag{2.5}$$

The two extreme cases are (1) fixed time, $t_k = t$ for all k, and (2) fixed particle number, $n_k = n$ for all k. This last situation is known as *tagged particle* problem. Since the analysis is of the same degree of difficulty for any space-like path, we will consider the general situation.

Consider any smooth function π, $w^0 = \pi(w^1)$, in the forward light cone of the origin that satisfies

$$|\pi'| \leq 1, \quad |w^1| \leq \pi(w^1). \tag{2.6}$$

These are space-like paths in $\mathbb{R}\times\mathbb{R}_+$, see Figure 1. The first condition (the space-like property) is related to the applicability of our result to sequences of particles in $\mathscr{S}$. The second condition just reflects the choice of having $t \geq 0$ and $n \geq 0$. Time and particle number are connected with the variables w^1 and w^0 by a rotation of 45 degrees. To avoid unnecessary $\sqrt{2}$'s, we set

$$\left\{\begin{array}{lcl} w^1 & = & \frac{t-n}{2} \\ w^0 & = & \frac{t+n}{2} \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{lcl} t & = & w^0+w^1 \\ n & = & w^0-w^1 \end{array}\right\} \tag{2.7}$$

We want to study the joint distributions of particle positions in the limit of large time, where universal processes arise. Since we consider several times, we can not simply use t as large parameter. Instead, we consider a large parameter T. Particle numbers and times under investigation will have a leading term proportional to T. In the (w^1,w^0) plane, we consider w^1 around θT for a fixed θ, while $w^0 = T\pi(w^1/T)$. From KPZ we know that correlations are on $T^{2/3}$ scale. Therefore, we set the scaling as

$$\begin{cases} w^1(u) = \theta T - uT^{2/3}, \\ w^0(u) = \pi(\theta)T - \pi'(\theta)uT^{2/3} + \frac{1}{2}\pi''(\theta)u^2T^{1/3}. \end{cases} \tag{2.8}$$

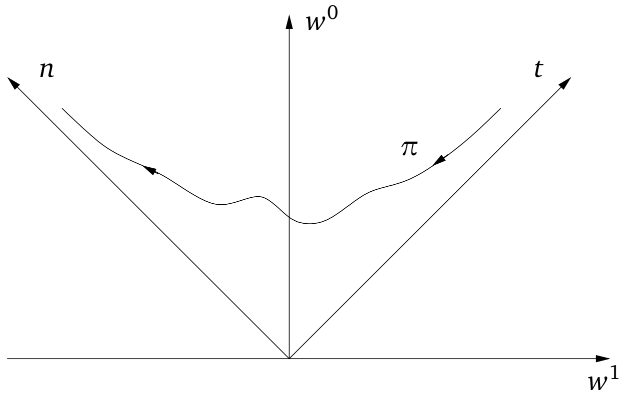

Figure 1: An example of a space-like path. Its slope is, in absolute value, at most 1.

Notice that $w^0(u)$ is equal to $T\pi(w^1(u)/T)$ up to terms that remain bounded, and they become irrelevant in the large T limit, since the fluctuations grow as $T^{1/3}$. Coming back to the (n,t) variables, we have

$$\begin{aligned} t(u) &= (\pi(\theta)+\theta)T - (\pi'(\theta)+1)uT^{2/3} + \tfrac{1}{2}\pi''(\theta)u^2T^{1/3}, \\ n(u) &= \big[(\pi(\theta)-\theta)T + (1-\pi'(\theta))uT^{2/3} + \tfrac{1}{2}\pi''(\theta)u^2T^{1/3}\big]. \end{aligned} \tag{2.9}$$

In particular, setting $\pi(\theta)=1-\theta$ we get the fixed time case with $t=T$, while setting $\pi(\theta)=\alpha+\theta$ we get the tagged particle situation with particle number $n=\alpha T$.

2.3 Scaling limits

Universality occurs in the large T limit. In Proposition 3.1 we obtain an expression for the joint distribution in the general setting. For the asymptotic analysis we consider the case where all particles have the same jump rates, i.e., we set

$$v_k = 1 \text{ for all } k \in I. \tag{2.10}$$

Moreover, we consider time-homogeneous case, i.e., we set $a(t)=Rt$ and $b(t)=Lt$ for some $R,L\geq 0$ (for time non-homogeneous case, one would just replace R and L by some time-dependent functions). Two important initial conditions are

(a) *flat initial condition*: particles start from $2\mathbb{Z}$,

(b) *step initial condition*: particles start from $\mathbb{Z}_- = \{\ldots,-3,-2,-1\}$.

In the first case, the macroscopic limit shape is flat, while in the second case it is curved, see [11] for a review on universality in the TASEP. For TASEP with step initial conditions and particle-dependent rates v_k, the study of tagged particle has been carried out in [13].

Flat initial conditions

For the flat initial condition, it is not very difficult to get the proper scaling limit as $T\to\infty$. The initial position of particle $n(u)$ is $-2n(u)$ and during time $t(u)$ it will have traveled around $\mathbf{v}\,t(u)$, where $\mathbf{v}$ is the mean speed of particles, given by

$$\mathbf{v} = -2L + R/2. \tag{2.11}$$

The reason is that the density of particle is $1/2$ and the particles jumps to the right with rate R but the site on its right has a $1/2$ chance to be empty. Moreover, particles move (and push) to the left with rate L but typically every second move to the left is due to a push from another particle. Therefore, the rescaled process is given by

$$u \mapsto X_T(u) = \frac{x_{n(u)}(t(u)) - (-2n(u) + \mathbf{v}\, t(u))}{-T^{1/3}}, \tag{2.12}$$

where $n(u)$ and $t(u)$ are defined in (2.9). The rescaled process X_T has a limit for large T given in terms of the Airy$_1$ process, $\mathscr{A}_1$ (see [4; 5; 11] and Section 2.4 for details on $\mathscr{A}_1$).

Theorem 2.2 (Convergence to the Airy$_1$ process)**.** *Let us set the vertical and horizontal rescaling*

$$S_v = ((8L+R)(\pi(\theta)+\theta))^{1/3}, \quad S_h = \frac{4((8L+R)(\pi(\theta)+\theta))^{2/3}}{(R+4L)(\pi'(\theta)+1)+4(1-\pi'(\theta))}. \tag{2.13}$$

Then

$$\lim_{T\to\infty} X_T(u) = S_v \mathscr{A}_1(u/S_h) \tag{2.14}$$

in the sense of finite dimensional distributions.

The proof of this theorem is in Section 5. The specialization for fixed time $t = T$ is

$$S_v = (8L+R)^{1/3}, \quad S_h = \frac{(8L+R)^{2/3}}{2}, \tag{2.15}$$

and the one for tagged particle $n = \alpha T$ at times $t(u) = T - 2uT^{2/3}$, obtained by setting $\theta = (1-\alpha)/2$, is

$$S_v = (8L+R)^{1/3}, \quad S_h = \frac{2(8L+R)^{2/3}}{4L+R}. \tag{2.16}$$

Step initial condition

The proper rescaled process for step initial condition is quite intricate. Denote by βt the typical position of particle with number around αt at time t. In the situations previously studied in the literature, there was a nice function $\beta = \beta(\alpha)$. In the present situation this is not anymore true, but we can still describe the limit shape. More precisely, α and β are parametrized by a $\mu \in (0,1)$ via

$$\alpha(\mu) = (1-\mu)^2(R + L/\mu^2), \quad \beta(\mu) = -((1-2\mu)R + L/\mu^2). \tag{2.17}$$

The parameter μ comes from the asymptotic analysis in Section 5.2, where it represents the position of the double critical point. To see that it is a proper parametrization, we have to verify that for a given point on the space-like curve $(\theta, \pi(\theta))$ there corresponds exactly one value of μ. From (2.9) we have $n \simeq t(\pi(\theta)-\theta)/(\pi(\theta)+\theta)$ and, since we have set $n \simeq \alpha t$, we have

$$\alpha(\mu) = \frac{\pi(\theta)-\theta}{\pi(\theta)+\theta}. \tag{2.18}$$

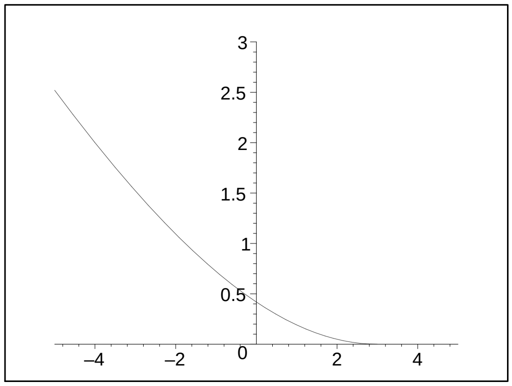

Figure 2: Parametric plot of $(\beta(\mu), \alpha(\mu))$, for $L = 1, R = 4$.

For any given θ, there exists only one μ such that (2.18) holds, because α is strictly monotone in μ. Some computations are needed, but finally we get the rescaling of the position x as a function of u, namely,

$$x(u) = \sigma_0 T - \sigma_1 u T^{2/3} + \sigma_2 u^2 T^{1/3}, \tag{2.19}$$

where

$$\begin{aligned} \sigma_0 &= (\pi(\theta)+\theta)\beta(\mu) \\ \sigma_1 &= 1 + (\pi'(\theta)+1)\left(\mu R - \tfrac{L}{\mu}\right) + (1-\pi'(\theta))\frac{1}{1-\mu} \\ \sigma_2 &= \tfrac{1}{2}\pi''(\theta)\left(\mu R + \tfrac{L}{\mu} - \tfrac{1}{1-\mu}\right) + \frac{(\pi'(\theta)(1-\alpha(\mu)) - (1+\alpha(\mu)))^2}{4(1-\mu)^3(\pi(\theta)+\theta)(R+L/\mu^3)}. \end{aligned} \tag{2.20}$$

The rescaled process is then given by

$$u \mapsto X_T(u) = \frac{x_{n(u)}(t(u)) - (\sigma_0 T - \sigma_1 u T^{2/3} + \sigma_2 u^2 T^{1/3})}{-T^{1/3}}, \tag{2.21}$$

with $n(u)$ and $t(u)$ given in (2.9). Define the constants

$$\begin{aligned} \kappa_0 &= \frac{(\pi(\theta)+\theta)(R+L/\mu^3)}{\mu(1-\mu)}, \\ \kappa_1 &= \frac{(\pi'(\theta)+1)(R+L/\mu^2)}{2\mu} - \frac{\pi'(\theta)-1}{2\mu(1-\mu)^2}. \end{aligned} \tag{2.22}$$

Then, a detailed asymptotic analysis would lead to,

$$\lim_{T\to\infty} X_T(u) = \mu \kappa_0^{1/3} \mathscr{A}_2(\kappa_1 \kappa_0^{-2/3} u), \tag{2.23}$$

in the sense of finite dimensional distributions, where $\mathscr{A}_2$ is the Airy$_2$ process (see [18; 14; 11] and Section 2.4 for details on $\mathscr{A}_2$). As for the flat PNG, special cases are tagged particle and fixed time. In Section 5.2 we obtain (2.23) by looking at the contribution coming from the series expansion around a double critical point. To get (2.23) rigorously, one has to control (1) the error terms in the convergence on bounded sets and (2) get some bounds to get convergence of the Fredholm determinants. This is what we actually do in the flat initial condition setting.

2.4 Limit processes

For completeness, we shortly recall the definitions of the limit processes $\mathscr{A}_1$ and $\mathscr{A}_2$ appearing above. The notation $\mathrm{Ai}(x)$ below stands for the classical Airy function [1].

Definition 2.3 (The Airy$_1$ process). *The* Airy$_1$ process $\mathscr{A}_1$ *is the process with m-point joint distributions at* $u_1 < u_2 < \ldots < u_m$ *given by the Fredholm determinant*

$$\mathbb{P}\Big(\bigcap_{k=1}^{m}\{\mathscr{A}_1(u_k) \le s_k\}\Big) = \det(\mathbb{1} - \chi_s K_{\mathscr{A}_1} \chi_s)_{L^2(\{u_1,\ldots,u_m\}\times\mathbb{R})}, \tag{2.24}$$

where $\chi_s(u_k, x) = \mathbb{1}(x > s_k)$ *and the kernel* $K_{\mathscr{A}_1}$ *is given by*

$$\begin{aligned} K_{\mathscr{A}_1}(u_1,s_1;u_2,s_2) = &-\frac{1}{\sqrt{4\pi(u_2-u_1)}}\exp\left(-\frac{(s_2-s_1)^2}{4(u_2-u_1)}\right)\mathbb{1}(u_2>u_1) \\ &+\mathrm{Ai}(s_1+s_2+(u_2-u_1)^2)\exp\left((u_2-u_1)(s_1+s_2)+\frac{2}{3}(u_2-u_1)^3\right). \end{aligned} \tag{2.25}$$

Definition 2.4 (The Airy$_2$ process). *The* Airy$_2$ process $\mathscr{A}_2$ *is the process with m-point joint distributions at* $u_1 < u_2 < \ldots < u_m$ *given by the Fredholm determinant*

$$\mathbb{P}\Big(\bigcap_{k=1}^{m}\{\mathscr{A}_2(u_k) \le s_k\}\Big) = \det(\mathbb{1} - \chi_s K_{\mathscr{A}_1} \chi_s)_{L^2(\{u_1,\ldots,u_m\}\times\mathbb{R})}, \tag{2.26}$$

where $\chi_s(u_k, x) = \mathbb{1}(x > s_k)$ *and the kernel* $K_{\mathscr{A}_2}$ *is given by*

$$K_{\mathscr{A}_2}(u_1,s_1;u_2,s_2) = \begin{cases} \int_{\mathbb{R}_+} e^{-\lambda(u_2-u_1)}\mathrm{Ai}(s_1+\lambda)\mathrm{Ai}(s_2+\lambda), & u_2 \ge u_1, \\ -\int_{\mathbb{R}_-} e^{-\lambda(u_2-u_1)}\mathrm{Ai}(s_1+\lambda)\mathrm{Ai}(s_2+\lambda), & u_2 < u_1. \end{cases} \tag{2.27}$$

3 Finite time kernel

In this section we first derive an expression for the joint distributions of particle positions in a finite system. They are given by Fredholm determinants of a kernel, which is first stated for general jump rates and initial positions. After that, we specialize to the cases of uniform jump rates in the case of step and flat initial conditions. Flat initial conditions are obtained via a limit of finite systems.

3.1 General kernel for PushASEP

To state the following result, proven in Section 4, we introduce a space of functions V_n. Consider the set of numbers $\{v_1,\ldots,v_n\}$ and let $\{u_1 < u_2 < \ldots < u_\nu\}$ be their different values, with α_k being the multiplicity of u_k (v_k is the jump rate of particle with label k). Then we define the space

$$V_n = \mathrm{span}\{x^l u_k^x, 1 \le k \le \nu, 0 \le l \le \alpha_k - 1\}. \tag{3.1}$$

Recall that the evolution of particle indexed by n is independent of the particles with index $m > n$.

Proposition 3.1. *Consider a system of particles with indices $n = 1, 2, \dots$ starting from positions $y_1 > y_2 > \dots$. Denote by $x_n(t)$ the position of particle with index n at time t. Then the joint distribution of particle positions is given by the Fredholm determinant*

$$\mathbb{P}\Big(\bigcap_{k=1}^{m}\{x_{n_k}(t_k) \geq s_k\}\Big) = \det\big(\mathbb{1} - \tilde{\chi}_s K \tilde{\chi}_s\big)_{\ell^2(\{(n_1,t_1),\dots,(n_m,t_m)\}\times\mathbb{Z})} \tag{3.2}$$

with $((n_1, t_1), \dots, (n_m, t_m)) \in \mathscr{S}$, and $\tilde{\chi}_s((n_k, t_k))(x) = \mathbb{1}(x < s_k)$. The kernel K is given by

$$K((n_1,t_1),x_1;(n_2,t_2),x_2) = -\phi^{((n_1,t_1),(n_2,t_2))}(x_1,x_2) + \sum_{k=1}^{n_2} \Psi^{n_1,t_1}_{n_1-k}(x_1)\Phi^{n_2,t_2}_{n_2-k}(x_2) \tag{3.3}$$

where

$$\Psi^{n,t}_{n-l}(x) = \frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_0} \mathrm{d}z z^{x-y_l-1} e^{a(t)/z+b(t)z} \frac{(1-v_1 z)\cdots(1-v_n z)}{(1-v_1 z)\cdots(1-v_l z)}, \quad l = 1, 2, \dots, \tag{3.4}$$

the functions $\{\Phi^{n,t}_{n-k}\}_{k=1}^{n}$ are uniquely determined by the orthogonality relations

$$\sum_{x\in\mathbb{Z}} \Psi^{n,t}_{n-l}(x)\Phi^{n,t}_{n-k}(x) = \delta_{k,l}, \quad 1 \leq k, l \leq n, \tag{3.5}$$

and by the requirement $\mathrm{span}\{\Phi^{n,t}_{n-l}(x), l = 1, \dots, n\} = V_n$. *The first term in (3.3) is given by*

$$\phi^{((n_1,t_1),(n_2,t_2))}(x,y) = \frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_0} \frac{\mathrm{d}z}{z^{y-x+1}} \frac{e^{(a(t_1)-a(t_2))/z} e^{(b(t_1)-b(t_2))z}}{(1-v_{n_1+1}z)\cdots(1-v_{n_2}z)} \mathbb{1}_{[(n_1,t_1)\prec(n_2,t_2)]}. \tag{3.6}$$

The notation Γ_0 stands for any anticlockwise oriented simple loop including only the pole at 0.

