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Quasi stationary distributions and Fleming-Viot processes in countable spaces*

Pablo A. Ferrari and Nevena Marić
Instituto de Matemática e Estatística
Universidade de São Paulo,
Caixa Postal 66281, 05311-970 São Paulo, Brazil
pablo@ime.usp.br, nevena@ime.usp.br
http://www.ime.usp.br/~pablo
http://www.ime.usp.br/~nevena

Abstract

We consider an irreducible pure jump Markov process with rates $Q = (q(x, y))$ on $\Lambda \cup \{0\}$ with Λ countable and 0 an absorbing state. A *quasi stationary distribution* (QSD) is a probability measure ν on Λ that satisfies: starting with ν, the conditional distribution at time t, given that at time t the process has not been absorbed, is still ν. That is, $\nu(x) = \nu P_t(x)/(\sum_{y\in\Lambda} \nu P_t(y))$, with P_t the transition probabilities for the process with rates Q.
A *Fleming-Viot* (FV) process is a system of N particles moving in Λ. Each particle moves independently with rates Q until it hits the absorbing state 0; but then instantaneously chooses one of the $N - 1$ particles remaining in Λ and jumps to its position. Between absorptions each particle moves with rates Q independently.
Under the condition $\alpha := \sum_{x\in\Lambda} \inf Q(\cdot, x) > \sup Q(\cdot, 0) := C$ we prove existence of QSD for Q; uniqueness has been proven by Jacka and Roberts. When $\alpha > 0$ the FV process is ergodic for each N. Under $\alpha > C$ the mean normalized densities of the FV unique stationary measure converge to the QSD of Q, as $N \to \infty$; in this limit the variances vanish .

Key words: Quasi stationary distributions; Fleming-Viot process.

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1 Introduction

Let Λ be a countable set and Z_t be a pure jump regular Markov process on $\Lambda \cup \{0\}$ with transition rates matrix $Q = (q(x,y))$, transition probabilities $P_t(x,y)$ and with absorbing state 0; that is $q(0,x) = 0$ for all $x \in \Lambda$. Assume that the exit rates are uniformly bounded above: $\bar{q} := \sup_x \sum_{y \in \{0\} \cup \Lambda \setminus \{x\}} q(x,y) < \infty$, that $P_t(x,y) > 0$ for all $x, y \in \Lambda$ and $t > 0$ and that the absorption time is almost surely finite for any initial state. The process Z_t is ergodic with a unique invariant measure δ_0, the measure concentrating mass in the state 0. Let μ be a probability on Λ. The law of the process at time t starting with μ conditioned to non absorption until time t is given by

$$\varphi_t^\mu(x) = \frac{\sum_{y \in \Lambda} \mu(y) P_t(y,x)}{1 - \sum_{y \in \Lambda} \mu(y) P_t(y,0)}, \quad x \in \Lambda. \tag{1.1}$$

A *quasi stationary distribution* (QSD) is a probability measure ν on Λ satisfying $\varphi_t^\nu = \nu$. Since P_t is honest and satisfies the forward Kolmogorov equations we can use an equivalent definition of QSD, according Nair and Pollett (12). Namely, a QSD (and only a QSD) is a left eigenvector ν for the restriction of the matrix Q to Λ with eigenvalue $-\sum_{y \in \Lambda} \nu(y) q(y,0)$: ν must satisfy the system

$$\sum_{y \in \Lambda} \nu(y) \left[q(y,x) + q(y,0)\nu(x) \right] = 0, \quad \forall x \in \Lambda. \tag{1.2}$$

(recall $q(x,x) = -\sum_{y \in \Lambda \cup \{0\} \setminus \{x\}} q(x,y)$.)
The *Yaglom limit* for the measure μ is defined by

$$\lim_{t \to \infty} \varphi_t^\mu(y), \quad y \in \Lambda \tag{1.3}$$

if the limit exists and it is a probability on Λ.

When Λ is finite, Darroch and Seneta (1967) prove that there exists a unique QSD ν for Q and that the Yaglom limit equals ν for any initial distribution μ. When Λ is infinite the situation is more complex. Neither existence nor uniqueness of QSD are guaranteed. An example is the asymmetric random walk, $\Lambda = \mathbb{N}$, $p = q(x, x+1) = 1 - q(x, x-1)$, for $x \geq 1$. In this case there are infinitely many QSD when $p < 1/2$ and none when $p \geq 1/2$ (see Cavender (2) and Ferrari, Martinez and Picco (6) for birth and death more general examples). For $\Lambda = \mathbb{N}$ under the condition $\lim_{x \to \infty} \mathbb{P}(R < t | Z_0 = x) = 0$, where R is the absorption time of Z_t, Ferrari, Kesten, Martínez and Picco (5) prove that the existence of QSD is equivalent to the existence of a positive exponential moment for R, i.e. $\mathbb{E}e^{\theta R} < \infty$ for some $\theta > 0$. When the Yaglom limit exists, it is known to be a QSD, but existence of the limit is not known in general for infinite state space. Phil Pollett maintains an updated bibliography on QSD in the site http://www.maths.uq.edu.au/˜pkp/papers/qsds/qsds.html.

Define the *ergodicity coefficient* of the chain Q by

$$\alpha = \alpha(Q) := \sum_{z \in \Lambda} \inf_{x \in \Lambda \setminus \{z\}} q(x,z). \tag{1.4}$$

If $\alpha(z) := \inf_{x \neq z} q(x,z) > 0$, then z is called Doeblin state. Define the *maximal absorbing rate* of Q by

$$C = C(Q) := \sup_{x \in \Lambda} q(x,0)\,. \tag{1.5}$$

Since the chain is absorbed with probability one, $C > 0$. On the other hand, $C \leq \bar{q}$, the maximal rate.

Jacka and Roberts (9) proved that if there exists a Doeblin state $z \in \Lambda$ such that $\alpha(z) > C$ and *if* there exists a QSD ν for Q, then ν is the unique QSD for Q and the Yaglom limit equals ν for any initial measure μ; their proof also works under the weaker assumption $\alpha > C$. We show that $\alpha > C$ is a *sufficient* condition for the existence of a QSD for Q.

Theorem 1.1. *If $\alpha > C$ then there exists a unique* QSD *ν for Q and the Yaglom limit converges to ν for any initial measure μ.*

The condition $\alpha > C$ is complementary to the condition $\lim_{x\to\infty} \mathbb{P}(R > t | Z_0 = x) = 1$, under which (5) show existence of QSD. On the other hand, $\alpha > 0$ implies that R has a positive exponential moment.

The Fleming-Viot process (FV). Let N be a positive integer and consider a system of N particles evolving on Λ. The particles move independently, each of them governed by the transition rates Q until absorption. Since there cannot be two simultaneous jumps, at most one particle is absorbed at any given time. When a particle is absorbed to 0, it goes instantaneously to a state in Λ chosen with the empirical distribution of the particles remaining in Λ. In other words, it chooses one of the other particles uniformly and jumps to its position. Between absorption times the particles move independently governed by Q. This process has been introduced by Fleming and Viot (7) and studied by Burdzy, Holyst and March (1), Grigorescu and Kang (8) and Löbus (11) in a Brownian motion setting. The original process introduced by Fleming Viot is a model for a population with constant number of individuals which also encodes the positions of particles. When individuals die randomly independently of their position, the scaling limit is a fractal ("measure valued diffusion"). In the case studied in (1) and here the particles die only on some region of the state space (the boundary of a domain in $\mathbb{R}^d$ in (1) and the absorbing state here); in both cases the scaling limit is a deterministic measure. We agree with Burdzy that the two models are sufficiently similar to be called Fleming-Viot.

