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Ergodic Properties of Multidimensional Brownian Motion with Rebirth

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Abstract

In a bounded open region of the d dimensional space we consider a Brownian motion which is reborn at a fixed interior point as soon as it reaches the boundary. The evolution is invariant with respect to a density equal, modulo a constant, to the Green function of the Dirichlet Laplacian centered at the point of return. We calculate the resolvent in closed form, study its spectral properties and determine explicitly the spectrum in dimension one. Two proofs of the exponential ergodicity are given, one using the inverse Laplace transform and properties of analytic semigroups, and the other based on Doeblin's condition. Both methods admit generalizations to a wide class of processes.

Key words: .

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1 Introduction

The paper studies a process analyzed in [10]. This type of processes were first introduced in [8] and also investigated in [15]. Let $\mathcal{R}$ be a bounded open region in $\mathbb{R}^d$ with a smooth boundary (to make things precise, of class C^2) such that the origin $O \in \mathcal{R}$. For $x \in \mathbb{R}^d$, let $W_x = (w_x(t,\omega), \{\mathcal{F}_t\}_{t\geq 0})$ be a Brownian motion on $\mathbb{R}^d$ such that $P(w_x(0,\omega) = x) = 1$. On the region $\mathcal{R}$, for any $x \in \mathcal{R}$, we define a process $\{z_x(t,\omega)\}_{t\geq 0}$ with values in $\mathcal{R}$ which is identical to a standard d dimensional Brownian motion until the almost surely finite time τ when it reaches the boundary, then instantaneously returns to the origin O at τ and repeats the same evolution indefinitely. This is the multidimensional version of the problem described in [10], which may be called *Brownian motion with rebirth*, since after emulating the Brownian motion with absorbing boundary conditions (in other words, *killed* at the boundary) it is reborn at the origin. The state space can be shown to be compact with the topology described in (14) creating a shunt at the return point. As a consequence, the dynamics has an invariant measure. We identify it as the Green function $G(\cdot,\cdot)$ for the Laplacian with pole at $\xi = 0$, modulo a normalizing factor. The average time a Brownian motion starting at x spends in the set $B \subset \mathcal{R}$ before hitting the boundary is determined ([13], Section 7.4) as $\int_B G(x,y)dy$. Our particle will repeat the trip from the origin to the boundary indefinitely and will stabilize in time, by ergodicity, towards the measure which gives the mean value over all configurations, at exponential rate.

Theorem 1 provides an explicit formula for the transition probabilities of the process and shows that the associated semigroup is a strongly continuous compact Feller semigroup on the domain X where the boundary and the return point are glued together (14). In any dimension, Theorem 2 gives a closed formula of the resolvent and the spectrum of the generator $\sigma = \{\lambda_i\}_{i\geq 0}$ is described as a subset of singularities appearing in the resolvent formula (23), ordered according to the real part $0 = \lambda_0 > \Re(\lambda_1) \geq \Re(\lambda_2) \geq \ldots$. The exponential ergodicity with the exact convergence rate $\Re(\lambda_1)$ is given in Theorem 3.

We note that the compactness of the semigroup is used to validate the sharp rate of convergence (lower bound) in the exponential ergodicity limit (26). The strongly continuous Feller semigroup property is crucial for the inversion formula of the resolvent.

The proof is based on the inversion formula for the Laplace transform (Proposition 6) and the estimates of Propositions 7 and 8, which make essential use of a contour integration over the boundary of a sector (18) of the complex plane with angle $2\phi > \pi$. Whenever the semigroup is analytic (Theorem 7.7 in [17]), the integral exists and provides an inverse formula giving an exact exponential rate for the error term.

It is natural that a key part of the proof consists in establishing that the semigroup corresponding to the rebirth process is analytic, more specifically, proving estimate (24). To begin with, we need the analytic semigroup results for operators on the space of continuous functions with the topology of uniform convergence found in [18] - [19]. In our case, they are applied to the Dirichlet Laplacian on a regular domain $\mathcal{R}$, but more general dynamics can be considered with the same method.

Another advantage of the method is that once Theorem 1 has been established for delta functions, one can readily generalize to arbitrary redistribution measure μ. The closed formula (23) captures the renewal mechanism imbedded in the process. The estimates needed for the Laplace transform inversion formula are not harder to obtain in an L^p norm than in the uniform norm. A reference in that direction is again [17]. The proofs presented in this paper are easy to modify in order to include the L^p case. For example, when the measure $\mu(dx)$ has a density in $L^2(\mathcal{R})$, then part of the analysis carried out for establishing (40)-(37) is simplified due to the explicit form of the resolvent of the Dirichlet Laplacian in the square norm. Recently, more detailed results on the spectral properties of diffusions with rebirth are worked out, with different methods, in [2].

Section 6 looks at the one dimensional case. We are able to calculate the spectrum of the generator of the process exactly, and show that the spectral gap is equal to the second eigenvalue of the Dirichlet Laplacian $\lambda_1 = \lambda_2^{abs} < \lambda_1^{abs}$, by identifying a set of removable singularities in (23) which includes the point λ_1^{abs}. This corrects the spectral gap estimate from [10], which is correct in the sense of an upper bound only, as $\lambda_1 \leq \lambda_1^{abs}$.

In Section 7 we give an alternative proof of the exponential ergodicity, based on the Doeblin condition. For explicit computations as in Section 6, the Doeblin condition is not helpful. However, the Doeblin condition argument leads to various generalizations, and is very short.

There are two venues for applications of the rebirth process. The first originates in a variant of the Fleming-Viot branching process introduced in [3] and studied further in [11]- [12]. Assume that the singular measure $\mu(dx)$ giving the distribution of the rebirth location of the Brownian particle is replaced by a time-dependent deterministic measure $\mu(t, dx)$. The tagged particle process from [12] is an example in the case when $\mu(t, dx)$ is the deterministic macroscopic limit of the empirical measures of a large system of Brownian particles with branching confined to the region $\mathcal{R}$. In particular, in equilibrium, the updating measure $\mu(t, dx)$ is constant in time, being equal to $\mu(dx) = \Phi_1(x)dx$, the probability measure with density equal to the first eigenfunction of the Dirichlet Laplacian (normalized).

The second application is coming from mathematical finance. If $\{S(t)\}_{t\geq 0}$ denotes the asset process in a model for the derivative markets, then $S(t)$ is typically assumed to follow the path of a geometric Brownian motion (see [6], also [9]). The *double knock-out barrier option* has payoff equal to $S(t)$ as long as it belongs to a region $\mathcal{R}$ with the prescription that it falls back to one (zero rate of return, or a prescribed value) as soon as the barrier or boundary is reached and then resumes its evolution. In that case, $\log S(t)$ is a diffusion with rebirth.

2 Main results

We shall denote by $(\Omega, \mathcal{F}, P)$ a probability space supporting the law of the family of d-dimensional coupled Brownian motions indexed by their starting points $x \in \mathcal{R}$. Let $\mathcal{A}$ be an open region in $\mathbb{R}^d$ and $x \in \mathcal{A}$. In general we shall use the notation

$$T_x(\mathcal{A}) = \inf\{t > 0 \,:\, w_x(t,\omega) \notin \mathcal{A}\}\,, \tag{1}$$

the *exit time* from the region $\mathcal{A}$ for the Brownian motion starting at x. Occasionally we shall suppress either x or the set $\mathcal{A}$ if they are unambiguously defined in a particular context. We shall define inductively the increasing sequence of stopping times $\{\tau_n\}_{n\geq 0}$, together with a family of adapted nondecreasing point processes $\{N_x(t,\omega)\}_{t\geq 0}$ and the process $\{z_x(t,\omega)\}_{t\geq 0}$, starting at $x \in \mathcal{R}$. Let $\tau_0 = T_x = \inf\{t \,:\, w_x(t,\omega) \notin \mathcal{R}\}$, while for $t \leq \tau_0$ we set $N_x(t,\omega) = 1_{\{\partial\mathcal{R}\}}(w_x(t,\omega))$ and $z_x(t,\omega) = w_x(t,\omega) - \int_0^t w_x(s,\omega)dN_x(s,\omega)$. We notice that $z_x(\tau_0-,\omega) = w_x(\tau_0,\omega) \in \partial\mathcal{R}$. By induction on $n \geq 0$,

$$\tau_{n+1} = \inf\{t > \tau_n \,:\, w_x(t,\omega) - \int_0^{\tau_n} z_x(s-,\omega)dN_x(s,\omega) \notin \mathcal{R}\} \tag{2}$$

which enables us to define, for $\tau_n < t \leq \tau_{n+1}$,

$$N_x(t,\omega) = N_x(\tau_n,\omega) + 1_{\{\partial\mathcal{R}\}}(z_x(t-,\omega))\,, \tag{3}$$

as well as

$$z_x(t,\omega) = w_x(t,\omega) - \int_0^t z_x(s-,\omega)dN_x(s,\omega)\,. \tag{4}$$

We notice that $z_x(t,\omega) = 0$ for all $t = \tau_n$. The construction and the summations present in (2) and (4) are finite due to the following result.

Proposition 1. *The sequence of stopping times $\tau_0 < \tau_1 < \ldots < \tau_n < \ldots$ are finite for all n and $\lim_{n\to\infty}\tau_n = \infty$, both almost surely. Also, the integer-valued processes $N_x(t,\omega)$ defined for $t \geq 0$ have the properties (i) they are nondecreasing, piecewise constant, progressively measurable and right-continuous, and (ii) for any $x \in \mathcal{R}$, $P(N_x(t,\omega) < \infty) = 1$.*

Proof. It is easy to see that τ_n , $n \geq 0$ are i.i.d. with finite moments, and the conclusion follows from the law of large numbers (more details are given in [10]). Since the processes $N_x(t,\omega) \geq 0$ are clearly integer valued, non-decreasing and right-continuous by construction (3), they automatically preserve the same value until the next boundary hit. Progressive measurability is a consequence of the fact that the first exit times $\{\tau_n\}$ are stopping times. $\square$

Let $f \in C(\overline{\mathcal{R}})$ and $p_{abs}(t,x,y)$ denote the *absorbing Brownian kernel* generating the semigroup $\{S_t^{abs}\}_{t\geq 0}$

$$S_t^{abs}f(x) = \int_{\overline{\mathcal{R}}} f(y)p_{abs}(t,x,y)dy = E[f(w_x(t,\omega))\,,\, t < T_x(\mathcal{R})]\,. \tag{5}$$

The operator Δ with Dirichlet boundary conditions on $\partial\mathcal{R}$ has a countable spectrum $\{\lambda_i^{abs}\}_{i\geq 1}$

$$0 > \lambda_1^{abs} > \lambda_2^{abs} \geq \ldots \tag{6}$$

with corresponding eigenfunctions $\{\Phi_n(x)\}$ and

$$p_{abs}(t,x,y) = \sum_{n=1}^{\infty} \exp\Big(\frac{\lambda_n^{abs}t}{2}\Big)\Phi_n(x)\Phi_n(y)\,. \tag{7}$$

The functions $\{\Phi_n(x)\}$ are smooth and form an orthonormal basis of $L^2(\mathcal{R})$ (reference [13], or [7], (6.5)). The resolvent of the absorbing Brownian motion applied to $f \in C(\mathcal{R})$ will be denoted by

$$R_\alpha^{abs}f(x) = \int_0^\infty \int_{\mathcal{R}} e^{-\alpha t}p_{abs}(t,x,y)f(y)dydt\,. \tag{8}$$

In the following, the Laplace transform of the first exit time $T_x(\mathcal{R})$ from the domain $\mathcal{R}$ of a Brownian motion starting at x will be denoted by

$$\widehat{h^x}(\alpha) = E_x[e^{-\alpha T_x(\mathcal{R})}] = \int_0^\infty e^{-\alpha t}h^x(t)dt\,. \tag{9}$$

where $h^x(t)$ is the density function of $T_x(\mathcal{R})$. The Laplace transform (9) exists on the complex plane for all α with $\Re(\alpha) > \lambda_1^{abs}$ and can be extended (page 211, [20]) via analytic continuation to the resolvent set of the Dirichlet Laplacian. Also $\widehat{h^x}(\alpha)$ can be re-written directly in terms of the resolvent R_α^{abs} as shown in equations (34)-(35).

