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Infinitely divisible random probability distributions with an application to a random motion in a random environment

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Abstract

The infinite divisibility of probability distributions on the space $\mathcal{P}(\mathbb{R})$ of probability distributions on $\mathbb{R}$ is defined and related fundamental results such as the Lévy-Khintchin formula, representation of Itô type of infinitely divisible RPD, stable RPD and Lévy processes on $\mathcal{P}(\mathbb{R})$ are obtained. As an application we investigate limiting behaviors of a simple model of a particle motion in a random environment

Key words: random probability distribution, infinitely divisibility, Lévy- Khintchin representation, Lévy-Itô representation, random environment

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1 Introduction

Let w be a random parameter defined on a probability space (W, P), which corresponds to an environment. Suppose that for each w we have a sequence of families of random variables $\{X_{nk}^w(\omega),\ 1 \le k \le n\}, n = 1, 2, \cdots$ defined on a probability space (Ω, P^w). We are interested in the limiting behavior of $\sum_{k=1}^n X_{nk}^w(\omega)$ under suitable centering. The usual problem is to find a limit distribution of $\sum_{k=1}^n X_{nk}^w(\omega)$, after suitable centering, under the joint probability law $\overline{P}(dwd\omega) = P(dw)P_0^w(d\omega)$.

On the other hand, for each w let Ξ_n^w denote the distribution of $\sum_{k=1}^n X_{nk}^w(\omega)$ under the probability law P^w. Then Ξ_n^w is a *random variable with values in the space* $\mathcal{P}(\mathbb{R})$ *of probability distributions on* $\mathbb{R}$ (shortly, random probability distributions and abbreviated as RPD). The problem is now to investigate the law convergence of Ξ_n^w as $n \to \infty$, after suitable shift of location, with respect to P. Obtaining this latter result requires a finer structure of the model and more delicate analysis than the former. If $X_{nk}^w(\omega)$, $n \ge 1$, are independent for each fixed w and n, and if we denote by q_{nk}^w, $1 \le k \le n$ the probability distribution of $X_{nk}^w(\omega)$ under P^w, then $\Xi_n^w = q_{n1}^w * \cdots * q_{nn}^w$. Suppose that q_{nk}^w, $1 \le k \le n$ are also independent. In this case, if the law convergence of Ξ_n^w to some Ξ^w holds, after suitable shift of location, then the limit Ξ^w in law is expected to be an infinitely divisible RPD in the sense described later provided that a certain infinitesimal condition for $\{q_{nk}^w\}$, $1 \le k \le n$ is satisfied. Such an example appears in the study of limiting behavior of a simple model of a particle motion in a random environment, which was considered probably as the simplest model that exhibits many of limiting behaviors similar to those of Solomon (7), Kesten-Kozlov-Spitzer (5), Tanaka (9) and Hu-Shi-Yor (1). The analysis of this simple model was one of the motivation of the present investigation,

We thus formulate several notions concerning the infinite divisibility of RPDs and then find fundamental results related to them. Next we investigate limiting behavior of the simple model of a particle motion in a random environment in the framework of our general theory.

A $\mathcal{P}(\mathbb{R})$-valued random variable Ξ is called infinitely divisible if for any $n \ge 2$ there exist i.i.d. RPDs $\Xi_1, \cdots, \Xi_n$ such that

$$\Xi \overset{(d)}{=} \Xi_1 * \cdots * \Xi_n, \tag{1.1}$$

where $\overset{(d)}{=}$ means equivalence in distribution.

Let $\mathcal{P}(\mathcal{P}(\mathbb{R}))$ denote the space of probability distributions on $\mathcal{P}(\mathbb{R})$. For Q and R in $\mathcal{P}(\mathcal{P}(\mathbb{R}))$ we define $Q \circledast R \in \mathcal{P}(\mathcal{P}(\mathbb{R}))$ by

$$Q \circledast R(E) = \int_{\mathcal{P}(\mathbb{R})} \int_{\mathcal{P}(\mathbb{R})} Q(d\mu)R(d\nu)I_E(\mu * \nu) \quad (E \in \mathcal{B}(\mathcal{P}(\mathbb{R})).$$

An element Q of $\mathcal{P}(\mathcal{P}(\mathbb{R}))$ is called *infinitely divisible* if for any $n \ge 2$ there exists a $Q_n \in \mathcal{P}(\mathcal{P}(\mathbb{R}))$ such that

$$Q = Q_n^{\circledast n} = Q_n \circledast \cdots \circledast Q_n. \tag{1.2}$$

For $Q \in \mathcal{P}(\mathcal{P}(\mathbb{R}))$ we define a moment measure $M_Q \in \mathcal{P}(\mathbb{R}^{\mathbb{N}})$ by

$$M_Q = \int_{\mathcal{P}(\mathbb{R})} \mu^{\otimes \mathbb{N}} Q(d\mu), \tag{1.3}$$

where $\mathbb{N}$ is the set of positive integers and

$$\mathbb{R}^{\mathbb{N}} = \mathbb{R} \times \mathbb{R} \times \cdots, \qquad \mu^{\otimes \mathbb{N}} = \mu \otimes \mu \otimes \cdots.$$

Then M_Q is an exchangeable probability measure on $\mathbb{R}^{\mathbb{N}}$ (shortly, $M_Q \in \mathcal{P}^{ex}(\mathbb{R}^{\mathbb{N}})$).

Conversely any $M \in \mathcal{P}^{ex}(\mathbb{R}^{\mathbb{N}})$ can be represented as

$$M = \int_{\mathcal{P}(\mathbb{R})} \mu^{\otimes \mathbb{N}} Q(d\mu)$$

with a unique $Q \in \mathcal{P}(\mathcal{P}(\mathbb{R}))$. Thus there is a $1-1$ correspondence between $\mathcal{P}(\mathcal{P}(\mathbb{R}))$ and $\mathcal{P}^{ex}(\mathbb{R}^{\mathbb{N}})$. Clearly $M_{Q \circledast R} = M_Q * M_R$, and Q is infinitely divisible if and only if M_Q is infinitely divisible (in the sense that any finite dimensional projection of M_Q has the property). The characteristic function Φ_Q of M_Q is defined by

$$\Phi_Q(\boldsymbol{z}) = \int_{\mathcal{P}(\mathbb{R})} \prod_{j \in \mathbb{N}} \widehat{\mu}(z_j) Q(d\mu), \tag{1.4}$$

where $\widehat{\mu}$ is the characteristic function of μ and $\boldsymbol{z} = \{z_j\} \in \mathbb{R}^{\mathbb{N}}$ with $z_j = 0$ except for finitely many $j \in \mathbb{N}$.

One of the main theorems is to represent Φ_Q as the 'Lévy-Khintchin formula'; naturally it resembles the 'classical Lévy-Khintchin formula'. There are three parts, a deterministic part, a Gaussian part and a Poisson-type part; those are connected by convolution.

Another main interest is to obtain a representation of Lévy-Itô type for an infinitely divisible RPDs. To do this we need to define two types of integrals by means of a Poisson random measure Π on $\mathcal{P}_*(\mathbb{R}) = \mathcal{P}(\mathbb{R}) \setminus \{\delta_0\}$. The first one is defined simply by $\mu_1 * \mu_2 * \cdots$ for a realization $\sum_j \delta_{\mu_j}$ of Π, which is convergent almost surely under a certain condition for Π (see (3.1)), and denoted by $\int \mu * \Pi(d\mu)$. The second one called a renormalized Poisson integral and denoted by $\int \mu * \Pi^{reno}(d\mu)$, is defined for a wider class of Π than the first one (see the condition (3.2)). In a case where $\int \mu * \Pi(d\mu)$ is defined, the renormalized Poisson integral is defined as $\left\{ \int \mu * \Pi(d\mu) \right\} * \delta_{-\alpha}$ where $\alpha = \int_{\mathcal{P}_*(\mathbb{R})} \int_{[|x| \le 1]} x \mu(dx) m(d\mu)$, m being the intensity measure of Π.

For the definition in a general case we have to take a limit procedure starting from those already defined. This definition is a highlight in our arguments. It corresponds to a compensated Poisson integrals in an ordinary case and also corresponds exactly to the third part of the Lévy-Khintchin formula. By virtue of renormalized Poisson integrals and the Lévy-Khinchin formula our tasks go somewhat in a smooth way. In fact, the representation of Lévy-Itô type of an infinitely divisible RPD is obtained. It consists of the deterministic part, the Gaussian part and the renormalized Poisson integral part, all being connected by convolution.

The definition of a stable RPD and its general structure are given. Examples of (not necessarily stable) RPDs are also given, some of which are useful in the section 7.

We now go to Lévy processes on $\mathcal{P}(\mathbb{R})$. First we define a $\mathcal{P}(\mathbb{R})$-valued Lévy process $(\Xi(t))_{t \ge 0}$ in law and then give its representation of Lévy-Itô type. But at this moment the representation is realized on some probability space which, in general, differs from the one on which $(\Xi(t))_{t \ge 0}$ is defined. The represented process is continuous in probability but lacks the right continuity with left limits (cadlag) of the paths. So we have to make a cadlag modification of the process represented as Lévy-Itô type. We can do this by partly imitating [(6): Lemma 20.2, Lemma 20.4]. Thus we arrive at what is appropriate to be called a Lévy process on $\mathcal{P}(\mathbb{R})$. It is a $\mathcal{P}(\mathbb{R})$-valued cadlag process with a sort of 'stationary independent increments' in the sense described in the section 6.

Finally we apply the above general method to the study of limiting behavior, as $x \to \infty$, of the hitting time $\sigma^w(x)$ to $x > 0$ for a particle performing a simple movement in a random environment. Let $w = \{w_n\}_{n\geq 0}$ be an increasing sequence with $w_0 = 0$ and set $N(x,w) = \max\{n \geq 0 : w_n \leq x\}$. We consider a Markov process $(X(t), P_x^w)$ on $\mathbb{R}_+$ with generator $A^w f(x) = f'(x) + f(w_{N(x,w)}) - f(x)$. A visual description of the movement of the particle will be explained in the section 7. Now suppose that $w = \{w_n\}_{n\geq 0}$ is a random sequence with the property that $w_n' = w_n - w_{n-1}, n \geq 1$ are i.i.d. random variables with $P(w_n' > t) = e^{-\lambda t}, t \geq 0$, for some parameter $\lambda > 0$. Then w is called a random environment under which the particle moves randomly. Thus randomeness is two-fold, one is for the random environment and the other is for the random movement of the particle under a frozen environment. T! he limiting behavior of $\sigma^w(x)$ as $x \to \infty$ is usually studied under the probability law $\overline{P}(dwd\omega) = P(dw)P_0^w(d\omega)$ (see (7) and (5) for more complicated models). When $\lambda > 2$ it can be proved that for almost all frozen environment w the distribution of $\{x^{-1/2}(\sigma^w(x) - E^w(\sigma^w(x)))\}$ under P^w converges to a normal distribution $\mathcal{N}(0, c_\lambda)$ as $x \to \infty$ with some constant $c_\lambda > 0$ in Theorem 7.1 (iv) of the section 7 (see [(9), p.374] for a similar model). But this type of result does not hold for $0 < \lambda < 2$. So we pose the following problem:

Denote by $\Xi^{\sigma^w(x)}$ the probability distribution of $\sigma^w(x)$ under the probability law P_0^w and regard it as an RPD. The problem is to find, depending on λ, various scaling limits *in law* of $\Xi^{\sigma^w(x)}$ as $x \to \infty$. In the special case $\lambda > 2$ in the above $\Xi = \mathcal{N}(0, c_\lambda)$ a.s., i.e., non-random. In general case, scaling limits in law are stable RPD taking values of infinitely divisible distributions on $\mathbb{R}$. In the method and stating the results we mainly use Poisson integrals and renormalized Poisson integrals. Owing to the simplicity of our model some explicit computations are possible and so we can obtain finer limiting results in the case of hitting times. The detailed results depend on λ, and will be stated in several cases.

2 Infinitely divisible RPDs

We begin by stating some fundamental notions concerning infinitely divisible RPD, some of which may overlope with those already given in the introduction.

Let $\mathcal{P}(\mathbb{R})$ be the set of all probability distributions on $\mathbb{R}$, which is equipped with the topology of weak convergence. We note that the topology is compatible with the following metric

$$d(\mu_1, \mu_2) = \sum_{m=1}^{\infty} 2^{-m} \sup_{|z|\leq m} |\widehat{\mu}_1(z) - \widehat{\mu}_2(z)| \quad (\mu_1,\, \mu_2 \in \mathcal{P}(\mathbb{R})), \tag{2.1}$$

which makes $(\mathcal{P}(\mathbb{R}), d)$ a complete separable metric space. On the other hand $\mathcal{P}(\mathbb{R})$ is a commutative semigroup with convolution $*$, i.e. for $\mu,\, \nu \in \mathcal{P}(\mathbb{R})$

$$\mu * \nu(A) = \int_{\mathbb{R}2} \mu(dx)\nu(dy) I_A(x+y) = \int_{\mathbb{R}} \mu(A-x)\nu(dx) \quad (A \in \mathcal{B}(\mathbb{R})).$$

We define a shift operator θ_b $(b \in \mathbb{R})$ and a scaling operator τ_c $(c > 0)$ on $\mathcal{P}(\mathbb{R})$ by

$$\theta_b\mu(A) = \mu(A + b) \quad (A \in \mathcal{B}(\mathbb{R})),$$

and

$$\tau_c\mu(A) = \mu(c^{-1}A) \quad (A \in \mathcal{B}(\mathbb{R})).$$

Let Ξ be a random probability distribution (shortly, RPD), namely, Ξ is a $\mathcal{P}(\mathbb{R})$-valued random variable. Ξ is called *infinitely divisible* if for any $n \geq 2$ there exist i.i.d. RPDs $\Xi_1, \cdots, \Xi_n$ such that (1.1) holds.

The operations $*$, θ_b $(b \in \mathbb{R})$ and τ_c $(c > 0)$ are extended to operators $\circledast$, Θ_b and T_b on $\mathcal{P}(\mathcal{P}(\mathbb{R}))$ by

$$Q \circledast R(E) = \int_{\mathcal{P}(\mathbb{R})} \int_{\mathcal{P}(\mathbb{R})} Q(d\mu)R(d\nu)I_E(\mu * \nu) \quad (E \in \mathcal{B}(\mathcal{P}(\mathbb{R})),$$

$$\Theta_b Q(E) = Q(\theta_b E) \quad (E \in \mathcal{B}(\mathcal{P}(\mathbb{R})),$$

and

$$T_c Q(E) = Q(\tau_c E) \quad (E \in \mathcal{B}(\mathcal{P}(\mathbb{R})).$$

Let $Q \in \mathcal{P}(\mathcal{P}(\mathbb{R}))$. We say that Q is *infinitely divisible* if for any $n \geq 2$ there exists a $Q_n \in \mathcal{P}(\mathcal{P}(\mathbb{R}))$ such that (1.2) holds.

Let Ξ be an RPD with the distribution $Q \in \mathcal{P}(\mathcal{P}(\mathbb{R}))$. It is clear that Ξ is infinitely divisible if and only if Q is infinitely divisible.

Let denote by $\ell_0(\mathbb{N})$ the totality of $\boldsymbol{z} = \{z_j\} \in \mathbb{R}^{\mathbb{N}}$ with $z_j = 0$ except for finitely many $j \in \mathbb{N}$. We use the following notations:

$$\langle \boldsymbol{z}, \boldsymbol{x} \rangle = \sum_{j \in \mathbb{N}} z_j x_j \quad (\boldsymbol{x} \in \mathbb{R}^{\mathbb{N}}, \ \boldsymbol{z} \in \ell_0(\mathbb{N})),$$

and for $\mu \in \mathcal{P}(\mathbb{R})$ and bounded measureble functions $F(x)$ and $G(\boldsymbol{x})$ defined on $\mathbb{R}$ and $\mathbb{R}^{\mathbb{N}}$ respectively, we use the same notation

$$\langle \mu, F \rangle = \int_{\mathbb{R}} \mu(dx)F(x), \quad \langle \mu^{\otimes \mathbb{N}}, G \rangle = \int_{\mathbb{R}^{\otimes \mathbb{N}}} \mu^{\otimes \mathbb{N}}(d\boldsymbol{x})G(\boldsymbol{x}),$$

which will make no confusion with $\langle \boldsymbol{z}, \boldsymbol{x} \rangle$.

For $Q \in \mathcal{P}(\mathcal{P}(\mathbb{R}))$ we define the moment measure $M_Q \in \mathcal{P}(\mathbb{R}^{\mathbb{N}})$ by (1.3), and the characteristic function $\Phi_Q(\boldsymbol{z})$ of M_Q by (1.4) for $\boldsymbol{z} \in \ell_0(\mathbb{N})$.