The generator of the FV process acts on functions $f : \Lambda^{(1,\dots,N)} \to \mathbb{R}$ as follows

$$\mathcal{L}^N f(\xi) = \sum_{i=1}^{N} \sum_{y \in \Lambda \setminus \{\xi(i)\}} \Big[q(\xi(i),y) + q(\xi(i),0)\,\frac{\eta(\xi,y)}{N-1}\Big](f(\xi^{i,y}) - f(\xi)), \tag{1.6}$$

where $\xi^{i,y}(j) = y$ for $j = i$ and $\xi^{i,y}(j) = \xi(j)$ otherwise and

$$\eta(\xi,y) := \sum_{i=1}^{N} \mathbf{1}\{\xi(i) = y\}. \tag{1.7}$$

We call ξ_t the process in $\Lambda^{(1,\dots,N)}$ with generator (1.6) and $\eta_t = \eta(\xi_t,\cdot)$ the corresponding unlabeled process on $\{0,1,\dots\}^\Lambda$; $\eta_t(x)$ counts the number of ξ particles in state x at time t. For μ a measure on Λ, we denote $\xi_t^{N,\mu}$ the process starting with independent identically μ-distributed random variables $(\xi_0^{N,\mu}(i),\ i=1,\dots,N)$; the corresponding variables $\eta_0^{N,\mu}(x)$ have multinomial law with parameters N and $(\mu(x),\ x\in\Lambda)$. The profile of the FV process at time t converges as $N\to\infty$ to the conditioned evolution of the chain Z_t:

Theorem 1.2. *Let μ be a probability measure on Λ. Assume that $(\xi_0^{N,\mu}(i),\ i=1,\dots,N)$ are i.i.d. with law μ. Then, for $t>0$ and $x\in\Lambda$,*

$$\lim_{N\to\infty}\mathbb{E}\Big(\frac{\eta_t^{N,\mu}(x)}{N}-\varphi_t^\mu(x)\Big)^2=0\,. \tag{1.8}$$

Since the functions are bounded (by 1), this is equivalent to convergence in probability. The convergence in probability has been proven for Brownian motions in a compact domain in (1). Extensions of this result and the process induced in the boundary have been studied in (8) and (11).

When Λ is finite, the FV process is an irreducible pure-jump Markov process on a finite state space. Hence it is ergodic (that is, there exists a unique stationary measure for the process and starting from any measure, the process converges to the stationary measure). When Λ is infinite, general conditions for ergodicity are still not established. We prove the following result

Theorem 1.3. *If $\alpha>0$, then for each N the* FV *process with N particles is ergodic.*

Assume $\alpha>0$. Let η^N be a random configuration distributed with the unique invariant measure for the FV process with N particles. Our next result says that the empirical profile of the invariant measure for the FV process converges in L_2 to the unique QSD for Q.

Theorem 1.4. *Assume $\alpha>C$. Then there exists a probability measure ν on Λ such that for all $x\in\Lambda$,*

$$\lim_{N\to\infty}\mathbb{E}\Big(\frac{\eta^N(x)}{N}-\nu(x)\Big)^2=0\,. \tag{1.9}$$

Furthermore ν is the unique QSD *for Q.*

Sketch of proofs The existence part of Theorem 1.1 is a corollary of Theorem 1.4. The rest is a consequence of Jacka and Roberts' theorem (stated later as Theorem 5.1).

Theorem 1.3 is proven by constructing a stationary version of the process "from the past" as in perfect simulation. We do it in Section 2.

Theorems 1.2 and 1.4 are both based on the asymptotic independence of the ξ particles, as $N\to\infty$. Lemma 5.1 later shows that φ_t is the unique solution of the Kolmogorov forward equations

$$\frac{d}{dt}\varphi_t^\mu(x)=\sum_{y\in\Lambda}\varphi_t^\mu(y)[q(y,x)+q(y,0)\varphi_t^\mu(x)],\qquad x\in\Lambda\,. \tag{1.10}$$

From a generator computation, taking $f(\xi)=\eta(\xi,x)$ in (1.6),

$$\frac{d}{dt}\mathbb{E}\Big(\frac{\eta_t^{N,\mu}(x)}{N}\Big)=\frac{\mathbb{E}\mathcal{L}^N\eta^{N,\mu}}{N}=\sum_{y\in\Lambda}\mathbb{E}\Big(\frac{\eta_t^{N,\mu}(y)}{N}\Big(q(y,x)+q(y,0)\frac{\eta_t^{N,\mu}(x)}{N-1}\Big)\Big)\,. \tag{1.11}$$

If solutions of (1.11) converge along subsequences as $N \to \infty$, then the limits equal the unique solution of (1.10). In fact, we prove in Proposition 3.1 that for $x, y \in \Lambda$,

$$\mathbb{E}\Big(\eta_t^{N,\mu}(y)\,\eta_t^{N,\mu}(x) - \mathbb{E}\eta_t^{N,\mu}(y)\,\mathbb{E}\eta_t^{N,\mu}(x)\Big) = O(N)\,. \tag{1.12}$$

This argument shows the convergence of the means $\mathbb{E}\eta_t^{N,\mu}(x)/N$ to $\varphi_t^\mu(x)$. Since the variances ((1.12), with $x = y$) divided by N^2 go to zero, the L_2 convergence follows.

The stationary case is proven analogously. If η^N is distributed with the invariant measure for the FV process, from (1.11),

$$0 = \sum_{y\in\Lambda} \mathbb{E}\Big(\frac{\eta^N(y)}{N}\Big(q(y,x) + q(y,0)\frac{\eta^N(x)}{N-1}\Big)\Big)\,. \tag{1.13}$$

Under the hypothesis $\alpha > C$ we show a result for η^N analogous to (1.12) to conclude that solutions of (1.13) converge to the unique solution of (1.2).

To show that the limits are probability measures it is necessary to show that the families of measures $(\frac{1}{N}\mathbb{E}\eta_t^{N,\mu},\ N \in \mathbb{N})$ and $(\frac{1}{N}\mathbb{E}\eta^N,\ N \in \mathbb{N})$ are tight; we do it in Section 4.

Comments One interesting point of the Fleming-Viot approach is that it permits to show the existence of a QSD in the $\alpha > C$ case, a new result as far as we know.

Compared with the results for Brownian motion in a bounded region with absorbing boundary (Burdzy, Holyst and March (1), Grigorescu and Kang (8) and Löbus (11) and other related works), we do not have trouble with the existence of the FV process, it is immediate here. On the other hand those works prove the convergence in probability without computing the correlations. We prove that the fact that the correlations vanish asymptotically is sufficient to show convergence in probability. For the moment we are able to show that the correlations vanish for the stationary state under the hypothesis $\alpha > C$.

