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On the Exponentials of Fractional Ornstein-Uhlenbeck Processes

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Abstract

We study the correlation decay and the expected maximal increment (Burkholder-Davis-Gundy type inequalities) of the exponential process determined by a fractional Ornstein-Uhlenbeck process. The method is to apply integration by parts formula on integral representations of fractional Ornstein-Uhlenbeck processes, and also to use Slepian's inequality. As an application, we attempt Kahane's T-martingale theory based on our exponential process which is shown to be of long memory.

Key words: Long memory (Long range dependence), Fractional Brownian motion, Fractional Ornstein-Uhlenbeck process, Exponential process, Burkholder-Davis-Gundy inequalities.

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1 Introduction

We begin with a review on the definition and properties of fractional Brownian motion (*FBM* for short).

Definition 1.1. *Let $0 < H < 1$. A fractional Brownian motion $\{B_t^H\}_{t\in\mathbb{R}}$ is a centered Gaussian process with $B_0^H = 0$ and* $\mathrm{Cov}\left(B_s^H, B_t^H\right) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$, $(t,s) \in \mathbb{R}^2$.

It is well known that *FBM* has stationary increments and self-similarity with index H, i.e., for any $c > 0$ $\{B_{ct}^H\}_{t\in\mathbb{R}} \stackrel{d}{=} \{c^H B_t^H\}_{t\in\mathbb{R}}$ where $\stackrel{d}{=}$ denotes equality in all finite dimensional distributions.
Another important property is long memory. Although there exist several notions, we use the following definition which is widely known.

Definition 1.2. *Let $(X_1, X_2, \ldots)$ be a zero mean stationary stochastic sequence with a finite variance. Write $\Gamma(n) = \mathrm{Cov}(X_1, X_n)$. We say that the process has long memory if $\sum_{n=0}^{\infty} \Gamma(n) = \infty$.*

Note that $E[(X_1)^2] < \infty$ implies that the correlation and the covariance are different only in multiplication of some constant and we use the latter here. The definition is related to the classical invariance principle, i.e., if $\sum_{n=0}^{\infty} |\Gamma(n)| < \infty$, the properly normalized sum of $(X_1, X_2, \ldots)$ converges weakly to Brownian motion (*BM* for short). By contrast, if the process has a long memory, this may not hold (see p.191 of [18] or p.336 of [19]). For other definitions and their relations one can consult [18].
FBM with $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ has the following incremental property. For any $0 < h < s$, $t \in \mathbb{R}$ and $N = 1, 2, \ldots$,

$$\begin{aligned}
\Gamma_h(s) &:= \mathrm{Cov}\left(B_{t+h}^H - B_t^H, B_{t+s+h}^H - B_{t+s}^H\right) = \mathrm{Cov}\left(B_h^H, B_{s+h}^H - B_s^H\right) \\
&= \sum_{n=1}^{\infty} \frac{h^{2n}}{(2n)!} \left(\prod_{k=0}^{2n-1} (2H-k)\right) s^{2H-2n} \\
&= \sum_{n=1}^{N} \frac{h^{2n}}{(2n)!} \left(\prod_{k=0}^{2n-1} (2H-k)\right) s^{2H-2n} + O\left(s^{2H-2N-2}\right), \text{ as } s \to \infty.
\end{aligned}$$

Thus $\sum_{n=0}^{\infty} \Gamma_h(nh) = \infty$ and the incremental process of *FBM* with $H \in (\frac{1}{2}, 1)$ is proved to have a long memory property. Concerning the maximal inequality of *FBM* there have been several results; see, for example, Chapter 4.4 of [5]. In particular [13] proved the maximal inequality for *FBM* with $H \in (\frac{1}{2}, 1)$ which corresponds to the Burkholder-Davis-Gundy inequality for martingale: For $p > 0$ there exist constants $c(p,H)$ and $C(p,H)$ such that

$$c(p,H) E[\tau^{pH}] \le E\left[\max_{s \le \tau} \left|B_s^H\right|^p\right] \le C(p,H) E[\tau^{pH}],$$

where τ is a stopping time. Extending this, [11] have obtained inequalities for the moment of integrals with respect to *FBM*. Regarding other properties, we refer to [5] or [19] which give conclusive introduction to *FBM*. We also refer to recently published [12] which gives nice summary on stochastic calculus for *FBM*.

Now we turn to Ornstein-Uhlenbeck processes driven by *FBM* with $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ (*FOU* for short), which is defined by

$$Y_t^{H,\xi} = e^{-\lambda t}\left(\xi + \sigma \int_0^t e^{\lambda u} dB_u^H\right), \tag{1}$$

where $\lambda > 0, \sigma > 0$ and ξ is $a.s.$ finite random variable. This process appears firstly in [4]. They show that *FOU* $\{Y_t^{H,\xi}\}_{t\geq 0}$ is the unique $a.s.$ continuous-path process which solves

$$Z_t = \xi - \lambda \int_0^t Z_s ds + \sigma B_t^H, \quad t \geq 0,$$

and is strictly stationary if

$$\xi = \sigma \int_{-\infty}^0 e^{\lambda u} dB_u^H.$$

We mainly study this stationary version, as follows

$$Y_t^H := \sigma \int_{-\infty}^t e^{-\lambda(t-u)} dB_u^H,$$

where the random function $t \to Y_t^H$ now can be and will be extended to the whole $t \in \mathbb{R}$. Let

$$c_H := \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi}$$

a constant. The correlation decay of $\{Y_t^H\}_{t\in\mathbb{R}}$ with $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ as $s \to \infty$ satisfies, see p.289 of [15],

$$\begin{aligned}
\mathrm{Cov}(Y_t^H, Y_{t+s}^H) &= \mathrm{Cov}(Y_0^H, Y_s^H) \\
&= 2c_H\sigma^2 \int_0^\infty \cos(sx)\frac{x^{1-2H}}{\lambda + x^2} dx & (2)\\
&= \frac{1}{2}\sigma^2 \sum_{n=1}^N \lambda^{-2n} \left(\prod_{k=0}^{2n-1}(2H-k)\right) s^{2H-2n} + O(s^{2H-2N-2}),\ N = 1, 2, \ldots, & (3)
\end{aligned}$$

so that the decay is similar to that of $\mathrm{Cov}\left(B_{t+h}^H - B_t^H, B_{t+s+h}^H - B_{t+s}^H\right)$ as $s \to \infty$. In particular, for $H \in (\frac{1}{2}, 1)$ it exhibits long range dependence, which contrasts with the exponential decay of Ornstein-Uhlenbeck process driven by *BM*. Regarding distribution of the maximum of *FOU*, [17] more generally obtained estimates of the tail of the maximum of stochastic integrals with respect to *FBM*, of which we shall make use. Other interesting results are given in Chapters 1.9 and 1.10 of [12].