When $\Phi_Q(\boldsymbol{z})$ can be expressed as $\exp \Psi(\boldsymbol{z})$ with a unique function $\Psi(\boldsymbol{z})$ with $\Psi(\mathbf{0}) = 0$ and such that $\Psi(\boldsymbol{z})$ is continuous in $\boldsymbol{z}$ of length n for each $n = 1, 2, \cdots$, we call $\Psi(\boldsymbol{z})$ *the moment characteristics* of Q or of an RPD Ξ with distribution Q and write $\Psi(\boldsymbol{z}) = \log \Phi_Q(\boldsymbol{z})$. For example if Q is infinitely divisible, its moment characteristics $\log \Phi_Q(\boldsymbol{z})$ exists.

Example(Gaussian distributions).

To know what distribution $Q \in \mathcal{P}(\mathcal{P}(\mathbb{R}))$ is to be called a Gaussian distribution, we start from the corresponding moment measure M_Q. We call M_Q is Gaussian if any finite dimensional projection of M_Q is Gaussian. We call Q a Gaussian distribution if the corresponding M_Q is Gaussian. If a $\mathcal{P}(\mathbb{R})$-valued random variable Ξ has a Gaussian distribution Q, then Ξ is also called Gaussian. A general form of a Gaussian RPD Ξ will be shown to be identical in law to $\mathcal{N}(c^{1/2}\xi+b,a)$, where $\mathcal{N}(m,a)$ stands for the normal distribution with mean $m\in\mathbb{R}$ and variance $a\geq 0$, and ξ is a Gaussian random variable with mean 0 and variance 1. The corresponding moment characteristics is

$$\log\Phi_Q(\boldsymbol{z}) = -\frac{a}{2}\parallel \boldsymbol{z}\parallel^2 + ib\left(\sum_{j\in\mathbb{N}} z_j\right) - \frac{c}{2}\left(\sum_{j\in\mathbb{N}} z_j\right)^2 .$$

These will be clarified soon.

We first establish the Lévy-Khintchin representation for the moment characteristics of arbitrary infinitely divisible RPDs.

Theorem 2.1. *$Q\in\mathcal{P}(\mathcal{P}(\mathbb{R}))$ is infinitely divisible if and only if its moment characteristics of Q has the following representation: for $\boldsymbol{z}\in\ell_0(\mathbb{N})$*

$$\begin{aligned}&\log\Phi_Q(\boldsymbol{z}) \\ =\ &\sum_{j\in\mathbb{N}}\left(-\frac{\alpha}{2}z_j^2 + i\gamma z_j + \int_{\mathbb{R}\setminus\{0\}}\left(e^{iz_jx}-1-iz_jxI_{[|x|\leq 1]}\right)\rho(dx)\right) \\ &-\frac{\beta}{2}\left(\sum_{j\in\mathbb{N}} z_j\right)^2 \\ &+\int_{\mathcal{P}_*(\mathbb{R})}\langle\mu^{\otimes\mathbb{N}}, e^{i\langle z,x\rangle}-1-i\left(\sum_{j\in\mathbb{N}} z_jx_jI_{[|x_j|\leq 1]}\right)\rangle m(d\mu)\end{aligned} \tag{2.2}$$

where $\alpha\geq 0$, $\beta\geq 0$ and $\gamma\in\mathbb{R}$ are constants, ρ is a σ-finite measure on $\mathbb{R}\setminus\{\mathbf{0}\}$ satisfying

$$\int_{\mathbb{R}\setminus\{0\}}\left(x^2\wedge 1\right)\rho(dx)<\infty, \tag{2.3}$$

and m is a σ-finite measure on $\mathcal{P}_*(\mathbb{R})=\mathcal{P}(\mathbb{R})\setminus\{\delta_0\}$ satisfying

$$\int_{\mathcal{P}_*(\mathbb{R})}\langle\mu, x^2\wedge 1\rangle m(d\mu)<\infty. \tag{2.4}$$

We remark that in the Lévy-Khintchin representation (2.2), the first term corresponds to a deterministic part, the second term to a Gaussian random part and the third term to a non-Gaussian random part, which will be seen in Theorem 3.3.

The proof of Theorem 2.1 is divided into several steps. Suppose that $Q\in\mathcal{P}(\mathcal{P}(\mathbb{R}))$ is infinitely divisible.

Step 1. For each $n\geq 1$ let

$$M_n = \int_{\mathcal{P}(\mathbb{R})}\mu^{\otimes n}Q(d\mu).$$

Since M_n is an infinitely divisible distribution on $\mathbb{R}^n$, its characteristic function

$$\Phi_n(\boldsymbol{z}) = \widehat{M_n}(\boldsymbol{z}) = \int_{\mathbb{R}^n} e^{i\langle z,x\rangle} M_n(d\boldsymbol{x}) \quad (\boldsymbol{z} \in \mathbb{R}^n)$$

has the following Lévy-Khintchin representation:

$$\begin{aligned} \log \Phi_n(\boldsymbol{z}) &= -\frac{1}{2}\langle A^{(n)}\boldsymbol{z}, \boldsymbol{z}\rangle_n + i\langle \boldsymbol{b}^{(n)}, \boldsymbol{z}\rangle_n \\ &+ \int_{\mathbb{R}^n\setminus\{0\}} \left(e^{i\langle z,x\rangle_n} - 1 - i\langle \boldsymbol{z}, \boldsymbol{x}\rangle_n I_{[|x|_n\le 1]}\right) \nu^{(n)}(d\boldsymbol{x}) \\ &= -\frac{1}{2}\langle A^{(n)}\boldsymbol{z}, \boldsymbol{z}\rangle_n + i\langle \boldsymbol{\gamma}^{(n)}, \boldsymbol{z}\rangle_n \\ &+ \int_{\mathbb{R}^n\setminus\{0\}} \left(e^{i\langle z,x\rangle_n} - 1 - i\sum_{j=1}^n z_j x_j I_{[|x_j|\le 1]}\right) \nu^{(n)}(d\boldsymbol{x}) \end{aligned} \tag{2.5}$$

where $A^{(n)}$ is an $n \times n$ symmetric nonnegative definite matrix, $\boldsymbol{b}^{(n)} \in \mathbb{R}^n$ and $\boldsymbol{\gamma}^{(n)} \in \mathbb{R}^n$ satisfy

$$\gamma_j^{(n)} = b_j^{(n)} + \int_{\mathbb{R}^n\setminus\{0\}} x_j \left(I_{[|x_j|\le 1]} - I_{[|x|_n\le 1]}\right) \nu^{(n)}(d\boldsymbol{x}) \quad (1 \le j \le n),$$

$\langle \boldsymbol{x}, \boldsymbol{z}\rangle_n = \sum_{j=1}^n x_j z_j$, $|\boldsymbol{x}|_n^2 = \sum_{j=1}^n x_j^2$ and $\nu^{(n)}$ is a Radom measure on $\mathbb{R}^n \setminus \{\boldsymbol{0}\}$ satisfying

$$\int_{\mathbb{R}^n\setminus\{0\}} \left(|\boldsymbol{x}|_n^2 \wedge 1\right) \nu^{(n)}(d\boldsymbol{x}) < \infty.$$

From the exchangeability of M_n and the uniqueness of the Lévy-Khintchin representation it follows that $A^{(n)}$, $\boldsymbol{\gamma}^{(n)}$ and $\nu^{(n)}$ have the following properties;

$$\begin{aligned} A_{jj}^{(n)} &= a_n \quad (1 \le j \le n), \quad A_{jk}^{(n)} = b_n \quad (1 \le j \ne k \le n), \\ \gamma_j^{(n)} &= \gamma_n \quad (1 \le j \le n), \\ \sigma \cdot \nu^{(n)} &= \nu^{(n)} \quad \text{for any permutation } \sigma \text{ of } \{1, \cdots, n\}. \end{aligned}$$

The consistency condition among $\{M_n\}$ implies that a_n, b_n, γ_n are constants in n, i.e.

$$a_n = a, \quad b_n = b, \quad \gamma_n = \gamma.$$

Moreover by the non-negative definiteness of $A^{(n)}$, a and b satisfy

$$\langle A^{(n)}\boldsymbol{z}, \boldsymbol{z}\rangle_n = b\left(\sum_{j=1}^n z_j\right)^2 + (a-b)|\boldsymbol{z}|_n^2 \ge 0.$$

From this it is obvious that $a - b \ge 0$. Setting $\boldsymbol{z} = (1/n, \cdots, 1/n)$ with $n \to \infty$ we see that $b \ge 0$. So, setting $\alpha = a - b$, $\beta = b$ we obtain

$$\begin{aligned} &\log \Phi_n(\boldsymbol{z}) \\ =\ &\sum_{j=1}^n \left(-\frac{\alpha}{2} z_j^2 + i\gamma z_j\right) - \frac{\beta}{2}\left(\sum_{j=1}^n z_j\right)^2 \\ +\ &\int_{\mathbb{R}^n\setminus\{0\}} \left(e^{i\langle z,x\rangle_n} - 1 - i\sum_{j=1}^n z_j x_j I_{[|x_j|\le 1]}\right) \nu^{(n)}(d\boldsymbol{x}). \end{aligned} \tag{2.6}$$

Step 2. For $n > m \geq 1$ denote by $\pi_{n,m}$ the projection from $\mathbb{R}^n$ to $\mathbb{R}^m$, i.e.

$$\pi_{n,m}\boldsymbol{x} = (x_1, \cdots, x_m) \quad \text{for} \quad \boldsymbol{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n,$$

and by $\pi_{\infty,m}$ the projection from $\mathbb{R}^{\mathbb{N}}$ to $\mathbb{R}^m$, i.e.

$$\pi_{\infty,m}\boldsymbol{x} = (x_1, \cdots, x_m) \quad \text{for} \quad \boldsymbol{x} = (x_1, \cdots, x_n, \cdots) \in \mathbb{R}^{\mathbb{N}}.$$

Then the consistency condition among $\{M_n\}$ and the uniqueness of the representation (2.5) imply that for $n > m \geq 1$

$$\nu^{(n)}(\pi_{n,m}^{-1}(B)) = \nu^{(m)}(B) \quad (B \in \mathcal{B}(\mathbb{R}^m \setminus \{\mathbf{0}\})).$$

By modifying the Kolmogorov extension theorem, one can show that there exists a unique exchangeable σ-finite measure $\nu^{(\infty)}$ on $\mathbb{R}^{\mathbb{N}} \setminus \{\mathbf{0}\}$ satisfying that

$$\nu^{(\infty)}(\pi_{\infty,m}^{-1}(B)) = \nu^{(m)}(B) \quad (B \in \mathcal{B}(\mathbb{R}^n \setminus \{\mathbf{0}\})),$$

and

$$\int_{\mathbb{R}^{\mathbb{N}}\setminus\{0\}} \left(x_j^2 \wedge 1\right) \nu^{(\infty)}(d\boldsymbol{x}) < \infty \quad (j \in \mathbb{N}). \tag{2.7}$$

Step 3.

Lemma 2.2. *Suppose that $\nu^{(\infty)}$ is an exchangeable σ-finite measure on $\mathbb{R}^{\mathbb{N}}\setminus\{\mathbf{0}\}$ satisfying (2.7). Then there exists a Radon measure ρ on $R \setminus \{\mathbf{0}\}$ satisfying (2.3) and a σ-finite measure m on $\mathcal{P}_*(\mathbb{R})$ satisfying (2.4) such that*

$$\nu^{(\infty)} = \sum_{j\in\mathbb{N}} \int_{\mathbb{R}\setminus\{0\}} \left(\delta_x^j \otimes \prod_{k\in\mathbb{N}\setminus\{j\}}^{\otimes} \delta_0^k \right) \rho(dx) + \int_{\mathcal{P}_*(\mathbb{R})} \mu^{\otimes\mathbb{N}} m(d\mu), \tag{2.8}$$

where δ_x^k stands for the point mass at x for the k-coordinate, and the integrand of the first integral of (2.8) is an infinite product of δ_x^k and $\{\delta_0^j\}_{j\in\mathbb{N}\setminus\{k\}}$.

(*Proof.*) Let

$$\xi_n(\boldsymbol{x}) = \frac{1}{n}\sum_{j=1}^{n}(|x_j| \wedge 1) \quad (n \in \mathbb{N}).$$

By (2.7), $\xi_n \in L^2(\nu^{(\infty)})$. Note that for $n \geq m \geq 1$

$$\xi_n - \xi_m = \left(\frac{1}{n} - \frac{1}{m}\right)\sum_{j=1}^{m}|x_j| \wedge 1 + \frac{1}{n}\sum_{m<j\leq n}|x_j| \wedge 1.$$

Setting

$$a = \int_{\mathbb{R}^{\mathbb{N}}} \left(|x_1|^2 \wedge 1\right) \nu^{(\infty)}(d\boldsymbol{x}), \quad b = \int_{\mathbb{R}^{\mathbb{N}}} \left(|x_1| \wedge 1\right)\left(|x_2| \wedge 1\right) \nu^{(\infty)}(d\boldsymbol{x}),$$

by the exchangeability of $\nu^{(\infty)}$ we have

$$\begin{aligned}\| \xi_n - \xi_m \|^2_{L^2(\nu^{(\infty)})} &= \left(\frac{1}{n} - \frac{1}{m}\right)^2 (ma + m(m-1)b) \\ &+ \frac{2}{n}\left(\frac{1}{n} - \frac{1}{m}\right) m(n-m)b \\ &\quad + \frac{1}{n^2}((n-m)a + (n-m)(n-m-1)b) \\ &= \left(\left(\frac{1}{n} - \frac{1}{m}\right)^2 m + \frac{n-m}{n^2}\right)(a-b) \\ &\to 0 \qquad (m, n \to \infty).\end{aligned}$$

Hence there exists an exchangeable function $\xi \in L^2(\nu^{(\infty)})$ such that

$$\lim_{n\to\infty} \| \xi_n - \xi \|_{L^2(\nu^{(\infty)})} = 0.$$

For each integer $p \geq 1$, define a finite measure ν_p on $\mathbb{R}^{\mathbb{N}} \setminus \{\mathbf{0}\}$ by

$$\nu_p(A) = \nu^{(\infty)}\left(A \cap [\frac{1}{p+1} < \xi \leq \frac{1}{p}]\right) \quad (A \in \mathcal{B}(\mathbb{R}^{\mathbb{N}} \setminus \{\mathbf{0}\}),$$

and define a σ-finite measure ν_0 on $\mathbb{R}^{\mathbb{N}} \setminus \{\mathbf{0}\}$ by

$$\nu_0(A) = \nu^{(\infty)}(A \cap [\xi = 0]) \quad (A \in \mathcal{B}(\mathbb{R}^{\mathbb{N}} \setminus \{\mathbf{0}\}).$$

Note that ν_p is an exchangeable finite measure on $\mathbb{R}^{\mathbb{N}} \setminus \{0\}$. Hence by De Finetti's theorem (cf. (Ch)) there exists a finite measure m_p on $\mathcal{P}_*(\mathbb{R})$ such that

$$\nu_p = \int_{\mathcal{P}_*(\mathbb{R})} \mu^{\otimes\mathbb{N}} m_p(d\mu) \quad (p \geq 1).$$

On the other hand, noting that ν_0 is an exchangeable σ-finite measure on $\mathbb{R}^{\mathbb{N}}$ and $\xi = 0$ ($\nu_0 - a.e.$), we have

$$\lim_{n\to\infty} \| \xi_n \|^2_{L^2(\nu_0)} = \int_{\mathbb{R}^{\mathbb{N}}} (|x_1| \wedge 1)(|x_2| \wedge 1)\nu_0(d\boldsymbol{x}),$$

and

$$\lim_{n\to\infty} \| \xi_n \|^2_{L^2(\nu_0)} = \lim_{n\to\infty} \| \xi_n - \xi \|^2_{L^2(\nu_0)} \leq \lim_{n\to\infty} \| \xi_n - \xi \|^2_{L^2(\nu^{(\infty)})} = 0.$$

Thus we obtain

$$\int_{\mathbb{R}^{\mathbb{N}}} (|x_1| \wedge 1)(|x_2| \wedge 1)\nu_0(d\boldsymbol{x}) = 0,$$

from which and the exchangeability of ν_0 it follows

$$\nu_0(x_j \neq 0) = \nu_0(x_j \neq 0,\ x_k = 0) \quad (k \neq j). \tag{2.9}$$

This means that ν_0 is concentrated on the union of the x_j-axis over all $j \in \mathbb{N}$ excluding $\mathbf{0}$. Now we set

$$\rho = \pi_{\infty,1} \cdot \nu_0,$$

then ρ satisfies (2.3). (2.9) and (2.7) imply that

$$\nu_0 = \sum_{j\in\mathbb{N}} \int_{\mathbb{R}\setminus\{0\}} \left(\delta_x^j \otimes \prod_{k\in\mathbb{N}\setminus\{j\}}^{\otimes} \delta_0^k \right) \rho(dx).$$

Finally, setting

$$m = \sum_{p=1}^{\infty} m_p,$$

we see that m satisfies (2.4), hence

$$\begin{aligned} \nu^{(\infty)} &= \nu_0 + \sum_{p=1}^{\infty} \nu_p \\ &= \sum_{j\in\mathbb{N}} \int_{\mathbb{R}\setminus\{0\}} \left(\delta_x^j \otimes \prod_{k\in\mathbb{N}\setminus\{j\}}^{\otimes} \delta_0^k \right) \rho(dx) + \int_{\mathcal{P}_*(\mathbb{R})} \mu^{\otimes\mathbb{N}} m(d\mu), \end{aligned}$$

completeing the proof of Lemma 2.2. □

Step 4. Applying Lemma 2.2 to the last term of (2.6) we have

$$\begin{aligned} &\int_{\mathbb{R}^n\setminus\{0\}} \left(e^{i\langle z,x\rangle_n} - 1 - i \sum_{j=1}^{n} z_j x_j I_{[|x_j|\le 1]} \right) \nu^{(n)}(d\boldsymbol{x}) \\ &= \sum_{j=1}^{n} \int_{\mathbb{R}\setminus\{0\}} \left(e^{iz_jx} - 1 - iz_j x I_{[|x|\le 1]} \right) \rho(dx) \\ &\quad + \int_{\mathcal{P}_*(\mathbb{R})} \left(\prod_{j=1}^{n} \widehat{\mu}(z_j) - 1 - i \left(\sum_{j=1}^{n} z_j \right) \langle \mu, x I_{[|x|\le 1]} \rangle \right) m(d\mu). \end{aligned}$$

Thus (2.2) with (2.3) and (2.4) follows from (2.6) and (2.7). □

3 Poisson integrals on $\mathcal{P}(\mathbb{R})$

In order to construct an infinitely divisible RPD we introduce a Poisson integral on $\mathcal{P}(\mathbb{R})$.