The law of the process $\{z_x(t,\omega)\}_{t\geq 0}$, adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ will be denoted by Q_x and the family of processes $\{Q_x\}_{x\in\mathcal{R}}$ will be denoted simply by $\{Q\}$. The construction described by equations (2) through (4) can be made deterministically for any $x \in \mathcal{R}$ and each path $w_x(\cdot) \in C([0,\infty),\mathbb{R}^d)$ resulting in a mapping preserving the progressive measurability

$$\Phi(w_x(\cdot)) = w_x(\cdot) - \int_0^{\cdot} w_x(s,\omega)dN_x(s,\omega)\,. \tag{10}$$

With this notation $\Phi : C([0,\infty),\mathbb{R}^d) \to D([0,\infty),\mathcal{R})$ and $Q_x = W_x \circ \Phi^{-1}$ is the law of the process $\{z_x(t,\omega)\}_{t\geq 0}$ with values in the region $\mathcal{R}$, a measure on the Skorohod space $D([0,\infty),\mathcal{R})$.

Let $m \in \mathbb{Z}_+$, $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_d) \in \mathbb{Z}_+^d$ be a d dimensional multi-index vector and we write $|\alpha| = \sum_{i=1}^d \alpha_i$. If $\mathcal{A} \subseteq \mathbb{R}^d$ and $f : \mathcal{A} \to \mathbb{R}$, we use the standard notation

$$\partial^{(\alpha)} f(x) = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_d^{\alpha_d}}(x)$$

if the derivative exists. Naturally $C^m(\mathcal{A})$ is the set of functions for which all derivatives with multi-indices α such that $|\alpha| \leq m$ exist and are continuous. We recall that the process $\{z_x(t,\omega)\}_{t\geq 0}$ is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ corresponding to the underlying standard d-dimensional Brownian motion.

Proposition 2. *If $f \in C^2(\mathcal{R}) \cap C(\overline{\mathcal{R}})$, then*

$$f(z_x(t,\omega)) - f(x) - \int_0^t \frac{1}{2}\Delta f(z_x(s,\omega))ds - \int_0^t (f(0) - f(z_x(s-,\omega))dN_x(s,\omega) \tag{11}$$

is a $\mathcal{F}_t$ - martingale with respect to Q_x.

Proof. The proof is identical to the $d = 1$ case from [10]. □

Let

$$\begin{gathered}\mathcal{D} = \left\{ f \in C^2(\mathcal{R}) \,:\, \forall\, |\alpha| \leq 2\,,\, \forall\, b \in \partial\mathcal{R}\,,\, \exists \lim_{x\to b} \partial^{(\alpha)} f(x) \in \mathbb{R} \right\} \\ \mathcal{D}_0 = \left\{ f \in \mathcal{D} \,:\, \forall\, b \in \partial\mathcal{R}\,,\, \lim_{x\to b} f(x) = f(0) \right\}.\end{gathered} \tag{12}$$

Corollary 1. *If $f \in \mathcal{D}_0$ then*

$$f(z_x(t,\omega)) - f(x) - \int_0^t \frac{1}{2}\Delta f(z_x(s,\omega))ds \tag{13}$$

is a $\mathcal{F}_t$ - martingale with respect to Q_x.

The next result allows us to regard $\{z_x(t,\omega)\}_{t\geq 0}$ as a process with continuous paths on the compact state space X obtained by identifying the boundary $\partial\mathcal{R}$ and the origin O.

Below $B(x,r) \subset \mathbb{R}^d$ is an open ball centered at $x \in \mathbb{R}^d$ of radius $r > 0$. Let $X = \mathcal{R}$ with the topology $\mathcal{T}$ generated by the neighborhood basis

$$\begin{aligned} V_{x,r} &= \Big\{ B(x,r) \,:\, B(x,r) \subset \mathcal{R} \setminus \{0\} \Big\} && \text{if} \quad x \neq 0\,,\, r > 0 \\ V_{0,r} &= \Big\{ B(0,r) \cup \Big(\cup_{b \in \partial\mathcal{R}} \left(B(b,r) \cap \mathcal{R} \right) \Big) \Big\} && \text{if} \quad x = 0\,,\, r \in (0, \tfrac{1}{2} d(x, \partial\mathcal{R}))\,. \end{aligned} \tag{14}$$

We define the class of functions of class C^2 up to the boundary $\{0\}$ of $X \setminus \{0\}$

$$\mathcal{D}(X) = \{ f \in C^2(X \setminus \{0\}) \,:\, \exists \lim_{x \to y} \partial^{(\alpha)} f(x) \in \mathbb{R}\,,\ \forall\, |\alpha| \leq 2\,,\, \forall\, y \in \{0\} \cup \partial\mathcal{R} \} \tag{15}$$

with the notational convention that the one-sided limit $\lim_{x \to y} g(x)$ is defined as $\lim_{x \to y} g(x)$ in the topology inherited from $\mathbb{R}^d$ by the set $B(0,r) \subseteq \mathcal{R}$, $r > 0$, in the case of $y = 0$ and $\mathcal{R} \cap B(y,r)$, if $y \in \partial\mathcal{R}$.

The inclusion mapping $\mathcal{I} \,:\, \mathcal{D}(X) \to \mathcal{D}$ is defined as $\mathcal{D}(X) \ni f \longrightarrow \mathcal{I}(f) \in \mathcal{D}$, where $\mathcal{I}(f)(x) = f(i(x))$ and $i(x) = x$ is the identification mapping from $\mathcal{R}$ to X.

Under the inclusion mapping $\mathcal{I} \,:\, \mathcal{D}(X) \to \mathcal{D}$ we define the domain

$$\mathcal{D}_0(X) = \Big\{ f \in \mathcal{D}(X) \,:\, \forall b \in \partial\mathcal{R} \quad \lim_{x \to 0} f(x) = \lim_{x \to b} f(x) \Big\}\,. \tag{16}$$

Corollary 2. *Let $\widehat{Q_x} = Q_x \circ i^{-1}$ be the measure induced on $C([0,\infty), X)$ by $i \,:\, \mathcal{R} \to X$. Then, $\widehat{Q_x}$ solves the martingale problem for the Markov pregenerator*

$$\mathcal{L} = (\frac{1}{2}\Delta_X, \mathcal{D}_0(X)) \tag{17}$$

with the convention that $\Delta_X f = \Delta \mathcal{I}(f)$ for any $f \in \mathcal{D}_0(X)$.

Remark 1. The space $(X, \mathcal{T})$ is compact and homeomorphic to a sphere in $\mathbb{R}^{d+1}$ with the North and South poles identified. The boundary conditions from below introduce a shunt at the origin which is responsible for the intrinsic asymmetry of the evolution. The domain is composed of functions which are C^2 up to the boundary $\{0\}$, yet the one-sided limits on the South pole neighborhood are equal, ensuring C^2 regularity on the lower sheet of the domain, while the one-sided limits on the North pole (that is, the boundary inherited from $\partial\mathcal{R}$) are non necessarily equal, with the exception of the multi-index $|\alpha| = 0$ which ensures continuity.

Remark 2. We note that the domain $\mathcal{D}_0$ of the original process on $\mathcal{R}$ is not dense in $C(\overline{\mathcal{R}})$.

Proof. The argument does not change with $d > 1$ and is presented in [10]. We refer to [14] for the definition of a Markov pregenerator. The properties of $f \in \mathcal{D}_0(X)$ ensure that $\overline{\mathcal{D}_0(X)} = C(X)$. In addition, we have to show that if x is a maximum point for f, then $\Delta f(x) \leq 0$. If $x \neq 0$, this is a consequence of Taylor's formula around x. At $x = 0$ we can still apply the standard argument which shows that $\nabla f(0) = 0$ because it only depends on the ball $B(0,r)$, which is a subset of a neighborhood of the origin in $(X, \mathcal{T})$ as well, and then necessarily $\Delta_X f(0) \leq 0$. The rest is immediate from Proposition 2. $\square$

In the following we shall use the notation $\|f\|_{C(\overline{\mathcal{R}})}$ for the supremum norm of the bounded function f and we assume that the domain $\mathcal{R}$ has boundary $\partial\mathcal{R} \in C^2$. We also recall that the Laplace transform of the first boundary hit (9) is analytic on the resolvent set of the Dirichlet Laplacian (6) due to the analyticity of the resolvent [20].

Given $\zeta \in \mathbb{R}$ and $\phi \in (\frac{\pi}{2}, \pi)$, we denote by

$$U_\zeta(\phi) = \left\{\alpha \in \mathbb{C} \,:\, |\arg(\alpha - \zeta)| < \phi\right\}, \qquad L(\zeta) = \partial U_\zeta(\phi) \tag{18}$$

the sector of angle 2ϕ with vertex at ζ and its boundary contour. For $R > 0$, we denote by $U_\zeta(R, \phi)$ the truncated sector

$$U_\zeta(R, \phi) = \left\{\alpha \,:\, |arg(\alpha - \zeta)| < \phi \,,\, |\alpha - \zeta| > R\right\}. \tag{19}$$

In the following, the angle ϕ will be omitted whenever not necessary.