In recent years, it has been of great interest to study the exponential functionals and the exponential processes determined by *BM* and Lévy processes, see [2] and [3], with the view toward application in financial economics. In this paper, we study the exponential process determined by $\{Y_t^H\}_{t\in\mathbb{R}}$,

$$X_t^H := e^{Y_t^H}. \tag{4}$$

We shall call the process to be a *geometric fractional Ornstein-Uhlenbeck processes* (*gFOU*, for short). We study two fundamentally important properties of *gFOU*s. The correlation decay $\mathrm{Cov}\left(X_t^H, X_{t+s}^H\right)$ as $s \to \infty$, and the expected maximal increments

$$E\left(\max_{t\le s\le t+r}\left|X_s^H - X_t^H\right|\right) \text{ as } r \downarrow 0.$$

The first result is useful to understand the spectral structure of the process. The second result is of intrinsic importance to the path variation (and hence toward various applications) of the process. In case $\{Y_t^H\}$ with $H = 1/2$ (*BM* case), some weaker form of the results appears very recently in a paper by Anh, Leonenko and Shieh (2007), whose methods are based on Hermite orthogonal expansion and the Itô's calculus for martingales. However, both tools are lack for *FBM* case, since *FBM* is not a semimartingale. Thus we need to use other devices, which are mainly precise calculations based on the Gaussian properties, the integral representations of *FOU*s, and the Slepian's inequality. We remark that the main results Theorems 2.1 and 2.2 in this paper are new *even* in the *BM* case, to our knowledge.

This paper is organized as follows. In Section 2 we state the main results. In Section 3 we treat Kahane's T-martingale theory as an application. We present all proofs of our results in Section 4.

2 The main results

From now on we treat *gFOU* and *FOU* with $\lambda = \sigma = 1$ for convenience. Moreover, we consider these process on $\mathbb{R}$. The notation $\stackrel{d}{=}$ denotes equality in distributional sense, for processes also for random variables or vectors. All proofs of our results are given in the final section.

As a preliminary step we confirm the following basic result.

Lemma 2.1. *Let $H \in (0,1)$ and $\{X_t^H\}_{t\in\mathbb{R}} := \{e^{Y_t^H}\}_{t\in\mathbb{R}}$ be gFOU constructed by the stationary process $\{Y_t^H\}_{t\in\mathbb{R}}$. Then $\{X_t^H\}_{t\in\mathbb{R}}$ is also stationary.*

Our first study is about correlation decay of *gFOU*, which holds for all *gFOUs* with full range $H \in (0,\frac{1}{2}) \cup (\frac{1}{2},1)$. Note that, although the following Propositions 2.1 and 2.2 are stated for covariances, results for correlations are routinely obtained when covariances are divided by $\mathrm{Var}(X_0^H) = \exp\left(2\mathrm{Var}(Y_0^H)\right)$ and $\mathrm{Var}((Y_0^H)^m)$ respectively.

Proposition 2.1. *Let $H \in (0,\frac{1}{2})\cup(\frac{1}{2},1)$ and $\{X_t^H\}_{t\in\mathbb{R}} := \{e^{Y_t^H}\}_{t\in\mathbb{R}}$ be gFOU constructed by the stationary process $\{Y_t^H\}_{t\in\mathbb{R}}$. Then for fixed $t \in \mathbb{R}$ and $s \to \infty$,*

$$\begin{aligned}&\mathrm{Cov}\left(X_t^H, X_{t+s}^H\right)\\ &= \exp\left[\mathrm{Var}(Y_0^H)\right]\left[H(2H-1)s^{2H-2} + \frac{1}{2}H^2(2H-1)^2 s^{4H-4} + O\left(\max\{s^{6H-6}, s^{2H-4}\}\right)\right].\end{aligned}$$

Remark 2.1. *Proposition 2.1 asserts that the gFOU with $H \in (\frac{1}{2},1)$, like the FOU case as shown below in Proposition 2.2 (with $m = 1$ there), exhibits the long range dependence; this contrasts to both the O-U and the geometric O-U processes driven by BM.*

We also analyze correlation decay of m-power of *FOU*, the process $\{(Y_t^H)^m\}_{t\in\mathbb{R}}$.

Proposition 2.2. *Let $\{Y_t^H\}_{t\in\mathbb{R}}$ be FOU with $H \in (0,\frac{1}{2})\cup(\frac{1}{2},1)$ and $m = 1,2,\ldots$. Then for fixed $t \in \mathbb{R}$ and $s \to \infty$,*

$$
\begin{aligned}
&\operatorname{Cov}\left((Y_t^H)^m,(Y_{t+s}^H)^m\right)\\
&=\begin{cases}
m^2\left((m-2)!!\right)^2\left(\operatorname{Var}(Y_0^H)\right)^{m-1}\operatorname{Cov}\left(Y_0^H,Y_s^H\right)+O\left(\left(\operatorname{Cov}\left(Y_0^H,Y_s^H\right)\right)^2\right) & \text{if } m \text{ is odd}\\
\frac{1}{2}\left(\frac{m!(m-3)!!}{(m-2)!}\right)^2\left(\operatorname{Var}(Y_0^H)\right)^{m-2}\left(\operatorname{Cov}\left(Y_0^H,Y_s^H\right)\right)^2+O\left(\left(\operatorname{Cov}\left(Y_0^H,Y_s^H\right)\right)^4\right) & \text{if } m \text{ is even}
\end{cases}\\
&=\begin{cases}
m^2\left((m-2)!!\right)^2\left(\operatorname{Var}(Y_0^H)\right)^{m-1}H(2H-1)s^{2H-2}+O(s^{4H-4}) & \text{if } m \text{ is odd}\\
\frac{1}{2}\left(\frac{m!(m-3)!!}{(m-2)!}\right)^2\left(\operatorname{Var}(Y_0^H)\right)^{m-2}H^2(2H-1)^2s^{4H-4}+O(s^{4H-6})+O(s^{8H-8}) & \text{if } m \text{ is even.}
\end{cases}
\end{aligned}
$$

Remark 2.2. *The correlation decay of $\{(Y_t^H)^m\}_{t\in\mathbb{R}}$ for all odd m is the same as that of $\{Y_t^H\}_{t\in\mathbb{R}}$, which exhibits the long range dependence for all $H \in (\frac{1}{2},1)$. While for even m, $\{(Y_t^H)^m\}$ with $H \in (\frac{1}{2},\frac{3}{4})$ no longer has the long memory property. We also observe that the situation depends entirely on m being even or being odd, rather than the actual value of m.*

Now we turn to the more difficult second part, namely to study expected maximal increment of *gFOU*. For the upper bound inequality, we only consider those *gFOU*s with $H \in (\frac{1}{2},1)$, and we are not able to obtain the case of $H < 1/2$, yet this latter case is of less interest in view that the process if not of long range dependence then. Before analyzing, we present three lemmas which we think themselves to be interesting in future researches. Indeed Lemmas 2.3 and 2.4 deal with maximal inequalities for *FOU*s. For the consistency, in all the following statements, we always include the $H = 1/2$ case.

