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The Exit Place of Brownian Motion in the Complement of a Horn

Dante DeBlassie
Department of Mathematics
Texas A&M University
3368 TAMU
College Station, TX 77843-3368

deblass@math.tamu.edu

Abstract

Consider the domain $\Omega = \{(\tilde{x}, x_{d+1}) \in \mathbb{R}^d \times \mathbb{R} \colon \ |\tilde{x}| > 1 \quad \text{and} \quad |x_{d+1}| < A|\tilde{x}|^\alpha\}$, where $0 < \alpha \leq 1$. We determine asymptotics of the logarithm of the chance that Brownian motion in Ω has a large exit place. The behavior is given explicitly in terms of the geometry of the domain. It is independent of the dimension when $\alpha < 1$ and dependent on the dimension when $\alpha = 1$. Analytically, the result is equivalent to estimating the harmonic measure of $\partial\Omega$ outside a cylinder with large diameter .

Key words: Horn-shaped domain, h-transform, Feynman-Kac representation, exit place of Brownian motion, harmonic measure.

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1 Introduction

For an open set $D \subseteq \mathbb{R}^d$, let τ_D be the first exit time of d-dimensional Brownian motion B_t from D. It is well-known that the exit place of two-dimensional Brownian motion from the upper half plane U has a Cauchy distribution (Durrett (1984)). Thus the chance that the exit place is large decays as the reciprocal of the distance to the origin:

$$P_x(|B_{\tau_U}| > r) \sim C(x) r^{-1} \quad \text{as} \quad r \to \infty.$$

In particular, $|B_{\tau_U}|$ has a moment of order $p > 0$ iff $p < 1$.

The half-plane is a special case of a two-dimensional wedge with angle $\theta = \frac{\pi}{2}$. In the case of an axially symmetric cone C with angle $\theta \in (0, \pi)$ in dimension $d \geq 2$, Burkholder (1977) showed there is a critical power $p(d, \theta)$ such that for $p > 0$,

$$E_x[|B_{\tau_C}|^p] < \infty \quad \text{iff} \quad p < p(d, \theta).$$

For $d \geq 3$, the critical power can be expressed in terms of the first positive zero of a certain hypergeometric function and the function $\theta \in (0, \pi) \mapsto p(d, \theta)$ is continuous, strictly decreasing, onto $(0, \infty)$ and $p\big(d, \frac{\pi}{2}\big) = 1$. For $d = 2$, $p(2, \theta) = \frac{\pi}{2\theta}$. From classical estimates of harmonic measure (Essén and Haliste (1984) and Haliste (1984)),

$$C_1(x) r^{-p(d,\theta)} \leq P_x(|B_{\tau_C}| > r) \leq C_2(x) r^{-p(d,\theta)}, \quad r \text{ large}.$$

More recently, in the course of their study of iterated Brownian motion in a cone, Bañuelos and DeBlassie (2006) proved

$$P_x(|B_{\tau_C}| > r) \sim C(x) r^{-p(d,\theta)} \quad \text{as} \quad r \to \infty,$$

where $C(x)$ was explicitly identified. Consequently,

$$\lim_{r \to \infty} (\log r)^{-1} \log P_x(|B_{\tau_C}| > r) = -p(d, \theta).$$

In addition to this, we showed there is an analogous result for more general cones. Since the result will be used below, we state it explicitly. An open set $G \subseteq \mathbb{R}^d$ is a cone if for each $c > 0$, $cG = G$. If G is a cone and $G \cap S^{d-1}$ is C^3, then we showed that

$$\lim_{r \to \infty} (\log r)^{-1} \log P_x(|B_{\tau_G}| > r) = -p(d, G), \tag{1}$$

where

$$p(d, G) = \frac{-(d-2) + \sqrt{(d-2)^2 + 4\gamma_1}}{2}$$

and γ_1 is the smallest positive Dirichlet eigenvalue of the Laplace-Beltrami operator $\Delta_{S^{d-1}}$ on $G \cap S^{d-1}$.

It is natural to ask how the geometry of other unbounded domains will influence the chance of a large exit position. Note that an equivalent analytic formulation is to determine how the harmonic measure of the part of the domain outside a large ball depends on the geometry.

For example, Bañuelos and Carroll (2005) used Carleman-type estimates and a conformal technique to study the parabolic-type domain

$$\mathcal{P}_\alpha = \{(\tilde{x}, x_{d+1}) \in \mathbb{R}^d \times \mathbb{R} \colon x_{d+1} > 0 \quad \text{and} \quad |\tilde{x}| < A x_{d+1}^\alpha\},$$

where $0 < \alpha < 1$ and $A > 0$. (Note that $\alpha = 1$ corresponds to the axially symmetric cone C described above). One of their principal results was that the chance of a large exit place is subexponential:

$$\lim_{t\to\infty} t^{\alpha-1} \log P_x(|B(\tau_\alpha)| > t) = -\frac{\sqrt{\lambda_1}}{A(1-\alpha)},$$

where λ_1 is the smallest positive eigenvalue of the Dirichlet Laplacian on the unit ball of $\mathbb{R}^d$. Thus the chance depends on both the dimension and the geometry of the domain. Note when $d = 1$, $\sqrt{\lambda_1} = \frac{\pi}{2}$. The authors actually derived more refined results; please see that paper for the details.

In this article, we study the corresponding question for the domain

$$\Omega = \{(\tilde{x}, x_{d+1}) \in \mathbb{R}^d \times \mathbb{R} \colon \ |\tilde{x}| > 1 \quad \text{and} \quad |x_{d+1}| < A|\tilde{x}|^\alpha\}, \tag{2}$$

where $A > 0$ and $0 < \alpha \le 1$. To understand what the domain looks like, it is useful to pass to the cylindrical coordinates (ρ, z, θ) of $x = (\tilde{x}, x_{d+1}) \in (\mathbb{R}^d \backslash \{0\}) \times \mathbb{R}$:

$$(\rho, z, \theta) = (|\tilde{x}|, x_{d+1}, \tilde{x}/|\tilde{x}|).$$

Indeed, we can write

$$\Omega = \Omega_D = D \times S^{d-1}, \tag{3}$$

where

$$D = \{(\rho, z) \colon \rho > 1, \quad |z| < A\rho^\alpha\}.$$

With this in mind, when $d = 2$, Ω is obtained by revolving the region

$$\{(\rho, z) \colon \rho > 1, \quad |z| < A\rho^\alpha\}$$

about the z-axis. In contrast, the parabolic-type region studied by Bañuelos and Carroll would be obtained by revolving

$$\{(\rho, z) \colon \rho > 0, \quad |\rho| < A z^\alpha\}$$

about the z-axis.

Since the $(d+1)$-dimensional Laplacian in cylindrical coordinates is given by

$$\Delta_{\mathbb{R}^{d+1}} = \frac{\partial^2}{\partial\rho^2} + \frac{d-1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2} + \frac{1}{\rho^2}\Delta_{S^{d-1}},$$

a further advantage of the representation (3) is that by symmetry, the chance of a large exit place $|B(\tau_{\Omega_D})|$ of $(d+1)$-dimensional Brownian motion from Ω_D is the same as the chance of a large exit place of the diffusion corresponding to the operator

$$\frac{1}{2}\left[\frac{\partial^2}{\partial\rho^2} + \frac{d-1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2}\right] \tag{4}$$

from D. We will make use of this fact below.

The domain Ω is a special case of the following more general domain studied by several authors:

$$\Omega_a = \{(\tilde{x}, x_{d+1}) \in \mathbb{R}^d \times \mathbb{R} \colon \ |\tilde{x}| > x_0 \quad \text{and} \quad |x_{d+1}| < a(|\tilde{x}|)\},$$

where $x_0 \geq 0$, $a : [x_0, \infty) \to [0, \infty)$ is continuous and positive on (x_0, ∞) with $a(x_0) \geq 0$.

For example, under some regularity conditions on the function $a(\cdot)$, Ioffe and Pinsky (1994) completely identified the positive harmonic functions in Ω_a vanishing continuously at the boundary. Aikawa and Murata (1996), Murata (2002) and Murata (2005) weakened the conditions on $a(\cdot)$ and also extended the result to nonsymmetric domains where the condition $|x_{d+1}| < a(|\tilde{x}|)$ is replaced by $b(|\tilde{x}|) < |x_{d+1}| < a(|\tilde{x}|)$. Collet et al (2006) studied ratio limit theorems for the Dirichlet heat kernel in Ω_a with $a(t) = \sqrt{t}$; they also found the chance of a long lifetime of Brownian motion in that domain. Using another method, DeBlassie (2007) found the chance of a long lifetime of Brownian motion for $a(t) = t^\alpha$, $0 < \alpha < 1$. Although all these authors call Ω_a a horn-shaped domain, as we saw above in three dimensions, it is better to call it the complement of a horn-shaped domain to conform to more common usage in the literature.

Now we dispense with the case $\alpha = 1$ in the following example.

Example. Let

$$D = \{(\rho, z) \colon \ \rho > 0, \quad |z| < A\rho\},$$

where $A > 0$, and define

$$\Omega_D = D \times S^{d-1}.$$

Since we are interested in the chance of a large exit place $|B(\tau_{\Omega_D})|$ of $(d+1)$-dimensional Brownian motion from Ω_D, it is intuitively clear that modifying D within a ball will have little effect on the chance of a large exit place (this idea is made precise below in the proof of our main result for the case $0 < \alpha < 1$; we omit the details for the example). Thus Ω_D is more or less the case of Ω from (2) with $\alpha = 1$.

On the other hand, since $c\Omega_D = \Omega_D$ for any $c > 0$, Ω_D is a generalized cone in $\mathbb{R}^{d+1}$ and so the analogue of the formula (1) holds:

$$\lim_{r \to \infty} (\log r)^{-1} \log P_x(|B(\tau_{\Omega_D})| > r) = -p(d), \tag{5}$$

where

$$p(d) = \frac{-(d-1) + \sqrt{(d-1)^2 + 4\gamma_1(d)}}{2}$$

and $\gamma_1(d)$ is the smallest positive Dirichlet eigenvalue of the Laplace-Beltrami operator Δ_{S^d} on $\Omega_D \cap S^d$. Below in section 6 we will show that $p(d)$ is not independent of d. Thus the limit in (5) is not independent of the dimension. $\square$

Our main result is the following theorem.

