Electronic Journal of Probability

Vol. 10 (2005), Paper no. 27, pages 925-947.

Journal URL
http://www.math.washington.edu/~ejpecp/

On the Increments of the Principal Value of Brownian Local Time

Endre Csáki[1]
Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences,
H-1364 Budapest, P.O.B. 127, Hungary
csaki@renyi.hu

and

Yueyun Hu
Département de Mathématiques, Institut Galilée (L.A.G.A. UMR 7539)
Paris XIII, 99 Avenue J-B Clément, 93430 Villetaneuse, France
yueyun@math.univ-paris13.fr

Abstract: Let W be a one-dimensional Brownian motion starting from 0. Define $Y(t) = \int_0^t \frac{ds}{W(s)} := \lim_{\epsilon\to 0} \int_0^t 1_{(|W(s)|>\epsilon)} \frac{ds}{W(s)}$ as Cauchy's principal value related to local time. We prove limsup and liminf results for the increments of Y.

Keywords: Brownian motion, local time, principal value, large increments.

AMS 2000 Subject Classification: 60J65 60J55 60F15

Submitted to EJP on January 10, 2005. Final version accepted on June 1, 2005.

[1] Research supported by the Hungarian National Foundation for Scientific Research, Grant No. T 037886 and T 043037.

1. Introduction

Let $\{W(t);\, t \geq 0\}$ be a one-dimensional standard Brownian motion with $W(0) = 0$, and let $\{L(t,x);\, t \geq 0,\, x \in \mathbb{R}\}$ denote its jointly continuous local time process. That is, for any Borel function $f \geq 0$,

$$\int_0^t f(W(s))\,\mathrm{d}s = \int_{-\infty}^{\infty} f(x)L(t,x)\,\mathrm{d}x, \qquad t \geq 0.$$

We are interested in the process

$$Y(t) := \int_0^t \frac{\mathrm{d}s}{W(s)}, \qquad t \geq 0. \tag{1.1}$$

Rigorously speaking, the integral $\int_0^t \mathrm{d}s/W(s)$ should be considered in the sense of Cauchy's principal value, i.e., $Y(t)$ is defined by

$$Y(t) := \lim_{\varepsilon \to 0^+} \int_0^t \frac{\mathrm{d}s}{W(s)} \mathbb{1}_{\{|W(s)| \geq \varepsilon\}} = \int_0^{\infty} \frac{L(t,x) - L(t,-x)}{x}\,\mathrm{d}x. \tag{1.2}$$

Since $x \mapsto L(t,x)$ is Hölder continuous of order ν, for any $\nu < 1/2$, the integral on the extreme right in (1.2) is almost surely absolutely convergent for all $t > 0$. The process $\{Y(t),\, t \geq 0\}$ is called the principal value of Brownian local time.

It is easily seen that $Y(\cdot)$ inherits a scaling property from Brownian motion, namely, for any fixed $a > 0$, $t \mapsto a^{-1/2}Y(at)$ has the same law as $t \mapsto Y(t)$. Although some properties distinguish $Y(\cdot)$ from Brownian motion (in particular, $Y(\cdot)$ is not a semimartingale), it is a kind of folklore that the asymptotic behaviors of Y are somewhat like that of a Brownian motion. For detailed studies and surveys on principal value, and relation to Hilbert transform see Biane and Yor [4], Fitzsimmons and Getoor [13], Bertoin [2], [3], Yamada [20], Boufoussi et al. [5], Ait Ouahra and Eddahbi [1], Csáki et al. [11] and a collection of papers [22] together with their references. Biane and Yor [4] presented a detailed study on Y and determined a number of distributions for principal values and related processes.

Concerning almost sure limit theorems for Y and its increments, we summarize the relevant results in the literature. It was shown in [17] that the following law of the iterated logarithm holds:

Theorem A. (Hu and Shi [17])

$$\limsup_{T \to \infty} \frac{Y(T)}{\sqrt{T \log\log T}} = \sqrt{8}\,, \qquad \text{a.s.} \tag{1.3}$$

This was extended in [10] to a Strassen-type [18] functional law of the iterated logarithm.

Theorem B. (Csáki et al. [10]) With probability one the set

$$\left\{ \frac{Y(xT)}{\sqrt{8T \log\log T}},\, 0 \le x \le 1 \right\}_{T \ge 3} \tag{1.4}$$

is relatively compact in $C[0,1]$ with limit set equal to

$$\mathcal{S} := \left\{ f \in C[0,1] :\ f(0) = 0,\ f \text{ is absolutely continuous and } \int_0^1 (f'(x))^2 \, \mathrm{d}x \le 1 \right\}. \tag{1.5}$$

Concerning Chung-type law of the iterated logarithm, we have the following result:

Theorem C. (Hu [16])

$$\liminf_{T\to\infty} \sqrt{\frac{\log\log T}{T}} \sup_{0\le s\le T} |Y(s)| = K_1\,, \qquad \text{a.s.} \tag{1.6}$$

with some (unknown) constant $K_1 > 0$.

The large increments were studied in [7] and [8]:

Theorem D. (Csáki et al. [7]) *Under the conditions*

$$\begin{cases} 0 < a_T \le T, \\ \\ T \mapsto a_T \text{ and } T \mapsto T/a_T \text{ are both non-decreasing,} \\ \\ \lim\limits_{T\to\infty} \dfrac{\log(T/a_T)}{\log\log T} = \infty, \end{cases} \tag{1.7}$$

we have

$$\lim_{T\to\infty} \frac{\sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s) - Y(t)|}{\sqrt{a_T \log(T/a_T)}} = 2, \qquad \text{a.s.} \tag{1.8}$$

Wen [19] studied the lag increments of Y and among others proved the following results.

Theorem E. (Wen [19])

$$\limsup_{T\to\infty} \sup_{0\le t\le T} \frac{\sup_{t\le s\le T} |Y(s) - Y(s-t)|}{\sqrt{t(\log(T/t) + 2\log\log t)}} = 2, \qquad \text{a.s.} \tag{1.9}$$

Under the conditions $0 < a_T \le T$, $a_T \to \infty$ *as* $T \to \infty$, *we have*

$$\limsup_{T\to\infty} \sup_{0\le t\le T-a_T} \frac{\sup_{0\le s\le a_T} |Y(t+s) - Y(t)|}{\sqrt{a_T(\log((t+a_T)/a_T) + 2\log\log a_T)}} \le 2, \qquad \text{a.s.} \tag{1.10}$$

If a_T *is onto, then we have equality in* (1.10).

In this note our aim is to investigate further limsup and liminf behaviors of the increments of Y.

Theorem 1.1. *Assume that $T \mapsto a_T$ is a function such that $0 < a_T \le T$, and both a_T and T/a_T are non-decreasing. Then*

(i)

$$\limsup_{T\to\infty} \frac{\sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s)-Y(t)|}{\sqrt{a_T\left(\log\sqrt{T/a_T}+\log\log T\right)}} = \sqrt{8}, \qquad \text{a.s.} \tag{1.11}$$

(iia) *If $a_T > T(\log T)^{-\alpha}$ for some $\alpha < 2$, then*

$$\liminf_{T\to\infty} \sqrt{\frac{\log\log T}{a_T}} \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s)-Y(t)| = K_2, \qquad \text{a.s.} \tag{1.12}$$

(iib) *If $a_T \le T(\log T)^{-\alpha}$ for some $\alpha > 2$, then*

$$\liminf_{T\to\infty} \frac{\sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s)-Y(t)|}{\sqrt{a_T\log(T/a_T)}} = K_3, \qquad \text{a.s.} \tag{1.13}$$

with some positive constants K_2, K_3. If, moreover,

$$\lim_{T\to\infty} \frac{\log(T/a_T)}{\log\log T} = \infty,$$

then $K_3 = 2$.

Theorem 1.2. *Assume that $T \mapsto a_T$ is a function such that $0 < a_T \le T$, and both a_T and T/a_T are non-decreasing. Then*

(i)

$$\liminf_{T\to\infty} \frac{\sqrt{T\log\log T}}{a_T} \inf_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s)-Y(t)| = K_4, \qquad \text{a.s.} \tag{1.14}$$

with some positive constant K_4. If $\lim_{T\to\infty}(a_T/T) = 0$, then $K_4 = 1/\sqrt{2}$.

(iia) *If $0 < \lim_{T\to\infty}(a_T/T) = \rho \le 1$, then*

$$\limsup_{T\to\infty} \frac{\inf_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s)-Y(t)|}{\sqrt{T\log\log T}} = \rho\sqrt{8}, \qquad \text{a.s.} \tag{1.15}$$

(iib) *If*

$$\lim_{T\to\infty} \frac{a_T(\log\log T)^2}{T} = 0,$$

then

$$\limsup_{T\to\infty} \frac{\sqrt{T}}{a_T\sqrt{\log\log T}} \inf_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s)-Y(t)| = K_5, \qquad \text{a.s.} \tag{1.16}$$

with some positive constant K_5.

