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# ON THE APPROXIMATE SOLUTIONS OF THE STRATONOVITCH EQUATION

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**Abstract** We present new methods for proving the convergence of the classical approximations of the Stratonovitch equation. We especially make use of the fractional Liouville-valued Sobolev space $W^{r,p}(\mathcal{J}_{\alpha,p})$. We then obtain a support theorem for the capacity $c_{r,p}$.

**Keywords:** Stratonovitch equations, Kolmogorov lemma, quasi-sure analysis .

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**Introduction**

This paper is a contribution to the study of the approximations of solutions of the Stratonovitch equation

$$\text{(S)} \qquad X_t = x_0 + \int_0^t \sigma(X_s)\circ dW_s + \int_0^t \beta(X_s)\, ds$$

Many authors and especially Ikeda-Watanabe [8] have studied this problem by means of piecewise linear approximations of the Brownian motion. Here we introduce a method which simplifies and shortens the calculations in three ways.
a) We use the notion of (strong) approximate solution of (S), which eliminates the need to have simultaneously the approximate solution and the exact solution in the calculations.
b) We use the Liouville space $\mathcal{J}_{\alpha,p}$, where it turns out that the calculations are simpler even than with uniform convergence.
The main point is the isomorphism $\mathcal{J}_{\alpha,p}(L^p) \approx L^p(\mathcal{J}_{\alpha,p})$. Moreover this isomorphism is a sharpening of the Kolmogorov lemma (*cf.* [5, 6]).
c) With the classical regularity conditions on $\sigma$ and $\beta$, we prove convergence of approximate solutions in each space $W^{r,p}(\mathcal{J}_{\alpha,p})$ for suitable values of $\alpha$ and $p$. Without using truncation property, this improves some results of [8].
d) The $p$-admissibility (*cf.* [4]) of the vector-valued Sobolev space $W^{r,p}(\mathcal{J}_{\alpha,p})$ allows us to obtain easily convergence in the space $\mathcal{L}^1(\Omega, c_{r,p}, \mathcal{J}_{\alpha,p})$ which is the natural space of $\mathcal{J}_{\alpha,p}$-valued quasi-continuous functions on the Wiener space $\Omega$ (*cf.* Cor.4 below). This is a sharpening of the preceding known results ([3, 8, 10, 12]).

As a corollary we not only see that the image measure $X(\mu)$ is carried by the closure of the skeleton $X(H)$ in the space $\mathcal{J}_{\alpha,p}$, but that this is also true for the image measure $X(\xi)$ for every measure $\xi$ majorized by the capacity $c_{r,p}$.

In fact, in the same way as for the Hölder support theorem (*cf.* [1, 2, 3, 7, 9, 10, 11, 14]), we obtain a support theorem for capacity : the support of the image capacity $X(c_{r,p})$ is exactly the closure of the skeleton.

**I. Preliminaries**

Let $f : [0,1] \to \mathbb{R}$ a Borel function. For $0 < \alpha \leq 1$ the Liouville integral is

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt = \frac{1}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} f(x-t)\, dt$$

Recall that $I^\alpha(L^p(dx)) \subset L^p(dx)$ and that $I^\alpha$ is one to one (*cf.* [5, 13]). The range $\mathcal{J}_{\alpha,p} = I^\alpha(L^p)$ is a separable Banach space under the norm

$$N_{\alpha,p}(I^\alpha f) = N_p(f)$$

where $N_p$ stands for the $L^p$-norm. Denote $\mathcal{H}_\alpha$ the space of $\alpha$-Hölder continuous functions vanishing at 0 with its natural norm. This is not a separable space.

Nevertheless, for $\alpha > 1/p \geq 0$ and $\beta > \gamma \geq 0$, we have the following inclusions (*cf.* [5, 6, 13])

$$\mathcal{J}_{\alpha,p} \subset \mathcal{H}_{\alpha-1/p} \quad \& \quad \mathcal{H}_\beta \subset \mathcal{J}_{\gamma,\infty}$$

These definitions and these inclusions extend to the case of $B$-valued functions where $B$ is a separable Banach space endowed with norm $|.|$ (*cf.* [5, 6]).

Particularly, taking $B = L^p(\Omega, \mu)$ where $\mu$ is a measure, we get the Kolmogorov theorem:

if $(X_t)_{t\in[0,1]}$ is a $\mathbb{R}^m$-valued process satisfying $N_p(X_t - X_s) \leq c|t-s|^\alpha$, then for $\alpha > \beta > 1/p$, this process has a modification with $(\beta - 1/p)$-Hölder continuous trajectories. Indeed, it suffices to point out that $X - X_0$ belongs to the space $\mathcal{H}_\alpha(L^p) \subset \mathcal{J}_{\beta,p}(L^p) \approx L^p(\mathcal{J}_{\beta,p}) \subset L^p(\mathcal{H}_{\beta-1/p})$. Observe that if $Y$ is a process, and $Y_\cdot(\omega) = I^\beta(Z_\cdot(\omega))$ then the norm of $Y$ in both spaces $\mathcal{J}_{\beta,p}(L^p)$ and $L^p(\mathcal{J}_{\beta,p})$ is worth $[\mathbb{E}\int |Z_t|^p\, dt]^{1/p}$.