Theorem 1.1. *Let B_t be $(d+1)$-dimensional Brownian motion. Then for Ω as in* (2) *with $0 < \alpha < 1$ and $A > 0$,*

$$\lim_{N \to \infty} N^{\alpha - 1} \log P_x(|B(\tau_\Omega)| > N) = -\frac{\pi}{2A(1-\alpha)}. \qquad \square$$

In light of the example above, the fact that the limit is independent of the dimension is interesting and surprising. The reason for the independence is that there is a delicate balance of competing effects between the geometry of the domain and the transience of the process. More precisely, recall that the problem reduces to the study of the diffusion corresponding to the operator (4) in the region

$$D = \{(\rho, z) \colon \rho > 1, \quad |z| < A|\rho|^\alpha\}.$$

The Bessel-type drift $\frac{d-1}{\rho}$ in (4) pushes the process away from the boundary and tends to increase the chance of a large exit place. Although the drift is small when the process is far away, when $\alpha = 1$, it is not negligible as indicated by the dimensional dependence of the limit (5). As an aside, remember that a Bessel-type drift added to a one-dimensional Brownian motion cannot be neglected even when the process is is far away: the drift can completely change the recurrence/transience of the process.

On the other hand, Theorem 1.1 tells us that in the case when $\alpha < 1$, the effect of the drift is negligible. The key is that the $\alpha < 1$ domain expands more slowly than the $\alpha = 1$ domain. Indeed, because of this difference in expansion, the vertical component of the process tends to force a quicker exit, hence a smaller exit place, from the $\alpha < 1$ domain. The effect is to reduce the chance of a large exit place. Thus there are two competing effects on the chance of a large exit place. When $\alpha < 1$ the tendency to reduce the chance of a large exit place completely overwhelms the tendency of the drift to increase the chance and so we get dimensional independence of the limit in Theorem 1.1. It is interesting to note there is a "phase transition" from power law decay to subexponential decay in the asymptotic behavior of the chance of a large exit place as the growth of the domain passes from linear to sublinear.

The organization of the article follows. In Section 2 we state some preliminaries and reduce the problem to two dimensions. A Feynman–Kac representation of an h-transform is given in Section 3. In Section 4 we derive an upper bound for $d \geq 3$ and a lower bound for $d = 2$, both using an h-transform. In Section 5 we use results of Carroll and Hayman on a conformal transformation to a strip together with an h-transform to derive a lower bound for $d \geq 3$ and an upper bound for $d = 2$. In Section 6 we prove the dimensional dependence of the limit in (5).

2 Preliminaries

Suppose $G \subseteq \mathbb{R}^n$ is open and $b\colon\ G \to \mathbb{R}^n$ is bounded and measurable. When $G = \mathbb{R}^n$, by the diffusion associated with $\frac{1}{2}\Delta_{\mathbb{R}^n} + b \cdot \nabla$ on G we will mean the diffusion associated with the corresponding martingale problem, which is well-posed in this context (Stroock and Varadhan (1979)). When $G \neq \mathbb{R}^n$, extend b to be bounded and measurable on all of $\mathbb{R}^n$ and let X be the diffusion associated with the extended operator on $\mathbb{R}^n$. The diffusion associated with $\frac{1}{2}\Delta_{\mathbb{R}^n} + b\cdot\nabla$ on G is then defined to be X killed upon exiting G.

For $x = (\tilde{x}, x_{d+1}) \in (\mathbb{R}^d\backslash\{0\}) \times \mathbb{R}$, recall that

$$(\rho, z, \theta) = (|\tilde{x}|, x_{d+1}, \tilde{x}/|\tilde{x}|)$$

denote the cylindrical coordinates of x and in these coordinates,

$$\Delta_{\mathbb{R}^{d+1}} = \frac{\partial^2}{\partial \rho^2} + \frac{d-1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} + \frac{1}{\rho^2}\Delta_{S^{d-1}}.$$

Define

$$L = \frac{1}{2}\left[\frac{\partial^2}{\partial\rho^2} + \frac{d-1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2}\right],$$
$$D = \{(\rho, z) \in \mathbb{R}^2 \colon\ \rho > 1, |z| < A\rho^\alpha\}$$

and denote by X the diffusion associated with L on D.

By symmetry, for $x = (\tilde{x}, x_{d+1}) \in \Omega$, the function $P_x(|B(\tau_\Omega)| > N)$ is independent of the angular coordinate $\theta = \tilde{x}/|\tilde{x}|$. Consequently if

$$\tau_D = \tau_D(X) = \inf\{t > 0 \colon\ X_t \notin D\}$$

we have for $(\rho, z) = (|\tilde{x}|, x_{d+1})$,

$$P_x(|B(\tau_\Omega)| > N) = P_{(\rho,z)}(|X(\tau_D)| > N).$$

Thus Theorem 1.1 is an immediate consequence of the following theorem.

Theorem 2.1. *Let $x = (\rho, z) \in D$. Then*

$$\lim_{N\to\infty} N^{\alpha-1} \log P_x(|X(\tau_D)| > N) = -\frac{\pi}{2A(1-\alpha)}. \qquad \square$$

The next reduction is analogous to a reduction in Bañuelos and Carroll (2005). It shows that the heart of the matter is in the behavior of the first coordinate of the exit place.

Lemma 2.2. *For large N,*

$$P_x(X_1(\tau_D) > N) \le P_x(|X(\tau_D)| > N) \le P_x(X_1(\tau_D) > N[1 - |o(1)|]).$$

Proof. Now if $|X(\tau_D)| > N > 1$ then

$$\begin{aligned} |X(\tau_D)|^2 &= X_1^2(\tau_D) + X_2^2(\tau_D) \\ &= X_1^2(\tau_D) + A^2 X_1^{2\alpha}(\tau_D). \end{aligned}$$

Thus

$$\begin{aligned} P_x(X_1(\tau_D) > N) &\le P_x(|X(\tau_D)|^2 > N^2 + A^2 N^{2\alpha}) \\ &\le P_x(|X(\tau_D)| > N). \end{aligned}$$

As for the other inequality, choose $x_1(N)$ such that $[x_1(N)]^2 + A^2[x_1(N)]^{2\alpha} = N^2$. Then $x_1(N)$ is the first coordinate of the intersection of the curve $z = A\rho^\alpha$ with the circle $z^2 + \rho^2 = N^2$ and so it is easy to see

$$x_1(N) = N[1 - |o(1)|] \quad \text{as} \quad N \to \infty.$$

In particular, if $|X(\tau_D)| > N$, then X_{τ_D} is outside the circle $\rho^2 + z^2 = N^2$ and so $X_1(\tau_D) > x_1(N) = N[1 - |o(1)|]$. Thus

$$P_x(|X(\tau_D)| > N) \le P_x(X_1(\tau_D) > N[1 - |o(1)|]). \qquad \square$$

Combining Lemma 2.2 with the next result, we immediately obtain Theorem 2.1.

Theorem 2.3. *Let $x = (\rho, z) \in D$. Then*

$$\lim_{N\to\infty} N^{\alpha-1} \log P_x(X_1(\tau_D) > N) = -\frac{\pi}{2A(1-\alpha)}. \qquad \square$$

The rest of the article, except for the last section, will be devoted to proving Theorem 2.3.

We will need the following theorem due to Bañuelos and Carroll (2005).

Theorem 2.4. *For the two-dimensional Brownian motion B_t,*

$$\lim_{N\to\infty} N^{\alpha-1} \log P_x(|B(\tau_D)| > N) = -\frac{\pi}{2A(1-\alpha)} = \lim_{N\to\infty} N^{\alpha-1} \log P_x(B_1(\tau_D) > N). \qquad \square$$

We close this section with a result needed to obtain the lim sup behavior in the case $d = 2$. Roughly speaking, it says that the part of D inside a bounded set does not have too much influence on the exit place being far away.

Let Γ be a non-intersecting Lipschitz curve, lying in D except for its endpoints, which lie on ∂D. Then Γ divides D into two components; let D_Γ be the unbounded one.

Lemma 2.5. *Assume D_Γ is Lipschitz. If $x \in D_\Gamma$ then for some $c > 0$,*

$$P_x(X_1(\tau_D) > N) \le c P_x(X_1(\tau_{D_\Gamma}) > N)$$

for all sufficiently large N.