We consider a Poisson random measure. For its definition see (6), p.119. A basic formula for a Poisson random measure Π is

$$E\left(e^{-\langle \Pi, f\rangle}\right) = \exp\left(\int (e^{-f(\mu)} - 1) m(d\mu) \right)$$

for any non-negative measurable function f on the state space of Π where m is the intensity measure of Π. It determines the law of Π and characterizes a Poisson random measure in the sense that if Π is a counting random measure satisfying the above equation, then Π is a Poisson random measure. Our use of a Poisson random measure is mainly to represent various quantities in terms of Π.

Let m be a σ-finite measure on $\mathcal{P}_*(\mathbb{R})$, and let Π_m be a Poisson random measure on $\mathcal{P}_*(\mathbb{R})$ with intensity measure m, which is defined on a complete probability space $(\Omega, \mathcal{F}, P)$.

We now introduce the following two conditions;

$$\int_{\mathcal{P}_*(\mathbb{R})} \langle \mu, |x| \wedge 1 \rangle m(d\mu) < \infty, \tag{3.1}$$

and

$$\int_{\mathcal{P}_*(\mathbb{R})} \langle \mu, x^2 \wedge 1 \rangle m(d\mu) < \infty. \tag{3.2}$$

3.1 Poisson integral $\int_{\mathcal{P}_*(\mathbb{R})} \mu * \Pi_m(d\mu)$

Let us first define a stochastic integral as a $\mathcal{P}(\mathbb{R})$-valued random variable under the condition (3.1) by

$$\Xi = \int_{\mathcal{P}_*(\mathbb{R})} \mu * \Pi_m(d\mu) = *_j \mu_j \quad \text{for} \quad \Pi_m = \sum_j \delta_{\mu_j}, \tag{3.3}$$

where $*_j \mu_j = \mu_1 * \mu_2 * \cdots * \mu_n * \cdots$ stands for the convolution of $\{\mu_j\}$. In particular, if $\Pi_m = \mathbf{0}$ (the zero-meazure), we define $\Xi = \delta_0$.

Since (3.1) implies that

$$\sum_j \langle \mu_j, |x| \wedge 1 \rangle < \infty \quad \text{for} \quad \Pi_m = \sum_j \delta_{\mu_j} \quad P - a.s.,$$

and

$$\sum_j |1 - \widehat{\mu_j}(z)| \le (2 \vee |z|) \sum_j \langle \mu_j, |x| \wedge 1 \rangle < \infty,$$

$$\widehat{(*_j \mu_j)}(z) = \prod_j \widehat{\mu}_j(z)$$

is convergent uniformly in z on each compact interval. Hence $*_j \mu_j$ is convergent in the metric space $(\mathcal{P}(\mathbb{R}), d)$. Thus Ξ of (3.3) is well-defined as a $\mathcal{P}(\mathbb{R})$-valued random variable.

Lemma 3.1. *Under the condition (3.1) let*

$$\Xi = \int_{\mathcal{P}_*(\mathbb{R})} \mu * \Pi_m(d\mu).$$

Then its moment characteristics is given by

$$\log \Phi(\boldsymbol{z}) = \int_{\mathcal{P}_*(\mathbb{R})} \langle \mu^{\otimes \mathbb{N}}, e^{i\langle z, x \rangle} - 1 \rangle m(d\mu) \quad (\boldsymbol{z} \in \ell_0(\mathbb{N})). \tag{3.4}$$

(*Proof.*) Assuming that m is a finite measure, we show (3.4), since the extension to a general m satisfying (3.1) is straightforward. If m is the zero measure, (3.4) is trivial, so we assume $m(\mathcal{P}_*(\mathbb{R})) > 0$. Let $\{\xi_j\}$ be an i.i.d. sequence of $\mathcal{P}(\mathbb{R})$-valued random variables with the common distribution $m/m(\mathcal{P}_*(\mathbb{R}))$, and let N be a Poisson random variable with parameter $m(\mathcal{P}_*(\mathbb{R}))$, which is independent of $\{\xi_j\}$. Then

$$\Pi_m \stackrel{(d)}{=} \sum_{1\le j\le N} \delta_{\xi_j},$$

and

$$\int_{\mathcal{P}_*(\mathbb{R})} \mu * \Pi_m(d\mu) = *_{j=1}^N \xi_j.$$

Recall that if $N = 0$, the r.h.s. is δ_0. For $\boldsymbol{z} \in \ell_0(\mathbb{N})$,

$$\begin{aligned}
\Phi(\boldsymbol{z}) &= E\left(\prod_{k\in\mathbb{N}}\prod_{j=1}^N \widehat{\xi}_j(z_k)\right) \\
&= e^{-m(\mathcal{P}_*(\mathbb{R}))}\left(1+\sum_{p=1}^\infty \frac{m(\mathcal{P}_*(\mathbb{R}))^p}{p!}\left(\int_{\mathcal{P}_*(\mathbb{R})}\prod_{k\in\mathbb{N}}\widehat{\mu}(z_k)\frac{m(d\mu)}{m(\mathcal{P}_*(\mathbb{R}))}\right)^p\right) \\
&= \exp\int_{\mathcal{P}_*(\mathbb{R})}\left(\prod_{k\in\mathbb{N}}\widehat{\mu}(z_k)-1\right)m(d\mu),
\end{aligned}$$

which yields (3.4). □

3.2 Renormalized Poisson integral $\int_{\mathcal{P}_*(\mathbb{R})} \mu * \Pi_m^{reno}(d\mu)$

Next, we define a stochastic integral on $\mathcal{P}(\mathbb{R})$ under the condition (3.2). Let $A \in \mathcal{B}(\mathcal{P}_*(\mathbb{R}))$. If

$$\int_A \langle \mu, |x|\wedge 1\rangle m(d\mu) < \infty, \tag{3.5}$$

then $m|_A$, the restriction of m on A, satisfies (3.1), so we set

$$\int_A \mu * \Pi_m(d\mu) = \int_{\mathcal{P}_*(\mathbb{R})} \mu * \Pi_{m|_A}(d\mu),$$

and

$$\int_A \mu * \Pi_m^{reno}(d\mu) = \theta_{a_A}\cdot \int_A \mu * \Pi_m(d\mu),$$

where

$$a_A = \int_A \langle \mu, xI_{[|x|\le 1]}\rangle m(d\mu).$$

This integral also defines a $\mathcal{P}(\mathbb{R})$-valued random variable. Noting that for $\varepsilon > 0$, $A_\varepsilon = [\langle \mu, |x| \wedge 1\rangle \ge \varepsilon]$ satisfies (3.5) because of

$$\langle \mu, |x|\wedge 1\rangle \le \varepsilon^{-1}\langle \mu, |x|\wedge 1\rangle^2 \le \varepsilon^{-1}\langle \mu, x^2\wedge 1\rangle \quad \text{on} \quad A_\varepsilon,$$

we set

$$\Xi_\varepsilon = \int_{A_\varepsilon} \mu * \Pi_m^{reno}(d\mu). \tag{3.6}$$

Lemma 3.2. *Under the condition (3.2), there exists an RPD Ξ defined on $(\Omega, \mathcal{F}, P)$ such that Ξ_ε converges to an RDP Ξ in probability as $\varepsilon \to 0+$. The limit is denoted by*

$$\Xi = \int_{\mathcal{P}_*(\mathbb{R})} \mu * \Pi_m^{reno}(d\mu).$$

Then the moment characteristics of Ξ is given by

$$\log \Phi(\boldsymbol{z}) = \int_{\mathcal{P}_*(\mathbb{R})} \langle \mu^{\otimes \mathbb{N}}, e^{i\langle z,x\rangle} - 1 - \sum_{j\in\mathbb{N}} z_j x_j I_{[|x_j|\le 1]} \rangle m(d\mu). \tag{3.7}$$

(*Proof.*) Recall that $\mathcal{P}(\mathbb{R})$ is a complete separable metric space with the metric d of (2.1). For $0 < \varepsilon < \varepsilon_0$, set

$$\Xi_{\varepsilon,\varepsilon_0} = \int_{[\varepsilon \le \langle \mu, |x|\wedge 1\rangle \le \varepsilon_0]} \mu * \Pi_m^{reno}(d\mu).$$

Since, by Lemma 3.1

$$\begin{aligned} \widehat{E\Xi}_{\varepsilon,\varepsilon_0}(z) &= E\left(\widehat{\Xi}_{\varepsilon,\varepsilon_0}(z)\right) \\ &= \exp \int_{[\varepsilon \le \langle \mu, |x|\wedge 1\rangle < \varepsilon_0]} \langle \mu, e^{izx} - 1 - izx I_{[|x|\le 1]}\rangle m(d\mu) \\ &\to 1 \quad (0 < \varepsilon < \varepsilon_0 \to 0), \end{aligned}$$

it holds that

$$E\Xi_{\varepsilon,\varepsilon_0} \overset{(w)}{\Longrightarrow} \delta_0 \quad (0 < \varepsilon < \varepsilon_0 \to 0). \tag{3.8}$$

Noting that

$$\Xi_\varepsilon = \Xi_{\varepsilon_0} * \Xi_{\varepsilon,\varepsilon_0},$$

we see for any $\eta > 0$

$$\begin{aligned} & \sup_{|z|\le m} |\widehat{\Xi}_\varepsilon(z) - \widehat{\Xi}_{\varepsilon_0}(z)| \qquad (3.9) \\ \le\ & \sup_{|z|\le m} |\widehat{\Xi}_{\varepsilon,\varepsilon_0}(z) - 1| \\ =\ & \sup_{|z|\le m} \left| \int_{[|y|\le\eta]} (e^{izy} - 1)\Xi_{\varepsilon,\varepsilon_0}(dy) + \int_{[|y|>\eta]} (e^{izy} - 1)\Xi_{\varepsilon,\varepsilon_0}(dy) \right| \\ \le\ & m\eta + 2\Xi_{\varepsilon,\varepsilon_0}(\mathbb{R} \setminus [-\eta,\eta]). \end{aligned}$$

Thus, by (3.8)

$$\lim_{\varepsilon,\varepsilon_0\to 0+} E\left(\sup_{|z|\le m} |\widehat{\Xi}_\varepsilon(z) - \widehat{\Xi}_{\varepsilon_0}(z)| \right) = 0 \quad (\forall m \ge 1),$$

which yields

$$\lim_{\varepsilon,\varepsilon_0\to 0+} E\left(d(\Xi_\varepsilon, \Xi_{\varepsilon_0})\right) = 0.$$

Hence there exists an RPD Ξ defined on $(\Omega, \mathcal{F}, P)$ such that

$$\lim_{\varepsilon\to 0+} E\left(d(\Xi_\varepsilon, \Xi)\right) = 0, \tag{3.10}$$

which proves the first part of Lemma 3.2. (3.7) can be easily verified using (3.4) for Ξ_ε. □

Remark 3.1 Althought Lemma 3.2 asserts that Ξ_ε converges to the limit in probability, using the arguments of the section 6 we can show that the convergence holds P-almost surely.

Using this Poisson integral we obtain Itô-type representation for arbitrary infinitely divisible RPDs.

Theorem 3.3. *Let Ξ be an infinitely divisible RPD of which moment characteristics is given by (2.2). Then*

$$\Xi \overset{(d)}{=} \nu * \delta_{\beta^{1/2}\xi} * \left(\int_{\mathcal{P}(\mathbb{R})} \mu * \Pi_m^{reno}(d\mu) \right), \tag{3.11}$$

where ξ is a normal random variable with mean 0 and variance 1, independent of Π_m, and ν is an infinitely divisible distribution on $\mathbb{R}$ with characteristic function

$$\widehat{\nu}(z) = \exp\left(-\frac{\alpha}{2} z^2 + i\gamma z + \int_{\mathbb{R}\setminus\{0\}} \left(e^{izx} - 1 - izxI_{[|x|\le 1]}\right) \rho(dx) \right).$$

(*Proof.*) It is immediate from Lemma 3.2. □

We remark that the Itô-type representation (3.11) consists of three terms, and each term corresponds to that of the moment characteristics (2.2) in Theorem 2.1.

Let $\mathcal{P}^{id}(\mathbb{R})$ denote the totality of infinitely divisible probability distributions on $\mathbb{R}$. It should be noted that an arbitrary RPD is not a $\mathcal{P}^{id}(\mathbb{R})$-valued RPD in general. However, it is true if the measure m of (2.2) is supported in $\mathcal{P}^{id}(\mathbb{R})$.

Theorem 3.4. *Let Ξ be an infinitely dibisible RPD. Suppose that the correponding m of (2.2) is supported in $\mathcal{P}^{id}(\mathbb{R})$. Then Ξ is an infinitely divisible RPD, taking values in $\mathcal{P}^{id}(\mathbb{R})$.*

(*Proof.*) It is obvious that for any $\varepsilon > 0$, Ξ_ε of (3.6) is $\mathcal{P}^{id}(\mathbb{R})$-valued since Ξ_ε is defined by convolutions of finite number of elements of $\mathcal{P}^{id}(\mathbb{R})$ or δ_0. Moreover Ξ is a limit of such Ξ_ε P-a.s. by (3.10), which implies the conclusion of Theorem 3.4. □

Let $(U, \mathcal{B}_U, \alpha)$ be a σ-finite measure space, and let $U \ni u \mapsto \mu_u \in \mathcal{P}_*(\mathbb{R})$ be a measurable mapping. Suppose that

$$\int_U \langle \mu_u, |x| \wedge 1 \rangle \alpha(du) < \infty. \tag{3.12}$$

Then for a Poisson random measure Π_α with intensity measure α, one can define a Poisson integral $\int_U \mu_u * \alpha(du)$ on $\mathcal{P}(\mathbb{R})$ analogously to (3.3). Moreover, under the condition

$$\int_U \langle \mu_u, x^2 \wedge 1 \rangle \alpha(du) < \infty. \tag{3.13}$$

the renormalized Poisson integral $\int_U \mu_u * \Pi_\alpha^{reno}(du)$ is defined by

$$\int_U \mu_u * \Pi_\alpha^{reno}(du) = \lim_{\varepsilon \to +0} \theta_{a_\varepsilon} \cdot \int_{A_\varepsilon} \mu_u * \Pi_\alpha(du),$$

where

$$A_\varepsilon = \{u \in U : \langle \mu_u, |x| \wedge 1 \rangle \ge \varepsilon\}, \quad a_\varepsilon = \int_{A_\varepsilon} \langle \mu_u, xI_{[|x|\le 1]} \rangle \alpha(du).$$

4 Stable RPDs

Let Ξ be an RPD on $\mathbb{R}$. Ξ is called *stable* if for any $n \geq 2$ and n independent copies $\Xi_1, \cdots, \Xi_n$ of Ξ there exists constants $b_n \in \mathbb{R}$ and $c_n > 0$ such that

$$\Xi_1 * \cdots * \Xi_n \overset{(d)}{=} \theta_{b_n} \tau_{c_n} \Xi. \tag{4.1}$$

In particular, if $b_n = 0$ in (4.1) for any $n \geq 1$, Ξ is called *strictly stable.*
Let Q be the distribution of Ξ. Ξ is a stable RPD if and only if $Q \in \mathcal{P}(\mathcal{P}(\mathbb{R}))$ is stable, i.e. for any $n \geq 2$ there exists $b_n \in \mathbb{R}$ and $c_n > 0$ such that

$$Q^{\circledast n} = Q \circledast \cdots \circledast Q = \Theta_{b_n} T_{c_n} Q.$$

Since (4.1) implies that $E(\Xi)$ is a stable distribution on $\mathbb{R}$, it holds that $c_n = n^{1/\alpha}$ with an $0 < \alpha \leq 2$. (cf. (Fe)) In this case Ξ and Q are called *α-stable.*

Theorem 4.1. *(1) Let Ξ be a 2-stable RPD. Then its moment characteristics is of the following form:*

$$\log \Phi(\boldsymbol{z}) = \sum_{j \in \mathbb{N}} \left(i\gamma z_j - \frac{\alpha}{2} z_j^2 \right) - \frac{\beta}{2} \left(\sum_{j \in \mathbb{N}} z_j \right)^2 \quad (\boldsymbol{z} \in \ell_0(\mathbb{N})), \tag{4.2}$$

where $\alpha \geq 0$, $\beta \geq 0$ and $\gamma \in \mathbb{R}$ are constants. Moreover it holds that

$$\Xi \overset{(d)}{=} \delta_{\beta^{1/2}\xi} * \mathcal{N}(\gamma, \alpha) = \mathcal{N}(\beta^{1/2}\xi + \gamma, \alpha), \tag{4.3}$$

where $\mathcal{N}(m, a)$ denotes the normal distribution with mean m and variance $a \geq 0$, and ξ is a random variable with a standard normal distribution $\mathcal{N}(0, 1)$.