The conditioned distribution φ_t^μ is not necessarily the same as $\frac{1}{N}\mathbb{E}\eta_t^{N,\mu}$, the expected proportion of particles in the FV process with N particles. This has been proven in Example 2.1 of (1) for $\Lambda = \{1,2\}$ and $q(1,0) = q(1,2) = q(2,1) = 1$. The QSD ν for this chain (the unique solution of (1.2)) is $\nu(1) = (3-\sqrt{5})/2$ and $\nu(2) = (-1+\sqrt{5})/2$. The unlabeled FV process with two particles η_t^2 assumes values in $\{(1,1),\ (2,0),\ (0,2)\}$ and evolves with rates $a((0,2),(1,1)) = a((1,1),(0,2)) = a((2,0),(1,1)) = 2$ and $a((1,1),(2,0)) = 1$. The invariant measure for η_t^2 gives weight $2/5$ to $(1,1)$ and $(0,2)$ and weight $1/5$ to $(2,0)$. This implies that in equilibrium the mean proportion of particles in states 1 and 2 are $\rho^2(1) = 2/5$ and $\rho^2(2) = 3/5$ respectively. Our values for ν and ρ^2 do not agree with those of (1), but the conclusion is the same: $\nu \neq \rho^2$, which in turn implies $\frac{1}{2}\mathbb{E}\eta_t^{2,\nu} \neq \varphi_t^\nu = \nu$ for sufficiently large t, as $\frac{1}{2}\mathbb{E}\eta_t^{2,\nu}$ converges to ρ^2 as t grows. More generally, for rational rates q, the equilibrium mean proportions ρ^N have rational components, as they come from the solution of a linear system with rational coefficients, while those of ν may be irrational, as ν is the solution of a nonlinear system.

To prove tightness we have classified the ξ particles in types. This already appears in Burdzy, Holyst and March (1) to show the convergence result. Our application here is somehow simpler. Curiously our tightness proof needs the same condition ($\alpha > C$) as the vanishing correlations proof.

2 Construction of FV process

In this section we perform the graphic construction of the FV process ξ_t^N. Recall $C < \infty$ and $\alpha \geq 0$. Recall $\alpha(z) = \inf_{x\in\Lambda\setminus\{z\}} q(x,z)$.

For each $i = 1,\dots,N$, we define independent stationary marked Poisson processes (PP's) on $\mathbb{R}$:

- Regeneration times. PP rate α: $(a_n^i)_{n\in\mathbb{Z}}$, with marks $(A_n^i)_{n\in\mathbb{Z}}$
- Internal times. PP rate $\bar{q}-\alpha$: $(b_n^i)_{n\in\mathbb{Z}}$, with marks $((B_n^i(x),\, x\in\Lambda),\, n\in\mathbb{Z})$
- Voter times. PP rate C: $(c_n^i)_{n\in\mathbb{Z}}$, with marks $((C_n^i,\, (F_n^i(x),\, x\in\Lambda)),\, n\in\mathbb{Z})$.

The marks are independent of the PP's and mutually independent. The denominations will be transparent later. The marginal laws of the marks are:

- $\mathbb{P}(A_n^i = y) = \alpha(y)/\alpha$, $y\in\Lambda$;
- $\mathbb{P}(B_n^i(x) = y) = \dfrac{q(x,y)-\alpha(y)}{\bar{q}-\alpha}$, $x\in\Lambda$, $y\in\Lambda\setminus\{x\}$;
  $\mathbb{P}(B_n^i(x) = x) = 1 - \sum_{y\in\Lambda\setminus\{x\}} \mathbb{P}(B_n^i(x) = y)$.
- $\mathbb{P}(F_n^i(x) = 1) = \dfrac{q(x,0)}{C} = 1 - \mathbb{P}(F_n^i(x) = 0)$, $x\in\Lambda$.
- $\mathbb{P}(C_n^i = j) = \dfrac{1}{N-1}$, $j\neq i$.

Denote $(\Omega,\mathcal{F},\mathbb{P})$ the space on which the marked Poisson processes have been constructed. Discard the null event corresponding to two simultaneous events at any given time.

We construct the process in an arbitrary time interval $[s,t]$. Given the mark configuration $\omega\in\Omega$ we construct $\xi_{[s,t]}^{N,\xi} (= \xi_{[s,t],\omega}^{N,\xi})$ in the time interval $[s,t]$ as a function of the Poisson times and its respective marks and the initial configuration ξ at time s.

The relation of this notation with the one in Theorem 1.2 is the following:

$$\xi_t^{N,\mu} = \xi_{[s,s+t]}^{N,\xi} \tag{2.14}$$

where $\xi = (\xi(1),\dots,\xi(N))$ is a random vector with iid coordinates, each distributed according to μ on Λ. That is, for any function $f:\Lambda^N\to\mathbb{R}$,

$$\mathbb{E}f(\xi_t^{N,\mu}) = \sum_{\xi} \left[\textstyle\prod_i \mu(\xi(i))\right] \mathbb{E}f(\xi_{[s,s+t]}^{N,\xi}). \tag{2.15}$$

Construction of $\xi_{[s,t]}^{N,\xi} = \xi_{[s,t],\omega}^{N,\xi}$

Since for each particle i there are three Poisson processes with rates C, α and $\bar{q}-\alpha$, the number of events in the interval $[s,t]$ is Poisson with mean $N(C+\bar{q})$. So the events can be ordered from the earliest to the latest.

At time s the initial configuration is ξ. Then, proceed event by event following the order as follows:

The configuration does not change between Poisson events.

At each regeneration time a^i_n particle i jumps to state A^i_n regardless of the current configuration.

If at the internal time b^i_n- the state of particle i is x, then at time b^i_n particle i jumps to state $B^i_n(x)$ regardless of the position of the other particles.

If at the voter time c^i_n- the state of particle i is x and $F^i_n(x) = 1$, then at time c^i_n particle i jumps to the state of particle C^i_n; if $F^i_n(x) = 0$, then particle i does not jump.

The configuration obtained after using all events is $\xi^{N,\xi}_{[s,t]}$. The denominations are now clear. At regeneration times a particle jumps to a new state independently of the current configuration. At voter times a particle either jumps to the state of another particle chosen at random or does not jump. At internal times the particle jumps are indifferent to the position of the other particles.

Lemma 2.1. *For each $s \in \mathbb{R}$, the process $(\xi^{N,\xi}_{[s,t]}, t \geq s)$ is Markov with generator (1.6) and initial condition $\xi^{N,\xi}_{[s,s]} = \xi$.*

Proof This follows from the Markov properties of the Poisson processes; the rate for particle i to jump from x to y is the sum of three terms: (a) $\alpha\frac{\alpha(y)}{\alpha}$ (the rate of a regeneration event times the probability that the corresponding mark takes the value y), (b) $(\bar{q} - \alpha)\frac{q(x,y)-\alpha(y)}{\bar{q}-\alpha}$ (the maximal rate of internal events times the probability that the corresponding mark takes the value y) and (c) $C\frac{q(x,0)}{C}\sum_{j\neq i}\mathbf{1}\{\xi(j) = y\}\frac{1}{N-1}$ (the maximal absorption rate times the probability the absorption rate from state x divided by the maximal absorption rate times the empirical probability of state y for the particles different from i). The sum of these three rates is the rate indicated by the generator (the square brackets in (1.6) with $\xi(i) = x$). □

Generalized duality For each realization ω of the marked Poisson processes and each interval $[s, t]$ we construct a set $\Psi^i_\omega[s, t] \subset \{1, \dots, N\}$ corresponding to the particles involved at time s in the definition of $\xi^{N,\xi}_{[s,t],\omega}(i)$. We drop the label ω in the notation.