Proof. Define

$$\sigma_\Gamma = \inf\{t > 0\colon\ X(t) \in \Gamma\}.$$

Let γ be a curve lying in D, except for its endpoints, which lie on ∂D. We also assume $x \in \gamma$, $\Gamma \cap \gamma = \emptyset$ and

$$G := D \backslash \overline{D_\gamma}$$

is Lipschitz. Then for N so large that $\gamma \cap \{\rho = N\} = \emptyset$,

$$\begin{aligned}\sup_{y\in\gamma} P_y(X_1(\tau_D) > N) &\le \sup_{y\in\gamma} P_y(X_1(\tau_D) > N, \sigma_\Gamma \ge \tau_D) + \sup_{y\in\gamma} P_y(X_1(\tau_D) > N, \sigma_\Gamma < \tau_D) \\ &\le \sup_{y\in\gamma} P_y(X_1(\tau_D) > N, \sigma_\Gamma \ge \tau_D) + \sup_{y\in\gamma} E_y[I(\sigma_\Gamma < \tau_D) P_{X(\sigma_\Gamma)}(X_1(\tau_D) > N)]\end{aligned}$$

(by the strong Markov property)

$$\le \sup_{y\in\gamma} P_y(X_1(\tau_D) > N, \sigma_\Gamma \ge \tau_D) + \sup_{z\in\Gamma} P_z(X_1(\tau_D) > N) \cdot \sup_{y\in\gamma} P_y(\sigma_\Gamma < \tau_D). \tag{6}$$

Now if

$$\sup_{y\in\gamma} P_y(\sigma_\Gamma < \tau_D) = 1,$$

then by continuity and that $P_z(\sigma_\Gamma < \tau_D) = 0$ when $z \in \partial D$, the L-harmonic function $P_\bullet(\sigma_\Gamma < \tau_D)$ would take on an interior maximum on D_Γ, contrary to the maximum principle. Thus

$$c := \sup_{y\in\gamma} P_y(\sigma_\Gamma < \tau_D) < 1$$

and (6) yields

$$\sup_{y\in\gamma} P_y(X_1(\tau_D) > N) \le \sup_{y\in\gamma} P_y(X_1(\tau_D) > N, \sigma_\Gamma \ge \tau_D) + c \sup_{z\in\Gamma} P_z(X_1(\tau_D) > N). \tag{7}$$

Observe that the function

$$u(z) = P_z(X_1(\tau_D) > N)$$

is L-harmonic on D and vanishes continuously on $\partial D \cap \{\rho < N\}$. Then since $\tau_D \wedge \sigma_\gamma = \tau_G$ a.s. P_z for $z \in \Gamma$, we have

$$\begin{aligned}\sup_{z\in\Gamma} u(z) &= \sup_{z\in\Gamma} E_z[u(X(\tau_G))] \\ &= \sup_{z\in\Gamma} E_z[u(X(\sigma_\gamma))I(\sigma_\gamma < \tau_D)] \\ &\le \sup_{y\in\gamma} u(y),\end{aligned}$$

which is to say

$$\sup_{z\in\Gamma} P_z(X_1(\tau_D) > N) \le \sup_{y\in\gamma} P_y(X_1(\tau_D) > N).$$

Using this in (7) we end up with

$$\sup_{y\in\gamma} P_y(X_1(\tau_D) > N) \le \frac{1}{1-c} \sup_{y\in\gamma} P_y(X_1(\tau_D) > N, \sigma_\Gamma > \tau_D). \tag{8}$$

The functions

$$\begin{aligned} v(z) &= P_z(X_1(\tau_D) > N, \sigma_\Gamma > \tau_D) \\ w(z) &= P_z(\sigma_\Gamma < \tau_D) \end{aligned}$$

are positive and L-harmonic on a neighborhood in D of $\gamma \cap D$ and both vanish continuously on a neighborhood in ∂D of the endpoints of γ. Hence by the boundary Harnack principle (Pinsky (1995), Theorem 8.0.1 on page 333), for some $c_1 > 0$,

$$\frac{v(z)}{w(z)} \le c_1 \frac{v(x)}{w(x)}, \qquad z \in \gamma.$$

Thus for some $c_2 > 0$ depending on x,

$$\begin{aligned}\sup_{z\in\gamma} v(z) &\le c_2 \sup_{z\in\gamma} w(z) v(x) \\ &\le c_2 v(x)\end{aligned}$$

since $w \le 1$. Using the definition of v, when combined with (8), this gives

$$\begin{aligned} P_x(X_1(\tau_D) > N) &\le \sup_{y\in\gamma} P_y(X_1(\tau_D) > N) \\ &\le \frac{1}{1-c} c_2 P_x(X_1(\tau_D) > N, \sigma_\Gamma > \tau_D) \\ &\le \frac{c_2}{1-c} P_x(X_1(\tau_{D_\Gamma}) > N), \end{aligned}$$

since $\tau_D = \tau_{D_\Gamma}$ when $\sigma_\Gamma > \tau_D$ under P_x. □

Remark 2.6. Let $B = (B_1, B_2)$ be two-dimensional Brownian motion and set $S = \{(r,w)\colon\ |w| < \frac{\pi}{2}\}$ and $S_M = \{(r,w)\colon\ r > M, |w| < \frac{\pi}{2}\}$, $M > 0$. Then the same proof as in Lemma 2.5 gives that if $x \in S_M$ then for some $c > 0$,

$$P_x(B_1(\tau_S) > N) \le cP_x(B_1(\tau_{S_M}) > N)$$

for all large N.

3 Feyman–Kac representation of an h-transform

In this section we will prove a result used repeatedly in the sequel. In essence, it allows us to convert the study of the exit distribution of a process into that of an h-transform of the process.

We are going to apply some results of Pinsky (1995). In order to do so, we review some definitions and facts found there. In what follows, $\beta \in (0,1)$.

Assumption *H*. The operator

$$L = \frac{1}{2}\Delta_{\mathbb{R}^2} + b\cdot\nabla + V$$

defined on the closure $\overline{U}$ of a domain $U \subseteq \mathbb{R}^2$ has bounded coefficients and satisfies $b, V \in C^\beta(\overline{U})$. □

Assumption $\widetilde{\boldsymbol{H}}$. The operator

$$L = \frac{1}{2}\Delta_{\mathbb{R}^2} + b\cdot\nabla + V$$

defined on the closure $\overline{U}$ of a domain $D \subseteq \mathbb{R}^2$ satisfies $b \in C^{1,\beta}(\overline{U})$ and $V \in C^\beta(\overline{U})$. □

Assumption $\widetilde{\boldsymbol{H}}_{\boldsymbol{loc}}$. The operator

$$L = \frac{1}{2}\Delta_{\mathbb{R}^2} + b\cdot\nabla + V$$

defined on a domain $U \subseteq \mathbb{R}^2$ satisfies Assumption $\widetilde{H}$ on every subdomain $U' \subseteq\subseteq U$. □

Let $U \subseteq \mathbb{R}^2$ be a bounded open set with $C^{2,\beta}$ boundary and suppose

$$L = \frac{1}{2}\Delta_{\mathbb{R}^2} + b\cdot\nabla + V$$

satisfies Assumption H on $\overline{U}$. Then by Theorem 3.6.1 in Pinsky (1995), the principal eigenvalue of L on U is given by

$$\lambda_0(L,U) = \lim_{t\to\infty}\frac{1}{t}\log E_x\left[I(\tau_U > t)\exp\left(\int_0^t V(X_s)ds\right)\right],$$

where X_t is the diffusion associated with $L - V = \frac{1}{2}\Delta + b\cdot\nabla$ on U.

An operator L on a domain G is said to be *subcritical* if it possesses a Green function. Probabilistically, this corresponds to transience of the diffusion associated with L on G. If $H \in C^{2,\beta}(G)$ with $H > 0$ on G, then define the H-transform of L to be the operator L^H given by

$$L^H f = \frac{1}{H}L(Hf).$$

Theorem 3.1. *Let $G \subseteq \mathbb{R}^2$ be Lipschitz domain with piecewise $C^{2,\beta}$ boundary and denote the $C^{2,\beta}$ part of ∂G by T. Suppose $H, b \in C^\infty(\overline{G})$ and $H > 0$ on $\overline{G}$. Let X_t and Y_t be the diffusions associated with the operators*

$$L = \frac{1}{2}\Delta_{\mathbb{R}^2} + b \cdot \nabla$$
$$L^H - \frac{LH}{H} = \frac{1}{2}\Delta_{\mathbb{R}^2} + \left(b + \frac{\nabla H}{H}\right) \cdot \nabla$$

on G, respectively, where L^H is the H-transform of L. Assume the exit times $\tau_G(X)$ and $\tau_G(Y)$ from G are a.s. finite, LH has constant sign, and $\frac{LH}{H}$ is bounded and continuous on $\overline{G}$. Finally, assume for any bounded open set $U \subseteq G$ with $C^{2,\beta}$ boundary, $\lambda_0(L, U) < 0$. Then for all $A \subseteq T$,

$$P_x(X(\tau_G) \in A) = H(x) E_x \left[H(Y(\tau_G))^{-1} I(Y(\tau_G) \in A) \exp\left(\int_0^{\tau_G} \frac{LH}{H}(Y_s) ds \right) \right].$$

Proof. It suffices to show for each nonnegative smooth function f on $\mathbb{R}^2$ with compact support and supp $f \cap \partial G \subseteq T$,

$$E_x[f(X_{\tau_G})] = H(x) E_x \left[\frac{f}{H}(Y_{\tau_G}) \exp\left(\int_0^{\tau_G} \frac{LH}{H}(Y_s) ds \right) \right]. \tag{9}$$

To this end, let $U \subseteq G$ be a bounded open set with $C^{2,\beta}$ boundary such that

$$\partial U \cap \partial G \subseteq T$$
$$\text{supp } f \cap \partial G = \text{supp } f \cap \partial U.$$

Notice then that

$$Y(\tau_U) \in \text{supp } f \cap \partial U \Rightarrow Y(\tau_U) \in \text{supp } f \cap \partial G \quad \text{and} \quad \tau_U = \tau_G. \tag{10}$$

By our hypotheses, L satisfies Assumption H on $\overline{U}$, hence we can apply an existence/uniqueness theorem for the Dirichlet problem (Pinsky (1995), Theorem 3.3.1) to get a unique solution $g \in C^{2,\beta}(\overline{U})$ to

$$\begin{cases} Lg = 0 & \text{on } U \\ g = f & \text{on } \partial U. \end{cases} \tag{11}$$