**1 Proposition:** *(The Kolmogorov-Ascoli lemma) Let $X^n \in \mathcal{H}_\alpha(L^p)$ with $p \in ]1, +\infty[$ a sequence of processes. Assume that $N_p(X^n_t - X^n_s) \leq c|t-s|^\alpha$ and that $\operatorname*{Lim}_{n\to\infty} N_p(X_t - X^n_t) = 0$ for every $t$. Then $X^n$ converges to $X$ in the space $L^p(\mathcal{J}_{\beta,p}) \subset L^p(\mathcal{H}_{\beta-1/p})$ ($\alpha > \beta > 1/p > 0$).*

Proof : First it is easily seen that $N_p(X_t - X^n_t)$ converges to 0 uniformly with respect to $t$ as $n \to \infty$. Take $\alpha'$ such that $\alpha > \alpha' > \beta > 1/p$ and $\eta > 0$. We get

$$\frac{N_p(X_t - X^n_t - X_s + X^n_s)}{|t-s|^{\alpha'}} \leq \frac{\varepsilon_n}{\eta^{\alpha'}}$$

for $|t-s| \geq \eta$, and

$$\frac{N_p(X_t - X^n_t - X_s + X^n_s)}{|t-s|^{\alpha'}} \leq 2c\eta^{\alpha-\alpha'}$$

for $|t-s| \leq \eta$. That is

$$\operatorname*{Lim}_{n\to\infty} \operatorname*{Sup}_{s,t} \frac{N_p(X_t - X^n_t - X_s + X^n_s)}{|t-s|^{\alpha'}} = 0$$

Hence, convergence holds in the space $\mathcal{H}_{\alpha'}(L^p) \subset \mathcal{J}_{\beta,p}(L^p)$, and we are done.

**2 Remarks:** a) One can only assume that $\operatorname*{Lim}_{n\to\infty} N_p(X_t - X^n_t) = 0$ for every $t$ in a dense subset $D \subset [0,1]$.

b) We can prove more precisely the estimate $\|X - X^n\|_{\mathcal{H}_{\alpha'}(L^p)} \leq K\varepsilon_n^{1-\alpha'/\alpha}$. This gives a criterion for the convergence of the series $\Sigma_n(X - X^n)$.

Now assume that $(\Omega, \mu)$ is a Gaussian vector space, and let $W^{r,p}(\Omega, \mu)$ be the $(r,p)$ Sobolev space endowed with the norm $\|f\|_{r,p} = N_p\left((I-L)^{r/2} f\right)$ where $L$ is the Ornstein-Uhlenbeck operator. Recall that we have the isomorphism

$W^{r,p}(\Omega, \mathcal{J}_{\beta,p}) \approx U^r(L^p(\Omega, \mathcal{J}_{\beta,p}))$ where $U = (I-L)^{-1/2}$ according to [4], th.25 ($p$-admissibility of the space $\mathcal{J}_{\beta,p}$ which is a closed subspace of an $L^p$-space). In view of the above proposition, we obtain

**3 Proposition:** *(The Sobolev-Kolmogorov-Ascoli lemma) For $p \in ]1, +\infty[$ and $r \in ]0, +\infty[$, we have $\mathcal{J}_{\alpha,p}(W^{r,p}(\Omega, \mu)) \approx W^{r,p}(\Omega, \mu, \mathcal{J}_{\alpha,p})$ as above. Moreover, let $X^n \in \mathcal{H}_\alpha(W^{r,p})$ a sequence of processes. Assume that $\|X^n_t - X^n_s\|_{r,p} \le c|t-s|^\alpha$ and that $\operatorname*{Lim}_{n\to\infty} \|X_t - X^n_t\|_{r,p} = 0$ for every $t$. Then $X^n$ converges in the space $W^{r,p}(\mathcal{J}_{\beta,p})$ ($\alpha > \beta > 1/p$).*

Proof : As above, if $Y$ is a process, and $Y_.(\omega) = I^\beta(Z_.(\omega))$ then the norm of $Y$ in both spaces $\mathcal{J}_{\beta,p}(W^{r,p})$, and $W^{r,p}(\mathcal{J}_{\beta,p})$ is worth $[\mathbb{E}\int |(I-L)^{r/2} Z_t(\omega)|^p\, dt]^{1/p}$. The first isomorphism is obvious (*cf.* [5]). Put $Y^n_t = (I-L)^{r/2} X^n_t$ and apply the previous proposition to $X^n$ and $Y^n$.