3.2 Kernel for step initial condition

We set all the jump rates to 1: $v_1 = v_2 = \dots = 1$. The transition function (3.6) does not depend on initial conditions. It is useful to rewrite it in a slightly different form.

Lemma 3.2. *The transition function can be rewritten as*

$$\begin{aligned} &\phi^{((n_1,t_1),(n_2,t_2))}(x,y) \\ = &\frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_{0,1}} \mathrm{d}w \frac{1}{w^{x-y+1}} \left(\frac{w}{w-1}\right)^{n_2-n_1} \frac{e^{a(t_1)w+b(t_1)/w}}{e^{a(t_2)w+b(t_2)/w}} \mathbb{1}_{[(n_1,t_1)\prec(n_2,t_2)]}. \end{aligned} \tag{3.7}$$

Proof of Lemma 3.2. The proof follows by the change of variable $z = 1/w$ in (3.6). □

Lemma 3.3. *Let $y_i = -i, i \geq 1$. Then, the functions Φ and Ψ are given by*

$$\begin{aligned} \Psi^{n,t}_k(x) &= \frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_{0,1}} \mathrm{d}w \frac{(w-1)^k}{w^{x+n+1}} e^{a(t)w+b(t)/w}, \\ \Phi^{n,t}_j(x) &= \frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_1} \mathrm{d}z \frac{z^{x+n}}{(z-1)^{j+1}} e^{-a(t)z-b(t)/z}. \end{aligned} \tag{3.8}$$

Proof of Lemma 3.3. $\Psi_k^{n,t}(x)$ comes from the change of variable $z = 1/w$ in (3.4). For $k \geq 0$, the pole at $w = 1$ is irrelevant, but in the kernel $\Psi_k^{n,t}$ enters also for negative values of k.

We have to verify that the function $\Phi_j^{n,t}(x)$ satisfy the orthogonal condition (3.5) and span the space V_n given in (3.1). For $v_1 = \cdots = v_n = 1$, $V_n = \mathrm{span}(1, x, \ldots, x^{n-1})$. By the residue's theorem, the function $\Phi_j^{n,t}(x)$ is a polynomial of degree j in x. Thus, $\mathrm{span}(\Phi_j^{n,t}(x), j = 0, \ldots, n-1) = V_n$.

The second step is to compute $\sum_{x\in\mathbb{Z}} \Phi_j^{n,t}(x)\Psi_k^{n,t}(x)$ for $0 \leq j, k \leq n-1$. We divide it into the sum over $x \geq 0$ and the one over $x < 0$. We have

$$\sum_{x\geq 0} \Phi_j^{n,t}(x)\Psi_k^{n,t}(x) = \sum_{x\geq 0} \frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_1} \mathrm{d}z \oint_{\Gamma_0} \mathrm{d}w \frac{e^{a(t)w+b(t)/w}}{e^{a(t)z+b(t)/z}} \frac{(w-1)^k}{(z-1)^{j+1}} \frac{z^{x+n}}{w^{x+n+1}}. \tag{3.9}$$

We choose the paths Γ_0 and Γ_1 satisfying $|z| < |w|$, so that we can take the sum inside the integrals and use

$$\sum_{x\geq 0} \frac{z^x}{w^{x+1}} = \frac{1}{w-z}, \tag{3.10}$$

to get

$$(3.9) = \frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_1} \mathrm{d}z \oint_{\Gamma_{0,z}} \mathrm{d}w \frac{e^{a(t)w+b(t)/w}}{e^{a(t)z+b(t)/z}} \frac{(w-1)^k}{(z-1)^{j+1}} \frac{z^n}{w^n} \frac{1}{w-z}, \tag{3.11}$$

where the subscript z in $\Gamma_{0,z}$ reminds that z is a pole for the integral over w.

Next consider the sum over $x < 0$. We have

$$\sum_{x<0} \Phi_j^{n,t}(x)\Psi_k^{n,t}(x) = \sum_{x<0} \frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} \mathrm{d}w \oint_{\Gamma_1} \mathrm{d}z \frac{e^{a(t)w+b(t)/w}}{e^{a(t)z+b(t)/z}} \frac{(w-1)^k}{(z-1)^{j+1}} \frac{z^{x+n}}{w^{x+n+1}}. \tag{3.12}$$

This time we choose the paths Γ_0 and Γ_1 satisfying $|z| > |w|$ and then take the sum inside the integrals. Using

$$\sum_{x<0} \frac{z^x}{w^{x+1}} = -\frac{1}{w-z} \tag{3.13}$$

we obtain

$$(3.12) = -\frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_0} \mathrm{d}w \oint_{\Gamma_{1,w}} \mathrm{d}z \frac{e^{a(t)w+b(t)/w}}{e^{a(t)z+b(t)/z}} \frac{(w-1)^k}{(z-1)^{j+1}} \frac{z^n}{w^n} \frac{1}{w-z}, \tag{3.14}$$

where now w is a pole for the integral over z. Thus,

$$\sum_{x\in\mathbb{Z}} \Phi_j^{n,t}(x)\Psi_k^{n,t}(x) = (3.11) + (3.14). \tag{3.15}$$

We can deform the paths of integration in (3.14) so that they become as the integration paths of (3.11) up to correcting the contribution of the residue at $z = w$. Thus we finally get, for $0 \leq j, k \leq n-1$,

$$\sum_{x\in\mathbb{Z}} \Phi_j^{n,t}(x)\Psi_k^{n,t}(x) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_1} \mathrm{d}z (z-1)^{k-j-1} = \delta_{j,k}. \tag{3.16}$$

As a side remark, the same computations would not hold for $k, j < 0$, for which $\Phi_j^{n,t}(x) \equiv 0$, because it is not possible to choose the paths with $|z| > |w|$ without introducing an extra pole at $z = 0$. □

Proposition 3.4 (Step initial condition, finite time kernel).
The kernel for $y_i = -i$, $i \geq 1$, is given by

$$\begin{aligned} & K((n_1,t_1),x_1;(n_2,t_2),x_2) \\ = & -\frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_0} \mathrm{d}w \frac{1}{w^{x_1-x_2+1}}\left(\frac{w}{1-w}\right)^{n_2-n_1} \frac{e^{a(t_1)w+b(t_1)/w}}{e^{a(t_2)w+b(t_2)/w}} \mathbb{1}_{[(n_1,t_1)\prec(n_2,t_2)]} \\ + & \frac{1}{(2\pi \mathrm{i})^2}\oint_{\Gamma_0} \mathrm{d}w \oint_{\Gamma_1} \mathrm{d}z \frac{e^{b(t_1)/w+a(t_1)w}}{e^{b(t_2)/z+a(t_2)z}} \frac{(1-w)^{n_1}}{w^{x_1+n_1+1}} \frac{z^{x_2+n_2}}{(1-z)^{n_2}} \frac{1}{w-z}. \end{aligned} \tag{3.17}$$

The contours Γ_0 and Γ_1 include the poles $w = 0$ and $z = 1$ and no other poles. This means in particular that Γ_0 and Γ_1 are disjoints, because of the term $1/(w-z)$.

Proof of Proposition 3.4. Consider the main term of the kernel, namely

$$\begin{aligned} \sum_{k=1}^{n_2} \Psi^{n_1,t_1}_{n_1-k}(x_1)\Phi^{n_2,t_2}_{n_2-k}(x_2) = \sum_{k=1}^{n_2} \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_{0,1}} \mathrm{d}w \frac{(w-1)^{n_1-k}}{w^{x_1+n_1+1}} e^{a(t_1)w+b(t_1)/w} \\ \times \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_1} \mathrm{d}z \frac{z^{x_2+n_2}}{(z-1)^{n_2-k+1}} e^{-a(t_2)z-b(t_2)/z}. \end{aligned} \tag{3.18}$$

First we extend the sum to $+\infty$, since the second term is identically equal to zero for $k > n_2$. We choose the integration paths so that $|z-1| < |w-1|$. Then, we can take the sum inside the integral. The k-dependent terms are

$$\sum_{k\geq 1} \frac{(z-1)^{k-1}}{(w-1)^k} = \frac{1}{w-z}. \tag{3.19}$$

Thus, we get

$$(3.18) = \frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_1} \mathrm{d}z \oint_{\Gamma_{0,z}} \mathrm{d}w \frac{e^{a(t_1)w+b(t_1)/w}}{e^{a(t_2)z+b(t_2)/z}} \frac{(w-1)^{n_1}}{w^{x_1+n_1+1}} \frac{z^{x_2+n_2}}{(z-1)^{n_2}} \frac{1}{w-z}. \tag{3.20}$$

Notice now we have a new pole at $w = z$, but the one at $w = 1$ vanished. The contribution of the pole at $w = z$ is exactly equal to the contribution of the pole at $z = 1$ in the transition function (3.7). Therefore in the final result the first term coming from (3.7) has the integral only around $z = 0$, and the second term is (3.20) but with the integral over w only around the pole at $w = 0$ and does not contain z. Finally, a conjugation by a factor $(-1)^{n_1-n_2}$ gives the final result. □

3.3 Kernel for flat initial condition

The kernel for the flat initial condition is obtained as a limit of those for systems with finitely many particles as follows. We first compute the kernel for a finite number of particles starting from $y_i = -2i$, $i \geq 1$. Then we shift the focus by N particles, i.e., we consider particles with numbers $N + n_i$ instead of those with numbers n_i. For any finite time t, we then take the $N \to \infty$ limit, in which the deviations due to the finite number of particles on the right tend to zero. The limiting kernel is what we call the kernel for the flat initial condition ($y_i = -2i$ with $i \in \mathbb{Z}$).

For this case we also consider the homogeneous jump rates, $v_1 = v_2 = \cdots = 1$.

Lemma 3.5. *Let $y_i = -2i$, $i \geq 1$. Then, the functions Φ and Ψ are given by*

$$\begin{aligned}
\Psi_k^{n,t}(x) &= \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_{0,1}} \mathrm{d}w \frac{(w(w-1))^k}{w^{x+2n+1}} e^{a(t)w+b(t)/w}, \\
\Phi_j^{n,t}(x) &= \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_1} \mathrm{d}z \frac{(2z-1)z^{x+2n}}{(z(z-1))^{j+1}} e^{-a(t)z-b(t)/z}.
\end{aligned} \tag{3.21}$$

Proof of Lemma 3.5. The proof is almost identical to the one of Lemma 3.3. The only difference is that contribution of the residue is in this case given by

$$\sum_{x\in\mathbb{Z}} \Phi_j^{n,t}(x)\Psi_k^{n,t}(x) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_1} \mathrm{d}z(2z-1)(z(z-1))^{k-j-1} = \delta_{j,k} \tag{3.22}$$

by the change of variable $w = z(z-1)$. □

Proposition 3.6 (Flat initial conditions, finite time kernel)**.**
The kernel for $y_i = -2i$, $i \in \mathbb{Z}$, is given by

$$\begin{aligned}
&K((n_1,t_1),x_1;(n_2,t_2),x_2) \\
&= -\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}w \frac{1}{w^{x_1-x_2+1}} \left(\frac{w}{1-w}\right)^{n_2-n_1} \frac{e^{a(t_1)w+b(t_1)/w}}{e^{a(t_2)w+b(t_2)/w}} \mathbb{1}_{[(n_1,t_1)\prec(n_2,t_2)]} \\
&+ \frac{-1}{2\pi \mathrm{i}} \oint_{\Gamma_1} \mathrm{d}z \frac{e^{a(t_1)(1-z)+b(t_1)/(1-z)}}{e^{a(t_2)z+b(t_2)/z}} \frac{z^{n_1+n_2+x_2}}{(1-z)^{n_1+n_2+x_1+1}}.
\end{aligned} \tag{3.23}$$

Proof of Proposition 3.6. The first step is to get the kernel for $y_i = -2i$, $i \geq 1$. This step is similar to the one of Proposition 3.4. The integration paths are taken to satisfy $|z(z-1)| < |w(w-1)|$. Then this time, the sum in k is

$$\sum_{k\geq 1} \frac{(z(z-1))^{k-1}}{(w(w-1))^k} = \frac{1}{(w-z)(w-1+z)} \tag{3.24}$$

and we get

$$\frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_1} \mathrm{d}z \oint_{\Gamma_{0,1-z,z}} \mathrm{d}w \frac{e^{a(t_1)w+b(t_1)/w}}{e^{a(t_2)z+b(t_2)/z}} \frac{(w-1)^{n_1}}{(z-1)^{n_2}} \frac{z^{x_2+n_2}}{w^{x_1+n_1+1}} \frac{2z-1}{(w-z)(w-1+z)}. \tag{3.25}$$

Notice that the pole for $w = 1$ is now replaced by two simple poles, one at $w = z$ and one at $w = 1-z$. The pole at $w = z$ cancels with the one at $z = 1$ of (3.7). Therefore, the kernel for $y_i = -2i$, $i \geq 1$, is given by

$$\begin{aligned}
&-\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}w \frac{1}{w^{x_1-x_2+1}} \left(\frac{w}{1-w}\right)^{n_2-n_1} \frac{e^{a(t_1)w+b(t_1)/w}}{e^{a(t_2)w+b(t_2)/w}} \mathbb{1}_{[(n_1,t_1)\prec(n_2,t_2)]} \\
&\quad + \frac{1}{(2\pi \mathrm{i})^2} \oint_{\Gamma_1} \mathrm{d}z \oint_{\Gamma_{0,1-z}} \mathrm{d}w \frac{e^{a(t_1)w+b(t_1)/w}}{e^{a(t_2)z+b(t_2)/z}} \frac{(w-1)^{n_1}}{(z-1)^{n_2}} \frac{z^{x_2+n_2}}{w^{x_1+n_1+1}} \frac{2z-1}{(w-z)(w-1+z)}.
\end{aligned} \tag{3.26}$$

At this point, we pick a large N and shift the focus to particles around the Nth one. Accordingly, we shift the positions by $-2N$. More precisely, in (3.26), we replace

$$n_i \to n_i + N, \quad x_i \to x_i - 2N. \tag{3.27}$$

Then we get the kernel $K = K_0 + K_1 + K^{(N)}$ with $((n_1,t_1),x_1;(n_2,t_2),x_2)$-entries given by

$$\begin{aligned}
K_0 &= -\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_0}\mathrm{d}w\,\frac{1}{w^{x_1-x_2+1}}\left(\frac{w}{1-w}\right)^{n_2-n_1}\frac{e^{a(t_1)w+b(t_1)/w}}{e^{a(t_2)w+b(t_2)/w}}\mathbb{1}_{[(n_1,t_1)\prec(n_2,t_2)]}\\
K_1 &= \frac{(-1)^{n_1-n_2+1}}{2\pi\mathrm{i}}\oint_{\Gamma_1}\mathrm{d}z\,\frac{e^{a(t_1)(1-z)+b(t_1)/(1-z)}}{e^{a(t_2)z+b(t_2)/z}}\frac{z^{x_2+n_2+n_1}}{(1-z)^{x_1+n_1+n_2+1}},\\
K^{(N)} &= \frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_1}\mathrm{d}z\oint_{\Gamma_0}\mathrm{d}w\,\frac{e^{a(t_1)w+b(t_1)/w}}{e^{a(t_2)z+b(t_2)/z}}\frac{(w-1)^{n_1+N}}{(z-1)^{n_2+N}}\frac{z^{x_2+n_2-N}}{w^{x_1+n_1-N+1}}\frac{2z-1}{(w-z)(w-1+z)}.
\end{aligned} \tag{3.28}$$

The terms K_0 and K_1 are independent of N, while $K^{(N)}$ is not. We need to show that in the $N\to\infty$ limit, the contribution of $K^{(N)}$ vanishes, in the sense that the Fredholm determinant giving the joint distributions of Proposition 3.1 converges to the one with kernel $K_0 + K_1$.