Then by the martingale property and optional stopping, for any open $U' \subseteq \overline{U'} \subseteq U$,

$$g(x) = E_x[g(X(\tau_{U'}))], \qquad x \in U'.$$

Let $U' \uparrow U$ to get

$$g(x) = E_x[g(X_{\tau_U})] = E_x[f(X_{\tau_U})], \qquad x \in U. \tag{12}$$

Now b is bounded and continuous on $\overline{U}$, hence (L, U) is subcritical. Since $H \in C^{2,\beta}(\overline{U})$ and $H > 0$ on $\overline{U}$, and since L satisfies $\widetilde{H}_{\text{loc}}$ on U, by Theorem 4.3.3 (iv) in Pinsky (1995) the generalized principal eigenvalues $\lambda_C(L, U)$ and $\lambda_C(L^H, U)$ of L and L^H, respectively, on U

coincide. But L and L^H satisfy Assumption $\widetilde{H}$ on U, hence by Theorem 4.3.2 in Pinsky (1995), $\lambda_C(L,U) = \lambda_0(L,U)$ and $\lambda_C(L^H,U) = \lambda_0(L^H,U)$. Thus we conclude

$$\lambda_0(L^H,U) = \lambda_0(L,U) < 0,$$

by hypothesis. As a consequence, because we also have $\frac{f}{H} \in C^{2,\beta}(\overline{U})$ by positivity of H on $\overline{G}$, we can apply another existence/uniqueness theorem (Theorem 3.6.5 in Pinsky (1995)) to get a unique solution $v \in C^{2,\beta}(\overline{U})$ to

$$\begin{cases} L^H v = 0 & \text{in } U \\ v = \dfrac{f}{H} & \text{on } \partial U. \end{cases} \tag{13}$$

Applying a version of the Feynman–Kac formula (Theorem 3.6.6 (iii) in Pinsky (1995)) to L^H, in our notation we get

$$v(x) = E_x\left[\frac{f}{H}(Y_{\tau_U})\exp\left(\int_0^{\tau_U}\frac{LH}{H}(Y_s)ds\right)\right]$$

(there is a misprint in the statement of that Theorem: $\bar{L}$ there should be $\overline{L_0}$). On the other hand, by (11) the function $\tilde{v} = \frac{g}{H} \in C^{2,\beta}(\overline{U})$ also solves (13). Uniqueness then forces $\frac{g}{H} = \tilde{v} = v$; that is, by (12),

$$\frac{1}{H(x)}E_x[f(X_{\tau_U})] = E_x\left[\frac{f}{H}(Y_{\tau_U})\exp\left(\int_0^{\tau_U}\frac{LH}{H}(Y_s)ds\right)\right], \quad x \in U. \tag{14}$$

Since $\tau_G(X)$ and $\tau_G(Y)$ are a.s. finite,

$$\lim_{U\uparrow G}\frac{f}{H}(Y_{\tau_U})\exp\left(\int_0^{\tau_U}\frac{LH}{H}(Y_s)ds\right) = \frac{f}{H}(Y_{\tau_G})\exp\left(\int_0^{\tau_G}\frac{LH}{H}(Y_s)ds\right) \quad \text{a.s.}$$

Hence by Fatou's lemma and dominated convergence in (14),

$$E_x\left[\frac{f}{H}(Y_{\tau_G})\exp\left(\int_0^{\tau_G}\frac{LH}{H}(Y_s)ds\right)\right] \le \frac{1}{H(x)}E_x[f(X_{\tau_G})] < \infty. \tag{15}$$

By (10),

$$\begin{aligned} \frac{f}{H}(Y_{\tau_U}) &= \frac{f}{H}(Y_{\tau_U})I(Y_{\tau_U} \in \text{supp } f \cap \partial U) \\ &\le \frac{f}{H}(Y_{\tau_U})I(Y_{\tau_G} \in \text{supp } f \cap \partial G)I(\tau_G = \tau_U) \\ &\le \frac{f}{H}(Y_{\tau_G}). \end{aligned} \tag{16}$$

Since LH has constant sign,

$$\frac{f}{H}(Y_{\tau_U})\exp\left(\int_0^{\tau_U}\frac{LH}{H}(Y_s)ds\right) \le \begin{cases} \dfrac{f}{H}(Y_{\tau_G}), & LH \le 0 \text{ always} \\ \dfrac{f}{H}(Y_{\tau_G})\exp\left(\displaystyle\int_0^{\tau_G}\frac{LH}{H}(Y_s)ds\right), & LH \ge 0 \text{ always} \end{cases}$$

and each of the latter is in L^1 (for $LH \le 0$ use that f/H is bounded and for $LH \ge 0$ use (15)). Thus by boundedness and continuity of f, $\frac{f}{H}$ and $\frac{LH}{H}$ on $\overline{G}$, we can apply dominated convergence to (14), letting $U \uparrow G$ to end up with

$$\frac{1}{H(x)} E_x[f(X_{\tau_G})] = E_x\left[\frac{f}{H}(Y_{\tau_G}) \exp\left(\int_0^{\tau_G} \frac{LH}{H}(Y_s) ds\right)\right], \quad x \in G.$$

This completes the proof of Theorem 3.1. □

4 Upper bound for $d \ge 3$, lower bound for $d = 2$

Throughout this section we will use the notation of Section 2. In particular,

$$D = \{(\rho, z) : \ \rho > 1, |z| < A\rho^\alpha\}$$

and

$$L = \frac{1}{2}\left[\frac{\partial^2}{\partial \rho^2} + \frac{d-1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2}\right]$$

is the nonangular part of $\frac{1}{2}\Delta_{\mathbb{R}^{d+1}}$ expressed in cylindrical coordinates. We will prove the following theorem.

Theorem 4.1. *Let $x = (\rho, z) \in D$. Then*

a) $\limsup_{N\to\infty} N^{\alpha-1} \log P_x(X_1(\tau_D) > N) \le -\frac{\pi}{2A(1-\alpha)}$ *if $d \ge 3$ and*

b) $\liminf_{N\to\infty} N^{\alpha-1} \log P_x(X_1(\tau_D) > N) \ge -\frac{\pi}{2A(1-\alpha)}$ *if $d = 2$.* □

We make an H-transform to eliminate the drift in L and use Theorem 3.1 to convert the problem into a question about two-dimensional Brownian motion in D. To this end, let

$$p = \frac{d-1}{2} \quad \text{and} \quad H(\rho, z) = \rho^{-p}, \qquad (\rho, z) \in D.$$

Then a simple computation yields that

$$\frac{LH}{H} = -\frac{1}{2}\frac{p(p-1)}{\rho^2}$$

and

$$L^H = \frac{1}{2}\left[\frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial z^2} - \frac{p(p-1)}{\rho^2}\right].$$

Notice then that the diffusion associated with $L^H - \frac{LH}{H}$ on D is just two-dimensional Brownian motion $B = (B_1, B_2)$. It is known that $P_x(\tau_D(B) < \infty) = 1$ (Bañuelos et al. (2001), Li (2003) or Lifshits and Shi (2000)). Since the exit time of X from D is the same as the exit time of $(d + 1)$-dimensional Brownian motion W from Ω, and the latter is known to be a.s. finite (DeBlassie (2007)), we get $P_x(\tau_D(X) < \infty) = 1$.

Moreover, if $U \subseteq D$ is a bounded open set with $C^{2,\beta}$ boundary, then the exit time $\tau_U(X)$ is the same as $\tau_{\widetilde{U}}(W)$, where

$$\widetilde{U} = \{(\tilde{x}, x_{d+1}) \in \mathbb{R}^d \times \mathbb{R}\colon\ (|\tilde{x}|, x_{d+1}) \in U\}.$$

But $\widetilde{U}$ is bounded, so for some $\lambda(\widetilde{U}) > 0$,

$$P_x(\tau_{\widetilde{U}}(W) > t) \le e^{-\lambda(\widetilde{U})t} \quad \text{for large} \quad t.$$

Since L satisfies Assumption H on $\overline{U}$,

$$\begin{aligned}
\lambda_0(L, U) &= \lim_{t\to\infty} \frac{1}{t} \log P_x(\tau_U(X) > t) \\
&= \lim_{t\to\infty} \frac{1}{t} \log P_{(x,\theta)}(\tau_{\widetilde{U}}(W) > t) \\
&< 0.
\end{aligned}$$

It is easy to verify all the other hypotheses of Theorem 3.1 hold for G there replaced by D and Y replaced by B. Thus we can conclude that for large N,

$$P_x(X_1(\tau_D) > N) = \rho^{-p} E_x\left[B_1(\tau_D)^p I(B_1(\tau_D) > N) \exp\left(-\frac{p(p-1)}{2}\int_0^{\tau_D} B_1(s)^{-2} ds\right)\right] \tag{17}$$

Proof of Theorem 4.1 a). Since $d \ge 3$, we have $p(p-1) \ge 0$ and (17) yields that

$$P_x(X_1(\tau_D) > N) \le \rho^{-p} E_x[B_1(\tau_D)^p I(B_1(\tau_D) > N)].$$

By Theorem 2.4,

$$E_x[B_1(\tau_D)^q] < \infty \quad \text{for all} \quad q > 0.$$

Consider any $\eta, \xi > 1$ with $\frac{1}{\xi} + \frac{1}{\eta} = 1$. Then

$$\begin{aligned}
P_x(X_1(\tau_D) > N) &\le \rho^{-p}(E_x[B_1(\tau_D)^{p\eta}])^{1/\eta}(P_x(B_1(\tau_D) > N))^{1/\xi} \\
&= C(P_x(B_1(\tau_D) > N))^{1/\xi},
\end{aligned}$$

where C is independent of N. Another application of Theorem 2.4 yields

$$\limsup_{N\to\infty} N^{\alpha-1} \log P_x(X_1(\tau_D) > N) \le -\frac{1}{\xi}\frac{\pi}{2A(1-\alpha)}.$$

Let $\xi \to 1^+$ to finish. □

Proof of Theorem 4.1 b). Since $d = 2$, $p(p-1) = -\frac{1}{4}$. Then by (17), since $B_1(\tau_D) \ge 1$,

$$\begin{aligned}
P_x(X_1(\tau_D) > N) &= \rho^{-p} E_x\left[B_1(\tau_D)^p I(B_1(\tau_D) > N) \exp\left(\frac{1}{8}\int_0^{\tau_D} B_1(s)^{-2} ds\right)\right] \\
&\ge \rho^{-p} P_x(B_1(\tau_D) > N).
\end{aligned}$$

The desired lim inf behavior follows from Theorem 2.4. □

5 Lower bound for $d \geq 3$, upper bound for $d = 2$

We continue using the notation of Section 2. To complete the proof of Theorem 2.3, in light of Theorem 4.1 it suffices to prove the following theorem.