(2) Let Ξ be an α-stable RPD with $0 < \alpha < 2$. Then the moment characteristics is of the following form:

$$\begin{aligned} & \log \Phi(\boldsymbol{z}) \\ = & \sum_{j \in \mathbb{N}} \left(i\gamma z_j + \int_{\mathbb{R}\setminus\{0\}} \left(e^{iz_j x} - 1 - iz_j x I_{[|x| \leq 1]} \right) \rho(dx) \right) \\ + & \int_{\mathcal{P}_*(\mathbb{R})} \langle \mu^{\otimes \mathbb{N}}, e^{i\langle z, x \rangle} - 1 - i \left(\sum_{j \in \mathbb{N}} z_j x_j I_{[|x_j| \leq 1]} \right) \rangle m(d\mu), \end{aligned} \tag{4.4}$$

where $\gamma \in \mathbb{R}$ is a constant, ρ and m are σ-finite measures on $\mathbb{R} \setminus \{0\}$ and $\mathcal{P}_*(\mathbb{R})$ satisfying (2.3), (2.4) respectively, and

$$\tau_r \cdot \rho = r^\alpha \rho, \qquad T_r \cdot m = r^\alpha m \quad (\forall r > 0). \tag{4.5}$$

In this case it holds that

$$\Xi \overset{(d)}{=} \nu * \left(\int_{\mathcal{P}_*(\mathbb{R})} \mu * \Pi_m^{reno}(d\mu) \right), \tag{4.6}$$

where $\nu \in \mathcal{P}^{id}(\mathbb{R})$ *has the characteristic function*

$$\widehat{\nu}(z) = \exp\left(\int_{\mathbb{R}\setminus\{0\}} \left(e^{izx} - 1 - izxI_{[|x|\le 1]}\right)\rho(dx) + i\gamma z\right),$$

and Π_m *is a Poisson random measure on* $\mathcal{P}_*(\mathbb{R})$ *with the intensity measure* m.

(*Proof.*) If Ξ is 2-stable, the n-moment measure $E(\Xi^{\otimes n})$ is a 2-stable distribution, so that $\nu^{(n)}$ in (2.6) vanishes, proving (4.2). (4.3) immediately. Next, suppose that Ξ is α-stable with $0 < \alpha < 2$. Then by the same reason α and β in (2.6) vanish, and $\nu^{(n)}$ enjoys the scaling invariance for any $n \ge 1$. Furthermore, it is easy to see that the scaling invariance inherits to (4.5) for ρ and m. □

5 Examples

Throughout this section we always take $\boldsymbol{z} \in \ell_0(\mathbb{N})$.

Example 1 (non-random case) Let μ_0 be an infinitely divisible distribution on $\mathbb{R}$ with characteristic exponent

$$\log\widehat{\mu}_0(z) = -\frac{\alpha}{2}z^2 + iz\gamma + \int_{\mathbb{R}\setminus\{0\}} \left(e^{izx} - 1 - izxI_{[|x|\le 1]}\right)\rho(dx), \tag{5.1}$$

and set

$$\Xi = \mu_0.$$

Then Ξ is an infinitely divisible RPD with the moment characteristics

$$\log\Phi(\boldsymbol{z}) = \sum_{j\in\mathbb{N}}\left(-\frac{\alpha}{2}z_j^2 + i\gamma z_j + \int_{\mathbb{R}\setminus\{0\}} \left(e^{iz_jx} - 1 - iz_jxI_{[|x|\le 1]}\right)\rho(dx)\right),$$

which is the first term of the Lévy-Khintchin representation (2.2).

Example 2 Let X a random variable with the distribution μ_0 of (5.1), and set

$$\Xi = \delta_X.$$

Then Ξ is an infinitely divisible RPD with the moment characteristics

$$\begin{aligned}\log\Phi(\boldsymbol{z}) &= -\frac{\alpha}{2}\left(\sum_{j\in\mathbb{N}} z_j\right)^2 + i\left(\sum_{j\in\mathbb{N}} z_j\right)\gamma \\ &\quad + \int_{\mathbb{R}\setminus\{0\}}\left(e^{i(\sum_{j\in\mathbb{N}} z_j)x} - 1 - i\left(\sum_{j\in\mathbb{N}} z_j\right)xI_{[|x|\le 1]}\right)\rho(dx).\end{aligned}$$

The last term corresponds to the third term of the Lévy-Khintchin representation (2.2) with

$$m = \int_{\mathbb{R}\setminus\{0\}} \delta_{\delta_x}\rho(dx).$$

Example 3. Let e_t be the exponential distribution with mean $t > 0$, and for $\lambda > 0$ set

$$\rho_\lambda(dt) = I_{[t>0]}\lambda^2 t^{-\lambda-1}dt. \tag{5.2}$$

Let Π_{ρ_λ} be a Poisson random measure on $(0,\infty)$ with intensity measure ρ_λ. If $0 < \lambda < 1$, (3.12) is satisfied and

$$\Xi = \int_{(0,\infty)} e_t * \Pi_{\rho_\lambda}(dt), \tag{5.3}$$

defines an infinitely divisible RPD which takes values in $\mathcal{P}^{id}(\mathbb{R})$ by Theorem 3.4, having the characteristic function

$$\widehat{\Xi}(z) = \exp\int_{(0,\infty)} -\log(1-izt)\Pi_{\rho_\lambda}(dt). \tag{5.4}$$

Noting that

$$-\log(1-izt) = \int_0^\infty (e^{izx}-1)x^{-1}e^{-t^{-1}x}dx, \tag{5.5}$$

we have the Lévy-Khintchin representation for Ξ

$$\widehat{\Xi}(z) = \exp\int_{(0,\infty)} (e^{izx}-1)\nu_\Xi(dx),$$

where ν_Ξ is given by

$$\nu_\Xi(dx) = I_{[x>0]}x^{-1}\left(\int_{(0,\infty)} e^{t^{-1}x}\Pi_{\rho_\lambda}(dt)\right)dx, \tag{5.6}$$

which satisfies that if $0 < \lambda < 1$

$$\int_{(0,\infty)} x\nu_\Xi(dx) = \int_{(0,\infty)} t\Pi_{\rho_\lambda}(dt) < \infty.$$

Note that by (3.4) the moment characteristics

$$\log\Phi(\boldsymbol{z}) = \int_{(0,\infty)} \langle e_t^{\otimes\mathbb{N}}, e^{i\langle z,x\rangle}-1\rangle\rho_\lambda(dt) \quad (\boldsymbol{z}\in\ell_0(\mathbb{N})), \tag{5.7}$$

and the measure m in (2.2) is given by

$$m = \int_{(0,\infty)} \delta_{e_t}\rho_\lambda(dt),$$

so by Theorem 4.1 Ξ is a λ-stable RPD.

Let us denote by F_λ the distribution function of the first moment $E(\Xi)\in\mathcal{P}(\mathbb{R})$. Then using (5.7) and

$$\int_{(0,\infty)} e_t(dx)\rho_\lambda(dt) = \lambda^2\Gamma(\lambda+1)I_{[x>0]}x^{-\lambda-1}dx, \tag{5.8}$$

we obtain the Laplace transform of F_λ

$$\begin{aligned}\int_0^\infty e^{-\xi x}dF_\lambda(x) &= \exp\left(\int_{(0,\infty)} \langle e_t, e^{-\xi x}-1\rangle\rho_\lambda(dt)\right)\\ &= e^{-c_\lambda\xi^\lambda} \quad (\xi>0),\end{aligned} \tag{5.9}$$

with $c_\lambda = \lambda\Gamma(\lambda+1)\Gamma(1-\lambda)$. Thus F_λ is a λ-stable distribution on $(0,\infty)$. The stable RPD (5.3) and the distribution function F_λ will appear in Theorem 7.1 and Corollary 7.3.

In Example 3, if $1 \le \lambda < 2$, the condition (3.12) fails, but (3.13) holds. Let

$$\Xi = \int_{(0,\infty)} e_t * \Pi^{reno}_{\rho_\lambda}(dt). \tag{5.10}$$

Then by Lemma 3.2 and Theorem 4.1 Ξ is a λ-stable RPD with the moment characteristics is

$$\log \Phi(\boldsymbol{z}) = \int_{(0,\infty)} \langle e_t^{\otimes \mathbb{N}}, e^{i\langle z,x\rangle} - 1 - i\sum_{j\in\mathbb{N}} z_j x_j I_{[x_j\le 1]}\rangle \rho_\lambda(dt). \tag{5.11}$$

The characteristic function of (5.10) is

$$\begin{aligned}\widehat{\Xi}(z) &= \exp\int_{(0,\infty)} -\log(1-izt)\tilde{\Pi}_{\rho_\lambda}(dt) \\ &\quad -\int_{(0,\infty)} \left(\log(1-izt) + iztI_{[t\le 1]}\right)\rho_\lambda(dt),\end{aligned} \tag{5.12}$$

where $\tilde{\Pi}_{\rho_\lambda}$ is the compensated Poisson random measure of Π_{ρ_λ}, i.e. $\tilde{\Pi}_{\rho_\lambda} = \Pi_{\rho_\lambda} - \rho_\lambda$. Since ν_Ξ of (5.6) satisfies

$$\int_{(0,\infty)} x^2 \nu_\Xi(dx) = \int_{(0,\infty)} t^2 \Pi_{\rho_\lambda}(dt) < \infty,$$

it follows from (5.12) that

$$\widehat{\Xi}(z) = \exp\int_{(0,\infty)} (e^{izx} - 1 - izxI_{[x\le 1]})\nu_\Xi(dx) + izb^\omega, \tag{5.13}$$

with

$$b^\omega = \left(\int_{(0,1)} t\tilde{\Pi}_{\rho_\lambda}(dt) + \int_{(0,\infty)} t(I_{[t\ge 1]} - e^{-t^{-1}})\Pi_{\rho_\lambda}(dt)\right). \tag{5.14}$$

Let G_λ denote the distribution function of the first moment measure $E(\Xi)$. Then by making use of (5.11) and (5.8) we obtain the characteristic function of G_λ as follows.

$$\widehat{G_\lambda}(z) = \begin{cases} \exp -\frac{\pi}{2}|z| - iz\log|z| + i\gamma_1 z & (\lambda = 1) \\ \exp -c_\lambda |z|^\lambda (1 - i\tan\frac{\pi\lambda}{2}\mathrm{sign}(z)) + i\gamma_\lambda z & (1<\lambda<2)\end{cases}$$

where

$$\gamma_1 = \int_1^\infty \sin u u^{-2} du + \int_0^1 (\sin u - u)u^{-2} du,$$

$$c_\lambda = -\lambda^2\Gamma(\lambda+1)\Gamma(-\lambda)\cos\frac{\pi\lambda}{2}, \quad \gamma_\lambda = \frac{\lambda^2\Gamma(\lambda+1)}{\lambda-1}.$$

The stable RPD of (5.10) and the stable distribution function G_λ will appear in Theorem 7.2 and Corollary 7.3.

Example 4. For $t>0$ let $\theta_t \cdot e_t$ be the shifted exponential distribution of e_t with mean 0, i.e.

$$\theta_t \cdot e_t(dx) = I_{[x+t>0]} t^{-1} e^{-t^{-1}(x+t)} dx.$$

Let ρ_λ be of (5.2). Then it satisfies the condition (3.12) if and only if $0 < \lambda < 1$, which is easily verified using

$$\begin{aligned}\langle \theta_t \cdot e_t, |x| \wedge 1\rangle &= \langle e_t, |x-t| \wedge 1\rangle \\ &= t\langle e_1, |x-1| \wedge t^{-1}\rangle \\ &\sim t\langle e_1, |x-1|\rangle \quad (t \to 0).\end{aligned}$$

However, even in the case $1 \le \lambda < 2$, it is possible to define the Poisson integral

$$\Xi = \int_{(0,\infty)} (\theta_t \cdot e_t) * \Pi_{\rho_\lambda}(dt). \tag{5.15}$$

Namely, for $\varepsilon > 0$, we set

$$\Xi_\varepsilon = \int_{(\varepsilon,\infty)} (\theta_t \cdot e_t) * \Pi_{\rho_\lambda}(dt).$$

Then the characteristic function is

$$\widehat{\Xi}_\varepsilon(z) = \exp \int_{(\varepsilon,\infty)} \left(\log \frac{1}{1-izt} - izt \right) \Pi_{\rho_\lambda}(dt).$$

Since for any $1 \le \lambda < 2$

$$\int_{(0,\infty)} \sup_{|z|\le m} \left(\left| \log \frac{1}{1-izt} - izt \right| \wedge 1 \right) \rho_\lambda(dt) < \infty \quad (\forall m > 0),$$

it holds that P-a.s.

$$\lim_{\varepsilon\to 0} \widehat{\Xi}_\varepsilon(z) = \exp \int_{(0,\infty)} \left(\log \frac{1}{1-izt} - izt \right) \Pi_{\rho_\lambda}(dt)$$

uniformly in each bounded interval. Thus there exists an RPD Ξ, taking values in $\mathcal{P}^{id}(\mathbb{R})$, such that

$$\lim_{\varepsilon\to 0+} d(\Xi_\varepsilon, \Xi) = 0 \quad P-a.s.,$$

and the characteristic function of Ξ is given by

$$\widehat{\Xi}(z) = \exp \int_{(0,\infty)} \left(\log \frac{1}{1-izt} - izt \right) \Pi_{\rho_\lambda}(dt). \tag{5.16}$$

We identify the Poisson integral (5.15) with this limit Ξ.