The Fredholm determinant (3.2) is projected onto $x_i < s_i$. Therefore, for any given $s_1,\dots,s_m$ we need to get bounds on the kernel for x_i's bounded from above, say for $x_i \le \ell$ for an $\ell \in \mathbb{Z}$ fixed. In the simplest case of pure TASEP dynamics ($b(t)\equiv 0$), the limit turns out to be easy because for $x_1 + n_1 < N$, the pole at $w=0$ vanishes. However, in our model, $b(t)$ is generically non-zero and the integrand has an essential singularity at $w=0$. In what follows, we choose the indices $k\in\{1,\dots,m\}$ so that $(n_k,t_k)\prec(n_{k+1},t_{k+1})$, $k=1,\dots,m-1$. Also, we simplify the notation by writing $k\in\{1,\dots,m\}$ instead of (n_k,t_k) in the arguments of the kernel. Then, the Fredholm determinant becomes

$$(3.2) = \sum_{n\ge 0}\frac{(-1)^n}{n!}\sum_{i_1,\dots,i_n=1}^{m}\sum_{x_1<s_{i_1}}\cdots\sum_{x_n<s_{i_n}}\det\left(K(i_k,x_k;i_l,x_l)\right)_{1\le k,l\le n}. \tag{3.29}$$

We apply the following conjugation of the kernel, which keeps unchanged the above expression,

$$\widetilde{K}(i_k,x_k;i_l,x_l) = K(i_k,x_k;i_l,x_l)e^{\varepsilon i_l x_l - \varepsilon i_k x_k}e^{(x_l-x_k)/2}. \tag{3.30}$$

Using the bound of Lemma 3.7, for any choice of ε in $(0,(8m)^{-1}]$ and for x_k, x_l bounded from above, we have

$$\begin{aligned}
|\widetilde{K}_0(i_k,x_k;i_l,x_l)| &\le \mathrm{const}\, e^{\varepsilon x_l},\\
|\widetilde{K}_1(i_k,x_k;i_l,x_l)| &\le \mathrm{const}\, e^{(x_l+x_k)/8} \le \mathrm{const}\, e^{\varepsilon x_l},\\
|\widetilde{K}^{(N)}(i_k,x_k;i_l,x_l)| &\le \mathrm{const}\, e^{(x_l+x_k)/8}\kappa^N \le \mathrm{const}\, e^{\varepsilon x_l}\kappa^N,
\end{aligned} \tag{3.31}$$

where $\kappa\in[0,1)$. In the above bounds, we use the same symbol 'const' for all the constants. With the choice of ordering of the (n_k,t_k)'s, we have that $K_0 = 0 = \widetilde{K}_0$ if $i_l \le i_k$, thus the bound holds trivially. For the case $i_l > i_k$, Lemma 3.7 implies the estimate

$$|\widetilde{K}_0| \le \mathrm{const}\, e^{\varepsilon i_k(x_l-x_k)}e^{-|x_l-x_k|/4}e^{\varepsilon x_l}, \tag{3.32}$$

for x_k, x_l bounded from above. The bound in (3.31) is then obtained by choosing $\varepsilon \le (4m)^{-1}$, since then $\varepsilon i_k \le \varepsilon m \le 1/4$. The other bounds on $\widetilde{K}_1$ and $\widetilde{K}^{(N)}$ are satisfied for $\varepsilon m \le 1/8$.

Therefore, the summand in the multiple sums of (3.29) is uniformly bounded by

$$\left| \frac{(-1)^n}{n!} \det\left(\widetilde{K}(i_k, x_k; i_l, x_l)\right)_{1\le k,l\le n} \right| \le \frac{1}{n!} e^{\varepsilon(x_1+\ldots+x_n)} \mathrm{const}^n (1+\kappa^N)^n n^{n/2}, \tag{3.33}$$

the term $n^{n/2}$ being Hadamard bound on the value of a $n \times n$ determinant whose entries have modulus bounded by 1. Since $\kappa < 1$, replacing $1+\kappa^N$ by 2 yields a uniform bound, which is summable. Thus, by dominated convergence we can take the $N \to \infty$ limit inside the Fredholm series. Since $\kappa < 1$, we have $\lim_{N\to\infty} K^{(N)} = 0$, thus the result is proven. Finally, just for convenience, we conjugate the kernel by $(-1)^{n_1-n_2}$, which however has no impact on the Fredholm determinant in question. □

Lemma 3.7. *Let K_0, K_1, $K^{(N)}$ be as in (3.28). Then, for $x_1, x_2 \le \ell$, we have the following bounds.*

$$\begin{aligned} |K_0((n_1,t_1),x_1;(n_2,t_2),x_2)| &\le \mathrm{const}\, e^{(x_1-x_2)/2} e^{-|x_2-x_1|/4} \mathbb{1}_{[(n_1,t_1)\prec(n_2,t_2)]}, \\ |K_1((n_1,t_1),x_1;(n_2,t_2),x_2)| &\le \mathrm{const}\, e^{(x_1-x_2)/2} e^{(x_1+x_2)/4}, \\ |K^{(N)}((n_1,t_1),x_1;(n_2,t_2),x_2)| &\le \mathrm{const}\, e^{(x_1-x_2)/2} e^{(x_1+x_2)/4} \kappa^N, \end{aligned} \tag{3.34}$$

for some $\kappa \in [0,1)$. The constants const *and κ are uniform in N and depend only on ℓ and n_i, t_i's.*

Proof of Lemma 3.7. For K_0 and $x_2 - x_1 \ge 0$, we can just choose the integration path as $\Gamma_0 = \{|w| = e^{-1}\}$, from which we have $|K_0| \le \mathrm{const}\, e^{-(x_2-x_1)} \le \mathrm{const}\, e^{-(x_2-x_1)3/4}$. In the case $x_1 - x_2 \ge 0$, we choose the integration path as $\Gamma_0 = \{|w| = e^{-1/4}\}$. Then, $|K_0| \le \mathrm{const}\, e^{(x_1-x_2)/4}$.

For K_1, we choose $\Gamma_1 = \{|1-z| = e^{-2}\}$. Then,

$$|K_1| \le \mathrm{const}\, \frac{\max_{\Gamma_1} |z|^{x_2}}{\min_{\Gamma_1} |1-z|^{x_1}}. \tag{3.35}$$

Along Γ_1, $|1-z|$ is constant, thus $(\min_{\Gamma_1} |1-z|^{x_1})^{-1} = e^{2x_1} \le \mathrm{const}\, e^{3x_1/4}$ for x_1 bounded from above. Remark that const depends on the upper bound, ℓ, for x_1. In this case, we can take const $= e^{5\ell/4}$. Also, for x_2 bounded from above, $\max_{\Gamma_1} |z|^{x_2} \le \mathrm{const}\,(1-1/e^2)^{x_2} \le \mathrm{const}\, e^{-x_2/4}$.

For $K^{(N)}$, we use the path $\Gamma_0 = \{|w| = e^{-4}\}$ and $\Gamma_1 = \{|1-z| = e^{-2}\}$. As required, these paths do not intersect because $1/e^4 < 1 - 1/e^2$. Then,

$$|K^{(N)}| \le \mathrm{const}\, \frac{\max_{\Gamma_1} |z|^{x_2}}{\min_{\Gamma_0} |w|^{x_1}} \kappa^N, \quad \kappa = \frac{\max_{\Gamma_0} |w(w-1)|}{\min_{\Gamma_1} |z(z-1)|}. \tag{3.36}$$

For x_1, x_2 bounded from above, we have $\max_{\Gamma_1} |z|^{x_2} \le \mathrm{const}\, e^{-x_2/4}$ (as above), and $(\min_{\Gamma_0} |w|^{x_1})^{-1} = e^{4x_1} \le \mathrm{const}\, e^{3x_1/4}$. Finally, it is not difficult to obtain $\kappa = (1+1/e^4)/(1-e^2) = 0.159\ldots$, since the maximum of $|w(w-1)|$ is obtained at $w = -e^{-4}$ and the minimum of $|z(z-1)|$ at $z = 1 - e^{-2}$. □

4 Determinantal measures

In this section we first prove Proposition 2.1. Then, we use it to extend the measure to space-like paths. More precisely, we first obtain a general determinantal formula in Theorem 4.1. Then, in Theorem 4.2, we prove that the measure has determinantal correlations and obtain an expression of the associated kernel.

Proof of Proposition 2.1. We first prove that the initial condition is satisfied. We have

$$F_{k,l}(x,0)=\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_0}\mathrm{d}z z^{x-1}\frac{\prod_{i=1}^{k-1}(1-v_{N+1-i}z)}{\prod_{j=1}^{l-1}(1-v_{N+1-j}z)}. \tag{4.1}$$

(a) $F_{k,l}(x,0)=0$ for $x\geq 1$ because the pole at $z=0$ vanishes.
(b) $F_{k,l}(x,0)=0$ for $k\geq l$ and $x<l-k$, because then

$$F_{k,l}(x,0)=\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_0}\mathrm{d}z z^{x-1}(1-v_l z)\cdots(1-v_{k-1}z) \tag{4.2}$$

and the residue at infinity equals to zero for $x<l-k$.

Assume that $x_N<\cdots<x_1$. If $x_N>y_N$, also $x_l>y_N$ for $l=1,\dots,N-1$. Thus $F_{1,l}(x_{N+1-l}-y_N,0)=0$ using (a). Therefore $G(x_N,\dots,x_1;0)=0$. On the other hand, if $x_N<y_N$, then $x_N<y_k-N+k$, $k=1,\dots,N-1$. Thus $F_{k,1}(x_N-y_{N+1-k},0)=0$ using (b) and the fact that $x_N-y_{N+1-k}<1-k$. Therefore we conclude that $G(x_N,\dots,x_1;0)=0$ if $x_N\neq y_N$. For $x_N=y_N$, $F_{1,1}(0,0)=1$ and by (a) $F_{1,l}(x_{N+1-l}-y_N,0)=0$ for $l=2,\dots,N$. This means that

$$G(x_N,\dots,x_1;0)=\delta_{x_N,y_N}G(x_{N-1},\dots,x_1;0). \tag{4.3}$$

By iterating the procedure we obtain

$$G(x_N,\dots,x_1;0)=\prod_{k=1}^{N}\delta_{x_k,y_k}. \tag{4.4}$$

Notice that the prefactor in (2.2) is equal to one at $t=0$.

The initial condition being settled, we need to prove that (2.2) satisfies the PushASEP dynamics. For that purpose, let us first compute $\frac{\mathrm{d}F_{k,l}(x,t)}{\mathrm{d}t}$.

$$\frac{\mathrm{d}F_{k,l}(x,t)}{\mathrm{d}t}=\dot a(t)F_{k,l}(x-1,t)+\dot b(t)F_{k,l}(x+1,t), \tag{4.5}$$

from which it follows, by differentiating the prefactor and the determinant column by column,

$$\begin{aligned}\frac{\mathrm{d}G(x_N,\dots,x_1;t)}{\mathrm{d}t} &= -\Big(\dot a(t)\sum_{k=1}^{N}v_k+\dot b(t)\sum_{k=1}^{N}\frac{1}{v_k}\Big)G(x_N,\dots,x_1;t)\\ &\quad+\dot a(t)\sum_{k=1}^{N}v_k G(\dots,x_k-1,\dots;t)\\ &\quad+\dot b(t)\sum_{l=1}^{N}\frac{1}{v_l}G(\dots,x_l+1,\dots;t).\end{aligned} \tag{4.6}$$

To proceed, we need an identity. Using

$$\frac{z^x}{1-v_{N+1-l}z} = \frac{v_{N+1-l}z^{x+1}}{1-v_{N+1-l}z} + z^x \tag{4.7}$$

it follows that

$$F_{k,l+1}(x,t) = F_{k,l}(x,t) + v_{N+1-l}F_{k,l+1}(x+1,t). \tag{4.8}$$

Therefore, for $j = 2,\dots,N$, by setting $\tilde{y}_k = y_{N+1-k}$,

$$\begin{aligned} G(\dots,x_j,x_{j-1}=x_j,\dots;t) &= \frac{1}{Z_N}\det\Big[v_{N+1-l}^{x_{N+1-l}}F_{k,l}(x_{N+1-l}-\tilde{y}_k,t)\Big]_{1\le k,l\le N} \\ &= \frac{1}{Z_N}\det\Big[\dots \quad v_j^{x_j}F_{k,N+1-j}(x_j-\tilde{y}_k,t) \quad v_{j-1}^{x_j}F_{k,N+2-j}(x_{j-1}-\tilde{y}_k,t)\cdots\Big]. \end{aligned} \tag{4.9}$$

Here Z_N does not depend on the x_j's.

Using (4.8) we have

$$\begin{aligned} & v_{j-1}^{x_j}F_{k,N+2-j}(x_j-\tilde{y}_k,t) \\ = \;& v_{j-1}^{x_j}F_{k,N+1-j}(x_j-\tilde{y}_k,t) + v_{j-1}^{x_j+1}F_{k,N+2-j}(x_j+1-\tilde{y}_k,t)\frac{v_j}{v_{j-1}}. \end{aligned} \tag{4.10}$$

Using this identity in the previous formula, the first term cancels being proportional to its left column, and the second term yields

$$G(\dots,x_j,x_{j-1}=x_j,\dots;t) = \frac{v_j}{v_{j-1}}G(\dots,x_j,x_{j-1}=x_j+1,\dots;t). \tag{4.11}$$

With (4.11) we can go back to (4.6). First, consider all the terms in (4.6) which are proportional to $\dot{a}(t)$. They are given by

$$-\sum_{k=1}^{N} v_k G(\dots;t) + \sum_{k=1}^{N} v_k G(\dots,x_k-1,\dots;t) \tag{4.12}$$

$$= \quad -v_1G(\dots;t) - \sum_{k=2}^{N} v_k(1-\delta_{x_{k-1},x_k+1})G(\dots;t) \tag{4.13}$$

$$+v_NG(x_N-1,\dots;t) + \sum_{k=1}^{N-1} v_k(1-\delta_{x_{k+1},x_k})G(\dots,x_k-1,\dots;t) \tag{4.14}$$

$$-\sum_{k=2}^{N} v_kG(\dots,x_k,x_{k-1}=x_k+1,\dots;t) \tag{4.15}$$

$$+\sum_{k=1}^{N-1} v_kG(\dots,x_{k+1}=x_k,x_k,\dots;t). \tag{4.16}$$

The notation means that the first term of (4.12) has been subdivided into (4.13), which contains non-zero terms when $x_{k-1} \neq x_k+1$, and (4.15), whose terms are non-zero only when $x_{k-1} = x_k+1$.

Similarly for the second term of (4.12). By using (4.11) and shifting the summation index by one, we get that (4.16) equals

$$\sum_{k=2}^{N} v_{k-1} G(\ldots, x_k, x_{k-1} = x_k + 1, \ldots; t) \frac{v_k}{v_{k-1}}, \tag{4.17}$$

which cancels (4.15). The expression (4.13) is the contribution in the master equation of the particles jumping to the right and leaving the state $(x_N, \ldots, x_1)$ with jump rate $\dot{a}(t) v_k$, while (4.14) is the contribution of the particles arriving to the state $(x_N, \ldots, x_1)$. Therefore, the jumps to the right satisfy the exclusion constraint.

Secondly, consider all the terms in (4.6) which are proportional to $\dot{b}(t)$. They are

$$-\sum_{k=1}^{N} \frac{1}{v_k} G(\ldots; t) + \sum_{k=1}^{N} \frac{1}{v_k} G(\ldots, x_k + 1, \ldots; t). \tag{4.18}$$

Let us denote by $m(k)$ the index of the last particle to the right of particle k such that particle $m(k)$ belongs to the same block of particles as particle k (we say that two particles are in the same block if between them all sites are occupied). Then, (4.18) takes the form

$$(4.18) = -\sum_{k=1}^{N} \frac{1}{v_k} G(\ldots; t) + \sum_{k=1}^{N} \frac{1}{v_k} G(\ldots, x_k + 1, x_k + 1, \ldots, x_k + k - m(k), \ldots; t). \tag{4.19}$$

Using (4.11) we get

$$\begin{aligned} & \frac{1}{v_k} G(\ldots, x_k + 1, x_k + 1, \ldots, x_k + k - m(k), \ldots; t) \\ = \quad & \frac{1}{v_k} \frac{v_k}{v_{k-1}} G(\ldots, x_k + 1, x_k + 2, \ldots, x_k + k - m(k), \ldots; t) \qquad (4.20) \\ = \quad & \frac{1}{v_{k-1}} G(\ldots, x_k + 1, x_{k-1} + 1, \ldots, x_k + k - m(k), \ldots; t). \qquad (4.21) \end{aligned}$$

By iterations we finally obtain

$$(4.18) = -\sum_{k=1}^{N} \frac{1}{v_k} G(\ldots; t) + \sum_{k=1}^{N} \frac{1}{v_{m(k)}} G(\ldots, x_k + 1, x_{k-1} + 1, \ldots, x_{m(k)} + 1, \ldots; t). \tag{4.22}$$

The first term in (4.22) is the contribution of particles pushing to the left and leaving the state $(x_N, \ldots, x_1)$, while the second term is the contribution of particles arriving at the state $(x_N, \ldots, x_1)$ because they were pushed, and the particle number k pushes to the left with rate $\dot{b}(t)/v_k$. □

We would like to obtain the joint distribution of particle N_k at time t_k for $N_1 \geq N_2 \geq \ldots \geq N_m \geq 1$ and $0 \leq t_1 \leq t_2 \leq \ldots \leq t_m$. By Proposition 2.1, this can be written as an appropriate marginal of a product of m determinants (by summing over all variables except the $x_1^{N_k}(t_k)$, $k = 1, \ldots, N_m$ under consideration).