Theorem 5.1. *Let $x = (\rho, z) \in D$. Then*

a) $\liminf_{N\to\infty} N^{\alpha-1} \log P_x(X_1(\tau_D) > N) \geq -\frac{\pi}{2A(1-\alpha)}$ *if $d \geq 3$*

and

b) $\limsup_{N\to\infty} N^{\alpha-1} \log P_x(X_1(\tau_D) > N) \leq -\frac{\pi}{2A(1-\alpha)}$ *if $d = 2$.* □

Unlike in Section 4, we do not have the confluence of events where the exponential in the Feynman–Kac representation of the h-transform does not come into play. Here it does and the analysis is much more delicate. So instead of h-transforming first, we conformally map D into a strip and then h-transform. Using explicit properties of the conformal map we can analyze the exponential in the Feynman–Kac representation.

In the first subsection below we will describe properties of the conformal map, make the appropriate h-transform and apply Theorem 3.1 to obtain a Feynman–Kac representation. We prove Theorem 5.1 in the subsequent two subsections.

5.1 Preliminaries

Let f be the conformal mapping of

$$\mathcal{P} = \{(\rho, z) \colon \ \rho > 0, |z| < A\rho^\alpha\}$$

onto the strip

$$S = \left\{(r, w) \colon \ |w| < \frac{\pi}{2}\right\}$$

such that f is real on the real axis and f' is positive there. Write $g\colon \ S \to \mathcal{P}$ for the inverse map and set $h = \mathrm{Re}(g)$. Bañuelos and Carroll (2005) used this map to derive their results on the distribution of the exit place of Brownian motion from a parabolic-type domain. We now state some properties of the map f.

Theorem 5.2. (1) *The inverse map g extends to a homeomorphism of $\overline{S}$ onto $\overline{\mathcal{P}}\backslash\{0\}$ and the derivative g' has a continuous nonzero extension to $\overline{S}$.*

(2) *As $\rho \to \infty$,*

$$f(\rho, z) \sim \frac{\pi}{2A(1-\alpha)}\rho^{1-\alpha},$$

uniformly for $|z| < A\rho^\alpha$.

(3) *As $r \to \infty$,*

$$h(r, w) \sim \left[\frac{2A(1-\alpha)}{\pi} r\right]^{\frac{1}{1-\alpha}},$$

uniformly for $|w| < \frac{\pi}{2}$.

(4) *As* $r \to \infty$,

$$|\nabla h(r,w)| = |g'(r,w)| = 2A\left[\frac{1}{\pi} + o(1)\right] h(r,w)^{\alpha},$$

uniformly for $|w| < \frac{\pi}{2}$.

Proof. By Proposition 4 in Bañuelos and Carroll (2005), part (4) holds.

Part (2) holds by Theorem X in Warschawski (1942) or Theorem 1 in Carroll and Hayman (2004).

A simple modification of the proof of Proposition 4 in Bañuelos and Carroll (2005) shows g' has a continuous nonzero extension to $\overline{S}$. Then the extension of g to a homeomorphism on $\overline{S}$ follows. By symmetry, the extension maps $-\infty$ to 0. This proves part (1).

To prove part (3), note by part (1), f extends to a homeomorphism of $\overline{\mathcal{P}}\backslash\{0\}$ onto $\overline{S}$. Then given $N > 0$ we can choose r_N such that

$$f(\{\rho \le N\} \cap \overline{\mathcal{P}}) \cap \left\{(r,w):\ \ r \ge r_N, |w| \le \frac{\pi}{2}\right\} = \emptyset.$$

Consequently

$$g\left(\left\{(r,w):\ \ r \ge r_N, |w| < \frac{\pi}{2}\right\}\right) \subseteq \{(\rho,z):\ \ \rho > N, |z| < A\rho^{\alpha}\};$$

in other words, for $r \ge r_N$ and $|w| < \frac{\pi}{2}$,

$$h(r,w) = \mathrm{Re}(g(r,w)) > N.$$

Thus $h(r,w) \to \infty$ as $r \to \infty$, uniformly for $|w| < \frac{\pi}{2}$. Hence by part (2), as $r \to \infty$, uniformly for $|w| < \frac{\pi}{2}$ we have

$$\begin{aligned} r &= \mathrm{Re}\ f(g(r,w)) \\ &\sim \frac{\pi}{2A(1-\alpha)}(\mathrm{Re}\ g(r,w))^{1-\alpha} \\ &= \frac{\pi}{2A(1-\alpha)}(h(r,w))^{1-\alpha}. \qquad \square \end{aligned}$$

Corollary 5.3. *As* $r \to \infty$,

$$\left|\frac{\nabla h}{h}(r,w)\right| = \frac{1}{(1-\alpha)r}[1 + o(1)],$$

uniformly for $|w| < \frac{\pi}{2}$.

Proof. Let $\varepsilon > 0$. By Theorem 5.2 (4), choose $R > 0$ such that

$$(1-\varepsilon)\frac{2A}{\pi}h(r,w)^{\alpha-1} \le \left|\frac{\nabla h}{h}(r,w)\right| \le (1+\varepsilon)\frac{2A}{\pi}h(r,w)^{\alpha-1}, \qquad r \ge R, |w| < \frac{\pi}{2}. \tag{18}$$

By Theorem 5.2 (2), choose $\rho_0 > 0$ such that

$$(1-\varepsilon)\frac{\pi}{2A(1-\alpha)}\rho^{1-\alpha} \le \text{Re } f(\rho,z) \le (1+\varepsilon)\frac{\pi}{2A(1-\alpha)}\rho^{1-\alpha}, \qquad \rho \ge \rho_0, |z| < A\rho^\alpha, \tag{19}$$

and by part (3) choose $R_0 > R$ such that

$$h(r,w) \ge \rho_0 \quad \text{for} \quad r \ge R_0, |w| < \frac{\pi}{2}.$$

Then since Re $f(g(r,w)) = r$, (19) gives

$$(1-\varepsilon)\frac{\pi}{2A(1-\alpha)}h(r,w)^{1-\alpha} \le r \le (1+\varepsilon)\frac{\pi}{2A(1-\alpha)}h(r,w)^{1-\alpha}, \qquad r \ge R_0, |w| < \frac{\pi}{2}.$$

Combined with (18) we get

$$\frac{(1-\varepsilon)^2}{1-\alpha}\frac{1}{r} \le \left|\frac{\nabla h}{h}(r,w)\right| \le \frac{(1+\varepsilon)^2}{1-\alpha}\frac{1}{r}, \qquad r \ge R_0, |w| < \frac{\pi}{2}. \qquad \square$$

Theorem 5.4. *The function $h = Re(g)$ is C^∞ on $\overline{S}$.*

Proof. The function

$$v(\rho,z) = \text{Im } f(\rho,z)$$

is harmonic on $\mathcal{P}$ and by Theorem 5.2 (1) it is continuous on $\overline{\mathcal{P}}\backslash\{0\}$. On the upper part of $\partial\mathcal{P}\backslash\{0\}$, v has value $\frac{\pi}{2}$ and on the lower part it has value $-\frac{\pi}{2}$. Then by the Elliptic Regularity Theorem (see the comment after Theorem 6.19 on page 111 in Gilbarg and Trudinger (1983)) $v \in C^\infty(\overline{\mathcal{P}}\backslash\{0\})$. By the Cauchy–Riemann equations we get that $u = \text{Re}(f) \in C^\infty(\overline{\mathcal{P}}\backslash\{0\})$. Thus $f \in C^\infty(\overline{\mathcal{P}}\backslash\{0\})$. By Theorem 5.2 (1), $f' \neq 0$ on $\overline{\mathcal{P}}\backslash\{0\}$ and then using the formula $g'(r,w) = \frac{1}{f'(g(r,w))}$ we get that $g \in C^\infty(\overline{S})$. $\square$

In what follows, for $T > 0$ we will write

$$S_T = S \cap \{(r,w) \colon \ r > T\}.$$

Let $0 < \varepsilon < 1$ and by Corollary 5.3, choose large $M > 1$ so that

$$\frac{1-\varepsilon}{(1-\alpha)^2 r^2} \le \left|\frac{\nabla h}{h}(r,w)\right|^2 \le \frac{1+\varepsilon}{(1-\alpha)^2 r^2}, \qquad r > M, |w| < \frac{\pi}{2} \tag{20}$$

and

$$f(D) \supseteq S_M. \tag{21}$$

Making M larger (as we will below) does not change the validity of (20)–(21).

Let $N > 1$ and set

$$Q_1 = Q_1(N) = (N, AN^\alpha)$$
$$Q_2 = Q_2(N) = (N, -AN^\alpha).$$

Then for some $R(N)$,

$$\begin{cases} f(Q_1) = \left(R(N), \dfrac{\pi}{2}\right), \\ f(Q_2) = \left(R(N), -\dfrac{\pi}{2}\right), \\ f(\partial D \cap \{\rho \geq N, z = A\rho^\alpha\}) = \partial S_{R(N)} \cap \left\{w = \dfrac{\pi}{2}\right\}, \\ f(\partial D \cap \{\rho \geq N, z = -A\rho^\alpha\}) = \partial S_{R(N)} \cap \left\{w = -\dfrac{\pi}{2}\right\}. \end{cases} \tag{22}$$

Lemma 5.5. *As $N \to \infty$,*

$$R(N) \sim \frac{\pi}{2A(1-\alpha)} N^{1-\alpha}.$$

Proof. By Theorem 5.2 (2),

$$R(N) = \text{Re } f(N, AN^\alpha) \sim \frac{\pi}{2A(1-\alpha)} N^{1-\alpha}$$

as $N \to \infty$. □

Now let us convert our problem to the strip S. Define

$$\mathcal{L} = \frac{1}{2}\left[\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial w^2} + (d-1)\frac{\nabla h}{h}(r, w) \cdot \nabla\right]$$

and let Y be the diffusion corresponding to $\mathcal{L}$ in S. Notice by the Cauchy–Riemann equations the operator $|\nabla h|^{-2}\mathcal{L}$ is just the operator L from Section 2 expressed in the coordinates $(r, w) = f(\rho, z)$. In particular, Y and $f(X)$ in $f(D)$ have the same exit distribution on ∂S and so we can write

$$P_x(X_1(\tau_D) > N) = P_y(Y_1(\tau_{f(D)}) > R(N)), \qquad y = f(x). \tag{23}$$

Moreover, for any open $U \subseteq D$, Y and $f(X)$ have the same exit distribution on $\partial f(U)$ and so

$$P_x(X(\tau_U) \in A) = P_y(Y(\tau_{f(U)}) \in f(A)), \qquad y = f(x), A \subseteq \partial U. \tag{24}$$

Next, let us h-transform and get a Feynman–Kac representation.