Theorem 1. *Let $P(t, x, dy)$ be the transition probability for the process $\{Q_x\}_{x\in\mathcal{R}}$. For any $t > 0$ the measure $P(t, x, dy)$ is absolutely continuous with respect to the Lebesgue measure on $\mathcal{R}$ and, if $N_x(t, \omega)$ is the total number of visits to the boundary up to time $t > 0$, its probability density function $p(t, x, y)$ for $t > 0$ is given by*

$$p(t, x, y) = p_{abs}(t, x, y) + \int_0^t p_{abs}(t - s, 0, y) dE[N_x(s, \omega)] \tag{20}$$

where

$$E[N_x(s, \omega)] = \sum_{n=1}^{\infty} P(N_x(s, \omega) \geq n) = \int_0^s \sum_{n=1}^{\infty} (h^x * (h^0)^{*,n-1})(r) dr \tag{21}$$

with the convention that $(h^0)^{*,k} = \delta_0$ for $k = 0$. Moreover, the semigroup it generates on $f \in C(X)$

$$S_t f(x) = \int_{\mathcal{R}} p(t, x, y) f(y) dy \tag{22}$$

is a strongly continuous, compact Feller semigroup.

Theorem 2. *(i) The generator $\mathcal{L}$ of the semigroup S_t has a pure point spectrum σ included in the union of the eigenvalues of the Dirichlet Laplacian (6) and the zeros of $1 - \widehat{h^0}(\alpha)$, and there exist $\phi \in (\frac{\pi}{2}, \pi)$, $R > 0$ such that the resolvent set ϱ includes the union of $(\lambda_1^{abs}, \infty) \setminus \{0\}$, the right half-plane $\Re(\alpha) > 0$ and the truncated sector $U_0(R, \phi)$ from (19).*

(ii) The resolvent R_α of the semigroup S_t is a meromorphic function on the resolvent set of the Dirichlet Laplacian given by

$$R_\alpha f(x) = R_\alpha^{abs} f(x) + R_\alpha^{abs} f(0) \frac{\widehat{h^x}(\alpha)}{1 - \widehat{h^0}(\alpha)} \tag{23}$$

and there exists $M > 0$ such that

$$\|R_\alpha f\|_{C(\overline{\mathcal{R}})} \leq \frac{M}{|\alpha|} \|f\|_{C(\overline{\mathcal{R}})} \quad \forall\, \alpha \in U_0(R, \phi)\,. \tag{24}$$

Theorem 3. *The resolvent R_α has a simple pole at $\alpha = 0$ with residue equal to the continuous operator with kernel*

$$\rho(y) = \frac{G(0,y)}{\int_{\mathcal{R}} G(0,y)dy} \tag{25}$$

where $G(x,y)$ is the Green function of the Laplacian with Dirichlet boundary conditions. Moreover, if α^* is one of the nonzero elements of the spectrum σ with maximal real part, then

$$\sup_{\alpha\in\sigma\setminus\{0\}} \Re(\alpha) = \Re(\alpha^*) < 0$$

and

$$\lim_{t\to\infty} \frac{1}{t} \log\Big(\sup_{\|f\|_{C(X)}\leq 1} \|S_t f(\cdot) - \int_{\mathcal{R}} \rho(z) f(z) dz\|_{C(X)}\Big) = \Re(\alpha^*)\,. \tag{26}$$

Corollary 3. *The process $\{Q\}$ is exponentially ergodic.*

3 Proof of Theorem 1

The derivation of (20) does not depend on the dimension $d \in \mathbb{Z}_+$ hence we can refer to the proof of Theorem 1 in [10] directly. To prove that S_t is a strongly continuous Feller semigroup, we have to show that (F1) $S_t f \in C(X)$ for all $f \in C(X)$, and (F2) $\lim_{t\downarrow 0} S_t f = f$ in the norm of uniform convergence on X. In addition, S_t is a compact semigroup if the operator S_t is compact for any $t > 0$, which is property (F3) to prove. In the following we shall use $\mathcal{R}$ instead of X where there is no possibility of confusion. Without loss of generality, we shall assume that $t \in [0,T]$, $T > 0$ arbitrary but fixed.

Denote $v(t) = \sum_{n=1}^{\infty} (h^0)^{*,n-1}(t)$. Then (20) reads

$$S_t f(x) = \int_{\mathcal{R}} p_{abs}(t,x,y) f(y) dy + \int_0^t \int_{\mathcal{R}} p_{abs}(t-s,0,y) f(y) dy (h^x * v)(s) ds \tag{27}$$

and if we set

$$U(t) = \int_0^t S^{abs}_{t-s} f(0) v(s) ds = [S^{abs}_{\cdot} f(0) * v(\cdot)](t)\,, \tag{28}$$

with $S^{abs}_{\cdot}$ defined in (5), we obtain

$$S_t f(x) = S^{abs}_t f(x) + \int_0^t U(t-s) h^x(s) ds\,, \tag{29}$$

where S^{abs}_t denotes the semigroup corresponding to p_{abs}.

Proposition 3. *For any $T > 0$, the function $U(t)$ is continuous on $[0,T]$, bounded by $C_{U,T}||f||$, with $C_{U,T}$ independent of $t \in [0,T]$ and $f \in C(X)$. In addition, it is differentiable for $t > 0$, and for any $\epsilon \in (0,T)$, there exists $C_1(\epsilon,T) > 0$ such that*

$$\sup_{s\in[\epsilon,T]} |U'(s)| \leq C_1(\epsilon,T)||f||\,. \tag{30}$$

Proof. We have

$$U(t) = S^{abs}_t f(0) + [S^{abs}_{\cdot} f(0) * \sum_{n=2}^{\infty} (h^0)^{*,n-1}(\cdot)](t)\,. \tag{31}$$

Denote $g(t) = \sum_{n=2}^{\infty}(h^0)^{*,n-1}(t)$. The first term in $U(t)$ is continuous for $t \geq 0$ and bounded by $||f||$. For any $t \in [0,T]$ we have

$$\int_0^t g(s)ds = E[N_0(t)] < \infty\,, \qquad E[N_0(t)] = \sum_{k=1}^{\infty} P(N_0(t) \geq k)\,,$$

where $N_0(t)$ is the number of boundary hits up to time t from (21), when starting at $x=0$. The second term is bounded by $||f||E[N_0(T)]$. The constant $C_{U,T} = 1 + E[N_0(T)] < \infty$ due to the renewal theorem.

To prove continuity at $t_0 \in [0,T]$, we observe that the convolution of a bounded continuous function with an integrable function is continuous. More precisely, in our context, we write the second term in (31)

$$\int_0^T \mathbf{1}_{[0,t]}(s) S^{abs}_{t-s} f(0) g(s) ds\,.$$

For sake of detail, we formally extend continuously S^{abs} by setting $S^{abs}_s f(0) = f(0)$ for $s \in [-T,0]$. We note that the integrand converges to $\mathbf{1}_{[0,t_0]}(s) S^{abs}_{t_0-s} f(0) g(s)$ a.e. in $s \in [0,T]$ as $t \to t_0$. At the same time,

$$|\mathbf{1}_{[0,t]}(s) S^{abs}_{t-s} f(0) g(s)| \leq ||f|| g(s)$$

and g is integrable. Dominated convergence implies that the second part of (31) is continuous.

The proof of smoothness for $t > 0$ is based on the inversion formula given in Proposition 7 for the Laplace transformation

$$\widehat{U}(\alpha) = \int_0^{\infty} e^{-\alpha t} U(t) dt = R^{abs}_{\alpha} f(0)(1 - \widehat{h^0}(\alpha))^{-1}\,.$$

We can apply relation (35) and the lower bound (43) from Proposition 5 to obtain a bound away from zero uniform in α on a truncated sector $U_0(R',\phi')$ for $1-\widehat{h^0}(\alpha)$. Without loss of generality, the bound from the analytic semigroup properties (40) for p_{abs} stating that $|\alpha R^{abs}_{\alpha} f(0)| \leq M^{abs}\|f\| < \infty$ is valid on the same $U_0(R',\phi')$. Putting the bounds together, we have that $|\alpha \widehat{U}(\alpha)| \leq M_2||f|| < \infty$. We mention that both bounds (37) and (43) are based on independent proofs regarding the resolvent R^{abs} of the absorbing Brownian kernel p_{abs}.

The inversion formula from Proposition 6, together with Proposition 7 show that

$$\frac{d^n}{dt^n} U(t) = \frac{1}{2\pi i} \int_{L(\zeta_1)} \alpha^n e^{\alpha t} \widehat{U}(\alpha) d\alpha \tag{32}$$

where the contour is chosen as in Proposition 7, with $\zeta_1 > 0$ and $U_{\zeta_1}(\phi'')$. We note that it is always possible to choose $\phi'' \in (\frac{\pi}{2}, \phi']$ such that, on the domain to the right hand side of $L(\zeta_1/2)$, $\widehat{U}(\alpha)$ is analytic and there exists $M_{21} < \infty$ such that $|\alpha\widehat{U}(\alpha)| \leq M_{21}\|f\|$. For the differentiation under the integral we refer the reader to Lemma 2 in [10].

Let $\epsilon > 0$ be a small number. We want to show (30). Formula (32) with $n = 1$ gives a bound

$$|U'(s)| \leq \frac{M_{21}||f||}{2\pi} \int_{L(\zeta_1)} |e^{\alpha s}|\, d|\alpha| \leq \frac{M_{21}||f||e^{\zeta_1 s}}{\pi|\cos(\phi'')|s}.$$

The proof is done by setting $C_1(\epsilon, T) = \sup_{s\in[\epsilon,T]} \frac{M_{21}e^{\zeta_1 s}}{\pi|\cos(\phi'')|s}$. □

Proof of (F1). Let $t > 0$ be fixed. We shall prove a stronger statement, namely equicontinuity of the family $\{S_t f(x)\}$ for $f \in C(X)$ with $||f|| \leq 1$, more precisely the limit

$$\lim_{x\to x_0} \sup_{||f||\leq 1} |S_t f(x) - S_t f(x_0)| = 0. \tag{33}$$

Without loss of generality, we may further assume that $f(0) = 0$, since the difference in (33) remains identical under constant addition. Since $f \in C(X)$, this means that the function f satisfies the Dirichlet boundary condition.

Step 1. First, we prove continuity at $x_0 \neq 0$. The first term of (29) is the solution to the heat equation with Dirichlet b.c. for the half-Laplacian which is continuous in (t, x), $t \geq 0$, $x \in \mathcal{R}$. Since the kernel $p_{abs}(t, x, y)$, $t > 0$ is continuous in x and y on $\mathcal{R}$ up to the boundary, the semigroup S_t^{abs} is compact, which implies that $\lim_{x\to x_0} \sup_{||f||\leq 1} |S_t^{abs} f(x) - S_t^{abs} f(x_0)| = 0$ by the equicontinuity of the family $\{S_t^{abs} f(x)\}$ for $||f|| \leq 1$, $f \in C(X)$ with $f(0) = 0$.

For the second term, we split the difference $\int_0^t U(t-s)(h^x(s)-h^{x_0}(s))ds$ in two parts, an integral on $[0, \epsilon]$ and another on $[\epsilon, t]$, for an arbitrary but sufficiently small fixed ϵ.