Notational remark: Below there is an abuse of notation. For example, $x_l^n(t_i)$ and $x_l^n(t_{i+1})$ are considered different variables even if $t_i = t_{i+1}$. One could call them simply $x_l^n(i)$ and $x_l^n(i+1)$, but then one loses the connection with the times t_i's. In this sense, t_i is considered as a symbol, not as a number.

Theorem 4.1. *Let us set $t_0 = 0$, $a(t_0) = b(t_0) = 0$, and $N_{m+1} = 0$. The joint distribution of PushASEP particles is a marginal of a (generally speaking, signed) measure, obtained by summation of the variables in the set*

$$D = \{x_k^l(t_i), 1 \le k \le l, 1 \le l \le N_i, 0 \le i \le m\} \setminus \{x_1^{N_i}(t_i), 1 \le i \le m\}; \tag{4.23}$$

the range of summation for any variable in this set in $\mathbb{Z}$. Precisely,

$$\begin{aligned}
&\mathbb{P}(x_{N_i}(t_i) = x_1^{N_i}(t_i), 1 \le i \le m | x_k(0) = y_k, 1 \le k \le N_1) \\
&= \text{const} \times \sum_D \det\left[\Psi_{N_1-l}^{N_1}(x_k^{N_1}(t_0))\right]_{1\le k,l\le N_1} \\
&\times \prod_{i=1}^m \Bigg[\det[\mathcal{T}_{t_i,t_{i-1}}(x_l^{N_i}(t_i), x_k^{N_i}(t_{i-1}))]_{1\le k,l\le N_i} \\
&\qquad \times \prod_{n=N_{i+1}+1}^{N_i} \det[\phi_n(x_k^{n-1}(t_i), x_l^n(t_i))]_{1\le k,l\le n} \Bigg]
\end{aligned} \tag{4.24}$$

where

$$\begin{aligned}
\mathcal{T}_{t_j,t_i}(x,y) &= \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z z^{x-y-1} e^{(a(t_j)-a(t_i))/z} e^{(b(t_j)-b(t_i))z}, && (4.25)\\
\Psi_{N_1-l}^{N_1}(x) &= \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z z^{x-y_l-1}(1-v_{l+1}z)\cdots(1-v_{N_1}z), && (4.26)\\
\phi_n(x,y) &= v_n^{y-x}\mathbb{1}_{[y\ge x]} \quad \textit{and} \quad \phi_n(x_n^{n-1},y) = v_n^y. && (4.27)
\end{aligned}$$

The normalizing constant in (4.24) is chosen so that the sum over all variables $\{x_k^l(t_i)$, $1 \le k \le l$, $1 \le l \le N_i$, $0 \le i \le m\}$ equals 1.

Remark: the variables x_n^{n-1} participating in the last factor of (4.24) are fictitious, cf. (4.27), and are used for convenience of notation only.

We illustrate the determinantal structure in Figure 3.

Proof of Theorem 4.1. Since the evolution is Markovian, we have

$$\begin{aligned}
&\mathbb{P}(x_{N_i}(t_i) = x_1^{N_i}(t_i), 1 \le i \le m | x_k(0) = y_k, 1 \le k \le N_1) \\
= &\sum \mathbb{1}(x_1^k(0) = y_k, 1 \le k \le N_1) \\
\times &\prod_{i=1}^m \mathbb{P}(x_k(t_i) = x_1^k(t_i), 1 \le k \le N_i | x_k(t_{i-1}) = x_1^k(t_{i-1}), 1 \le k \le N_i)
\end{aligned} \tag{4.28}$$

where the sum is over $x_1^k(0)$, $1 \le k \le N_1$, and $x_1^k(t_i)$, $1 \le k \le N_i - 1$, $i = 1, \ldots, m$. Note that so far the lower index of all variables x_l^k is identically equal to 1.

The continuation of the proof requires a series of Lemmas collected at the end of this section, see Section 4.1. We apply Proposition 2.1 to the $m+1$ factors in (4.28) (including the indicator function, which corresponds to the value $t = 0$ in Proposition 2.1). Namely,

$$\begin{aligned}
&\mathbb{P}(x_k(t_i) = x_1^k(t_i), 1 \le k \le N_i | x_k(t_{i-1}) = x_1^k(t_{i-1}), 1 \le k \le N_i) \qquad (4.29)\\
&= \text{const} \times \left(\prod_{n=1}^{N_i} v_n^{x_1^n(t_i)-x_1^n(t_{i-1})}\right) \det\left[F_{k,l}(x_1^{N_i+1-l}(t_i) - x_1^{N_i+1-k}(t_{i-1}), a_i, b_i)\right]_{1\le k,l\le N_i},
\end{aligned}$$

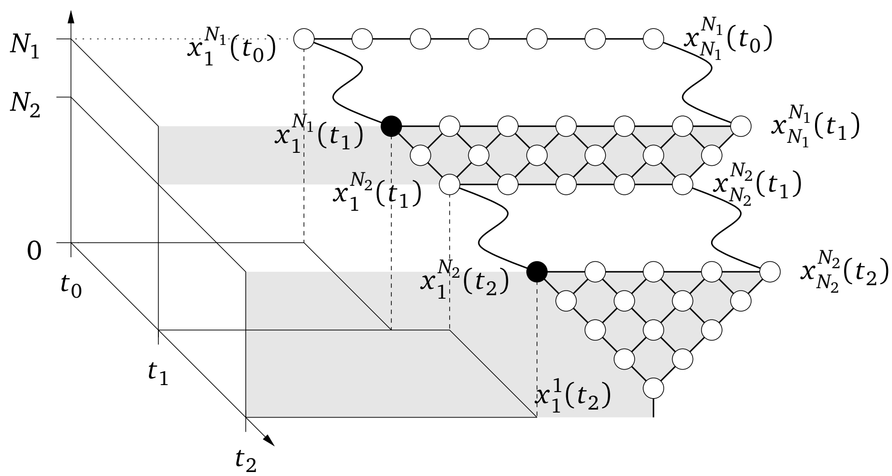

Figure 3: A graphical representation of variables entering in the determinantal structure, illustrated for $m=2$. The wavy lines represents the time evolution between t_0 and t_1 and from t_1 to t_2. The rest is the interlacing structure on the variables induced by the $\det[\phi_n(\cdots)]$. The black dots are the only variables which are not in the summation set $D=D(0)\cup D^*(t_1)\cup\cdots\cup D^*(t_m)$ (see Figure 4 too). The variables of the border of the interlacing structures are explicitly indicated.

where we introduced the notation $a_i := a(t_i)-a(t_{i-1})$, and $b_i := b(t_i)-b(t_{i-1})$.

First we collect all the factors coming from the $\prod_{n=1}^{N_i} v_n^{x_1^n(t_i)-x_1^n(t_{i-1})}$. We have the factor

$$\begin{aligned}
&\Big(\prod_{n=1}^{N_1} v_n^{x_1^n(0)-y_n}\Big)\prod_{k=1}^{m}\prod_{n=1}^{N_n} v_n^{x_1^n(t_k)-x_1^n(t_{k-1})}\\
=\;&\Big(\prod_{n=1}^{N_1} v_n^{-y_n}\Big)\Big(\prod_{i=1}^{m-1}\prod_{n=N_{i+1}+1}^{N_i} v_n^{x_1^n(t_i)}\Big)\prod_{n=1}^{N_m} v_n^{x_1^n(t_m)}.
\end{aligned}\tag{4.30}$$

Then we apply Lemma 4.4 to all the factors $\det[F_{k,l}(\cdots)]$. For the initial condition we have

$$\sum_{\widetilde{D}(0)}\det\Big[F_{N_1+1-l,1}(x_k^{N_1}(0)-y_l,0,0)\Big]_{1\le k,l\le N_1}\prod_{n=2}^{N_1}\det\Big[\varphi_n(x_k^{n-1}(0),x_l^n(0))\Big]_{1\le k,l\le n}.\tag{4.31}$$

For the other terms, $i=1,\ldots,m$, we get

$$\begin{aligned}
&\sum_{\widetilde{D}(t_i)}\det\Big[F_{N_i+1-l,1}(x_k^{N_i}(t_i)-x_1^l(t_{i-1}),a_i,b_i)\Big]_{1\le k,l\le N_i}\\
&\times\prod_{n=2}^{N_i}\det\Big[\varphi_n(x_k^{n-1}(t_i),x_l^n(t_i))\Big]_{1\le k,l\le n}.
\end{aligned}\tag{4.32}$$

Thus, the probability we want to compute in (4.28) is obtained by a marginal of a measure on $m+1$ interlacing triangles, when we sum over all the variables in $D(0),D^*(t_1),\ldots,D^*(t_m)$, see Figure 4 for the definitions of these sets. At this point we apply Lemma 4.5 as follows. For $i=1,\ldots,m-1$ we do the sum over the variables in $\widehat{D}(t_i)$. Notice that the remaining variables in (4.30) do not belong to the $\widehat{D}(t_i)$, thus we factorize them out. So, r.h.s. of (4.28) is, up to a constant, equal to

$$
\begin{aligned}
&\sum (4.30) \times \det\Big[F_{N_1+1-l,1}(x_k^{N_1}(0)-y_l,0,0)\Big]_{1\le k,l\le N_1} \\
\times\ &\Bigg[\prod_{i=0}^{m-1}\Big(\prod_{n=2}^{N_i}\det\big[\varphi_n(x_k^{n-1}(t_i),x_l^n(t_i))\big]_{1\le k,l\le n}\Big) \\
\times\ &\det\Big[F_{N_{i+1}+1-l,1}(x_k^{N_{i+1}}(t_{i+1})-x_1^l(t_i),a_{i+1},b_{i+1})\Big]_{1\le k,l\le N_{i+1}}\Bigg] \\
\times\ &\prod_{n=2}^{N_m}\det\big[\varphi_n(x_k^{n-1}(t_m),x_l^n(t_m))\big]_{1\le k,l\le n} \qquad (4.33)
\end{aligned}
$$

with the sum is over the variables described just above. By summing over the $\widehat{D}(t_i)$, the determinant with $F_{N_{i+1}+1-l,1}$ becomes a determinant with $F_{1,1}$ and the product of the $\det[\varphi_n(\cdots)]$ is restricted to $n=N_{i+1}+1,\ldots,N_i$. Thus,

$$
\begin{aligned}
(4.28) \quad = \quad & \text{const} \times \sum (4.30) \times \det\Big[F_{N_1+1-l,1}(x_k^{N_1}(0)-y_l,0,0)\Big]_{1\le k,l\le N_1} \\
\times & \prod_{i=1}^{m}\Big(\det\Big[F_{1,1}(x_k^{N_i}(t_i)-x_l^{N_i}(t_{i-1}),a_i,b_i)\Big]_{1\le k,l\le N_i} \\
& \times \prod_{n=N_{i+1}+1}^{N_i}\det\big[\varphi_n(x_k^{n-1}(t_i),x_l^n(t_i))\big]_{1\le k,l\le n}\Big) \qquad (4.34)
\end{aligned}
$$

where we set $N_{m+1}=0$ (the contribution from $n=1$ is 1). Finally, by using Lemma 4.6 we can include the terms in (4.30) into the φ_n's by modifying the last row, i.e., by setting it equal to ν_n^y. Thus,

$$
\begin{aligned}
(4.28) \quad = \quad & \text{const} \times \det\Big[F_{N_1+1-l,1}(x_k^{N_1}(0)-y_l,0,0)\Big]_{1\le k,l\le N_1} \\
\times & \prod_{i=1}^{m}\Big(\det\Big[F_{1,1}(x_k^{N_i}(t_i)-x_l^{N_i}(t_{i-1}),a_i,b_i)\Big]_{1\le k,l\le N_i} \\
& \times \prod_{n=N_{i+1}+1}^{N_i}\det\big[\phi_n(x_k^{n-1}(t_i),x_l^n(t_i))\big]_{1\le k,l\le n}\Big). \qquad (4.35)
\end{aligned}
$$

The identification to the expressions in Theorem 4.1 uses the representations (2.3) and (3.4). □

The first line represent the initial condition at $t_0=0$, the term with $\Psi_{N_1-l}^{N_1}$ in Theorem 4.1. These N_1 variables evolves until time t_1 and this is represented by the first line (term $\mathcal{T}_{t_1,t_0}$). After that, there is a reduction of the number of variables from N_1 to N_2 by the interlacing structure, which is followed by the time evolution from t_1 to t_2. This is repeated $m-1$ times. Finally it ends with an interlacing structure. If $N_1=N_2$, then the first interlacing structure is trivial (not present), while if for example $t_2=t_1$, then the time evolution is just the identity.

In what follows, the picture to keep in mind consists of reading Figure 3 from bottom to top, i.e., in the reversed order with respect to the original decomposition. Then n_i increases and t_i decreases.

This corresponds to having a sort of vicious walkers with increasing number of walkers when the transition is made by the ϕ's, and with constant number of walkers if the transition is the temporal one made by $\mathscr{T}$.

The measure in (4.24) is written with the outer product over time moments but it can be rewritten by taking the outer product over the index n in the variables x_k^n's. Let us introduce the following notations. For any level n there is a number $c(n) \in \{0,\dots,m+1\}$ of products of terms $\mathscr{T}$ which are the time evolution of n particles between consecutive times in the set $\{t_1,\dots,t_m\}$ (in other words $c(n)$ is $\#\{i|N_i = n\}$). Let us denote them by $t_0^n < \dots < t_{c(n)}^n$. Notice that $t_0^n = t_{c(n+1)}^{n+1}$, $t_0^{N_1} = t_0$, $t_1^{N_1} = t_1$, and $t_0^0 = t_{c(0)}^0 = t_m$. Then, the measure in (4.24) takes the form

$$\mathrm{const} \times \prod_{n=1}^{N_1} \left[\det[\phi_n(x_k^{n-1}(t_0^{n-1}), x_l^n(t_{c(n)}^n))]_{1\le k,l\le n} \right. \tag{4.36}$$

$$\left. \times \prod_{a=1}^{c(n)} \det[\mathscr{T}_{t_a^n,t_{a-1}^n}(x_k^n(t_a^n), x_l^n(t_{a-1}^n))]_{1\le k,l\le n} \right] \det[\Psi_{N_1-l}^{N_1}(x_k^{N_1}(t_0^{N_1}))]_{1\le k,l\le N_1}.$$

In Theorem 4.2 we show that a measure on the $x_k^n(t_a^n)$ of the form (4.36) is determinantal and we give the expression for the kernel. Then we particularize it in case of the PushASEP with particle dependent jump rates. For this purpose, we introduce a couple of notations. For any two time moments $t_{a_1}^{n_1}, t_{a_2}^{n_2}$, we define the convolution over all the transitions between them by $\phi^{(t_{a_1}^{n_1}, t_{a_2}^{n_2})}$ (backwards in time, since forward in the n's). For $(n_1,t_1) \prec (n_2,t_2)$ (see the definition in (2.5)), we set

$$\phi^{(t_{a_1}^{n_1}, t_{a_2}^{n_2})} = \mathscr{T}_{t_{a_1}^{n_1}, t_0^{n_1}} * \phi_{n_1+1} * \mathscr{T}^{n_1+1} * \dots * \phi_{n_2} * \mathscr{T}_{t_{c(n_2)}^{n_2}, t_{a_2}^{n_2}} \tag{4.37}$$

where

$$\mathscr{T}^n = \mathscr{T}_{t_{c(n)}^n, t_0^n}. \tag{4.38}$$

For $(n_1,t_1) \not\prec (n_2,t_2)$ we set $\phi^{(t_{a_1}^{n_1}, t_{a_2}^{n_2})} = 0$. Above we used

$$\mathscr{T}_{t_3,t_2} * \mathscr{T}_{t_2,t_1} = \mathscr{T}_{t_3,t_1}, \tag{4.39}$$

which is an immediate corollary of (4.25). In a more general case considered in Theorem 4.2 below, if (4.39) does not holds, then $\mathscr{T}^n$ is just the convolution of the transitions between $t_{c(n)}^n$ and t_0^n by definition. Moreover, define the matrix M with entries $M_{k,l}$, $1 \le k,l \le N_1$,

$$M_{k,l} = \left(\phi_k * \mathscr{T}^k * \dots * \phi_{N_1} * \mathscr{T}^{N_1} * \Psi_{N_1-l}^{N_1}\right)(x_k^{k-1}) \tag{4.40}$$

and the vector

$$\Psi_{n-l}^{n,t_a^n} = \phi^{(t_a^n, t_0^{N_1})} * \Psi_{N_1-l}^{N_1}. \tag{4.41}$$

We remind that the variables x_k^{k-1} in $M_{k,l}$ are fictitious, compare with (4.27).

