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Statistics of a Vortex Filament Model

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Abstract

A random incompressible velocity field in three dimensions composed by Poisson distributed Brownian vortex filaments is constructed. The filaments have a random thickness, length and intensity, governed by a measure γ. Under appropriate assumptions on γ we compute the scaling law of the structure function of the field and show that, in particular, it allows for either K41-like scaling or multifractal scaling.

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1 Introduction

Isotropic homogeneous turbulence is phenomenologically described by several theories, which usually give us the scaling properties of moments of velocity increments. If $u(x)$ denotes the velocity field of the fluid and $S_p(\varepsilon)$, the so called *structure function*, denotes the p-moment of the velocity increment over a distance ε (often only its longitudinal projection is considered), then one expect a behavior of the form

$$S_p(\varepsilon) = \langle |u(x+\varepsilon) - u(x)|^p \rangle \sim \varepsilon^{\zeta_p}. \tag{1}$$

Here, in our notations, ε is not the dissipation energy, but just the spatial scale parameter (see remark 5). Let us recall two major theories: the Kolmogorov-Obukov scaling law (K41) (see [14]) says that

$$\zeta_2 = \frac{2}{3}$$

probably the best result compared with experiments; however the heuristic basis of the theory also implies $\zeta_p = \frac{p}{3}$ which is not in accordance with experiments. Intermittency corrections seem to be important for larger p's. A general theory which takes them into account is the multifractal scaling theory of Parisi and Frisch [11], that gives us ζ_p in the form of a Fenchel-Legendre transform:

$$\zeta_p = \inf_{h \in I} [hp + 3 - D(h)].$$

This theory is a sort of container, which includes for instance the striking particular case of She and Leveque [18]. We do not pretend to go further in the explanation of this topic and address the reader to the monograph [9].

The foundations of these theories, in particular of the multifractal one, are usually mathematically poor, based mainly on very good intuition and a suitable "mental image" (see the beginning of Chapter 7 of [9]). Essentially, the scaling properties of $S_p(\varepsilon)$ are given *a priori*, after an intuitive description of the mental image. The velocity field of the fluid is not mathematically described or constructed, but some crucial aspects of it are described only in plain words, and then $S_p(\varepsilon)$ is given (or heuristically "deduced").

We do not pretend to remedy here to this extremely difficult problem, which ultimately should start from a Navier-Stokes type model and the analysis of its invariant measures.

The contribution of this paper is only to construct rigorously a random velocity field which has two interesting properties: i) its realizations have a geometry inspired by the pictures obtained by numerical simulations of turbulent fluids; ii) the asymptotic as $\varepsilon \to 0$ of $S_p(\varepsilon)$ can be explicitly computed and the multifractal model is recovered with a suitable choice of the measures defining the random field. Its relation with Navier-Stokes models and their invariant measures is obscure as well (a part from some vague conjectures, see [7]), so it is just one small step beyond pure phenomenology of turbulence.

Concerning (i), the geometry of the field is that of a collection of vortex filaments, as observed for instance by [19] and many others. The main proposal to model vortex filaments by paths of stochastic processes came from A. Chorin, who made several considerations about their statistical mechanics, see [5]. The processes considered in [5] are self-avoiding walks, hence discrete. Continuous processes like Brownian motion, geometrically more natural, have been considered by [10], [16], [6], [8], [17] and others. We do not report here numerical results, but we have observed in simple simulations that the vortex filaments of the present paper, with the tubular smoothening due to the parameter ℓ (see below), have a shape that reminds very strongly the simulations of [3].

Concerning (ii), we use stochastic analysis, properties of stochastic integrals and ideas related to the theory of the Brownian sausage and occupation measure. We do not know whether it is possible to reach so strict estimates on $S_p(\varepsilon)$ as those proved here in the case when stochastic processes are replaced by smooth curves. The power of stochastic calculus seem to be important.

Ensembles of vortex structures with more stiff or artificial geometry have been considered recently by [1] and [12]. They do not stress the relation with multifractal models and part of their results are numerical, but nevertheless they indicate that scaling laws can be obtained by models based on many vortex structures. Probably a closer investigation of simpler geometrical models like that ones, in spite of the less appealing geometry of the objects, will be important to understand more about this approach to turbulence theory. Let us also say that these authors introduced that models also for numerical purposes, so the simplicity of the structures has other important motivations.