In the second integral on $[\epsilon, t]$, $h^x(s)$ is continuous in $(s, x) \in [\epsilon, T] \times \overline{B}(x_0, r_1)$ for a sufficiently small r_1, and hence uniformly continuous. Since $|U(t)| \leq C_{U,T}||f||$ for all $t \leq T$, we have shown $\lim_{x\to x_0} \sup_{||f||\leq 1} |\int_\epsilon^t U(t-s)(h^x(s) - h^{x_0}(s))ds| = 0$.

The integral on $[0, \epsilon]$ is bounded above by $C_{U,T}||f||[P_x(\tau_0 \leq \epsilon) + P_{x_0}(\tau_0 \leq \epsilon)]$. As $x \to x_0$ and then $\epsilon \to 0$ it also vanishes, proving uniform continuity at x_0 on $\{f \in C(X) : ||f|| \leq 1\}$.

Step 2. We prove continuity at $x_0 = 0$. If $x \to x_0$ from a ball centered at x_0, the proof is identical to the case $x_0 \neq 0$. Let $x \to \partial\mathcal{R}$. The first term in (27) approaches zero. Formally, $h^x(t) \to \delta_0(t)$ in the sense of distributions, so $\lim_{x\to\partial\mathcal{R}} S_t f(x) = U(t)$.

On the other hand, $S_t f(0) = S_t^{abs} f(0) + [S_\cdot^{abs} f(0) * v(\cdot)](t) - S_t^{abs} f(0) = U(t)$ due to the fact that the summation in $v(\cdot)$ starts at $n = 2$ when $x = 0$. To conclude the argument, we have to prove rigorously that $h^x(t) \to \delta_0(t)$ and that the convergence is uniform in f. We know from Proposition 3 that U is bounded for $t \geq 0$ and has continuous derivative for $t > 0$. Choose arbitrary but fixed $\epsilon \in (0, t)$. Then $\int_0^t h^x(s)U(t-s)ds - U(t) = A + B - U(t)$ with A and B the integrals on $[0, t-\epsilon]$ and $[t-\epsilon, t]$, respectively.

$$A - U(t) = \int_0^{t-\epsilon} h^x(s)U(t-s)ds - U(t)$$

$$= P_x(\tau_0 \leq s)U(t-s)\Big|_0^{t-\epsilon} + \int_0^{t-\epsilon} P_x(\tau_0 \leq s)U'(t-s)ds - U(t)$$

$$= (P_x(\tau_0 \leq t-\epsilon) - 1)U(\epsilon) + U(\epsilon)$$

$$+\int_0^{t-\epsilon}(P_x(\tau_0 \le s)-1)U'(t-s)ds-(U(\epsilon)-U(t))-U(t)$$

such that

$$|A-U(t)| \le P_x(\tau_0 > t-\epsilon)|U(\epsilon)| + \int_0^{t-\epsilon} P_x(\tau_0 > s)|U'(t-s)|ds$$

showing that

$$\sup_{||f||\le 1}|A-U(t)| \le C_{U,T}P_x(\tau_0 > t-\epsilon) + C_1(\epsilon,T)\int_0^{t-\epsilon} P_x(\tau_0 > s)ds\,.$$

The last limit converges to zero as $x \to \partial\mathcal{R}$ due to the regularity of the domain $\mathcal{R}$ and by dominated convergence in the case of the time integral.

Finally $B \le C_{U,T}||f||P_x(t-\epsilon < \tau_0 \le t)$ that vanishes as x approaches the boundary. This concludes the proof.

Proof of (F2). Without loss of generality we can assume $f(0) = f(x)|_{x\in\partial\mathcal{R}} = 0$ and we prove continuity at $t = 0$. We write $|S_tf(x) - f(x)| \le A_1 + A_2 + A_3$ with $A_1 = E_x[f(z(t))\,,\, t < \tau_0] - f(x)$, $A_2 = E_x[f(z(t))\,,\, \tau_0 \le t < \tau_1]$, $A_3 = E_x[f(z(t))\,,\, \tau_0 < \tau_1 \le t]$. By construction, $A_1 = S_t^{abs}f(x) - f(x)$ converges in the sup norm to zero as $t \to 0$ due to the strong continuity of S^{abs} and f being zero on the boundary. Again by construction and the strong Markov property we see that $A_3 \le ||f||P_0(\tau_0 \le t)$ uniformly in x, which approaches zero as $t \to 0$. There are two cases for A_2, if $x \in \overline{\mathcal{R}_\delta} = \{x \in \mathcal{R} : d(x,\partial\mathcal{R}) \ge \delta\}$, and $x \in \mathcal{R}\setminus\overline{\mathcal{R}_\delta}$. The first case is bounded above by $||f||\sup_{x\in\overline{\mathcal{R}_\delta}} P_x(\tau_0 \le t)$, with limit zero as $t \to 0$. Finally we have the second case bounded above by $\sup_{s\in[0,t]} E_0[f(w(s))]$. Here we replaced the process $z(t)$ by the Brownian motion $w(t)$ after using the strong Markov property. Given a small ϵ, this is bounded by

$$\sup_{s\in[0,t]} E_0[f(w(s))] \le ||f||P_0(\sup_{s\in[0,t]}|w(s)| \ge \epsilon) + \sup_{x\in\overline{B}(0,\epsilon)}|f(x)|\,.$$

The martingale inequality shows that the first term vanishes as $t \to 0$. The second term is independent of x and vanishes as $\epsilon \to 0$ since $f(0) = 0$. This concludes the proof that S_t is a strongly continuous Feller semigroup on $C(X)$.

Proof of (F3). Since $||S_t|| \le 1$ and the space X is compact, we shall apply the Arzelà-Ascoli theorem to prove that the family of functions $\{S_tf\}$ with $||f|| \le 1$ is equicontinuous. The statement (33) shown in (F1) is equivalent to equicontinuity, proving that the semigroup is compact.

4 Proof of Theorem 2

By definition, the Laplace transform of a function $g(t)$ is equal to $\hat{g}(\alpha) = \int_0^\infty e^{-\alpha t}g(t)dt$ whenever the integral converges. From equation (5) $P(T_x > t) = \int_{\mathcal{R}} p_{abs}(t,x,y)dy$ we see that

$$\widehat{h^x}(\alpha) = E\Big[e^{-\alpha T_x}\Big] = -\int_0^\infty e^{-\alpha t}dP(T_x > t) = -\int_{\mathcal{R}}\int_0^\infty e^{-\alpha t}dp_{abs}(t,x,y)dy\,. \tag{34}$$

For $\Re(\alpha) > 0$, we derive

$$\widehat{h^x}(\alpha) = -\left(\int_{\mathcal{R}} e^{-\alpha t} p_{abs}(t,x,y)dy\right)\Big|_0^\infty - \alpha R_\alpha^{abs}\mathbf{1}(x) = 1 - \alpha R_\alpha^{abs}\mathbf{1}(x) \tag{35}$$

where $\mathbf{1}(x)$ is the constant function equal to 1 and R_α^{abs} is the resolvent of the half Laplacian with Dirichlet boundary conditions (the infinitesimal generator of the absorbing Brownian motion) from (8). If $\Re(\alpha) > 0$ and arbitrary $x \in \mathcal{R}$ we immediately have $|\widehat{h^x}(\alpha)| < 1$.

We shall use the results on analytic semigroups generated by strongly elliptic operators under Dirichlet boundary condition from [18] and [19]. More precisely, Theorem 1 in [18], adapted to the simpler case of the Dirichlet half-Laplacian, shows that there exist $R_0 > 0$, $\phi_0 \in (\frac{\pi}{2}, \pi)$ and $M^{abs} > 0$, such that, for all $u \in C^2(\mathcal{R}) \cap C(\overline{\mathcal{R}})$ vanishing on the boundary $\partial\mathcal{R}$, we have the bound

$$\|u\|_{C(\overline{\mathcal{R}})} \leq \frac{M^{abs}}{|\alpha|}\, \|(\frac{1}{2}\Delta - \alpha)u\|_{C(\overline{\mathcal{R}})} \quad \text{for all } \alpha \in U_0(R_0, \phi_0)\,. \tag{36}$$

Proposition 4. *For any radius $R' \in [R_0, \infty)$ and any angle $\phi' \in (\frac{\pi}{2}, \phi_0]$ we have*

$$\sup_{\alpha \in U_0(R',\phi')} \sup_{x \in \mathcal{R}} \left|1 - \alpha R_\alpha^{abs}\mathbf{1}(x)\right| < \infty\,. \tag{37}$$

Proof. For α, β in the resolvent set ϱ (a subset of the resolvent set of the Dirichlet half-Laplacian) the resolvent identity reads

$$R_\alpha^{abs} - R_\beta^{abs} = (\beta - \alpha)\, R_\alpha^{abs}\, (R_\beta^{abs})\,. \tag{38}$$

Let $\beta \in \varrho$ with $\Re(\beta) > 0$. For such β the semigroup S_t^{abs} defined in (5) and its resolvent R_β^{abs} can be applied to any $f \in C(\overline{\mathcal{R}})$. The function $R_\beta^{abs} f(x)$ belongs to $C^2(\mathcal{R}) \cap C(\overline{\mathcal{R}})$ and vanishes at the boundary. Moreover, it satisfies the inversion formula $(\beta I - \frac{1}{2}\Delta)R_\beta^{abs} f(x) = f(x)$. Thus, consistently with (38), R_α^{abs} can be applied to all $f \in C(\overline{\mathcal{R}})$ for arbitrary $\alpha \in \varrho$

$$R_\alpha^{abs} f(x) = [I + (\beta - \alpha)\, R_\alpha^{abs}](R_\beta^{abs} f(x))\,. \tag{39}$$

Notice that R_α^{abs} is applied to a continuous vanishing on the boundary. Relation (39) also implies that $(\alpha I - \frac{1}{2}\Delta)R_\alpha^{abs} f(x) = f(x)$.