Note that Ξ is a λ-stable RPD, taking values in $\mathcal{P}^{id}(\mathbb{R})$, with the moment characteristics

$$\begin{aligned}\log \Phi(\boldsymbol{z}) &= \int_{(0,\infty)} \langle (\theta_t \cdot e_t)^{\otimes \mathbb{N}}, e^{i\langle z,x\rangle} - 1\rangle \rho_\lambda(dt) \\ &= \int_{(0,\infty)} \left(\prod_{j\in\mathbb{N}} \frac{e^{-iz_j t}}{1-iz_j t} - 1 \right) \rho_\lambda(dt), \end{aligned} \tag{5.17}$$

and the corresponding m of (2.2) is given by

$$m = \int_{(0,\infty)} \delta_{\theta_t \cdot e_t} \rho_\lambda(dt).$$

Since by (5.5), (5.16) turns to

$$\widehat{\Xi}(z) = \exp \int_{(0,\infty)} (e^{izx} - 1 - izxI_{[x\le 1]})\nu_\Xi(dx) \\ -iz\left(\int_{(0,\infty)} te^{-t^{-1}}\Pi_{\rho_\lambda}(dt)\right), \tag{5.18}$$

with ν_Ξ of (5.6), by (5.13), (5.14) and (5.18) we obtain the following relation between Example 3 and Example 4,

$$\int_{(0,\infty)} e_t * \Pi^{reno}_{\rho_\lambda}(dt) \tag{5.19}$$
$$= \left(\int_{(0,\infty)} (\theta_t \cdot e_t) * \Pi_{\rho_\lambda}(dt)\right) * \delta_{\left\{\int_{(0,1]} t\tilde{\Pi}_{\rho_\lambda}(dt) + \int_{(1,\infty)} t\Pi_{\rho_\lambda}(dt)\right\}}$$

Let H_λ denote the distribution function of the first moment $E(\Xi)$. Then H_λ is a λ-stable distribution on $\mathbb{R}$, and after tedious computations we have the characteristic function

$$\widehat{H_\lambda}(z) = \begin{cases} \exp -c_1|z| + i\gamma z & (\lambda = 1) \\ \exp -c_\lambda|z|^\lambda(1 - i\beta\tan\frac{\pi\lambda}{2}\mathrm{sgn}(z)) & (1 < \lambda < 2) \end{cases}$$

where $c_1 = \pi e^{-1}$, $\gamma < 0$, $c_\lambda > 0$ and $\beta \in (-1,1)$ are given by

$$\gamma = \int_0^1 (e^{-(1-t)} - e^{-(1+t)})dt - \int_1^\infty e^{-(1+t)}t^{-1}dt - 1,$$

$$c_\lambda = (c_+ + c_-)\Gamma(-\lambda)\cos\frac{\pi\lambda}{2}, \quad \beta = \frac{c_+ - c_-}{c_+ + c_-},$$

with

$$c_+ = e^{-1}\lambda^2\Gamma(\lambda+1), \quad c_- = e^{-1}\lambda^2\int_0^1 e^u u^\lambda du.$$

It may be of interest that the Lévy measure ν of H_λ is supported fully by $\mathbb{R}\setminus\{0\}$ as follows,

$$\nu(dx) = \begin{cases} c_+|x|^{-\lambda-1}dx & (x > 0) \\ c_-|x|^{-\lambda-1}dx & (x < 0), \end{cases}$$

although the Lévy measure ν_Ξ of Ξ is supported in $(0,\infty)$ by (5.6).
The stable RPD of (5.15) with $1 \le \lambda < 2$ and the stable distribution function H_λ will appear in Theorem 7.1 and Corollary 7.3.

Example 5. For a positive infinitely divisible random variable ζ, we set

$$\Xi = \mathcal{N}(0,\zeta).$$

Then it is obvious that the RPD Ξ is infinitely divisible. Note that if ζ satisfies

$$\zeta = \int_{(0,\infty)} t\Pi_\rho(dy),$$

where Π_ρ is a Poisson random measure on $(0,\infty)$ with an intensity measure ρ satisfying

$$\int_{(0,\infty)} (t \wedge 1)\rho(dt) < \infty.$$

Then it holds that

$$\Xi = \int_{(0,\infty)} \mathcal{N}(0,t) * \Pi_\rho(dt).$$

In particular, ζ is a positive α-stable random variable such that for some $0 < \alpha < 1$ and $c > 0$

$$E(e^{-z\zeta}) = e^{-cz^\alpha} \quad (z > 0),$$

the RPD Ξ is a 2α-stable RPD, taking valued in $\mathcal{P}^{id}(\mathbb{R})$, with the moment characteristics

$$\log \Phi(\boldsymbol{z}) = -c2^{-\alpha} \parallel \boldsymbol{z} \parallel^{2\alpha} = -c2^{-\alpha} \left(\sum_{j\in\mathbb{N}} z_j^2 \right)^\alpha.$$

6 Lévy processes on $\mathcal{P}(\mathbb{R})$

Let $(\Xi(t))_{t\geq 0}$ be a $\mathcal{P}(\mathbb{R})$-valued stochastic process. $(\Xi(t))_{t\geq 0}$ is called *a Lévy process on $\mathcal{P}(\mathbb{R})$ in law*, if

(i) $\Xi(0) = \delta_0$,

(ii) There exists a family of $\mathcal{P}(\mathbb{R})$-valued random variables $\{\Xi_{s,t}\}_{0\leq s<t}$ satisfying that for any $n \geq 2$ and $0 = t_0 < t_1 < \cdots < t_n$, $\{\Xi_{t_{i-1},t_i}\}_{1\leq i\leq n}$ are independent, and

$$\Xi(t_n) = \Xi_{t_0,t_1} * \cdots * \Xi_{t_{n-1},t_n} \quad P-a.s.$$

(iii) For $0 < s < t$

$$\Xi_{s,t} \stackrel{(d)}{=} \Xi(t-s).$$

(iv) $(\Xi(t))_{t\geq 0}$ is stochastically continuous.

It is obvious that for each $t > 0$, $\Xi(t)$ is an infinitely devisible RPD, and any finite dimensional distributions of $(\Xi(t))$ are uniquely determined by the moment characteristics of $\Xi(1)$.

In addition to the conditions (i), (ii), (iii) and (iv), if $t \in [0,\infty) \mapsto \Xi(t) \in \mathcal{P}(\mathbb{R})$ is right continuous with left limit in $t \geq 0$ P-a.s., namely $(\Xi(t))$ is a $\mathcal{P}(\mathbb{R})$-valued cadlag process, $(\Xi(t))$ is called *a Lévy process on $\mathcal{P}(\mathbb{R})$*.

The basic results for classical Lévy processes are

(i) for a given Lévy process in law, there exists a modification of a Lévy process,

(ii) for a given Lévy process, the Lévy-Itô representation for the sample path is valid, that is,

(a) a collection of jump times and jump sizes forms a Poisson point process, and the jump part of the process is recovered from the Poisson point process.

(b) the continuous part is nothing but a Gaussian Lévy process independent of the the jump part.

For Lévy processes on $\mathcal{P}(\mathbb{R})$ in law, the first one holds essentially in the same way as for the classical case. However for Lévy processes on $\mathcal{P}(\mathbb{R})$, the second one seems difficult. The reason is that for the sample path $\Xi(t)$ the jump size μ should be defined as

$$\Xi(t) = \Xi(t-) * \mu,$$

but existence of such μ does not follow automatically, and the uniqueness of μ does not hold in general.

In this section for a given infinitely divisible probability distribution Q on $\mathcal{P}(\mathcal{P}(\mathbb{R}))$, we will obtain a Lévy process $(\Xi(t))$ on $\mathcal{P}(\mathbb{R})$ such that $\Xi(1)$ has the distribution Q, by constructing a Lévy-Itô-type representation.

To do this the main task is to define a Poisson integral as a $\mathcal{P}(\mathbb{R})$-valued cadlag process.

For $T > 0$ let $D([0,T],\mathcal{P}(\mathbb{R}))$ be the totality of $\mathcal{P}(\mathbb{R})$-valued cadlag paths on $[0,T]$, and for $U = (U_t), V = (V_t) \in D([0,T],\mathcal{P}(\mathbb{R}))$ we set

$$d_T(U,V) = \sup_{0\le t\le T} d(U_t,V_t).$$

Then $(D([0,T],\mathcal{P}(\mathbb{R})), d_T)$ is a complete metric space. Recalling the definition of d in (2.1), we see easily that for $U = (U_t), V = (V_t) \in D([0,T],\mathcal{P}(\mathbb{R}))$

$$d_T(U * V, V) \le d_T(U, \boldsymbol{\delta}_0), \tag{6.1}$$

where $\boldsymbol{\delta}_0$ is the path identically equal to δ_0. The inequality (6.1) will be used frequently.

Let m be a σ-finite measure on $\mathcal{P}_*(\mathbb{R}) = \mathcal{P}(\mathbb{R})\setminus\{\delta_0\}$ satisfying the condition (3.2), and let $\Pi_{\lambda\times m}$ be a Poisson random measure on $[0,\infty)\times\mathcal{P}_*(\mathbb{R})$ with intensity measure $\lambda\times m$ defined on a probability space $(\Omega,\mathcal{F},P)$, where λ stands for the Lebesgue measure on $[0,\infty)$. For $\varepsilon > 0$ and $0 < \varepsilon < \varepsilon'$ let

$$A^{(\varepsilon)} = \{\mu \in \mathcal{P}_*(\mathbb{R}) : \langle \mu, |x|\wedge 1\rangle > \varepsilon\},$$

and

$$A^{(\varepsilon,\varepsilon')} = \{\mu \in \mathcal{P}_*(\mathbb{R}) : \varepsilon < \langle \mu, |x|\wedge 1\rangle \le \varepsilon'\}.$$

Noting that $m(A^{(\varepsilon)}) < \infty$ and $m(A^{(\varepsilon,\varepsilon')}) < \infty$, we set

$$\Xi_t^{(\varepsilon)} = \theta_{a_\varepsilon t}\cdot \int_{[0,t]\times A^{(\varepsilon)}} \mu * \Pi_{\lambda\times m}(dsd\mu),$$

and

$$\Xi_t^{(\varepsilon,\varepsilon')} = \theta_{a_{\varepsilon,\varepsilon'} t}\cdot \int_{[0,t]\times A^{(\varepsilon,\varepsilon')}} \mu * \Pi_{\lambda\times m}(dsd\mu),$$

where

$$a_\varepsilon = \int_{A^{(\varepsilon)}} \langle \mu, xI_{[|x|\le 1]}\rangle m(d\mu),$$

and

$$a_{\varepsilon,\varepsilon'} = \int_{A^{(\varepsilon,\varepsilon')}} \langle \mu, xI_{[|x|\le 1]}\rangle m(d\mu).$$

It is clear that $\Xi^{(\varepsilon)} = (\Xi_t^{(\varepsilon)})$ and $\Xi^{(\varepsilon,\varepsilon')} = (\Xi_t^{(\varepsilon,\varepsilon')})$ are $\mathcal{P}(\mathbb{R})$-valued cadlag processes.

Theorem 6.1. *There exists a $\mathcal{P}(\mathbb{R})$-valued cadlag process $\Xi = (\Xi(t))$ defined on $(\Omega, \mathcal{F}, P)$ such that*

$$\lim_{\varepsilon\to 0+} d_T(\Xi^{(\varepsilon)}, \Xi) = 0 \quad (\forall T > 0) \quad P - a.s.$$

We denote the limit Ξ in Theorem 6.1 by

$$\int_{[0,t]\times\mathcal{P}_*(\mathbb{R})} \mu * \Pi^{reno}_{\lambda\times m}(dsd\mu),$$

which is a $\mathcal{P}(\mathbb{R})$-valued cadlag process.

For the proof of Theorem 6.1 we start with the following lemma, which is a $\mathcal{P}(\mathbb{R})$-version of Theorem 20.2 of (6).

Lemma 6.2. *Let $\{U_j(t), t \in [0,T]\}_{1\le j\le n}$ be an independent $\mathcal{P}(\mathbb{R})$-valued cadlag processes on $[0,T]$, and for $1 \le m < n$ let*

$$\Xi_{m,n}(t) = U_m(t) * \cdots * U_n(t).$$

Then for any $\eta > 0$ and $n \ge 1$

$$P(\max_{1\le j\le n} d_T(\Xi_{1,j}, \boldsymbol{\delta}_0) \ge 3\eta) \le 3 \max_{1\le j\le n} P(d_T(\Xi_{1,j}, \boldsymbol{\delta}_0) \ge \eta). \tag{6.2}$$

(*Proof.*) Note that

$$\Xi_{1,n} = \Xi_{1,k} * \Xi_{k+1,n} \quad (1 \le k < n),$$

and by (6.1)

$$d_T(\Xi_{1,k}, \Xi_{1,k} * \Xi_{k+1,n}) \le d_T(\boldsymbol{\delta}_0, \Xi_{k+1,n}),$$

$$\begin{aligned} d_T(\Xi_{1,k}, \boldsymbol{\delta}_0) &\le d_T(\Xi_{1,n}, \boldsymbol{\delta}_0) + d_T(\Xi_{1,k}, \Xi_{1,n}) \\ &\le d_T(\Xi_{1,n}, \boldsymbol{\delta}_0) + d_T(\boldsymbol{\delta}_0, \Xi_{k+1,n}). \end{aligned}$$

Since $d_T(\Xi_{1,k}, \boldsymbol{\delta}_0) \ge 3\eta$ implies either $d_T(\Xi_{1,n}, \boldsymbol{\delta}_0) \ge \eta$ or $d_T(\boldsymbol{\delta}_0, \Xi_{k+1,n}) \ge 2\eta$ $(1 \le k \le n)$ with convention that $\Xi_{n+1,n} = \boldsymbol{\delta}_0$,

$$\begin{aligned}
& P(\max_{1\le j\le n} d_T(\Xi_{1,j},\boldsymbol{\delta}_0)\ge 3\eta)\\
= & \sum_{k=1}^{n} P(\max_{1\le j\le k-1} d_T(\Xi_{1,j},\boldsymbol{\delta}_0)<3\eta, d_T(\Xi_{1,k},\boldsymbol{\delta}_0)\ge 3\eta)\\
\le & P(d_T(\Xi_{1,n},\boldsymbol{\delta}_0)\ge \eta)\\
& +\sum_{k=1}^{n-1} P(\max_{1\le j\le k-1} d_T(\Xi_{1,j},\boldsymbol{\delta}_0)<3\eta, d_T(\Xi_{1,k},\boldsymbol{\delta}_0)\ge 3\eta, d_T(\boldsymbol{\delta}_0,\Xi_{k+1,n})\ge 2\eta)\\
\le & P(d_T(\Xi_{1,n},\boldsymbol{\delta}_0)\ge \eta)\\
& +\sum_{k=1}^{n-1} P(\max_{1\le j\le k-1} d_T(\Xi_{1,j},\boldsymbol{\delta}_0)<3\eta, d_T(\Xi_{1,k},\boldsymbol{\delta}_0)\ge 3\eta)P(d_T(\boldsymbol{\delta}_0,\Xi_{k+1,n})\ge 2\eta)\\
\le & P(d_T(\Xi_{1,n},\boldsymbol{\delta}_0)\ge \eta)+\max_{1\le k\le n-1} P(d_T(\boldsymbol{\delta}_0,\Xi_{k+1,n})\ge 2\eta).
\end{aligned}$$

Then (6.2) follows from

$$\begin{aligned}
d_T(\boldsymbol{\delta}_0,\Xi_{k+1,n}) & \le d_T(\boldsymbol{\delta}_0,\Xi_{1,n})+d_T(\Xi_{1,n},\Xi_{k+1,n})\\
& \le d_T(\boldsymbol{\delta}_0,\Xi_{1,n})+d_T(\Xi_{1,k},\boldsymbol{\delta}_0).
\end{aligned}$$

□

Lemma 6.3. *For* $\varepsilon_0>0$ *and* $\eta>0$

$$P(\sup_{0<\varepsilon<\varepsilon_0} d_T(\Xi^{(\varepsilon)},\Xi^{(\varepsilon_0)})\ge 3\eta)\le 3\sup_{0<\varepsilon<\varepsilon_0} P(d_T(\Xi^{(\varepsilon,\varepsilon_0)},\boldsymbol{\delta}_0)\ge\eta). \tag{6.3}$$

(*Proof.*) Note that $\Xi^{(\varepsilon)}$ is right continuous in $\varepsilon\in(0,\varepsilon_0)$. For $\mathbb{Q}\cap(0,\varepsilon_0)=\{\varepsilon_n\}_{n\ge1}$

$$\begin{aligned}
\sup_{0<\varepsilon<\varepsilon_0} d_T(\Xi^{(\varepsilon)},\Xi^{(\varepsilon_0)}) & = \sup_{\varepsilon\in\mathbb{Q}\cap(0,\varepsilon_0)} d_T(\Xi^{(\varepsilon)},\Xi^{(\varepsilon_0)})\\
& = \lim_{n\to\infty}\max_{1\le j\le n} d_T(\Xi^{(\varepsilon_j)},\Xi^{(\varepsilon_0)}),
\end{aligned}$$

so we see that

$$\begin{aligned}
& P(\sup_{0<\varepsilon<\varepsilon_0} d_T(\Xi^{(\varepsilon)},\Xi^{(\varepsilon_0)})>3\eta)\\
= & \lim_{n\to\infty} P(\max_{1\le j\le n} d_T(\Xi^{(\varepsilon_j)},\Xi^{(\varepsilon_0)})>3\eta)\\
\le & \lim_{n\to\infty} P(\max_{1\le j\le n} d_T(\Xi^{(\varepsilon_j,\varepsilon_0)},\boldsymbol{\delta}_0)\ge 3\eta).
\end{aligned}$$

