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Asymptotic Growth Of Spatial Derivatives Of Isotropic Flows

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Abstract

It is known from the multiplicative ergodic theorem that the norm of the derivative of certain stochastic flows at a previously fixed point grows exponentially fast in time as the flows evolves. We prove that this is also true if one takes the supremum over a bounded set of initial points. We give an explicit bound for the exponential growth rate which is far different from the lower bound coming from the Multiplicative Ergodic Theorem.

Key words: stochastic flows, isotropic Brownian flows, isotropic Ornstein-Uhlenbeck flows, asymptotic behavior of derivatives.

AMS 2000 Subject Classification: Primary 60G15, 60G60, 60F99.

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1 Introduction

The evolution of the diameter of a bounded set under the evolution of a stochastic flow has been studied since the 1990's (see [5], [6], [11], [17], [12] and the survey article [16] to name just a few references). It is known to be linearly growing in time if the flow has a positive Lyapunov exponent. Of course the considered diameter links to the the supremum of $|\phi_t(x)|$ ranging over x in a subset of $\mathbb{R}^d$. In the following we will consider the case where the flow is replaced by its spatial derivative. We emphasize that we consider the asymptotics in time (the spatial asymptotics for a fixed time horizon have been considered in [8] in a very general setting and in [2] in the particular one treated here). If the flow has a positive top exponent it is known that the growth is at least exponentially fast which is then true even for a singleton (this follows directly from the Multiplicative Ergodic Theorem). We will show in the case of an isotropic Brownian flow (IBF) or an isotropic Ornstein-Uhlenbeck flow (IOUF) that $\sup_x |\log \left\| Dx_t \right\| |$ grows at most linearly in time t where the supremum is taken over x in a bounded subset of $\mathbb{R}^d$ no matter what the top Lyapunov exponent is. This shows that the growth of the norm of the derivative is indeed at most exponentially fast but it also gives some insight into the distance of $D\phi_t(x)$ to singularity by bounding $\sup_{0\le t\le T} t^{-1} \inf_x \log \left\| Dx_t \right\|$ from below in the lim inf sense. This excludes super-exponential decay to singularity which might be of interest especially if the top exponent is negative. Exponential bounds on the growth of spatial derivatives play a role in the proof of Pesin's formula for stochastic flows (see [13]). It has also been conjectured that this should yield a new proof of the fact that the diameter grows at most linearly in time (but there are much simpler proofs known for this - see the references given above). Despite the fact that we can come up with an upper bound for the exponential growth rate we make no claims about its optimality (and we conjecture that our bound is far from optimal).

1.1 Definition And Prerequisites

In this section we will recall the definition of an isotropic Brownian Flow (IBF) from [4] and an Isotropic Ornstein-Uhlenbeck Flow (IOUF) from [2]. We will keep the convention of speaking of an IOUF only if its drift c is not equal to zero. (see [2] or [3] for a discusion of this issue).

Definition 1.1 (IBF and IOUF).
Let $c > 0$ and $F(t,x,\omega)$ be an isotropic Brownian field with a C^4-covariance tensor i.e. $\left\langle F^i(\cdot,x), F^j(\cdot,y)\right\rangle_t = t b_{ij}(x-y)$ where the function $b(\cdot) = b_{ij}(\cdot) : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is four times continuously differentiable with bounded derivatives up to order four and preserved by rigid motions (see [4] or [7]). We define the semimartingale field $V(t,x,\omega) := F(t,x,\omega) - cxt$ and an IOUF to be the solution $\phi = \phi_{s,t}(x,\omega)$ of the Kunita-type stochastic differential equation (SDE)

$$\phi_{s,t}(x) = x + \int_s^t V(du, \phi_{s,u}(x)) = x + \int_s^t F(du, \phi_{s,u}(x)) - c\int_s^t \phi_{s,u}(x)du. \tag{1}$$

Note that the definition of b and [9, Theorem 3.1.3] imply

$$\left\langle \partial_l F^i(\cdot, x_\cdot), \partial_k F^j(\cdot, y_\cdot)\right\rangle_t = -\int_0^t \partial_l\partial_k b^{i,j}(x_s - y_s)ds. \tag{2}$$

If one puts $c = 0$ in (1) one gets the definition of an IBF.

We call c the drift of ϕ and b the covariance tensor of ϕ. Note that we write x_t for $\phi_t(x) = \phi_{0,t}(x) = \phi_{0,t}(x,\omega)$ and $x_t^1, \dots, x_t^d$ for its components. We will write the spatial derivatives as $\partial_i x_t^j := \frac{\partial}{\partial x^i}\phi_{0,t}^j(x)$ and $\left\|Dx_t\right\| := \left\|D\phi_t(x)\right\|$. The same notations are used for $y_t = \phi_{0,t}(y,\omega)$ etc.. The assumptions on the smoothness of b ensure that the local characteristics $(b, c\cdot)$ are smooth enough to guarantee the existence of a solution flow to (1) which can be shown to be of class $C^{3,\delta}$ for arbitrary $1 > \delta > 0$ (see [9, Theorem 3.4.1]). In fact one has to choose a modification to get the mentioned smoothness (which we do without change of notation). IOUFs have been studied in [7] and in [2] and we will recall some facts that we use later.

Lemma 1.2 (some finite-dimensional marginals).
Let ϕ be an IOUF with drift c and covariance tensor b and let $x, y \in \mathbb{R}^d$. Then we have

1. *ϕ is a Brownian flow (i.e. it has independent increments) and its law is invariant under orthogonal transformations.*
2. *The distance process $\{|x_t - y_t| : t \in \mathbb{R}_+\}$ solves the SDE*

$$|x_t - y_t| = |x - y| + \int_0^t \sqrt{2\left[1 - B_L(|x_s - y_s|)\right]}\, dW_s + \int_0^t (d-1)\frac{1 - B_N(|x_s - y_s|)}{|x_s - y_s|} - c|x_s - y_s| ds \tag{3}$$

for a standard Brownian motion $(W_t)_{t\geq 0}$. Therein B_L and B_N are the longitudinal and normal correlation functions of b respectively (see [4] or Lemma 1.3). There are constants $\lambda > 0$ and $\bar\sigma > 0$ such that the following is true.

   (a) *There is a standard Brownian motion $(W_t)_{t\geq 0}$ such that we have a.s. for all $t \geq 0$ that $|x_t - y_t| \leq |x - y| e^{\bar\sigma \sup_{0\leq s\leq t} W_s + \lambda t}$.*

   (b) *We have for each $x, y \in \mathbb{R}^d$, $T > 0$ and $q \geq 1$ that*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |x_t - y_t|^q\right]^{1/q} \leq 2|x - y| e^{(\lambda + \frac{1}{2}q\bar\sigma^2)T}. \tag{4}$$

3. *For $x \in \mathbb{R}^d$ the spatial derivative $\partial_j x_t^i := \frac{\partial \phi_t^i(x)}{dx_j}$ solves the following SDE.*

$$\partial_j x_t^i = \delta_{ij} + \int_0^t \sum_k \partial_j x_s^k \partial_k F^i(ds, x_s) - c\int_0^t \partial_j x_t^i ds. \tag{5}$$

We will use the symbol $\sum_k$ as a shorthand for $\sum_{k=1}^d$ (also for multiple summation indeces).

Proof: [7, Proposition 7.1.1, Corollary 7.1.1 and p. 139] and [16, condition (H)]. □
Observe that we write $\mathbb{E}[X]^q$ for $(\mathbb{E}[X])^q$ which is different from $\mathbb{E}[X^q]$. We will use this convention throughout the whole paper. The following lemma gives some insight into the structure of b which is known since [18].

Lemma 1.3 (local properties of b).
We have for $i, j, k, l = 1, \dots, d$ that:

1. *b can be written as* $b^{i,j}(x) = \begin{cases} (B_L(|x|) - B_N(|x|))\frac{x_i x_j}{|x|2} & : \text{for } x \neq 0 \\ \delta_{i,j} B_L(0) = \delta_{i,j} B_N(0) = \delta_{i,j} & : \text{else} \end{cases}$ *where* B_L *and* B_N *are the so-called longitudinal and transversal correlation functions defined by* $B_L(r) = b^{i,i}(re_i), r \geq 0$ *and* $B_N(r) = b^{i,i}(re_j), r \geq 0, i \neq j$. B_N *and* B_L *are bounded* C^4*- functions from* $[0,\infty)$ *to* $\mathbb{R}$ *with bounded derivatives up to order four. Letting* $\beta_{L,N} := -B''_{L,N}(0)$ *denote their right-hand derivatives at zero we have the Taylor-expansions* $B_{L,N}(r) = 1 - \frac{1}{2}\beta_{L,N} r^2 + O(r^4)$ *for* $r \to +0$. *Both* β_L *and* β_N *are strictly positive. Observe that our definition assumes the rigid-motion-part of b to vanish (which is different from the definition in [10] where one has to assume* $\alpha = 1$ *to be consistent with our notation).*

2. *The partial derivatives of b at* 0 *satisfy*
$$\partial_k \partial_l b^{i,j}(0) = \frac{1}{2}(\beta_N - \beta_L)(\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li}) - \beta_N \delta_{kl}\delta_{ij}.$$

3. *There is* $0 < \bar{r} < 1$ *and* $C > 0$ *such that for* $x \in \mathbb{R}^d$ *with* $|x| < \bar{r}$ *we have*
$$|\partial_k \partial_l b^{i,j}(x) - \partial_k \partial_l b^{i,j}(0)| \leq C|x|^2.$$

Proof: [7, Proposition 1.2.2]. □

Lemma 1.4 (a lemma on real functions)**.**
Let $f, g : [0,\infty) \to [0,\infty)$ *be increasing functions that are differentiable on* $(0,\infty)$. *Let further* f *be convex and* g *be concave. If we have for some* $t > 0$ *that* $f(t) \geq g(t)$ *and* $f'(t) \geq g'(t)$ *then we have for all* $s \geq t$ *that* $f(s) \geq g(s)$*.*

Proof: easy undergraduate exercise □
The following result is the main tool that allows for the estimation of suprema of the derivatives. Observe that $r_+ = r \vee 0$ denotes the positive part of $r \in \mathbb{R}$.

Theorem 1.5 (Chaining Growth Theorem)**.**
Let $\psi : [0,\infty) \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ *be a continuous random field satisfying*

1. *There are* $A > 0$ *and* $B \geq 0$ *such that for all* $k > 0$ *and bounded* $S \subset \mathbb{R}^d$ *we have* $\limsup_{T\to\infty} \frac{1}{T} \log \sup_{x\in S} \mathbb{P}\left[\sup_{0\leq t\leq T} |\psi_t(x)|^q > kT\right] \leq -\frac{(k-B)_+^2}{2A^2}$.