Theorem 4.2. *Assume that the matrix M is invertible. Then the normalizing constant in (4.36) is equal to $(\det M)^{-1}$, the normalized measure*[2] *of the form (4.36) viewed as $(N_1 + \dots + N_m)$-point process is*

[2] With *normalized measure* we mean that all weights add up to one. If all weights are non-negative, it is a *probability measure* (this is the case for example for PushASEP with step initial conditions).

determinantal, and the correlation kernel can be computed as follows

$$\begin{aligned} K(t^{n_1}_{a_1},x_1;t^{n_2}_{a_2},x_2) &= -\phi^{(t^{n_1}_{a_1},t^{n_2}_{a_2})}(x_1,x_2) \\ &+ \sum_{k=1}^{N_1}\sum_{l=1}^{n_2}\Psi^{n_1,t^{n_1}_{a_1}}_{n_1-k}(x_1)[M^{-1}]_{k,l}(\phi_l*\phi^{(t^l_{c(l)},t^{n_2}_{a_2})})(x^{l-1}_l,x_2). \end{aligned} \tag{4.42}$$

In the case when the matrix M is upper triangular, there is a simpler way to write the kernel. Set

$$\Phi^{n,t^n_a}_{n-k}(x)=\sum_{l=1}^{n}[M^{-1}]_{k,l}\big(\phi_l*\phi^{(t^l_{c(l)},t^n_a)}\big)(x^{l-1}_l,x) \tag{4.43}$$

for all $n=1,\dots,N_1$ and $k=1,\dots,n$. Then, $\{\Phi^{n,t^n_a}_{n-k}\}_{k=1,\dots,n}$ is the unique basis of the linear span of

$$\Big\{(\phi_1*\phi^{(t^1_{c(1)},t^n_a)})(x^0_1,x),\dots,(\phi_n*\phi^{(t^n_{c(n)},t^n_a)})(x^{n-1}_n,x)\Big\} \tag{4.44}$$

that is biorthogonal to $\{\Psi^{n,t^n_a}_{n-k}\}$:

$$\sum_{x\in\mathbb{Z}}\Phi^{n,t^n_a}_i(x)\Psi^{n,t^n_a}_j(x)=\delta_{i,j},\quad i,j=0,\dots,n-1. \tag{4.45}$$

The correlation kernel can then be written as

$$K(t^{n_1}_{a_1},x_1;t^{n_2}_{a_2},x_2)=-\phi^{(t^{n_1}_{a_1},t^{n_2}_{a_2})}(x_1,x_2)+\sum_{k=1}^{n_2}\Psi^{n_1,t^{n_1}_{a_1}}_{n_1-k}(x_1)\Phi^{n_2,t^{n_2}_{a_2}}_{n_2-k}(x_2). \tag{4.46}$$

Moreover, one has the identity

$$\phi^{(t^{n_1}_{a_1},t^{n_2}_{a_2})}*\Phi^{n_2,t^{n_2}_{a_2}}_{n_2-l}=\Phi^{n_1,t^{n_1}_{a_1}}_{n_1-l} \tag{4.47}$$

for $n_1\geq n_2$ and $a_1\leq a_2$ for $n_1=n_2$.

Proof of Theorem 4.2. The proof is similar to the one of Lemma 3.4 in [4], which is in its turn based on the formalism of [8]. The only place where the argument changes substantially is the definition of the matrix L, see [4], formula (3.32). We need to construct the matrix L in such a way that its suitable minors reproduce, up to a common constant, the weights (4.36) of the measure. Then our measure turns into a *conditional L-ensemble* in the terminology of [8].

The variables of interest live in the space $\mathfrak{Y}=\mathfrak{X}^{(1)}\cup\dots\cup\mathfrak{X}^{(N_1)}$, with $\mathfrak{X}^{(n)}=\mathfrak{X}^{(n)}_0\cup\dots\cup\mathfrak{X}^{(n)}_{c(n)}$, where $\mathfrak{X}^{(n)}_a=\mathbb{Z}$ is the space where the n variables at time t^n_a live. Let us also denote $I=\{1,\dots,N_1\}$. Then, the matrix L written with the order given by the entries in the set of all variables $\mathfrak{X}=I\cup\mathfrak{Y}$ becomes

$$L=\begin{pmatrix} 0 & E_0 & 0 & E_1 & 0 & E_2 & 0 & \cdots & E_{N_1-1} & 0 \\ 0 & 0 & -T_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -W_{[1,2)} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & -T_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -W_{[2,3)} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -T_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -W_{[N_1-1,N_1)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -T_{N_1} \\ \Psi^{(N_1)} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{4.48}$$

with the matrix blocks in L have the following entries:

$$[\Psi^{(N_1)}]_{x,j} = \Psi^{N_1}_{N_1-j}(x), \quad x \in \mathfrak{X}^{(N_1)}_0, j \in I, \tag{4.49}$$

$$[E_n]_{i,y} = \begin{cases} \phi_{n+1}(x^n_{n+1}, y), & i = n+1, y \in \mathfrak{X}^{(n+1)}_{c(n+1)}, \\ 0, & i \in I \setminus \{n+1\}, y \in \mathfrak{X}^{(n+1)}_{c(n+1)}, \end{cases} \tag{4.50}$$

$$[W_{[n,n+1)}]_{x,y} = \phi_{n+1}(x,y), \quad x \in \mathfrak{X}^{(n)}_0, y \in \mathfrak{X}^{(n+1)}_{c(n+1)}, \tag{4.51}$$

and T_n is the matrix made of blocks

$$T_n = \begin{pmatrix} T_{n,1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & T_{n,c(n)} \end{pmatrix}, \tag{4.52}$$

where

$$[T_{n,a}]_{x,y} = \mathscr{T}_{t^n_a, t^n_{a-1}}(x,y), \quad x \in \mathfrak{X}^{(n)}_a, y \in \mathfrak{X}^{(n)}_{a-1}. \tag{4.53}$$

The rest of the proof is along the same lines as that of Lemma 3.4 in [4].

Although the argument gives a proof in the case when all variables $x^n_a(t^n_b)$ vary over finite sets, a simple limiting argument immediately extends the statement to any discrete sets, provided the series that defines $M_{k,l}$ are absolutely convergent, which is certainly true in our case. □

A special case of Theorem 4.2 is Proposition 3.1 stated in Section 3, which we prove below.

Proof of Proposition 3.1. We first prove the statement in the case the jump rates are ordered, $v_1 > v_2 > \dots$, and then use analytic continuation in v_j's.

For $v_1 > v_2 > \dots$, the claim is a specialization of Theorem 4.2. The kernel depends only on the actual times and particle numbers, therefore we might drop the label a_i of $t^{n_i}_{a_i}$. Equivalently, we can use the notation (n_i, t_i) instead of $t^{n_i}_{a_i}$, to go back to the natural notations of the model. For PushASEP we have $\Psi^{N_1}_{N_1-l}(x) = F_{N_1+1-l,1}(x - y_l, 0, 0)$ and

$$\mathscr{T}_{t_j,t_i}(x,y) = F_{1,1}(x-y, a(t_j) - a(t_i), b(t_j) - b(t_i)). \tag{4.54}$$

First of all, we sum over the $\{x^{N_1}_k(0), 1 \le k \le N_1\}$ variables, since we are not interested in the initial conditions (being fixed). When applied to the $F_{k,l}(x, a(t_i), b(t_i))$, the time evolution $\mathscr{T}_{t_j,t_i}$ changes it into $F_{k,l}(x, a(t_j), b(t_j))$,

$$\sum_{y\in\mathbb{Z}} \mathscr{T}_{t_j,t_i}(x,y) F_{k,l}(y, a(t_i), b(t_i)) = F_{k,l}(x, a(t_j), b(t_j)). \tag{4.55}$$

This implies that Theorem 4.2 still holds but with $t^{N_1}_0 = t_1$ and

$$\Psi^{N_1}_{N_1-l}(x) = F_{N_1+1-l,1}(x - y_l, a(t_1), b(t_1)). \tag{4.56}$$

We have, see (4.69), that

$$(\phi_k * F_{l,N_1+1-k})(x,a,b) = F_{l,N_1+2-k}(x,a,b). \tag{4.57}$$

Using (4.55) and (4.57) repeatedly one then gets

$$\Psi_{n-l}^{n,t_k^n}(x) = F_{N_1+1-l,N_1+1-n}(x - y_l, a(t_k^n), b(t_k^n)) \tag{4.58}$$

which can be rewritten as (3.4).

Next we show that the matrix M is upper triangular. Once again, (4.55) and (4.57) are applied several times, leading to

$$M_{k,l} = \sum_{y\in\mathbb{Z}} v_k^y F_{N_1+1-l,N_1+1-k}(y - y_l, a(t_{c(k)}^k), b(t_{c(k)}^k)). \tag{4.59}$$

Set $a_k = a(t_{c(k)}^k)$ and $b_k = b(t_{c(k)}^k)$. Then,

$$M_{k,l} = \sum_{y\in\mathbb{Z}} v_k^y \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z z^{y-y_l-1} e^{a_k/z+b_k z} \frac{(1-v_{l+1}z)\cdots(1-v_{N_1}z)}{(1-v_{k+1}z)\cdots(1-v_{N_1}z)}. \tag{4.60}$$

(Note that we need the assumption $v_k > \max\{v_l\}_{l>k}$ in order for this sum to converge.) We divide the sum over y in two regions, $\{y \geq 0\}$ and $\{y < 0\}$. The sum over $y \geq 0$ can be taken into the integral provided that $|v_k z| < 1$ and then we use $\sum_{y\geq 0}(v_k z)^y = \frac{1}{1-v_k z}$. Similarly, the sum over $y < 0$ is taken into the integrals provided that $|v_k z| > 1$ and one uses $\sum_{y\geq 0}(v_k z)^y = -\frac{1}{1-v_k z}$. For $k > l$ the new term in the denominator, $1 - v_k z$, is canceled so that this is not a pole and we can deform the contours to be the same. Thus for $k > l$ the net result is zero. This is not the case for $k \leq l$, since in that case the new pole at $1/v_k$ does not have to vanish. The diagonal term is easy to compute, since the pole at $1/v_k$ is simple. Computing its residue we obtain $M_{k,k} = v_k^{y_l+1} e^{v_k a_k + b_k/v_k}$ and

$$\det M = \prod_{k=1}^{N_1} v_k^{y_l+1} e^{v_k a_k + b_k/v_k} \neq 0. \tag{4.61}$$

Next, we need to determine the space V_n where the orthogonalization has to be made. We have

$$(\phi_k * \phi^{(t_{c(k)}^k, t_1)})(x_k^{k-1}, x) = \sum_{y\in\mathbb{Z}} v_k^y \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z z^{y-x-1} \frac{e^{a_k/z+b_k z}}{(1-v_{k+1}z)\cdots(1-v_{N_1}z)}. \tag{4.62}$$

Once again we divide the sum over $\{y \geq 0\}$, $\{y < 0\}$ and then deform the paths so that the only remaining contribution is the residue at $z = 1/v_k$, which is equal to $const \cdot v_k^x$. Thus, $V_n = \mathrm{span}(v_1^x, \ldots, v_n^x)$, $n = 1, \ldots, N_1$.

Finally, we need an expression for the transition between two times, which is given by (4.37). Every time that we convolute a ϕ_k with $\mathscr{T}$, we get an extra factor $1/(1 - v_k z)$ in the integral. Therefore, if $t_{a_2}^{n_2} \leq t_{a_1}^{n_1}$ and $n_2 \geq n_1$, then

$$\phi^{(t_{a_1}^{n_1}, t_{a_2}^{n_2})}(x,y) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z z^{x-y-1} \frac{e^{(a(t_{a_1}^{n_1})-a(t_{a_2}^{n_2}))/z} e^{(b(t_{a_1}^{n_1})-b(t_{a_2}^{n_2}))z}}{(1-v_{n_1+1}z)\cdots(1-v_{n_2}z)}, \tag{4.63}$$

while $\phi^{(t_{a_1}^{n_1}, t_{a_2}^{n_2})}(x,y) = 0$ otherwise.

Now let us do the analytic continuation.

As all the functions $\Psi_k^{n,t}$, see (3.4), can be estimated as

$$|\Psi_k^{n,t}(x)| \le \text{const} \cdot q^{|x|}, \qquad x \in \mathbb{Z}, \tag{4.64}$$

for any $q > 0$ and $v_1, v_2, \dots$ varying in a compact set, the weights (4.36) can be majorated by a convergent series for $v_1, v_2, \dots$ varying in a compact set. Further, the normalizing constant $(\det M)^{-1}$ is analytic as long as v_j's are nonzero, see (4.61). Thus, the correlation functions of our measure are analytic in v_j's.

Set, for $k = 0, \dots, n-1$,

$$f_k(x) = \frac{1}{2\pi\mathrm{i}} \oint \frac{\mathrm{d}z\, z^{-x-1}}{(1 - v_{n-k}z)(1 - v_{n-k+1}z)\cdots(1 - v_n z)}, \tag{4.65}$$

where the integration contour includes the poles $v_{n-k}^{-1}, \dots, v_n^{-1}$. Note that $f_k(x)$ is a linear combination of $v_{n-k}^x, \dots, v_n^x$. Denote by $G = [G_{k,l}]_{k,l=0,\dots,n-1}$ the Gram matrix

$$G_{k,l} = \sum_{x \in \mathbb{Z}} f_k(x) \Psi_l^{n,t}(x). \tag{4.66}$$

Then for $v_1 > v_2 > \dots$ we have

$$\Phi_k^{n,t}(x) = \sum_{l=0}^{n-1} [G^{-1}]_{k,l} f_l(x). \tag{4.67}$$

Since the matrix M is triangular, G is also triangular. Its diagonal elements are easy to compute: $G_{k,k} = e^{a(t)v_k + b(t)/v_k} v_k^{y_k+1}$. Hence, (4.67) gives a formula for Φ's that is analytic in v_j's as long as they stay away from zero. This implies that the corresponding expression for the correlation kernel (3.3) is also analytic in v_j's, and thus both sides of the determinantal formula for the correlation functions can be analytically continued. Finally, it is not difficult to see that the functions (4.65) span the space V_n given by (3.1), which implies the statement. □

4.1 Some lemmas

In this subsection we state and prove the Lemmas used in the proof of Theorem 4.2.

Lemma 4.3. *Let us define the function*

$$\varphi_n(x, y) = \begin{cases} v_n^{y-x}, & y \ge x, \\ 0, & y < x. \end{cases} \tag{4.68}$$

Then the following recurrence relations holds

$$F_{k,l+1}(x, a, b) = (\varphi_{N+1-l} * F_{k,l})(x, a, b) \tag{4.69}$$

and

$$F_{k-1,l}(x, a, b) = (\varphi_{N+2-k} * F_{k,l})(x, a, b). \tag{4.70}$$

From (4.70) and $\varphi_n(x, y) = \varphi_n(0, y - x) = \varphi_n(-y, -x)$ it follows

$$F_{k-1,l}(-x, a, b) = \sum_{y \in \mathbb{Z}} F_{k,l}(-y, a, b) \varphi_{N+2-k}(y, x). \tag{4.71}$$

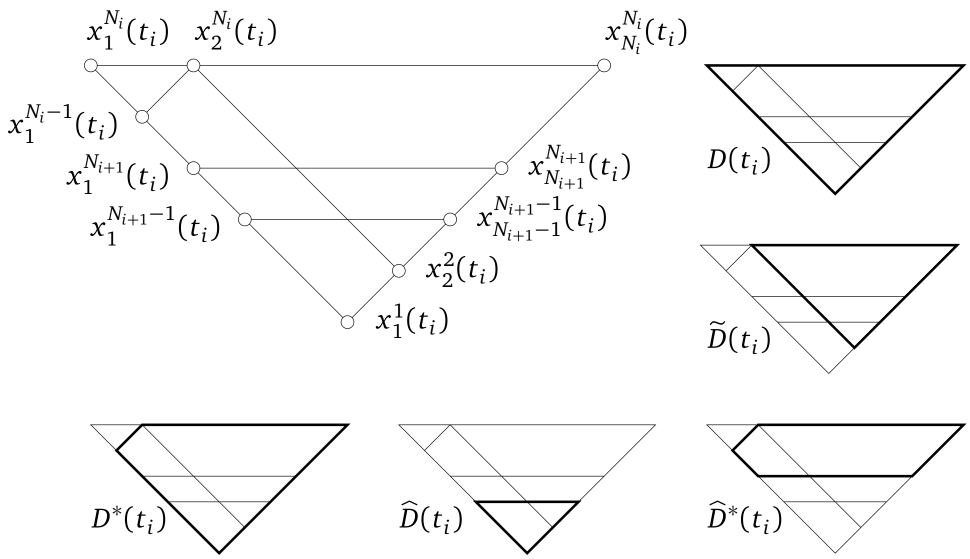

Figure 4: A graphical representation of the summation domains that occurs in the next lemmas and theorem. The bold lines passes through the border of the domains.