Remark 1. *The exact values of the constants K_i, $i = 2, 3, 4, 5$ are unknown in general and it seems difficult to determine them except in certain particular cases. In the proofs we establish different upper and lower bounds. It follows however by 0-1 law for Brownian motion that the limsup's and liminf's considered here are non-random constants.*

Remark 2. *Plainly we recover some previous results on the path properties of Y by considering particular cases of Theorems 1.1 and 1.2. For instance, Theorems A and C follow from (1.11) and (1.12) respectively by taking $a_T = T$, and (1.8) follows from (1.11) combining with (1.13). However in Theorem 1.1(ii) and Theorem 1.2(ii) there are still small gaps in a_T.*

The organization of the paper is as follows: In Section 2 some facts are presented needed in the proofs. Section 3 contains the necessary probability estimates. Theorem 1.1(i) and Theorem 1.1(iia,b) are proved in Sections 4 and 5, resp., while Theorem 1.2(i) and Theorem 1.2(iia,b) are proved in Sections 6 and 7, resp.

Throughout the paper, the letter K with subscripts will denote some important but unknown finite positive constants, while the letter c with subscripts denotes some finite and positive universal constants not important in our investigations. When the constants depend on a parameter, say δ, they are denoted by $c(\delta)$ with subscripts.

2. Facts

Let $\{W(t),\, t \geq 0\}$ be a standard Brownian motion and define the following objects:

$$g := \sup\{t :\, t \leq 1,\, W(t) = 0\} \tag{2.1}$$

$$B(s) := \frac{W(sg)}{\sqrt{g}}, \qquad 0 \leq s \leq 1, \tag{2.2}$$

$$m(s) := \frac{|W(g + s(1-g))|}{\sqrt{1-g}}, \qquad 0 \leq s \leq 1. \tag{2.3}$$

Here we summarize some well-known facts needed in our proofs.

Fact 2.1. (Biane and Yor [4])

$$\frac{\mathbb{P}(Y(1) \in \mathrm{d}x)}{\mathrm{d}x} = \sqrt{\frac{2}{\pi^3}} \sum_{k=0}^{\infty} (-1)^k \exp\left(-\frac{(2k+1)^2 x^2}{8}\right), \quad x \in \mathbb{R}. \tag{2.4}$$

Consequently we have the estimate: for $\delta > 0$

$$c_1 \exp\left(-\frac{z^2}{8(1-\delta)}\right) \leq \mathbb{P}(Y(1) \geq z) \leq \exp\left(-\frac{z^2}{8}\right), \qquad z \geq 1 \tag{2.5}$$

with some positive constant $c_1 = c_1(\delta)$. Moreover, g, $\{B(s),\, 0 \le s \le 1\}$ and $\{m(s),\, 0 \le s \le 1\}$ are independent, g has arcsine distribution, B is a Brownian bridge and m is a Brownian meander.

$$\begin{aligned}&\mathbb{P}\left(\int_0^1 \frac{\mathrm{d}v}{m(v)} < z \,\Big|\, m(1) = 0\right)\\ &= \sum_{k=-\infty}^{\infty} (1 - k^2 z^2) \exp\left(-\frac{k^2 z^2}{2}\right) = \frac{8\pi^2\sqrt{2\pi}}{z^3} \sum_{k=1}^{\infty} \exp\left(-\frac{2k^2\pi^2}{z^2}\right), \quad z > 0.\end{aligned} \tag{2.6}$$

$$\mathbb{P}(m(1) > x) = e^{-x^2/2}, \qquad x > 0. \tag{2.7}$$

Fact 2.2. (Yor [21, Exercise 3.4 and pp. 44]) *Let $Q^{\delta}_{x\to 0}$ be the law of the square of a Bessel bridge from x to 0 of dimension $\delta > 0$ during time interval $[0, 1]$. The process $(m^2(1 - v), 0 \le v \le 1)$ conditioned on $\{m^2(1) = x\}$ is distributed as $Q^3_{x\to 0}$. Furthermore, we have*

$$Q^{\delta}_{x\to 0} = Q^{\delta}_{0\to 0} * Q^0_{x\to 0}, \quad \forall\, \delta > 0,\, x > 0, \tag{2.8}$$

where $*$ denotes convolution operator. Consequently, for any $x > 0$

$$\mathbb{P}\left(\int_0^1 \frac{\mathrm{d}v}{m(v)} < z \,\Big|\, m(1) = x\right) \ge \mathbb{P}\left(\int_0^1 \frac{\mathrm{d}v}{m(v)} < z \,\Big|\, m(1) = 0\right). \tag{2.9}$$

Fact 2.3. (Hu [16]) *For $0 < z \le 1$*

$$c_2 \exp\left(-\frac{c_3}{z^2}\right) \le \mathbb{P}(\sup_{0\le s\le 1} |Y(s)| < z) \le c_4 \exp\left(-\frac{c_5}{z^2}\right) \tag{2.10}$$

with some positive constants c_2, c_3, c_4, c_5.

Fact 2.4. (Csörgő and Révész [12]) *Assume that $T \mapsto a_T$ is a function such that $0 < a_T \le T$, and both a_T and T/a_T are non-decreasing. Then*

$$\limsup_{T\to\infty} \frac{\sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |W(t+s) - W(t)|}{\sqrt{a_T\,(\log(T/a_T) + \log\log T)}} = \sqrt{2}, \qquad \text{a.s.} \tag{2.11}$$

Fact 2.5. (Strassen [18]) *If $f \in \mathcal{S}$ defined by (1.5), then for any partition $x_0 = 0 < x_1 < \ldots < x_k < x_{k+1} = 1$ we have*

$$\sum_{i=1}^{k+1} \frac{(f(x_i) - f(x_{i-1}))^2}{x_i - x_{i-1}} \le 1. \tag{2.12}$$

Fact 2.6. (Chung [6])

$$\liminf_{t\to\infty} \sqrt{\frac{\log\log t}{t}} \sup_{0\le s\le t} |W(s)| = \frac{\pi}{\sqrt{8}}, \qquad \text{a.s.} \tag{2.13}$$

Define $g(T) := \max\{s \le T : W(s) = 0\}$. A joint lower class result for $g(T)$ and $M(T) := \sup_{0\le s\le T} |W(s)|$ reads as follows.

Fact 2.7. (Grill [15]) *Let $\beta(t)$, $\gamma(t)$ be positive functions slowly varying at infinity, such that $0 < \beta(t) \le 1$, $0 < \gamma(t) \le 1$, $\beta(t)$ is non-increasing, $\beta(t)\sqrt{t} \uparrow \infty$, $\gamma(t)$ is monotone, $\gamma(t)t \uparrow \infty$, $\gamma(t)/\beta^2(t)$ is monotone. Then*

$$\mathbb{P}\left(M(T) \le \beta(T)\sqrt{T},\, g(T) \le \gamma(T)T \quad \text{i.o.}\right) = 0 \quad \text{or} \quad 1$$

according as $I(\beta,\gamma) < \infty$ or $= \infty$, where

$$I(\beta,\gamma) = \int_1^\infty \frac{1}{t\beta^2(t)} \left(1 + \frac{\beta^2(t)}{\gamma(t)}\right)^{-1/2} \exp\left(-\frac{(4-3\gamma(t))\pi^2}{8\beta^2(t)}\right) \mathrm{d}t.$$

Now define $d(T) := \min\{s \ge T :\, W(s) = 0\}$. Since $\{d(T) > t\} = \{g(t) < T\}$, we deduce from Fact 2.7 the following estimate on $d(T)$ when $T \to \infty$.

Fact 2.8. *With probability 1*

$$d(T) = O(T(\log T)^3), \qquad T \to \infty.$$

3. Probability estimates

Lemma 3.1. *For $T \ge 1$, $\delta, z > 0$ we have*

$$\begin{aligned} &\mathbb{P}\left(\sup_{0\le t\le T-1} \sup_{0\le s\le 1} |Y(t+s) - Y(t)| > z\right) \\ &\qquad \le c_6 \left(\sqrt{T}\exp\left(-\frac{z^2}{8(1+\delta)}\right) + T\exp\left(-\frac{z^2}{2(1+\delta)}\right)\right) \end{aligned} \tag{3.1}$$

with some positive constant $c_6 = c_6(\delta)$.

For the proof see Csáki et al. [7], Lemma 2.8.

Lemma 3.2. *For $T > 1$, $0 < \delta < 1/2$, $z > 1$ we have*

$$\begin{aligned} &\mathbb{P}\left(\sup_{0\le t\le T-1} (Y(t+1) - Y(t)) \ge z\right) \\ &\qquad \ge \min\left(\frac{1}{2},\, \frac{c_7\sqrt{T-1}}{z}\exp\left(-\frac{z^2}{8(1-\delta)}\right)\right) - \exp\left(-z^2\right) \end{aligned} \tag{3.2}$$

with some positive constant $c_7 = c_7(\delta) > 0$.