2. *There exist* $\Lambda \geq 0, \sigma > 0, q_0 \geq 1$ *and* $\bar{c} > 0$ *such that for each* $x, y \in \mathbb{R}^d$, $T > 0$ *and even* $q \geq q_0$ *we have that* $\mathbb{E}\left[\sup_{0\leq t\leq T} |\psi_t(x) - \psi_t(y)|^q\right]^{1/q} \leq$
$\bar{c}|x-y|e^{(\Lambda + \frac{1}{2}q\sigma^2)T}$. $\bar{c}$ *may depend on* q *and* d *but neither on* $|x-y|$ *nor on* T*.*

Let Ξ *be a compact subset of* $\mathbb{R}^d$ *with box (see [16, page 19] for a definition; just note that a closed ball in* $\mathbb{R}^d$ *has box dimension d) dimension* $\Delta > 0$. *Then we have* $\limsup_{T\to\infty} \sup_{0\leq t\leq T} \sup_{x\in\Xi} \frac{1}{T}|\psi_t(x)| \leq K$ *a.s. where for* $\Lambda_0 := \frac{\sigma^2 d}{\Delta}\left(\frac{d}{2} - \Delta\right)$ *we put* $K :=$
$$\begin{cases} B + A\sqrt{2\Delta\left(\Lambda + \sigma^2\Delta + \sqrt{\sigma^4\Delta^2 + 2\Delta\Lambda\sigma^2}\right)} & : \text{if } \Lambda \geq \Lambda_0 \\ B + A\sqrt{2\Delta\frac{d}{d-\Delta}\left(\Lambda + \frac{1}{2}\sigma^2 d\right)} & : \text{otherwise} \end{cases}.$$

Proof: A complete proof can be found in [16] as Theorem 5.1. Observe that the change of „$q \geq 1$“ to „even $q \geq q_0$“ does not alter the statement at all, because the assumptions above guarantee the same assumptions for $q \geq 1$ with a change only in the value of $\bar{c}$. The fact that we allow $\bar{c}$ to depend on q does not play any role because the proof in [16] is perfectly valid with q-depending $\bar{c}$. We will only briefly indicate what is involved in it. Choosing $\epsilon > 0$ and a sufficiently small $r_0 > 0$ one can cover Ξ with at most $e^{\gamma T(\Delta+\epsilon)}$ subsets of $\mathbb{R}^d$ of diameter at most $e^{-\gamma T} < r_0$. Denote these subsets by $\Xi_1, \ldots, \Xi_{e^{\gamma T(\Delta+\epsilon)}}$ For fixed T the assumption $\sup_{0\leq t\leq T} \sup_{x\in\Xi} \frac{1}{T}|\psi_t(x)| > \kappa$ implies one of the following to occur up to time T: one of the Ξ_i gets diameter at least one or the center of a Ξ_i reaches a distance of $\kappa T - 1$ from its original position. Hence we have for $\delta > 0$ that

$$\mathbb{P}\left[\sup_{0\leq t\leq T} \sup_{x\in\Xi} \frac{|x_t|}{T} > \kappa T + \delta\right] \leq S_1 + S_2 \tag{6}$$

$$\text{with } S_1 := e^{\gamma T(\Delta+\epsilon)} \max_{x\in\Xi} \mathbb{P}\left[\sup_{0\leq t\leq T} |x_t| \geq \kappa T - 1\right]$$

$$\text{and } S_2 := e^{\gamma T(\Delta+\epsilon)} \max_{i} \mathbb{P}\left[\sup_{0\leq t\leq T} \operatorname{diam}\phi_t(\Xi_i) \geq 1\right].$$

The chaining based result [16, Theorem 3.1] now shows that the two-point condition is sufficient to control S_2 and the one-point-condition covers S_1. In this way one obtains that the sum over $T \in \mathbb{N}$ of the right hand side of (6) is finite for some γ and κ which completes the proof via the Borel-Cantelli-Lemma. The formula for the constant K is obtained via an optimization over γ and κ. ☐

2 The Main Result

We are now ready to state the main result.

Theorem 2.1 (exponential growth of spatial derivatives).
Let ϕ be an IOUF or an IBF with b and c as above and let Ξ be a compact subset of $\mathbb{R}^d$ with box dimension $\Delta > 0$. Then

$$\limsup_{T\to\infty} \left(\sup_{0\leq t\leq T} \sup_{x\in\Xi} \frac{1}{T} |\log \left\|Dx_t\right\| |\right) \leq K \text{ a.s.} \tag{7}$$

$$\textit{where } K := \begin{cases} B + A\sqrt{2\Delta\left(\Lambda + \sigma^2\Delta + \sqrt{\sigma^4\Delta^2 + 2\Delta\Lambda\sigma^2}\right)} & : \textit{if } \Lambda \geq \Lambda_0 \\ B + A\sqrt{2\Delta\frac{d}{d-\Delta}\left(\Lambda + \frac{1}{2}\sigma^2 d\right)} & : \textit{ otherwise} \end{cases}$$

for $\Lambda_0 := \frac{\sigma^2 d}{\Delta}\left(\frac{d}{2} - \Delta\right)$, $A := \sqrt{\beta_L}$, $B := \frac{\beta_L}{2} + \frac{|(d-1)\beta_N - 2c|}{2}$, $\Lambda := \Lambda_1 \vee \Lambda_2 \vee \Lambda_6$ and $\sigma := \sigma_1 \vee \sigma_2 \vee \sigma_6$. The Λ_i and σ_i depend only on b and d and will be specified later.

Proof: This follows directly from Theorem 1.5 applied to $\psi : [0, \infty) \times \mathbb{R}^d \times \Omega \to \mathbb{R}$;
$\psi_t(x, \omega) := \log\left\|Dx_t\right\|$ if we can verify the following lemmas. We will only use ψ in the meaning given above from now on. Note that we choose the matrix norm to be the Frobenius norm $||(a_{i,j})_{1\leq i,j\leq d}|| := (\sum_{i,j} a_{i,j}^2)^{1/2}$ for its computational simplicity although the special choice of a norm is irrelevant because of to their equivalence.

Lemma 2.2 (condition on the one-point motion).
We have for each bounded $S \subset \mathbb{R}^d$ that

$$\limsup_{T\to\infty} \frac{1}{T} \log \sup_{x\in S} \mathbb{P}\left[\sup_{0\le t\le T} |\psi_t(x)| > kT\right] \le -\frac{(k-B)_+^2}{2A^2} \tag{8}$$

for A and B as given in Theorem 2.1.

Lemma 2.3 (condition on the two-point motion).
We have for each $x, y \in \mathbb{R}^d$, $T > 0$ and even $q \ge q_0 := \frac{4\sqrt{\beta_L}\left[2(d-1)\beta_N - c - 2\beta_L\right]}{128\beta_L} \vee 3$ that

$$\mathbb{E}\left[\sup_{0\le t\le T} |\psi_t(x) - \psi_t(y)|^q\right]^{1/q} \le \bar{c}|x-y|\sqrt{q}e^{(\Lambda+\frac{1}{2}q\sigma^2)T} \tag{9}$$

for Λ and σ as given in Theorem 2.1 and $\bar{c} := \bar{c}_1 + \bar{c}_2 + \bar{c}_6$. The $\bar{c}_i$ are constants that depend on b and d and will be specified later.

The proofs of these lemmas will be given in the next sections. Observe that $\bar{c}$ does not enter into the constant K, so we do not need to pay attention to get a small value for it.

3 Proof Of Lemma 2.2: The One-Point Condition

Before proving Lemma 2.2 we will need to establish some facts on $\left\|Dx_t\right\|^2$.

Lemma 3.1 (SDE for $\left\|Dx_t\right\|^2$).
Let $x \in \mathbb{R}^d$ and put $M_t := 2\int_0^t \sum_{i,j,k} \partial_k F^i\left(ds, x_s\right) \frac{\partial_j x_s^k \partial_j x_s^i}{\left\|Dx_s\right\|^2}$. Then we have

1. *$(M_t)_{t\ge0}$ is a continuous local martingale with* $2(\beta_L - \beta_N)t + 2(\beta_L + \beta_N)\int_0^t \sum_{i,j,k,m} \frac{\partial_j x_s^k \partial_j x_s^i \partial_m x_s^k \partial_m x_s^i}{\left\|Dx_s\right\|^4} ds = \langle M\rangle_t \le 4\beta_L t$ *a.s. for all $t > 0$ and hence a true martingale.*

2. *We have the SDE* $\left\|Dx_t\right\|^2 = d + 2\int_0^t \sum_{i,j,k} \partial_k F^i\left(ds, x_s\right) \partial_j x_s^k \partial_j x_s^i$ $+ \left[(d-1)\beta_N + \beta_L - 2c\right] \int_0^t \left\|Dx_s\right\|^2 ds$ $= d + \int_0^t \left\|Dx_s\right\|^2 dM_s + \left[(d-1)\beta_N + \beta_L - 2c\right] \int_0^t \left\|Dx_s\right\|^2 ds.$

3. *We have* $\left\langle \left\|Dx_\cdot\right\|^2 \right\rangle_t = 2(\beta_L - \beta_N)\int_0^t \left\|Dx_s\right\|^4 ds$ $+2(\beta_N + \beta_L)\int_0^t \sum_{i,j,k,m} \partial_j x_s^k \partial_j x_s^i \partial_m x_s^k \partial_m x_s^i ds.$

4. *$\psi_t(x) = \log\left\|Dx_t\right\|$ solves the SDE* $\psi_t(x) = \frac{1}{2}\log d + \frac{1}{2}M_t + \left[\frac{(d-1)\beta_N + \beta_L}{2} - c\right] t - \frac{1}{4}\langle M\rangle_t.$

Proof: Lemma 1.2 and Itô's formula imply for $\left\|Dx_t\right\|^2 = \sum_{i,j}(\partial_j x_t^i)^2$ that

$$\left\|Dx_t\right\|^2 = d + 2\sum_{i,j}\int_0^t \partial_j x_s^i d\partial_j x_s^i + \sum_{i,j}\left\langle \partial_j x_\cdot^i\right\rangle_t. \tag{10}$$

By Lemma 1.2 this is equal to

$$\begin{aligned}
&d + 2\sum_{i,j,k}\int_0^t \partial_j x_s^i \partial_k F^i\left(ds, x_s\right)\partial_j x_s^k - 2c\sum_{i,j}\int_0^t (\partial_j x_s^i)^2 ds \\
&+\sum_{i,j}\int_0^t \sum_{k,l}\partial_j x_s^l \partial_j x_s^k d\left\langle \partial_l F^i\left(d\cdot, x_\cdot\right), \partial_k F^i\left(d\cdot, x_\cdot\right)\right\rangle_s \\
=&d + 2\sum_{i,j,k}\int_0^t \partial_k F^i\left(ds, x_s\right)\partial_j x_s^i \partial_j x_s^k - 2c\int_0^t \left\|Dx_s\right\|^2 ds \\
&-\sum_{i,j,k,l}\int_0^t \partial_j x_s^l \partial_j x_s^k \partial_k \partial_l b^{i,i}(0)\, ds.
\end{aligned} \tag{11}$$

Since we have by Lemma 1.3 that $\sum_{i,j,k,l}\int_0^t \partial_j x_s^l \partial_j x_s^k \partial_k \partial_l b^{i,i}(0)\, ds$
$= \int_0^t \sum_{i,j,k,l}\partial_j x_s^l \partial_j x_s^k \left[(\beta_N - \beta_L)\delta_{ki}\delta_{li} - \beta_N \delta_{kl}\right] ds$ which is nothing but
$-\left[\beta_L + (d-1)\beta_N\right]\int_0^t \left\|Dx_s\right\|^2 ds$ we also get from (11) that

$$\begin{aligned}
\left\|Dx_t\right\|^2 =& d + 2\sum_{i,j,k}\int_0^t \partial_k F^i\left(ds, x_s\right)\partial_j x_s^i \partial_j x_s^k \\
&- 2c\int_0^t \left\|Dx_s\right\|^2 ds + \left[\beta_L + (d-1)\beta_N\right]\int_0^t \left\|Dx_s\right\|^2 ds \\
=& d + 2\int_0^t \left\|Dx_s\right\|^2 dM_s + \left[\beta_L + (d-1)\beta_N - 2c\right]\int_0^t \left\|Dx_s\right\|^2 ds.
\end{aligned}$$

Thus 2. is proved. 4. follows from this and Itô's formula since $\psi_t(x) = \frac{1}{2}\log(\left\|Dx_t\right\|^2)$. To prove 1. and 3. we observe that by (2) and Lemma 1.3 we have that $\langle M\rangle_t$ equals

$$\begin{aligned}
&-4\sum_{i,j,k,l,m,n}\int_0^t \frac{\partial_j x_s^k \partial_j x_s^i \partial_m x_s^n \partial_m x_s^l}{\left\|Dx_s\right\|^4}\partial_k \partial_n b^{i,l}(0)\, ds \\
=&2(\beta_L - \beta_N)\int_0^t \sum_{i,j,l,m}\frac{(\partial_j x_s^i)^2(\partial_m x_s^n)^2}{\left\|Dx_s\right\|^4} ds \\
&+ 2(\beta_L - \beta_N)\int_0^t \sum_{i,j,k,m}\frac{\partial_j x_s^k \partial_j x_s^i \partial_m x_s^i \partial_m x_s^k}{\left\|Dx_s\right\|^4} ds \\
&+ 4\beta_N \int_0^t \sum_{i,j,k,m}\frac{\partial_j x_s^k \partial_j x_s^i \partial_m x_s^k \partial_m x_s^i}{\left\|Dx_s\right\|^4} ds \\
=&2(\beta_L - \beta_N)t + 2(\beta_L + \beta_N)\int_0^t \sum_{i,j,k,m}\frac{\partial_j x_s^k \partial_j x_s^i \partial_m x_s^k \partial_m x_s^i}{\left\|Dx_s\right\|^4} ds.
\end{aligned} \tag{12}$$

This together with 2. proves 3. and 1. also follows from this and the next proposition. $\square$

Proposition 3.2 (a simple estimate).