Initially $\Psi^i[t, t] = \{i\}$ and look backwards in time for the most recent i-Poisson event, at some time τ, in the past of t but more recent than s. If τ is a regeneration event, then we do not need to go further in the past to know the state of the i particle, so we erase the i particle from $\Psi^i[\tau-, t]$. If τ is the voter event c^i_n, its C^i_n mark pointing to particle j, say, then we need to know the state of the particle i at time $\tau-$ to see which F^i_n will be used to decide if the i particle effectively takes the value of particle j or not. Hence, we need to follow backwards particles i and j and we add the j particle to $\Psi^i[\tau-, t]$. Then continue this procedure starting from each of the particles in $\Psi^i[\tau-, t]$. The process backwards finishes if $\Psi^i[r, t]$ is empty for some r smaller than s or if we have processed all marks involving i in the time interval $[s, t]$. More rigorously:

Construction of $\Psi^i[s, t]$

We construct $\Psi^i[s, t]$ backwards in time. Changes occur at Poisson events and $\Psi^i[s, t]$ is constant between two Poisson events. The construction of $\Psi^i[s, t]$ depends only on the regeneration and voter events. It ignores the internal events.

Initially $\Psi^i[t, t] = \{i\}$.

Assume $\Psi^i[r', t]$ has been constructed for all $r' \in [\tau', t]$. Let τ be the time of the latest Poisson event before τ'.

Set $\Psi^i[r'', t] = \Psi^i[\tau', t]$ for all $r'' \in (\tau, \tau']$.

If $\tau < s$ stop, we have constructed $\Psi^i[r, t]$ for all $r \in [s, t]$. If not, proceed as follows.

If τ is a regeneration event involving particle j (that is, $\tau = a_n^j$ for some n), then set $\Psi^i[\tau, t] = \Psi^i[\tau', t] \setminus \{j\}$.

If τ is a voter event whose mark points to particle j (that is, $\tau = c_n^\ell$ for some ℓ and n and $C_n^\ell = j$), then set $\Psi^i[\tau, t] = \Psi^i[\tau', t] \cup \{j\}$.

This ends the iterative step of the construction.

For a generic Poisson marked event m let time(m) be the time it occurs and label(m) its label; for instance time$(c_n^i) = c_n^i$, label$(c_n^i) = i$. For a realization ω of the Poisson marks let $\omega^i[s, t]$ be the marks involved in the definition of $\Psi^i[s, t]$, given by

$$\omega^i[s,t] = \big\{m \in \omega \,:\, (\text{label}(m), \text{time}(m)+) \in \{(\Psi^i_\omega[r,t], r),\ r \in [s,t]\}\big\}\,, \tag{2.16}$$

the set of marked events in ω involved in the value of $\xi^{N,\xi}_{[s,t],\omega}(i)$ and

$$\xi^i[s,t] = (\xi(j),\ j \in \Psi^i_\omega[s,t]), \tag{2.17}$$

the initial particles involved in the value of $\xi^{N,\xi}_{[s,t],\omega}(i)$.

The generalized duality equation is

$$\xi^{N,\xi}_{[s,t],\omega}(i) = H(\omega^i[s,t], \xi^i[s,t]). \tag{2.18}$$

There is no explicit formula for H but the important point is that for any real time s, $\xi^{N,\xi}_{[s,t]}(i)$ depends only on a *finite* number of Poisson events contained in $\omega^i[s, t]$ and on the initial state $\xi(j)$ of the particles $j \in \Psi^i_\omega[s, t]$. The internal marks involved in the definition of ξ depend on the initial configuration ξ and the evolution of the process but in any case are bounded by a Poisson random variable with mean $\bar{q}|\Psi^i[s, t]|$.

Proof of Theorem 1.3 If the number of marks in $\omega^i[-\infty, t]$ is finite with probability one, then the process

$$\xi^N_{t,\omega}(i) = \lim_{s\to-\infty} H(\omega^i[s,t], \xi^i[s,t]), \quad i \in \{1, \dots, N\}, \;\; t \in \mathbb{R} \tag{2.19}$$

is well defined with probability one and does not depend on ξ. By construction $(\xi^N_t,\ t \in \mathbb{R})$ is a stationary Markov process with generator (1.6). Since the law at time t does not depend on the initial configuration ξ, the process admits a unique invariant measure, the law of ξ^N_t. See (4) for more details about this argument.

The number of points in $\omega^i[-\infty, t]$ is finite if and only if for some finite $s < t$, $\Psi^i[s, t] = \emptyset$. But since there are $3N$ stationary finite-intensity Poisson processes, with probability one, for almost all ω there is an interval $[s(\omega), s(\omega) + 1]$ in the past of t such that there is at least one regeneration mark for all particle k and there are no voter marks in that interval. We have used here that the regeneration rate $\alpha > 0$. This guarantees that $\Psi^i[s(\omega), t] = \emptyset$. To conclude notice that if $\Psi^i[s, t] = \emptyset$, then $\Psi^i[s', t] = \emptyset$ for $s' < s$. □

3 Particle correlations in the FV process

In this section we show that the particle-particle correlations in the FV process with N particles is of the order $1/N$.

Proposition 3.1. *Let $x, y \in \Lambda$. For all $t > 0$*

$$\Big|\mathbb{E}\Big(\frac{\eta_t^{N,\mu}(x)\eta_t^{N,\mu}(y)}{N^2}\Big) - \mathbb{E}\Big(\frac{\eta_t^{N,\mu}(x)}{N}\Big)\,\mathbb{E}\Big(\frac{\eta_t^{N,\mu}(y)}{N}\Big)\Big| \;<\; \frac{1}{N}\,e^{2Ct}\,. \tag{3.20}$$

Assume $\alpha > C$. Let η^N be distributed according to the unique invariant measure for the FV *process with N particles. Then*

$$\Big|\mathbb{E}\Big(\frac{\eta^{N}(x)\eta^{N}(y)}{N^2}\Big) - \mathbb{E}\Big(\frac{\eta^{N}(x)}{N}\Big)\,\mathbb{E}\Big(\frac{\eta^{N}(y)}{N}\Big)\Big| \;<\; \frac{1}{N}\,\frac{\alpha}{\alpha - C}\,. \tag{3.21}$$

We introduce a 4-fold coupling $(\Psi^i[s,t], \Psi^j[s,t], \hat\Psi^i[s,t], \hat\Psi^j[s,t])$ with $\Psi^i[s,t] = \hat\Psi^i[s,t]$ with the property "$\hat\Psi^j[s,t] \cap \Psi^i[s,t] = \emptyset$ implies $\Psi^j[s,t] = \hat\Psi^j[s,t]$" and such that the marginal process $(\hat\Psi^i[s,t], \hat\Psi^j[s,t])$ have the same law as two independent processes with the same marginals as $(\Psi^i[s,t], \Psi^j[s,t])$. The construction is analogous to the one in Fernández, Ferrari and Garcia (4).

We use two independent families of marked Poisson processes each with the same law as the Poisson family used in the graphic construction; the marked events are called *red* and *green*. We augment the probability space and continue using $\mathbb{P}$ and $\mathbb{E}$ for the probability and the expectation with respect to the product space generated by the red and green events. With these marked events we construct simultaneously the processes $(\Psi^i[s,t], \Psi^j[s,t], \hat\Psi^i[s,t], \hat\Psi^j[s,t])$ and a new process $\mathrm{I}[s,t]$ as follows.