Proof. Let us construct an increasing sequence of stopping times by $\eta_0 := 0$ and

$$\eta_{k+1} := \inf\{t > \eta_k + 1 :\, W(t) = 0\}, \qquad k = 0, 1, 2, \ldots$$

Let

$$\nu_t := \min\{i \geq 1 :\ \eta_i > t\}$$

$$Z_i := Y(\eta_{i-1}+1) - Y(\eta_{i-1}), \qquad i = 1, 2, \ldots$$

Then $(Z_i,\, \eta_i - \eta_{i-1})_{i\geq 1}$ are i.i.d. random vectors with

$$\eta_i - \eta_{i-1} \stackrel{\text{law}}{=} 1 + \tau^2, \qquad Z_i \stackrel{\text{law}}{=} Y(1),$$

where τ has Cauchy distribution. Clearly, for $t > 0$,

$$\sup_{0\leq s\leq t} (Y(s+1) - Y(s)) \geq \max_{1\leq i\leq \nu_t} Z_i = \overline{Z}_{\nu_t},$$

with $\overline{Z}_k := \max_{1\leq i\leq k} Z_i$. First consider the Laplace transform ($\lambda > 0$):

$$\begin{aligned}
&\lambda \int_0^\infty e^{-\lambda u} \mathbb{P}\left(\overline{Z}_{\nu_u} < z\right)\ \mathrm{d}u \\
&= \lambda \sum_{k=1}^\infty \mathbb{E} \int_0^\infty e^{-\lambda u} 1_{\{\eta_{k-1}\leq u < \eta_k\}} 1_{\{\overline{Z}_k < z\}}\ \mathrm{d}u \\
&= \sum_{k=1}^\infty \mathbb{E}\left(\left[e^{-\lambda\eta_{k-1}} - e^{-\lambda\eta_k}\right] 1_{\{\overline{Z}_k < z\}}\right) \\
&= \sum_{k=1}^\infty \left(\mathbb{E}\left[1_{\{\overline{Z}_k < z\}} e^{-\lambda\eta_{k-1}}\right] - \mathbb{E}\left[1_{\{\overline{Z}_k < z\}} e^{-\lambda\eta_k}\right]\right) \\
&= \sum_{k=1}^\infty \left(\mathbb{E}\left[1_{\{\overline{Z}_{k-1} < z\}} e^{-\lambda\eta_{k-1}}\right] - \mathbb{E}\left[1_{\{\overline{Z}_{k-1} < z,\, Z_k \geq z\}} e^{-\lambda\eta_{k-1}}\right] - \mathbb{E}\left[1_{\{\overline{Z}_k < z\}} e^{-\lambda\eta_k}\right]\right) \\
&= 1 - \sum_{k=1}^\infty \mathbb{E}\left[1_{\{\overline{Z}_{k-1} < z,\, Z_k \geq z\}}\, e^{-\lambda\eta_{k-1}}\right] \\
&= 1 - \sum_{k=1}^\infty \mathbb{E}\left[1_{\{\overline{Z}_{k-1} < z\}} e^{-\lambda\eta_{k-1}}\right] \mathbb{P}(Y(1) \geq z) \\
&= 1 - \sum_{k=1}^\infty \left(\mathbb{E}\left[1_{\{Z_1 < z\}} e^{-\lambda\eta_1}\right]\right)^{k-1} \mathbb{P}(Y(1) \geq z) \\
&= 1 - \frac{\mathbb{P}(Y(1) \geq z)}{1 - \mathbb{E}\left[1_{\{Z_1 < z\}} e^{-\lambda\eta_1}\right]},
\end{aligned}$$

i.e.,

$$\lambda \int_0^\infty e^{-\lambda u} \mathbb{P}\left(\overline{Z}_{\nu_u} \geq z\right)\ \mathrm{d}u = \frac{\mathbb{P}(Y(1) \geq z)}{1 - \mathbb{E}\left[1_{\{Z_1 < z\}} e^{-\lambda\eta_1}\right]}. \tag{3.3}$$

But (recalling that $Z_1 = Y(1)$)

$$1 - \mathbb{E}\left[1_{\{Z_1 < z\}} e^{-\lambda\eta_1}\right] \leq 1 - \mathbb{E}(e^{-\lambda\eta_1}) + \mathbb{P}(Y(1) \geq z)$$

and (cf. [14], 3.466/1)

$$1 - \mathbb{E}e^{-\lambda\eta_1} = 1 - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-\lambda(1+x^2)}}{1+x^2}\,\mathrm{d}x = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{\lambda}} e^{-x^2}\,\mathrm{d}x \le 2\sqrt{\lambda},$$

hence

$$\lambda\int_0^{\infty} e^{-\lambda u}\mathbb{P}\left(\overline{Z}_{\nu_u} \ge z\right)\,\mathrm{d}u \ge \frac{\mathbb{P}(Y(1)\ge z)}{2\sqrt{\lambda}+\mathbb{P}(Y(1)\ge z)}.$$

On the other hand, for any $u_0 > 0$ we have

$$\begin{aligned}\lambda\int_0^{\infty} e^{-\lambda u}\mathbb{P}\left(\overline{Z}_{\nu_u} \ge z\right)\,\mathrm{d}u &= \lambda\int_0^{u_0} e^{-\lambda u}\mathbb{P}\left(\overline{Z}_{\nu_u} \ge z\right)\,\mathrm{d}u + \lambda\int_{u_0}^{\infty} e^{-\lambda u}\mathbb{P}\left(\overline{Z}_{\nu_u} \ge z\right)\,\mathrm{d}u \\ &\le \mathbb{P}\left(\overline{Z}_{\nu_{u_0}} \ge z\right) + e^{-\lambda u_0}.\end{aligned}$$

It turns out that

$$\mathbb{P}\left(\overline{Z}_{\nu_{u_0}} \ge z\right) \ge \frac{\mathbb{P}(Y(1)\ge z)}{2\sqrt{\lambda}+\mathbb{P}(Y(1)\ge z)} - e^{-\lambda u_0} \ge \min\left(\frac{1}{2},\, \frac{\mathbb{P}(Y(1)\ge z)}{4\sqrt{\lambda}}\right) - e^{-\lambda u_0}, \tag{3.4}$$

where the inequality

$$\frac{x}{y+x} \ge \min\left(\frac{1}{2},\, \frac{x}{2y}\right), \qquad x>0,\, y>0$$

was used. Choosing $u_0 = T-1$, $\lambda = z^2/u_0$, and applying (2.5) of Fact 2.1, we finally get

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\le t\le T-1}(Y(t+1)-Y(t)) \ge z\right) \\ &\quad\ge \min\left(\frac{1}{2},\, \frac{c_8(\delta)\sqrt{T-1}}{z}\exp\left(-\frac{z^2}{8(1-\delta)}\right)\right) - \exp\left(-z^2\right).\end{aligned} \tag{3.5}$$

This proves Lemma 3.2. □

Lemma 3.3. *For $T \ge 2$, $0 \le \kappa < 1$ and $\delta, z > 0$ we have*

$$\mathbb{P}\left(\sup_{0\le t\le T-1}(Y(t+1)-Y(t)) < z\right) \le \frac{5}{T^{\kappa/2}} + \exp\left(-c_9 T^{(1-\kappa)/2}e^{-(1+\delta)z^2/8}\right) \tag{3.6}$$

with some positive constant $c_9 = c_9(\delta)$.

See Csáki et al. [7], Lemma 3.1.

Lemma 3.4. *For $T \ge 1$, $0 < z \le 1/2$ we have*

$$\mathbb{P}\left(\sup_{0\le t\le T-1}\sup_{0\le s\le 1}|Y(t+s)-Y(t)| < z\right) \ge \frac{c_{10}}{\sqrt{T}}\exp\left(-\frac{c_{11}}{z^2}\right) \tag{3.7}$$

with some positive constants c_{10}, c_{11}.

Proof. Define the events

$$A := \left\{ \sup_{0\le s\le 1} |Y(s)| < \frac{z}{4},\, W(1) \ge \frac{4}{z},\, \inf_{1\le u\le T} W(u) \ge \frac{2}{z} \right\}$$

and

$$\widetilde{A} := \left\{ \sup_{0\le t\le T-1} \sup_{0\le s\le 1} |Y(t+s) - Y(t)| < z \right\}.$$

Then $A \subset \widetilde{A}$, since if A occurs and $t < 1$, $t + s \le 1$, then

$$|Y(t+s) - Y(t)| \le 2 \sup_{0\le s\le 1} |Y(s)| \le \frac{z}{2} < z.$$

If A occurs and $t < 1$, $s \le 1$, $1 < t + s \le T$, then

$$|Y(t+s) - Y(t)| \le Y(t+s) - Y(1) + |Y(t) - Y(1)| \le \int_1^{t+s} \frac{\mathrm{d}u}{W(u)} + \frac{z}{2} < z.$$

Moreover, if A occurs and $1 \le t$, $s \le 1$, $t + s \le T$, then

$$|Y(t+s) - Y(t)| = \int_t^{t+s} \frac{\mathrm{d}u}{W(u)} \le \frac{z}{2} < z.$$

Hence $A \subset \widetilde{A}$ as claimed. But by the Markov property of W,

$$\mathbb{P}(A) = \int_{4/z}^{\infty} \mathbb{P}\left(\sup_{0\le s\le 1} |Y(s)| < \frac{z}{4} \,\middle|\, W(1) = x \right) \mathbb{P}\left(\inf_{1\le u\le T} W(u) \ge \frac{2}{z} \,\middle|\, W(1) = x \right) \varphi(x)\, \mathrm{d}x, \tag{3.8}$$

where φ denotes the standard normal density function.