**4 Corollary:** *Under the same conditions, the process $X^n$ converges to $X$ in the space $\mathcal{L}^1(\Omega, c_{r,p}, \mathcal{J}_{\beta,p})$.*

Proof : Recall (*cf.* [4]) that $\mathcal{L}^1(\Omega, c_{r,p}, \mathcal{J}_{\beta,p})$ is the functional completion of $\mathcal{J}_{\beta,p}$-valued bounded continuous functions with the norm

$$c_{r,p}(\varphi) = \operatorname{Inf}\,\{N_p(f) \;/\; f(\omega) \ge N_{\beta,p}(\varphi(\omega))\}$$

The results follows from the inclusion $U^r(L^p(\Omega, \mathcal{J}_{\beta,p})) \subset \mathcal{L}^1(\Omega, c_{r,p}, \mathcal{J}_{\beta,p})$ (*cf.* [4]).

## II The Stratonovitch equation.

Now let $\Omega = \mathcal{C}([0,1], \mathbb{R}^l)$ endowed with the Wiener mesure $\mu$. Let

$$\text{(S)} \qquad X_t = x_0 + \int_0^t \sigma(X_s) \circ dW_s + \int_0^t \beta(X_s)\, ds$$

a Stratonovitch SDE. In this formula, $W_t$ is the $\ell$-dimensional Brownian motion, $\circ dW_s$ stands for the Stratonovitch differential, $\sigma(x)$ is an $(m,\ell)$-matrix, $\beta(x)$ an $(m,1)$-column, $\sigma$ and $\beta$ are Lipschitz, and $x_0 \in \mathbb{R}^m$.

If $X$ is a Borel process, we denote $\widehat{X}$ its predictable projection. Note that we have $\widehat{X}_t = \mathbb{E}(X_t \mid \mathcal{F}_t)$ for every $t \in [0,1]$.

Let $\varepsilon > 0$, we say that a Borel process $X$ is an $\varepsilon$-approximate solution in $L^p$ of (S) if we have

$$N_p\left(X_t - x_0 - \int_0^t \sigma(\widehat{X}_s)\, dW_s - \frac{1}{2}\int_0^t \operatorname{Tr}\varphi(\widehat{X}_s)\, ds - \int_0^t \beta(\widehat{X}_s)\, ds\right) \le \varepsilon$$

for every $t \in [0,1]$. In this formula, the stochastic integral is to be taken in the Ito sense. Moreover $\varphi = \sigma' \cdot \sigma$ is the contracted tensor product $\varphi^i_{j,\ell} = \Sigma_k(\partial_k \sigma^i_j)\sigma^k_\ell$ and $\operatorname{Tr}\varphi$ stands for the convenient vector-valued trace $\Sigma_{j,k}(\partial_k \sigma^i_j)\sigma^j_k$.

In fact we will suppose in the following that $\beta = 0$. Indeed, the case $\beta \neq 0$ does not bring any other difficulty.

**5 Proposition:** *Let $\varepsilon_n$ be a sequence tending to 0, and let $X^n$ be a sequence of $\varepsilon_n$-approximate solutions in $L^p$. Assume that $\sigma$ and $\varphi$ are Lipschitz. Then, $X_t^n$ converges in $L^p$ towards the solution of (S).*
*In addition suppose that we have*

$$N_p(X_t^n - X_s^n) \leq K\sqrt{t-s}$$

*for every $0 \leq s \leq t \leq 1$ then $X^n$ converges in the space $L^p(\mathcal{J}_{\alpha,p})$ for $1/p < \alpha < \frac{1}{2}$.*
With this additional condition, we say that $X_n$ is a sequence of *strong* $\varepsilon_n$-approximate solutions.

Proof : Burkholder's inequality gives

$$N_p\left(X_t^n - X_t^m\right)^2 \leq K \int_0^t N_p\left(X_s^n - X_s^m\right)^2 \, ds + K(\varepsilon_n^2 + \varepsilon_m^2)$$

so that by Gronwall's lemma we have

$$N_p\left(X_t^n - X_t^m\right) \leq K'(\varepsilon_n + \varepsilon_m)$$

Now under the additional hypothesis, in view of the Kolmogorov-Ascoli lemma, the convergence holds in the space $L^p(\mathcal{J}_{\alpha,p})$ for $1/p < \alpha < 1/2$.

**Searching approximate solutions**

Now the problem is to find a sequence of strong $\varepsilon_n$-approximate solutions of (S) with $\varepsilon_n \to 0$.

Consider a partition $\pi = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ of $[0,1]$. Put $\delta_i = t_{i+1} - t_i$, $\delta = \mathrm{Sup}_i\, \delta_i$, $\Delta W_i = W_{t_{i+1}} - W_{t_i}$, and $\widetilde{t} = t_i$ for $t \in [t_i, t_{i+1}[$. If $f$ is a function, we define $\widetilde{f}(t) = f(\widetilde{t})$.
Let $W_t^\pi$ be the linear interpolation defined by

$$W_t^\pi = W_{t_i}(\omega) + (t - t_i)\frac{\Delta W_i}{\delta_i}$$

for $t \in [t_i, t_{i+1}[$.
Note $X_t$ the unique solution of the ODE

$$X_t = x_0 + \int_0^t \sigma(X_s)\, dW_s^\pi$$

For $t \in \pi$ we also have

$$(S^\pi) \qquad X_t = x_0 + Z_t + \int_0^t \left[\sigma(X_s) - \sigma(\widetilde{X}_s)\right] \, dW_s^\pi$$