The first result (Lemma 2.3) is based on Statement 4.8 of [17]. It will be useful to give a clean statement of this since the definition of FBM is different from ours and there are minor mistakes in [17] (e.g., regarding his $q_f(s,t)$ a constant is lacking, he referred to Theorem 4.1 in Statement 4.2 but we can not find Theorem 4.1 in his paper).

Lemma 2.2 (Statement 4.8 (2) of [17])**.** *Let $H \in (\frac{1}{2},1)$ and $0 < \beta < 2H-1$. Write*

$$q_f(s,t) = H(2H-1)\int_s^t\int_s^t f(u)f(v)|u-v|^{2H-2}dudv.$$

If $f \in L^{2/(1+\beta)}([0,1])$ then for every real $r \in [0,1]$

$$P\left(\max_{0\le t\le 1}\int_0^t f(s)dB_s^H > \lambda\right) \le \int_{\lambda r/\sqrt{q_{f+}(0,1)}}^{\infty} + \int_{\lambda(1-r)/\sqrt{q_{f-}(0,1)}}^{\infty}\sqrt{\frac{2}{\pi}}e^{-x^2/2}dx, \tag{5}$$

where $f_\pm = \frac{|f|\pm f}{2}$.

Lemma 2.3. *Let $H \in [\frac{1}{2},1)$. Then for any $\lambda \ge 0, r \ge 0$ and $t \in \mathbb{R}$ we have*

$$P\left(\max_{t\le s\le t+1} B_s^H \ge \lambda\right) \le \sqrt{\frac{2}{\pi}}\int_\lambda^\infty e^{-x^2/2}dx.$$

$$P\left(\max_{t\le s\le t+1}\left|B_s^H\right|\ge\lambda\right)\le\frac{2\sqrt{2}}{\sqrt{\pi}}\int_\lambda^\infty e^{-x^2/2}dx. \tag{6}$$

$$E\left[\left(\max_{t\le s\le t+r}\left|B_s^H\right|\right)^m\right]\le\begin{cases} r^{Hm}\frac{2\sqrt{2}}{\sqrt{\pi}}(m-1)!! & \text{if } m \text{ is odd}\\ r^{Hm}2(m-1)!! & \text{if } m \text{ is even.}\end{cases}$$

Lemma 2.4. *Let $H\in[\frac{1}{2},1)$ and $m=1,2,\ldots$. Then FOU $\{Y_t^H\}_{t\in\mathbb{R}}$ has following bound for m-th moment of maximal increments for all $r\ge0$ and all $t\in\mathbb{R}$.*

$$\frac{E\left[\max_{t\le s\le t+r}\left|Y_s^H-Y_t^H\right|^m\right]}{m!}\le c_1r^{Hm}, \tag{7}$$

where c_1 is an universal constant, which does not depend on any m or H.

Lemma 2.5. *Let $H\in(0,1)$ and $p>0$. Then FOU $\{Y_t^H\}_{t\in\mathbb{R}}$ has the following lower bound for p-th moment of maximal increments for all $0\le r\le T$ and all $t\in\mathbb{R}$.*

$$E\left[\max_{t\le s\le t+r}\left|Y_s^H-Y_t^H\right|^p\right]\ge c_2(p,T,H)r^{pH}, \tag{8}$$

where $c_2(p,T,H)$ is a constant depending on parameters p,T and H.

Now we state our main results. The upper bound inequality is given as follows.

Theorem 2.1. *Let $H\in[\frac{1}{2},1)$ and $\{X_t^H\}_{t\in\mathbb{R}}:=\{e^{Y_t^H}\}_{t\in\mathbb{R}}$ be gFOU constructed by the stationary process $\{Y_t^H\}_{t\in\mathbb{R}}$. Then there exists a constant $\overline{C}(H)$ such that for all r with $r^H<1/2$ and all $t\in\mathbb{R}$,*

$$E\left[\max_{t\le s\le t+r}\left|X_s^H-X_t^H\right|\right]\le\overline{C}(H)r^H.$$

The lower bound inequality is given as follows.

Theorem 2.2. *Let $H\in(0,1)$ and $\{X_t^H\}_{r\in\mathbb{R}}:=\{e^{Y_t^H}\}_{t\in\mathbb{R}}$ be gFOU constructed by the stationary process $\{Y_t^H\}_{t\in\mathbb{R}}$. Then there exists a constant $\underline{c}(T,H)$ such that for all $t\in\mathbb{R},0\le r\le T$,*

$$E\left[\max_{t\le s\le t+r}\left|X_s^H-X_t^H\right|\right]\ge\underline{c}(T,H)r^H.$$

Remark 2.3. *In the above, we always include $H=1/2$ (the BM case); we should keep in mind that for this critical case the proofs can be done separately by stochastic analysis of martingales (see, for example, [7]). We would like to show that the methods in this paper are powerful enough to obtain Theorems 2.1 and 2.2.*

3 An application to Kahane's T-martingale Theory

J.-P. Kahane established T-martingale Theory as a mathematical formulation of Mandelbrot's turbulence cascades; see [8] and [14] for inspiring surveys. To our knowledge, the theory is only applied to independent or Markovian cascades. Using the results in Section 2 we are able to give an *dependent* attempt to this theory. To describe our result, let $\overline{X}$ be a normalized *gFOU* which is defined to be, for a given fixed $H \in (\frac{1}{2}, 1)$ and its corresponding stationary *FOU* Y_t^H,

$$\overline{X}(t) := e^{Y_t^H - c^H}, \quad t \in \mathbb{R},$$

where c^H is chosen so that the resulting positive-valued stationary process is of mean 1. Note that the process $\overline{X}$ is non-Markovian (indeed, it is of long range dependence). Now let a sequence of independent *gFOU* $\overline{X}_n$, defined on a common probability space (Ω, P); each process $\{\overline{X}_n(t)\}_{t\in\mathbb{R}}$ is of continuous paths and is distributed as $\{\overline{X}(b^n \cdot t)\}_{t\in\mathbb{R}}$, $n = 0, 1, 2, \ldots$, where the scaling factor

$$b > 1 + \mathrm{Var}(\overline{X}(0)) = 1 + \frac{\exp\left(2\mathrm{Var}(Y_0^H)\right)}{\exp\left(2c^H\right)}.$$

We consider the integrated process of the $n+1$ products,

$$A_n(t, \omega) := \int_0^t \prod_{i=0}^n \overline{X}_i(s, \omega) ds, \quad t \geq 0, \omega \in \Omega.$$

We note that, for each t, $A_n(t, \omega)$ is well-defined as an integral for path-wise ω, since the integrand is a positive-valued continuous function in s for path-wise ω. The following two facts are basic to the theory:

1. for each t fixed, the sequence $A_n(t)$ form a martingale in n.
2. for each n fixed, $t \to A_n(t)$ is continuous and increasing.