Using reflection principle and $x \ge 4/z$, $z \le 1/2$, we get

$$\begin{aligned} &\mathbb{P}\left(\inf_{1\le u\le T} W(u) \ge \frac{2}{z} \,\middle|\, W(1) = x \right) = 2\Phi\left(\frac{x - 2/z}{\sqrt{T-1}} \right) - 1 \\ &\ge 2\Phi\left(\frac{2}{z\sqrt{T-1}} \right) - 1 \ge 2\Phi\left(\frac{4}{\sqrt{T}} \right) - 1 \ge \frac{c_{12}}{\sqrt{T}}, \end{aligned} \tag{3.9}$$

with some constant $c_{12} > 0$, where $\Phi(\cdot)$ is the standard normal distribution function. Hence

$$\mathbb{P}(\widetilde{A}) \ge \mathbb{P}(A) \ge \frac{c_{12}}{\sqrt{T}} \mathbb{P}\left(\sup_{0\le s\le 1} |Y(s)| \le \frac{z}{4},\, W(1) \ge \frac{4}{z} \right). \tag{3.10}$$

To get a lower bound of the probability on the right-hand side, define g, $(m(v),\, 0 \le v \le 1)$, $(B(u),\, 0 \le u \le 1)$ by (2.1), (2.2) and (2.3), respectively. Recall (see Fact 2.1) that these three objects are independent, g has arc sine distribution, m is a Brownian meander and B is a Brownian

bridge. Moreover, (g, m, B) are independent of $\mathtt{sgn}(W(1))$ which is a Bernoulli variable. Observe that

$$\sup_{0\le s\le g}|Y(s)| = \sqrt{g}\sup_{0\le s\le 1}\left|\int_0^s \frac{\mathrm{d}u}{B(u)}\right|,$$
$$\sup_{g\le s\le 1}|Y(s)| = |Y(1)-Y(g)| = \sqrt{1-g}\int_0^1 \frac{\mathrm{d}v}{m(v)},$$
$$|W(1)| = \sqrt{1-g}\,m(1).$$

Then

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\le s\le 1}|Y(s)|\le\frac{z}{4},\, W(1)\ge\frac{4}{z}\right)\\
&\ge \mathbb{P}\left(\sup_{0\le s\le g}|Y(s)|\le\frac{z}{8},\, Y(1)-Y(g)\le\frac{z}{8},\, W(1)\ge\frac{4}{z}\right)\\
&\ge \mathbb{P}\left(\sqrt{g}\sup_{0\le s\le 1}\left|\int_0^s\frac{\mathrm{d}u}{B(u)}\right|\le\frac{z}{8},\,\sqrt{1-g}\int_0^1\frac{\mathrm{d}v}{m(v)}\le\frac{z}{8},\,\sqrt{1-g}\,m(1)\ge\frac{4}{z},\, W(1)>0,\, g<z^2\right)\\
&\ge \mathbb{P}\left(\sup_{0\le s\le 1}\left|\int_0^s\frac{\mathrm{d}u}{B(u)}\right|\le\frac{1}{8},\,\int_0^1\frac{\mathrm{d}v}{m(v)}\le\frac{z}{8},\, m(1)\ge\frac{4}{z\sqrt{1-z^2}},\, W(1)>0,\, g<z^2\right)\\
&= \mathbb{P}\left(\sup_{0\le s\le 1}\left|\int_0^s\frac{\mathrm{d}u}{B(u)}\right|\le\frac{1}{8}\right)\mathbb{P}\left(\int_0^1\frac{\mathrm{d}v}{m(v)}\le\frac{z}{8},\, m(1)\ge\frac{4}{z\sqrt{1-z^2}}\right)\mathbb{P}(W(1)>0)\mathbb{P}(g<z^2)\\
&\ge c_{13}z\mathbb{P}\left(\int_0^1\frac{\mathrm{d}v}{m(v)}\le\frac{z}{8},\, m(1)\ge\frac{4}{z\sqrt{1-z^2}}\right)\\
&= c_{13}z\int_{4/(z\sqrt{1-z^2})}^{\infty}\mathbb{P}\left(\int_0^1\frac{\mathrm{d}v}{m(v)}\le\frac{z}{8}\,\middle|\, m(1)=x\right)\mathbb{P}(m(1)\in\mathrm{d}x).
\end{aligned}$$

It follows from Facts 2.1 and 2.2 that for $x>0$, $z>0$

$$\mathbb{P}\left(\int_0^1\frac{\mathrm{d}v}{m(v)}\le\frac{z}{8}\,\middle|\, m(1)=x\right)\ge\mathbb{P}\left(\int_0^1\frac{\mathrm{d}v}{m(v)}\le\frac{z}{8}\,\middle|\, m(1)=0\right)\ge\frac{c_{14}}{z^3}\exp\left(-\frac{c_{15}}{z^2}\right) \tag{3.11}$$

and

$$\mathbb{P}\left(m(1)>\frac{4}{z\sqrt{1-z^2}}\right)=\exp\left(-\frac{8}{z^2(1-z^2)}\right). \tag{3.12}$$

Putting (3.10), (3.11), (3.12) together, we get (3.7). □

Lemma 3.5. *For $T>1$, $0<z\le 1/2$, $0<\delta\le 1/2$ we have*

$$\begin{aligned}
&\mathbb{P}\left(\inf_{0\le t\le T-1}\sup_{0\le s\le 1}|Y(t+s)-Y(t)|<z\right)\\
&\le c_{16}\left(\exp\left(-\frac{(1-\delta)^2}{2(1+\delta)^2z^2T}\right)+\exp\left(-\frac{c_5\delta}{4(1+\delta)^2z^2}\right)+\exp\left(\frac{c_{17}}{z^2}-\frac{c_{18}z^2}{T}e^{c_{19}/z^2}\right)\right)
\end{aligned} \tag{3.13}$$

with some positive constants c_{16}, $c_{17} = c_{17}(\delta)$, $c_{18} = c_{18}(\delta)$, $c_{19} = c_{19}(\delta)$.

Proof. Consider a positive integer N to be given later, $h = (T-1)/N$, $t_k = kh$, $k = 0, 1, 2, \ldots, N$. Then for $0 < \delta \le 1/2$ we have

$$\begin{aligned}
&\mathbb{P}\left(\inf_{0\le t\le T-1}\sup_{0\le s\le 1}|Y(t+s)-Y(t)| < z\right)\\
&\le \mathbb{P}\left(\inf_{0\le k\le N}\sup_{0\le s\le 1}|Y(t_k+s)-Y(t_k)| \le (1+\delta)z\right) + \mathbb{P}\left(\sup_{0\le t\le T-1}\sup_{0\le s\le h}|Y(t+s)-Y(t)| > \delta z\right)\\
&=: P_1 + P_2.
\end{aligned}$$

By scaling and Lemma 3.1

$$\begin{aligned}
P_2 &= \mathbb{P}\left(\sup_{0\le t\le (T-1)/h}\sup_{0\le s\le 1}|Y(t+s)-Y(t)| > \frac{\delta z}{\sqrt{h}}\right)\\
&\le c_6\left(\sqrt{\frac{T-1}{h}+1}\,\exp\left(-\frac{\delta^2 z^2}{8h(1+\delta)}\right) + \left(\frac{T-1}{h}+1\right)\exp\left(-\frac{\delta^2 z^2}{2h(1+\delta)}\right)\right)\\
&\le 2c_6(N+1)\exp\left(-\frac{\delta^2 z^2}{8h(1+\delta)}\right).
\end{aligned}$$