Setting $u(x) = R_\alpha^{abs} f(x)$ in (36) for $f \in C(\overline{\mathcal{R}})$, we obtain that the main estimate for analytic semigroups (from [17; 20])

$$\|R_\alpha^{abs} f\|_{C(\overline{\mathcal{R}})} \leq \frac{M^{abs}}{|\alpha|}\, \|f\|_{C(\overline{\mathcal{R}})} \quad \text{for all } \alpha \in U_0(R_0, \phi_0) \tag{40}$$

is valid for any $f \in C(\overline{\mathcal{R}})$, in particular the constant $f(x) = \mathbf{1}(x)$. Based on this observation, the uniform bound (37) is satisfied. □

The resolvent identity (38) applied to the constant function $\mathbf{1}$ for $\alpha, \beta \in \varrho$ (we switched α and β) is

$$R_\beta^{abs}\mathbf{1} - R_\alpha^{abs}\mathbf{1} = (\alpha - \beta)\, R_\beta^{abs}\, (R_\alpha^{abs}\mathbf{1})$$

and implies

$$\big(I-\big(1-\frac{\beta}{\alpha}\big)\,\alpha R_{\alpha}^{abs}\big)\big(\beta R_{\beta}^{abs}\mathbf{1}-\mathbf{1}\big) \;=\; \alpha R_{\alpha}^{abs}\mathbf{1}-\mathbf{1}\,. \tag{41}$$

Let $\beta=|\alpha|$. Since we have $\|\alpha R_{\alpha}^{abs}\|\le M^{abs}$ in the operator norm from $C(\overline{\mathcal{R}})$ to $C(\overline{\mathcal{R}})$, then for all α in the truncated sector $U_0(R_0,\phi_0)$ from (40),

$$\|\big(I-\big(1-\frac{\beta}{\alpha}\big)\,\alpha R_{\alpha}^{abs}\big)\|\le\big(1+2M^{abs}\big)=M_1\,.$$

Therefore,

$$\|\alpha R_{\alpha}^{abs}\mathbf{1}-1\|_{C(\overline{\mathcal{R}})}=\|\big(I-\big(1-\frac{\beta}{\alpha}\big)\,\alpha R_{\alpha}^{abs}\big)\big(\beta R_{\beta}^{abs}\mathbf{1}-1\big)\;\|_{C(\overline{\mathcal{R}})}\le M_1\,\|\beta R_{\beta}^{abs}\mathbf{1}-1\|_{C(\overline{\mathcal{R}})}\,, \tag{42}$$

showing that we can bound the left hand side of (42), depending on arbitrary α in the truncated cone by the right hand side, which depends on β with $\Re(\beta)>0$, an identity needed in the next proposition.

Proposition 5. *There exist a radius $R'\in[R_0,\infty)$, an angle $\phi'\in(\frac{\pi}{2},\phi_0]$ such that*

$$\inf_{\alpha\in U_0(R',\phi')}\Big|\alpha R_{\alpha}^{abs}\mathbf{1}(0)\Big|>0\,. \tag{43}$$

Proof. Assume that (43) is false for any $U_0(R',\phi(R'))$ where $R'>R_0$ and $\phi(R')\in(\frac{\pi}{2},\phi_0]$ is of the form $\phi(R')=\frac{\pi}{2}+\arcsin(\frac{1}{R'})$. Let $R_n\to\infty$. Then, there exists a subsequence $\{n_k\}_{k\ge1}$ such that $\{\alpha_{n_k}\}\in U_0(R_{n_k},\phi(R_{n_k}))$ violates the lower bound (43). The domain $U_0(R',\phi(R'))$ is closed under complex conjugation and the complex norm from (43) is invariant under conjugation. This shows that we can assume, without loss of generality, that $\Im(\alpha_{n_k})>0$. For simplicity we subindex the subsequence by n as well.

We write $\alpha_n=r_n\exp\Big(i(\frac{\pi}{2}+\epsilon_n)\Big)$. Naturally $r_n\ge R_n\to\infty$ and also $\epsilon_n<\arcsin(\frac{1}{R_n})$. On the other hand, we can select a subsequence such that $\liminf\epsilon_n=0$. Otherwise there exists $\epsilon>0$ such that $\epsilon_n\le-\epsilon$ for large enough n. This, together with the inequality $|\alpha_n R_{\alpha_n}^{abs}\mathbf{1}(0)|\ge 1-|\widehat{h^0}(\alpha_n)|$ derived from (35) and

$$\lim_{n\to\infty}|\widehat{h^0}(\alpha_n)|\le\lim_{n\to\infty}E[e^{-r_n\cos(\frac{\pi}{2}-\epsilon)T_0}]=0$$

would imply a contradiction with the assumption on $\{\alpha_n\}$. We have shown that $\epsilon_n\to0$.

Equation (41) can be re-written in the form

$$\frac{\beta}{\alpha}\big(\beta R_{\beta}^{abs}\mathbf{1}(x)-1\big)+\Big(1-\frac{\beta}{\alpha}\Big)\Big(I-\alpha R_{\alpha}^{abs}\Big)\big(\beta R_{\beta}^{abs}\mathbf{1}(x)-1\big)=\alpha R_{\alpha}^{abs}\mathbf{1}(x)-1\,. \tag{44}$$

Choose

$$\beta_n=r_n\exp\Big(i(\frac{\pi}{2}-\delta_n)\Big)$$

with $\delta_n=\arcsin(\frac{1}{\sqrt{r_n}})$. Then the second term from (44) applied to $x=0$ with $\alpha=\alpha_n$ and $\beta=\beta_n$,

$$\Big|\Big(1-\frac{\beta}{\alpha}\Big)\Big(I-\alpha R_{\alpha}^{abs}\Big)\big(\beta R_{\beta}^{abs}\mathbf{1}(0)-1\big)\Big|$$

has the upper bound

$$\left|1-\frac{\beta}{\alpha}\right|\|I-\alpha R_\alpha^{abs}\|\|\beta R_\beta^{abs}\mathbf{1}(\cdot)-1\|_{C(\overline{\mathcal{R}})}\le\left|1-\frac{\beta}{\alpha}\right|(1+M^{abs})M_1$$

where we used (40) for the operator norm and (42) for the uniform norm. This term vanishes as $n\to\infty$ since $\left|1-\frac{\beta_n}{\alpha_n}\right|\le 2\left|\sin(\frac{\delta_n+\epsilon_n}{2})\right|$.

The first term in (44) at $x=0$ satisfies the bound

$$\left|\frac{\beta_n}{\alpha_n}\big(\beta_n R_{\beta_n}^{abs}\mathbf{1}(0)-1\big)\right|\le\left|\beta_n R_{\beta_n}^{abs}\mathbf{1}(0)-1\right|\le$$

$$\le E\Big[e^{-\Re(\beta_n)T_0}\Big]=E\Big[e^{-(r_n\sin\delta_n)T_0}\Big]=E\Big[e^{-\sqrt{r_n}T_0}\Big]\;\to\;0\,.$$

These estimates show that as $n\to\infty$, the left hand side of (44) with $\alpha=\alpha_n$, $\beta=\beta_n$ vanishes meanwhile the right hand side approaches -1 by the assumption made on the sequence $\{\alpha_n\}$, which is a contradiction. This concludes the proof of (43). □

We now resume the proof of Theorem 2. With (20) and (21) in mind, for $\Re(\alpha)>0$, we obtain

$$\int_{\mathcal{R}}\hat p(\alpha,x,y)f(y)dy=\int_{\mathcal{R}}\widehat{p_{abs}}(\alpha,x,y)f(y)dy+$$

$$+\int_{\mathcal{R}}\widehat{p_{abs}}(\alpha,0,y)f(y)dy\Big(\widehat{\sum_{n=1}^{\infty}(h^x*(h^0)^{*,n-1})}\Big)(\alpha)$$

$$=\int_{\mathcal{R}}\widehat{p_{abs}}(\alpha,x,y)f(y)dy+\int_{\mathcal{R}}\widehat{p_{abs}}(\alpha,0,y)f(y)dy\Big(\sum_{n=1}^{\infty}\widehat{h^x}(\alpha)(\widehat{h^0}(\alpha))^{n-1}\Big)$$

which proves (23) on $\{\alpha:\Re(\alpha)>0\}$ in the form

$$R_\alpha f(x)=R_\alpha^{abs}f(x)+R_\alpha^{abs}f(0)\,\frac{1-\alpha R_\alpha^{abs}\mathbf{1}(x)}{\alpha R_\alpha^{abs}\mathbf{1}(0)}\,. \tag{45}$$

For any $f\in\mathcal{D}_0(X)$ the resolvent $R_\alpha^{abs}f(x)$ is analytic on $\mathbb{C}\setminus\{\lambda_n^{abs}:n\ge1\}$ ([20], page 211, applied to the generator of a semigroup), which implies that (45) can be extended as a meromorphic function outside the spectrum (6) of the Dirichlet Laplacian.

We want to extend the estimate (40) to the resolvent (45) to obtain (24). There are two parts in the right hand side of the equation (45) multiplied by α. For the first part $\alpha R_\alpha^{abs}f(x)$, (40) grants that there exist $R_0>0$ and $\phi_0\in(\frac{\pi}{2},\pi)$ such that $\alpha R_\alpha^{abs}f(x)$ stays bounded by $M^{abs}\|f\|_{C(\overline{\mathcal{R}})}$ for $\alpha\in U_0(R_0,\phi_0)$. For the second part, we need to show that there exist a radius $R'>0$, an angle $\phi'\in(\frac{\pi}{2},\pi)$ and a constant $\tilde M>0$ such that

$$\sup_{\alpha\in U_0(R',\phi')}\sup_{x\in\mathcal{R}}\left|\alpha R_\alpha^{abs}f(0)\frac{1-\alpha R_\alpha^{abs}\mathbf{1}(x)}{\alpha R_\alpha^{abs}\mathbf{1}(0)}\right|\le\tilde M\|f\|_{C(\overline{\mathcal{R}})}\,. \tag{46}$$

Since $|\alpha R_\alpha^{abs}f(0)|$ can be bounded by $M^{abs}\|f\|$ using (40), the goal is achieved based on relation (37) from Proposition 4 and the lower bound (43) from Proposition 5. Without of loss of generality we can choose $R' \in [R_0, \infty)$ and an angle $\phi' \in (\frac{\pi}{2}, \phi_0]$ so as to satisfy both Propositions 4 and 5. Now by letting $R = R'$, $\phi = \phi'$, we have proved (24).

On the real axis, the function $\widehat{h^0}(\alpha)$ is the Laplace transform of the first hitting time to the boundary, equal to 1 at $\alpha = 0$ and non-increasing on $(\lambda_1^{abs}, \infty)$. The function is analytic wherever R_α^{abs} is analytic, hence $1 - \widehat{h^0}(\alpha)$ has no other zeros on a neighborhood of $(\lambda_1^{abs}, \infty)$.

Since $R_\alpha^{abs} f$ is analytic in the union of U_0 with the right half-plane $\Re(\alpha) > \lambda_1^{abs}$, the denominator $1 - \widehat{h^0}(\alpha) = \alpha R_\alpha^{abs} \mathbf{1}(0)$ from (45) has only isolated zeros. We conclude that all singularities of the resolvent R_α contained in the resolvent set of the Dirichlet Laplacian are poles coinciding with the zeros of the denominator.

5 Proof of Theorem 3

We can compute the residue at $\alpha = 0$. Multiplying (23) by α, we get

$$\alpha R_\alpha f(x) = \alpha R_\alpha^{abs} f(x) + \frac{\alpha R_\alpha^{abs} f(0)}{\alpha R_\alpha^{abs} \mathbf{1}(0)} \left(1 - \alpha R_\alpha^{abs} \mathbf{1}(x)\right)$$

Since $R_\alpha^{abs} f$ is analytic in a neighborhood of $\alpha = 0$, it is enough to figure out the limit of $\alpha R_\alpha f(x)$ as $\alpha \to 0$ along the positive real axis. By dominated convergence, or directly from the continuity of the resolvent R_α^{abs} at $\alpha = 0$, we see that $\lim_{\alpha \to 0+} \alpha R_\alpha^{abs} f(x) = 0$, and that

$$\lim_{\alpha \to 0+} \frac{\alpha R_\alpha^{abs} f(0)}{\alpha R_\alpha^{abs} \mathbf{1}(0)} = \frac{\int_{\mathcal{R}} G(0,y)\, f(y)\, dy}{\int_{\mathcal{R}} G(0,y)\, dy} = \int_{\mathcal{R}} \rho(y)\, f(y)\, dy$$

where $\rho(y) = G(0,y)(\int_{\mathcal{R}} G(0,y)\, dy)^{-1}$.