Rearranging $\{\varepsilon_1,\cdots,\varepsilon_n\}$ we may assume $0<\varepsilon_n<\cdots<\varepsilon_1<\varepsilon_0$ and $\Xi^{(\varepsilon_j,\varepsilon_0)}=\Xi^{(\varepsilon_j,\varepsilon_{j-1})}*\cdots*\Xi^{(\varepsilon_1,\varepsilon_0)}$, so applying Lemma 6.2 we have

$$\begin{aligned}
& P(\max_{1\le j\le n} d_T(\Xi^{(\varepsilon_j,\varepsilon_0)},\boldsymbol{\delta}_0)\ge 3\eta)\\
\le & 3\max_{1\le j\le n} P(d_T(\Xi^{(\varepsilon_j,\varepsilon_0)},\boldsymbol{\delta}_0)\ge\eta)\\
\le & 3\sup_{0<\varepsilon<\varepsilon_0} P(d_T(\Xi^{(\varepsilon,\varepsilon_0)},\boldsymbol{\delta}_0)\ge\eta),
\end{aligned}$$

completing the proof of (6.3). □

Lemma 6.4. *(i) For $f \in C_b^2(\mathbb{R})$,*

$$\langle \Xi_t^{(\varepsilon,\varepsilon_0)}, f\rangle - f(0) = \int_{[0,t]\times A^{(\varepsilon,\varepsilon_0)}} \langle \Xi_{s-}^{(\varepsilon,\varepsilon_0)}, \tilde{\mu} * f - f\rangle \tilde{\Pi}_{\lambda\times m}(dsd\mu) + \int_{[0,t]\times A^{(\varepsilon,\varepsilon_0)}} \langle \Xi_s^{(\varepsilon,\varepsilon_0)}, L_\mu f\rangle dsm(d\mu), \tag{6.4}$$

where $\tilde{\mu}(B) = \mu(-B) \quad (B \in \mathcal{B}(\mathbb{R}))$,

$$L_\mu f(x) = \int_{\mathbb{R}} (f(x+y) - f(x) - yI_{[|y|\le 1]} f'(x))\mu(dy),$$

and $\tilde{\Pi}_{\lambda\times m}$ stands for the compensated Poisson random measure of $\Pi_{\lambda\times m}$, i.e.

$$\tilde{\Pi}_{\lambda\times m} = \Pi_{\lambda\times m} - \lambda\times m.$$

(ii) Let $T > 0$. If $f \in C_b^2(\mathbb{R})$ satisfies $f(0) = 0$,

$$\lim_{\varepsilon_0\to 0+} \sup_{0<\varepsilon<\varepsilon_0} E\left(\sup_{0\le t\le T} \left|\langle \Xi_t^{(\varepsilon,\varepsilon_0)}, f\rangle\right|\right) = 0. \tag{6.5}$$

(*Proof.*) The proof of (i) is straightforward, since $\Xi_t^{(\varepsilon,\varepsilon_0)}$ has finite jump times in each finite interval. For (ii), using (6.4), the maximal enequality for martingales,

$$|\tilde{\mu} * f(x) - f(x)|^2 \le C\langle \mu, |y|^2 \wedge 1\rangle,$$

$$|L_\mu f(x)| \le C\langle \mu, |y|^2 \wedge 1\rangle,$$

and

$$\lim_{\varepsilon_0\to 0} \sup_{0<\varepsilon<\varepsilon_0} \int_{A^{(\varepsilon,\varepsilon_0)}} \langle \mu, |y|^2 \wedge 1\rangle m(d\mu) = 0,$$

we obtain (6.5). □

Lemma 6.5.

$$\lim_{\varepsilon_0\to 0} \sup_{0<\varepsilon<\varepsilon_0} E\left(d_T(\Xi^{(\varepsilon,e_0)}, \boldsymbol{\delta}_0)\right) = 0. \tag{6.6}$$

(*Proof.*) By a similar argument to (3.9) we see that for every $\eta > 0$

$$\sup_{0\le t\le T} \sup_{|z|\le m} |\widehat{\Xi}_t^{(\varepsilon,\varepsilon_0)}(z) - 1| \le m\eta + 2 \sup_{0\le t\le T} \Xi_t^{(\varepsilon,\varepsilon_0)}(\mathbb{R}\setminus[-\eta,\eta]).$$

By Lemma 6.4,

$$\lim_{\varepsilon_0\to 0} \sup_{0<\varepsilon<\varepsilon_0} E\left(\sup_{0\le t\le T} \Xi_t^{(\varepsilon,\varepsilon_0)}(\mathbb{R}\setminus[-\eta,\eta])\right) = 0,$$

hence we have

$$\lim_{\varepsilon_0\to 0} \sup_{0<\varepsilon<\varepsilon_0} E\left(\sup_{0\le t\le T} \sup_{|z|\le m} |\widehat{\Xi}_t^{(\varepsilon,\varepsilon_0)}(z) - 1|\right) = 0,$$

which leads to (6.6). □

(*Proof of Theorem 6.1.*)

By Lemma 6.3 and Lemma 6.5 it holds that for every $\eta > 0$

$$\lim_{\varepsilon_0 \to 0+} P(\sup_{0<\varepsilon<\varepsilon_0} d_T(\Xi^{(\varepsilon)}, \Xi^{(\varepsilon_0)}) \geq 3\eta) = 0,$$

so that

$$\begin{aligned} & P(\lim_{\varepsilon_0 \to 0+} \sup_{0<\varepsilon,\varepsilon'<\varepsilon_0} d_T(\Xi^{(\varepsilon)}, \Xi^{(\varepsilon')}) \geq 3\eta) \\ &= \lim_{\varepsilon_0 \to 0+} P(\sup_{0<\varepsilon,\varepsilon'<\varepsilon_0} d_T(\Xi^{(\varepsilon)}, \Xi^{(\varepsilon')}) \geq 3\eta) = 0. \end{aligned}$$

Thus it holds that P-a.s.

$$\lim_{\varepsilon_0 \to 0+} \sup_{0<\varepsilon,\varepsilon'<\varepsilon_0} d_T(\Xi^{(\varepsilon)}, \Xi^{(\varepsilon')}) = 0.$$

Hence there exists a $\mathcal{P}(\mathbb{R})$-valued cadlag process $\Xi = (\Xi_t)$ such that

$$\lim_{\varepsilon \to 0+} d_T(\Xi^{(\varepsilon)}, \Xi) = 0 \quad (\forall T > 0) \quad P - a.s.,$$

completing the proof of Theorem 6.1. □

Let Q be the distribution of an infinitely divisible RPD, of which moment characteristics Φ_Q is represented by Theorem 2.1 in terms of the characteristic quantities $(\alpha, \gamma, \rho, \beta, m)$. Now we construct a Lévy process $\Xi = (\Xi(t))$ on $\mathcal{P}(\mathbb{R})$ such that $\Xi(1)$ has the distribution Q.

Let $(B(t))$ be a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$, and $\Pi_{\lambda\times m}$ be a Poisson random measure on $[0,\infty)\times\mathcal{P}_*(\mathbb{R})$ with intensity measure $\lambda\times m$ defined on $(\Omega, \mathcal{F}, P)$. We assume $(B(t))$ and $\Pi_{\lambda\times m}$ are independent. For $t \geq 0$ we set

$$\Xi(t) = \nu(t) * \delta_{\beta^{1/2}B(t)} * \left(\int_{[0,t]\times\mathcal{P}_*(\mathbb{R})} \mu * \Pi^{reno}_{\lambda\times m}(dsd\mu) \right), \tag{6.7}$$

where $\nu(t)$ is an infinitely divisible distribution on $\mathbb{R}$ with chatacteristic function

$$\widehat{\nu(t)}(z) = \exp t \left(-\frac{\alpha z^2}{2} + iz\gamma + \int_{\mathbb{R}\setminus\{0\}} (e^{izx} - 1 - izxI_{[|x|\leq 1]})\rho(dx) \right).$$

Theorem 6.6. *Let $\Xi = (\Xi(t))$ be defined by (6.7). Then $(\Xi(t))$ is a Lévy process on $\mathcal{P}(\mathbb{R})$ such that the moment characteristics of $\Xi(1)$ coincides with (2.2).*

The proof is immediate from Theorem 6.1 and Lemma 3.2.

7 A simple particle motion in a random environment

Given a strictly increasing sequence $w = \{w_n\}_{n\geq 0}$ with $w_0 = 0$, we introduce a Markov process $(X(t) = X(t,\omega), P_x^w))$ on $\mathbb{R}_+$, generated by

$$A^w f(x) = f'(x) + f(w_{N(x,w)}) - f(x), \tag{7.1}$$

where

$$N(x,w) = \max\{n \geq 0 : \ w_n \leq x\},$$

and $\{P_x^w\}_{x\geq 0}$ is a Markov family of probability measures on $(\Omega, \mathcal{F})$. The sequence $w = \{w_n\}_{n\geq 0}$ is called *an environment* and each w_n *a ladder point.*

A visual description of the path of the Markov process is as follows. A particle moves to the right with constant speed 1 and at an exponential random time T_1 with mean 1 it jumps back to the left adjacent ladder point $w_{N(X(T_1-),w)}$. Here $X(t)$ denotes the position of the particle at t. After jumping back to $w_{N(X(T_1-),w)}$ the particle starts afresh and continues the same uniform motion with speed 1 until the next exponential random time, at which the particle again continues similar movements.

We now consider the environment to be random. Specifically we consider the case where the randomness is introduced by the assumption that $w_n' = w_n - w_{n-1}$, $n \geq 1$ are i.i.d. random variables with exponential distribution of parameter $\lambda > 0$ defined on a probability space (W, P). In other words, $\{w_n\}_{n\geq 1}$ forms a Poisson point process on $(0,\infty)$ with intensity $\lambda > 0$.

Let $\sigma^w(x)$ be the hitting time to $x > 0$ for $(X(t), P_x^w)$. For $x > 0$, let $\Xi^{\sigma^w(x)}$ be the distribution of the hitting time $\sigma^w(x)$ under the probability law P_0^w, so $\Xi^{\sigma^w(x)}$ is an RPD. Our problem is to investigate a scaling limit of $\Xi^{\sigma^w(x)}$ as $x \to \infty$.

In order to describe scaling limit results of the RPD $\Xi^{\sigma^w(x)}$, we first introduce a Poisson random measure. For $\lambda > 0$, let Π_{ρ_λ} be a Poisson random measure on $(0,\infty)$ with intensity measure

$$\rho_\lambda(dt) = I_{[t>0]}\lambda^2 t^{-\lambda-1}dt, \tag{7.2}$$

and we denote by $\tilde{\Pi}_{\rho_\lambda}$ the compensated Poisson random measure of Π_{ρ_λ}, i.e.

$$\tilde{\Pi}_{\rho_\lambda} = \Pi_{\rho_\lambda} - \rho_\lambda.$$

For a sequence of RPDs $\{\Xi_n\}$ and an RPD Ξ_∞, let denote the distributions of Ξ_n and Ξ_∞ by Q_{Ξ_n} and Q_{Ξ_∞} respectively. If Q_{Ξ_n} converges weakly to Q_{Ξ_∞} as $n \to \infty$, we write

$$\Xi_n \overset{(d)}{\Longrightarrow} \Xi_\infty.$$

Recall that for $t > 0$, e_t is the exponential distribution with mean $t > 0$, and $\theta_t \cdot e_t$ is the shifted exponential distribution with mean 0, which appears in Example 4 of the section 5.

Denote by $m^w(x)$ the expectation of $\sigma^w(x)$ with respect to P_0^w. Then we obtain the following results.

Theorem 7.1. *(i) Let* $0 < \lambda < 1$. *Then*

$$\tau_{x^{-1/\lambda}} \cdot \Xi^{\sigma^w(x)} \overset{(d)}{\Longrightarrow} \int_{(0,\infty)} e_t * \Pi_{\rho_\lambda}(dt) \quad (x \to \infty). \tag{7.3}$$

(ii) Let $1 \le \lambda < 2$. *Then*

$$\tau_{x^{-1/\lambda}} \cdot \theta_{m^w(x)} \cdot \Xi^{\sigma^w(x)} \overset{(d)}{\Longrightarrow} \int_{(0,\infty)} (\theta_t \cdot e_t) * \Pi_{\rho_\lambda}(dt) \quad (x \to \infty). \tag{7.4}$$

(iii) Let $\lambda = 2$. *Then*

$$\lim_{x\to\infty} d(\tau_{(2x\log x)^{-1/2}} \cdot \theta_{m^w(x)} \cdot \Xi^{\sigma^w(x)}, \mathcal{N}(0,1)) = 0 \quad \textit{in probability.} \tag{7.5}$$

(iv) Let $\lambda > 2$. *Then*

$$\lim_{x\to\infty} d(\tau_{x^{-1/2}} \cdot \theta_{m^w(x)} \cdot \Xi^{\sigma^w(x)}, \mathcal{N}(0,c_\lambda)) = 0 \quad P-a.s., \tag{7.6}$$

where

$$c_\lambda = \frac{2\lambda}{(\lambda-2)(\lambda-1)^2}. \tag{7.7}$$

Let $\lambda > 1$. Then by (8.3) in Lemma 8.1 below it holds that

$$\overline{m}(x) \equiv E(m^w(x)) \sim \frac{\lambda x}{\lambda - 1} \quad (x \to \infty).$$

So we next discuss another scaling limit for the RPD $\Xi^{\sigma^w(x)}$, replacing the random centering $m^w(x)$ in $\tau_{x^{-1/\lambda}} \cdot \theta_{m^w(x)} \cdot \Xi^{\sigma^w(x)}$ by a constant centering $\overline{m}(x) \sim \frac{\lambda x}{\lambda-1}$.

Theorem 7.2. *(i) Let* $1 < \lambda < 2$. *Then as* $x \to \infty$,

$$\tau_{x^{-1/\lambda}} \cdot \theta_{\frac{\lambda x}{\lambda-1}} \cdot \Xi^{\sigma^w(x)} \overset{(d)}{\Longrightarrow} \theta_{\lambda^2(\lambda-1)^{-1}} \cdot \int_{(0,\infty)} e_t * \Pi^{reno}_{\rho_\lambda}(dt).$$

(ii) Let $\lambda = 2$. *Then as* $x \to \infty$,

$$\tau_{(2x\log x)^{-1/2}} \cdot \theta_{\frac{\lambda x}{\lambda-1}} \cdot \Xi^{\sigma^w(x)} \overset{(d)}{\Longrightarrow} \mathcal{N}(\xi, 1),$$

where ξ *is a normal random variable with mean* 0 *and variance* 1.

(iii) Let $\lambda > 2$. *Then as* $x \to \infty$,

$$\tau_{x^{-1/2}} \cdot \theta_{\frac{\lambda x}{\lambda-1}} \cdot \Xi^{\sigma^w(x)} \overset{(d)}{\Longrightarrow} \mathcal{N}(\xi_\lambda, c_\lambda),$$

where c_λ *is of (7.7) and* ξ_λ *is a normal random variable with mean* 0 *and variance*

$$a_\lambda = \frac{2\lambda}{(\lambda-2)(\lambda-1)^3}. \tag{7.8}$$

We remark that the limiting RPDs in Theorems 7.1 and Theorem 7.2 are stable RPDs taking values in $\mathcal{P}^{id}(\mathbb{R})$ as seen in Examples 3 and 4 of the section 5.

Recall the probability measure $\overline{P}$ on $W \times \Omega$, which is defined by

$$\overline{P}(dwd\omega) = P(dw)P_0^w(d\omega).$$

We regard $\sigma^w(x) = \sigma^w(x,\omega)$ as a random variable on the probability space $(W \times \Omega, \overline{P})$. From Theorem 7.1 and 7.2 it follows immediately that

Corollary 7.3. *(i) Let $0 < \lambda < 1$. Then for any $a > 0$*

$$\lim_{x\to\infty} \overline{P}(x^{-1/\lambda}\sigma^w(x,\omega) \le a) = F_\lambda(a),$$

where F_λ is a λ-stable distribution function with the Laplace transform (5.9).

(ii) Let $1 \le \lambda < 2$. Then for any $a \in \mathbb{R}$

$$\lim_{x\to\infty} \overline{P}(x^{-1/\lambda}(\sigma^w(x,\omega) - m^w(x)) \le a) = H_\lambda(a).$$

In particular, if $1 < \lambda < 2$, for any $a \in \mathbb{R}$

$$\lim_{x\to\infty} \overline{P}(x^{-1/\lambda}(\sigma^w(x,\omega) - \frac{\lambda x}{\lambda - 1}) \le a) = G_\lambda(a),$$

where H_λ and G_λ are the λ-stable distribution functions which appear in Example 4 and Example 3 of the section 5.

(iii) Let $\lambda = 2$. Then for $a \in \mathbb{R}$

$$\lim_{x\to\infty} \overline{P}\left(2^{-1}(x\log x)^{-1/2}(\sigma^w(x,\omega) - \frac{\lambda x}{\lambda - 1}) \le a\right) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy.$$

(iv) Let $\lambda > 2$. Then for $a \in \mathbb{R}$

$$\lim_{x\to\infty} \overline{P}\left(((a_\lambda + c_\lambda)x)^{-1/2}(\sigma^w(x,\omega) - \frac{\lambda x}{\lambda - 1}) \le a\right) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy.$$

where c_λ and a_λ are given by (7.8) and (7.7).