We have $\left|\sum_{i,j,k,m} \frac{\partial_j x_s^k \partial_j x_s^i \partial_m x_s^k \partial_m x_s^i}{\|Dx_s\|^4}\right| \leq 1.$

Proof: The proof is just the combination of the triangle inequality with Schwarz' inequality. We leave the details to the reader. The reason why we state this fact as a proposition of its own is that we will use it again. $\square$

Now we can turn to the proof of Lemma 2.2. Since we can write $M_t = W_{\langle M\rangle_t}$ for a standard Brownian Motion $(W_t)_{t\geq 0}$ we get with Lemma 3.1 that

$$\begin{aligned}\psi_t(x) =& \frac{1}{2}\log d + \left[\frac{(d-1)\beta_N+\beta_L}{2} - c\right] t + \frac{1}{2}\left(W_{\langle M\rangle_t} - \frac{1}{2}\langle M\rangle_t\right)\\ \leq& \frac{1}{2}\log d + \left[\frac{(d-1)\beta_N+\beta_L}{2} - c\right] t + \frac{1}{2}\sup_{0\leq s\leq 4\beta_L t}\left(W_s - \frac{1}{2}s\right)\\ \stackrel{d}{=}& \frac{1}{2}\log d + \left[\frac{(d-1)\beta_N+\beta_L}{2} - c\right] t + \sqrt{\beta_L t}\sup_{0\leq s\leq 1}\left(W_s - \sqrt{\beta_L t}s\right)\end{aligned} \tag{13}$$

where the latter means equality in distribution. Therefore we get for any $k>0$ that

$$\begin{aligned} I :=& \mathbb{P}\left[\sup_{0\leq t\leq T}\psi_t(x) \geq kT\right] \\ \leq& \mathbb{P}\left[\frac{\log d}{2} + \sup_{0\leq t\leq T}\left\{\left[\frac{(d-1)\beta_N+\beta_L}{2} - c\right] t + \sqrt{\beta_L t}\sup_{0\leq s\leq 1}\left(W_s - \sqrt{\beta_L t s})\right)\right\} \geq kT\right]\\ \leq& \mathbb{P}\left[\sup_{0\leq t\leq T}\left\{\left[\frac{(d-1)\beta_N+\beta_L-2c}{2\sqrt{\beta_L}}\right]_+ t + \sqrt{t}\sup_{0\leq s\leq 1}\left(W_s - \sqrt{\beta_L t s})\right)\right\} \geq \frac{k}{\sqrt{\beta_L}}T - \frac{\log d}{2\sqrt{\beta_L}}\right].\end{aligned} \tag{14}$$

Here we distinguish between two cases to treat (14). If $(d-1)\beta_N+\beta_L-2c\leq 0$ then we immediately get (using $1-\Phi(t)\leq \exp(-1/2t^2)$)

$$\begin{aligned} I \leq& \mathbb{P}\left[\sqrt{T}\sup_{0\leq s\leq 1}W_s \geq \frac{k}{\sqrt{\beta_L}}T - \frac{\log d}{2\sqrt{\beta_L}}\right] = \mathbb{P}\left[\sup_{0\leq s\leq 1}W_s \geq \frac{k}{\sqrt{\beta_L}}\sqrt{T} - \frac{\log d}{2\sqrt{\beta_L T}}\right]\\ =& 2\left[1-\Phi\left(\frac{k\sqrt{T}}{\sqrt{\beta_L}} - \frac{\log d}{2\sqrt{\beta_L T}}\right)\right] \leq 2e^{-\frac{1}{2}\left(\frac{k^2 T}{\beta_L} - \frac{k\log d}{\beta_L} + \frac{(\log d)^2}{4\beta_L T}\right)}\end{aligned} \tag{15}$$

which gives $\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{P}\left[\sup_{0\leq t\leq T}\psi_t(x)\geq kT\right] \leq -\frac{1}{2\beta_L}k^2 = -\frac{1}{2\beta_L}k_+^2$.

If $\left[(d-1)\beta_N+\beta_L\right]-2c>0$ we get similarly to (15) that

$$\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{P}\left[\sup_{0\leq t\leq T}\psi_t(x)\geq kT\right] \leq -\frac{1}{2\beta_L}\left[k - \frac{(d-1)\beta_N+\beta_L-2c}{2}\right]_+^2.$$

We now only have to exclude the possibility that the modulus of the logarithm in $\psi_t(x)$ might become large due to a very small $\|Dx_t\|$. Observe, that (13) implies $\psi_t(x)\geq 1/2((d-1)\beta_N-\beta_L-2c)t+1/2W_{\langle M\rangle_t}$. Distinguishing between the signs of $(d-1)\beta_N-\beta_L-2c$

we get with a completely analogous computation $\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{P}\left[\inf_{0\le t\le T}\psi_t(x)\le -kT\right]\le -\frac{1}{2\beta_L}\left[k-\frac{\beta_L-(d-1)\beta_N+2c}{2}\right]_+^2$. This completes the proof of Lemma 2.2. □

4 Proof Of Lemma 2.3: The Two-Point Condition

4.1 General Estimates And Preparation

We now turn to the proof of Lemma 2.3 and start with a sketch of proof since it is rather technical. It consists of four steps.

1. Derivation of a SDE for $\psi_t(x)-\psi_t(y)$ involving $X_t^{(ij)} := \frac{\partial_j x_s^i}{\|Dx_s\|} - \frac{\partial_j y_s^i}{\|Dy_s\|}$.
2. Derivation of a SDE for $(X_t^{(ij)})^q$ for $q\ge 2$.
3. Obtaining estimates of the type required in Lemma 2.3 for $X_t^{(ij)}$. This is done for small $|x-y|$ estimating the probability for very fast increase in the beginning and with a Grönwall type argument in the case this increase does not occur.
4. Obtaining the estimate for $\psi_t(x)-\psi_t(y)$ as integrated versions of the ones on the $X^{(ij)}$ using the Burkholder-Davies-Gundy inequality.

Observe that we have by Lemma 3.1

$$\begin{aligned}\psi_t(x)-\psi_t(y) =& \int_0^t \sum_{i,j,k}\partial_k F^i\left(ds,x_s\right)\frac{\partial_j x_s^k\partial_j x_s^i}{\left\|Dx_s\right\|^2} - \partial_k F^i\left(ds,y_s\right)\frac{\partial_j y_s^k\partial_j y_s^i}{\left\|Dy_s\right\|^2}\\ &+\frac{(\beta_L+\beta_N)}{2}\int_0^t\sum_{i,j,k,m}\frac{\partial_j y_s^k\partial_j y_s^i\partial_m y_s^k\partial_m y_s^i}{\left\|Dy_s\right\|^4}-\frac{\partial_j x_s^k\partial_j x_s^i\partial_m x_s^k\partial_m x_s^i}{\left\|Dx_s\right\|^4}ds\\ =:&\tilde{M}_t+A_t. \end{aligned}\tag{16}$$

To further analyze the latter we first prove the following lemma.

Lemma 4.1 (general estimates for A_t and $\tilde{M}_t$).
With $\tilde{M}_t$ and A_t defined as above the following holds.

1. *A.s. we have for all $t\ge 0$ that $A_t\le(\beta_N+\beta_L)t$.*
2. *Introducing the abbreviation $\tilde{b}_{k,n}^{i,l}(z):=\partial_k\partial_n b^{i,l}(z)-\partial_k\partial_n b^{i,l}(0)$ we can write for the quadratic variation of $\tilde{M}$*

$$\begin{aligned}\left\langle\tilde{M}\right\rangle_t =&\frac{\beta_L+\beta_N}{2}\int_0^t\sum_{i,k}\left(\sum_j\frac{\partial_j x_s^i\partial_j x_s^k}{\left\|Dx_s\right\|^2}-\frac{\partial_j y_s^i\partial_j y_s^k}{\left\|Dy_s\right\|^2}\right)^2 ds\\ &+2\sum_{i,j,k,l,m,n}\int_0^t\frac{\partial_j x_s^i\partial_j x_s^k\partial_m y_s^l\partial_m y_s^n}{\left\|Dx_s\right\|^2\left\|Dy_s\right\|^2}\tilde{b}_{k,n}^{i,l}\left(x_s-y_s\right)de\le\tilde{c}t\end{aligned}$$

wherein we put $\tilde{c}:=2\beta_L+2d^6\max_{i,l,k,n}\sup_{z\in\mathbb{R}^d}\partial_k\partial_n b^{i,l}(z)<\infty$.

Proof: 1. is clear by definition of A_t and Proposition 3.2. Clearly $\left\langle \tilde{M} \right\rangle_t$ equals

$$
\begin{aligned}
& -\sum_{i,j,k,l,m,n} \int_0^t \frac{\partial_j x_s^k \partial_j x_s^i \partial_m x_s^n \partial_m x_s^l}{\left\| Dx_s \right\|^4} \partial_k \partial_n b^{i,l}(0)\, ds \\
& + \sum_{i,j,k,l,m,n} \int_0^t \frac{\partial_j x_s^k \partial_j x_s^i \partial_m y_s^n \partial_m y_s^l}{\left\| Dx_s \right\|^2 \left\| Dy_s \right\|^2} \partial_k \partial_n b^{i,l}\left(x_s - y_s\right) ds \\
& + \sum_{i,j,k,l,m,n} \int_0^t \frac{\partial_j y_s^k \partial_j y_s^i \partial_m x_s^n \partial_m x_s^l}{\left\| Dy_s \right\|^2 \left\| Dx_s \right\|^2} \partial_k \partial_n b^{i,l}\left(y_s - x_s\right) ds \\
& - \sum_{i,j,k,l,m,n} \int_0^t \frac{\partial_j y_s^k \partial_j y_s^i \partial_m y_s^n \partial_m y_s^l}{\left\| Dy_s \right\|^4} \partial_k \partial_n b^{i,l}(0)\, ds \\
= & - \sum_{i,j,k,l,m,n} \int_0^t \left[\frac{1}{2}(\beta_N - \beta_L)(\delta_{ki}\delta_{nl} + \delta_{kl}\delta_{ni}) - \beta_N \delta_{kn}\delta_{il} \right] \\
& \qquad \left(\frac{\partial_j x_s^k \partial_j x_s^i \partial_m x_s^n \partial_m x_s^l}{\left\| Dx_s \right\|^4} + \frac{\partial_j y_s^k \partial_j y_s^i \partial_m y_s^n \partial_m y_s^l}{\left\| Dy_s \right\|^4} \right) ds \\
& + 2 \sum_{i,j,k,l,m,n} \int_0^t \frac{\partial_j x_s^k \partial_j x_s^i \partial_m y_s^n \partial_m y_s^l}{\left\| Dx_s \right\|^2 \left\| Dy_s \right\|^2} \partial_k \partial_n b^{i,l}\left(x_s - y_s\right) ds =: II + III.
\end{aligned}
\tag{17}
$$

Using $II = (\beta_L - \beta_N)t + \frac{\beta_L + \beta_N}{2} \int_0^t \sum_{i,j,k,m} \frac{\partial_j x_s^k \partial_j x_s^i \partial_m x_s^k \partial_m x_s^i}{\left\| Dx_s \right\|^4} + \frac{\partial_j y_s^k \partial_j y_s^i \partial_m y_s^k \partial_m y_s^i}{\left\| Dy_s \right\|^4} ds$ Proposition 3.2 now yields $II \leq (\beta_L - \beta_N)t + \frac{(\beta_N + \beta_L)}{2} 2t = 2\beta_L t$ and since $III \leq 2d^6 \max_{i,l,k,n} \sup_{z \in \mathbb{R}^d} \partial_k \partial_n b^{i,l}(z)\, t$ is clear the first part of 2. follows. To prove the second part we have to rearrange (17) using the symmetry and isotropy of of b.

$$
\begin{aligned}
& \left\langle \tilde{M} \right\rangle_t \\
= & - \sum_{i,j,k,l,m,n} \int_0^t \left[\frac{\partial_j x_s^i \partial_j x_s^k}{\left\| Dx_s \right\|^2} - \frac{\partial_j y_s^i \partial_j y_s^k}{\left\| Dy_s \right\|^2} \right] \left[\frac{\partial_m x_s^n \partial_m x_s^l}{\left\| Dx_s \right\|^2} - \frac{\partial_m y_s^n \partial_m y_s^l}{\left\| Dy_s \right\|^2} \right] \partial_k \partial_n b^{i,l}(0)\, ds \\
& + 2 \sum_{i,j,k,l,m,n} \int_0^t \frac{\partial_j x_s^i \partial_j x_s^k \partial_m y_s^l \partial_m y_s^n}{\left\| Dx_s \right\|^2 \left\| Dy_s \right\|^2} \tilde{b}_{k,n}^{i,l}\left(x_s - y_s\right) ds =: IV + V.
\end{aligned}
\tag{18}
$$