All singularities, with the exception of zero, have negative real part since S_t is a Feller semigroup. Notice that the singularities must be among the zeros of the denominator $\alpha R_\alpha^{abs} \mathbf{1}(0) = 1 - \widehat{h^0}(\alpha)$. Suppose $\alpha = ik$ with $k \in \mathbb{R}$ is a zero to $1 - \widehat{h^0}(\alpha)$. We need to show that the Fourier transform of a probability density function $f(t)$ can never attain the value one except at $k = 0$. The transform is $\int_0^\infty e^{-ikt} f(t) dt = 1$ and this implies that $\int_0^\infty (1 - \cos(kt)) f(t) dt = 0$, a contradiction unless $k = 0$.

We recall from the proof of (46) that there exists $R > 0$ and $\phi \in (\frac{\pi}{2}, \pi)$ such that $U_0(R, \phi)$ belongs to the resolvent set and the bound (24) is satisfied. There are finitely many elements of the spectrum in $|\alpha| \leq R$. After ordering the elements of the spectrum (except the origin) according to their real values, let α^* be a representative of the values with largest real part and α^{**} be a representative of those with second largest real part, that is $0 > \Re(\alpha^*) > \Re(\alpha^{**})$. Pick $\alpha_0 > 0$.

For all sufficiently small $\epsilon > 0$, there exists an angle $\phi^* \in (\frac{\pi}{2}, \phi)$ such that the following conditions are satisfied.

1) The domain $U_{\frac{\alpha_0}{2}}(\phi^*)$ is included in the resolvent set,

2) All elements of the spectrum with real part less than or equal to $\Re(\alpha^{**})$ are in the complement of $U_{\Re(\alpha^{**}) + \frac{\epsilon}{2}}(\phi^*)$

3) All elements of the spectrum with real part greater than or equal to $\Re(\alpha^*)$ are in $U_{\Re(\alpha^{**})+\epsilon}(\phi^*)\setminus \overline{U}_{\frac{\alpha_0}{2}}(\phi^*)$

For a given $f \in C(\overline{\mathcal{R}})$ and $x \in \mathcal{R}$, Proposition 6 provides an inversion formula for the resolvent $\alpha \to R_\alpha f(x)$. For $\alpha_0 > 0$ (plays the role of ζ_0 in the proposition) and $t > 0$,

$$S_t f(x) = \frac{1}{2\pi i} P.V. \int_{\alpha_0 - i\infty}^{\alpha_0 + i\infty} e^{\alpha t} R_\alpha f(x) d\alpha\,. \tag{47}$$

From the bound that characterizes the analytic semigroup (24) we see that Proposition 7 can be applied to $\alpha \to R_\alpha f(x)$ with $\zeta_1 = \alpha_0/2$ (or any positive number less than α_0).

Apply Proposition 8 below to $\alpha \to R_\alpha f(x)$ with $\zeta_2' = \Re(\alpha^{**}) + \frac{\epsilon}{2}$, $\zeta_2 = \Re(\alpha^{**}) + \epsilon < \Re(\alpha^*)$, $\zeta_1 = \alpha_0/2$, and $\zeta_1' = \alpha_0$. Then, by construction of the contour $L(\zeta_2) = \partial U_{\zeta_2}(\phi^*)$, the simple pole $\beta_0 = 0$ and the poles β_j, $1 \le j \le l$ with $\Re(\beta_j) = \Re(\alpha^*)$ and multiplicities m_j are situated in the corresponding domain $U_{\zeta_2} \setminus \overline{U}_{\zeta_1}$ of Proposition 8. At this point, we set δ in the proposition equal to $\epsilon/4$. Let

$$G_*(t,f)(x) = \sum_{j=1}^{l} e^{\beta_j t}\Big(C_{j1} + \frac{C_{j2}}{1!}t + \ldots + \frac{C_{jm_j}}{(m_j-1)!}t^{m_j-1}\Big) = e^{\Re(\alpha^*)t} P(t,f)(x)\,,\; t \ge T$$

with coefficients

$$C_{jk} f(x) = \frac{1}{2\pi i}\int_{\partial B(\beta_j,\rho)} (\alpha - \beta_j)^{k-1} R_\alpha f(x)\, d\alpha\,, \tag{48}$$

$1 \le k \le m_j$ and $\rho > 0$ sufficiently small so that β_j is the only singularity in the ball $B(\beta_j, \rho)$. Then C_{jk} are bounded linear operators applied to f at the point x, and the operator norm $||P(t,\cdot)||$ is of polynomial order.

Denote $A_1 = S_t f(x)$, $A_2 = Res(0; e^{\alpha t} R_\alpha f(x))$, $A_3 = G_*(t,f)(x)$ and the improved error bound given after (53) $A_4 = CM_{\frac{\epsilon}{4}} e^{(\Re(\alpha^{**})+\frac{3\epsilon}{4})t}$ with the particular choice of $\zeta = \zeta_2' + \delta$ with the current values $\zeta_2' = \Re(\alpha^{**}) + \frac{\epsilon}{2}$ and $\delta = \epsilon/4$. We notice that for $g(\alpha) = R_\alpha f(x)$ there exists $M_\epsilon' > 0$ independent of f and x such that $M_{\frac{\epsilon}{4}}$ from A_4 satisfies $M_{\frac{\epsilon}{4}} \le M_\epsilon' ||f||$. Since $||A_1 - A_2| - |A_3|| \le |A_1 - A_2 - A_3| \le A_4$ we have, for all f with $||f|| \le 1$, the double inequality

$$|A_3| - CM_\epsilon' e^{(\Re(\alpha^{**})+\frac{3\epsilon}{4})t} \le |A_1 - A_2| \le |A_3| + CM_\epsilon' e^{(\Re(\alpha^{**})+\frac{3\epsilon}{4})t}\,. \tag{49}$$

Taking the supremum over $x \in \overline{\mathcal{R}}$ and f with $||f|| \le 1$, the upper and lower bounds for (49) become

$$e^{\Re(\alpha^*)t}||P(t,\cdot)|| \pm CM_\epsilon' e^{(\Re(\alpha^{**})+\frac{3\epsilon}{4})t}\,.$$

Upper bound. The norms $||P(t,\cdot)||$ are of polynomial order, as mentioned before.

Lower bound. The lower bound is nontrivial. It is sufficient to find a function $\phi(x)$ with $||\phi||_{C(\overline{\mathcal{R}})} = 1$ such that $||P(t,\phi)||_{C(\overline{\mathcal{R}})}$ is bounded away from zero uniformly in t. Since the semigroup S_t (22) is a strongly continuous and compact Feller semigroup, it follows from Theorem 3.3 in [17] that the corresponding resolvent R_α (for $\Re(\alpha) > 0$) is compact, and as a consequence the infinitesimal generator $\mathcal{L}$ has a pure point spectrum. Let β_j, $1 \le j \le l$ be one of the eigenvalues with $\Re(\beta_j) = \Re(\alpha^*)$ and let ϕ be a corresponding eigenfunction normalized to have norm one. Since $R_\alpha \phi = (\alpha - \beta_j)^{-1}\phi$, all $C_{j'k}\phi$ are zero except for $j' = j$ and $k = 1$, when it is exactly ϕ, making $P(t,\phi)(x) = e^{i\Im(\beta_j)t}\phi(x)$ which gives a lower bound $||\phi|| = 1$ uniform over t.

After taking the logarithm, dividing by t and passing to the limit as $t \to \infty$, we have proven (26).

Finally, we give the statement of the classical inversion theorems for the Laplace transform (Proposition 3, Chapter 4 in [5]).

Proposition 6. *Let $G(t)$ be a continuous function defined for $t > 0$ such that there exists an $\zeta_0 \in \mathbb{R}$ with the property that*

$$\int_0^\infty e^{-\zeta_0 t}|G(t)|dt < \infty\,.$$

Then, the Laplace transform $g(\alpha)$ is analytic in the half-plane $Re(\alpha) > \zeta_0$ and the following inversion formula is valid

$$G(t) = \frac{1}{2\pi i} P.V. \int_{\zeta - i\infty}^{\zeta + i\infty} e^{\alpha t} g(\alpha) d\alpha \tag{50}$$

where $\zeta \geq \zeta_0$ is arbitrary.

For a given angle $\phi \in (\frac{\pi}{2}, \pi)$, we recall the definition of the sector $U_\zeta = U_\zeta(\phi)$ from (18). Let $L(\zeta)$ be the contour $\Re(\alpha) = \zeta + R\cos\phi$ and $\Im(\alpha) = \pm R\sin\phi$ for all $R \geq 0$.

Proposition 7. *Let $\zeta_1 < \zeta_0$ be real numbers and $g(\alpha)$ be an analytic function in an open set including the domain $\overline{U}_{\zeta_1} \setminus \{\alpha \in \mathbb{C} : \Re(\alpha) > \zeta_0\}$ such that there exist $R_0 > 0$ and $\phi \in (\frac{\pi}{2}, \pi)$ and $M_g > 0$ such that $|\alpha g(\alpha)| \leq M_g$ for all $\alpha \in U_{\zeta_1}(R_0, \phi)$. Suppose that the integral (50) is finite, then for $t > 0$, (50) is equal to*

$$\frac{1}{2\pi i} \int_{L(\zeta_1)} e^{\alpha t} g(\alpha) d\alpha \tag{51}$$

in the sense of principal value along $L(\zeta_1)$.

Proof. First we pick a simple closed curve in $\mathbb{C}$ consisting of piecewise straight line segments connecting A^-, C, A^+, B^+, D, B^- and coming back to A^- with counterclockwise orientation where the coordinates of the points are $C = (\zeta_0, 0)$, $D = (\zeta_1, 0)$, $A^\pm = (\zeta_0, \pm R\sin\phi)$ and $B^\pm = (\zeta_1 + R\cos\phi, \pm R\sin\phi)$. Choose $R > 0$ large enough so that $R_0 < R\sin\phi$. By the residue theorem, we have

$$\int_{A^-CA^+} e^{\alpha t} g(\alpha) d\alpha = \int_{B^-DB^+} e^{\alpha t} g(\alpha) d\alpha + \mathcal{E}_R$$

and the error term $\mathcal{E}_R$ is given by

$$\mathcal{E}_R = -\int_{A^+B^+} e^{\alpha t} g(\alpha) d\alpha - \int_{B^-A^-} e^{\alpha t} g(\alpha) d\alpha\,.$$

From $R_0 < R\sin\phi$, we have the line segments A^+B^+ and B^-A^- inside $U_{\zeta_1}(R_0, \phi)$. Hence we reach an upper bound on $\mathcal{E}_R$

$$|\mathcal{E}_R| \leq C(t, \zeta_0, \phi) M_g R^{-1}$$

where $C(t, \zeta_0, \phi)$ is a constant that only depends on t, ζ_0 and ϕ. Letting R to infinity we see that (51) exists and is equal to (50) in principal value sense. □

We can now give the inversion formula in the case of multiple poles.