Initially set $\mathrm{I}[t,t] = 0$, $\hat\Psi^i[t,t] = \Psi^i[t,t] = i$ and $\hat\Psi^j[t,t] = \Psi^j[t,t] = j$

Go backwards in time as in the construction of Ψ^i in Section 2 proceeding event by event as follows. Assume $\mathrm{I}[r',t]$, $\hat\Psi^i[r',t]$, $\Psi^i[r',t]$, $\hat\Psi^j[r',t]$ and $\Psi^j[r',t]$ have been constructed for all $r' \in [\tau',t]$. Let τ be the time of the latest Poisson event before τ'.

If $\mathrm{I}[\tau',t] = 1$ then: (a) if the event is green, use it to update $\hat\Psi^i[\tau,t]$, $\Psi^i[\tau,t]$ and $\Psi^j[\tau,t]$ only; (b) if the event is red, use it only to update $\hat\Psi^j[\tau,t]$.

If $\mathrm{I}[\tau',t] = 0$ then:

(a) if the event is green, then use it to update $\hat\Psi^i[\tau,t]$, $\Psi^i[\tau,t]$ and $\Psi^j[\tau,t]$. Use it also to update $\hat\Psi^j[\tau,t]$ only if (after the updating) $\hat\Psi^j[\tau,t] \cap \hat\Psi^i[\tau,t] = \emptyset$. Otherwise do not update $\hat\Psi^i[\tau,t]$ and set $\mathrm{I}[\tau,t] = 1$.

(b) if the event is red do not use it to update $\hat\Psi^i[\tau,t]$, $\Psi^i[\tau,t]$ and $\Psi^j[\tau,t]$. Use it to update $\hat\Psi^j[\tau,t]$ only if after the updating $\hat\Psi^j[\tau,t] \cap \hat\Psi^i[\tau,t] \neq \emptyset$; in this case set $\mathrm{I}[\tau,t] = 1$. Otherwise do not update $\hat\Psi^i[\tau,t]$ and keep $\mathrm{I}[\tau,t] = 0$.

The processes so constructed satisfy

1. $\mathrm{I}[s,t]$ indicates if the hated processes intersect:

$$\mathrm{I}[s,t] = \mathbf{1}\{\hat\Psi^j[s,t] \cap \hat\Psi^i[s,t] \neq \emptyset\}. \tag{3.22}$$

2. $\Psi^i[s,t]$ and $\Psi^j[s,t]$ are constructed using only the *green* events.

3. $\hat\Psi^i[s,t]$ is also constructed using the *green* events, hence it coincides with $\Psi^i[s,t]$.

4. $\hat\Psi^j[s,t]$ is constructed with a combination of the red and green events in such a way that it coincides with $\Psi^j[s,t]$ as long as possible, it is independent of $\hat\Psi^i[s,t]$ and has the same marginal distribution of $\Psi^j[s,t]$.

We use the coupling processes to estimate the covariances of $\xi^{N,\mu}_{[s,t]}$. Call $\omega^j[s,t]$, $\omega^i[s,t]$, $\hat\omega^j[s,t]$ and $\hat\omega^i[s,t]$ the set of marked events defined with (2.16) using $\Psi^j[s,t]$, $\Psi^i[s,t]$, $\hat\Psi^j[s,t]$ and $\hat\Psi^i[s,t]$ respectively. Take two independent random vectors X and Y with the same distribution as in (2.14), that is, i.i.d. coordinates with law μ. Denote the initial particles defined as in (2.17) by $X^j[s,t]$, $X^i[s,t]$, $\hat X^j[s,t]$ and $\hat Y^i[s,t]$ as function of $\Psi^j[s,t]$, $\Psi^i[s,t]$, $\hat\Psi^j[s,t]$ and $\hat\Psi^i[s,t]$ respectively. Denote ω^i instead of $\omega^i[s,t]$, X^i instead of $X^i[s,t]$, etc.; we have

$$\begin{aligned}
&\mathbb{P}(\xi^{N,\mu}_{[s,t]}(j)=x,\xi^{N,\mu}_{[s,t]}(i)=y)-\mathbb{P}(\xi^{N,\mu}_{[s,t]}(j)=x)\mathbb{P}(\xi^{N,\mu}_{[s,t]}(i)=y)\\
&\quad=\mathbb{P}(\xi^{N,X}_{[s,t]}(j)=x,\xi^{N,X}_{[s,t]}(i)=y)-\mathbb{P}(\xi^{N,X}_{[s,t]}(j)=x)\mathbb{P}(\xi^{N,Y}_{[s,t]}(i)=y) \qquad (3.23)\\
&\quad=\mathbb{E}\Big(\mathbf{1}\{H(\omega^j,X^j)=x,\ H(\omega^i,X^i)=y)\}-\mathbf{1}\{H(\hat\omega^j,\hat X^j)=x),\ H(\hat\omega^i,\hat Y^i)=y)\}\Big)\,.
\end{aligned}$$

If $\mathrm{I}[s,t]=0$ then $\Psi^j[s',t]=\hat\Psi^j[s',t]$ and $\Psi^i[s',t]=\hat\Psi^i[s',t]$ for all $s'\in[s,t]$ and the same holds for the corresponding ω's. Also, given $\mathrm{I}[s,t]=0$, X^j and Y^i depend on disjoint sets of initial particles. This implies that we can couple X^i and Y^i in such a way that in the event $\mathrm{I}[s,t]=0$, $X^i=Y^i$. Hence, taking absolute values in (3.23) we get

$$|\mathbb{P}(\xi^{N,\mu}_{[s,t]}(j)=x,\xi^{N,\mu}_{[s,t]}(i)=y)-\mathbb{P}(\xi^{N,\mu}_{[s,t]}(j)=x)\mathbb{P}(\xi^{N,\mu}_{[s,t]}(i)=y)|\ \le\ \mathbb{P}(\mathrm{I}[s,t]=1). \qquad (3.24)$$

Lemma 3.1. *For $t\ge 0$ and different particles $i,j\in\{1,\dots,N\}$*

$$\mathbb{P}(\mathrm{I}[s,t]=1)\ \le\ \frac{1}{N-1}\,\frac{C}{\alpha-C}\,(1-e^{2(C-\alpha)(t-s)})\,. \qquad (3.25)$$

Proof: At time s the process $\mathrm{I}[s,t]$ jumps from 0 to 1 at a rate depending on $\hat\Psi^i[s,t]$ and $\hat\Psi^j[s,t]$ which is bounded above by

$$\frac{2C}{N-1}\hat\Psi^i[s,t]\hat\Psi^j[s,t]\,\mathbf{1}\{\mathrm{I}[s,t]=0\}.$$

Dominating the indicator function by one:

$$\mathbb{P}(\mathrm{I}[s,t]=0\,|\,\mathcal{F}_{[s,t]})\ \ge\ \exp\Big\{-\frac{2C}{N-1}\int_s^t\hat\Psi^i[s',t]\hat\Psi^j[s',t]ds'\Big\} \qquad (3.26)$$

where $\mathcal{F}_{[s,t]}$ is the sigma field generated by $((\hat\Psi^i[s',t],\ \hat\Psi^j[s',t]),\ s<s'<t)$. From (3.26), using $1-e^{-a}\le a$ and taking expectations,

$$\mathbb{P}(\mathrm{I}[s,t]=1)\ \le\ \frac{2C}{N-1}\int_s^t\mathbb{E}\hat\Psi^i[s',t]\,\mathbb{E}\hat\Psi^j[s',t]ds'\,. \qquad (3.27)$$