To bound P_1, we denote by $d(t) := \inf\{s \ge t : W(s) = 0\}$ the first zero of W after t. Consider those k for which $\sup_{0\le s\le 1}|Y(t_k+s)-Y(t_k)| \le (1+\delta)z$. If, moreover, $d(t_k) \ge t_k + 1 - \delta$, which means that the Brownian motion W does not change sign over $[t_k, t_k+1-\delta)$, then

$$(1+\delta)z \ge |Y(t_k+1-\delta)-Y(t_k)| = \int_0^{1-\delta}\frac{\mathrm{d}s}{|W(t_k+s)|} \ge \frac{1-\delta}{\sup_{0\le s\le T}|W(s)|},$$

and it follows that

$$\begin{aligned}
P_1 &\le \mathbb{P}\left(\sup_{0\le s\le T}|W(s)| > \frac{(1-\delta)}{z(1+\delta)}\right)\\
&+ \mathbb{P}\left(\exists k \le N : \sup_{0\le s\le 1}|Y(t_k+s)-Y(t_k)| \le (1+\delta)z; d(t_k) < t_k+1-\delta\right)\\
&\le 4\exp\left(-\frac{(1-\delta)^2}{2(1+\delta)^2 z^2 T}\right)\\
&+ \sum_{k=0}^{N}\mathbb{P}\left(\sup_{0\le s\le 1}|Y(t_k+s)-Y(t_k)| \le (1+\delta)z; d(t_k) < t_k+1-\delta\right).
\end{aligned}$$

Let $\widehat{W}(s) = W(d(t_k)+s)$ for $s \ge 0$ and $\widehat{Y}(s)$ be the associated principal values. Observe that on $\{\sup_{0\le s\le 1}|Y(t_k+s)-Y(t_k)| \le (1+\delta)z; d(t_k) < t_k+1-\delta\}$, we have $\sup_{0\le u\le\delta}|\widehat{Y}(u) + (Y(d(t_k))-Y(t_k))| < (1+\delta)z$, and $|Y(d(t_k))-Y(t_k)| \le (1+\delta)z$ which imply that

$$\sup_{0\le u\le\delta}|\widehat{Y}(u)| < 2(1+\delta)z.$$

By scaling and Fact 2.3 we have

$$\mathbb{P}\left(\sup_{0\le u\le\delta}|\widehat{Y}(u)|<2(1+\delta)z\right)\le c_4\exp\left(-\frac{c_5\delta}{4(1+\delta)^2z^2}\right).$$

Therefore, we obtain:

$$P_1\le 4\exp\left(-\frac{(1-\delta)^2}{2(1+\delta)^2z^2T}\right)+c_4(N+1)\exp\left(-\frac{c_5\delta}{4(1+\delta)^2z^2}\right).$$

Hence

$$\begin{aligned}P_1+P_2\le{}&4\exp\left(-\frac{(1-\delta)^2}{2(1+\delta)^2z^2T}\right)+c_4(N+1)\exp\left(-\frac{c_5\delta}{4(1+\delta)^2z^2}\right)\\&+2c_6(N+1)\exp\left(-\frac{\delta^2z^2}{8h(1+\delta)}\right).\end{aligned}$$

By taking $N=[e^{c_5\delta/(4(1+\delta)^2z^2)}]+1$, we get

$$\begin{aligned}&P_1+P_2\\&\le c_{16}\left(\exp\left(-\frac{(1-\delta)^2}{2(1+\delta)^2z^2T}\right)+\exp\left(-\frac{c_5\delta}{4(1+\delta)^2z^2}\right)+\exp\left(\frac{c_{17}}{z^2}-\frac{c_{18}z^2}{T}e^{c_{19}/z^2}\right)\right)\end{aligned}$$

with relevant constants c_{16}, c_{17}, c_{18}, c_{19}, proving (3.13). □

4. Proof of Theorem 1.1(i)

The upper estimation, i.e.

$$\limsup_{T\to\infty}\frac{\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}|Y(t+s)-Y(t)|}{\sqrt{8a_T\left(\log\sqrt{T/a_T}+\log\log T\right)}}\le 1,\qquad\text{a.s.}\tag{4.1}$$

follows easily from Wen's Theorem E.

Now we prove the lower bound, i.e.

$$\limsup_{T\to\infty}\frac{\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}|Y(t+s)-Y(t)|}{\sqrt{8a_T\left(\log\sqrt{T/a_T}+\log\log T\right)}}\ge 1,\qquad\text{a.s.}\tag{4.2}$$

In the case when $a_T=T$, (4.2) follows from the law of the iterated logarithm (1.3) of Theorem A. Now we assume that $a_T/T\le\rho<1$, with some constant ρ for all $T>0$.

By scaling, (3.2) of Lemma 3.2 is equivalent to

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\le t\le T-a}(Y(t+a)-Y(t))\ge z\sqrt{a}\right)\\&\qquad\ge\min\left(\frac{1}{2},\frac{c_7\sqrt{T/a-1}}{z}\exp\left(-\frac{z^2}{8(1-\delta)}\right)\right)-\exp\left(-z^2\right)\end{aligned}\tag{4.3}$$

for $0 < a < T$, $0 < \delta < 1/2$, $z > 1$.

Define the sequences

$$t_k := e^{7k \log k}, \qquad k = 1, 2, \dots \tag{4.4}$$

and $\theta_0 := 0$,

$$\theta_k := \inf\{t > T_k : W(t) = 0\}, \qquad k = 1, 2, \dots, \tag{4.5}$$

where $T_k := \theta_{k-1} + t_k$. For $0 < \delta < \min(1/2, 1 - \rho)$ define the events

$$A_k := \left\{ \sup_{0 \le t \le t_k(1-\delta) - a_{t_k}} (Y(\theta_{k-1} + t + a_{t_k}) - Y(\theta_{k-1} + t)) \ge (1-\delta)\beta_k \right\}, \quad k = 1, 2, \dots$$

with

$$\beta_k := \sqrt{8 a_{t_k} \left(\log \sqrt{\frac{t_k}{a_{t_k}}} + \log \log t_k \right)}.$$

Applying (4.3) with $T = t_k(1-\delta)$, $a = a_{t_k}$, $z = (1-\delta)\sqrt{8(\log \sqrt{t_k/a_{t_k}} + \log \log t_k)}$, we have for k large

$$\begin{aligned} \mathbb{P}(A_k) &= \mathbb{P}\left(\sup_{0 \le t \le t_k(1-\delta) - a_{t_k}} (Y(t + a_{t_k}) - Y(t)) \ge (1-\delta)\beta_k \right) \\ &\ge \min\left(\frac{1}{2}, \frac{b_k}{(\log t_k)^{1-\delta}} \right) - \frac{1}{(\log t_k)^{8(1-\delta)^2}} \end{aligned}$$

with

$$b_k = \frac{c_7 \sqrt{t_k(1-\delta)/a_{t_k} - 1}}{(t_k/a_{t_k})^{(1-\delta)/2} \sqrt{\log \sqrt{t_k/a_{t_k}} + \log \log t_k}} \ge \frac{c_{20}}{\sqrt{\log k}}.$$

Hence $\sum_k \mathbb{P}(A_k) = \infty$ and since A_k are independent, Borel-Cantelli lemma yields

$$\mathbb{P}(A_k \text{ i.o.}) = 1.$$

It follows that

$$\limsup_{k \to \infty} \frac{\sup_{0 \le t \le t_k(1-\delta) - a_{t_k}} (Y(\theta_{k-1} + t + a_{t_k}) - Y(\theta_{k-1} + t))}{\sqrt{8 a_{t_k} \left(\log \sqrt{\frac{t_k}{a_{t_k}}} + \log \log t_k \right)}} \ge 1 - \delta, \qquad \text{a.s.} \tag{4.6}$$

It can be seen (cf. [9]) that we have almost surely for large enough k

$$t_k \le T_k \le t_k \left(1 + \frac{1}{k} \right),$$

consequently

$$\lim_{k\to\infty}\frac{t_k}{T_k}=1, \qquad \text{a.s.} \tag{4.7}$$

Since by our assumptions

$$\frac{t_k}{T_k}\le\frac{a_{t_k}}{a_{T_k}}\le 1,$$

we have also

$$\lim_{k\to\infty}\frac{a_{t_k}}{a_{T_k}}=1, \qquad \text{a.s.} \tag{4.8}$$

On the other hand, for any $\delta>0$ small enough we have almost surely for large k

$$a_{T_k}\le(1+\delta)a_{t_k}\le t_k\delta+a_{t_k},$$

thus

$$T_k-a_{T_k}\ge T_k-t_k\delta-a_{t_k},$$

consequently

$$\begin{aligned}&\sup_{0\le t\le T_k-a_{T_k}}\sup_{0\le s\le a_{T_k}}|Y(t+s)-Y(t)|\\ &\ge\sup_{0\le t\le t_k(1-\delta)-a_{t_k}}(Y(\theta_{k-1}+t+a_{t_k})-Y(\theta_{k-1}+t)),\end{aligned} \tag{4.9}$$

hence we have also

$$\limsup_{k\to\infty}\frac{\sup_{0\le t\le T_k-a_{T_k}}\sup_{0\le s\le a_{T_k}}|Y(t+s)-Y(t)|}{\sqrt{8a_{t_k}\left(\log\sqrt{\frac{t_k}{a_{t_k}}}+\log\log t_k\right)}}\ge 1-\delta, \qquad \text{a.s.} \tag{4.10}$$

and since $\delta>0$ can be arbitrary small, (4.2) follows by combining (4.7), (4.8), (4.9) and (4.10). □

5. Proof of Theorem 1.1(ii)

First assume that

$$a_T>\frac{T}{(\log T)^\alpha} \qquad \text{for some} \quad \alpha<2. \tag{5.1}$$

By Theorem C,

$$\begin{aligned}&\liminf_{T\to\infty}\sqrt{\frac{\log\log T}{a_T}}\sup_{0\le t\le T-a_T}\sup_{0\le s\le a_T}|Y(t+s)-Y(t)|\\ &\ge\liminf_{T\to\infty}\sqrt{\frac{\log\log a_T}{a_T}}\sup_{0\le s\le a_T}|Y(s)|\ge K_1, \qquad \text{a.s.},\end{aligned} \tag{5.2}$$

proving the lower bound in (1.12).