with the martingale

$$Z_t = \int_0^t \sigma(\widetilde{X}_s)\, dW_s$$

We remark that $X$ is not an adapted process, we only have $X_t \in \mathcal{F}_t$ for $t \in \pi$ so that $\widetilde{X}$ is an adapted process.
For $t \in [t_i, t_{i+1}[$ we have

$$\frac{d}{dt}\sigma(X_t) = \sigma'(X_t)\cdot\sigma(X_t)\frac{\Delta W_i}{\delta_i} \qquad \Rightarrow \qquad \left|\frac{d}{dt}\sigma(X_t)\right| \le k|\sigma(X_t)|\frac{|\Delta W_i|}{\delta_i}$$

so that we obtain the following inequalities

$$(1) \qquad \big|\sigma(X_t)\big| \le |\sigma(X_{t_i})|\, \mathrm{e}^{k|\Delta W_i|}$$

$$(2) \qquad |X_t - X_{t_i}| \le |\sigma(X_{t_i})||\Delta W_i|\, \mathrm{e}^{k|\Delta W_i|}$$

$$(3) \qquad |X_{t_{i+1}} - X_{t_i} - Z_{t_{i+1}} + Z_{t_i}| \le k|\sigma(X_{t_i})||\Delta W_i|^2\, \mathrm{e}^{k|\Delta W_i|}$$

The next lemma proves that if as $\delta \to 0$ $X$ is an approximate solution of $(S)$, then it defines a strong approximate solution.

**6 Lemma:** *If $\sigma$ is Lipschitz, and $p > 1$, there exists a constant $K$ such that $N_p(X_t - X_s) \le K|\sigma(x_0)|\sqrt{t-s}$, for every $s, t \in [0,1]$ and every $\pi$.*
Proof : First, for $a, b \in \pi$ we get from (3)

$$|X_b - X_a - Z_b + Z_a| \le k \sum_{a \le t_i < b} |\sigma(X_{t_i})||\Delta W_i|^2\, \mathrm{e}^{k|\Delta W_i|}$$

$$N_p(X_b - X_a - Z_b + Z_a) \le k' \sum_{a \le t_i < b} N_p(\sigma(X_{t_i}))\delta_i = k' \int_a^b N_p(\sigma(\widetilde{X}_s))\, ds$$

By Burkholder's and Cauchy-Schwarz inequalities

$$N_p(X_b - X_a)^2 \le K_1 \int_a^b N_p(\sigma(\widetilde{X}_s))^2\, ds$$

and by Gronwall's lemma, as $\sigma$ is Lipschitz

$$N_p(X_b - X_a) \le K_2 N_p(\sigma(X_a))\sqrt{b-a} \le K_3|\sigma(x_0)|\sqrt{b-a}$$

In view of (2) this last inequality extends to every $a, b \in [0,1]$, with a constant $K$ independent of $\pi$.

**7 Lemma:** *If $\sigma$ and $\varphi$ are Lipschitz, we have for $t \in \pi$*

$$N_p\left(X_t - x_0 - Z_t - \frac{1}{2}\sum_{t_i<t}\varphi(X_{t_i})\cdot(\Delta W_i)^{(2)}\right) \leq K|\sigma(x_0)|t\sqrt{\delta}$$

*where the symbol $\varphi(X_{t_i})\cdot(\Delta W_i)^{(2)}$ stands for $\sum_{k,\ell}\varphi^j_{k,\ell}(X_{t_i})\Delta W^k_i\Delta W^\ell_i$*

Proof : First by the fundamental theorem of calculus we get

$$X_{t_{i+1}} - X_{t_i} - \sigma(X_{t_i}).\Delta W_i - \frac{1}{2}\varphi(X_{t_i})\cdot(\Delta W_i)^{(2)} = \int_{t_i}^{t_{i+1}}(t_{i+1}-s)\left[\varphi(X_s) - \varphi(X_{t_i})\right]\cdot(\Delta W_i)^{(2)}\frac{ds}{\delta_i^2}$$

$$\left|X_{t_{i+1}} - X_{t_i} - \sigma(X_{t_i})\cdot\Delta W_i - \frac{1}{2}\varphi(X_{t_i})\cdot(\Delta W_i)^{(2)}\right| \leq K|\sigma(X_{t_i})||\Delta W_i|^3\,\mathrm{e}^{k|\Delta W_i|}$$

$$N_p\left(X_t - x_0 - Z_t - \frac{1}{2}\sum_{t_i<t}\varphi(X_{t_i})\cdot(\Delta W_i)^{(2)}\right) \leq K\sum_i|\sigma(x_0)|\delta_i^{3/2} \leq K|\sigma(x_0)|t\sqrt{\delta}$$