On the other hand, $\hat{\Psi}^i[s', t]$ is dominated by the position at time $t - s$ of a random walk that grows by one with rate C and decreases by one with rate α. Hence its expectation is bounded above by $e^{(t-s')(C-\alpha)}$. Substituting this bound in (3.27),

$$\mathbb{P}(\mathrm{I}[s,t] = 1) \le \frac{2C}{N-1} \int_s^t e^{2(C-\alpha)(t-s')} ds' \tag{3.28}$$

which gives (3.25). □

Proof of Proposition 3.1 Defining

$$\eta^{N,\mu}_{[s,t]}(x) = \sum_{i=1}^{N} \mathbf{1}\{\xi^{N,\mu}_{[s,t]} = x\}$$

Then $\eta^{N,\mu}_{[s,t]}$ has the same law as $\eta^{N,\mu}_{t-s}$ and η^N has the same law as $\eta^{N,\mu}_{[-\infty,t]}$. Hence

$$\begin{aligned}
\mathbb{E}\Big(\frac{\eta^{N,\mu}_{[s,t]}(x)\, \eta^{N,\mu}_{[s,t]}(y)}{N^2}\Big) &= \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{P}(\xi^{N,\mu}_{[s,t]}(i) = x, \xi^{N,\mu}_{[s,t]}(j) = y) \\
\frac{\mathbb{E}\eta^{N,\mu}_{[s,t]}(x)\, \mathbb{E}\eta^{N,\mu}_{[s,t]}(y)}{N^2} &= \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{P}(\xi^{N,\mu}_{[s,t]}(i) = x)\mathbb{P}(\xi^{N,\mu}_{[s,t]}(j) = y)\,.
\end{aligned}$$

Using this, (3.24) and (3.25) with $s = 0$ and $\alpha = 0$ we get (3.20).
If $\alpha > C$, $\eta^{N,\eta}_{[s,t]}$ converges as $s \to -\infty$ to η^N_t a configuration distributed with the unique invariant measure, as in Theorem 1.3, see (2.19) for the corresponding statement for ξ^N_t. Hence the left hand side of (3.21) is bounded above by $\mathbb{P}(\mathrm{I}[-\infty, t] = 1)$. Taking $s = -\infty$ in (3.25) we get (3.21). □

4 Tightness

In this section we prove tightness for the mean densities as probability measures in Λ, indexed by N.

Proposition 4.1. *For all $t > 0$, $x \in \Lambda$, $i = 1, \ldots, N$ and probability μ on Λ it holds*

$$\frac{\mathbb{E}\eta^{N,\mu}_t(x)}{N} \le e^{Ct} \sum_{z \in \Lambda} \mu(z) P_t(z, x). \tag{4.1}$$

As a consequence the family of measures $(\mathbb{E}\eta^{N,\mu}_t/N,\ N \in \mathbb{N})$ is tight.

Assume $\alpha > 0$ and define the probability measure μ_α on Λ by

$$\mu_\alpha(x) = \frac{\alpha(x)}{\alpha}, \quad x \in \Lambda,$$

(recall $\alpha(x) = \inf_{z\in\Lambda\setminus\{x\}} q(z,x)$). Let $\tilde{Q}$ be the matrix on $\Lambda \cup \{0\}$ with entries $\tilde{q}(x,y) = q(x,y) - \alpha(y)$ for $x \neq y$ and $\tilde{q}(x,x) = -\sum_{y\neq x} \tilde{q}(x,y)$. Let $\tilde{P}_t$ be the corresponding semigroup and $\tilde{Z}_t$ the corresponding process.

For $z, x \in \Lambda$ define

$$R_\lambda(z,x) = \int_0^\infty \lambda e^{-\lambda t} \tilde{P}_t(z,x)dt. \tag{4.2}$$

The matrix R_λ represents the semigroup $\tilde{P}_t$ evaluated at a random time T_λ exponentially distributed with rate λ independent of $(\tilde{Z}_t)$. $R_\lambda(z,x)$ is the probability the process $(\tilde{Z}_t)$ with initial state z be in x at time T_λ. The matrix R is substochastic: $\sum_{x\in\Lambda} R_\lambda(z,x)$ is just the probability of non absorption of $(\tilde{Z}_t)$ with initial state z at the random time T_λ.

Proposition 4.2. *Assume $\alpha > C$ and let $\rho^N(x)$ be the mean proportion of particles in state x under the unique invariant measure for the* FV *process with N particles. Then for $x \in \Lambda$,*

$$\rho^N(x) \;\leq\; \frac{\alpha}{\alpha - C}\, \mu_\alpha R_{(\alpha-C)}(x)\,. \tag{4.3}$$

As a consequence, the family of measures $(\rho^N,\, N \in \mathbb{N})$ is tight.

Types To prove the propositions we introduce the concept of types. We say that particle i is *type* 0 at time t if it has not been absorbed in the time interval $[0,t]$. Particles may change type only at absorption times. If at absorption time s particle i jumps over particle j which has type k, then at time s particle i changes its type to $k+1$. Hence, at time t a particle has type k if at its last absorbing time it jumped over a particle of type $k-1$. We write

$$\text{type}(i,t) := \text{type of particle } i \text{ at time } t.$$

The marginal law of $\xi_t^{N,\mu}(i)\mathbf{1}\{\text{type}(i,t)=0\}$ is the law of the process Z_t^μ:

$$\mathbb{P}(\xi_t^{N,\mu}(i) = x, \text{type}(i,t) = 0) = \sum_{z\in\Lambda} \mu(z) P_t(z,x). \tag{4.4}$$

Proof of Proposition 4.1 Since $\frac{\mathbb{E}\eta_t^{N,\mu}(x)}{N} = \mathbb{P}(\xi_t^{N,\mu}(i) = x)$, it suffices to show that for $k \geq 0$

$$\mathbb{P}(\xi_t^{N,\mu}(i) = x, \text{type}(i,t) = k) \leq \frac{(Ct)^k}{k!} \sum_{z\in\Lambda} \mu(z) P_t(z,x)\,. \tag{4.5}$$

We proceed by induction. By (4.4) the statement is true for $k = 0$. Assume (4.5) holds for some $k \geq 0$. We prove it holds for $k+1$. Time is partitioned according to the last absorption time s of the ith particle. The absorption occurs at rate bounded above by C. The particle jumps at time s to a particle j with probability $1/(N-1)$, this particle has type k and state y. Then it must go from y to x in the time interval $[s,t]$ without being absorbed. Using the Markov property, we get:

$$\mathbb{P}(\xi_t^{N,\mu}(i) = x, \text{type}(i,t) = k+1) \tag{4.6}$$

$$\leq \int_0^t C \frac{1}{N-1} \sum_{j\neq i} \sum_{y\in\Lambda} \mathbb{P}(\xi_s^{N,\mu}(j) = y, \text{type}(j,s) = k)\, P_{t-s}(y,x)\, ds. \tag{4.7}$$

The symmetry of the particles allows to cancel the sum over j with $(N-1)^{-1}$. The recursive hypothesis (4.5) implies that (4.7) equals

$$= \int_0^t C\,\frac{(Cs)^k}{k!}\sum_{z\in\Lambda}\mu(z)\sum_{y\in\Lambda}P_s(z,y)P_{t-s}(y,x)ds \;=\; \frac{(Ct)^{k+1}}{(k+1)!}\sum_{z\in\Lambda}\mu(z)P_t(z,x) \qquad (4.8)$$

by Chapman-Kolmogorov. This completes the induction step. □

Proof of Proposition 4.2 If ξ^N is distributed according to the unique invariant measure for the FV process then $\rho^N(x) = \mathbb{P}(\xi^N(i) = x)$. Since $\alpha > 0$ we can construct a version of the stationary process ξ^N_s such that $\mathbb{P}(\xi^N(i) = x) = \mathbb{P}(\xi^N_s(i) = x)$, $\forall s$. We analyze the marginal law of the particle distribution for each type, as in the proof of Proposition 4.1. Define the types as before, but when a particle meets a regeneration mark, then the particle type is reset to 0. In the construction, at that time the state of the particle is chosen with law μ_α.