To get an upper bound, note that by scaling, (3.7) of Lemma 3.4 is equivalent to

$$\mathbb{P}\left(\sup_{0\le t\le T-a}\sup_{0\le s\le a}|Y(s+t)-Y(t)|<z\sqrt{a}\right)\ge c_{10}\sqrt{\frac{a}{T}}\exp\left(-\frac{c_{11}}{z^2}\right) \tag{5.3}$$

for $T\ge a$, $0<z\le 1/2$.

Let t_k and θ_k be defined by (4.4) and (4.5), resp., $T_k=\theta_{k-1}+t_k$ as in the proof of Theorem 1.1(i). Let c_{11} be the constant as in (5.3) and choose $\delta>0$ such that $\alpha/2+c_{11}/\delta^2<1$. For $\varepsilon>0$ define the events

$$E_k:=\left\{\sup_{0\le t\le(1+\varepsilon)t_k-a_{t_k(1+\varepsilon)}}\sup_{0\le s\le a_{t_k(1+\varepsilon)}}|Y(\theta_{k-1}+t+s)-Y(\theta_{k-1}+t)|\le\delta\sqrt{\frac{a_{t_k}}{\log\log t_k}}\right\}.$$

Then putting $T=(1+\varepsilon)t_k$, $a=a_{(1+\varepsilon)t_k}$, $z=\delta/\sqrt{\log\log t_k}$ into (5.3), we get

$$\mathbb{P}(E_k)=\mathbb{P}\left(\sup_{0\le t\le(1+\varepsilon)t_k-a_{t_k(1+\varepsilon)}}\sup_{0\le s\le a_{t_k(1+\varepsilon)}}|Y(t+s)-Y(t)|\le\delta\sqrt{\frac{a_{t_k}}{\log\log t_k}}\right)$$

$$\ge c_{10}\sqrt{\frac{a_{t_k}}{t_k}}\exp(-(c_{11}/\delta^2)\log\log(t_k)\ge\frac{c_{10}}{(\log t_k)^{\alpha/2+c_{11}/\delta^2}}=\frac{c_{10}}{(7k\log k)^{\alpha/2+c_{11}/\delta^2}},$$

hence $\sum_k\mathbb{P}(E_k)=\infty$, and since E_k are independent, we have $\mathbb{P}(E_k\,\text{i.o.})=1$, i.e.

$$\liminf_{k\to\infty}\sqrt{\frac{\log\log t_k}{a_{t_k}}}\sup_{0\le t\le(1+\varepsilon)t_k-a_{t_k(1+\varepsilon)}}\sup_{0\le s\le a_{t_k(1+\varepsilon)}}|Y(\theta_{k-1}+t+s)-Y(\theta_{k-1}+t)|\le\delta,\quad\text{a.s.} \tag{5.4}$$

for any $\varepsilon>0$. For large enough k by (4.7) and (4.8) we have $a_{T_k}\le(1+\varepsilon)a_{t_k}$, a.s. and $T_k-a_{T_k}\le\theta_{k-1}+(1+\varepsilon)t_k-(1+\varepsilon)a_{t_k}$, a.s. Thus given any $\varepsilon>0$, we have for large k

$$\begin{aligned}&\sup_{0\le t\le T_k-a_{T_k}}\sup_{0\le s\le a_{T_k}}|Y(t+s)-Y(t)|\\&\le 2\sup_{0\le t\le\theta_{k-1}}|Y(t)|+\sup_{0\le t\le(1+\varepsilon)t_k-a_{t_k(1+\varepsilon)}}\sup_{0\le s\le a_{t_k(1+\varepsilon)}}|Y(\theta_{k-1}+t+s)-Y(\theta_{k-1}+t)|.\end{aligned} \tag{5.5}$$

By Theorem A, Fact 2.8, (4.7), (5.1) and simple calculation,

$$\begin{aligned}&\sup_{0\le t\le\theta_{k-1}}|Y(t)|=O(\theta_{k-1}\log\log\theta_{k-1})^{1/2}\\&=O(t_{k-1}(\log t_{k-1})^3\log\log t_{k-1})^{1/2}=o\left(\frac{a_{t_k}}{\log\log t_k}\right)^{1/2},\qquad\text{a.s.}\end{aligned} \tag{5.6}$$

as $k\to\infty$. Assembling (5.4), (5.5) and (5.6), we get

$$\liminf_{k\to\infty}\sqrt{\frac{\log\log t_k}{a_{t_k}}}\sup_{0\le t\le T_k-a_{T_k}}\sup_{0\le s\le a_{T_k}}|Y(t+s)-Y(t)|$$

$$= \liminf_{k\to\infty} \sqrt{\frac{\log\log T_k}{a_{T_k}}} \sup_{0\le t\le T_k - a_{T_k}} \sup_{0\le s\le a_{T_k}} |Y(t+s) - Y(t)| \le \delta, \qquad \text{a.s.}$$

which together with (5.2) yields (1.12).

Now assume that

$$a_T \le \frac{T}{(\log T)^\alpha} \qquad \text{for some} \quad \alpha > 2. \tag{5.7}$$

By Theorem 1.1(i),

$$\begin{aligned} &\liminf_{T\to\infty} \frac{\sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s) - Y(t)|}{\sqrt{a_T \log(T/a_T)}} \\ &\le \limsup_{T\to\infty} \frac{\sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s) - Y(t)|}{\sqrt{a_T \log(T/a_T)}} \\ &\le \limsup_{T\to\infty} \frac{\sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s) - Y(t)|}{\sqrt{\frac{2\alpha a_T}{\alpha+2}\left(\log\sqrt{T/a_T} + \log\log T\right)}} \le 2\sqrt{\frac{\alpha+2}{\alpha}}, \end{aligned} \tag{5.8}$$

i.e., an upper bound in (1.13) follows.

To get a lower bound under (5.7), observe that by scaling, (3.6) of Lemma 3.3 is equivalent to

$$\mathbb{P}\left(\sup_{0\le t\le T-a} (Y(t+a) - Y(t)) < z\sqrt{a}\right) \le 5\left(\frac{a}{T}\right)^{\kappa/2} + \exp\left(-c_9\left(\frac{T}{a}\right)^{(1-\kappa)/2} e^{-(1+\delta)z^2/8}\right)$$

for $a \le T$, $0 \le \kappa < 1$, $0 < \delta$, $0 < z$. Using (5.7) we get further

$$\begin{aligned} &\mathbb{P}\left(\sup_{0\le t\le T-a} (Y(t+a) - Y(t)) < z\sqrt{a}\right) \\ &\le \frac{5}{(\log T)^{\alpha\kappa/2}} + \exp\left(-c_9 (\log T)^{\alpha(1-\kappa)/2} e^{-(1+\delta)z^2/8}\right). \end{aligned} \tag{5.9}$$

In the case when (1.7) holds, (1.13) was proved in [7]. In other cases the proof is similar. Let $T_k = e^k$ and define the events

$$F_k = \left\{\sup_{0\le t\le T_k - a_{T_k}} (Y(t + a_{T_k}) - Y(t)) \le C_1 \sqrt{a_{T_k} \log\frac{T_k}{a_{T_k}}}\right\}$$

with

$$C_1 = 2\sqrt{\frac{\alpha - 2 - 2\varepsilon\alpha}{(1+\delta)\alpha}}.$$

By (5.9) with $\kappa = 2/\alpha + \varepsilon$,

$$\mathbb{P}(F_k) \le \frac{5}{k^{\alpha\kappa/2}} + \exp\left(-c_9 k^{\alpha((1-\kappa)/2 - (1+\delta)C_1^2/8)}\right) \le \frac{5}{k^{1+\alpha\varepsilon/2}} + \exp\left(-c_9 k^{\alpha\varepsilon/2}\right).$$

One can easily see that with these choices $\sum_k \mathbb{P}(F_k) < \infty$, consequently

$$\liminf_{k\to\infty} \frac{\sup_{0\le t\le T_k - a_{T_k}} (Y(t+a_{T_k}) - Y(t))}{\sqrt{a_{T_k} \log \frac{T_k}{a_{T_k}}}} \ge C_1, \qquad \text{a.s.},$$

implying also

$$\liminf_{k\to\infty} \frac{\sup_{0\le t\le T_k - a_{T_k}} \sup_{0\le s\le a_{T_k}} |Y(t+s) - Y(t)|}{\sqrt{a_{T_k} \log \frac{T_k}{a_{T_k}}}} \ge 2\sqrt{\frac{\alpha-2}{\alpha}}, \qquad \text{a.s.},$$

for ε can be choosen arbitrary small.