Lemma 5.6. *Let $B = (B_1, B_2)$ be two-dimensional Brownian motion. Let $R > M$ and $y \in S_M$.*

(1) *For $p = \frac{d-1}{2}$,*

$$P_y(Y_1(\tau_S) > R) \geq h^{-p}(y) E_y\left[I(B_1(\tau_{S_M}) > R) \exp\left(-\frac{p(p-1)}{2}\int_0^{\tau_{S_M}} \left|\frac{\nabla h}{h}(B(s))\right|^2 ds\right)\right].$$

(2) *If $d = 2$ then*

$$P_y(Y_1(\tau_{S_M}) > R) = h^{-1/2}(y) E_y\left[h^{1/2}(B(\tau_{S_M})) I(B_1(\tau_{S_M}) > R) \exp\left(\frac{1}{8}\int_0^{\tau_{S_M}} \left|\frac{\nabla h}{h}(B(s))\right|^2 ds\right)\right].$$

Proof. We are going to verify the hypotheses of Theorem 3.1 for

$$\begin{aligned} G &= S_M \\ T &= \partial G \backslash \left\{ \left(M, \pm\frac{\pi}{2} \right) \right\} \\ L &= \mathcal{L} \\ H &= h^{-p} \\ b &= \frac{d-1}{2}\frac{\nabla h}{h}. \end{aligned}$$

Since h is harmonic on S,

$$\mathcal{L}H = \mathcal{L}(h^{-p}) = -\frac{p(p-1)}{2}\left|\frac{\nabla h}{h}\right|^2 h^{-p},$$

and $\mathcal{L}H$ has constant sign. By Theorem 5.2 (1), $\inf_{\overline{G}} h > 0$ and so by Theorem 5.4, H and b are in $C^\infty(\overline{G})$ and

$$\frac{\mathcal{L}H}{H} = -\frac{p(p-1)}{2}\left|\frac{\nabla h}{h}\right|^2$$

is continuous on $\overline{G}$. Moreover, by Corollary 5.3 and Theorem 5.2, $\frac{\mathcal{L}H}{H}$ is bounded on $\overline{G}$. Write $\widetilde{X}$ and $\widetilde{Y}$ for the diffusions associated with $\mathcal{L}$ and $\mathcal{L}^H - \frac{\mathcal{L}H}{H} = \frac{1}{2}\left[\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial w^2}\right]$, respectively, on G. Notice $\widetilde{X} = Y$ from the statements prior to Lemma 5.6 and $\widetilde{Y}$ is two-dimensional Brownian motion. We need to show $\tau_G(\widetilde{X})$ and $\tau_G(\widetilde{Y})$ are a.s. finite. Write $I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and note $\tau_G(\widetilde{Y}) \le \tau_I(\widetilde{Y}_2)$. But $\widetilde{Y}_2$ is one-dimensional Brownian motion and it is known that for some positive constants c_1 an c_2,

$$P_{y_2}(\tau_I(\widetilde{Y}_2) > t) \le c_1 e^{-c_2 t}.$$

Hence

$$P_y(\tau_G(\widetilde{Y}) > t) \le c_1 e^{-c_2 t}.$$

Also, since $\widetilde{X}$ is two-dimensional Brownian motion with bounded drift, by the Cameron–Martin–Girsanov Theorem, $P_y(\tau_G(\widetilde{X}) > t)$ satisfies a similar bound. In particular both $\tau_G(\widetilde{X})$ and $\tau_G(\widetilde{Y})$ a.s. finite. Moreover, if $U \subseteq G$ is any bounded open set with $C^{2,\beta}$ boundary then $\tau_U(\widetilde{X}) \le \tau_G(\widetilde{X})$, and so by the exponential decay of the tail distribution of the latter, $\lambda_0(\mathcal{L}, U) < 0$. Thus all the hypotheses of Theorem 3.1 hold and we can conclude that for $R > M$

$$\begin{aligned} P_y(\widetilde{X}_1(\tau_{S_M}) > R) = h^{-p}(y) E_y \Bigg[& h^p(\widetilde{Y}(\tau_{S_M})) I(\widetilde{Y}_1(\tau_{S_M}) > R) \cdot \\ & \cdot \exp\left(-\frac{p(p-1)}{2} \int_0^{\tau_{S_M}} \left|\frac{\nabla h}{h}(Y(s))\right|^2 ds \right) \Bigg]. \end{aligned}$$

Recalling that $\widetilde{X} = Y$ and $\widetilde{Y}$ is two-dimensional Brownian motion, this translates into

$$\begin{aligned} P_y(Y_1(\tau_{S_M}) > R) = h^{-p}(y) E_y \Bigg[& h^p(B(\tau_{S_M})) I(B_1(\tau_{S_M}) > R) \cdot \\ & \cdot \exp\left(-\frac{p(p-1)}{2} \int_0^{\tau_{S_M}} \left|\frac{\nabla h}{h}(B(s))\right|^2 ds \right) \Bigg]. \end{aligned} \tag{25}$$

By (21),

$$\begin{aligned} h^p(B(\tau_{S_M})) &= [\text{Re } f^{-1}(B(\tau_{S_M}))]^p \\ &\geq 1 \end{aligned}$$

and so (25) leads to

$$\begin{aligned} P_y(Y_1(\tau_S) > R) &\geq P_y(Y_1(\tau_{S_M}) > R) \\ &\geq h^{-p}(y) E_y \left[I(B_1(\tau_{S_M}) > R) \exp\left(-\frac{p(p-1)}{2} \int_0^{\tau_{S_M}} \left| \frac{\nabla h}{h}(B(s)) \right|^2 ds \right) \right]. \end{aligned}$$

This gives part (1) of the lemma.

Now assume $d = 2$. Then $p = \frac{1}{2}$ and (25) yields part (2). ☐

Lemma 5.7. *For two-dimensional Brownian motion B and $y \in S_M$,*

$$E_y[B_1(\tau_{S_M})^q] < \infty \quad \textit{for all} \quad q > 0.$$

Proof. Choose $x \in D$ such that $f(x) = y$. By conformal invariance of two-dimensional Brownian motion and (21)–(22), for large N

$$\begin{aligned} P_y(B_1(\tau_{S_M}) > R(N)) &= P_x(B_1(\tau_{g(S_M)}) > N) \\ &\leq P_x(B_1(\tau_D) > N) \end{aligned}$$

and

$$P_y(B_1(\tau_S) > R(N)) \geq P_x(B_1(\tau_D) > N).$$

Then by Remark 2.6, for some $c > 0$, if N is large we have

$$\begin{aligned} \frac{1}{c} P_x(B_1(\tau_D) > N) &\leq \frac{1}{c} P_y(B_1(\tau_S) > R(N)) \\ &\leq P_y(B_1(\tau_{S_M}) > R(N)) \\ &\leq P_x(B_1(\tau_D) > N). \end{aligned}$$

Applying Theorem 2.4, this yields

$$\lim_{N\to\infty} \frac{2A(1-\alpha)}{\pi} N^{\alpha-1} \log P_y(B_1(\tau_{S_M}) > R(N)) = -1.$$

Using Lemma 5.5, we get

$$\lim_{N\to\infty} \frac{1}{R(N)} \log P_y(B_1(\tau_{S_M}) > R(N)) = -1 \tag{26}$$

Thus

$$\lim_{T\to\infty} \frac{1}{T} \log P_y(B_1(\tau_{S_M}) > T) = -1$$

and the conclusion of the lemma holds. ☐

5.2 Proof of Theorem 5.1 a)

In this subsection we will assume $d \geq 3$. Consider any $\eta, \xi > 1$ with $\frac{1}{\eta} + \frac{1}{\xi} = 1$. Define

$$q = 2 + \sqrt{1 + \frac{4p(p-1)(1+\varepsilon)}{\xi(1-\alpha)^2}}$$

(recall $\varepsilon > 0$ is from Section 5.1—see just before (20)). This is real because $d \geq 3$ forces $p-1 \geq 0$. Notice q is chosen so that

$$\frac{(q-1)(q-3)}{4} = \frac{p(p-1)(1+\varepsilon)}{\xi(1-\alpha)^2}. \tag{27}$$

Let Z be the diffusion associated with the operator

$$\mathcal{M} = \frac{1}{2}\left[\frac{\partial^2}{\partial r^2} + \frac{q-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial w^2}\right]$$

for $(r, w) \in [0, \infty) \times \mathbb{R}$.

Lemma 5.8. *For two-dimensional Brownian motion B and $R > M$,*

$$P_y(Z_1(\tau_{S_M}) > R) \geq P_y(B_1(\tau_{S_M}) > R), \qquad y \in S_M.$$

Proof. For $B(0) = y$, we can write

$$\begin{aligned} dZ_1(t) &= dB_1(t) + \frac{q-1}{2Z_1(t)}dt \\ dZ_2(t) &= dB_2(t). \end{aligned}$$

Since $q > 2$, Z_1 never hits 0. By the Comparison Theorem (Ikeda and Watanabe (1989), Theorem VI.1.1)

$$Z_1(t) > B_1(t), \qquad t > 0.$$

Since $Z_2(t) = B_2(t)$ for all $t \geq 0$, we can conclude that either

(1) B exits S_M on $\{r = M\}$ and Z has not yet exited S_M

OR

(2) B exits S_M on ∂S and Z exits S_M at the same time.