(*Proof.*) Let $0 < \lambda < 1$. By Theorem 7.1 (i) it holds that

$$\lim_{x\to\infty} E\left(\tau_{x^{-1/\lambda}} \cdot \Xi^{\sigma^w(x)}\right) = E\left(\Xi_\lambda\right),$$

where

$$\Xi_\lambda = \int_{(0,\infty)} e_t * \Pi_{\rho_\lambda}(dt).$$

Hence by Theorem 7.1 we see that

$$\begin{aligned} \overline{P}(x^{-1/\lambda}\sigma^w(x,\omega) \le a) &= E\left(P_0^w(x^{-1/\lambda}\sigma^w(x,\omega) \le a)\right) \\ &= E\left(\tau_{x^{-1/\lambda}} \cdot \Xi^{\sigma^w(x)}\right)([0,a]) \\ &\to E\left(\Xi_\lambda\right)([0,a]), \end{aligned}$$

and by Example 3 of the section 5, the distribution of $E\left(\Xi_\lambda\right)$ coincides with F_λ, completing the proof of (i). The remaining cases can be proved in the same way, so we omit them. □

8 Proof of Theorem 7.1

In order to prove Theorem 7.1 we prepare several lemmas.

Lemma 8.1. *Let $x > 0$. (i)*

$$E_0^w\left(e^{iz\sigma^w(x)}\right) = \prod_{k=1}^{N(x,w)}\left(1-\frac{iz\left(e^{(1-iz)w'_k}-1\right)}{1-iz}\right)^{-1} \times\left(1-\frac{iz\left(e^{(1-iz)(x-w_{N(x,w)})}-1\right)}{1-iz}\right)^{-1} \tag{8.1}$$

(ii)

$$m^w(x) \equiv E_0^w(\sigma^w(x)) = \sum_{k=1}^{N(x,w)}\left(e^{w'_k}-1\right)+e^{x-w_{N(x,w)}}-1. \tag{8.2}$$

(iii) $\overline{m}(x) = E(m^w(x)) < \infty \quad (x > 0)$ *if and only if* $\lambda > 1$*. Moreover, if* $\lambda > 1$*,*

$$\overline{m}(x) = \frac{\lambda x}{\lambda - 1} + O(1) \quad (x \to \infty). \tag{8.3}$$

(*Proof.*) By (7.1) it is easy to show that if $w_n < x \leq w_{n+1}$,

$$E_{w_n}^w\left(e^{iz\sigma^w(x)}\right) = \left(1-\frac{iz\left(e^{(1-iz)(x-w_n)}-1\right)}{1-iz}\right)^{-1},$$

which and the strong Markov property yield (8.1). (ii) is trivial, and (iii) is easily verified by (8.2). □

Before proceeding to discuss convergence results we review the topology of $\mathcal{P}(\mathbb{R})$ and $\mathcal{P}(\mathcal{P}(\mathbb{R}))$. Recall that the metric d of $\mathcal{P}(\mathbb{R})$ is defined by (2.1), which is compatible with the topology of weak convergence and makes $\mathcal{P}(\mathbb{R})$ a complete separable metric space.

Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ be the one point compactification of $\mathbb{R}$. Then $\mathcal{P}(\overline{\mathbb{R}})$ is compact with respect to the weak topology. Since $\mathcal{P}(\mathbb{R})$ is regarded as a subset of $\mathcal{P}(\overline{\mathbb{R}})$ and the topology of $\mathcal{P}(\mathbb{R})$ coincides with the relative topology induced by $\mathcal{P}(\overline{\mathbb{R}})$, it holds that

$$\{A \cap \mathcal{P}(\mathbb{R}))|\, A \in \mathcal{B}(\mathcal{P}(\overline{\mathbb{R}}))\} = \mathcal{B}(\mathcal{P}(\mathbb{R})).$$

Then each $Q \in \mathcal{P}(\mathbb{R})$ is identified with $\overline{Q} \in \mathcal{P}(\overline{\mathbb{R}})$ defined by

$$\overline{Q}(A) = Q(A \cap \mathcal{P}(\mathbb{R})) \quad (A \in \mathcal{B}(\mathcal{P}(\overline{\mathbb{R}})).$$

Lemma 8.2. *Let Q_n, $Q_\infty \in \mathcal{P}(\mathcal{P}(\mathbb{R}))$, and denote their moment characteristic functions by Φ_{Q_n} and Φ_Q respectively. Then Q_n converges to Q_∞ weakly as $n \to \infty$ if and only if*

$$\lim_{n\to\infty}\Phi_{Q_n}(\boldsymbol{z}) = \Phi_{Q_\infty}(\boldsymbol{z}) \quad (\boldsymbol{z} \in \ell_0(\mathbb{N})). \tag{8.4}$$

($Proof.$) The "only if" part is trivial, since $\prod_{j=1}^{m}\widehat{\mu}(z_j)$ is a bounded continuous function on $\mathcal{P}(\mathbb{R})$. Conversely, (8.4) means

$$\lim_{n\to\infty}\int_{\mathcal{P}(\mathbb{R})}\prod_{j=1}^{m}\widehat{\mu}(z_j)Q_n(d\mu)=\int_{\mathcal{P}(\mathbb{R})}\prod_{j=1}^{m}\widehat{\mu}(z_j)Q_\infty(d\mu),$$

so that by Perseval's identity, for every $f_1, f_2, \cdots, f_m \in \mathcal{S}(\mathbb{R})$ (the totality of rapidly decreasing C^∞-functions)

$$\lim_{n\to\infty}\int_{\mathcal{P}(\mathbb{R})}\prod_{j=1}^{m}\langle\mu, f_j\rangle Q_n(d\mu)=\int_{\mathcal{P}(\mathbb{R})}\prod_{j=1}^{m}\langle\mu, f_j\rangle Q_\infty(d\mu). \tag{8.5}$$

For $f\in\mathcal{S}(\mathbb{R})$, denote by $\bar{f}$ the extension of f on $\overline{\mathbb{R}}$ with $\bar{f}(\infty)=0$. By the Stone-Weierstrass theorem, the linear hull of the family consisting of functions of μ of the form $\prod_{j=1}^{m}\langle\mu,\bar{f}_j\rangle\,;\, m\geq 1, f_j\in\mathcal{S}(\mathbb{R})$ where f_j $(1\leq j\leq m)$ are real and $\in\mathcal{S}(\mathbb{R})$, $m\geq 1$, and of constant functions is dense in $C(\mathcal{P}(\overline{\mathbb{R}}))$. Hence (8.5) implies that $\overline{Q}_n$ converges weakly to $\overline{Q}$ in $\mathcal{P}(\mathcal{P}(\overline{\mathbb{R}}))$ and $\overline{Q}_n(\mathcal{P}(\mathbb{R}))=\overline{Q}_\infty(\mathcal{P}(\mathbb{R}))=1$. Finally since the topology of $\mathcal{P}(\mathbb{R})$ coincides with the relative topology induced by $\mathcal{P}(\overline{\mathbb{R}})$,we obtain

$$Q_n \xrightarrow{(w)} Q_\infty \quad (n\to\infty).$$

□

For two sequences of random variables $\{X_n\}$ and $\{Y_n\}$ if

$$X_n = Y_n Z_n \quad (n\geq 1),$$

and Z_n converges to 1 in probability, as $n\to\infty$, we use the notation

$$X_n \overset{(p)}{\sim} Y_n \quad (n\to\infty).$$

For $x>0$ let

$$M(x,w)=\min\{n\geq 1: w_n>n\}=N(x,w)+1.$$

In the sequel we use $M(x,w)$ instead of $N(x,w)$, since $M(x,w)$ is an $\mathcal{F}_n^w$-stopping time, where $\mathcal{F}_n^w$ is the σ-algebra generated by $\{w_k\}_{1\leq k\leq n}$. The following lemma makes this replacement clear. For example, the lemma justifies to write

$$x^{-1/\lambda}m^w(x)=x^{-1/\lambda}\sum_{k=1}^{M(x,w)}\left(e^{w'_k}-1\right)+o(1).$$

Lemma 8.3. *For every* $\lambda>0$

$$\lim_{x\to\infty}x^{-1/\lambda}e^{w'_{M(x,w)}}=0 \quad \textit{in probability.} \tag{8.6}$$

In particular, if $\lambda>2$,

$$\lim_{x\to\infty}x^{-1/2}e^{w'_{M(x,w)}}=0 \quad P-a.s. \tag{8.7}$$

(*Proof.*) It is well-known in the renewal theory (eg. see (Fe), p.386, problem 10) that the distribution of $w'_{M(x,w)}$ converges as $x \to \infty$ to a gamma distribution, which yields (8.6). If $\lambda > 2$, by the Borel-Cantelli lemma we have

$$\lim_{n\to\infty} n^{-1/2} e^{w'_n} = 0 \quad P - a.s.,$$

which, combined with the fact $\lim_{x\to\infty} x^{-1} M(x,w) = \lambda$, P-a.s., yields (8.7). □

For each $x > 0$ we introduce a random counting measure $\Pi^{(x)}$ on $(0,\infty)$ and its compensated one $\tilde{\Pi}^{(x)}$ by

$$\langle \Pi^{(x)}, f \rangle = \sum_{k=1}^{M(x,w)} f(x^{-1/\lambda}(e^{w'_k} - 1)),$$

and

$$\langle \tilde{\Pi}^{(x)}, f \rangle = \sum_{k=1}^{M(x,w)} \left(f(x^{-1/\lambda}(e^{w'_k} - 1)) - E(f(x^{-1/\lambda}(e^{w'_k} - 1))) \right).$$

Since $M(x,w)$ is an $\mathcal{F}_n^w$-stopping time, it holds that

$$E\left(\langle \Pi^{(x)}, f \rangle\right) = E(M(x,w)) E\left(f(x^{-1/\lambda}(e^{w'_1} - 1))\right), \tag{8.8}$$

$$E\left(\langle \tilde{\Pi}^{(x)}, f \rangle^2\right) = E(M(x,w)) Var\left(f(x^{-1/\lambda}(e^{w'_1} - 1))\right), \tag{8.9}$$

which will be used later.

Lemma 8.4. *Let $0 < \lambda < 2$. (i) For $U_x^w = \tau_{x^{-1/\lambda}} \cdot \Xi^{\sigma^w(x)}$,*

$$\begin{aligned} \widehat{U_x^w}(z) &= E_0^w\left(e^{izx^{-1/\lambda}\sigma^w(x)}\right) \\ &\overset{(p)}{\sim} \exp - \left(\int_{(0,\infty)} \log(1 - izt) \Pi^{(x)}(dt) \right). \end{aligned} \tag{8.10}$$

(ii) For $V_x^w = \tau_{x^{-1/\lambda}} \cdot \theta_{m^w(x)} \cdot \Xi^{\sigma^w(x)}$,

$$\begin{aligned} \widehat{V_x^w}(z) &= E_0^w\left(e^{izx^{-1/\lambda}(\sigma^w(x) - m^w(x))}\right) \\ &\overset{(p)}{\sim} \exp\left(\int_{(0,\infty)} \left(\log \frac{1}{1 - izt} - izt \right) \Pi^{(x)}(dt) \right). \end{aligned} \tag{8.11}$$

(Proof.) Using Lemma 8.1 and Lemma 8.3 we see that

$$\begin{aligned}
\widehat{U_x^w}(z) &= \prod_{k=1}^{N(x,w)} \left(1 - \frac{izx^{-1/\lambda}\left(e^{(1-izx^{-1/\lambda})w'_k} - 1\right)}{1 - izx^{-1/\lambda}}\right)^{-1} \\
&\quad \times \left(1 - \frac{izx^{-1/\lambda}\left(e^{(1-izx^{-1/\lambda})(x-w_{N(x,w)})} - 1\right)}{1 - izx^{-1/\lambda}}\right)^{-1} \\
&\overset{(p)}{\sim} \prod_{k=1}^{M(x,w)} \left(1 - \frac{izx^{-1/\lambda}\left(e^{(1-izx^{-1/\lambda})w'_k} - 1\right)}{1 - izx^{-1/\lambda}}\right)^{-1} \\
&= \prod_{k=1}^{M(x,w)} \left(1 - izx^{-1/\lambda}\left(e^{w'_k} - 1\right)\right)^{-1} \prod_{k=1}^{M(x,w)} \left(1 + \theta(x, w'_k)\right)^{-1}, \\
&= \exp\left(\int_{(0,\infty)} - \log(1 - izt)\Pi^{(x)}(dt)\right) \prod_{k=1}^{M(x,w)} \left(1 + \theta(x, w'_k)\right)^{-1}
\end{aligned}$$

where

$$\theta(x, w'_k) = \frac{izx^{-1/\lambda}e^{w'_k}(1 - e^{-izx^{-1/\lambda}w'_k}) - (izx^{-1/\lambda})^2(e^{w'_k} - 1)}{(1 - izx^{-1/\lambda})(1 - izx^{-1/\lambda}(e^{w'_k} - 1))}.$$

Notice that

$$|\theta(x, w'_k)| \le |z|^2 x^{-2/\lambda}(e^{w'_k}w'_k + e^{w'_k} - 1),$$

and

$$\lim_{x\to\infty} \frac{M(x,w)}{x} = \lambda, \quad P - a.s. \tag{8.12}$$

Furthermore, it holds that for any $0 < \lambda < 2$,

$$\lim_{n\to\infty} n^{-2/\lambda} \sum_{k=1}^{n} (e^{w'_k}w'_k + e^{w'_k} - 1) = 0 \quad P - a.s.$$

Because, note that for $\lambda/2 < \alpha < \lambda \wedge 1$, $\left(e^{w'_k}w'_k + e^{w'_k} - 1\right)^\alpha$ has finite expectation, and

$$\begin{aligned}
&\left(n^{-2/\lambda} \sum_{k=1}^{n} (e^{w'_k}w'_k + e^{w'_k} - 1)\right)^\alpha \\
&\le n^{-2\alpha/\lambda} \sum_{k=1}^{n} \left(e^{w'_k}w'_k + e^{w'_k} - 1)\right)^\alpha,
\end{aligned}$$

which vanishes as $x \to \infty$ by the strong law of large numbers. Thus we have

$$\lim_{x\to\infty} \sum_{k=1}^{M(x,w)} |\theta(x, w'_k)| = 0 \quad P - a.s.,$$

which implies

$$\lim_{x\to\infty}\prod_{k=1}^{M(x,w)}(1+\theta(x,w_k'))=1 \quad P-a.s.$$

Hence we obtain (8.10), and (8.11) follows from (8.10). $\square$

Lemma 8.5. *Let $0<\lambda<2$, and let f be a measurable function defined on $(0,\infty)$.*

(i) If f satisfies that for some $\eta>0$

$$\int_0^\eta |f(t)|t^{-\lambda-1}dt<\infty, \tag{8.13}$$

then

$$\langle \Pi^{(x)}, f\rangle \overset{(d)}{\Longrightarrow} \langle \Pi_{\rho_\lambda}, f\rangle \quad (x\to\infty). \tag{8.14}$$

(ii) If f satisfies that for some $\eta>0$

$$\int_0^\eta |f(t)|^2 t^{-\lambda-1}dt<\infty, \tag{8.15}$$

then

$$\langle \tilde{\Pi}^{(x)}, f\rangle \overset{(d)}{\Longrightarrow} \langle \tilde{\Pi}_{\rho_\lambda}, f\rangle \quad (x\to\infty). \tag{8.16}$$

(*Proof.*) (i) For the proof we may assume that f is nonnegative. For any $\xi>0$ and $\varepsilon>0$,

$$\begin{aligned}
&E\left(e^{-\xi\langle\Pi^{(x)},f\rangle}\right)\\
\le\ & P(M(x,w)\le(1-\varepsilon)\lambda x)+E\left(\prod_{k=1}^{[(1-\varepsilon)\lambda x]} e^{-\xi f(x^{-1/\lambda}(e^{w_k'}-1))}\right)\\
=\ & P(M(x,w)\le(1-\varepsilon)\lambda x)+\left(\int_0^\infty e^{-\xi f(x^{-1/\lambda}(e^u-1))}\lambda e^{-\lambda u}du\right)^{[(1-\varepsilon)\lambda x]}.
\end{aligned}$$

The first term vanishes as $x\to\infty$, and by (8.13) the second term turns to

$$\begin{aligned}
&\left(1-\frac{1}{x}\int_0^\infty\left(1-e^{-\xi f(t)}\right)\lambda(t+x^{-1/\lambda})^{-\lambda-1}dt\right)^{[(1-\varepsilon)\lambda x]}\\
&\overset{(x\to\infty)}{\to}\ \exp-(1-\varepsilon)\int_0^\infty(1-e^{-\xi f(t)})\rho_\lambda(dt).
\end{aligned}$$