Finally, let us remark about a difference with respect to the idea presented in [5] and also in [4]. There one put the attention on a single vortex and try to relate statistical properties of the paths of the process, like Flory exponents of 3D self-avoiding walk, with scaling law of the velocity field. Such an attempt is more intrinsic, in that it hopes to associate turbulent scalings with relevant exponents known for processes. On the contrary, here (and in [1] and [12]) we consider a fluid made of a multitude of vortex structures and extract statistics from the collective behavior. In fact, it this first work on the subject, we consider independent vortex structures only, having in mind the Gibbs couplings of [8] as a second future step. Due to the independence, at the end again it is just the single filament that determines, through the statistics of its parameters, the properties of $S_p(\varepsilon)$. However, the interpretation of the results and the conditions on the parameters are more in the spirit of the classical ideas of K41 and its variants, where one thinks to the 3D space more or less filled in by eddies or other structures. It is less natural to interpret multifractality, for instance, on a single filament (although certain numerical simulation on the evolution of a single filament suggest that multifractality could arise on a single filament by a non-uniform procedure of stretching and folding [2]).

1.1 Preliminary remarks on a single filament

The rigorous definitions will be given in section 2. Here we introduce less formally a few objects related to a *single* vortex filament.

We consider a 3d-Brownian motion $\{X_t\}_{t\in[0,T]}$ starting from a point X_0. This is the backbone of the vortex filament whose vorticity field is given by

$$\xi_{\mathrm{single}}(x) = \frac{U}{\ell^2}\int_0^T \rho_\ell(x - X_t) \circ dX_t. \tag{2}$$

The letter t, that sometimes we shall also call time, is not physical time but just the parameter of the curve. All our random fields are time independent, in the spirit of equilibrium statistical mechanics. We assume that $\rho_\ell(x) = \rho(x/\ell)$ for a radially symmetric measurable bounded function ρ with compact support in the ball $B(0,1)$ (the unit ball in 3d Euclidean space). To have an idea, consider the case $\rho = 1_{B(0,1)}$. Then $\xi_{\mathrm{single}}(x) = 0$ outside an ℓ-neighboor $\mathcal{U}_\ell$ of the support of the curve $\{X_t\}_{t\in[0,T]}$. Inside $\mathcal{U}_\ell$, $\xi_{\mathrm{single}}(x)$ is a time-average of the "directions" dX_t, with the pre-factor U/ℓ^2. More precisely, if $x \in \mathcal{U}_\ell$, one has to consider the time-set where $X_t \in B(x,\ell)$ and average dX_t on such time set. The resulting field $\xi_{\mathrm{single}}(x)$ looks much less irregular than $\{X_t\}_{t\in[0,T]}$, with increasing irregularity for smaller values of ℓ.

When ρ is only measurable, it is not a priori clear that the Stratonovich integral $\xi_{\mathrm{single}}(x)$ is well defined, since the quadratic variation corrector involves distributional derivatives of ρ (the Itô integral is more easily defined, but it is not the natural object to be considered, see the remarks in [6]). Since we shall never use explicitly $\xi_{\mathrm{single}}(x)$, this question is secondary and we may consider $\xi_{\mathrm{single}}(x)$ just as a formal expression that we introduce to motivate the subsequent definition of $u_{\mathrm{single}}(x)$. However, at least in some particular case ($\rho = 1_{B(0,1)}$) or under a little additional assumptions, the Stratonovich integral $\xi_{\mathrm{single}}(x)$ is well defined since the corrector has a meaning (in the case of $\rho = 1_{B(0,1)}$ the corrector involves the local time of 3d Brownian motion on sferical surfaces).

The factor U/ℓ^2 in the definition of $\xi_{\mathrm{single}}(x)$ is obscure at this level. Formally, it could be more natural just to introduce a parameter Γ, in place of U/ℓ^2, to describe the intensity of the vortex. However, we do not have a clear interpretation of Γ, a posteriori, from out theorems, while on the contrary it will arise that the parameter U has the meaning of a typical velocity intensity in the most active region of the filament. Thus the choice of the expression U/ℓ^2 has been devised a posteriori. The final interpretation of the three parameters is that T is the length of the filament, ℓ the thickness, U the typical velocity around the core.