Now if $B_1(\tau_{S_M}(B)) > R$ then (1) cannot hold and (2) then implies $\tau_{S_M}(B) = \tau_{S_M}(Z)$. Moreover, since $B_1 \leq Z_1$,

$$R < B_1(\tau_{S_M}) \leq Z_1(\tau_{S_M})$$

and we have

$$P_y(B_1(\tau_{S_M}) > R) \leq P_y(Z_1(\tau_{S_M}) > R),$$

as desired. ∎

We are going to compare $P_y(Y_1(\tau_{f(D)}) > R)$ and $P_y(B_1(\tau_{S_M}) > R)$ and in order to do so, we must have $y \in S_M$. In general though, it is possible that $y \notin S_M$ for $P_y(Y_1(\tau_{f(D)}) > R)$. The next lemma lets us get around this difficulty.

Lemma 5.9. *Suppose $y_1 \leq M$. Then for some $c > 0$,*

$$P_y(Y_1(\tau_{f(D)}) > R) \geq cP_{(2M,0)}(Y_1(\tau_{f(D)}) > R), \qquad R > 2M.$$

Proof. Write $G = f(D)\backslash\overline{S_{2M}}$. By the strong Markov property, for $R > 2M$

$$\begin{aligned}
P_y(Y_1(\tau_{f(D)}) > R) &\geq E_y\left[I\left(Y_2(\tau_G) \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]\right) I(Y_1(\tau_{f(D)}) > R)\right] \\
&= E_y\left[I\left(Y_2(\tau_G) \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]\right) P_{Y(\tau_G)}(Y_1(\tau_{f(D)}) > R)\right] \\
&\geq P_y\left(Y_2(\tau_G) \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]\right) cP_{(2M,0)}(Y_1(\tau_{f(D)}) > R)
\end{aligned}$$

(by the Harnack inequality)

$$= \tilde{c}P_{(2M,0)}(Y_1(\tau_{f(D)}) > R),$$

where $\tilde{c}$ is independent of $R > 2M$. □

Now we can complete the proof of Theorem 5.1 a). Recall we are writing $y = f(x)$ for $x = (\rho, z) \in D$. Depending on the choice of ε made just before (20), it is possible that $y_1 < M$. Since we can make M larger without affecting the validity of (20)–(21), it is no loss to assume that we have $y_1 < M$. For large N, by (23) and Lemma 5.9, for some $c > 0$ independent of N,

$$\begin{aligned}
P_x(X_1(\tau_D) > N) &= P_y(Y_1(\tau_{f(D)}) > R(N)) \\
&\geq cP_{(2M,0)}(Y_1(\tau_{f(D)}) > R(N)) \\
&\geq ch^{-p}(2M,0)E_{(2M,0)}\Bigg[I(B_1(\tau_{S_M}) > R(N)) \cdot \\
&\quad \cdot \exp\left(-\frac{p(p-1)}{2}\int_0^{\tau_{S_M}}\left|\frac{\nabla h}{h}(B(s))\right|^2 ds\right)\Bigg]
\end{aligned}$$

(by Lemma 5.6 (1))

$$\begin{aligned}
&\geq ch^{-p}(2M,0)E_{(2M,0)}\Bigg[I(B_1(\tau_{S_M}) > R(N)) \cdot \\
&\quad \cdot \exp\left(-\frac{p(p-1)}{2}\frac{1+\varepsilon}{(1-\alpha)^2}\int_0^{\tau_{S_M}} B_1(s)^{-2} ds\right)\Bigg] \qquad (28)
\end{aligned}$$

(by (20) and that $p - 1 \geq 0$).

We are going to apply Theorem 3.1 to

$$\begin{aligned}
&G = S_M \\
&T = \partial G\backslash\left\{\left(M, \pm\frac{\pi}{2}\right)\right\} \\
&\mathcal{M} \\
&H(r,w) = r^{-(q-1)/2}.
\end{aligned}$$

Noting that

$$\frac{\mathcal{M}H}{H} = -\frac{(q-1)(q-3)}{8r^2}$$

and

$$\mathcal{M}^H - \frac{\mathcal{M}H}{H} = \frac{1}{2}\left[\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial w^2}\right],$$

we can repeat the argument in the proof of Lemma 5.6 to show the hypotheses of Theorem 3.1 are met. Thus that theorem with (X, Y) there replaced by (Z, B) yields

$$\begin{aligned}
P_{(2M,0)}(Z_1(\tau_{S_M}) > R(N)) &= H(2M,0)E_{(2M,0)}\Bigg[H(B(\tau_{S_M}))^{-1}\cdot \\
&\quad \cdot I(B_1(\tau_{S_M}) > R(N)) \exp\left(\int_0^{\tau_{S_M}} \frac{\mathcal{M}H}{H}(B(s))ds\right)\Bigg] \\
&= H(2M,0)E_{(2M,0)}\left[B_1(\tau_{S_M})^{(q-1)/2} I(B_1(\tau_{S_M}) > R(N)) \cdot \exp\left(-\frac{(q-1)(q-3)}{8}\int_0^{\tau_{S_M}} B_1(s)^{-2}ds\right)\right] \\
&\le H(2M,0)(E_{(2M,0)}[B_1(\tau_{S_M})^{\eta(q-1)/2}])^{1/\eta}\cdot \\
&\quad \cdot \left(E_{(2M,0)}\left[I(B_1(\tau_{S_M}) > R(N)) \exp\left(-\frac{(q-1)(q-3)\xi}{8}\int_0^{\tau_{S_M}} B_1(s)^{-2}ds\right)\right]\right)^{1/\xi} \\
&= c\left(E_{(2M,0)}\left[I(B_1(\tau_{S_M}) > R(N)) \exp\left(-\frac{p(p-1)(1+\varepsilon)}{2(1-\alpha)^2}\int_0^{\tau_{S_M}} B_1(s)^{-2}ds\right)\right]\right)^{1/\xi}
\end{aligned}$$

(by Lemma 5.7 and (27)), where c is independent of large N. Using this in (28), we get for some $\tilde{c} > 0$ independent of N,

$$\begin{aligned}
P_x(X_1(\tau_D) > N) &\ge \tilde{c}(P_{(2M,0)}(Z_1(\tau_{S_M}) > R(N)))^\xi \\
&\ge \tilde{c}(P_{(2M,0)}(B_1(\tau_{S_M}) > R(N)))^\xi
\end{aligned}$$

(by Lemma 5.8). With Lemma 5.5 and (26) this gives

$$\liminf_{N\to\infty} N^{\alpha-1}\log P_x(X_1(\tau_D) > N) \ge -\xi\frac{\pi}{2A(1-\alpha)}.$$

Let $\xi \to 1^+$ to get part a) of Theorem 5.1. □

5.3 Proof of Theorem 5.1 b)

We continue using the notation of Section 2. It will be necessary to make M larger so that $f(x) \notin S_M$. We will compare $X_1(\tau_D)$ with $X_1(\tau_{D_\Gamma})$ where $D_\Gamma = g(S_M)$ and this requires the starting point x to be in D_Γ or equivalently, $f(x) \in S_M$. The corollary following the next lemma will let us get around this problem.

Lemma 5.10. *Suppose $z = (z_1, z_2) \in D$. Then for some constant $c > 0$,*

$$P_z(X_1(\tau_D) > N) \le cP_{(z_1,0)}(X_1(\tau_D) > N), \qquad N > 2z_1.$$

Proof. Let $R_1 < z_1 < R_2 < 2z_1$ and set

$$G = \{(\rho, z) \in D\colon\ R_1 < \rho < R_2\}.$$

Notice $z \in G$. The functions

$$\begin{aligned} u(w) &= P_w(X_1(\tau_D) > N) \\ v(w) &= P_w(X(\tau_G) \in \partial G \backslash \partial D) \end{aligned}$$

are positive and L-harmonic on G. Since they vanish continuously at $\partial D \cap \{(\rho, z)\colon\ R_1 < \rho < R_2\}$, by the boundary Harnack principle there is $c > 0$, independent of N, such that

$$\frac{u(w)}{v(w)} \leq c \frac{u(z_1, 0)}{v(z_1, 0)}, \qquad w \in D \text{ with } w_1 = z_1.$$

Thus

$$\begin{aligned} P_z(X_1(\tau_D) > N) &\leq c \frac{v(z)}{v(z_1, 0)} P_{(z_1,0)}(X_1(\tau_D) > N) \\ &= \tilde{c} P_{(z_1,0)}(X_1(\tau_D) > N) \end{aligned}$$

where $\tilde{c}$ is independent of $N > 2z_1$. ☐

Next, choose $K = K(M)$ so large that the curve

$$\Gamma = g\left(\left\{(r, w)\colon\ r = M, |w| \leq \frac{\pi}{2}\right\}\right) \tag{29}$$

lies to the left of the vertical line $\{(\rho, z)\colon\ \rho = K\}$.