On the other hand

$$\begin{aligned}
&E\left(e^{-\xi\langle\Pi^{(x)},f\rangle}\right)\\
\ge\ & E\left(\prod_{k=1}^{[(1+\varepsilon)\lambda x]} e^{-\xi f(x^{-1/\lambda}(e^{w_k'}-1))}\ :\ M(x,w)\le(1+\varepsilon)\lambda x\right)\\
\ge\ & E\left(\prod_{k=1}^{[(1+\varepsilon)\lambda x]} e^{-\xi f(x^{-1/\lambda}(e^{w_k'}-1))}\right)-P\left(M(x,w)>(1+\varepsilon)\lambda x\right)\\
\overset{(x\to\infty)}{\to}\ & \exp-(1+\varepsilon)\int_0^\infty(1-e^{-\xi f(t)})\rho_\lambda(dt).
\end{aligned}$$

Thus we obtain

$$
\begin{aligned}
\lim_{x\to\infty} E\left(e^{-\xi\langle\Pi^{(x)},f\rangle}\right) &= \exp - \int_0^\infty (1-e^{-\xi f(t)})\rho_\lambda(dt) \\
&= E\left(e^{-\xi\langle\Pi_{\rho_\lambda},f\rangle}\right),
\end{aligned}
$$

which yields (8.14). Next suppose that f satisfies (8.15). From (i) it follows that for $\varepsilon > 0$,

$$
\begin{aligned}
&\langle\tilde{\Pi}^{(x)}, I_{(\varepsilon,\infty)}f\rangle \\
&= \langle\Pi^{(x)}, I_{(\varepsilon,\infty)}f\rangle - \frac{M(x,w)}{x}\int_{(\varepsilon,\infty)} f(t)\lambda(x^{-1/\lambda}+t)^{-\lambda-1}dt \\
&\overset{(d)}{\Longrightarrow} \langle\Pi_{\rho_\lambda}, I_{(\varepsilon,\infty)}f\rangle - \int_{(\varepsilon,\infty)} f(t)\rho_\lambda(dt) \\
&= \langle\tilde{\Pi}_{\rho_\lambda}, I_{(\varepsilon,\infty)}f\rangle.
\end{aligned}
$$

Moreover using (8.9) and (8.15) one can easily verify

$$
\lim_{\varepsilon\to 0+}\sup_{x\geq 1} E\left(\langle\tilde{\Pi}^{(x)}, I_{(0,\varepsilon]}f\rangle^2\right) = 0,
$$

so that we obtain

$$
\begin{aligned}
\lim_{x\to\infty} E\left(e^{iz\langle\tilde{\Pi}^{(x)},f\rangle}\right) &= \lim_{\varepsilon\to 0+}\lim_{x\to\infty} E\left(e^{iz\langle\tilde{\Pi}^{(x)},I_{(\varepsilon,\infty)}f\rangle}\right) \\
&= \lim_{\varepsilon\to 0+} E\left(e^{iz\langle\tilde{\Pi}_{\rho_\lambda},I_{(\varepsilon,\infty)}f\rangle}\right) \\
&= E\left(e^{iz\langle\tilde{\Pi}_{\rho_\lambda},f\rangle}\right),
\end{aligned}
$$

completing the proof of (ii). □

We use the following facts, which are verified by elementary calculations.

Lemma 8.6. *Let $\{w'_k\}$ be i.i.d. random variables with exponential distribution of patameter* $\lambda = 2$.

(i)

$$
\lim_{n\to\infty}(n\log n)^{-1}\sum_{k=1}^n e^{2w'_k} = 1 \quad \textit{in probability.}
$$

(ii)

$$
\lim_{n\to\infty}(n\log n)^{-3/2}\sum_{k=1}^n e^{3w'_k} = 0 \quad \textit{in probability.}
$$

(iii)

$$
(n\log n)^{-1/2}\sum_{k=1}^n \left(e^{w'_k}-1-2w'_k\right) \overset{(d)}{\Longrightarrow} \mathcal{N}(0,1).
$$

(*Proof of Theorem 7.1.*)

(i) Let $z_1, \cdots, z_n \in \mathbb{R}$. Noting that $\log(1 - izt)$ satisfies (8.13), apply Lemma 8.5 and Lemma 8.4. Then for $U_x^w = \tau_{x^{-1/\lambda}} \cdot \Xi^{\sigma^w(x)}$

$$\{\widehat{U_x^w}(z_j)\}_{1\le j\le n} \overset{(p)}{\sim} \left\{ \exp - \left(\int_{(0,\infty)} \log(1 - iz_j t)\Pi^{(x)}(dt) \right) \right\}_{1\le j\le n}$$

$$\overset{(d)}{\Longrightarrow} \left\{ \exp - \left(\int_{(0,\infty)} \log(1 - iz_j t)\Pi_{\rho_\lambda}(dt) \right) \right\}_{1\le j\le n},$$

which coincides with (5.4) of Example 3. Thus we obtain (7.3) by Lemma 8.2.

(ii) Let $1 \le \lambda < 2$. Noting that $\log \frac{1}{1-izt} - izt$ satisfies (8.13), by Lemma 8.4 (ii) we see

$$\{\widehat{V_x^w}(z_j)\}_{1\le j\le n} \overset{(p)}{\sim} \left\{ \exp \left(\int_{(0,\infty)} \left(\log \frac{1}{1 - iz_j t} - izt \right) \Pi^{(x)}(dt) \right) \right\}_{1\le j\le n}$$

$$\overset{(d)}{\Longrightarrow} \left\{ \exp \left(\int_{(0,\infty)} \left(\log \frac{1}{1 - iz_j t} - izt \right) \Pi_{\rho_\lambda}(dt) \right) \right\}_{1\le j\le n},$$

which coincides with (5.16) of Example 4. Thus we obtain (7.4).

(iii) Let $\lambda = 2$. For $V_x^w = \tau_{\varepsilon(x)} \cdot \theta_{m^w(x)} \cdot \Xi^{\sigma^w(x)}$ with $\varepsilon(x) = (2x \log x)^{-1/2}$ one can use the same argument as Lemma 8.4 to get

$$\begin{aligned}
\widehat{V_x^w}(z) &\overset{(p)}{\sim} \prod_{j=1}^{M(x,w)} \left(1 - iz\varepsilon(x)\left(e^{w'_k} - 1\right)\right)^{-1} \prod_{k=1}^{M(x,w)} e^{-iz\varepsilon(x)(e^{w'_k}-1)} \\
&= \exp \sum_{k=1}^{M(x,w)} - \left(\log(1 - iz\varepsilon(x)(e^{w'_k} - 1)) + iz\varepsilon(x)(e^{w'_k} - 1)\right) \\
&= \exp \sum_{k=1}^{M(x,w)} \frac{(iz\varepsilon(x))^2}{2}(e^{w'_k} - 1)^2 + O\left(\sum_{k=1}^{M(x,w)} \varepsilon(x)^3(e^{w'_k} - 1)^3 \right).
\end{aligned}$$

Hence, using Lemma 8.6 (i), (ii) and (8.12) we obtain

$$\lim_{n\to\infty} \widehat{V_x^w}(z) = e^{-z^2/2} \quad \text{in probability,}$$

which completes the proof of (iii).

(iv) Let $\lambda > 2$. Using (8.7), we see that for $V_x^w = \tau_{x^{-1/2}} \cdot \theta_{m^w(x)} \cdot \Xi^{\sigma^w(x)}$

$$\begin{aligned}
\widehat{V_x^w}(z) &\sim \prod_{k=1}^{M(x,w)} \left(1 - \frac{izx^{-1/2}\left(e^{(1-izx^{-1/2})w'_k} - 1\right)}{1 - izx^{-1/2}}\right)^{-1} e^{-izx^{-1/2}m^w(x)} \\
&= \prod_{k=1}^{M(x,w)} \left\{ \left(1 - izx^{-1/2}(e^{w'_k} - 1)\right) e^{izx^{-1/2}(e^{w'_k}-1)} \right\}^{-1} \\
&\quad \times \prod_{k=1}^{M(x,w)} \left(1 + \eta(x, w'_k)\right)^{-1},
\end{aligned}$$

where

$$\eta(x, w'_k) = \frac{ix^{-1/2}ze^{w'_k}(1 - e^{-izx^{-1/2}w'_k}) - (izx^{-1/2})^2(e^{w'_k} - 1)}{(1 - ix^{-1/2}z)(1 - izx^{-1/2}(e^{w'_k} - 1))}.$$

Applying the law of large numbers together with (8.7) we obtain

$$\lim_{x\to\infty} \sum_{k=1}^{M(x,w)} \eta(x, w'_k) = z^2\lambda E\left(w_1e^{w_1} + e^{w_1} - 1\right). \tag{8.17}$$

Furthermore, it is easily verified that P-a.s.

$$\lim_{x\to\infty} \max_{1\le k\le M(x,w)} |\eta(x, w'_k)| \le \lim_{x\to\infty} \frac{1}{x} \max_{1\le k\le M(x,w)} (1 + w'_k)e^{w'_k} = 0,$$

so that by (8.17)

$$\lim_{x\to\infty} \prod_{k=1}^{M(x,w)} \left(1 + \eta(x, w'_k)\right) = \exp\left(z^2\lambda E\left(w_1e^{w_1} + e^{w_1} - 1\right)\right).$$

The same argument is applied to get

$$\lim_{x\to\infty} \prod_{k=1}^{M(x,w)} \left\{\left(1 - izx^{-1/2}(e^{w'_k} - 1)\right) e^{izx^{-1/2}(e^{w'_k}-1)}\right\}^{-1}$$
$$= \exp\left(-\frac{z^2}{2}\lambda E\left((e^{w_1} - 1)^2\right)\right) \quad P - a.s.$$

Thus we obtain that for each $z \in \mathbb{R}$, P-a.s.

$$\begin{aligned}\lim_{x\to\infty} \widehat{V_x^w}(z) &= \exp -\frac{z^2}{2}\lambda E\left(e^{2w_1} - 1 + 2w_1e^{w_1}\right) \\ &= \exp -\frac{\lambda z^2}{(\lambda - 2)(\lambda - 1)^2},\end{aligned}$$

which completes the proof of (iv). □

9 Proof of Theorem 7.2

Theorem 7.2 will be reduced to Theorem 7.1 and the following Lemma.

Lemma 9.1. *(i) If* $1 < \lambda < 2$,

$$x^{-1/\lambda}(m^w(x) - \overline{m}(x)) \stackrel{(d)}{\Longrightarrow} \int_{(0,\infty)} t\tilde{\Pi}_{\rho_\lambda}(dt) \quad (x \to \infty).$$

(ii) If $\lambda = 2$

$$(2x\log x)^{-1/2}(m^w(x) - \overline{m}(x)) \stackrel{(d)}{\Longrightarrow} \mathcal{N}(0, 1) \quad (x \to \infty).$$

(iii) If $\lambda > 2$,

$$x^{-1/2}(m^w(x) - \overline{m}(x)) \stackrel{(d)}{\Longrightarrow} \mathcal{N}(0, a_\lambda) \quad (x \to \infty),$$

where a_λ *is of (7.8).*

(*Proof.*) (i) Using Lemma 8.1 and Lemma 8.5 we obtain

$$\begin{aligned} x^{-1/\lambda}(m^w(x)-\overline{m}(x)) &\sim x^{-1/\lambda}\sum_{k=1}^{M(x,w)}(e^{w'_k}-1)-\frac{1}{\lambda-1}x^{-1/\lambda}E(M(x,w)) \\ &= \int_{(0,\infty)} t\tilde{\Pi}^{(x)}(dt)-\frac{1}{\lambda-1}x^{-1/\lambda}\left(M(x,w)-E(M(x,w))\right) \\ &\overset{(d)}{\Longrightarrow} \int_{(0,\infty)} t\tilde{\Pi}_{\rho_\lambda}(dt) \quad (x\to\infty), \end{aligned}$$

completing the proof of (i).

For the proof of (ii) and (iii) let $\varepsilon(x)=(2x\log x)^{-1/2}$ for $\lambda=2$ and $\varepsilon(x)=x^{-1/2}$ for $\lambda>2$. Use

$$E(M(x,w))=E(N(x,w))+1=\lambda x+1,$$

and Lemma 8.3, then

$$\begin{aligned} &\varepsilon(x)(m^w(x)-\overline{m}(x)) \\ &\sim \varepsilon(x)\sum_{k=1}^{M(x,w)}(e^{w'_k}-1)-\frac{1}{\lambda-1}\varepsilon(x)E(M(x,w)) \\ &\sim \varepsilon(x)\sum_{k=1}^{M(x,w)}(e^{w'_k}-1-\frac{\lambda}{\lambda-1}w'_k)+\varepsilon(x)\frac{\lambda}{\lambda-1}(w_{M(x,w)}-x) \\ &\sim \varepsilon(x)\sum_{k=1}^{M(x,w)}(e^{w'_k}-1-\frac{\lambda}{\lambda-1}w'_k). \end{aligned} \tag{9.1}$$

Let $\varphi(z)$ be the characteristic function of $e^{w'_k}-1-\frac{\lambda}{\lambda-1}w'_k$. If $\lambda=2$, by Lemma 8.6 it holds that

$$\lim_{x\to\infty}\varphi(\varepsilon(x)z)^{[x]}=\exp-\frac{z^2}{4}, \tag{9.2}$$

and if $\lambda>2$, by the classical CLT,

$$\lim_{x\to\infty}\varphi(\varepsilon(x)z)^{[x]}=\exp-\frac{a_\lambda}{2\lambda}z^2. \tag{9.3}$$

For $n\geq 1$ we set

$$Y_n=\varphi(\varepsilon(x)z)^{-n}\exp\left(iz\varepsilon(x)\sum_{k=1}^{n}(e^{w'_k}-1-\frac{\lambda}{\lambda-1}w'_k)\right).$$

Then Y_n is an $\mathcal{F}^w_n$-martingale with mean 1. Since $M(x,w)$ is an $\mathcal{F}^w_n$-stopping time for each $x>0$ and the optional sampling theorem is applicable, we have

$$E\left(\varphi(\varepsilon(x)z)^{-M(x,w)}\exp\left(iz\varepsilon(x)\sum_{k=1}^{M(x,w)}(e^{w'_k}-1-\frac{\lambda}{\lambda-1}w'_k)\right)\right)=1. \tag{9.4}$$

By (9.2), (9.3) and the law of large number,

$$\lim_{x\to\infty} \varphi(\varepsilon(x)z)^{M(x,w)} = \exp -\frac{a_\lambda}{2}z^2 \quad P-a.s., \tag{9.5}$$

where $a_2 = 1$ and a_λ is of (7.8) for $\lambda > 2$. Since $N(x,w) = M(x,w) - 1$ has the Poisson distribution of parameter λx, using (9.2) and (9.3) we can verify that $\varphi(\varepsilon(x)z)^{-M(x,w)}$ is uniformly integrable as $x \to \infty$. Hence by (9.4) and (9.5) we obtain

$$E\left(\exp\left(iz\varepsilon(x)\sum_{k=1}^{M(x,w)}(e^{w'_k} - 1 - \frac{\lambda}{\lambda-1}w'_k)\right)\right) = \exp -\frac{a_\lambda}{2}z^2. \tag{9.6}$$

Thus (9.1) and (9.6) conclude (ii) and (iii). □

(*Proof of Therem 7.2.*)

(i) Setting $U_x^w \equiv \tau_{x^{-1/\lambda}} \cdot \theta_{\overline{m}(x)} \cdot \Xi^{\sigma^w(x)}$ and $V_x^w \equiv \theta_{x^{-1/\lambda}} \cdot \theta_{m^w(x)} \cdot \Xi^{\sigma^w(x)}$, we have

$$U_x^w = \theta_{-x^{-1/\lambda}(m^w(x)-\overline{m}(x))} \cdot V_x^w = V_x^w * \delta_{x^{-1/\lambda}(m^w(x)-\overline{m}(x))}.$$

By a careful observation of the proofs of Theorem 7.1 and Lemma 9.1 we obtain

$$(x^{-1/\lambda}(m^w(x) - \overline{m}(x)), V_x^w) \stackrel{(d)}{\Longrightarrow} \left(\int_{(0,\infty)} t\tilde{\Pi}_{\rho_\lambda}(dt), \int_{(0,\infty)} (\theta_t \cdot e_t) * \Pi_{\rho_\lambda}(dt)\right).$$

Hence it holds that as $x \to \infty$

$$U_x^w \stackrel{(d)}{\Longrightarrow} \left(\int_{(0,\infty)} (\theta_t \cdot e_t) * \Pi_{\rho_\lambda}(dt)\right) * \ \delta_{\left\{\int_{(0,\infty)} t\tilde{\Pi}_{\rho_\lambda}(dt)\right\}}.$$

Moreover, by (5.19) the limit is identified with

$$\int_{(0,\infty)} e_t * \Pi_{\rho_\lambda}^{reno}(dt) * \ \delta_{\{-\lambda^2(\lambda-1)^{-1}\}},$$

which completes the proof of (i). (ii) and (iii) follows immediately from Theorem 7.1 and Lemma 9.1. □

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