Proposition 8. *Let $\zeta_2' < \zeta_2 < \zeta_1 < \zeta_1'$ and let $g(\alpha)$ be analytic in the domain $V = U_{\zeta_2'} \setminus \overline{U}_{\zeta_1'}$ with the exception of $\alpha = \beta_j$, $0 \le j \le l$ which are poles of order $m_j \in Z_+$, all contained in $U_{\zeta_2} \setminus \overline{U}_{\zeta_1}$ with the principal part of the Laurent expansion about β_j equal to*

$$\frac{c_{j1}}{(\alpha-\beta_j)} + \ldots + \frac{c_{jm_j}}{(\alpha-\beta_j)^{m_j}}\,. \tag{52}$$

Assume that $g(\alpha)$ is bounded outside of a ball of radius $R_0 > 0$ centered at the origin in V. Then for any given $T > 0$ the integral (in principal value sense)

$$G(t) = \frac{1}{2\pi i}\int_{L(\zeta_1)} e^{\alpha t} g(\alpha) d\alpha$$

is uniformly convergent for $t \ge T$, and we have

$$\Big|G(t) - \sum_{j=0}^{l} e^{\beta_j t}\Big(c_{j1} + \frac{c_{j2}}{1!}t + \ldots + \frac{c_{jm_j}}{(m_j-1)!}t^{m_j-1}\Big)\Big| \le Ce^{\zeta_2 t}\sup_{\alpha\in L(\zeta_2)}|g(\alpha)|\,, \tag{53}$$

where C does not depend on g, t and ζ_2 for $t \ge T$. Moreover, for any sufficiently small $\delta > 0$, if M_δ is an upper bound of $|g(\alpha)|$ on $\overline{U}_{\zeta_2'+\delta} \setminus U_{\zeta_2}$, then for any $\zeta_2' + \delta \le \zeta \le \zeta_2$, the right-hand side of (53) can be replaced by $CM_\delta e^{\zeta t}$.

Remark. The supremum of g over $L(\zeta_2)$ (and $\overline{U}_{\zeta_2'+\delta} \setminus U_{\zeta_2}$) is achieved due to the fact that g is continuous and bounded far from the origin.

Proof. Consider a contour made of piecewise line segments connecting A^-, C, A^+, B^+, D, B^- and coming back to A^- with counterclockwise orientation where the coordinates of these points are $A^\pm = (\zeta_1 + R\cos\phi, \pm R\sin\phi)$, $B^\pm = (\zeta_2 + R\cos\phi, \pm R\sin\phi)$, $C = (\zeta_1, 0)$ and $D = (\zeta_2, 0)$.
Choose $R > 0$ large enough so that the line segments A^+B^+ and B^-A^- are outside $B(0, R_0)$, ball of radius R_0 centered at 0 and also that $\zeta_1 + R\cos\phi < 0$. For instance, one can choose $R > \max\{R_0 + |\zeta_1| + |\zeta_2|\,,\,|\zeta_1|/|\cos\phi|\}$. Let $c_g = \sup\{|g(\alpha)| : \alpha \in V\,,\,|\alpha| \ge R_0\}$.
First we prove the uniform convergence of $G(t)$ on $[T, \infty)$ for any given $T > 0$. Let $G_R(t) = \frac{1}{2\pi i}\int_{A^-CA^+} e^{\alpha t} g(\alpha)\,d\alpha$. Then for R large enough to satisfy the condition mentioned above,

$$|G(t) - G_R(t)| \le \frac{c_g}{\pi}\int_R^\infty e^{(\zeta_1 + r\cos\phi)t}dr = \frac{c_g}{\pi t|\cos\phi|}e^{(\zeta_1+R\cos\phi)t}\,.$$

Hence for $t \ge T$, we have

$$|G(t) - G_R(t)| \le \frac{c_g}{\pi T|\cos\phi|}e^{(\zeta_1+R\cos\phi)T}$$

which tends to zero as R goes to infinity. Therefore $G(t)$ converges uniformly on $[T, \infty)$.
The residue theorem for the integral around our contour provides

$$G_R(t) - \sum_{j=0}^{\ell} e^{\beta_j t}\sum_{k=1}^{m_j}\frac{c_{jk}}{(k-1)!}t^{k-1} = \frac{1}{2\pi i}\int_{B^-DB^+} e^{\alpha t} g(\alpha) d\alpha + \mathcal{E}_R \tag{54}$$

where the error term is

$$\mathcal{E}_R = -\frac{1}{2\pi i}\Big(\int_{A^+B^+} e^{\alpha t} g(\alpha) d\alpha + \int_{B^-A^-} e^{\alpha t} g(\alpha) d\alpha \Big)\,.$$

It is easy to see that $\int_{B^-DB^+} e^{\alpha t} g(\alpha) d\alpha$ converges to $\int_{L_{\zeta_2}} e^{\alpha t} g(\alpha) d\alpha$ as R goes to infinity uniformly on $[T,\infty)$ via the analogous argument used for $G_R(t)$ converging to $G(t)$ uniformly on $[T,\infty)$.

On the other hand, for large enough R as described at the beginning of the proof and for $t \geq T$,

$$|\mathcal{E}_R| \leq \frac{c_g}{\pi t}\, e^{(\zeta_1 + R\cos\phi)t} \leq \frac{c_g}{\pi T}\, e^{(\zeta_1 + R\cos\phi)T}$$

which converges to zero uniformly on $[T,\infty)$ as R goes to infinity. Sending R to infinity on both sides of the equation (54), we reach

$$G(t) - \sum_{j=0}^{\ell} e^{\beta_j t} \sum_{k=1}^{m_j} \frac{c_{jk}}{(k-1)!} t^{k-1} = \frac{1}{2\pi i} \int_{L(\zeta_2)} e^{\alpha t} g(\alpha) d\alpha\,.$$

Further estimate on the integral along $L(\zeta_2)$ reveals

$$\Big| \frac{1}{2\pi i} \int_{L(\zeta_2)} e^{\alpha t} g(\alpha) d\alpha \Big| \leq \frac{1}{\pi t |\cos\phi|} e^{\zeta_2 t} \sup_{\alpha \in L(\zeta_2)} |g(\alpha)| \leq C e^{\zeta_2 t} \sup_{\alpha \in L(\zeta_2)} |g(\alpha)| \tag{55}$$

where $C = \frac{1}{\pi T |\cos\phi|}$ for all $t \geq T$ independent of g, t and ζ_2.

Let $M_\delta = \sup\{\, |g(\alpha)| \,:\, \alpha \in \overline{U}_{\zeta_2' + \delta} \setminus U_{\zeta_2} \,\}$. If we replace the integral over $L(\zeta_2)$ by that over $L(\zeta)$ where $\zeta_2' + \delta \leq \zeta \leq \zeta_2$ and bound $|g(\alpha)|$ by M_δ, then we can improve the error estimate in (55) to $C M_\delta e^{\zeta t}$ with the same constant C appeared in (55). □

6 The one dimensional case

In $d = 1$, let $\mathcal{R} = (a,b)$, with $a < 0 < b$ as in [10]. Let $\lambda_k = k\pi/(b-a)$, $k = 1,2,\ldots$, and $\sigma_{abs} = \{-\lambda_k^2/2 \,:\, k = 1,2,\ldots\}$ be the spectrum of the half Laplacian with absorbing boundary conditions with transition kernel (7)

$$p_{abs}(t,x,y) = \frac{2}{b-a} \sum_{k=1}^{\infty} e^{-(\lambda_k^2/2)t} \sin\lambda_k(x-a) \sin\lambda_k(y-a)\,, \tag{56}$$

and resolvent kernel

$$\widehat{p_{abs}}(\alpha,x,y) = \frac{2}{b-a} \sum_{k=1}^{\infty} \frac{1}{\alpha + \lambda_k^2/2} \sin\lambda_k(x-a) \sin\lambda_k(y-a)\,. \tag{57}$$

The Laplace transform of the first exit time (34) can be written in two forms (see [10])

$$\widehat{h^x}(\alpha) = \frac{2\pi}{(b-a)^2} \sum_{k=1,\,odd} \frac{k}{\alpha + \lambda_k^2/2} \sin\lambda_k(x-a) = \frac{\cosh\sqrt{2\alpha}\Big(x - \frac{b+a}{2}\Big)}{\cosh\sqrt{2\alpha}\Big(\frac{b-a}{2}\Big)} \tag{58}$$

and the kernel of the resolvent (23) of the process given by the transition kernel (20) is then

$$\widehat{p}(\alpha, x, y) = \widehat{p_{abs}}(\alpha, x, y) + \widehat{p_{abs}}(\alpha, 0, y) H(\alpha, x) \tag{59}$$

where

$$H(\alpha, x) = \frac{\widehat{h^x}(\alpha)}{1 - \widehat{h^0}(\alpha)} = \frac{\cosh \sqrt{2\alpha}\left(x - \frac{b+a}{2}\right)}{\cosh \sqrt{2\alpha}\left(\frac{b-a}{2}\right) - \cosh \sqrt{2\alpha}\left(\frac{b+a}{2}\right)}. \tag{60}$$

Set $\gamma = -\lambda_1^2/2 = -\pi^2(\sqrt{2}(b-a))^{-2}$. Then we write $\sigma_{abs} = \{\gamma k^2 \, : \, k \in \mathbb{Z}_+\}$ for the spectrum of the absorbing Brownian kernel and

$$\sigma_H = \{0\} \cup \{4(1 + |a|/b)^2 \gamma k^2 \, , 4(1 + b/|a|)^2 \gamma k^2 \, : \, k \in \mathbb{Z}_+\}$$

for the zeros of $\cosh \sqrt{2\alpha}\left(\frac{b-a}{2}\right) - \cosh \sqrt{2\alpha}\left(\frac{b+a}{2}\right)$. In addition, σ_{abs}^{odd} and σ_{abs}^{even} denote the subsets of σ_{abs} for k odd and k even, respectively. It is easy to see that when $b/a \notin \mathbb{Q}$, then $\sigma_H \cap \sigma_{abs} = \emptyset$.