We state our T-martingale result for the *gFOU* process as follows. In the statement, we restrict the time parameter for the target process $A(t)$ to be $A(t), t \in [0, 1]$; though it can be defined for any compact time-interval $[0, T]$.

Proposition 3.1. *For each $t \in [0, 1]$, the random sequence $A_n(t)$ converges in $L^2(dP)$. Thus a limiting process $A(t), t \in [0, 1]$, is defined. The process $t \to A(t)$ is continuously increasing. Moreover, there exist $\overline{C}, \underline{C}$ such that the following mutual bounds hold for all $q \in [1, 2]$ and all $t \in [0, 1]$,*

$$\underline{C} \cdot t^{q - \log_2 E(\overline{X}(0))^q} \leq E(A(t))^q \leq \overline{C} \cdot t^{q - \log_2 E(\overline{X}(0))^q}.$$

Remark 3.1. *The close form of $E(\overline{X}(0))^q$ can be written out, since the random variable Y_t^H is Gaussian distributed; it is non-linear in q, which is the heart of the matter.*

Remark 3.2. *To our knowledge, in all the previous literatures on Kahane's theory, the initial process (in our case, $\overline{X}$) is assumed to be independent (for discrete cascades) or to be Markovian (for general cascades). The above result can be regarded to be a first attempt to apply the dependent process $\overline{X}$ to Kahane's theory, which theory aims to proceed some multi-scale analysis (usually termed as "multifractal analysis") on the atomless random measure induced by the continuously increasing process A.*

Remark 3.3. *We mention that, in [10] the authors adapt Kahane's formulation to stationary processes, and impose various conditions to enforce the validity of their re-formulation (in their eventual examples, one is a two-state Markov process and one is a Poisson process with random magnitudes). It has been a recent study to examine the validity of their re-formulation for several stationary exponential processes, see [1] and the references therein.*

4 The proofs

Proof of Lemma 2.1
It follows that for any real set $(t_1, t_2, \ldots, t_n)$ and all $h \in \mathbb{R}$,

$$(Y^H_{t_1}, \ldots, Y^H_{t_n}) \stackrel{d}{=} (Y^H_{t_1+h}, \ldots, Y^H_{t_n+h}).$$

Accordingly our assertion is implied by

$$\begin{aligned}(X^H_{t_1}, \ldots, X^H_{t_n}) &= (e^{Y^H_{t_1}}, \ldots, e^{Y^H_{t_n}}) \\ &\stackrel{d}{=} (e^{Y^H_{t_1+h}}, \ldots, e^{Y^H_{t_n+h}}) \\ &= (X^H_{t_1+h}, \ldots, X^H_{t_n+h}).\end{aligned}$$

□

Note that in Proofs of Proposition 2.1 and 2.1, we make full use of the stationarity both of $\{X_t\}_{t\in\mathbb{R}}$ and of $\{Y_t\}_{t\in\mathbb{R}}$, we namely use $\mathrm{Var}(Y^H_t) = \mathrm{Var}(Y^t_0)$ or $\mathrm{Cov}(Y^H_t, Y^H_{t+s}) = \mathrm{Cov}(Y^H_0, Y^H_s),\ t \in \mathbb{R}, s \geq 0$ without mentioning it.

Proof of Proposition 2.1
Since $\{Y^H_t\}_{t\in\mathbb{R}}$ is a Gaussian process, the distribution of $(Y^H_t, Y^H_{t+s}), t \in \mathbb{R}$ is a bivariate Gaussian. Then its moment generating function taken at 1 and the stationarity of $\{X_t\}_{t\in\mathbb{R}}$ yield

$$E[X^H_t] = E[X^H_0] = \exp\left\{\frac{1}{2}\mathrm{Var}\left(Y^H_0\right)\right\}$$

and

$$\begin{aligned}E[X^H_t X^H_{t+s}] &= E[X^H_0 X^H_s] \\ &= E[e^{Y^H_0 + Y^H_s}] \\ &= \exp\left\{\frac{1}{2}(1,1)\Sigma(1,1)'\right\},\end{aligned}$$

where

$$\Sigma = \begin{pmatrix} \mathrm{Var}\left(Y^H_0\right) & \mathrm{Cov}\left(Y^H_0, Y^H_s\right) \\ \mathrm{Cov}\left(Y^H_0, Y^H_s\right) & \mathrm{Var}\left(Y^H_s\right) \end{pmatrix}.$$

Then by using $\mathrm{Cov}\left(Y^H_0, Y^H_s\right) \to 0$ as $s \to \infty$ and $e^x = \sum_{n=0}^{\infty} x^n/n!$ for $|x| < \infty$, we have

$$\begin{aligned}&\mathrm{Cov}\left(X^H_t, X^H_{t+s}\right) \\ &= \exp\{\mathrm{Var}\left(Y^H_0\right)\}\left[\exp\left\{\mathrm{Cov}\left(Y^H_0, Y^H_s\right)\right\} - 1\right] \\ &= \exp\{\mathrm{Var}\left(Y^H_0\right)\}\left[\mathrm{Cov}\left(Y^H_0, Y^H_s\right) + \frac{\left(\mathrm{Cov}\left(Y^H_0, Y^H_s\right)\right)^2}{2!} + \frac{\left(\mathrm{Cov}\left(Y^H_0, Y^H_s\right)\right)^3}{3!} + \cdots\right].\end{aligned}$$

By aid of the equation (3), the equation above can be written as

$$\begin{aligned}
&\mathrm{Cov}\left(X_t^H, X_{t+s}^H\right)\\
&= \exp\{\mathrm{Var}\left(Y_0^H\right)\}\Bigg[H(2H-1)s^{2H-2} + \frac{1}{2}\prod_{k=0}^{3}(2H-k)s^{2H-4} + O(s^{2H-6})\\
&\qquad\qquad + \frac{1}{2}H^2(2H-1)^2 s^{4H-4} + O(s^{4H-6})\\
&\qquad\qquad + \frac{1}{6}H^3(2H-1)^3 s^{6H-6} + O(s^{6H-8})\\
&\qquad\qquad + O(s^{8H-8}) + \cdots\Bigg].
\end{aligned}$$