**8 Lemma:** *For $t \in \pi$ we have*

$$N_p\left(\sum_{t_i<t}\varphi(X_{t_i})\cdot(\Delta W_i)^{(2)} - \int_0^t \operatorname{Tr}\varphi(\widetilde{X}_s)\,ds\right) \leq K|\sigma(x_0)|\sqrt{t\delta}$$

Proof : Put

$$H_t = \sum_{t_i<t}\varphi(X_{t_i})\cdot(\Delta W_i)^{(2)} - \int_0^t \operatorname{Tr}\varphi(\widetilde{X}_s)\,ds$$

It is easy to see that $H_t$ is the martingale

$$H_t = 2\int_0^t \overline{\varphi}(\widetilde{X}_s)\cdot(W_s - \widetilde{W}_s)\cdot dW_s$$

with the symmetrized $\overline{\varphi}$, so that by Burkholder's inequality we get

$$N_p(H_t)^2 \leq K_1\int_0^t (s-\widetilde{s})N_p(\overline{\varphi}(\widetilde{X}_s))^2\,ds \leq K_2|\sigma(x_0)|^2\delta t$$

**9 Proposition:** *For $t \in [0,1]$ we have*

$$N_p\left(\widetilde{X}_t - x_0 - \int_0^t\sigma(\widetilde{X}_s)dW_s - \frac{1}{2}\int_0^t \operatorname{Tr}\varphi(\widetilde{X}_s)\,ds\right) \leq K|\sigma(x_0)|\operatorname{Inf}(\sqrt{\delta},\sqrt{t})$$

Proof : In view of the preceding lemmas, this is obvious for $t \in \pi$. If $t \in [t_i, t_{i+1}[$ we have to add a term which is (by formulas (1)–(3)) easily seen to be smaller than $K|\sigma(x_0)|\sqrt{t-t_i}$, and use the inequality $\sqrt{\delta t} + \sqrt{t-t_i} \leq 2\,\mathrm{Inf}(\sqrt{\delta}, \sqrt{t})$. ∎

Then, as $\delta \to 0$ we see that $\widetilde{X}_t$ is a convergent sequence of approximate solutions of (S) (it is strong by lemma 6).

**10 Theorem:** *If $\sigma$ and $\varphi$ are Lipschitz then we have*

$$N_p\left(X_t - x_0 - \int_0^t \sigma(\widehat{X}_s)\,dW_s - \frac{1}{2}\int_0^t \mathrm{Tr}\,\varphi(\widehat{X}_s)\,ds\right) \leq K|\sigma(x_0)|\sqrt{\delta}$$

*so that as $\delta \to 0$, $X_t$ is a sequence of approximate solutions of (S). Moreover we also have*

$$N_p(X_t - X_s) \leq K|\sigma(x_0)|\sqrt{t-s}$$

*so that the convergence holds in $L^p(\mathcal{J}_{\alpha,p})$ for $1/p < \alpha < 1/2$.*

Proof : It suffices to remark that $N_p(\widehat{X}_t - \widetilde{X}_t) \leq N_p(X_t - \widetilde{X}_t) \leq K|\sigma(x_0)|\sqrt{t-\widetilde{t}}$ and to bring it up in the inequality of proposition 5.
The second inequality is exactly lemma 6. ∎

**11 Remarks:** a) The theorem extends to the case where $\beta \neq 0$ and $\beta$ Lipschitz, as noted before proposition 5.
b) By remark 2b we can calculate the rate of decrease of a sequence $\delta_n$ in order to have for the corresponding series $\Sigma_n(X^n - X^{n+1})$ to converge normally.

### III. Approximate solutions in the Sobolev space

By Meyer's theorem, the Sobolev space $W^{1,p}(\Omega,\mu)$ exactly is the space of functions $f \in L^p$ such that the weak derivative $f'(x,y) \in L^p(\Omega \times \Omega, \mu \otimes \mu)$ with the norm

$$\left[\int |f|^p d\mu + \iint |f'|^p d(\mu \otimes \mu)\right]^{1/p}$$

Recall that

$$X_t = x_0 + \int_0^t \sigma(X_s)\,dW_s^\pi$$

the solution of an ODE. Its derivative $Y_t(\omega,\varpi) = X_t'(\omega,\varpi)$ in $W^{1,p}(\Omega,\mu)$ satisfies the following ODE

$$Y_t(\omega,\varpi) = \int_0^t \sigma'(X_s(\omega))\,dW_s^\pi(\omega) Y_s(\omega,\varpi) + \int_0^t \sigma(X_s(\omega))\,dW_s^\pi(\varpi)$$

This is a linear equation, we then have

$$Y_t(\omega,\varpi) = R_t(\omega)\int_0^t R_s^{-1}(\omega)\sigma(X_s(\omega))\,dW_s^\pi(\varpi)$$

where the resolvent $R_t(\omega)$ satisfies

$$R_t(\omega) = I + \int_0^t \sigma'(X_s(\omega))\, dW_s^\pi(\omega) R_s(\omega)$$

**12 Lemma:** *Assume that $\sigma$ and $\sigma'$ are Lipschitz and bounded. Then $R_t$ and $R_t^{-1}$ are bounded in $L^p$ independently of $\pi$.*