Under the hypothesis $\alpha > C$ the process

$$((\xi^N_t(i), \mathrm{type}(i,t)), i = 1,\dots,N),\ t\in\mathbb{R})$$

is Markovian and can be constructed in a stationary way as ξ^N_t. Hence

$$A_k(x) := \mathbb{P}(\xi^N_s(i) = x, \mathrm{type}(i,s) = k) \qquad (4.9)$$

does not depend on s.

The regeneration marks follow a Poisson process of rate α and the last regeneration mark of particle i before time s happened at time $s - T^i_\alpha$, where T^i_α is exponential of rate α. Then,

$$A_0(x) \;=\; \int_0^\infty \alpha e^{-\alpha t}\sum_{z\in\Lambda}\mu_\alpha(z)\tilde P_t(z,x)dt \;=\; \mu_\alpha R_\alpha(x). \qquad (4.10)$$

Here $\tilde P_t(z,x)$ is interpreted as the probability that the chain goes from z to x *given* that there was no regeneration marks in the time interval $(s-t, s]$.

A reasoning similar to (4.6)-(4.7) implies

$$A_k(x) \;\le\; \int_0^\infty e^{-\alpha t} C\sum_{z\in\Lambda}A_{k-1}(z)\tilde P_s(z,x)\,dt \qquad (4.11)$$

$$= \frac{C}{\alpha}A_{k-1}R_\alpha(x) \;\le\; \Big(\frac{C}{\alpha}\Big)^k\mu_\alpha R_\alpha^{k+1}(x). \qquad (4.12)$$

We interpret $R^k_\lambda(z,x)$ as the expectation of $\tilde P_{\tau_k}(z,x)$, where τ_k is a sum of k independent random variables with exponential distribution of rate λ. Summing (4.11), and multiplying and dividing by $\alpha(\alpha - C)$,

$$\mathbb{P}(\xi^N_s(i) = x) \;\le\; \frac{\alpha}{\alpha - C}\sum_{k=0}^\infty\Big(\frac{C}{\alpha}\Big)^k\Big(1-\frac{C}{\alpha}\Big)\mu_\alpha R_\alpha^{k+1}(x)\,. \qquad (4.13)$$

The sum can be interpreted as the expectation of $\mu_\alpha R^K_\alpha$, where K is a geometric random variable with parameter $p = 1 - (C/\alpha)$. Since an independent geometric(p) number of independent exponentials(α) is exponential(αp), we get

$$\mathbb{P}(\xi^N_s(i) = x) \;\le\; \frac{\alpha}{\alpha - C}\,\mu_\alpha R_{\alpha - C}(x). \qquad \square \qquad (4.14)$$

5 Proofs of theorems

In this section we prove Theorems 1.3 and 1.4. We start deriving the forward equations for φ_t^μ and show they have a unique solution.

Lemma 5.1. *The Kolmogorov forward equations for φ_t^μ are given by*

$$\frac{d}{dt}\varphi_t^\mu(x) = \sum_{y\in\Lambda} \varphi_t^\mu(y)[q(y,x) + q(y,0)\varphi_t^\mu(x)]\,. \tag{5.1}$$

These equations have a unique solution in the set of probability measures on Λ.

Proof: The Kolmogorov forward equations for P_t are:

$$\frac{d}{dt}P_t(z,x) = \sum_{y\in\Lambda} P_t(z,y)\; q(y,x), \qquad z\in\Lambda, \quad x \in \Lambda\cup\{0\}\,. \tag{5.2}$$

Write $\gamma_t = \sum_{z\in\Lambda}\mu(z)P_t(z,0)$ and differentiate (1.1) to get

$$\begin{aligned}
\frac{d}{dt}\varphi_t^\mu(x) &= \frac{\sum_{z\in\Lambda}\mu(z)\frac{d}{dt}P_t(z,x)}{1-\gamma_t} + \frac{(\frac{d}{dt}\gamma_t)}{1-\gamma_t}\cdot\frac{\sum_{z\in\Lambda}\mu(z)P_t(z,x)}{1-\gamma_t} \\
&= \frac{\sum_{z\in\Lambda}\mu(z)\sum_{y\in\Lambda}P_t(z,y)\;q(y,x)}{1-\gamma_t} \\
&\qquad + \frac{\sum_{z\in\Lambda}\mu(z)\sum_{y\in\Lambda}P_t(z,y)\;q(y,0)}{1-\gamma_t}\cdot\frac{\sum_{z\in\Lambda}\mu(z)P_t(z,x)}{1-\gamma_t}
\end{aligned} \tag{5.3}$$

which equals (5.1).

To show uniqueness let φ_t and ψ_t be two solutions of (1.10) and $\epsilon_t(x) = |\varphi_t(x)-\psi_t(x)|$. Then ϵ_t satisfies the inequation

$$\frac{d}{dt}\epsilon_t(y) \;\leq\; \sum_{z\in\Lambda}\epsilon_t(z)q(z,y) + \sum_{z\in\Lambda}|\varphi_t(z)\varphi_t(y) - \psi_t(z)\psi_t(y)|\,q(z,0). \tag{5.4}$$

Bound the modulus with $\varphi_t(z)\epsilon_t(y) + \epsilon_t(z)\psi_t(y)$, sum (5.4) in y, call $E_t = \sum_{y\in\Lambda}\epsilon_t(y)$ and use $\sum_{y\in\Lambda\setminus\{z\}} q(z,y) \leq \bar q$ and $q(z,0)\leq C$ to get

$$\frac{d}{dt}E_t \leq (\bar q + 2C)E_t\,. \tag{5.5}$$

This implies $E_t \leq E_0\, e^{(\bar q+2C)t}$. Since $E_t\geq 0$ and $E_0 = 0$, $E_t = 0$ for all $t\geq 0$. $\square$

Proof of Theorem 1.2 We first show convergence of the means

$$\lim_{N\to\infty}\mathbb{E}\Big(\frac{\eta_t^{N,\mu}(x)}{N}\Big) = \varphi_t^\mu(x). \tag{5.6}$$

Sum and subtract $\sum_{y\in\Lambda} q(y,0)\mathbb{E}(\frac{\eta_t^N(y)}{N})\mathbb{E}(\frac{\eta_t^N(x)}{N})$ to (1.11) to get

$$\mathbb{E}\mathcal{L}^N\Big(\frac{\eta_t^{N,\mu}(x)}{N}\Big) = \sum_{y\in\Lambda}\frac{\mathbb{E}\eta_t^{N,\mu}(y)}{N}\Big(q(y,x)+q(y,0)\frac{\mathbb{E}\eta_t^{N,\mu}(x)}{N}\Big) \tag{5.7}$$

$$+\sum_{y\in\Lambda} q(y,0)\Big[\mathbb{E}\Big(\frac{\eta_t^{N,\mu}(y)}{N}\frac{\eta_t^{N,\mu}(x)}{N-1}\Big)-\frac{\mathbb{E}\eta_t^{N,\mu}(y)}{N}\frac{\mathbb{E}\eta_t^{N,\mu}(x)}{N}\Big]. \tag{5.8}$$