Proposition 9. *The spectrum σ of the Brownian motion with return is $\sigma_{abs}^{even} \cup \sigma_H$. As a consequence, the largest nonzero point of the spectrum is $-\lambda_2^2/2 = 4\gamma$.*

Proof. Part 1. We prove that if $\beta \in \sigma_H$, then $\widehat{p_{abs}}(\alpha, 0, y)$ is analytic and nonzero at $\alpha = \beta$. For $\beta \in \sigma_H \setminus \sigma_{abs}$, we only have to show that $\widehat{p_{abs}}(\beta, 0, y) \neq 0$, which is evident since $\widehat{p_{abs}}(\beta, 0, y)$, as an element in $L^2[a, b]$, has nonzero Fourier coefficients. If $\beta \in \sigma_H \cap \sigma_{abs}$, then there exists a positive integer k such that $\beta = -k^2\gamma = -\lambda_k^2/2$, $\lambda_k = k\pi/(b-a)$ and from (60), $\cos \lambda_k(\frac{b-a}{2})(1 - \cos \lambda_k a) + \sin \lambda_k(\frac{b-a}{2}) \sin \lambda_k a = 0$. For k odd, $(-1)^{\frac{k-1}{2}} \sin \lambda_k a = 0$ and for k even $(-1)^{\frac{k}{2}}(1 - \cos \lambda_k a) = 0$. In all cases $\sin \lambda_k a = 0$. By taking the limit as α goes to β in (57) with $x = 0$, we see that β is a removable singularity of $\widehat{p_{abs}}(\cdot, 0, y)$ and that the limit is not zero, which proves the claim.

Part 2. Since $\sigma \subseteq \sigma_{abs} \cup \sigma_H$, we divide the proof in three steps.

(i) $\sigma_H \setminus \sigma_{abs} \subseteq \sigma$. Let $p = 1, 2, \ldots$ be the multiplicity of the pole denoted by β of $H(\alpha, x)$ such that $\beta \notin \sigma_{abs}$. Then

$$\lim_{\alpha \to \beta} \left\{ (\alpha - \beta)^p \widehat{p}_{abs}(\alpha, x, y) + \widehat{p}_{abs}(\alpha, 0, y) \left[(\alpha - \beta)^p H(\alpha, x) \right] \right\} = \widehat{p}_{abs}(\beta, 0, y) \bar{H}(\beta, x) \tag{61}$$

where $\lim_{\alpha \to \beta}(\alpha - \beta)^p H(\alpha, x) = \bar{H}(\beta, x) \neq 0$. Since $\widehat{p}_{abs}(\beta, 0, y) \neq 0$, β is a pole.

(ii) $\sigma_{abs} \cap \sigma_H \subseteq \sigma$. Let $\beta = -\lambda_k^2/2 \in \sigma_{abs} \cap \sigma_H$. We know that β is not a pole of $\widehat{p}_{abs}(\alpha, 0, y)$. If $p > 1$, since p_{abs} has only simple poles, the limit (61) is of the same type as in case (i). When $p = 1$, the limit (61) is

$$2(b-a)^{-1} \sin \lambda_k(x - a) \sin \lambda_k(y - a) + \widehat{p}_{abs}(-\lambda_k^2/2, 0, y) \bar{H}(-\lambda_k^2/2, x)$$

where the last factor is nonzero. As a function of y in $L^2[a, b]$ the limit is not identically zero, so β is a pole.

(iii) $\sigma_{abs} \setminus \sigma_H \subseteq \sigma$. Let $\beta = -\lambda_k^2/2 \in \sigma_{abs} \setminus \sigma_H$. The limit (61) with $p = 1$, based on (57), gives

$$2(b-a)^{-1} \sin \lambda_k(y - a) \Big[\sin \lambda_k(x - a) - \sin \lambda_k a \, H(-\lambda_k^2/2, x) \Big]. \tag{62}$$

In this case, we have

$$H(-\lambda_k^2/2,x) = \lim_{\alpha\to\beta} H(\alpha,x) = \frac{\cos\lambda_k(x-a)\cos\lambda_k(\frac{b-a}{2}) + \sin\lambda_k(x-a)\sin\lambda_k(\frac{b-a}{2})}{\cos\lambda_k(\frac{b-a}{2})(1-\cos\lambda_k a) + \sin\lambda_k(\frac{b-a}{2})\sin\lambda_k a}\,. \tag{63}$$

The bracket in (62) is equal to $(\sin\frac{\lambda_k a}{2})^{-1}\cos\lambda_k(x-\frac{a}{2}) \neq 0$ for k even and vanishes for k odd, which implies that $\sigma_{abs}^{even} \subset \sigma$ and $\sigma_{abs}^{odd} \cap \sigma = \emptyset$.

Part 3. We notice that $\sup\{\,\beta \in \sigma_H \setminus \{0\}\,\} < 4\gamma$, the second eigenvalue of p_{abs}, and 4γ corresponds to $k = 2$, the first even value in the spectrum. □

7 Exponential ergodicity via Doeblin's condition

Let $\mu(dx)$ be a probability measure concentrated on $\mathcal{R}$. We shall consider a generalization of the process $z_x(t,\omega)$ from Proposition 2, that upon reaching the boundary $\partial\mathcal{R}$ jumps to an interior random point $x_0 \in \mathcal{R}$ chosen with probability distribution $\mu(dx_0)$. The original process, *Brownian motion with rebirth*, corresponds to $\mu(dx_0) = \delta_0(dx_0)$. In this case, equations (20), (21), (23) are satisfied by replacing $p_{abs}(t,0,y)$ with $p_{abs}(t,\mu,y) = \langle p_{abs}(t,\cdot,y),\mu\rangle$, and $h^0(t)$ with $h^\mu = \langle h^{(\cdot)},\mu\rangle$, where the angle brackets denote the integration over $\mathcal{R}$. In the following, we shall give a proof of the exponential ergodicity based on Doeblin's theorem. The Lebesgue measure will be denoted by $\lambda(dx)$.

Proposition 10. *There exists a set $A \subseteq \mathcal{R}$ with $\lambda(A) > 0$, a time $T > 0$ and a constant $c > 0$ such that for all $x \in \mathcal{R}$ we have*

$$p(2T,x,y) \geq c \qquad \forall x \in \mathcal{R}\,, \forall y \in A\,, \tag{64}$$

where $p(t,x,y)$, $t \geq 0$ is the probability density of the transition kernel of the process $\{z_x(t,\omega)\}_{t\geq 0}$.

Proof. For sufficiently small $\delta > 0$, let $\mathcal{R}_\delta = \{x \in \mathcal{R} \,|\, d(x,\partial\mathcal{R}) > \delta\}$. Since μ is a probability measure concentrated on $\mathcal{R}$, there exists a $\delta > 0$ such that $\mu(\overline{\mathcal{R}}_\delta) > 0$. Let A be a compact subset of $\mathcal{R}$ (e.g. a closed ball centered at a given point of $\mathcal{R}$). Due to the regularity of the domain, it is possible to choose a sufficiently large $T > 0$ such that $\inf_{x\in\mathcal{R}\setminus\mathcal{R}_\delta} P(T_x(\mathcal{R}) \leq T) \geq 1/2$. Moreover, the density $p_{abs}(t,x_0,y)$ is continuous in all arguments for $t > 0$. Then there exists $c_1 > 0$ such that $p_{abs}(t,x_0,y) \geq c_1$ for all $T \leq t \leq 2T$, $x_0 \in \overline{\mathcal{R}}_\delta$ and $y \in A$. From (20), the density $p(t,x,y) \geq p_{abs}(t,x,y)$, and so

$$p(t,\mu,y) \geq \int_{\mathcal{R}} p_{abs}(t,x_0,y)\mu(dx_0) \geq \int_{\overline{\mathcal{R}}_\delta} p_{abs}(t,x_0,y)\mu(dx_0) \geq c_1\mu(\overline{\mathcal{R}}_\delta) = c_\mu > 0\,, \tag{65}$$

for all $T \leq t \leq 2T$ and $y \in A$.

Case 1. Suppose $x \in \overline{\mathcal{R}}_\delta$. Take $t = 2T$. From (20), for $x \in \overline{\mathcal{R}}_\delta$ and $y \in A$, we have

$$p(t,x,y) \geq p_{abs}(t,x,y) \geq c_1 > 0\,.$$

Case 2. Suppose $x \in \mathcal{R} \setminus \overline{\mathcal{R}}_\delta$. From (23), or directly from (20) we have

$$p(t,x,y) = p_{abs}(t,x,y) + \int_0^t p(t-s,\mu,y)h^x(s)\,ds\,. \tag{66}$$

Let $x \in \mathcal{R} \setminus \overline{\mathcal{R}}_\delta$. Then, using (66) and then (65), we see that for all $y \in A$,

$$p(2T,x,y) \geq \int_0^T p(2T-s,\mu,y)h^x(s)\,ds \geq \frac{c_\mu}{2}\,.$$

We conclude the proof by choosing $c = \min\{c_1, c_\mu/2\}$. ◻

We state the Doeblin condition, that ensures uniform exponential ergodicity. For the statement in discrete time, the reader is referred to [16] or [1], and Theorem 5.3 in [4] settles the case of continuous time processes. The Doeblin condition is much stronger than what is needed in Theorem 5.3, and its assumptions are trivially satisfied.

Theorem 4. *Let X be a locally compact metric space and Z_t, $t \geq 0$ be a continuous time Markov process with state space X. Assume there exists a time $T_0 > 0$, a probability measure $\nu(dx)$ on X and a positive constant $c < 1$ such that for all $x \in X$ and all $B \in \mathcal{B}(X)$ we have $P_x(Z_{T_0} \in B) \geq c\nu(B)$. Then, the process has an invariant measure $\eta(dx)$ and there exist positive constants C and $r < 1$ such that*

$$|||P_x(Z_t \in \cdot) - \eta(\cdot)||| \leq Cr^t\,, \tag{67}$$

where $|||\cdot|||$ denotes the total variation norm on the space of measures.

Theorem 5. *Under the same regularity conditions on $\mathcal{R}$ as in Theorem 1, the Brownian motion with rebirth at a random point with measure μ is uniformly exponentially ergodic, and the invariant probability measure has density given by $\langle G(\cdot,y),\mu\rangle$, modulo a normalizing constant.*

Proof. Let B be a Borel subset of $\mathcal{R}$ and let $\lambda(\cdot|A)$ be the probability measure defined by $\lambda(B|A) = \lambda(B\cap A)/\lambda(A)$ with the set A from Proposition 10. Then, for any $x \in \mathcal{R}$,

$$p(2T,x,B) \geq p(2T,x,B\cap A) \geq c\,\lambda(B\cap A) = c\,\lambda(A)\,\lambda(B|A)\,.$$

Setting $c_0 = \min\{c\,\lambda(A), 1\}$, we have proven that Doeblin's condition from Theorem 4 is satisfied for the Markov process $p(t,x,y)$ with $T_0 = 2T$, $\nu(\cdot) = \lambda(\cdot|A)$ and $c = c_0$. To conclude the proof, the invariant measure exists and is unique from Doeblin's condition and is easily identified in the same way as in the proof of Theorem 1. ◻

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