Hence we obtain the result. □

Proof of Proposition 2.2
As in Proof of Proposition 2.1, (Y_t^H, Y_{t+s}^H) is a bivariate Gaussian distribution and its moment generating function writes

$$\begin{aligned}
&E\left[e^{u_1 Y_t^H + u_2 Y_{t+s}^H}\right]\\
&= \exp\left\{\frac{1}{2}\left(u_1^2 \mathrm{Var}(Y_t^H) + 2u_1u_2 \mathrm{Cov}(Y_t^H, Y_{t+s}^H) + u_2^2 \mathrm{Var}(Y_{t+s}^H)\right)\right\}\\
&= \sum_{n=0}^{\infty} \frac{2^{-n}}{n!} \sum_{k,l\geq 0; k+l\leq n} \frac{n!}{k!\; l!\; (n-k-l)!}\\
&\qquad \times \left(u_1^2 \mathrm{Var}(Y_0^H)\right)^k \left(2u_1u_2 \mathrm{Cov}(Y_0^H, Y_s^H)\right)^l \left(u_2^2 \mathrm{Var}(Y_0^H)\right)^{n-l-k}\\
&= \sum_{n=0}^{\infty} \frac{2^{-n}}{n!} \sum_{k,l\geq 0; k+l\leq n} \frac{n!\; 2^l u_1^{2k+l} u_2^{2(n-l-k)+l}}{k!\; l!\; (n-k-l)!} \left(\mathrm{Var}(Y_0^H)\right)^{n-l} \left(\mathrm{Cov}(Y_0^H, Y_s^H)\right)^l. \qquad (9)
\end{aligned}$$

Here we use the expansion of e^x and the multinomial expansion. By using this representation we will calculate $E\left[(Y_t^H)^m (Y_{t+s}^H)^m\right]$. Recall that

$$E\left[(Y_t^H)^m (Y_{t+s}^H)^m\right] = \frac{\partial^{2m}}{(\partial u_1)^m (\partial u_2)^m} E\left[e^{u_1 Y_t^H + u_2 Y_{t+s}^H}\right]\Bigg|_{u_1=0, u_2=0}$$

and hence the remaining term of the sum in (9) is only that of $n = m$,

$$\frac{2^{-m}}{m!} \sum_{k,l\geq 0; k+l\leq m}^{m} \frac{m!\; 2^l u_1^{2k+l} u_2^{2(m-l-k)+l}}{k!\; l!\; (m-k-l)!} \left(\mathrm{Var}(Y_0^H)\right)^{m-l} \left(\mathrm{Cov}(Y_0^H, Y_s^H)\right)^l.$$

Then putting $m = 2k + l = 2(m - l - k) + l$, we have

$$\sum_{l=m-2k; 0\leq k\leq \lfloor m/2\rfloor} \frac{2^{-m+l} u_1^m u_2^m}{l!\, \{((m-l)/2)!\}^2} \left(\mathrm{Var}(Y_0^H)\right)^{m-l} \left(\mathrm{Cov}(Y_0^H, Y_s^H)\right)^l.$$

Thus

$$
\begin{aligned}
E\left[(Y_t^H)^m (Y_{t+s}^H)^m\right] &= \sum_{l=m-2k;0\le k\le\lfloor m/2\rfloor} \frac{2^{-m+l}}{l!}\left(\frac{m!}{((m-l)/2)!}\right)^2 \left(\mathrm{Var}(Y_0^H)\right)^{m-l}\left(\mathrm{Cov}(Y_0^H, Y_s^H)\right)^l \\
&= \sum_{l=m-2k;0\le k\le\lfloor m/2\rfloor} \frac{1}{l!}\left(\frac{m!(m-l-1)!!}{(m-l)!}\right)^2 \left(\mathrm{Var}(Y_0^H)\right)^{m-l}\left(\mathrm{Cov}\left(Y_0^H, Y_s^H\right)\right)^l .
\end{aligned}
$$

Here we use the formula $(2n-1)!! = (2n)!/(2^n n!)$. When m is odd only terms $l = 1, 3, 5, \ldots, m$ remain and it follows form the equation (3) that

$$
\begin{aligned}
E\left[(Y_t^H)^m (Y_{t+s}^H)^m\right] &= \left(\frac{m!(m-2)!!}{(m-1)!}\right)^2 \left(\mathrm{Var}(Y_0^H)\right)^{m-1} \mathrm{Cov}(Y_0^H, Y_s^H) + O\left(\left(\mathrm{Cov}(Y_0^H, Y_s^H)\right)^2\right) \\
&= m^2\left((m-2)!!\right)^2 \left(\mathrm{Var}(Y_0^H)\right)^{m-1} H(2H-1)s^{2H-2} + O(s^{4H-4}).
\end{aligned}
$$

When m is even only terms $l = 2, 4, \ldots, m$ remain and it follows from the equation (3) that

$$
\begin{aligned}
E\left[(Y_t^H)^m (Y_{t+s}^H)^m\right] &= ((m-1)!!)^2\left(\mathrm{Var}(Y_0^H)\right)^m \\
&\quad +\frac{1}{2}\left(\frac{m!(m-3)!!}{(m-2)!}\right)^2 \left(\mathrm{Var}(Y_0^H)\right)^{m-2}\left(\mathrm{Cov}(Y_0^H, Y_s^H)\right)^2 \\
&\quad +O\left(\left(\mathrm{Cov}(Y_0^H, Y_s^H)\right)^4\right) \\
&= ((m-1)!!)^2\left(\mathrm{Var}(Y_0^H)\right)^m \\
&\quad +\frac{1}{2}\left(\frac{m!(m-3)!!}{(m-2)!}\right)^2 \left(\mathrm{Var}(Y_t^H)\right)^{m-2} H^2(2H-1)^2 s^{4H-4} \\
&\quad +O(s^{4H-6}) + O(s^{8H-8}).
\end{aligned}
$$

Finally noticing $E[(Y_t^H)^m] = (m-1)!!\left(\mathrm{Var}(Y_0^H)\right)^{m/2}$, we can obtain the desired result. □

Proof of Lemma 2.3

The first equation follows from Statement 4.2 (2) of [17] with $f = 1$ and $r \uparrow 1$. However, since the definition of *FBM* in [17] is different from our definition, we briefly review the outline, which will help reader's understanding.

According to Lemma 5.7 of [17] there exists a Gaussian Markov process $\{\hat{B}_t^H\}_{t\in[0,1]}$ with independent increment such that $E[\hat{B}_t^H] = 0$ and $\mathrm{Cov}(\hat{B}_s^H, \hat{B}_t^H) = s^{2H}$ whenever $s \le t$. Then from Slepian's inequality (Lemma 5.6 of [17], see also [21] or Theorem 7.4.2 of [9]), we have

$$
P\left(\max_{0\le t\le 1} B_t^H \ge \lambda\right) \le P\left(\max_{0\le t\le 1} \hat{B}_t^H \ge \lambda\right).
$$

Finally by the reflection principle for Gaussian Markov processes (e.g. [16]), it follows that

$$
P\left(\max_{0\le t\le 1} B_t^H \ge \lambda\right) \le 2P\left(\hat{B}_t^H \ge \lambda\right).
$$

Second the symmetric property of *FBM* gives

$$
\begin{aligned}
P\left(\max_{0\le t\le 1} B_t^H \ge \lambda\right) &= P\left(-\max_{0\le t\le 1} B_t^H \le -\lambda\right)\\
&= P\left(\min_{0\le t\le 1} -B_t^H \le -\lambda\right)\\
&= P\left(\min_{0\le t\le 1} B_t^H \le -\lambda\right).
\end{aligned}
$$