Since (again by Lemma 1.3) we have $\partial_k\partial_n b^{i,l}(0) = \frac{1}{2}(\beta_N - \beta_L)(\delta_{ki}\delta_{nl} + \delta_{kl}\delta_{ni}) - \beta_N\delta_{kn}\delta_{il}$ we get that IV equals

$$\begin{aligned}
&\frac{\beta_L-\beta_N}{2}\sum_{i,j,l,m}\left(\frac{(\partial_j x_s^i)^2}{\|Dx_s\|^2}-\frac{(\partial_j y_s^i)^2}{\|Dy_s\|^2}\right)\left(\frac{(\partial_m x_s^l)^2}{\|Dx_s\|^2}-\frac{(\partial_m y_s^l)^2}{\|Dy_s\|^2}\right)ds\\
&+\frac{\beta_L-\beta_N}{2}\sum_{i,j,k,m}\int_0^t\left(\frac{\partial_j x_s^i\partial_j x_s^k}{\|Dx_s\|^2}-\frac{\partial_j y_s^i\partial_j y_s^k}{\|Dy_s\|^2}\right)\left(\frac{\partial_m x_s^i\partial_m x_s^k}{\|Dx_s\|^2}-\frac{\partial_m y_s^i\partial_m y_s^k}{\|Dy_s\|^2}\right)ds\\
&+\beta_N\sum_{i,j,k,m}\int_0^t\left(\frac{\partial_j x_s^i\partial_j x_s^k}{\|Dx_s\|^2}-\frac{\partial_j y_s^i\partial_j y_s^k}{\|Dy_s\|^2}\right)\left(\frac{\partial_m x_s^k\partial_m x_s^i}{\|Dx_s\|^2}-\frac{\partial_m y_s^k\partial_m y_s^i}{\|Dy_s\|^2}\right)ds\\
=&\frac{\beta_L+\beta_N}{2}\int_0^t\sum_{i,k}\left(\sum_j\frac{\partial_j x_s^i\partial_j x_s^k}{\|Dx_s\|^2}-\frac{\partial_j y_s^i\partial_j y_s^k}{\|Dy_s\|^2}\right)^2 ds. \qquad (19)
\end{aligned}$$

This completes the proof of Lemma 4.1. □

This Lemma shows that we have to control terms like $\frac{\partial_j x_s^i}{\|Dx_s\|} - \frac{\partial_j y_s^i}{\|Dy_s\|}$. We postpone this until we will have derived the following estimate from Lemma 4.1.

Lemma 4.2 (a priori bounds for the ψ-estimation)**.**

1. *We have for each $x, y \in \mathbb{R}^d$, $T > 0$ and $q \geq 1$that*
   $\mathbb{E}\left[\sup_{0\leq t\leq T}|\psi_t(x)-\psi_t(y)|^q\right]^{1/q} \leq (\beta_N+\beta_L)T + 2e^{-\frac{5}{12}}\sqrt{2\pi\tilde{c}}\sqrt{q+1}\sqrt{T}.$
2. *We have for each $x, y \in \mathbb{R}^d$, $T > 0$ and $q \geq 1$ that*
   $\mathbb{E}\left[\sup_{0\leq t\leq T}|\psi_t(x)-\psi_t(y)|^q\right]^{1/q} \leq \bar{C}_1 e^{\left(\Lambda_1+\frac{1}{2}\sigma_1^2 q\right)T}$
   with $\bar{C}_1 := \beta_N + \beta_L + 2e^{\frac{1}{4e}-\frac{5}{12}}\sqrt{2\pi\tilde{c}}$, $\Lambda_1 := 1/e$ and $\sigma_1 := \sqrt{2/e}$.
3. *Let $r > 0$ be fixed. We have for any $x, y \in \mathbb{R}^d$ with $|x-y| \geq r$, $T > 0$ and $q \geq 1$ that*
   $\mathbb{E}\left[\sup_{0\leq t\leq T}|\psi_t(x)-\psi_t(y)|^q\right]^{1/q} \leq \bar{c}_1|x-y|e^{\left(\Lambda_1+\frac{1}{2}\sigma_1^2\right)T}$ *for*
   $\bar{c}_1 := \frac{1}{r}\left[\beta_N+\beta_L+2e^{-\frac{5}{12}+\frac{1}{4e}}\sqrt{2\pi\tilde{c}}\right]$ *and Λ_1 and σ_1 as before.*

Proof: Once again observe that by the triangle inequality

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq t\leq T}|\psi_t(x)-\psi_t(y)|^q\right]^{1/q} \leq&\mathbb{E}\left[\sup_{0\leq t\leq T}A_t^q\right]^{1/q}+\mathbb{E}\left[\sup_{0\leq t\leq T}\tilde{M}_t^q\right]^{1/q}\\
\leq(\beta_N+\beta_L)T+\mathbb{E}\left[\sup_{0\leq t\leq T}\tilde{M}_t^q\right]^{1/q} =&:(\beta_N+\beta_L)T+VI. \qquad (20)
\end{aligned}$$

Since we can write $\tilde{M}_t = W_{\langle\tilde{M}\rangle_t}$ and $\left\langle\tilde{M}\right\rangle_t \leq \tilde{c}t$ a.s. we get using

$\mathbb{E}\left[\sup_{0\leq t\leq T}\tilde{M}_t^q\right] \leq \mathbb{E}\left[\sup_{0\leq t\leq \tilde{c}T}W_t^q\right] = (\tilde{c}T)^{\frac{q}{2}}\sqrt{\frac{2}{\pi}}\frac{\Gamma\left(\frac{q+1}{2}\right)}{2}2^{\frac{q+1}{2}}$

that Stirling's formula for the Gamma function implies

$$VI \leq (2\pi)^{-\frac{1}{2q}} 2^{\frac{q+1}{2q}} \Gamma\left(\frac{q+1}{2}\right)^{\frac{1}{q}} \sqrt{\tilde{c}T} \leq 2e^{-\frac{5}{12}} \sqrt{2\pi\tilde{c}} \sqrt{q+1} \sqrt{T}. \tag{21}$$

Thus 1. follows. For the proof of 2. it is sufficient to observe that since for any $t \geq 0$ we have $t \leq e^{t/e}$ and $\sqrt{t} \leq 1/4 + t$ we also get

$$\begin{aligned} (\beta_N+\beta_L)T + 2e^{-\frac{5}{12}}\sqrt{2\pi\tilde{c}}\sqrt{q+1}\sqrt{T} \leq \quad & (\beta_N+\beta_L)e^{\frac{T}{e}} + 2e^{-\frac{5}{12}}\sqrt{2\pi\tilde{c}}e^{\frac{\sqrt{q+1}\sqrt{T}}{e}} \leq \\ (\beta_N+\beta_L)e^{\frac{T}{e}} + 2e^{-\frac{5}{12}}\sqrt{2\pi\tilde{c}}e^{\frac{1/4+(q+1)T}{e}} \leq \quad & \left[\beta_N+\beta_L+2e^{\frac{1}{4e}-\frac{5}{12}}\sqrt{2\pi\tilde{c}}\right] e^{\frac{q+1}{E}T}. \end{aligned} \tag{22}$$

This proofs 2. and 3. follows from this by using $\frac{|x-y|}{r} \leq 1$. □

Since we can now control the moments of $\psi_t(x) - \psi_t(y)$ provided x and y are not too close to each other we introduce the following stopping time. Let $\bar{r}$ be chosen according to Lemma 1.3 and $\tilde{r} \leq \bar{r}$ to be specified later. Remember that we assumed $\bar{r} \leq 1$. We now define for $x, y \in \mathbb{R}^d$ with $|x-y| \leq r$ the stopping time

$$\tau := \inf_{t>0} \{|x_t - y_t| \geq \tilde{r}\}$$

and assume $r < \tilde{r}$ in the following (which ensures $\tau > 0$ a.s.).

4.2 Derivation Of Formula H

We now proceed to work on $\frac{\partial_j x_s^i}{\|Dx_s\|} - \frac{\partial_j y_s^i}{\|Dy_s\|}$ by proving the following proposition on $\frac{\partial_j x_s^i}{\|Dx_s\|}$.

Proposition 4.3 (SDE for the direction of the derivative).
We have the SDE

$$\begin{aligned} \frac{\partial_j x_s^i}{\|Dx_s\|} =& \frac{\delta^{ij}}{\sqrt{d}} + \int_0^t \sum_k \partial_k F^i(ds, x_s) \frac{\partial_j x_s^k}{\|Dx_s\|} - \int_0^t \sum_{k,l,m} \partial_m F^k(ds, x_s) \frac{\partial_l x_s^m \partial_l x_s^k \partial_j x_s^i}{\|Dx_s\|^3} \\ &+ \left[\left(\frac{1}{4} - \frac{d}{2}\right)\beta_N - \frac{1}{4}\beta_L\right] \int_0^t \frac{\partial_j x_s^i}{\|Dx_s\|} ds - \frac{\beta_N+\beta_L}{2} \int_0^t \sum_{k,l} \frac{\partial_j x_s^k \partial_l x_s^i \partial_l x_s^k}{\|Dx_s\|^3} ds \\ &+ \frac{3}{4}(\beta_N+\beta_L) \int_0^t \sum_{k,l,m,n} \frac{\partial_l x_s^k \partial_l x_s^n \partial_m x_s^k \partial_m x_s^n \partial_j x_s^i}{\|Dx_s\|^5} ds. \end{aligned} \tag{23}$$

Proof: Since by Itô's formula

$$\begin{aligned} \frac{\partial_j x_s^i}{\|Dx_s\|} =& \frac{\delta^{ij}}{\sqrt{d}} + \int_0^t \frac{d(\partial_j x_.^i)_s}{\|Dx_s\|} - \frac{1}{2}\int_0^t \frac{\partial_j x_s^i}{\|Dx_s\|^3} d\|Dx_.\|_s^2 \\ &- \frac{1}{2}\int_0^t \frac{d\left\langle \partial_j x_.^i, \|Dx_.\|^2 \right\rangle_s}{\|Dx_s\|^3} + \frac{3}{8}\int_0^t \frac{\partial_j x_s^i}{\|Dx_s\|^5} d\left\langle \|Dx_.\|^2 \right\rangle_s \end{aligned} \tag{24}$$

we only have to note that by Lemmas 1.2 and 3.1

$$\begin{aligned}\left\langle \partial_j x^i_\cdot, \left\|Dx_\cdot\right\|^2 \right\rangle_t =& -2\int_0^t \sum_{k,l,m,n} \partial_j x_s^k \partial_m x_s^l \partial_m x_s^n \partial_k \partial_l b^{i,n}(0)\, ds \\ =&(\beta_L-\beta_N)\int_0^t \partial_j x_s^i \left\|Dx_s\right\|^2 ds + (\beta_L+\beta_N)\int_0^t \sum_{k,m} \partial_j x_s^k \partial_m x_s^i \partial_m x_s^k ds \end{aligned} \tag{25}$$

which combined with Lemmas 1.2 and 3.1 and put into (24) yields

$$\begin{aligned}\frac{\partial_j x_t^i}{\left\|Dx_t\right\|} =& \frac{\delta^{ij}}{\sqrt{d}} + \int_0^t \sum_k \partial_k F^i(ds, x_s) \frac{\partial_j x_s^k}{\left\|Dx_s\right\|} - c\int_0^t \frac{\partial_j x_s^i}{\left\|Dx_s\right\|} ds \\ & - \int_0^t \sum_{k,l,m} \partial_m F^k(ds,x_s) \frac{\partial_l x_s^m \partial_l x_s^k \partial_j x_s^i}{\left\|Dx_s\right\|^3} \\ & - \frac{(d-1)\beta_N + \beta_L - 2c}{2}\int_0^t \frac{\partial_j x_s^i}{\left\|Dx_s\right\|} ds \\ & + \frac{\beta_N-\beta_L}{2}\int_0^t \frac{\partial_j x_s^i}{\left\|Dx_s\right\|} ds - \frac{\beta_N+\beta_L}{2}\int_0^t \sum_{k,l} \frac{\partial_j x_s^k \partial_l x_s^i \partial_l x_s^k}{\left\|Dx_s\right\|^3} ds \\ & + \frac{3}{4}(\beta_L-\beta_N)\int_0^t \frac{\partial_j x_s^i}{\left\|Dx_s\right\|} ds \\ & + \frac{3}{4}(\beta_N+\beta_L)\int_0^t \sum_{k,l,m,n} \frac{\partial_l x_s^k \partial_l x_s^n \partial_m x_s^k \partial_m x_s^n \partial_j x_s^i}{\left\|Dx_s\right\|^5} ds. \end{aligned} \tag{26}$$