Proof of Lemma 4.3. We have

$$F_{k,l}(x,a,b)=\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_0}\mathrm{d}z z^{x-1}e^{bz}e^{a/z}\frac{(1-v_N z)\cdots(1-v_{N+2-k}z)}{(1-v_N z)\cdots(1-v_{N+2-l}z)}. \tag{4.72}$$

Then applying $\sum_{y\geq x} v_{N+1-l}^{y-x} z^y = z^x/(1-v_{N+1-l}z)$ (for $|z|\ll 1$), we get that in the denominator we have an extra factor, which corresponds to increasing l by one. Similarly, applying φ_{N+2-k}, the extra factor in the denominator cancels the last one in the numerator, thus this is equivalent to decreasing k by one. □

We define the following domains, which will occurs several times in the following. A graphical representation is in Figure 4. Let us denote the set of interlacing variables at time t_i by

$$D(t_i)=\{x_k^n(t_i), 1\leq n\leq N_i, 1\leq k\leq n | x_k^{n+1}(t_i)<x_k^n(t_i)\leq x_{k+1}^{n+1}(t_i)\}. \tag{4.73}$$

Then let

$$\widetilde{D}(t_i)=\{x_k^n(t_i)\in D(t_i)|k\geq 2\},\quad \widehat{D}(t_i)=\{x_k^n(t_i)\in D(t_i)|n\leq N_{i+1}-1\}, \tag{4.74}$$

and

$$D^*(t_i)=D(t_i)\setminus\{x_1^{N_i}(t_i)\},\quad \widehat{D}^*(t_i)=D^*(t_i)\setminus\widehat{D}(t_i). \tag{4.75}$$

Lemma 4.4. *We have the identity*

$$\begin{aligned}
&\det\left[F_{k,l}(x_1^{N_i+1-l}(t_i)-x_1^{N_i+1-k}(t_{i-1}),a,b)\right]_{1\leq k,l\leq N_i}\\
=\;&\mathrm{const}\sum_{\widetilde{D}(t_i)}\left(\prod_{n=2}^{N_i}\det\left[\varphi_n(x_k^{n-1}(t_i),x_l^n(t_i))\right]_{1\leq k,l\leq n}\right)\\
\times\;&\det\left[F_{N_i+1-l,1}(x_k^{N_i}(t_i)-x_1^l(t_{i-1}),a,b)\right]_{1\leq k,l\leq N_i}
\end{aligned} \tag{4.76}$$

where we set $\varphi_n(x_n^{n-1}, x) = 1$.

Proof of Lemma 4.4. By changing the indices we get that l.h.s. of (4.76) is, up to a sign, equal to

$$\det\left[F_{N_i+1-l,k}(x_1^{N_i+1-k}(t_i) - x_1^l(t_{i-1}), a, b)\right]_{1\le k,l\le N_i} \tag{4.77}$$

Using repeatedly the identity (4.69) we have

$$F_{n,k}(x,a,b) = (\varphi_{N_i+2-k} * \cdots * \varphi_{N_i} * F_{n,1})(x,a,b). \tag{4.78}$$

Therefore,

$$(4.77) = \det\left[(\varphi_{N_i+2-k} * \cdots * \varphi_{N_i} * F_{N_i+1-l,1})(x_1^{N_i+1-k}(t_i) - x_1^l(t_{i-1}), a, b)\right]_{1\le i,j\le N_i} \tag{4.79}$$

We write explicitly the convolution by introducing explicit summation variables as follows

$$\begin{aligned} &(\varphi_{N_i+2-k} * \cdots * \varphi_{N_i} * F_{N_i+1-l,1})(x_1^{N_i+1-k}(t_i) - x_1^l(t_{i-1}), a, b) \\ = &\sum_{\substack{x_n^{N_i+1-k+n}, \\ 1\le n\le k-1}} \Big(\prod_{n=1}^{k-1} \varphi_{N_i+1-k+n}(x_n^{N_i-k+n}(t_i), x_{n+1}^{N_i+1-k+n}(t_i))\Big) \\ &\qquad \times F_{N_i+1-l,1}(x_k^{N_i}(t_i) - x_1^l(t_{i-1}), a, b), \end{aligned} \tag{4.80}$$

where we used the fact that $\varphi_m(x,y) = \varphi_m(x+c, y+c)$ for any $c \in \mathbb{Z}$. By multi-linearity of the determinant, we can take the sums and the factors φ's out of the determinant with the result

$$\begin{aligned} (4.77) \quad = &\sum_{\substack{x_k^n(t_i), \\ 2\le k\le n\le N_i}} \Big(\prod_{n=2}^{N_i}\prod_{k=1}^{n-1} \varphi_n(x_k^{n-1}(t_i), x_{k+1}^n(t_i))\Big) \\ \times &\det\left[F_{N_i+1-l,1}(x_j^{N_i}(t_i) - x_1^l(t_{i-1}), a, b)\right]_{1\le j,l\le N_i}. \end{aligned} \tag{4.81}$$

The product of the φ's is non-zero only if $x_k^{n-1}(t_i) \le x_{k+1}^n(t_i)$. Applying Lemma 3.3 in [4] we can further reduce the summation domain to $\widetilde{D}(t_i)$ without changing the result.

Finally, the product of the determinants of φ's in the right-hand side of (4.76) is either 1 or 0 depending on whether the variables interlace (belongs to $D(t_i)$) or not. This implies (4.76). ☐

Lemma 4.5. *We have the identity*

$$\begin{aligned} &\sum_{\widehat{D}(t_i)} \Big(\prod_{n=2}^{N_{i+1}} \det\left[\varphi_n(x_k^{n-1}(t_i), x_l^n(t_i))\right]_{1\le k,l\le n}\Big) \\ \times \quad &\det\left[F_{N_{i+1}+1-l,1}(x_k^{N_{i+1}}(t_{i+1}) - x_1^l(t_i), a, b)\right]_{1\le k,l\le N_{i+1}} \\ = \quad &\det\left[F_{1,1}(x_k^{N_{i+1}}(t_{i+1}) - x_l^{N_{i+1}}(t_i), a, b)\right]_{1\le k,l\le N_{i+1}}. \end{aligned} \tag{4.82}$$

Proof of Lemma 4.5. By an analogue (essentially inverse) procedure as in the proof of Lemma 4.4, we first get

$$
\begin{aligned}
(4.82) \quad = & \sum_{\substack{x_k^n(t_i),\\ 2\le n\le N_{i+1}-1,\\ 1\le k\le n}} \Big(\prod_{n=2}^{N_{i+1}}\prod_{k=1}^{n-1} \varphi_n(x_k^{n-1}(t_i), x_{k+1}^n(t_i))\Big) \\
& \times \ \det\Big[F_{N_{i+1}+1-l,1}(x_k^{N_{i+1}}(t_{i+1}) - x_1^l(t_i), a, b)\Big]_{1\le k,l\le N_{i+1}}. \qquad (4.83)
\end{aligned}
$$

Now we insert by linearity the factor $\prod_{n=l+1}^{N_{i+1}} \varphi_n(x_l^{n-1}(t_i), x_{l+1}^n(t_i))$ to terms $F_{N_{i+1}+1-l,1}(x_k^{N_{i+1}}(t_{i+1}) - x_1^l(t_i), a, b)$ as well as the sum over the corresponding variables. The sums are carried out by using (4.71), from which we get the r.h.s. of (4.82). □

Lemma 4.6. *Let us define*

$$\phi_n(x,y) = \varphi_n(x,y), \quad \phi_n(x_n^{n-1}, y) = v_n^y. \tag{4.84}$$

Then

$$v_n^{x_1^n} \det\Big[\varphi_n(x_k^{n-1}, x_l^n)\Big]_{1\le k,l\le n} = \det\Big[\phi_n(x_k^{n-1}, x_l^n)\Big]_{1\le k,l\le n} \tag{4.85}$$

Proof of Lemma 4.6. It is a consequence of the fact that both determinants are zero if the variables x_i^j do not interlace and when they do, the matrices are upper-triangular with diagonal equal to zero and with equal entries in the first $n-1$ rows. The only difference is for the last row, where the matrix in l.h.s. of (4.85) has entries 1 and r.h.s. of (4.85) has entries $v_n^{x_l^n}$. □

5 Asymptotic analysis

5.1 Flat initial conditions

To prove Theorem 2.2 we need the uniform convergence of the kernel in bounded sets as well as bounds uniform in T. These results are provided in the following Propositions 5.1, 5.2, 5.3.

Let us define the rescaled and conjugate kernel by

$$K_T^{\rm resc}(u_1,s_1;u_2,s_2) = K((n_1,t_1),x_1;(n_2,t_2),x_2)T^{1/3}\frac{e^{t_2(2L+R/2)}2^{x_2}}{e^{t_1(2L+R/2)}2^{x_1}} \tag{5.1}$$

where $n_i = n(u_i)$, $t_i = t(u_i)$, and

$$x_i = [-2n_i + \mathbf{v}\, t_i - s_i T^{1/3}]. \tag{5.2}$$

Proposition 5.1 (Uniform convergence in a bounded set)**.** *Fix u_1, u_2, then for any fixed $\ell > 0$, the rescaled kernel $K_T^{\rm resc}$ converges uniformly for $(s_1,s_2) \in [-\ell,\ell]^2$ as*

$$\lim_{T\to\infty} K_T^{\rm resc}(u_1,s_1;u_2,s_2) = S_v^{-1} K_{\mathscr{A}_1}(S_h^{-1}u_1, S_v^{-1}s_1; S_h^{-1}u_2, S_v^{-1}s_2), \tag{5.3}$$

with $K_{\mathscr{A}_1}$ the kernel of the Airy$_1$ process, see (2.25), and S_v, S_h are defined in (2.13).

Proof of Proposition 5.1. First we consider the term coming from the second integral in (3.23), namely

$$\frac{-T^{1/3}}{2\pi\mathrm{i}}\oint_{\Gamma_1}\mathrm{d}z\frac{e^{Rt_1(1-z)+Lt_1/(1-z)}}{e^{Rt_2 z+Lt_2/z}}\frac{z^{n_1+n_2+x_2}}{(1-z)^{n_1+n_2+x_1+1}}\frac{e^{t_2(2L+R/2)}2^{x_2}}{e^{t_1(2L+R/2)}2^{x_1}}. \tag{5.4}$$

Define the functions

$$\begin{aligned}
H(z) &= Rz+L/z-(R/2-2L)\ln(z),\\
g_0(z) &= (\pi(\theta)+\theta)H(z),\\
g_1(z,u) &= -u(\pi'(\theta)+1)H(z)+u(1-\pi'(\theta))\ln(z(1-z)),\\
g_2(z,u,s) &= u^2\pi''(\theta)[H(z)+\ln(z(1-z))]+s\ln(z),
\end{aligned} \tag{5.5}$$

from which we then set

$$\begin{aligned}
f_0(z) &= g_0(1-z)-g_0(z),\\
f_1(z) &= g_1(1-z,u_1)-g_1(z,u_2)-g_1(1/2,u_1)+g_1(1/2,u_2),\\
f_2(z) &= g_2(1-z,u_1,s_1)-g_2(z,u_2,s_2)-g_2(1/2,u_1,s_1)+g_2(1/2,u_2,s_2),\\
f_3(z) &= -\ln(1-z).
\end{aligned} \tag{5.6}$$

With these notations we get

$$(5.4)=\frac{-T^{1/3}}{2\pi\mathrm{i}}\oint_{\Gamma_1}\mathrm{d}z e^{Tf_0(z)+T^{2/3}f_1(z)+T^{1/3}f_2(z)+f_3(z)}. \tag{5.7}$$

The function $f_0(z)$ has a double critical point at $z=1/2$ and the contribution for large T will be dominated by the one close $z=1/2$. Thus we need to do series expansions around the critical point. Computations leads to

$$\begin{aligned}
f_0(z) &= \tfrac{1}{3}\kappa_0(z-1/2)^3+\mathcal{O}((z-1/2)^4),\\
f_1(z) &= -(u_1-u_2)\kappa_1(z-1/2)^2+\mathcal{O}((z-1/2)^3),\\
f_2(z) &= -2(s_1+s_2)(z-1/2)+\mathcal{O}((z-1/2)^2),\\
f_3(z) &= \ln(2)+\mathcal{O}((z-1/2))
\end{aligned} \tag{5.8}$$

with

$$\kappa_0=8(8L+R)(\pi(\theta)+\theta),\quad \kappa_1=(R+4L)(\pi'(\theta)+1)+4(1-\pi'(\theta)). \tag{5.9}$$

First we choose Γ_1 to be a steep descent path[3] for $f_0(z)$. Important for the later analysis is that the chosen steep descent path is, close to the critical point, the steepest descent one. We consider $\Gamma_1=\gamma\vee\gamma_c\vee\bar{\gamma}$, where $\gamma=\{1/2+e^{-\mathrm{i}\pi/3}\xi, 0\leq\xi\leq 1/2\}$, $\bar{\gamma}$ its image with respect to complex conjugation, and $\gamma_c=\{1-1/2e^{\mathrm{i}\phi},\pi/3\leq\phi\leq 2\pi-\pi/3\}$. We also have $f_0(z)=S_R(z)R(\pi(\theta)+\theta)+S_L(z)L(\pi(\theta)+\theta)$, with

$$S_R(z)=1-2z+\frac{1}{2}\ln(z/(1-z)),\quad S_L(z)=\frac{1}{1-z}-\frac{1}{z}-2\ln(z/(1-z)). \tag{5.10}$$

[3]For an integral $I=\int_\gamma \mathrm{d}z e^{tf(z)}$, we say that γ is a steep descent path if (1) $\mathrm{Re}(f(z))$ is maximum at some $z_0\in\gamma$: $\mathrm{Re}(f(z))<\mathrm{Re}(f(z_0))$ for $z\in\gamma\setminus\{z_0\}$, and (2) $\mathrm{Re}(f(z))$ is monotone along γ except at its maximum point z_0 and, if γ is closed, at a point z_1 where the minimum of $\mathrm{Re}(f)$ is reached.

On γ, simple computations leads to

$$
\begin{aligned}
\frac{\mathrm{dRe}(S_R(z))}{\mathrm{d}\xi} &= -\frac{8\xi^2(1+2\xi^2)}{((1+\xi^2)+2\xi^2)((1-\xi)^2+2\xi^2)}, \\
\frac{\mathrm{dRe}(S_L(z))}{\mathrm{d}\xi} &= -\frac{64\xi^2((1+2\xi^2)^2-12\xi^4)}{((1+\xi^2)+2\xi^2)^2((1-\xi)^2+2\xi^2)^2}
\end{aligned}
\tag{5.11}
$$

which are both strictly less than 0 for $\xi \in (0,1/2)$. Now consider the part of γ_c with $\phi \in [\pi/3,\pi]$. Then

$$
\begin{aligned}
\frac{\mathrm{dRe}(S_R(z))}{\mathrm{d}\phi} &= -\frac{4\sin(\phi)(1-\cos(\phi))}{5-4\cos(\phi)}, \\
\frac{\mathrm{dRe}(S_L(z))}{\mathrm{d}\phi} &= -\frac{32\sin(\phi)(1-\cos(\phi))(2-\cos(\phi))}{(5-4\cos(\phi))^2}
\end{aligned}
\tag{5.12}
$$

which are both strictly less than 0 for $\phi \in [\pi/3,\pi)$. For the piece of γ_c with $\phi \in (\pi, 2\pi - \pi/3]$, (5.12) is strictly positive, which is right since when ϕ increases we go closer to the critical point. Therefore the chosen Γ_1 is a steep descent path for $f_0(z)$.