From this we have

$$
\begin{aligned}
P\left(\max_{0\le t\le 1} \left|B_t^H\right| \ge \lambda\right) &\le P\left(\max_{0\le t\le 1} B_t^H \ge \lambda\right) + P\left(\min_{0\le t\le 1} B_t^H \le -\lambda\right)\\
&= 2P\left(\max_{0\le t\le 1} B_t^H \ge \lambda\right).
\end{aligned}
$$

Thus we get the second assertion. Next the self-similarity of *FBM* and $B_0^H = 0$ $a.s.$ yield

$$
\begin{aligned}
\max_{s\le t\le s+r} \left|B_t^H - B_s^H\right| &\stackrel{d}{=} \max_{0\le t\le r} \left|B_t^H - B_0^H\right|\\
&\stackrel{a.s.}{=} \max_{0\le t\le r} \left|B_t^H\right|\\
&\stackrel{d}{=} r^H \max_{0\le t\le 1} \left|B_t^H\right|.
\end{aligned}
$$

Hence

$$
P\left(\max_{s\le t\le s+r} \left|B_t^H - B_s^H\right| \ge x\right) = P\left(\max_{0\le t\le 1} \left|B_t^H\right| \ge r^{-H}x\right).
$$

Since for positive random variable X, we have

$$
E[X^m] = \int_0^\infty m\cdot y^{m-1} P(X > y)dy,
$$

it follows that

$$
\begin{aligned}
E\left[\left(\max_{s\le t\le s+r} \left|B_t^H\right|\right)^m\right] &\le \frac{2\sqrt{2}}{\sqrt{\pi}} \int_0^\infty m\cdot y^{m-1} \int_{r^{-H}y}^\infty e^{-x^2/2} dxdy\\
&= \frac{2\sqrt{2}}{\sqrt{\pi}} \int_0^\infty e^{-x^2/2} dx \int_0^{r^H x} m\cdot y^{m-1} dy\\
&= r^{Hm} \frac{2\sqrt{2}}{\sqrt{\pi}} \int_0^\infty x^m e^{-x^2/2} dx.
\end{aligned}
$$

Finally, we apply relations 2 and 3 of 3.461 of [6], i.e.,

$$
\begin{aligned}
\int_0^\infty x^{2n} e^{-px^2} dx &= \frac{(2n-1)!!}{2(2p)^n} \sqrt{\frac{\pi}{p}}, \quad p > 0,\ n = 1, 2, \ldots\\
\int_0^\infty x^{2n+1} e^{-px^2} dx &= \frac{n!}{2p^{n+1}}, \quad p > 0,
\end{aligned}
$$

to the integral of the last inequality above and obtain

$$\int_0^\infty x^m e^{-x^2/2} dx = \begin{cases} (m-1)!! & \text{if } m \text{ is odd} \\ \sqrt{\frac{\pi}{2}}(m-1)!! & \text{if } m \text{ is even.} \end{cases}$$

Thus the result holds. □

Proof of Lemma 2.4

By the stationary increment we consider $E[\max_{0\le s\le r}|Y_s^H - Y_0^H|^m]$. The increment of $\{Y_s^H\}_{s\ge 0}$ has the following representation via the integral by parts formula (see Theorem 2.21 of [22]).

$$\begin{aligned} Y_s^H - Y_0^H &= e^{-s}\int_{-\infty}^{s} e^u dB_u^H - \int_{-\infty}^{0} e^u dB_u^H \\ &= e^{-s}\int_0^s e^u dB_u^H + (e^{-s}-1)\int_{-\infty}^0 e^u dB_u^H \\ &\overset{a.s.}{=} e^{-s}\left[e^u B_u^H\right]_0^s - e^{-s}\int_0^s B_u^H e^u du + (e^{-s}-1)\int_{-\infty}^0 e^u dB_u^H \\ &= B_s^H - e^{-s}\int_0^s B_u^H e^u du + (e^{-s}-1)Y_0^H. \end{aligned}$$

Take absolute value of this to obtain

$$\left|Y_s^H - Y_0^H\right| \le \left|B_s^H\right| + \max_{0\le u\le s}\left|B_u^H\right|(1-e^{-s}) + (1-e^{-s})\left|Y_0^H\right|$$

and

$$\left|Y_s^H - Y_0^H\right|^m \le \sum_{i+j+k=m;\ i,j,k\ge 0} \frac{m!}{i!\ j!\ k!}\left|B_s^H\right|^i \left\{\max_{0\le u\le s}\left|B_u^H\right|(1-e^{-s})\right\}^j \left|Y_0^H\right|^k (1-e^{-s})^k.$$

Then taking expectation of maximum of this, we have

$$E\left[\max_{0\le s\le r}\left|Y_s^H - Y_0^H\right|^m\right] \le \sum_{i+j+k=m;\ i,j,k\ge 0}\frac{m!}{i!\ j!\ k!}E\left[\max_{0\le s\le r}\left|B_s^H\right|^{i+j}\left|Y_0^H\right|^k\right](1-e^{-s})^{j+k}. \tag{10}$$

On behalf of Lemma 2.3, the expectation in each term of the sum is bounded as

$$\begin{aligned} E\left[\max_{0\le s\le r}\left|B_s^H\right|^{i+j}\left|Y_0^H\right|^k\right] &\le \sqrt{E\left[\max_{0\le s\le r}\left|B_s^H\right|^{2(i+j)}\right]}\sqrt{E\left[\left|Y_0^H\right|^{2k}\right]} \\ &\le \sqrt{2\,(2(i+i)-1)!!r^{2H(i+j)}}\sqrt{(2k-1)!!M_1^{2k}} \\ &\le \sqrt{2^{i+j}(i+j)!r^{2H(i+j)}}\sqrt{2^k k!M_1^{2k}} \\ &= \sqrt{(i+j)!k!}(\sqrt{2}r^H)^{(i+j)}(\sqrt{2}M_1)^k. \end{aligned}$$

Here M_1 only depends on the variance of Gaussian random variable of Y_0^H, namely only depends on parameter H. However since $H \in [\frac{1}{2}, 1)$, we can make M_1 to attain a certain bound regardless of

H (for example, we may take $M_1 = \sqrt{Var(Y_0^{1/2})+2}$. Then substituting this relation into (10) and dividing by $m!$, we have

$$
\begin{aligned}
&\frac{E\left[\max_{0\le s\le r}\left|Y_s^H - Y_0^H\right|^m\right]}{m!}\\
&\le \frac{1}{\sqrt{m!}}\sum_{i+j+k=m;\ i,j,k\ge 0}\frac{m!}{i!\ j!\ k!}\sqrt{\frac{(i+j)!k!}{m!}}(\sqrt{2}r^H)^i(\sqrt{2}r^H(1-e^{-s}))^j(\sqrt{2}M_1(1-e^{-s}))^k\\
&\le \frac{1}{\sqrt{m!}}\left\{\sqrt{2}r^H+\sqrt{2}r^H(1-e^{-s})+\sqrt{2}M_1(1-e^{-s})\right\}^m\\
&= r^{Hm}\frac{\left\{\sqrt{2}+\sqrt{2}(1-e^{-s})+\sqrt{2}M_1\frac{1-e^{-s}}{r^H}\right\}^m}{\sqrt{m!}}.
\end{aligned}
$$