This proves Proposition 4.3. □

Of course the latter implies

$$\begin{aligned} & \frac{\partial_j x_s^i}{\left\|Dx_s\right\|} - \frac{\partial_j y_s^i}{\left\|Dy_s\right\|} \\ =& \int_0^t \sum_k \left(\partial_k F^i(ds,x_s) \frac{\partial_j x_s^k}{\left\|Dx_s\right\|} - \partial_k F^i(ds,y_s) \frac{\partial_j y_s^k}{\left\|Dy_s\right\|} \right) \\ & - \int_0^t \sum_{k,l,m} \left(\partial_m F^k(ds,x_s) \frac{\partial_l x_s^m \partial_l x_s^k \partial_j x_s^i}{\left\|Dx_s\right\|^3} - \partial_m F^k(ds,y_s) \frac{\partial_l y_s^m \partial_l y_s^k \partial_j y_s^i}{\left\|Dy_s\right\|^3} \right) \\ & + \left[\left(\frac{1}{4} - \frac{d}{2}\right)\beta_N - \frac{1}{4}\beta_L \right] \int_0^t \left(\frac{\partial_j x_s^i}{\left\|Dx_s\right\|} - \frac{\partial_j y_s^i}{\left\|Dy_s\right\|} \right) ds \\ & - \frac{\beta_N+\beta_L}{2}\int_0^t \sum_{k,l} \left(\frac{\partial_j x_s^k \partial_l x_s^i \partial_l x_s^k}{\left\|Dx_s\right\|^3} - \frac{\partial_j y_s^k \partial_l y_s^i \partial_l y_s^k}{\left\|Dy_s\right\|^3} \right) ds \\ & + \frac{3}{4}(\beta_N+\beta_L)\int_0^t \sum_{k,l,m,n} \left(\frac{\partial_l x_s^k \partial_l x_s^n \partial_m x_s^k \partial_m x_s^n \partial_j x_s^i}{\left\|Dx_s\right\|^5} - \frac{\partial_l y_s^k \partial_l y_s^n \partial_m y_s^k \partial_m y_s^n \partial_j y_s^i}{\left\|Dy_s\right\|^5} \right) ds. \end{aligned} \tag{27}$$

Letting

$$VII_t := \int_0^t \sum_k \left(\partial_k F^i \left(ds, x_s\right) \frac{\partial_j x_s^k}{\left\|Dx_s\right\|} - \partial_k F^i \left(ds, y_s\right) \frac{\partial_j y_s^k}{\left\|Dy_s\right\|} \right), \tag{28}$$

$$VIII_t := \int_0^t \sum_{k,l,m} \left(\partial_m F^k \left(ds, x_s\right) \frac{\partial_l x_s^m \partial_l x_s^k \partial_j x_s^i}{\left\|Dx_s\right\|^3} - \partial_m F^k \left(ds, y_s\right) \frac{\partial_l y_s^m \partial_l y_s^k \partial_j y_s^i}{\left\|Dy_s\right\|^3} \right) \tag{29}$$

we have to compute the cross variations to apply Itô's formula for powers to (27).

$$\begin{aligned}
&\langle VII \rangle_t \\
=& -\int_0^t \sum_{k,l} \left(\frac{\partial_j x_s^k \partial_j x_s^l}{\left\|Dx_s\right\|^2} + \frac{\partial_j y_s^k \partial_j y_s^l}{\left\|Dy_s\right\|^2} \right) \partial_k \partial_l b^{i,i}(0)\, ds \\
&+ \int_0^t \sum_{k,l} \left(\frac{\partial_j x_s^k \partial_j y_s^l}{\left\|Dx_s\right\| \left\|Dy_s\right\|} + \frac{\partial_j y_s^k \partial_j x_s^l}{\left\|Dy_s\right\| \left\|Dx_s\right\|} \right) \partial_k \partial_l b^{i,i}\left(x_s - y_s\right) ds \\
=& -\sum_{k,l} \int_0^t \left(\frac{\partial_j x_s^k}{\left\|Dx_s\right\|} - \frac{\partial_j y_s^k}{\left\|Dy_s\right\|} \right) \left(\frac{\partial_j x_s^l}{\left\|Dx_s\right\|} - \frac{\partial_j y_s^l}{\left\|Dy_s\right\|} \right) \left[(\beta_N - \beta_L) \delta_{ki} \delta_{li} - \beta_N \delta_{kl} \right] ds \\
&+ 2 \int_0^t \sum_{k,l} \frac{\partial_j x_s^k \partial_j y_s^l}{\left\|Dx_s\right\| \left\|Dy_s\right\|} \left[\partial_k \partial_l b^{i,i}\left(x_s - y_s\right) - \partial_k \partial_l b^{i,i}(0) \right] ds \\
=& (\beta_L - \beta_N) \int_0^t \left(\frac{\partial_j x_s^i}{\left\|Dx_s\right\|} - \frac{\partial_j y_s^i}{\left\|Dy_s\right\|} \right)^2 ds + \beta_N \int_0^t \sum_k \left(\frac{\partial_j x_s^k}{\left\|Dx_s\right\|} - \frac{\partial_j y_s^k}{\left\|Dy_s\right\|} \right)^2 ds \\
&+ 2 \int_0^t \sum_{k,l} \frac{\partial_j x_s^k \partial_j y_s^l}{\left\|Dx_s\right\| \left\|Dy_s\right\|} \tilde{b}_{k,l}^{i,i}\left(x_s - y_s\right) ds
\end{aligned} \tag{30}$$

and similarly

$$\begin{aligned}
\langle VIII \rangle_t =& \frac{\beta_L + \beta_N}{2} \int_0^t \sum_{k,n} \left(\sum_l \frac{\partial_l x_s^n \partial_l x_s^k \partial_j x_s^i}{\left\|Dx_s\right\|^3} - \frac{\partial_l y_s^n \partial_l y_s^k \partial_j y_s^i}{\left\|Dy_s\right\|^3} \right)^2 ds \\
&+ \frac{\beta_L - \beta_N}{2} \int_0^t \left(\frac{\partial_j x_s^i}{\left\|Dx_s\right\|} - \frac{\partial_j y_s^i}{\left\|Dy_s\right\|} \right)^2 ds \\
&+ 2 \sum_{k,l,m,n,p,r} \int_0^t \frac{\partial_l x_s^m \partial_l x_s^k \partial_j x_s^i \partial_p y_s^r \partial_p y_s^n \partial_j y_s^i}{\left\|Dx_s\right\|^3 \left\|Dy_s\right\|^3} \tilde{b}_{r,m}^{k,n}\left(x_s - y_s\right) ds
\end{aligned} \tag{31}$$

as well as

$$
\begin{aligned}
&\langle VII., VIII. \rangle_t \\
=&\frac{\beta_L - \beta_N}{2} \int_0^t \left(\frac{\partial_j x_s^i}{\|Dx_s\|} - \frac{\partial_j y_s^i}{\|Dy_s\|} \right)^2 ds \\
&+ \frac{\beta_L + \beta_N}{2} \int_0^t \sum_{k,l} \left(\frac{\partial_j x_s^k}{\|Dx_s\|} - \frac{\partial_j y_s^k}{\|Dy_s\|} \right) \left(\frac{\partial_l x_s^k \partial_l x_s^i \partial_j x_s^i}{\|Dx_s\|^3} - \frac{\partial_l y_s^k \partial_l y_s^i \partial_j y_s^i}{\|Dy_s\|^3} \right) ds \\
&+ \sum_{k,l,m,n} \int_0^t \left(\frac{\partial_j x_s^k \partial_m y_s^n \partial_m y_s^l \partial_j y_s^i}{\|Dx_s\| \|Dy_s\|^3} + \frac{\partial_j y_s^k \partial_m x_s^n \partial_m x_s^l \partial_j x_s^i}{\|Dy_s\| \|Dx_s\|^3} \right) \tilde{b}_{k,n}^{i,l} (x_s - y_s)\, ds. \qquad (32)
\end{aligned}
$$

The combination of (30), (31) and (32) with (27) and Itô's formula now yields for $X_t^{(ij)} := \frac{\partial_j x_t^i}{\|Dx_t\|} - \frac{\partial_j y_t^i}{\|Dy_t\|}$ the following (note that $X_t^{(ij)q}$ means $(X_t^{(ij)})^q$).

Proposition 4.4 (formula H).
We have for $q \geq q_0$ that

$$
\begin{aligned}
&X_t^{(ij)q} \\
=&q\int_0^t X_s^{(ij)q-1}\sum_k\left(\partial_k F^i\left(ds,x_s\right)\frac{\partial_j x_s^k}{\left\|Dx_s\right\|}-\partial_k F^i\left(ds,y_s\right)\frac{\partial_j y_s^k}{\left\|Dy_s\right\|}\right) \\
&-q\int_0^t X_s^{(ij)q-1}\sum_{k,l,m}\left(\partial_m F^k\left(ds,x_s\right)\frac{\partial_l x_s^m\partial_l x_s^k\partial_j x_s^i}{\left\|Dx_s\right\|^3}-\partial_m F^k\left(ds,y_s\right)\frac{\partial_l y_s^m\partial_l y_s^k\partial_j y_s^i}{\left\|Dy_s\right\|^3}\right) \\
&+\left[\frac{\beta_L-\beta_N}{4}q^2-\frac{d-1}{2}\beta_N q-\frac{1}{2}\beta_L q\right]\int_0^t X_s^{(ij)q}ds \\
&+\frac{q(q-1)}{2}\beta_N\int_0^t X_s^{(ij)q-2}\sum_k X_s^{(kj)2}ds \\
&-q\frac{\beta_N+\beta_L}{2}\int_0^t X_s^{(ij)q-1}\sum_{k,l}\left(\frac{\partial_j x_s^k\partial_l x_s^i\partial_l x_s^k}{\left\|Dx_s\right\|^3}-\frac{\partial_j y_s^k\partial_l y_s^i\partial_l y_s^k}{\left\|Dy_s\right\|^3}\right)ds \\
&+q(q-1)\frac{\beta_N+\beta_L}{4}\int_0^t X_s^{(ij)q-2}\sum_{k,n}\left(\sum_l\frac{\partial_l x_s^n\partial_l x_s^k\partial_j x_s^i}{\left\|Dx_s\right\|^3}-\frac{\partial_l y_s^n\partial_l y_s^k\partial_j y_s^i}{\left\|Dy_s\right\|^3}\right)^2ds \\
&-q(q-1)\frac{\beta_N+\beta_L}{2}\int_0^t X_s^{(ij)q-2}\sum_{k,l}X_s^{(kj)}\left(\frac{\partial_l x_s^k\partial_l x_s^i\partial_j x_s^i}{\left\|Dx_s\right\|^3}-\frac{\partial_l y_s^k\partial_l y_s^i\partial_j y_s^i}{\left\|Dy_s\right\|^3}\right)ds \\
&+\frac{3}{4}q(\beta_N+\beta_L)\int_0^t X_s^{(ij)q-1}\sum_{k,l,m,n}\left(\frac{\partial_l x_s^k\partial_l x_s^n\partial_m x_s^k\partial_m x_s^n\partial_j x_s^i}{\left\|Dx_s\right\|^5}\frac{\partial_l y_s^k\partial_l y_s^n\partial_m y_s^k\partial_m y_s^n\partial_j y_s^i}{\left\|Dy_s\right\|^5}\right)ds \\
&+q(q-1)\int_0^t X_s^{(ij)q-2}\sum_{k,l}\frac{\partial_j x_s^k\partial_j y_s^l}{\left\|Dx_s\right\|\left\|Dy_s\right\|}\tilde{b}_{k,l}^{i,i}\left(x_s-y_s\right)ds \\
&-q(q-1)\int_0^t X_s^{(ij)q-2}\sum_{k,l,m,n}\left(\frac{\partial_j x_s^k\partial_m y_s^n\partial_m y_s^l\partial_j y_s^i}{\left\|Dx_s\right\|\left\|Dy_s\right\|^3}+\frac{\partial_j y_s^k\partial_m x_s^n\partial_m x_s^l\partial_j x_s^i}{\left\|Dy_s\right\|\left\|Dx_s\right\|^3}\right)\tilde{b}_{k,n}^{i,l}\left(x_s-y_s\right)ds \\
&+q(q-1)\int_0^t X_s^{(ij)q-2}\sum_{k,l,m,n,p,r}\frac{\partial_l x_s^m\partial_l x_s^k\partial_j x_s^i\partial_p y_s^r\partial_p y_s^n\partial_j y_s^i}{\left\|Dx_s\right\|^3\left\|Dy_s\right\|^3}\tilde{b}_{r,m}^{k,n}\left(x_s-y_s\right)ds.
\end{aligned}
\tag{33}
$$

Proof: There is nothing left to show. □

Proposition 4.4 will be useful to estimate the expectation in Lemma 2.3 on the event $\{T \leq \tau\}$ but since this requires some additional preparations we will first consider the reversed case in the following intermezzo.