Since $\sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s) - Y(t)|$ is increasing in T, we obtain a lower bound in (1.13). This together with the 0-1 law for Brownian motion complete the proof of Theorem 1.1(ii). □

6. Proof of Theorem 1.2(i)

If $a_T = T$, then (1.14) is equivalent to Theorem C. Now assume that $\rho := \lim_{T\to\infty} a_T/T < 1$.

First we prove the lower bound, i.e.

$$\liminf_{T\to\infty} \frac{\sqrt{T \log\log T}}{a_T} \inf_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s) - Y(t)| \ge c, \qquad \text{a.s.} \tag{6.1}$$

By scaling, (3.13) of Lemma 3.5 is equivalent to

$$\begin{aligned} &\mathbb{P}\left(\inf_{0\le t\le T-a} \sup_{0\le s\le a} |Y(t+s) - Y(t)| < z\sqrt{a} \right) \\ &\le c_{16} \left(\exp\left(-\frac{a(1-\delta)^2}{2(1+\delta)^2 z^2 T} \right) + \exp\left(-\frac{c_5 \delta}{4(1+\delta)^2 z^2} \right) + \exp\left(\frac{c_{17}}{z^2} - \frac{c_{18} a z^2}{T} e^{c_{19}/z^2} \right) \right) \end{aligned} \tag{6.2}$$

for $a < T$, $0 < z \le 1/2$, $0 < \delta \le 1/2$.

Define the events

$$G_k = \left\{ \inf_{0\le t\le T_{k+1} - a_{T_k}} \sup_{0\le s\le a_{T_k}} |Y(t+s) - Y(t)| < z_k \sqrt{a_{T_k}} \right\} \quad k = 1, 2, \dots$$

Let $T_k = e^k$ and put $T = T_{k+1}$, $a = a_{T_k}$,

$$z = z_k = C_2 \sqrt{\frac{a_{T_k}}{T_{k+1} \log\log T_{k+1}}}$$

into (6.2). The constant C_2 will be choosen later. Denoting the terms on the right-hand side of (6.2) by I_1, I_2, I_3, resp., we have

$$\mathbb{P}(G_k) \le c_{16}(I_1^{(k)} + I_2^{(k)} + I_3^{(k)}),$$

where

$$I_1^{(k)} = \exp\left(-\frac{c_{21}}{C_2^2}\log\log T_{k+1}\right),$$

$$I_2^{(k)} = \exp\left(-\frac{c_{22}T_k}{C_2^2 a_{T_k}}\log\log T_{k+1}\right),$$

$$I_3^{(k)} = \exp\left(\frac{c_{23}T_k\log\log T_{k+1}}{C_2^2 a_{T_k}} - \frac{c_{24}C_2^2 a_{T_k}^2}{T_k^2\log\log T_{k+1}}\left(\log T_{k+1}\right)^{\frac{c_{25}T_k}{C_2^2 a_{T_k}}}\right)$$

with some constants $c_{21} = c_{21}(\delta)$, $c_{22} = c_{22}(\delta)$, c_{23}, c_{24}, c_{25}.

One can see easily that for any choice of positive C_2 and for all possible a_T (satisfying our conditions) we have $\sum_k I_3^{(k)} < \infty$. So we show that for appropriate choice of C_2 we have also $\sum_k I_j^{(k)} < \infty$, $j = 1, 2$.

First consider the case $0 < \rho$. Choosing a positive δ, one can select $C_2 < \min\left(\sqrt{c_{21}}, \sqrt{\frac{c_{22}}{\rho}}\right)$ and it is easy to verify that $\sum_k I_j^{(k)} < \infty$, $j = 1, 2$, hence also $\sum_k \mathbb{P}(G_k) < \infty$.

In the case $\rho = 0$ choose $C_2 < (1-\delta)/((1+\delta)\sqrt{2})$. With this choice we have $\sum_k I_1^{(k)} < \infty$ for arbitrary $\delta > 0$. Since $\lim_{k\to\infty}(T_k/a_{T_k}) = \infty$, we have also $\sum_k I_2^{(k)} < \infty$ and $\sum_k \mathbb{P}(G_k) < \infty$. The Borel-Cantelli lemma and interpolation between T_k's finish the proof of (6.1). We have also verified that in the case $\rho = 0$ one can choose $c = 1/\sqrt{2}$ in (6.1), since $\delta > 0$ can be choosen arbitrary small.

Now we turn to the proof of the upper bound, i.e.

$$\liminf_{T\to\infty} \frac{\sqrt{T\log\log T}}{a_T} \inf_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s) - Y(t)| \le C_3, \qquad \text{a.s.} \tag{6.3}$$

with some constant C_3.

If $\rho > 0$, then

$$\inf_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s) - Y(t)| \le \sup_{0\le s\le a_T} |Y(s)| \le \sup_{0\le s\le T} |Y(s)|$$

and hence (6.3) with some positive constant C_3 follows from Theorem C.

If $\rho = 0$, then let for any $\varepsilon > 0$

$$\lambda_T := \inf\{t : |W(t)| = \sup_{0\le s\le T(1-\varepsilon)} |W(s)|\}. \tag{6.4}$$

According to the law of the iterated logarithm, with probability one there exists a sequence $\{T_i,\ i \ge 1\}$ such that $\lim_{i\to\infty} T_i = \infty$ and

$$|W(\lambda_{T_i})| \ge \sqrt{2T_i(1-\varepsilon)\log\log T_i}. \tag{6.5}$$

But Fact 2.4 implies that for $\varepsilon > 0$

$$|W(\lambda_{T_i}) - W(s)| \le \sqrt{2(1+\varepsilon)\varepsilon T_i \log\log T_i}, \quad \lambda_{T_i} \le s \le \lambda_{T_i} + \varepsilon T_i, \quad i \ge 1. \tag{6.6}$$

Now assume that $W(\lambda_{T_i}) > 0$. The case when $W(\lambda_{T_i}) < 0$ is similar. Then (6.5) and (6.6) imply

$$W(s) \ge \left(\sqrt{1-\varepsilon} - \sqrt{\varepsilon(1+\varepsilon)}\right)\sqrt{2T_i \log\log T_i}, \quad \lambda_{T_i} \le s \le \lambda_{T_i} + \varepsilon T_i. \tag{6.7}$$

$\rho = 0$ implies that $a_T \le \varepsilon T$ for any $\varepsilon > 0$ and large enough T, hence we have from (6.7) for large i

$$\sup_{0\le s\le a_{T_i}} (Y(\lambda_{T_i}+s) - Y(\lambda_{T_i})) = Y(\lambda_{T_i}+a_{T_i}) - Y(\lambda_{T_i}) = \int_{\lambda_{T_i}}^{\lambda_{T_i}+a_{T_i}} \frac{\mathrm{d}s}{W(s)}$$

$$\le \frac{a_{T_i}}{\left(\sqrt{1-\varepsilon} - \sqrt{\varepsilon(1+\varepsilon)}\right)\sqrt{2T_i \log\log T_i}}.$$

Since $\varepsilon > 0$ is arbitrary, (6.3) follows with $C_3 = 1/\sqrt{2}$. This completes the proof of Theorem 1.2(i). □

7. Proof of Theorem 1.2(ii)

If $\rho = 1$, then (1.15) is equivalent to (1.3) of Theorem A. So we may assume that $0 < \rho < 1$. It suffices to show (1.15) when $a_T = \rho T$.

First we prove the upper bound

$$\limsup_{T\to\infty} \frac{\inf_{0\le t\le T-\rho T} \sup_{0\le s\le \rho T} |Y(t+s) - Y(t)|}{\sqrt{8T\log\log T}} \le \rho, \qquad \text{a.s.} \tag{7.1}$$

Let k be the largest integer for which $k\rho < 1$ and put $x_i = i\rho$, $i = 0, 1, \ldots, k$, $x_{k+1} = 1$. It suffices to show that if $f \in \mathcal{S}$ defined by (1.5), then

$$\min_{1\le i\le k+1} |f(x_i) - f(x_{i-1})| \le \rho.$$

Assume on the contrary that

$$|f(x_i) - f(x_{i-1})| > \rho, \qquad \forall i = 1, 2, \ldots, k+1.$$

Then

$$\sum_{i=1}^{k+1} \frac{(f(x_i) - f(x_{i-1}))^2}{x_i - x_{i-1}} > \sum_{i=1}^{k} \frac{\rho^2}{\rho} + \frac{\rho^2}{1-k\rho} = k\rho + \frac{\rho^2}{1-k\rho} \ge 1,$$

contradicting (2.12) of Fact 2.5. This proves (7.1).