Note that $(1-e^{-s})/r^H$ with $0\le s\le r$ and $H\in[\frac{1}{2},1)$, is uniformly bounded in $r\ge 0$. In addition $c^m/\sqrt{m!}$ with $c>0$ is also uniformly bounded in m. Hence we can take a universal constant $c_1>0$ and obtain

$$
\frac{E\left[\max_{0\le s\le r}\left|Y_s^H - Y_0^H\right|^m\right]}{m!}\le c_1 r^{Hm}.
$$

□

Proof of Lemma 2.5
We have easily

$$
E\left[\max_{t\le s\le t+r}\left|Y_s^H - Y_t^H\right|^p\right] = E\left[\max_{0\le s\le r}\left|Y_s^H - Y_0^H\right|^p\right]\ge E\left[\left|Y_r^H - Y_0^H\right|^p\right].
$$

Since $\{Y_t^H\}_{t\in\mathbb{R}}$ is a Gaussian process, a random vector (Y_0^H, Y_r^H) is a bivariate Gaussian distribution and its linear combination $(Y_r^H - Y_0^H)$ is also univariate Gaussian. We denote the square root of variance as

$$
\begin{aligned}
\hat{\sigma} &:= \sqrt{\mathrm{Var}(Y_r^H - Y_0^H)}\\
&= \sqrt{2\left(\mathrm{Var}(Y_0^H)-\mathrm{Cov}(Y_0^H, Y_r^H)\right)},
\end{aligned}
$$

and then

$$
\begin{aligned}
E\left[\left|Y_r^H - Y_0^H\right|^p\right] &= 2\int_0^\infty x^p\frac{1}{\sqrt{2\pi}\hat{\sigma}}\exp\left(-\frac{x^2}{2\hat{\sigma}^2}\right)dx\\
&= \frac{\hat{\sigma}^p}{\sqrt{\pi}}2^{p/2}\Gamma((p+1)/2). \qquad (11)
\end{aligned}
$$

Here we use the equality

$$
\int_0^\infty x^{\nu-1}e^{-\mu x^p}dx = \frac{1}{p}\mu^{-\nu/p}\Gamma(\nu/p), \quad \mathrm{Re}\ \mu>0, \mathrm{Re}\ \nu>0, p>0
$$

from 3.478 of [6]. An evaluation of $\hat{\sigma}^p$ is derived via the equation (2) with $\sigma=\lambda=1$.

$$\begin{aligned}
\hat{\sigma}^2 &:= 2\left(\mathrm{Var}(Y_0^H)-\mathrm{Cov}(Y_0^H,Y_r^H)\right)\\
&= 4c_H\left(\int_0^\infty \frac{x^{1-2H}}{1+x^2}dx-\int_0^\infty \cos(rx)\frac{x^{1-2H}}{1+x^2}dx\right)\\
&= 4c_H\left(\int_0^\infty (1-\cos(rx))\frac{x^{1-2H}}{1+x^2}dx\right)\\
&= 8c_H\int_0^\infty \sin^2(rx/2)\frac{x^{1-2H}}{1+x^2}dx\\
&\quad (\text{Change of variables}:\ \ y=rx)\\
&= 8c_H r^{2H}\int_0^\infty \sin^2(y/2)\frac{y^{1-2H}}{r^2+y^2}dy\\
&\geq 8c_H r^{2H}\int_0^\infty \sin^2(y/2)\frac{y^{1-2H}}{T^2+y^2}dy\\
&:= 8c_Hc_T r^{2H}.
\end{aligned}$$

Accordingly it follows that

$$\hat{\sigma}^p \geq (8c_Hc_T)^{p/2}r^{pH}.$$

Substituting this into (11), we can obtain

$$\begin{aligned}
E\left[\left|Y_r^H-Y_0^H\right|^p\right] &\geq \frac{2^{p/2}}{\sqrt{\pi}}\Gamma((p+1)/2)(8c_Hc_T)^{p/2}r^{pH}\\
&:= \underline{\mathrm{c}}(p,T,H)r^{pH}.
\end{aligned}$$

Hence Proof is over. $\square$

Proof of Theorem 2.1
The stationarity of $\{Y_t^H\}_{t\in\mathbb{R}}$ gives

$$\begin{aligned}
\max_{t\leq s\leq t+r}\left|X_s^H-X_t^H\right| &= \max_{t\leq s\leq t+r}\left|e^{Y_s^H}-e^{Y_t^H}\right|\\
&\overset{d}{=} \max_{0\leq s\leq r}\left|e^{Y_s^H}-e^{Y_0^H}\right|\\
&= e^{Y_0^H}\max_{0\leq s\leq r}\left|e^{Y_s^H-Y_0^H}-1\right|.
\end{aligned}$$

Taking expectation of both sides, we have

$$\begin{aligned}
E\left[\max_{t\leq s\leq t+r}\left|X_s^H-X_t^H\right|\right] &\leq \left(E\left[e^{2Y_0^H}\right]\right)^{\frac{1}{2}}\left(E\left[\left(\max_{0\leq s\leq r}\left|e^{Y_s^H-Y_0^H}-1\right|\right)^2\right]\right)^{\frac{1}{2}}\\
&= e^{\mathrm{Var}(Y_0^H)}\left(E\left[\max_{0\leq s\leq r}\left|e^{Y_s^H-Y_0^H}-1\right|^2\right]\right)^{\frac{1}{2}}. \qquad (12)
\end{aligned}$$

Since $\{Y_s^H\}_{s\geq 0}$ has a continuous version, it is bounded on $0 \leq s \leq r$ and we can use the expansion $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ on $\{X_s^H\}_{s\geq 0}$. Thus elementary calculations show that

$$
\begin{aligned}
&\max_{0\leq s\leq r} \left| e^{Y_s^H - Y_0^H} - 1 \right|^2 \\
&= \max_{0\leq s\leq r} \left| e^{2(Y_s^H - Y_0^H)} - 2e^{(Y_s^H - Y_0^H)} + 1 \right| \\
&\overset{a.s.}{=} \max_{0\leq s\leq r} \left| \sum_{m=1}^{\infty} \frac{(2(Y_s^H - Y_0^H))^m}{m!} - 2\sum_{m=1}^{\infty} \frac{(Y_s^H - Y_0^H)^m}{m!} \right| \\
&= \max_{0\leq s\leq r} \left| \left(Y_s^H - Y_0^H\right)^2 + \sum_{m=3}^{\infty} \frac{(2(Y_s^H - Y_0^H))^m}{m!} - 2\sum_{m=3}^{\infty} \frac{(Y_s^H - Y_0^H)^m}{m!} \right| \\
&\leq \max_{0\leq s\leq r} \left(Y_s^H - Y_0^H\right)^2 + \sum_{m=3}^{\infty} \frac{\max_{0\leq s\leq r}(2\left|Y_s^H - Y_0^H\right|)^m}{m!} + \sum_{m=3}^{\infty} \frac{\max_{0\leq s\leq r}\left|Y_s^H - Y_0^H\right|^m}{m!}.
\end{aligned}
$$