4.3 Treating Small $|x-y|$ And Large T

Since we obviously have from Schwarz' inequality that

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|\psi_t(x)-\psi_t(y)|^q 1\!\!1_{\{T\geq\tau\}}\right]^{\frac{1}{q}} \leq \mathbb{E}\left[\sup_{0\leq t\leq T}|\psi_t(x)-\psi_t(y)|^{2q}\right]^{\frac{1}{2q}}\mathbb{P}\left[T\geq\tau\right]^{\frac{1}{2q}} \tag{34}$$

it seems reasonable to compute some useful estimate for the tails of τ. We will also immediatly specify conditions on r and $\tilde{r}$. Assume first with Lemma 1.3 that $\tilde{r} < \bar{r}$ is small enough to ensure that for any $0 \leq r \leq \tilde{r}$ we have

$$\sqrt{2\left[1-B_L(r)\right]} \leq 2\sqrt{\beta_L}r \text{ and } (d-1)\frac{1-B_N(r)}{r}-cr \leq \left[2(d-1)\beta_N-c\right]r. \tag{35}$$

Lemma 4.5 (tails of τ).

If we assume $q \geq q_0$ then we have for $T \leq \frac{\log\frac{\tilde{r}}{|x-y|}}{128\beta_L q} \wedge \frac{\log\frac{\tilde{r}}{|x-y|}}{4\sqrt{\beta_L}\left[2(d-1)\beta_N - c - 2\beta_L\right]} = \frac{\log\frac{\tilde{r}}{|x-y|}}{128\beta_L q}$ that $\mathbb{P}\left[T \geq \tau\right] \leq \frac{2}{\tilde{r}^{2q}}|x-y|^{2q} \wedge \frac{2}{\tilde{r}^{4q}}|x-y|^{4q}$.

Proof: Let $X_t = |x-y| + \int_0^t 2\sqrt{\beta_L} X_s dW_s + \int_0^t \left[2(d-1)\beta_N - c\right] X_s ds$ i.e.
$X_t := |x-y|\exp\left\{2\sqrt{\beta_L}W_s + \left[2(d-1)\beta_N - c - 2\beta_L\right] t\right\}$ for some BM $(W_t)_{t\geq 0}$. Then we may start with (see (3) and (35))

$$\begin{aligned}
\mathbb{P}\left[T \geq \tau\right] =& \mathbb{P}\left[\sup_{0\leq t\leq T} |x_t - y_t| \geq \tilde{r}\right] \leq \mathbb{P}\left[\sup_{0\leq t\leq T} X_t \geq \tilde{r}\right] \qquad (36)\\
=& \mathbb{P}\left[\sup_{0\leq t\leq T} W_t \geq \frac{1}{2\sqrt{\beta_L}}\left(\log\frac{\tilde{r}}{|x-y|} - \left[2(d-1)\beta_N - 2\beta_L - c\right]_+ T\right)\right]\\
\leq& \mathbb{P}\left[\sup_{0\leq t\leq 1} W_t \geq \frac{\log\frac{\tilde{r}}{|x-y|}}{2\sqrt{\beta_L T}} - \left[2(d-1)\beta_N - 2\beta_L - c\right]_+ \sqrt{T}\right] =: IX.
\end{aligned}$$

Let first $2(d-1)\beta_N - 2\beta_L - c \leq 0$. Then we have using $1 - \Phi(t) \leq e^{-1/2t^2}$ that

$$\begin{aligned}
IX \leq& 2\left[1 - \Phi\left(\frac{\log\frac{\tilde{r}}{|x-y|}}{2\sqrt{\beta_L T}}\right)\right] \leq 2e^{-\frac{1}{8}\frac{\left(\log\frac{\tilde{r}}{|x-y|}\right)^2}{\beta_L T}}\\
=& 2\left(\frac{|x-y|}{\tilde{r}}\right)^{\frac{1}{8\beta_L T}\log\frac{\tilde{r}}{|x-y|}} \leq \frac{2}{\tilde{r}^{2q}}|x-y|^{2q} \qquad (37)
\end{aligned}$$

for $T \leq \frac{\log\frac{\tilde{r}}{|x-y|}}{16\beta_L q}$ (remember $|x-y| < r < \tilde{r}$). Let now $2(d-1)\beta_N - 2\beta_L - c > 0$. In this case we can use for $T \leq \frac{\log\frac{\tilde{r}}{|x-y|}}{4\sqrt{\beta_L}\left[2(d-1)\beta_N - c - 2\beta_L\right]} \wedge \frac{\log\frac{\tilde{r}}{|x-y|}}{128\beta_L q}$ that

$$\begin{aligned}
IX =& \mathbb{P}\left[\sup_{0\leq t\leq 1} W_t \geq \frac{\log\frac{\tilde{r}}{|x-y|}}{2\sqrt{\beta_L T}} - \left[2(d-1)\beta_N - 2\beta_L - c\right]\sqrt{T}\right]\\
\leq& \mathbb{P}\left[\sup_{0\leq t\leq 1} W_t \geq \frac{\log\frac{\tilde{r}}{|x-y|}}{4\sqrt{\beta_L T}}\right] = 2\left[1 - \Phi\left(\frac{\log\frac{\tilde{r}}{|x-y|}}{4\sqrt{\beta_L T}}\right)\right]\\
\leq& 2\exp\left\{-\frac{1}{2}\frac{1}{16\beta_L T}\left(\log\frac{\tilde{r}}{|x-y|}\right)^2\right\} = 2\left(\frac{|x-y|}{\tilde{r}}\right)^{\frac{\log\frac{\tilde{r}}{|x-y|}}{32\beta_L T}}\\
\leq& 2\frac{|x-y|^{2q}}{\tilde{r}^{2q}} \wedge 2\frac{|x-y|^{4q}}{\tilde{r}^{4q}}. \qquad (38)
\end{aligned}$$

The proof is complete. □

Lemma 4.6 (estimate for T after τ).
For $|x-y| \le r \le e^{-1}\tilde{r}$ and arbitrary $q \ge q_0$ and $T > 0$ we have

$$\mathbb{E}\left[\sup_{0\le t\le T} |\psi_t(x)-\psi_t(y)|^q 1\!\!1_{\{T\ge\tau\}}\right]^{1/q} \le \bar{c}_2|x-y|^2 e^{(\Lambda_2+\frac{1}{2}q\sigma_2^2)T} \tag{39}$$

with $\Lambda_2 := \Lambda_1 \vee 1$, $\sigma_2 := \sqrt{2}\sigma_1 \vee 2\sqrt{128\beta_L} \vee 2$ and
$\bar{c}_2 := \frac{\sqrt{2}\bar{C}_1}{\tilde{r}} \vee \frac{1}{\tilde{r}}\left[\frac{\beta_N+\beta_L}{128\beta_L} + \frac{\sqrt{2\pi\tilde{c}}e^{-\frac{5}{12}}}{\sqrt{16\beta_L}}\right] \vee \frac{2(\beta_L+\beta_N)}{\tilde{r}} \vee \frac{2e^{-\frac{5}{12}}\sqrt{256\tilde{c}\pi\beta_L}}{\tilde{r}}.$

Proof: If $T \le \frac{\log\frac{\tilde{r}}{|x-y|}}{128\beta_L q} \wedge \frac{\log\frac{\tilde{r}}{|x-y|}}{4\sqrt{\beta_L}\left[2(d-1)\beta_N - c - 2\beta_L\right]} = \frac{\log\frac{\tilde{r}}{|x-y|}}{128\beta_L q}$ then we just have to combine Lemmas 4.2 and 4.5 with (34) to obtain

$$\mathbb{E}\left[\sup_{0\le t\le T} |\psi_t(x)-\psi_t(y)|^q 1\!\!1_{\{T\ge\tau\}}\right]^{1/q} \le |x-y|\frac{\bar{C}_1}{\tilde{r}}2^{\frac{1}{2q}}e^{\left(\Lambda_1+\frac{1}{2}\sigma_1^2 2q\right)T}$$

which means we only have to consider the case $T \ge \frac{\log\frac{\tilde{r}}{|x-y|}}{128\beta_L q}$. By Lemma 4.2 we first observe for $T_0 := \frac{\log\frac{\tilde{r}}{|x-y|}}{128\beta_L q}$ that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T_0} |\psi_t(x)-\psi_t(y)|^q\right]^{1/q} \le (\beta_N+\beta_L)\frac{\log\frac{\tilde{r}}{|x-y|}}{128\beta_L q} \\
&+ 2e^{-\frac{5}{12}}\sqrt{2\pi\tilde{c}}\sqrt{q+1}\sqrt{\frac{\log\frac{\tilde{r}}{|x-y|}}{128\beta_L q}} \le \left[\frac{\beta_N+\beta_L}{128\beta_L} + \frac{\sqrt{2\pi\tilde{c}}e^{-\frac{5}{12}}}{\sqrt{16\beta_L}}\right]\frac{\tilde{r}}{|x-y|} \\
\le & \left[\frac{\beta_N+\beta_L}{128\beta_L} + \frac{\sqrt{2\pi\tilde{c}}e^{-\frac{5}{12}}}{\sqrt{16\beta_L}}\right]\left(\frac{\tilde{r}}{|x-y|}\right)^{\frac{\Lambda_2+\frac{1}{2}\sigma_2^2 q}{128\beta_L q}-1} \\
=&\frac{1}{\tilde{r}}\left[\frac{\beta_N+\beta_L}{128\beta_L} + \frac{\sqrt{2\pi\tilde{c}}e^{-\frac{5}{12}}}{\sqrt{16\beta_L}}\right]|x-y|e^{\frac{\Lambda_2+\frac{1}{2}\sigma_2^2 q}{128\beta_L q}\log\frac{\tilde{r}}{|x-y|}} \le \bar{c}_2|x-y|e^{(\Lambda_2+\frac{1}{2}q\sigma_2^2)T_0}.
\end{aligned} \tag{40}$$

Since $f : T \mapsto \bar{c}_2|x-y|e^{(\Lambda_2+\frac{1}{2}q\sigma_2^2)T}$ is a convex function and $g : T \mapsto (\beta_N+\beta_L)T + 2e^{-\frac{5}{12}}\sqrt{2\pi\tilde{c}}\sqrt{q+1}\sqrt{T}$ is concave one we may just check that $f'(T_0) \ge g'(T_0)$.

$$\begin{aligned}
&g'(T_0) = (\beta_N+\beta_L) + e^{-\frac{5}{12}}\sqrt{256\pi\tilde{c}\beta_L}\sqrt{q(q+1)}\frac{1}{\log\frac{\tilde{r}}{|x-y|}} \\
\le&\frac{\bar{c}_2\tilde{r}}{2}(\Lambda_2+\frac{1}{2}q\sigma_2^2)\left(\frac{\tilde{r}}{|x-y|}\right)^{\frac{\Lambda_2+\frac{1}{2}\sigma_2^2 q}{128\beta_L}-1} + \frac{\bar{c}_2\tilde{r}}{2}(\Lambda_2+\frac{1}{2}q\sigma_2^2)\left(\frac{\tilde{r}}{|x-y|}\right)^{\frac{\Lambda_2+\frac{1}{2}\sigma_2^2 q}{128\beta_L}-1} \\
=&\bar{c}_2(\Lambda_2+\frac{1}{2}q\sigma_2^2)\left(\frac{\tilde{r}}{|x-y|}\right)^{\frac{\Lambda_2+\frac{1}{2}\sigma_2^2 q}{128\beta_L}}|x-y| = f'(T_0).
\end{aligned} \tag{41}$$

Thus the proof of Lemma 4.6 is complete. □
To treat (33) we finally need the following proposition.