Proof : For $t \in \pi$, write

$$R_t = I + S_t + \sum_{t_i<t} \int_{t_i}^{t_{i+1}} \left[ \sigma'(X_s) \frac{\Delta W_i}{\delta_i} R_s - \sigma'(\widetilde{X}_s) \frac{\Delta W_i}{\delta_i} \widetilde{R}_s \right] ds$$

with the martingale $\qquad S_t = \int_0^t \sigma'(\widetilde{X}_s)\, dW_s \cdot \widetilde{R}_s$

$$\sigma'(X_s)\Delta W_i R_s - \sigma'(\widetilde{X}_s)\Delta W_i \widetilde{R}_s = [\sigma'(X_s) - \sigma'(X_{t_i})]\Delta W_i R_s + \sigma'(X_{t_i})\Delta W_i [R_s - R_{t_i}]$$

$$|R_t - I - S_t| \le K \sum_{t_i<t} |\Delta W_i|^2 \, \mathrm{e}^{k|\Delta W_i|} \, |R_{t_i}|$$

$$N_p(R_t - I - S_t) \le K' \sum_{t_i<t} N_p(R_{t_i})\delta_i \le K' \int_0^t N_p(\widetilde{R}_s)\, ds$$

as above. Burkholder's inequality yields for $t \in \pi$

$$N_p(R_t - I)^2 \le K'' \int_0^t N_p(\widetilde{R}_s)^2\, ds$$

By Gronwall's lemma we get as above

$$N_p(\widetilde{R}_t) \le K'''$$

This extends to every $t \in [0,1]$ for we have $|R_t - R_{t_i}| \le \mathrm{e}^{k|\Delta W_i|}\, |R_{t_i}||\Delta W_i|$ as in formula (2).

The same result holds for $R_t^{-1}$, as it satisfies

$$R_t^{-1} = I - \int_0^t R_s^{-1}\sigma'(X_s)\, dW_s^\pi$$

**13 Proposition:** *Assume that $\sigma$ is bounded with all of its derivatives up to order 3. Then $R_t$ converges in $L^p(\mathcal{J}_{\alpha,p})$ for every $1/2 > \alpha > 1/p$ as $\pi$ refines indefinitely.*

Proof : As in lemma 7, we get

$$N_p\left( R_t - I - S_t - \frac{1}{2}\sum_{t_i<t} \left[ (\sigma''\sigma)(X_{t_i})\cdot(\Delta W_i)^{(2)} + \sigma'(X_{t_i})\cdot\Delta W_i\cdot\sigma'(X_{t_i})\cdot\Delta W_i \right] R_{t_i} \right)$$
$$\le Kt\sqrt{\delta}$$

and as in proposition 9

$$N_p\left(R_t - I - S_t - \frac{1}{2}\int_0^t \left[\sigma''(\widetilde{X}_s)\cdot\sigma(\widetilde{X}_s) + \sigma'(\widetilde{X}_s)\cdot\sigma'(\widetilde{X}_s)\right]\cdot\widetilde{R}_s\,ds\right) \leq Kt\sqrt{\delta}$$

As in theorem 10 we infer that $R_t$ is an approximate solution of the Stratonovitch SDE

$$R_t = I + \int_0^t \sigma'(X_s)\circ dW_s \circ R_s$$

which in Ito form reads

$$R_t = I + \int_0^t \sigma'(X_s)\cdot dW_s\cdot R_s + \frac{1}{2}\int_0^t [\sigma''(X_s)\cdot\sigma(X_s) + \sigma'(X_s)\cdot\sigma'(X_s)]\cdot R_s\,ds$$

where $X_t$ is the solution of the corresponding SDE.

**14 Remark:** The analogous result holds for $R_t^{-1}$. As in theorem 10, the convergence holds in the space $L^p(\mathcal{J}_{\alpha,p})$ for every $1/2 > \alpha > 1/p$.

**15 Theorem:** *Assume that $\sigma$ is bounded with all of its derivatives up to order 3. Then as $\pi$ refines indefinitely, $Y_t(\omega, \varpi)$ converges in every $L^p(\mathcal{J}_{\alpha,p})$. In the same way $X_t(\omega)$ converges in every $W^{1,p}(\mathcal{J}_{\alpha,p})$.*

Proof : It suffices to remark that for almost every $\omega$, $Y_t(\omega, \varpi)$ converges to a Wiener integral in $\varpi$. The convergence takes place in $L^p(\mu \otimes \mu, \mathcal{J}_{\alpha,p})$.