By virtue of Lemma 2.4, it follows that

$$
\begin{aligned}
E\left[\max_{0\leq s\leq r}\left(e^{Y_s^H - Y_0^H} - 1\right)^2\right] &\leq E\left[\max_{0\leq s\leq r}\left(Y_s^H - Y_0^H\right)^2\right] + \sum_{m=3}^{\infty} \frac{E\left[\max_{0\leq s\leq r}(2\left|Y_s^H - Y_0^H\right|)^m\right]}{m!} \\
&\quad + \sum_{m=3}^{\infty} \frac{E\left[\max_{0\leq s\leq r}\left|Y_s^H - Y_0^H\right|^m\right]}{m!} \\
&= 2c_1 r^{2H} + \sum_{m=3}^{\infty} c_1 2^m r^{Hm} + \sum_{m=3}^{\infty} c_1 r^{Hm} \\
&= 2c_1 r^{2H} + \frac{c_1 8 r^{3H}}{1-2r^H} + \frac{c_1 r^{3H}}{1-r^H} \\
&\leq 2c_1 r^{2H}\left(1 + \frac{5r^H}{1-2r^H}\right) \\
&= 2c_1 r^{2H} + 2c_1' r^{3H},
\end{aligned}
$$

for all r such that $r^H < 1/2$. If we substitute this into (12), we observe that

$$
E\left[\max_{0\leq s\leq r}\left|X_s^H - X_0^H\right|\right] \leq c_2 \times e^{\mathrm{var}(Y_0^H)}\sqrt{2c_1 + 2c_1'}\, r^H,
$$

where $c_2 > 0$ is a constant. Then putting

$$
\overline{C}(H) := c_2 \times e^{\mathrm{var}(Y_0^H)}\sqrt{2c_1 + 2c_1'},
$$

we obtain the result. □

Proof of Theorem 2.2
As in Proof of Theorem 2.1, we have

$$
\max_{t\leq s\leq t+r}\left|X_s^H - X_t^H\right| \overset{d}{=} e^{Y_0^H}\max_{0\leq s\leq r}\left|e^{Y_s^H - Y_0^H} - 1\right|.
$$

According to the Cauchy-Schwartz inequality,

$$\left(E\left[\max_{0\le s\le r}\left|e^{Y_s^H-Y_0^H}-1\right|^{1/2}\right]\right)^2 \le E\left[e^{-Y_0^H}\right]E\left[e^{Y_0^H}\max_{0\le s\le r}\left|e^{Y_s^H-Y_0^H}-1\right|\right],$$

and thus

$$E\left[\max_{t\le s\le t+r}\left|X_s^H-X_t^H\right|\right] \ge e^{-\mathrm{Var}(Y_0^H)/2}\left(E\left[\max_{0\le s\le r}\left|e^{Y_s^H-Y_0^H}-1\right|^{1/2}\right]\right)^2.$$

Then by the symmetry of the distribution of $(Y_r^H-Y_0^H)$, we have

$$\begin{aligned}
E\left[\max_{0\le s\le r}\left|e^{Y_s^H-Y_0^H}-1\right|^{1/2}\right] &\ge E\left[\max_{0\le s\le r}\left|e^{Y_s^H-Y_0^H}-1\right|^{1/2}1_{\{Y_s^H-Y_0^H\ge 0\}}\right]\\
&\ge E\left[\max_{0\le s\le r}\left|Y_s^H-Y_0^H\right|^{1/2}1_{\{Y_s^H-Y_0^H\ge 0\}}\right]\\
&\ge E\left[\left|Y_r^H-Y_0^H\right|^{1/2}1_{\{Y_r^H-Y_0^H\ge 0\}}\right]\\
&\ge \frac{1}{2}E\left[\left|Y_r^H-Y_0^H\right|^{1/2}\right].
\end{aligned}$$

Now the conclusion is implied by Lemma 2.5. □

Proof of Proposition 3.1

The proof is based on the examination of the validity of several crucial assumptions imposed in [10], in which the authors adapt Kahane's formulation to stationary processes, Our results in Section 2 assert that the crucial conditions imposed in their paper hold for our exponential process $\overline{X}(t)$. Firstly, our Proposition 2.1 on the decay of the covariance function asserts the sufficiency conditions in their subsections 3.1 and 3.2 hold for $\overline{X}$. Namely, we apply our Proposition 2.1 to see that the $L^2(dP)$ norm of the martingale difference, $||A_n(1)-A_{n-1}(1)||_2$, is summable in n (their subsection 3.1), and thus a limiting process $A(t)$ exists, as the $L^2(dP)$ limit of $A_n(t)$ for each t. While it is obvious that $A(t)$ is path-wise increasing in t, we need apply again our Proposition 2.1 to see that it is indeed path-wise non-degenerate and continuous in t (their subsection 3.2). Secondly, our Theorem 2.1 asserts that, for all $q\in[1,2]$,

$$\sum_{n=0}^{\infty}E\left[\max_{0\le s\le b^{-n}}\left|\overline{X}(s)^q-\overline{X}(0)^q\right|\right] \le \overline{\overline{C}}(H)\sum_{n=0}^{\infty}b^{-nH},$$

where $\overline{\overline{C}}(H)$ is a constant derived from Theorem 2.1 and the normalizing factor defined in $\overline{X}(t)$. Since we have a dominating convergent series on the right-handed side of the above display, the main (and most crucial) assumption (10) in their Proposition 5 of subsection 3.3 indeed holds, and hence the inequality of Proposition 3.1 is established as a consequence of their Proposition 5 of subsection 3.3. □

Concluding remarks on some future works:

1. We may try to obtain the sharp bounds for $\overline{C}(H)$ and $\underline{c}(H)$ in Theorems 2.1 and 2.2. This will be a significant supplement to the works in [13] (in which some rather sharp bounds for B-D-G

inequalities of fractional Brownian motion is proved).
2. We use Slepian's inequality in this paper. Instead of this, we may try to consider some possible complements with the method in [13]. That is, we try to see the mutual benefit between our Slepian's inequality and their B-D-G inequalities.
3. We may consider similar results for for geometric fractional Lévy process or geometric fractional Ornstein-Uhlenbeck Lévy process, which are outside of the Gaussian realm.

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