Proposition 4.7 (expectation of zero-mean-martingales on rare events).
For $i,j \in \{1,\dots,d\}$ we define $\check{M}_t = \check{M}_t^{(ij)} := \int_0^t X_s^{(ij)q-1} dVII_t - \int_0^t X_s^{(ij)q-1} dVIII_t$. Then we have

1. *$\check{M}_t$ is a zero-mean-martingale and we have any $t \geq 0$ that $\left\langle \check{M} \right\rangle_t \leq \check{c}t$ a.s. for $\check{c} := 4(d^2 + 2d^4 + d^6) \max_{k,l,m,n} \sup_{z\in\mathbb{R}^d} \partial_k \partial_l b^{m,n}(z)$.*
2. *For all $0 \leq t \leq T$ and $q \geq q_0$ we get that $\left| \mathbb{E}\left[\check{M}_t 1\!\!1_{\{T\leq\tau\}} \right] \right|$ $\leq |x-y|^{2q} \frac{\bar{c}_3}{\tilde{r}^{2q}} e^{(\Lambda_3 + \frac{1}{2}\sigma_3^2 q + \sigma_4^2 q^2)T} =: f(T)$ for $\bar{c}_3 := \sqrt{2\check{c}} \vee \sqrt{\frac{\check{c}}{128\beta_L}} \vee 1$, $\Lambda_3 := \frac{1}{2e}$, $\sigma_3 := \sqrt{\frac{64\beta_L}{e}} \vee (128\beta_L \check{c})^{\frac{1}{4}}$ and $\sigma_4 := \sqrt{256\beta_L}$.*

Proof: 1. is clear from the definition of $\check{M}$ except for the value of $\check{c}$. This value follows from (30), (31) and (32) since $\left\langle \check{M} \right\rangle_t \leq \langle VII \rangle_i + \langle VIII \rangle_t + 2 |\langle VII, VIII \rangle_t|$. For the proof of 2. first assume $T \leq \frac{\log \frac{\tilde{r}}{|x-y|}}{128\beta_L q}$. From Lemma 4.5 we know that $\mathbb{P}[T > \tau] \leq |x-y|^{4q} \frac{2}{\tilde{r}^{4q}}$. This combined with Schwarz' inequality and 1. yields

$$\begin{aligned}\left| \mathbb{E}\left[\check{M}_t 1\!\!1_{\{T\leq\tau\}} \right] \right| &= \left| \mathbb{E}\left[\check{M}_t 1\!\!1_{\{T>\tau\}} \right] \right| \leq \sqrt{\mathbb{E}\left[\left\langle \check{M} \right\rangle_T \right]} \sqrt{\mathbb{P}[T>\tau]} \\ &\leq \sqrt{2\check{c}T} \frac{|x-y|^{2q}}{\tilde{r}^{2q}} \leq \sqrt{2\check{c}} \frac{|x-y|^{2q}}{\tilde{r}^{2q}} e^{\frac{T}{2e}} \end{aligned} \tag{42}$$

which proves Proposition 4.7 in this case. We always have as above that
$\left| \mathbb{E}\left[\check{M}_t 1\!\!1_{\{T\leq\tau\}} \right] \right| \leq \sqrt{\check{c}T} =: g(T)$ and so we can conclude for the same T_0 as before

$$\begin{aligned} g(T_0) &= \sqrt{\frac{\check{c}}{128\beta_L}} \sqrt{\log \frac{\tilde{r}}{|x-y|}} \leq \sqrt{\frac{\check{c}}{128\beta_L}} \left(\frac{\tilde{r}}{|x-y|} \right)^{\frac{1}{2e}} \\ &\leq \bar{c}_3 \left(\frac{\tilde{r}}{|x-y|} \right)^{\frac{\sigma_3^2}{128\beta_L}} \leq \bar{c}_3 \left(\frac{\tilde{r}}{|x-y|} \right)^{\frac{\Lambda_3 + \sigma_3^2 q + \sigma_4^2 q^2}{128\beta_L q}} \left(\frac{|x-y|}{\tilde{r}} \right)^{2q} \\ &= |x-y|^{2q} \frac{\bar{c}_3}{\tilde{r}^{2q}} e^{(\Lambda_3 + \frac{1}{2}\sigma_3^2 q + \sigma_4^2 q^2)T_0} =: f(T_0) \end{aligned} \tag{43}$$

so now (with Lemma 1.4) we only have to establish the inequality $g'(T_0) \leq f'(T_0)$ to complete the proof of Proposition 4.7. The proof of this is as follows.

$$g'(T_0) = \frac{1}{2} \sqrt{\frac{128\beta_L q \check{c}}{\log \frac{\tilde{r}}{|x-y|}}} \leq \frac{1}{2} \sqrt{128\beta_L q \check{c}} \tag{44}$$

$$\leq \left(\frac{|x-y|}{\tilde{r}} \right)^2 \left(\Lambda_3 + \frac{1}{2}\sigma_3^2 q + \sigma_4^2 q^2 \right) \left(\frac{\tilde{r}}{|x-y|} \right)^{\frac{\Lambda_3 + \frac{1}{2}\sigma_3^2 q + \sigma_4^2 q^2}{128\beta_L}} = f'(T_0).$$

The proof is complete. ☐

4.4 Evaluation Of Formula H

Now we are prepared to prove the following key result.

Lemma 4.8 (first estimate for τ after T).
We have for even $q \geq q_0$ if we assume $\tilde{r} \leq \frac{1}{\sqrt{2}}$ the following.

1. *For $h(t) := \mathbb{E}\left[(\max_{i,j} X_t^{(ij)q} \vee |x_t - y_t|^q) 11_{\{t \leq \tau\}}\right]$ the estimate holds*

$$h(t) \leq |x-y|^q \frac{2\bar{c}_3 d^2 q}{\tilde{r}^{2q}} \frac{e^{(\Lambda_3 + (\lambda \vee \frac{1}{2}\sigma_3^2)q + (\frac{1}{2}\bar{\sigma}^2 \vee \sigma_4^2)q^2 + d^2\kappa_q)t}}{\Lambda_3 + (\lambda \vee \frac{1}{2}\sigma_3^2)q + (\frac{1}{2}\bar{\sigma}^2 \vee \sigma_4^2)q^2 + d^2\kappa_q} \tag{45}$$

wherein κ_q behaves like a polynomial of degree 2 (w.r.t. growth of its modulus) in q and equals
$|\frac{\beta_L - \beta_N}{4} q^2 - \frac{d-1}{2}\beta_N q - \frac{1}{2}\beta_L q|$
$+\{Cd^6 + \left[\frac{9}{4}(\beta_N + \beta_L) + 2C\right] d^4 + \left[\frac{3}{2}(\beta_N + \beta_L) + C\right] d^2 + \frac{\beta_N}{2} d\} q^2$
$-\left\{Cd^6 + \left[2C - \frac{3}{2}(\beta_N + \beta_L)\right] d^4 + Cd^2 + \frac{1}{2}\beta_N d\right\} q.$

2.
$$\mathbb{E}\left[\max_{i,j} X_t^{(ij)q} 11_{\{T \leq \tau\}}\right] \leq h(t) \leq \bar{c}_5 \tilde{r}^{-2q} |x-y|^q e^{(\Lambda_5 q + \sigma_5^2 q^2)t} \tag{46}$$

for $\bar{c}_5 := \sup_{q \geq 3} \left\{\frac{2\bar{c}_3 d^2 q}{\Lambda_3 + (\lambda \vee \frac{1}{2}\sigma_3^2)q + (\frac{1}{2}\bar{\sigma}^2 \vee \sigma_4^2)q^2 + d^2\kappa_q}\right\}$,
$\Lambda_5 := \Lambda_3 + \lambda \vee \frac{1}{2}\sigma_3^2 - \left\{Cd^8 + \left[2C - \frac{3}{2}(\beta_N + \beta_L)\right] d^6 + Cd^4 - \frac{|\beta_N - \beta_L|}{2} d^2\right\}$ *and*
$\sigma_5^2 := \frac{1}{2}\bar{\sigma}^2 \vee \sigma_4^2 + \left\{Cd^8 + \left[\frac{9}{4}(\beta_N + \beta_L) + 2C\right] d^6 + \left[\frac{3}{2}(\beta_N + \beta_L) + C\right] d^4\right.$
$\left.+ \frac{\beta_N}{2} d^3 + \frac{|\beta_L - \beta_N|}{4} d^2\right\}.$

Proof: 2. follows obviously from 1. so we only have to prove this. Since the treatment of Proposition 4.4 leads to rather huge formulas we restrict ourselves to indicate how to computation is done omitting terms when treating similar ones. We first multiply (33) with $11_{\{t \leq \tau\}}$ then we take expectations and apply Fubini's theorem. Recalling the short-hand $\tilde{b}_{k,l}^{i,j}(z) := \partial_k \partial_l b^{i,j}(z) - \partial_k \partial_l b^{i,j}(0)$ we get

$$\begin{aligned}
\mathbb{E}\left[X_t^{(ij)q} 11_{\{t \leq \tau\}}\right] &= q\mathbb{E}\left[11_{\{t \leq \tau\}} \check{M}_t^{(ij)}\right] + \ldots - q\frac{\beta_N + \beta_L}{2} \int_0^t \mathbb{E}\left[X_s^{(ij)q-1}\right. \\
&\left.\sum_{k,l} \left(\frac{\partial_j x_s^k \partial_l x_s^i \partial_l x_s^k}{\|Dx_s\|^3} - \frac{\partial_j y_s^k \partial_l y_s^i \partial_l y_s^k}{\|Dy_s\|^3}\right) 11_{\{t \leq \tau\}}\right] ds + \ldots \\
&+ q(q-1)\int_0^t \mathbb{E}\left[X_s^{(ij)q-2} \sum_{k,l} \frac{\partial_j x_s^k \partial_j y_s^l}{\|Dx_s\| \|Dy_s\|} \tilde{b}_{k,l}^{i,i}(x_s - y_s) 11_{\{t \leq \tau\}}\right] ds
\end{aligned} \tag{47}$$

Applying the triangle inequality and using Lemma 1.3 we get

$$\mathbb{E}\left[X_t^{(ij)q}1\!\!1_{\{t\le\tau\}}\right]\le q\mathbb{E}\left[1\!\!1_{\{t\le\tau\}}\check{M}_t^{(ij)}\right]+\ldots+q\frac{\beta_N+\beta_L}{2}\int_0^t\mathbb{E}\left[\left|X_s^{(ij)q-1}\right|\right.$$
$$\left.\sum_{k,l}\left|\frac{\partial_j x_s^k\partial_l x_s^i\partial_l x_s^k}{\left\|Dx_s\right\|^3}-\frac{\partial_j y_s^k\partial_l y_s^i\partial_l y_s^k}{\left\|Dy_s\right\|^3}\right|1\!\!1_{\{t\le\tau\}}\right]ds+\ldots$$
$$+q(q-1)C\int_0^t\mathbb{E}\left[X_s^{(ij)q-2}\sum_{k,l}\frac{\left|\partial_j x_s^k\partial_j y_s^l\right|}{\left\|Dx_s\right\|\left\|Dy_s\right\|}|x_s-y_s|^2 1\!\!1_{\{t\le\tau\}}\right]ds \tag{48}$$

We recall the inequality $|\prod a_i-\prod b_i|\le\sum_i|a_i-b_i|\le d\max_i|a_i-b_i|$ (valid for $|a_i|\vee|b_i|\le 1$), use that the expectation of a positive random variable is growing in the domain of integration and conclude that the latter is less or equal to

$$q\mathbb{E}\left[1\!\!1_{\{t\le\tau\}}\check{M}_t^{(ij)}\right]+q\frac{\beta_N+\beta_L}{2}\int_0^t 3d^2\mathbb{E}\left[\left(\max_{i,j}X_s^{(ij)q}\vee|x_s-y_s|^q\right)1\!\!1_{\{s\le\tau\}}\right]ds$$
$$+q(q-1)Cd^2\int_0^t\mathbb{E}\left[\left(\max_{i,j}X_s^{(ij)q}\vee|x_s-y_s|^q\right)1\!\!1_{\{s\le\tau\}}\right]ds+\ldots \tag{49}$$