Take any $\delta > 0$ and set $\Gamma_1^\delta = \{z \in \Gamma_1 | |z-1/2| \leq \delta\}$. Then, if in (5.7) we integrate only along Γ_1^δ instead of integrating along Γ_1, the error made is just of order $\mathcal{O}(e^{-cT})$ for some $c > 0$ (more exactly, $c \sim \delta^3$ for δ small). Thus we now consider the integral on Γ_1^δ only. There, we can use the above series expansions to obtain

$$
\begin{aligned}
&\frac{-2T^{1/3}}{2\pi\mathrm{i}} \int_{\Gamma_1^\delta} \mathrm{d}z e^{\frac{1}{3}\kappa_0 T(z-1/2)^3 + (u_2-u_1)\kappa_1 T^{2/3}(z-1/2)^2 - 2(s_1+s_2)(z-1/2)} \\
&\times e^{\mathcal{O}\left(T(z-1/2)^4, T^{2/3}(z-1/2)^3, T^{1/3}(z-1/2)^2, (z-1/2)\right)}.
\end{aligned}
\tag{5.13}
$$

The difference between (5.13) and the same integral without the error term can be bounded by applying $|e^x - 1| \leq |x| e^{|x|}$ to $\mathcal{O}(\cdots)$. Thus, this error term can be bounded by

$$
\begin{aligned}
&\frac{2T^{1/3}}{2\pi} \int_{\Gamma_1^\delta} \mathrm{d}z \Big| e^{\frac{1}{3}c_0\kappa_0 T(z-1/2)^3 + (u_2-u_1)c_1\kappa_1 T^{2/3}(z-1/2)^2 - 2c_2(s_1+s_2)(z-1/2)} \\
&\times \mathcal{O}\left(T(z-1/2)^4, T^{2/3}(z-1/2)^3, T^{1/3}(z-1/2)^2, (z-1/2)\right) \Big|
\end{aligned}
\tag{5.14}
$$

for some c_0, c_1, c_2 which can be taken as close to 1 as needed by setting δ small enough. Then, by the change of variable $T^{1/3}(z-1/2) = w$ one gets that this error term is of order $\mathcal{O}(T^{-1/3})$ (what is needed is just $c_0 > 0$).

It remains to consider the leading term, namely (5.13) without the error terms. By extending the integral to infinity by continuing the two small straight segments forming Γ_1^δ, the error made is of order $\mathcal{O}(e^{-cT})$. Thus we obtained that (5.4) is, up to an error $\mathcal{O}(e^{-cT}, T^{-1/3})$ uniform for $s_1, s_2 \in [-\ell, \ell]^2$, equal to

$$
\frac{-2T^{1/3}}{2\pi\mathrm{i}} \int_{\gamma_\infty} \mathrm{d}z e^{\frac{1}{3}\kappa_0 T(z-1/2)^3 + (u_2-u_1)\kappa_1 T^{2/3}(z-1/2)^2 - 2(s_1+s_2)(z-1/2)},
\tag{5.15}
$$

where γ_∞ is a path going from $e^{i\pi/3}\infty$ to $e^{-i\pi/3}\infty$. By the change of variable $w=(\kappa_0 T)^{1/3}(z-1/2)$, we get

$$\begin{aligned}(5.15) &= \frac{-1}{2\pi i}\int_{\gamma_\infty} dw \frac{2}{\kappa_0^{1/3}} e^{\frac{1}{3}w^3+(u_2-u_1)w^2\kappa_1/\kappa_0^{2/3}-2(s_1+s_2)w/\kappa_0^{1/3}} \\ &= S_v^{-1}\mathrm{Ai}\big(S_h^{-2}(u_2-u_1)^2+S_v^{-1}(s_1+s_2)\big) \\ &\quad\times e^{\frac{2}{3}S_h^{-3}(u_2-u_1)^3+S_v^{-1}S_h^{-1}(u_2-u_1)(s_1+s_2)}\end{aligned} \tag{5.16}$$

with S_v and S_h defined in (2.13). Here we used the Airy function representation

$$\frac{-1}{2\pi i}\int_{\gamma_\infty} dv e^{v^3/3+av^2+bv} = \mathrm{Ai}(a^2-b)\exp(2a^3/3-ab). \tag{5.17}$$

To finish the proof, we need to consider the term coming from the first integral in (3.23), namely

$$-\frac{T^{1/3}}{2\pi i}\oint_{\Gamma_0} dw \frac{1}{w^{x_1-x_2+1}}\left(\frac{w}{1-w}\right)^{n_2-n_1} e^{(Rw+L/w)(t_1-t_2)}\frac{e^{t_2(2L+R/2)}2^{x_2}}{e^{t_1(2L+R/2)}2^{x_1}}. \tag{5.18}$$

This can be rewritten as

$$(5.18) = \frac{-T^{1/3}}{2\pi i}\oint_{\Gamma_0}\frac{dw}{w}e^{T^{2/3}(p_0(w)-p_0(1/2))+T^{1/3}(p_1(w)-p_1(1/2))} \tag{5.19}$$

with

$$\begin{aligned} p_0(w) &= (u_2-u_1)(\pi'(\theta)+1)H(w)-(u_2-u_1)(1-\pi'(\theta))\ln(w(1-w)), \\ p_1(w) &= -(u_2^2-u_1^2)\frac{\pi''(\theta)}{2}[H(w)+\ln(w(1-w))]-(s_2-s_1)\ln(w),\end{aligned} \tag{5.20}$$

where $H(w)$ is the function defined in (5.5). Remark that we need to do the analysis only for $u_2>u_1$. The function p_0 has critical point at $w=1/2$. The series expansions of p_0 and p_1 around $w=1/2$ are

$$\begin{aligned} p_0(w) &= p_0(1/2)+\kappa_1(u_2-u_1)(w-1/2)^2+\mathcal{O}((w-1/2)^3), \\ p_1(w) &= p_1(1/2)+2(s_1-s_2)(w-1/2)+\mathcal{O}((w-1/2)^2).\end{aligned} \tag{5.21}$$

We choose as path $\Gamma_0=\{\frac{1}{2}e^{i\phi},\phi\in(-\pi,\pi]\}$. This is a steep descent path for p_0. In fact, for $w\in\Gamma_0$,

$$\begin{aligned} \mathrm{Re}(H(w)) &= (R/2+2L)\cos(\phi)+(R/2-2L)\ln(2), \\ \mathrm{Re}(-\ln(w(1-w))) &= \ln(2)-\ln|1-w| = 2\ln(2)-\tfrac{1}{2}\ln(5-4\cos(\phi)),\end{aligned} \tag{5.22}$$

which are decreasing when $\cos(\phi)$ decreases. Thus, we can integrate only on $\Gamma_0^\delta=\{w\in\Gamma_0 \| w-1/2|\leq\delta\}$ and, for a small δ, the error term is just of order $\mathcal{O}(e^{-cT^{2/3}})$ with $c>0$ ($c\sim\delta^2$ as $\delta\ll 1$). The integral over Γ_0^δ is then given by

$$\begin{aligned}&\frac{-2T^{1/3}}{2\pi i}\int_{\Gamma_0^\delta} dw e^{\kappa_1(u_2-u_1)(w-1/2)^2T^{2/3}+2(s_1-s_2)(w-1/2)T^{1/3}} \\ &\times e^{\mathcal{O}((w-1/2)^3T^{2/3},(w-1/2)^2T^{1/3},(w-1/2))}.\end{aligned} \tag{5.23}$$

As above, we use $|e^x-1|\leq |x|e^{|x|}$, to control the difference between (5.23) and the same expression without the error terms. By taking $\delta\ll 1$ and the change of variable $\tilde w=(w-1/2)T^{1/3}$, we get that this difference is of order $\mathcal{O}(T^{-1/3})$ uniformly for s_1,s_2 in a bounded set. Once we have taken away the error terms in (5.23), we extend the integral to $1/2\pm \mathrm{i}\infty$. By this we make only an error of order $\mathcal{O}(e^{-cT^{2/3}})$. The integration path can be deformed to $1/2+\mathrm{i}\mathbb{R}$ without passing through any poles, therefore by setting $w=1/2+\mathrm{i}yT^{-1/3}$ we get

$$\begin{aligned} -\frac{1}{\pi}\int_{\mathbb{R}}\mathrm{d}y e^{-\kappa_1(u_2-u_1)y^2+2(s_1-s_2)} &= -\frac{1}{\sqrt{\pi\kappa_1(u_2-u_1)}}\exp\left(-\frac{(s_2-s_1)^2}{\kappa_1(u_2-u_1)}\right)\\ = \quad -\frac{S_v^{-1}}{\sqrt{4\pi(u_2-u_1)S_h^{-1}}}&\exp\left(-\frac{(s_2-s_1)^2S_v^{-2}}{4(u_2-u_1)S_h^{-1}}\right). \end{aligned}\tag{5.24}$$

Since all the error terms in the series expansions are uniform for $(s_1,s_2)\in[-\ell,\ell]^2$, the result of the Proposition is proven. □

Proposition 5.2 (Bound for the diffusion term of the kernel)**.**
For any $s_1,s_2\in\mathbb{R}$ and $u_2-u_1>0$ fixed, the bound

$$\begin{aligned} &\left|\frac{e^{t_2(2L+R/2)}2^{x_2}}{e^{t_1(2L+R/2)}2^{x_1}}\frac{T^{1/3}}{2\pi\mathrm{i}}\oint_{\Gamma_0}\mathrm{d}w\frac{1}{w^{x_1-x_2+1}}\left(\frac{w}{1-w}\right)^{n_2-n_1}e^{(Rw+L/w)(t_1-t_2)}\right|\\ &\leq \mathrm{const}\; e^{-|s_1-s_2|} \end{aligned}\tag{5.25}$$

holds for T large enough and const *independent of T. The* const *is uniform in s_1,s_2 but not in u_2-u_1.*

Proof of Proposition 5.2. From the analysis in Proposition 5.1, we just need a bound for $|s_2-s_1|\geq\ell$, $\ell>0$ fixed. We start with (5.19) but to obtain a decaying bound for large $|s_2-s_1|$ we consider another path Γ_0.

Consider an ε with $0<\varepsilon\ll 1$ and set $\Gamma_0=\{w=\rho e^{\mathrm{i}\phi},\phi\in[-\pi,\pi)\}$, with

$$\rho=\begin{cases} \frac{1}{2}+\frac{(s_2-s_1)T^{-1/3}}{(u_2-u_1)\kappa_1}, & \text{if } |s_2-s_1|\leq\varepsilon T^{1/3},\\ \frac{1}{2}+\frac{\varepsilon}{(u_2-u_1)\kappa_1}, & \text{if } s_2-s_1\geq\varepsilon T^{1/3},\\ \frac{1}{2}-\frac{\varepsilon}{(u_2-u_1)\kappa_1}, & \text{if } s_2-s_1\leq-\varepsilon T^{1/3}.\end{cases}\tag{5.26}$$

We have $\frac{\mathrm{d}}{\mathrm{d}\phi}\mathrm{Re}(w-\frac{1}{2}\ln(w))=-\rho\sin(\phi)$, $\frac{\mathrm{d}}{\mathrm{d}\phi}\mathrm{Re}(1/w+2\ln(w))=-\frac{4}{\rho}\sin(\phi)$, and $\frac{\mathrm{d}}{\mathrm{d}\phi}\mathrm{Re}(-\ln(w(1-w)))=-\frac{\rho\sin(\phi)}{1-2\rho\cos(\phi)+\rho^2}$. Thus Γ_0 is a steep descent path for $p_0(w)$. Moreover, since on Γ_0 we have $\mathrm{Re}(\ln(w))=\ln(\rho)$ is a constant, Γ_0 is also a steep descent path for $p_0(w)$ plus the term of $p_1(w)$ proportional to s_2-s_1. Let, for a small $\delta>0$ fixed, $\Gamma_0^\delta=\{w=\rho e^{\mathrm{i}\phi},\phi\in(-\delta,\delta)\}$. Then

$$\begin{aligned}(5.19) &= e^{T^{2/3}(p_0(\rho)-p_0(1/2))+T^{1/3}(p_1(\rho)-p_1(1/2))}\\ &\times\left(\mathcal{O}(e^{-cT^{2/3}})+\frac{-T^{1/3}}{2\pi\mathrm{i}}\int_{\Gamma_0^\delta}\frac{\mathrm{d}w}{w}e^{T^{2/3}(p_0(w)-p_0(\rho))+T^{1/3}(p_1(w)-p_1(\rho))}\right)\end{aligned}\tag{5.27}$$

for some $c>0$ (for small δ, $c\sim\delta^2$). On Γ_0^δ the s_i-dependent term in $\mathrm{Re}(p_1(w)-p_1(\rho))$ is equal to zero and the rest is of order $\mathcal{O}(\phi^2)$. Therefore the last integral can be bounded by

$$\frac{T^{1/3}}{2\pi}\int_{-\delta}^{\delta}\frac{\mathrm{d}\phi}{\rho}e^{-\frac{1}{2}T^{2/3}(u_2-u_1)\left[(\pi'(\theta)+1)(R\rho+L/\rho)+(1-\pi'(\theta))\rho/(1-\rho)^2\right]\phi^2+\mathcal{O}(T^{2/3}\phi^4,T^{1/3}\phi^2)}.\tag{5.28}$$

For δ small enough, and T large enough, the terms $\mathcal{O}(T^{2/3}\phi^4)$ and $\mathcal{O}(T^{1/3}\phi^2)$ are both controlled by the first term in the exponential. Then, by the change of variable $T^{1/3}\phi=\psi$ one sees that r.h.s. of (5.28) is bounded by a constant, uniformly in T.

What remains is therefore to bound the first term in the r.h.s. of (5.27). By the choice in (5.26) of ρ, $|\rho-1/2|\leq \varepsilon/((u_2-u_1)\kappa_1)\ll 1$ for ε small enough which can be still chosen. Series expansion for ρ close to $1/2$ leads to

$$\begin{aligned} p_0(\rho)-p_0(1/2) &= -2(s_2-s_1)(\rho-1/2)T^{1/3}(1+\mathcal{O}(\rho-1/2)) \\ &+ \kappa_1(u_2-u_1)(\rho-1/2)^2T^{2/3}(1+\mathcal{O}(\rho-1/2)). \end{aligned} \tag{5.29}$$

By (5.26) we obtain the bounds

$$\begin{aligned} p_0(\rho)-p_0(1/2) &= -\frac{(s_2-s_1)^2}{(u_2-u_1)\kappa_1}(1+\mathcal{O}(\varepsilon)),\ \text{if } |s_2-s_1|\leq \varepsilon T^{1/3}, \qquad (5.30)\\ p_0(\rho)-p_0(1/2) &= -\frac{|s_2-s_1|\varepsilon T^{1/3}}{(u_2-u_1)\kappa_1}(1+\mathcal{O}(\varepsilon)),\ \text{if } |s_2-s_1|\geq \varepsilon T^{1/3}. \end{aligned}$$

Combining the above result we have

$$|(5.25)|\leq \left[\mathcal{O}(e^{-cT^{2/3}})+\mathcal{O}(1)\right]\left[e^{-\frac{(s_2-s_1)^2}{(u_2-u_1)\kappa_1}(1+\mathcal{O}(\varepsilon))}+e^{-\frac{|s_2-s_1|\varepsilon T^{1/3}}{(u_2-u_1)\kappa_1}(1+\mathcal{O}(\varepsilon))}\right]. \tag{5.31}$$

Thus by taking an ε small enough and then T large enough the bound (5.31) implies the statement to be proven, since for any $\alpha>0$, there exists a $C_\alpha<\infty$ such that $e^{-\alpha(s_2-s_1)^2}\leq C_\alpha e^{-|s_2-s_1|}$. □

Proposition 5.3 (Bound on the main term of the kernel)**.**
Let u_1,u_2 be fixed. Then, for any $(s_1,s_2)\in[-\ell,\infty)^2$, the bound

$$\begin{aligned} &\left|\frac{-T^{1/3}}{2\pi \mathrm{i}}\oint_{\Gamma_1}\mathrm{d}z\frac{e^{Rt_1(1-z)+Lt_1/(1-z)}}{e^{Rt_2z+Lt_2/z}}\frac{z^{n_1+n_2+x_2}}{(1-z)^{n_1+n_2+x_1+1}}\frac{e^{t_2(2L+R/2)}2^{x_2}}{e^{t_1(2L+R/2)}2^{x_1}}\right| \\ &\leq \text{const } e^{-(s_1+s_2)} \end{aligned} \tag{5.32}$$

holds for T large enough, where const *is a constant independent of T.*

Proof of Proposition 5.3. Let $\tilde\ell$ be a constant independent from T, which can still be chosen large if needed. For $(s_1,s_2)\in[-\ell,\tilde\ell]^2$, the result is a consequence of the estimates in the proof of Proposition 5.1. Therefore we can consider just $(s_1,s_2)\in[-\ell,\infty)^2\setminus[-\ell,\tilde\ell]^2$. Introduce the notation $\tilde s_i=(s_i+\ell+\tilde\ell)T^{-2/3}$, which then belongs to $[\tilde\ell T^{-2/3},\infty)$.