**Higher order derivatives**

Now, assume that $\sigma$ is bounded with all of its derivatives. For a partition $\pi$ compute

$$Y_t^{(2)}(\omega,\omega^1,\omega^2)=\int_0^t \sigma'(X_s(\omega))\cdot dW_s^\pi(\omega)\cdot Y_s^{(2)}(\omega,\omega^1,\omega^2)+L_t^{(2)}(\omega,\omega^1,\omega^2)$$

where

$$\begin{aligned}L_t^{(2)}(\omega,\omega^1,\omega^2)=&\int_0^t \sigma''(X_s(\omega))\cdot Y_s(\omega,\omega^1)\cdot Y_s(\omega,\omega^2)\cdot dW_s^\pi(\omega)+\\ &+\int_0^t \sigma'(X_s(\omega))\cdot Y_s(\omega,\omega^1)\cdot dW_s^\pi(\omega^2)+\\ &+\int_0^t \sigma'(X_s(\omega))\cdot Y_s(\omega,\omega^2)\cdot dW_s^\pi(\omega^1)\end{aligned}$$

As in the preceding proof we can write

$$Y_t^{(2)}(\omega,\omega^1,\omega^2)=R_t(\omega)\int_0^t R_s^{-1}(\omega)\,dL_s^{(2)}(\omega,\omega^1,\omega^2)$$

where $R_t$ denotes the resolvent of this linear ODE, which is the same as in proposition 13.

**16 Lemma:** *Let $\pi$ be a partition of [0,1], let $v_t$ be a continuous process which is $\pi$–adapted, that is $v_t\in\mathcal{F}_t$ for $t\in\pi$. Assume that*

$$N_p(v_t-v_s)\le K_p\sqrt{t-s}$$

*where $K_p$ does not depend on $\pi$. Consider the following processes*

$$\begin{aligned}u_t&=\int_0^t v_s\cdot dW_s^\pi\\ s_t&=\int_0^t u_s\cdot dW_s^\pi\end{aligned}$$

*If for every $t$ $u_t$ and $v_t$ converges in every $L^p$, then $s_t$ converges in the space $L^p(\mathcal{J}_{\alpha,p})$ for $1/p<\alpha<1/2$.*

Proof : As in lemma 7, by the fundamental theorem of calculus, we get for $t\in\pi$

$$s_{t_i+1}-s_{t_i}-u_{t_i}\cdot\Delta W_i-\frac{1}{2}v_{t_i}\cdot(\Delta W_i)^{(2)}=\int_{t_i}^{t_{i+1}}(t_{i+1}-s)[v_s-v_{t_i}]\cdot(\Delta W_i)^{(2)}\frac{ds}{\delta_i}$$

As in lemma 7 and lemma 8

$$N_p\left(s_t - \int_0^t \widetilde{u}_s \cdot dW_s - \frac{1}{2}\int_0^t \mathrm{Tr}\,(v_s)\,ds\right) \le K\sqrt{t\delta}$$

where Tr stands for a suitable tensor-contraction of $v_t$. As $\pi$ refines indefinitely, we get the convergence of $s_t$ in every $L^p$, for every $t$.
It remains to prove the convergence in the space $L^p(\mathcal{J}_{\alpha,p})$. Replacing $[0,t]$ with $[t_i,t_j]$ and using Burkholder's inequality yield

$$N_p(s_{t_j} - s_{t_i}) \le K\sqrt{t_j - t_i}$$

Then for $s < t_i < t_j < t$ we get

$$N_p(s_t - s_s) \le K\sqrt{t_j - t_i}$$

Applying the Kolmogorov-Ascoli lemma (prop. 1) gives the result.

Then we get as above

**17 Theorem:** *Assume that $\sigma$ is bounded with all of its derivatives. Then as $\pi$ refines indefinitely, $X_t$ converges in every $W^{r,p}(\Omega,\mu,\mathcal{J}_{\alpha,p})$ $(r \ge 1, p > 2, 1/2 > \alpha > 1/p)$.*

Proof : First, for the second derivative, it suffices to apply the preceding lemma to the processes

$$s_t(\omega,\omega^1,\omega^2) = \int_0^t R_s^{-1}(\omega)\,dL_s^{(2)}(\omega,\omega^1,\omega^2)$$

which is an $L^p(d\omega^1 \otimes d\omega^2)$-valued process.
It is straightforward to verify that the lemma applies in the same way at any order of derivation.

**18 Corollary:** *$X_t$ converges in every $\mathcal{L}^1(\Omega, c_{r,p}, \mathcal{J}_{\alpha,p})$ $(r \ge 1, p > 2, 1/2 > \alpha > 1/p)$.*

Proof : Apply corollary 4.

**19 Remark:** As proved in [4], th.27, we find again that $X$ has a modification $\widetilde{X} : \mathcal{J}_{\alpha,p} \to \mathcal{J}_{\alpha,p}$ which is $c_{r,p}$-quasi-continuous for every $(r,p)$.