Summing up all the terms we suppressed we get that $\mathbb{E}\left[X_t^{(ij)q}1\!\!1_{\{t\le\tau\}}\right]$ is at most

$$\begin{aligned}
&q\mathbb{E}\left[1\!\!1_{\{t\le\tau\}}\check{M}_t^{(ij)}\right]\\
&+\left|\frac{\beta_L-\beta_N}{4}q^2-\frac{d-1}{2}\beta_N q-\frac{1}{2}\beta_L q\right|\int_0^t\mathbb{E}\left[\left(\max_{i,j}X_s^{(ij)q}\vee|x_s-y_s|^q\right)1\!\!1_{\{s\le\tau\}}\right]ds\\
&+\frac{q(q-1)}{2}\beta_N d\int_0^t\mathbb{E}\left[\left(\max_{i,j}X_s^{(ij)q}\vee|x_s-y_s|^q\right)1\!\!1_{\{s\le\tau\}}\right]ds\\
&+q\frac{\beta_N+\beta_L}{2}3d^2\int_0^t\mathbb{E}\left[\left(\max_{i,j}X_s^{(ij)q}\vee|x_s-y_s|^q\right)1\!\!1_{\{s\le\tau\}}\right]ds\\
&+q(q-1)\frac{\beta_N+\beta_L}{4}9d^4\int_0^t\mathbb{E}\left[\left(\max_{i,j}X_s^{(ij)q}\vee|x_s-y_s|^q\right)1\!\!1_{\{s\le\tau\}}\right]ds\\
&+q(q-1)\frac{\beta_N+\beta_L}{2}3d^2\int_0^t\mathbb{E}\left[\left(\max_{i,j}X_s^{(ij)q}\vee|x_s-y_s|^q\right)1\!\!1_{\{s\le\tau\}}\right]ds\\
&+\frac{3}{4}q(\beta_N+\beta_L)5d^4\int_0^t\mathbb{E}\left[\left(\max_{i,j}X_s^{(ij)q}\vee|x_s-y_s|^q\right)1\!\!1_{\{s\le\tau\}}\right]ds\\
&+q(q-1)(d^2+2d^4+d^6)C\int_0^t\mathbb{E}\left[\left(\max_{i,j}X_s^{(ij)q}\vee|x_s-y_s|^q\right)1\!\!1_{\{s\le\tau\}}\right]ds
\end{aligned}$$

$$=q\mathbb{E}\left[1\!\!1_{\{t\le\tau\}}\check{M}_t^{(ij)}\right]+\kappa_q\int_0^t\mathbb{E}\left[\left(\max_{i,j}X_s^{(ij)q}\vee|x_s-y_s|^q\right)1\!\!1_{\{s\le\tau\}}\right]ds.$$

This enables us to conclude with Proposition 4.7 and Lemma 1.2

$$
\begin{aligned}
&\mathbb{E}\left[\max_{i,j} X_t^{(ij)q} 11_{\{t\leq\tau\}} \vee |x_t - y_t|^q 11_{\{t\leq\tau\}}\right] \\
\leq &\sum_{i,j} \mathbb{E}\left[X_t^{(ij)q} 11_{\{t\leq\tau\}} 11_{\{t\leq\tau\}}\right] + \mathbb{E}\left[|x_t - y_t|^q\right] \leq \mathbb{E}\left[|x_t - y_t|^q\right] \\
&+ d^2 q \sum_{i,j} \mathbb{E}\left[11_{\{t\leq\tau\}} \check{M}_t^{(ij)}\right] + d^2\kappa_q \int_0^t \mathbb{E}\left[\left(\max_{i,j} X_s^{(ij)q} \vee |x_s - y_s|^q\right) 11_{\{s\leq\tau\}}\right] ds \\
\leq & d^2 q |x-y|^{2q} \frac{\bar{c}_3}{\tilde{r}^{2q}} e^{(\Lambda_3 + \frac{1}{2}\sigma_3^2 q + \sigma_4^2 q^2)t} + 2^q |x-y|^q e^{(\lambda q + \frac{1}{2} q^2 \bar{\sigma}^2)t} \\
&+ d^2\kappa_q \int_0^t \mathbb{E}\left[\left(\max_{i,j} X_s^{(ij)q} \vee |x_s - y_s|^q\right) 11_{\{s\leq\tau\}}\right] ds \\
\leq & d^2\kappa_q \int_0^t \mathbb{E}\left[\left(\max_{i,j} X_s^{(ij)q} \vee |x_s - y_s|^q\right) 11_{\{s\leq\tau\}}\right] ds \\
&+ |x-y|^q \left(\frac{2\bar{c}_3 d^2 q}{\tilde{r}^{2q}}\right) e^{(\Lambda_3 + (\Lambda \vee \frac{1}{2}\sigma_3^2) q + (\frac{1}{2}\bar{\sigma}^2 \vee \sigma_4^2) q^2)t} \qquad (50)
\end{aligned}
$$

since we assumed $\tilde{r} \leq 2^{-1/2}$. This now implies via Grönwall's inequality (see [15, I.§2.1] for an appropriate version) that

$$
h(t) \leq \qquad |x-y|^q 2 \frac{\bar{c}_3 d^2 q}{\tilde{r}^{2q}} \frac{e^{(\Lambda_3 + (\lambda \vee \frac{1}{2}\sigma_3^2) q + (\frac{1}{2}\bar{\sigma}^2 \vee \sigma_4^2) q^2 + d^2\kappa_q)t}}{\Lambda_3 + (\lambda \vee \frac{1}{2}\sigma_3^2) q + (\frac{1}{2}\bar{\sigma}^2 \vee \sigma_4^2) q^2 + d^2\kappa_q}. \qquad (51)
$$

The proof is complete. □

The next lemma will be the last ingredient to the proof of Lemma 2.3.

Lemma 4.9 (second estimate for τ after T).
We have for even $q \geq q_0$ that
$\mathbb{E}\left[\sup_{0\leq t\leq T} |\psi_t(x) - \psi_t(y)|^q 11_{\{T\leq\tau\}}\right]^{\frac{1}{q}} \leq \bar{c}_6 \sqrt{q} |x-y| e^{(\Lambda_6 + \frac{1}{2}\sigma_2^2 q)T}$
with $\Lambda_6 = \Lambda_5 + \frac{1}{e}$, $\sigma_6 := \sqrt{2}\sigma_5$ and
$\bar{c}_6 := \sup_{q\geq 3}\left(\frac{\bar{c}_5^{\frac{1}{q}}}{\tilde{r}^2} \frac{\sqrt{2} d^2 C_q^{\frac{1}{q}} q^{-\frac{1}{2}} \sqrt{\beta_L + \beta_N} + \sqrt{C} d^3 C_q^{\frac{1}{q}} q^{-\frac{1}{2}} \sqrt{2} + 2d^4(\beta_N + \beta_L)}{(\Lambda_5 q + \sigma_5^2 q^2)^{\frac{1}{q}}}\right)$*. Therein $C_q^{\frac{1}{q}}$ is the largest positive zero of the Hermite polynomial of order $2q$ and can be estimated via $C_q^{\frac{1}{q}} \leq k\sqrt{4q+1}$ for some constant $k > 0$ (see [1] and [14]).*

Proof: By the triangle inequality and the Burkholder-Davies-Gundy inequality, Lemma 4.1 and Jensen's inequality we have

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T} |\psi_t(x) - \psi_t(y)|^q 11_{\{T\leq\tau\}}\right]^{\frac{1}{q}} \leq \mathbb{E}\left[\sup_{0\leq t\leq T} \tilde{M}_t^q 11_{\{T\leq\tau\}}\right]^{\frac{1}{q}} \\
&+ \mathbb{E}\left[\sup_{0\leq t\leq T} A_t^q 11_{\{T\leq\tau\}}\right]^{\frac{1}{q}} \leq C_q^{\frac{1}{q}} \mathbb{E}\left[\langle M\rangle_{T\wedge\tau}^{\frac{q}{2}}\right]^{\frac{1}{q}} + \mathbb{E}\left[A_{T\wedge\tau}^q\right]^{\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C_q^{\frac{1}{q}}\sqrt{\frac{\beta_L+\beta_N}{2}}\mathbb{E}\left[\left(\left(\int_0^{T\wedge\tau} 4d^4\left(\max_{i,j} X_t^{(ij)}\vee|x_t-y_t|\right)^2 ds\right)^{\frac{q}{2}}\right]^{\frac{1}{q}}\\
&\quad + C_q^{\frac{1}{q}}\sqrt{2}\mathbb{E}\left[\left(Cd^6\int_0^{T\wedge\tau}\left(\max_{i,j} X_t^{(ij)}\vee|x_t-y_t|\right)^2 ds\right)^{\frac{q}{2}}\right]^{\frac{1}{q}}\\
&\quad + \frac{\beta_N+\beta_L}{2}\mathbb{E}\left[\left(4d^4\int_0^{T\wedge\tau}\max_{i,j} X_t^{(ij)}\vee|x_t-y_t|ds\right)^q\right]^{\frac{1}{q}}\\
&\leq \sqrt{2}d^2C_q^{\frac{1}{q}}\sqrt{\beta_L+\beta_N}T^{\frac{q-2}{2q}}\mathbb{E}\left[\int_0^T\left(\max_{i,j} X_t^{(ij)}\vee|x_t-y_t|\right)^q 1\!\!1_{\{t\leq\tau\}}ds\right]^{\frac{1}{q}}\\
&\quad + \sqrt{C}d^3C_q^{\frac{1}{q}}\sqrt{2}T^{\frac{q-2}{2q}}\mathbb{E}\left[\int_0^T\left(\max_{i,j} X_t^{(ij)}\vee|x_t-y_t|\right)^q 1\!\!1_{\{t\leq\tau\}}ds\right]^{\frac{1}{q}}\\
&\quad + 2d^4(\beta_N+\beta_L)T^{\frac{q-1}{q}}\mathbb{E}\left[\int_0^T\left(\max_{i,j} X_t^{(ij)}\vee|x_t-y_t|\right)^q 1\!\!1_{\{t\leq\tau\}}ds\right]^{\frac{1}{q}}. \qquad (52)
\end{aligned}
$$

Another application of Fubini's theorem now yields that the latter is less or equal to (remember that we chose even $q\geq 3$)

$$
\begin{aligned}
&\left(\sqrt{2}d^2C_q^{\frac{1}{q}}\sqrt{\beta_L+\beta_N}T^{\frac{q-2}{2q}}+\sqrt{C}d^3C_q^{\frac{1}{q}}\sqrt{2}T^{\frac{q-2}{2q}}+2d^4(\beta_N+\beta_L)T^{\frac{q-1}{q}}\right)\left(\int_0^T h(s)ds\right)^{\frac{1}{q}}\\
&\leq\left(\sqrt{2}d^2C_q^{\frac{1}{q}}\sqrt{\beta_L+\beta_N}+\sqrt{C}d^3C_q^{\frac{1}{q}}\sqrt{2}+2d^4(\beta_N+\beta_L)\right)\frac{\bar{c}_5^{\frac{1}{q}}|x-y|e^{(\Lambda_5+\frac{1}{e}+\sigma_5^2 q)T}}{(\Lambda_5 q+\sigma_5^2 q^2)^{\frac{1}{q}}\tilde{r}^2}.
\end{aligned}
$$

The proof is complete. $\square$

We now fix $\tilde{r}\leq\bar{r}\wedge 2^{-1/2}$ subject to (35) and $r\leq e^{-1}\tilde{r}$ and conclude that since $\sup_{0\leq t\leq T}|\psi_t(x)-\psi_t(y)|^q$ is the sum of the terms $\sup_{0\leq t\leq T}|\psi_t(x)-\psi_t(y)|^q 1\!\!1_{\{T\leq\tau\}}$ and $\sup_{0\leq t\leq T}|\psi_t(x)-\psi_t(y)|^q 1\!\!1_{\{T>\tau\}}$ another triangle inequality with Lemmas 4.2, 4.6 and 4.9 completes the proof of Lemma 2.3 and hence of Theorem 2.1. $\square$

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