The lower bound

$$\limsup_{T\to\infty} \frac{\inf_{0\le t\le T-\rho T}\ \sup_{0\le s\le \rho T}\ |Y(t+s)-Y(t)|}{\sqrt{8T\log\log T}} \ge \rho, \qquad \text{a.s.} \tag{7.2}$$

follows from the fact that by Theorem B the function $f(x)=x$, $0\le x\le 1$ is a limit point of

$$\frac{Y(xt)}{\sqrt{8T\log\log T}}$$

and for this function

$$\min_{0\le x\le 1-\rho} |f(x+\rho)-f(x)| = \rho.$$

This completes the proof of Theorem 1.2(iia). □

Now assume that

$$\lim_{T\to\infty} \frac{a_T(\log\log T)^2}{T} = 0. \tag{7.3}$$

Define λ_T as in (6.4). Then according to Chung's LIL (cf. Fact 2.6)

$$|W(\lambda_T)| \ge \frac{\pi}{\sqrt{8}}(1-\varepsilon)\sqrt{\frac{T}{\log\log T}} \tag{7.4}$$

for $\varepsilon>0$ and all T sufficiently large. But according to Fact 2.4,

$$\begin{aligned}&\sup_{0\le s\le a_T} |W(\lambda_T+s)-W(\lambda_T)|\\ &\le \sqrt{(2+\varepsilon)a_T(\log(T/a_T)+\log\log T)} \le \sqrt{\frac{(2+\varepsilon)\varepsilon T}{\log\log T}}.\end{aligned}$$

Assuming $W(\lambda_T)>0$, on choosing suitable $\varepsilon>0$ we get for some $c_{26}>0$

$$W(\lambda_T+s) \ge W(\lambda_T) - \sqrt{\frac{(2+\varepsilon)\varepsilon T}{\log\log T}} \ge c_{26}\sqrt{\frac{T}{\log\log T}}.$$

Hence

$$\begin{aligned}\inf_{0\le t\le T-a_T}\ \sup_{0\le s\le a_T} |Y(t+s)-Y(t)| &\le Y(\lambda_T+a_T)-Y(\lambda_T)\\ &= \int_0^{a_T} \frac{\mathrm{d}s}{W(\lambda_T+s)} \le \frac{a_T}{c_{26}}\sqrt{\frac{\log\log T}{T}}\end{aligned}$$

for all large T.

The case when $W(\lambda_T)<0$ is similar. This shows the upper bound in (1.16).

For the lower bound we use Fact 2.7: with probability one

$$g_T \le \frac{T}{(\log\log T)^2}, \quad \max_{0\le u\le T}|W(u)| \le \frac{\pi}{\sqrt{2}}\sqrt{\frac{T}{\log\log T}}, \quad \text{i.o.} \tag{7.5}$$

According to Theorem 1.2(i) we have for any $\varepsilon > 0$ and all large T

$$\begin{aligned}&\inf_{0\le t\le T(\log\log T)^{-2}} \sup_{0\le s\le a_T} |Y(t+s)-Y(t)| \\ &\ge \frac{(K_4-\varepsilon)a_T}{\sqrt{\left(\frac{T}{(\log\log T)^2}+a_T\right)\log\log T}} \ge \frac{(K_4-\varepsilon)a_T\sqrt{\log\log T}}{\sqrt{(1+\varepsilon)T}}.\end{aligned} \tag{7.6}$$

On the other hand, if $T(\log\log T)^{-2} \le t \le T - a_T$, and (7.5) is satisfied, then

$$|Y(t+a_T)-Y(t)| = \int_t^{t+a_T} \frac{\mathrm{d}s}{|W(s)|} \ge \frac{a_T\sqrt{2\log\log T}}{\pi\sqrt{T}}, \quad \text{i.o.} \tag{7.7}$$

Combining (7.6) and (7.7) for $\varepsilon > 0$ with probability one

$$\inf_{0\le t\le T-a_T} \sup_{0\le s\le a_T} |Y(t+s)-Y(t)| \ge \min\left(\frac{K_4-\varepsilon}{\sqrt{1+\varepsilon}}, \frac{\sqrt{2}}{\pi}\right)\frac{a_T\sqrt{\log\log T}}{T}, \quad \text{i.o.}$$

This shows the lower bound in (1.16). The proof of Theorem 1.2(iib) is complete by applying the 0-1 law for Brownian motion. □

Acknowledgements

The authors are indebted to Marc Yor for helpful remarks. We thank also the referee for useful suggestions. Cooperation between the authors was supported by the joint French–Hungarian Intergovernmental Grant "Balaton" (grant no. F-39/00).

References

[1] Ait Ouahra, M. and Eddahbi, M.: Théorèmes limites pour certaines fonctionnelles associées aux processus stables sur l'espace de Hölder. *Publ. Mat.* **45** (2001), 371–386.

[2] Bertoin, J.: On the Hilbert transform of the local times of a Lévy process. *Bull. Sci. Math.* **119** (1995), 147–156.

[3] Bertoin, J.: Cauchy's principal value of local times of Lévy processes with no negative jumps via continuous branching processes. *Electronic J. Probab.* **2** (1997), Paper No. 6, 1–12.

[4] Biane, P. and Yor, M.: Valeurs principales associées aux temps locaux browniens. *Bull. Sci. Math.* **111** (1987), 23–101.

[5] Boufoussi, B., Eddahbi, M. and Kamont, A.: Sur la dérivée fractionnaire du temps local brownien. *Probab. Math. Statist.* **17** (1997), 311–319.

[6] Chung, K.L.: On the maximum partial sums of sequences of independent random variables. *Trans. Amer. Math. Soc.* **64** (1948), 205–233.

[7] Csáki, E., Csörgő, M. Földes, A. and Shi, Z.: Increment sizes of the principal value of Brownian local time. *Probab. Th. Rel. Fields* **117** (2000), 515–531.

[8] Csáki, E., Csörgő, M. Földes, A. and Shi, Z.: Path properties of Cauchy's principal values related to local time. *Studia Sci. Math. Hungar.* **38** (2001), 149–169.

[9] Csáki, E. and Földes, A.: A note on the stability of the local time of a Wiener process. *Stoch. Process. Appl.* **25** (1987), 203–213.

[10] Csáki, E., Földes, A. and Shi, Z.: A joint functional law for the Wiener process and principal value. *Studia Sci. Math. Hungar.* **40** (2003), 213–241.

[11] Csáki, E., Shi, Z. and Yor, M.: Fractional Brownian motions as "higher-order" fractional derivatives of Brownian local times. In: *Limit Theorems in Probability and Statistics* (I. Berkes et al., eds.) Vol. I, pp. 365–387. János Bolyai Mathematical Society, Budapest, 2002.

[12] Csörgő, M. and Révész, P.: *Strong Approximations in Probability and Statistics.* Academic Press, New York, 1981.

[13] Fitzsimmons, P.J. and Getoor, R.K.: On the distribution of the Hilbert transform of the local time of a symmetric Lévy process. *Ann. Probab.* **20** (1992), 1484–1497.

[14] Gradshteyn, I.S. and Ryzhik, I.M.: *Table of Integrals, Series, and Products.* Sixth ed. Academic Press, San Diego, CA, 2000.

[15] Grill, K.: On the last zero of a Wiener process. In: *Mathematical Statistics and Probability Theory* (M.L. Puri et al., eds.) Vol. A, pp. 99–104. D. Reidel, Dordrecht, 1987.

[16] Hu, Y.: The laws of Chung and Hirsch for Cauchy's principal values related to Brownian local times. *Electronic J. Probab.* **5** (2000), Paper No. 10, 1–16.

[17] Hu, Y. and Shi, Z.: An iterated logarithm law for Cauchy's principal value of Brownian local times. In: *Exponential Functionals and Principal Values Related to Brownian Motion* (M. Yor, ed.), pp. 131–154. Biblioteca de la Revista Matemática Iberoamericana, Madrid, 1997.

[18] Strassen, V.: An invariance principle for the law of the iterated logarithm. *Z. Wahrsch. verw. Gebiete* **3** (1964), 211–226.

[19] Wen, Jiwei: Some results on lag increments of the principal value of Brownian local time. *Appl. Math. J. Chinese Univ. Ser. B* **17** (2002), 199–207.

[20] Yamada, T.: Principal values of Brownian local times and their related topics. In: *Itô's Stochastic Calculus and Probability Theory* (N. Ikeda et al., eds.), pp. 413–422. Springer, Tokyo, 1996.

[21] Yor, M.: *Some Aspects of Brownian Motion. Part 1: Some Special Functionals.* ETH Zürich Lectures in Mathematics. Birkhäuser, Basel, 1992.

[22] Yor, M., editor: *Exponential Functionals and Principal Values Related to Brownian Motion.* Biblioteca de la Revista Matemática Iberoamericana, Madrid, 1997.