The integral to be bounded is

$$\frac{-T^{1/3}}{2\pi\mathrm{i}}\oint_{\Gamma_1}\mathrm{d}z e^{Tf_0(z)+T^{2/3}f_1(z)+T^{1/3}f_2(z)+f_3(z)} \tag{5.33}$$

where $f_1(z)$ and $f_3(z)$ are given in (5.6), and $f_0(z)$ and $f_2(z)$ are just slight modifications of the functions in (5.6), namely

$$\begin{aligned} f_0(z) &= (\pi(\theta)+\theta)(H(1-z)-H(z))+\tilde s_1\ln(2(1-z))-\tilde s_2\ln(2z), \\ f_2(z) &= g_2(1-z,u_1,-\ell-\tilde\ell)-g_2(z,u_2,-\ell-\tilde\ell) \\ &\quad -g_2(1/2,u_1,-\ell-\tilde\ell)+g_2(1/2,u_2,-\ell-\tilde\ell). \end{aligned} \tag{5.34}$$

We put $\tilde{s}_1$ and $\tilde{s}_2$ in $f_0(z)$, because they are not restricted to be of order $T^{-2/3}$ (as it was the case in Proposition 5.1).

First we need to find a steep descent path for $f_0(z)$. We choose it as $\Gamma_1 = \{1-\rho e^{\mathrm{i}\phi}, \phi \in [-\pi,\pi)\}$ with $0<\rho\leq 1/2$, chosen as follows,

$$\rho = \begin{cases} \frac{1}{2} - ((\tilde{s}_1+\tilde{s}_2)/\kappa_0)^{1/2}, & \tilde{s}_1+\tilde{s}_2 \leq \varepsilon, \\ \frac{1}{2} - (\varepsilon/\kappa_0)^{1/2}, & \tilde{s}_1+\tilde{s}_2 \geq \varepsilon, \end{cases} \tag{5.35}$$

for some small $\varepsilon>0$ to be fixed later. Recall that the $\tilde{s}_i > 0$.

To see that Γ_1 is a steep descent path, we consider $f_0(z)$ term by term. First consider $\phi\in[0,\pi]$, the case $\phi\in[-\pi,0]$ is obtained by symmetry. The term proportional to $R(\pi(\theta)+\theta)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\phi}\mathrm{Re}(1-2z+\tfrac{1}{2}\ln(z/(1-z))) = -\frac{\rho(3-8\rho\cos(\phi)+4\rho^2)\sin(\phi)}{1-2\rho\cos(\phi)+\rho^2} \leq 0 \tag{5.36}$$

for all $0<\rho\leq 1/2$, with equality only at $\phi=0,\pi$. The term proportional to $L(\pi(\theta)+\theta)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\phi}\mathrm{Re}(1/(1-z)-1/z-2\ln(z/(1-z))) = -\frac{((1-2\rho\cos(\phi)+2\rho^2)^2-\rho^2)\sin(\phi)}{(1-2\rho\cos(\phi)+\rho^2)^2\rho} \leq 0 \tag{5.37}$$

for all $0<\rho\leq 1/2$, with equality only at $\phi=0,\pi$. Finally, $\mathrm{Re}(\ln(1-z))$ is constant on Γ_1 and $-\mathrm{Re}(\ln(2z)) = -\ln(2|z|)$ is strictly decreasing while moving on Γ_1 with $|\phi|$ increasing.

For a small $\delta>0$, $\Gamma_1^\delta = \{1-\rho e^{\mathrm{i}\phi}, \phi\in(-\delta,\delta)\}$. We also define

$$Q(\rho) = \exp\left(\mathrm{Re}\left(Tf_0(1-\rho)+T^{2/3}f_1(1-\rho)+T^{1/3}f_2(1-\rho)\right)\right). \tag{5.38}$$

Since Γ_1 is a steep descent path of $f_0(z)$, the integral over $\Gamma_1\setminus\Gamma_1^\delta$ is bounded by

$$Q(\rho)\mathcal{O}(e^{-cT}) \tag{5.39}$$

for some $c>0$ independent of T. The contribution of the integral over Γ_1^δ is bounded by

$$Q(\rho)\left|\frac{-T^{1/3}}{2\pi\mathrm{i}}\int_{\Gamma_1^\delta}\mathrm{d}z e^{T(f_0(z)-f_0(1-\rho))+T^{2/3}(f_1(z)-f_1(1-\rho))+T^{1/3}(f_2(z)-f_2(1-\rho))+f_3(z)}\right| \tag{5.40}$$

The series expansion around $\phi=0$ is

$$\mathrm{Re}(f_0(1-\rho e^{\mathrm{i}\phi})-f_0(1-\rho)) = -\gamma_1\phi^2(1+\mathcal{O}(\phi)) \tag{5.41}$$

with

$$\gamma_1 = \frac{\tilde{s}_2\rho}{2(1-\rho)^2} + \frac{(\pi(\theta)+\theta)(1-2\rho)}{(1-\rho)^2}\left(\frac{R\rho(3-2\rho)}{4}+\frac{L(1-\rho+2\rho^2)}{3\rho(1-\rho)}\right), \tag{5.42}$$

and

$$\mathrm{Re}(f_1(1-\rho e^{\mathrm{i}\phi})-f_1(1-\rho)) = \gamma_2\phi^2(1+\mathcal{O}(\phi)), \tag{5.43}$$

with

$$\gamma_2 = (u_2-u_1)\kappa_1+\mathcal{O}(\rho-1/2). \tag{5.44}$$

Finally, $\mathrm{Re}(f_2(1-\rho e^{\mathrm{i}\phi})-f_2(1-\rho))=\mathcal{O}(\phi^2)$. Thus, by the change of variable $z=1-\rho e^{\mathrm{i}\phi}$, the above estimates, and by setting $\gamma=\gamma_1+\gamma_2 T^{-1/3}$, we get

$$(5.40)=Q(\rho)\frac{T^{1/3}\rho}{2\pi(1-\rho)}\int_{-\delta}^{\delta}\mathrm{d}\phi\, e^{-\gamma\phi^2 T(1+\mathcal{O}(\phi))(1+\mathcal{O}(T^{-1/3}))}. \tag{5.45}$$

By choosing δ small enough (independent of T) and then T large enough, the factors with the error terms can be replaced by $1/2$, thus

$$(5.40)\le Q(\rho)\frac{T^{1/3}\rho}{2\pi(1-\rho)}\int_{-\delta}^{\delta}\mathrm{d}\phi\, e^{-\gamma\phi^2 T/2}\le Q(\rho)\frac{1}{\sqrt{2\pi\gamma T^{1/3}}}. \tag{5.46}$$

Remark that, the worse case is when γ becomes small, and this happens when $\rho\to 1/2$, i.e., it is the case of small values of $\tilde s_1+\tilde s_2$. But even in this case, $\gamma_1 T^{1/3}\sim(s_1+s_2+2\ell+2\tilde\ell)^{1/2}\ge(2\tilde\ell)^{1/2}$. Since γ_2 is of order one, $\gamma T^{1/3}=\gamma_1 T^{1/3}+\gamma_2>0$ for $\tilde\ell$ large enough. So, for by setting $\tilde\ell$ large enough, $(5.40)\le\mathrm{const}\,Q(\rho)$. This estimate, combined with (5.39), implies that the Proposition will be proven by showing that $Q(\rho)\le\mathrm{const}\,e^{-(s_1+s_2)}$. Since $1-\rho$ is close to $1/2$, we can apply the series expansion of f_i around $z=1/2$. The expansion of f_1 is in (5.8), while the one of f_2 is the same as in (5.8) with $s_1+s_2=-2\ell-2\tilde\ell$. Finally,

$$f_0(z)=\tfrac{1}{3}\kappa_0(z-1/2)^3(1+\mathcal{O}(z-1/2)^2)-(\tilde s_1+\tilde s_2)(z-1/2)(1+\mathcal{O}(z-1/2)). \tag{5.47}$$

First consider $\tilde s_1+\tilde s_2\le\varepsilon$. Then, with ρ chosen as in (5.35), we get

$$\begin{aligned}Q(\rho)&=e^{-\frac{2}{3}T(\tilde s_1+\tilde s_2)^{3/2}\kappa_0^{-1/2}T(1+\mathcal{O}(\sqrt{\varepsilon}))}e^{(u_2-u_1)\kappa_1(\tilde s_1+\tilde s_2)T^{2/3}\kappa_0^{-1}(1+\mathcal{O}(\sqrt{\varepsilon}))}\\&\quad\times e^{-2(\ell+\tilde\ell)(\tilde s_1+\tilde s_2)\kappa_0^{-1/2}T^{1/3}(1+\mathcal{O}(\sqrt{\varepsilon}))}\\&=e^{-\frac{2}{3}(s_1+s_2+2\ell+2\tilde\ell)^{3/2}\kappa_0^{-1/2}(1+\mathcal{O}(\sqrt{\varepsilon}))}e^{(u_2-u_1)\kappa_1(s_1+s_2+2\ell+2\tilde\ell)\kappa_0^{-1}(1+\mathcal{O}(\sqrt{\varepsilon}))}\\&\quad\times e^{-2(\ell+\tilde\ell)(s_1+s_2+2\ell+2\tilde\ell)\kappa_0^{-1/2}T^{-1/3}(1+\mathcal{O}(\sqrt{\varepsilon}))}.\end{aligned} \tag{5.48}$$

Recall that $s_1+s_2+2\ell+2\tilde\ell\ge 2\tilde\ell\gg 1$ for $\tilde\ell\gg 1$. Therefore by choosing $\tilde\ell$ large enough (depending only on the coefficients $\kappa_0,\kappa_1,u_1,u_2$ which are however fixed), all the terms are controlled by the first one, i.e.,

$$Q(\rho)\le e^{-\frac{1}{3}(s_1+s_2+2\ell+2\tilde\ell)^{3/2}\kappa_0^{-1/2}}\le e^{-\frac{1}{3}(s_1+s_2)^{3/2}\kappa_0^{-1/2}}. \tag{5.49}$$

Since this decays more rapidly that $\exp(-(s_1+s_2))$, the Proposition holds for $\tilde s_1+\tilde s_2\le\varepsilon$.

The last case is $\tilde s_1+\tilde s_2\ge\varepsilon$. In this case, with ρ chosen as in (5.35), we obtain

$$\begin{aligned}Q(\rho)&=e^{T\kappa_0^{-1/2}(1+\mathcal{O}(\sqrt{\varepsilon}))\sqrt{\varepsilon}(\varepsilon/3-(\tilde s_1+\tilde s_2))}e^{(u_2-u_1)\kappa_1\kappa_0^{-1}\varepsilon T^{2/3}(1+\mathcal{O}(\sqrt{\varepsilon}))}\\&\quad\times e^{-4(\ell+\tilde\ell)\kappa_0^{-1/2}\varepsilon T^{1/3}(1+\mathcal{O}(\sqrt{\varepsilon}))}.\end{aligned} \tag{5.50}$$

But now, $\varepsilon/3-(\tilde s_1+\tilde s_2)\le-\frac{2}{3}(\tilde s_1+\tilde s_2)$, thus the first term in the exponential is, up to a positive constant,$-\sqrt{\varepsilon}T^{1/3}(s_1+s_2+2\ell+2\tilde\ell)$, which dominates the second term $\sim\varepsilon T^{2/3}\le s_1+s_2+2\ell+2\tilde\ell$, and it also dominates the third term. Therefore, for any choice of ε and $\tilde\ell$ made before, we can take T large enough such that

$$Q(\rho)\le e^{-\frac{1}{3}\sqrt{\varepsilon}T^{1/3}(s_1+s_2)}, \tag{5.51}$$

which ends the proof of the Proposition. ☐

Proof of Theorem 2.2. The proof of Theorem 2.2 is the complete analogue of Theorem 2.5 in [3]. The results in Propositions 5.1,5.3,5.4, and 5.5 in [3] are replaced by the ones in Proposition 5.1, 5.2, 5.3. The strategy is to write the Fredholm series of the expression for finite T and, by using the bounds in Propositions 5.2 and 5.3, see that it is bounded by a T-independent and integrable function. Once this is proven, one can exchange the sums/integrals and the $T\to\infty$ limit by the theorem of dominated convergence. For details, see Theorem 2.5 in [3]. ☐

5.2 Sketch of the result (2.23)

With the rescaling (2.9) and (2.19), the rescaled kernel writes

$$K^{\rm resc}(u_1,s_1;u_2,s_2)=K((n_1,t_1),x_1;(n_2,t_2),x_2)T^{1/3}. \tag{5.52}$$

The main part of the kernel (the second term in (3.17)) writes

$$\frac{T^{1/3}}{(2\pi{\rm i})^2}\oint_{\Gamma_0}{\rm d}w\oint_{\Gamma_1}{\rm d}z\frac{e^{Tf_0(w)+T^{2/3}f_1(w;u_1)+T^{1/3}f_2(w;u_1,s_1)}}{e^{Tf_0(z)+T^{2/3}f_1(z;u_2)+T^{1/3}f_2(z;u_2,s_2)}}\frac{1}{w(w-z)} \tag{5.53}$$

with

$$\begin{aligned}
f_0(w)&=(\pi(\theta)+\theta)\left(Rw+\tfrac{L}{w}\right)+(\pi(\theta)-\theta)\ln\left(\tfrac{1-w}{w}\right)-\sigma_0\ln(w),\\
f_1(w;u_i)&=-\left[(\pi'(\theta)+1)\left(Rw+\tfrac{L}{w}\right)+(\pi'(\theta)-1)\ln\left(\tfrac{1-w}{w}\right)-\sigma_1\ln(w)\right]u_i,\\
f_2(w;u_i,s_i)&=\left[\tfrac12\pi''(\theta)\left(Rw+\tfrac{L}{w}+\ln\left(\tfrac{1-w}{w}\right)\right)-\sigma_2\right]u_i^2+s_i\ln(w).
\end{aligned} \tag{5.54}$$

The parameter μ is actually the position of the double critical point of $f_0(w)$. Series expansions gives

$$\begin{aligned}
f_0(w)&=f_0(\mu)-\frac{\kappa_0}{3}(w-\mu)^3+\mathcal{O}((w-\mu)^4),\\
f_1(w;u_1)&=f_1(\mu;u_1)-u_1\kappa_1(w-\mu)^2+\mathcal{O}((w-\mu)^3),\\
f_2(w;u_1,s_1)&=f_2(\mu;u_1,s_1)-\left(\frac{\kappa_1^2u_1^2}{\kappa_0}-\frac{s_1}{\mu}\right)(w-\mu)+\mathcal{O}((w-\mu)^2).
\end{aligned} \tag{5.55}$$

The terms $f_1(\mu;u_i)$ and $f_2(\mu;u_i,s_i)$ cancel out by an appropriate conjugation of the kernel (5.53). We denote by $\simeq$ an equality up to conjugation. Thus, asymptotically, (5.53) goes to

$$\frac{T^{1/3}}{\mu(2\pi{\rm i})^2}\oint_{\Gamma_0}{\rm d}w\oint_{\Gamma_1}\frac{{\rm d}z}{w-z}\frac{e^{-\kappa_0(w-\mu)^3T/3-u_1\kappa_1(w-\mu)^2T^{2/3}+T^{1/3}(w-\mu)(s_1/\mu-\kappa_1^2u_1^2/\kappa_0)}}{e^{-\kappa_0(z-\mu)^3T/3-u_2\kappa_1(z-\mu)^2T^{2/3}+T^{1/3}(z-\mu)(s_2/\mu-\kappa_1^2u_2^2/\kappa_0)}} \tag{5.56}$$

With the change of variable $(w-\mu)(\kappa_0T)^{1/3}=W$, $(z-\mu)(\kappa_0T)^{1/3}=Z$, we then obtain

$$(5.56)=\frac{\kappa_0^{-1/3}}{\mu(2\pi{\rm i})^2}\int{\rm d}W\int{\rm d}Z\frac{1}{W-Z}\frac{e^{\frac13Z^3+u_2Z^2\kappa_1/\kappa_0^{2/3}-Z(s_2/\mu-\kappa_1^2u_2^2/\kappa_0)/\kappa_0^{1/3}}}{e^{\frac13W^3+u_1W^2\kappa_1/\kappa_0^{2/3}-W(s_1/\mu-\kappa_1^2u_1^2/\kappa_0)/\kappa_0^{1/3}}}. \tag{5.57}$$

Let us denote by $\tilde{S}_v = \mu\kappa_0^{1/3}$ and $\tilde{S}_h = \kappa_1^{-1}\kappa_0^{2/3}$ the vertical and horizontal scaling. Then

$$(5.57) = \tilde{S}_v^{-1} K_{\mathscr{A}_2}(\tilde{S}_h^{-1}u_1, \tilde{S}_v^{-1}s_1; \tilde{S}_h^{-1}u_2, \tilde{S}_v^{-1}s_2) \tag{5.58}$$

where $K_{\mathscr{A}_2}$ is the extended Airy kernel associated to the Airy_2 process. An asymptotic analysis of large deviations similar to Propositions 5.2 and 5.3 above would then lead to the result of (2.23).

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