By Proposition 4.1, the family $(\mathbb{E}\eta_t^{N,\mu}(x)/N,\ N\in\mathbb{N})$ is tight. Use $\eta_t^{N,\mu}(x)/N\le 1$, $q(y,x)\le\bar q$, $q(y,0)\le C$ and (4.1) to bound the summands in (5.7) by $(\bar q + C)e^{Ct}\mu P_t(y)$, which is summable in y. Since $\eta_t^{N,\mu}(x)/N\in[0,1]$, the absolute value of the square brackets in (5.8) is bounded by $\mathbb{E}\eta_t^{N,\mu}(y)/(N-1)$ which in turn is bounded by $e^{Ct}\mu P_t(y)N/(N-1)$ by (4.1). Hence, for $N\ge 2$, the summands in (5.8) are bounded by $2Ce^{Ct}\mu P_t(y)$, which is summable in y. By dominated convergence we can take limits in N inside the sums. Proposition 3.1 implies (5.8) converges to zero as N goes to infinity for any subsequence. Take a subsequence of $\eta_t^{N,\mu}/N$ converging to some limit called ρ_t^μ. Along *this* subsequence, by the above considerations,

$$\lim_N \frac{\mathbb{E}\mathcal{L}^N\eta_t^{N,\mu}(x)}{N} = \sum_{y\in\Lambda}\rho_t^\mu(y)[q(y,x)+q(y,0)\rho_t^\mu(x)]\,. \tag{5.9}$$

(The right hand side of (5.9) is bounded by $\bar q + C$.) By (1.11),

$$\frac{\mathbb{E}\eta_t^{N,\mu}(x)}{N} = \frac{\mathbb{E}\eta_0^{N,\mu}(x)}{N} + \int_0^t \frac{\mathbb{E}\mathcal{L}^N\eta_s^{N,\mu}(x)}{N}ds. \tag{5.10}$$

From (5.9) we conclude that any limit ρ_t^μ must satisfy

$$\rho_t^\mu(x) = \mu(x) + \int_0^t \sum_{y\in\Lambda}\rho_s^\mu(y)[q(y,x)+q(y,0)\rho_s^\mu(x)]dt \tag{5.11}$$

which implies ρ_t^μ must satisfy (1.10), the forward equations for φ_t^μ. Since there is a unique solution for this equation, the limit exists and it is φ_t^μ.

Taking $y = x$ in (3.20), the variances asymptotically vanish:

$$\lim_{N\to\infty}\frac{\mathbb{E}[\eta_t^{N,\mu}(x)]^2-[\mathbb{E}\eta_t^{N,\mu}(x)]^2}{N^2} = 0. \tag{5.12}$$

This concludes the proof. □

Uniqueness and the Yaglom limit convergence of Theorem 1.4 is a consequence of the next theorem.

Theorem 5.1 (Jacka & Roberts)**.** *If $\alpha > C$ and there exists a* QSD *ν for Q, then ν is the unique* QSD *for Q and the Yaglom limit (1.3) converges to ν for any initial distribution μ.*

Jacka and Roberts (9) use the stronger hypothesis $\inf_{y\in\Lambda} q(y,x) > C$ for some $x\in\Lambda$ but the proof works under the hypothesis $\alpha > C$. We include their proof for completeness.

Proof: Assume ν is a QSD for Q. Construct a *Markov process* $\hat{Z}_t$ on Λ with rates

$$\hat{q}(x,y) = q(x,y) + q(x,0)\nu(y), \qquad \text{for } x \neq y\,. \tag{5.13}$$

In words, the chain $\hat{Z}_t$ moves with rates q until it jumps to 0; at this moment it is reset in Λ with distribution ν. Since the balance equations for $\hat{Z}_t$ coincide with (1.2) ν is stationary for $\hat{q}$ and the corresponding transition probability function $\hat{P}_t$ satisfies

$$\hat{P}_t(x,y) = P_t(x,y) + P_t(x,0)\nu(y), \qquad \text{for } x,y \in \Lambda\,. \tag{5.14}$$

Indeed, when it jumps to 0, it attains equilibrium. Recall $\varphi_t^\mu(y) = \mu P_t(y)/(1-\mu P_t(0))$ with $\mu P_t(y) = \sum_{z\in\Lambda}\mu(z)P_t(z,y)$ for $y \in \Lambda \cup \{0\}$. From (5.14),

$$\varphi_t^\mu(y) - \nu(y) \;=\; \frac{\mu\hat{P}_t(y) - \nu(y)}{1-\mu P_t(0)}, \qquad \text{for } y \in \Lambda\,. \tag{5.15}$$

The condition $\alpha > 0$ implies that $\hat{Z}_t$ is ergodic and converges exponentially fast at rate α in total variation to its unique stationary state ν starting from any μ:

$$\sum_{y\in\Lambda} |\mu\hat{P}_t(y) - \nu(y)| \le 2e^{-\alpha t}\,. \tag{5.16}$$

Since $1 - P_t(x,0) \ge e^{-Ct}$ for $x \in \Lambda$, (5.15) and (5.16) imply

$$\sum_{y\in\Lambda} |\varphi_t^\mu(y) - \nu(y)| \le 2e^{(C-\alpha)t}\,. \tag{5.17}$$

This implies uniqueness of ν and convergence of the Yaglom limit to ν. □

Proof of Theorem 1.4 Since $\alpha > 0$, the FV process governed by Q is ergodic by Theorem 1.3. Call η^N a random configuration chosen with the unique invariant measure. Since $\mathbb{E}\mathcal{L}^N\eta^N(x) = 0$, summing and subtracting $\sum_{y\in\Lambda} q(y,0)\frac{\mathbb{E}\eta^N(x)}{N}\frac{\mathbb{E}\eta^N(y)}{N}$ to (1.13) we get

$$\begin{aligned} 0 = \sum_{y\in\Lambda} \frac{\mathbb{E}\eta^N(y)}{N}\Big(q(y,x) + q(y,0)\frac{\mathbb{E}\eta^N(x)}{N}\Big) \\ + \sum_{y\in\Lambda} q(y,0)\Big[\mathbb{E}\Big(\frac{\eta^N(y)}{N}\frac{\eta^N(x)}{N-1}\Big) - \mathbb{E}\Big(\frac{\eta^n(y)}{N}\Big)\mathbb{E}\Big(\frac{\eta^N(x)}{N}\Big)\Big] \end{aligned} \tag{5.18}$$

which holds for any N and $x \in \Lambda$. By Proposition 4.2, $(\rho^N,\, N \in \mathbb{N})$ is tight and by (4.3) dominated uniformly in N by a summable sequence. Since $\frac{\eta^N(x)}{N} \in [0,1]$, the square bracket in (5.18) is bounded by $\rho^N(y)N/(N-1)$. Hence we can interchange limit with integral in (5.18) and use (3.21) to show that the second term in (5.18) vanishes as N goes to infinity. Then any limit ρ along a subsequence must satisfy the QSD equation (1.2). Since by Theorem 5.1 the solution is unique, the limit $\lim_N \frac{\mathbb{E}\eta^N}{N}$ exists and equals the unique QSD ν. The variances vanish by (3.21). □

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