**Application to the support theorem**

Assume that the hypotheses of theorem 15 hold, that is $\sigma$ and $\beta$ are bounded with all of its derivatives up to order 3. Denote $X^n$ the solution of the ODE

$$X_t^n = x_0 + \int_0^t \sigma(X_s^n)\, dW_s^n + \int_0^t \beta(X_s^n)\, ds$$

where $dW_s^n$ stands for the ordinary differential associated to the subdivision of maximum length $\delta_n$. We have seen that $X^n$ converges in the space $L^p(\mathcal{J}_{\alpha,p})$ for $1/2 > \alpha > 1/p$. Let $X$ be the limit, which is the solution of the Stratonovitch SDE. We also denote $X(h)$ the solution of the ODE

$$X_t(h) = x_0 + \int_0^t \sigma(X_s(h))\, h'(s)\, ds + \int_0^t \beta(X_s(h))\, ds$$

where $h(t) = \int_0^t h'(s)\, ds$ belongs to the Cameron-Martin space $H_l = W_0^{1,2}([0,1], dx, \mathbb{R}^l)$. Recall that $h \to X(h)$, which is a map from $H_l$ into $H_m$, is the so-called *skeleton* of $X$. Of course $X^n = X(\omega^n)$ where $\omega^n$ is the piecewise linear approximation of $\omega$. Then we have an improvement of the classical support theorem

**20 Theorem:** *The support of the image capacity $X(c_{1,p})$ is the closure in $\Omega = \mathcal{J}_{\alpha,p}$ of the skeleton $X(H)$.*

Proof : Let $\varphi$ be a continuous function on $\mathcal{J}_{\alpha,p}$ which vanishes on $X(H)$. Then $\varphi(X^n)$ vanishes for every $n$. We can assume that $X^n$ converges $c_{1,p}$-q.e. to $X$, so that as $n$ converges to $+\infty$, $\varphi(X^n) = 0$ converges to $\varphi(X)$. Then $\varphi(X) = 0$ q.e., so $X(c_{1,p})$ is carried by the closure of the skeleton.
Now notice that by the result of [3, 10], any point in the skeleton belongs to the support of $X(\mu)$ in the space $\mathcal{H}_\gamma$. As $\mathcal{H}_\gamma \subset \mathcal{J}_{\alpha,p}$ for $1/2 > \gamma > \alpha > 1/p$, (*cf.* [5, 6]), so by the obvious inclusion $\mathrm{Supp}\,(X(\mu)) \subset \mathrm{Supp}\,(X(c_{1,p}))$, we get the result.

Now let $\xi$ a measure belonging to the dual space of $\mathcal{L}^1(\Omega, c_{1,p})$. Then $X$ is $\xi$-measurable and we have

**21 Corollary:** *The image measure $X(\xi)$ is carried by the closure of the skeleton $X(H)$.*

**22 Remark:** If $\sigma$ is bounded with all of its derivatives, we can replace $c_{1,p}$ with $c_{r,p}$ in the preceding results.

**REFERENCES**


[1] S.Aida-S.Kusuoka-D.Stroock *On the support of Wiener functionals* In Asymptotic problems in Probability Theory; Wiener functionals and Asymptotics; p.3-34. Pitman Research Notes, Longman, 1993

[2] G.Ben Arous-G.Gradinaru *Normes höldériennes et support des diffusions* CRAS Paris, t.316, série I, n°3, 1993, p.283-286

[3] G.Ben Arous-G.Gradinaru-M.Ledoux *Hölder norms and the support theorem for diffusions.* Ann. Inst. Henri Poincaré, Proba. Stat. 30, n°3, p.415-436, 1994

[4] D.Feyel-A.de La Pradelle *Capacités gaussiennes* Ann. Inst. Fourier, t.41, f.1, p.49-76, 1991

[5] D.Feyel-A.de La Pradelle *Fractional integrals and Brownian Processes* Prépublication 39 de l'Université d'Evry Val d'Essonne, mars 96

[6] D.Feyel-A.de La Pradelle *On fractional Brownian processes* To appear in Potential Analysis, 1999

[7] I.Gyöngy-T.Pröhle *On the approximation of Stochastic Differential Equations and on Stroock-Varadhan's Support Theorem.* Computers Math. Applic. 19, 65-70, 1990

[8] N.Ikeda-S.Watanabe *Stochastic Differential Equations and Diffusion Processes* Amsterdam-Oxford-New-York: North Holland; Tokyo: Kodansha, 1981

[9] V.Mackevičius *On the support of the solution of Stochastic Differential Equations.* Litovskij Matematikow Rinkings XXXVI (1), 91-98, 1986

[10] A.Millet-M.Sanz-Solé *A simple proof of the support theorem for diffusion processes.* Sém. Proba. XXVIII, Lecture Notes in Math. n° 1583 p.36-48, 1994

[11] D. Nualart *The Malliavin calculus and related topics.* Springer-Verlag, 1995

[12] J.Ren *Analyse quasi-sûre des équations différentielles stochastiques* Bull. Soc. Math.,2ème série, 114, p.187-213, 1990

[13] S.G.Samko, A.A.Kilbas, O.I.Marichev *Fractional Integrals and Derivatives (Theory and Applications).* Gordon and Breach Science Publishers, 1987 (976 pages)

[14] D.W.Stroock-SRS.Varadhan *On the Support of Diffusion Processes with Applications to the strong maximum Principle.* Proc. sixth Berkeley Symp. Math. Stat. rob. 3, 333-359, Univ. California Press, Berkeley, 1972