Corollary 5.11. *Let $x \in D$ with $f(x) \notin S_M$. Then for some $c > 0$,*

$$P_x(X_1(\tau_D) > N) \leq c P_{(K,0)}(X_1(\tau_D) > N), \qquad N > 2K,$$

Proof. Define

$$\sigma_K = \inf\{t \geq 0\colon\ \mathrm{Re}(X_t) = K\}.$$

Since $f(x) \notin S_M$, x lies to the left of the curve Γ. Then for $N > 2K$

$$\begin{aligned} P_x(X_1(\tau_D) > N) &= P_x(\sigma_K < \tau_D, X_1(\tau_D) > N) \\ &= E_x[I(\sigma_K < \tau_D) P_{X(\sigma_K)}(X_1(\tau_D) > N)] \end{aligned}$$

(by the strong Markov property).
But $\mathrm{Re}(X(\sigma_K)) = K$, so by Lemma 5.10 this gives

$$\begin{aligned} P_x(X_1(\tau_D) > N) &\leq c P_x(\sigma_K < \tau_D) P_{(K,0)}(X_1(\tau_D) > N) \\ &\leq c P_{(K,0)}(X_1(\tau_D) > N), \end{aligned}$$

as desired. ☐

For two-dimensional Brownian motion $B = (B_1, B_2)$, let

$$\zeta = \inf\left\{t > 0\colon\ B_2(t) \notin \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}.$$

Then for some $\lambda_0 > 0$,

$$E_y[e^{\lambda\zeta}] < \infty \quad \text{for all} \quad \lambda < \lambda_0.$$

For $i = 1, 2$, let ξ_i and $\eta_i > 1$ satisfy $\frac{1}{\xi_i} + \frac{1}{\eta_i} = 1$. By making M larger if necessary, it is no loss to assume

$$\frac{(1+\varepsilon)\xi_1\eta_2}{8M^2(1-\alpha)^2} < \lambda_0$$

and

$$f(x) \notin S_M. \tag{30}$$

Now for large N, $R(N) > M$; hence if $B_1(\tau_{S_M}) > R(N)$, then $\tau_{S_M}(B) = \zeta$ and we have

$$E_y\left[I(B_1(\tau_{S_M}) > R(N)) \exp\left(\frac{(1+\varepsilon)\xi_1\eta_2}{8M^2(1-\alpha)^2}\tau_{S_M}\right)\right] \leq E_y\left[\exp\left(\frac{(1+\varepsilon)\xi_1\eta_2}{8M^2(1-\alpha)^2}\zeta\right)\right] < \infty. \tag{31}$$

In what follows, $c > 0$ will be a number whose exact value can change from line to line, but it will always be independent of N large. By Corollary 5.11 and (30), for large N we have

$$\begin{aligned} P_x(X_1(\tau_D) > N) &\leq cP_{(K,0)}(X_1(\tau_D > N) \\ &\leq cP_{(K,0)}(X_1(\tau_{D_\Gamma}) > N) \end{aligned}$$

(recall (29) and use Lemma 2.5)

$$= cP_{f(K,0)}(Y_1(\tau_{S_M}) > R(N))$$

(by (24), (22) and that $f(D_\Gamma) = S_M$)

$$\begin{aligned} &= cK^{-1/2}E_{f(K,0)}\Big[h^{1/2}(B(\tau_{S_M}))I(B_1(\tau_{S_M}) > R(N))\cdot \\ &\quad \cdot \exp\left(\frac{1}{8}\int_0^{\tau_{S_M}} \left|\frac{\nabla h}{h}(B(s))\right|^2 ds\right)\Big] \end{aligned}$$

(by Lemma 5.6 (2) and that $h(f(K,0)) = K$)

$$\begin{aligned} &\leq cE_{f(K,0)}\Big[B_1(\tau_{S_M})^{\frac{1}{2(1-\alpha)}} I(B_1(\tau_{S_M}) > R(N))\cdot \\ &\quad \cdot \exp\left(\frac{1}{8}\frac{1+\varepsilon}{(1-\alpha)^2}\int_0^{\tau_{S_M}} B_1(s)^{-2}ds\right)\Big] \end{aligned}$$

(by Theorem 5.2 (3) and (20))

$$\begin{aligned} &\leq c(E_{f(K,0)}[B_1(\tau_{S_M})^{\frac{\eta_1}{2(1-\alpha)}}])^{\frac{1}{\eta_1}}\Big(E_{f(K,0)}\Big[I(B_1(\tau_{S_M}) > R(N))\cdot \\ &\quad \cdot \exp\left(\frac{(1+\varepsilon)\xi_1}{8M^2(1-\alpha)^2}\tau_{S_M}\right)\Big]\Big)^{\frac{1}{\xi_1}} \end{aligned}$$

(since $B_1(s) \ge M$ for $s \le \tau_{S_M}$)

$$
\begin{aligned}
&= c\left(E_{f(K,0)}\left[I(B_1(\tau_{S_M}) > R(N))\exp\left(\frac{(1+\varepsilon)\xi_1}{8M^2(1-\alpha)^2}\tau_{S_M}\right)\right]\right)^{\frac{1}{\xi_1}}\\
&\le c(P_{f(K,0)}(B_1(\tau_{S_M}) > R(N)))^{\frac{1}{\xi_1\xi_2}}\left(E_{f(K,0)}\left[I(B_1(\tau_{S_M}) > R(N))\cdot\right.\right.\\
&\quad \left.\left.\cdot\exp\left(\frac{(1+\varepsilon)\xi_1\eta_2}{8M^2(1-\alpha)^2}\tau_{S_M}\right)\right]\right)^{\frac{1}{\xi_1\eta_2}}\\
&\le c(P_{f(K,0)}(B_1(\tau_{S_M}) > R(N)))^{\frac{1}{\xi_1\xi_2}}
\end{aligned}
$$

(by (31)).

Thus by Lemma 5.5

$$
\begin{aligned}
\limsup_{N\to\infty} N^{\alpha-1}\log P_x(X_1(\tau_D) > N) &\le \limsup_{N\to\infty}\frac{\pi}{2A(1-\alpha)}\frac{1}{R(N)}\frac{1}{\xi_1\xi_2}\log P_{f(K,0)}(B_1(\tau_{S_M}) > R(N))\\
&= -\frac{1}{\xi_1\xi_2}\frac{\pi}{2A(1-\alpha)}
\end{aligned}
$$

(by (26)). Let $\xi_1, \xi_2 \to 1^+$ to finish. $\square$

6 Dimensional dependence of the limit in (5)

In this section, we complete the argument for the Example in the Introduction. We continue to use the notation of the example.

The dimensional dependence of the limit in (5) will follow if we can show that $p(d)$ is strictly decreasing in the dimension d. Now for each fixed $a > 0$, the function

$$
f(y) = -y + \sqrt{y^2 + a}
$$

is strictly decreasing on $[0, \infty)$. Assuming that the eigenvalue $\gamma_1(d)$ is nonincreasing in d, we have for $d_1 < d_2$

$$
\begin{aligned}
2p(d_2) &= -(d_2 - 1) + \sqrt{(d_2-1)^2 + 4\gamma_1(d_2)}\\
&< -(d_1 - 1) + \sqrt{(d_1-1)^2 + 4\gamma_1(d_2)}\\
&\le -(d_1 - 1) + \sqrt{(d_1-1)^2 + 4\gamma_1(d_1)}\\
&= 2p(d_1),
\end{aligned}
$$

and the desired monotonicity holds.

Now we show $\gamma_1(d)$ is nonincreasing in d. Let $\varphi = \varphi(x)$ be the colatitude of a point $x = (\tilde{x}, x_{d+1}) \in (\mathbb{R}^d\backslash\{0\}) \times \mathbb{R}$; that is, φ is the angle between x and the x_{d+1}-axis. Since the function $f(y) = y/\sqrt{1+y^2}$ is monotone,

$$
|y| < A \iff |f(y)| < f(A),
$$

and we get that

$$\begin{aligned}
x = (\tilde{x}, x_{d+1}) \in \Omega_D \quad &\iff \quad \frac{|x_{d+1}|}{|\tilde{x}|} < A \\
&\iff \quad |f(x_{d+1}/|\tilde{x}|)| < f(A) \\
&\iff \quad \frac{|x_{d+1}|}{\sqrt{|\tilde{x}|^2 + x_{d+1}^2}} < f(A) \\
&\iff \quad |\cos\varphi(x)| < \tfrac{A}{\sqrt{1+A^2}}.
\end{aligned}$$

Thus

$$\Omega_D = \{x = (\tilde{x}, x_{d+1}) \colon |\cos\varphi(x)| < \tfrac{A}{\sqrt{1+A^2}}\},$$

and by symmetry, the first Dirichlet eigenfunction of Δ_{S^d} on $\Omega_D \cap S^d$ depends only on the colatitude φ. For such a function, Δ_{S^d} takes on the form

$$\begin{aligned}
2L &= (\sin\varphi)^{1-d}\frac{\partial}{\partial\varphi}\left[(\sin\varphi)^{d-1}\frac{\partial}{\partial\varphi}\right] \\
&= \frac{\partial^2}{\partial\varphi^2} + (d-1)(\cot\varphi)\frac{\partial}{\partial\varphi}
\end{aligned}$$

(Itô and McKean (1974), Section 7.15).

The grand conclusion is that $\frac{1}{2}\gamma_1(d)$ is the smallest positive Dirichlet eigenvalue of L on the interval

$$I = \left(\cos^{-1}\left(\tfrac{A}{\sqrt{1+A^2}}\right), \cos^{-1}\left(-\tfrac{A}{\sqrt{1+A^2}}\right)\right).$$

Since the coefficients of L are smooth on $\bar{I}$, by Theorem 4.3.2 in Pinsky (1995), and the definition after its proof, $-\frac{1}{2}\gamma_1(d)$ is the generalized principal eigenvalue of L on I. Let

$$h(\varphi) = (\sin\varphi)^{(1-d)/2}.$$

Then h is strictly positive and C^3 on I; hence by Theorem 4.3.3(iv) in Pinsky (1995), $-\frac{1}{2}\gamma_1(d)$ is also the generalized principal eigenvalue of the h-transformed operator L^h on I. It is easy to compute that

$$2L^h = \frac{\partial^2}{\partial\varphi^2} + V_d(\varphi),$$

where

$$V_d(\varphi) = \frac{(d-1)}{2}\left[1 + \frac{(d+1)}{2}\cot^2\varphi\right].$$

Since the operator L^h is well-behaved on $\bar{I}$, application of Theorems 4.3.2 and 3.6.1 in Pinsky (1995) yields that if B_t is one-dimensional Brownian motion,then

$$-\frac{1}{2}\gamma_1(d) = \lim_{t\to\infty} \log E_x\left[e^{\frac{1}{2}\int_0^t V_d(B_s)\,ds} I_{\tau_I > t}\right],$$

where τ_I is the first exit time of B_t from I. If $d_1 < d_2$, then we have $V_{d_1} < V_{d_2}$, and the last limit implies that $\gamma_1(d_2) \leq \gamma_1(d_1)$, as desired. $\square$

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