The velocity field u generated by ξ is given by the Biot-Savart relation

$$u_{\mathrm{single}}(x) = \frac{U}{\ell^2}\int_0^T K_\ell(x - X_t) \wedge \circ dX_t \tag{3}$$

where $\wedge$ stands for the vector product in $\mathbb{R}^3$, $\circ$ denote as usual Stratonovich integration of semimartingales and the vector kernel $K_\ell(x)$ is defined as

$$K_\ell(x) = \nabla V_\ell(x) = \frac{1}{4\pi}\int_{B(0,\ell)} \rho_\ell(y)\frac{x-y}{|x-y|^3}dy, \qquad V_\ell(x) = -\frac{1}{4\pi}\int_{B(0,\ell)} \frac{\rho_\ell(y)dy}{|x-y|}. \tag{4}$$

The scalar field V_ℓ satisfy the Poisson equation $\Delta V_\ell = \rho_\ell$ in all $\mathbb{R}^3$. Since ρ is radially symmetric and with compact support we can have two different situations according to the fact that the integral of ρ: $Q = \int_{B(0,1)} \rho(x)dx$ is zero or not. If $Q = 0$ then the field V_ℓ is identically zero outside the ball $B(0,\ell)$. Otherwise the fields V_ℓ, K_ℓ outside the ball $B(0,\ell)$ have the form

$$V_\ell(x) = Q\frac{\ell^3}{|x|}, \quad K_\ell(x) = -2Q\ell^3\frac{x}{|x|^3}, \qquad |x| \geq \ell. \tag{5}$$

Accordingly we will call the case $Q = 0$ *short range* and $Q \neq 0$ *long range*. The proof of the previous formula for K is given, for completeness, in the remark at the end of the section.

A basic result is that the Stratonovich integral in the expression of u_{single} can be replaced by an Itô integral:

Lemma 1 *Itô and Stratonovich integrals in the definition of $u_{single}(x)$ coincide:*

$$u_{single}(x) = \frac{U}{\ell^2}\int_0^T K_\ell(x - X_t) \wedge dX_t \tag{6}$$

where the integral is understood in Itô sense. Then u_{single} is a (local)-martingale with respect to the standard filtration of X.

About the proof, by an approximation procedure that we omit we may assume ρ Hölder continuous, so the derivatives of K_ℓ exist and are Hölder continuous by classical regularity theorems for elliptic equations. Under this regularity one may compute the corrector and prove that it is equal to zero, so, a posteriori, equation (6) holds true in the limit also for less regular ρ. About the proof that the corrector is zero, it can be done component-wise, but it is more illuminating to write the following heuristic computation: the corrector is formally given by

$$-\frac{1}{2}\int_0^T (\nabla K_\ell(x - X_t)dX_t) \wedge dX_t.$$

Now, from the property $dX_t^i dX_t^j = \delta_{ij}dt$ one can verify that

$$(\nabla K_\ell(x - X_t)dX_t) \wedge dX_t = (\text{curl } K_\ell)(x - X_t)dt.$$

Since K_ℓ is a gradient, we have curl $K_\ell = 0$, so the corrector is equal to zero.

Remark 1 *One may verify that div $u_{single} = 0$, so u_{single} may be the velocity fluid of an incompressible fluid. On the contrary, div ξ_{single} is different from zero and curl u_{single} is not ξ_{single} but its projection on divergence free fields. Therefore, one should think of ξ_{single} as an auxiliary field we start from in the construction of the model.*

Remark 2 *Let us prove (5), limited to K to avoid repetitions. We want to solve $\Delta V = \rho$ with ρ spherically symmetric. The gradient $K = \nabla V$ of V satisfies $\nabla \cdot K = \rho$ and, by spherical symmetry, it must be such that*

$$K(x) = \frac{x}{|x|} f(|x|)$$

for some scalar function $f(r)$. By Gauss theorem we have

$$\int_{B(0,r)} \nabla \cdot K(x) dx = \int_{\partial B(0,r)} K(x) \cdot d\sigma(x)$$

where $d\sigma(x)$ is the outward surface element of the sphere. So

$$f(r) 4\pi r^2 = \int_{B(0,r)} \rho(x) dx =: Q(r)$$

namely $f(r) = \frac{Q(r)}{4\pi r^2}$. Therefore

$$K(x) = Q(r) \frac{x}{4\pi |x|^3}$$

with $Q(r) = Q(1)$ if $r \geq 1$ (since ρ has support in $B(0,1)$).

2 Poisson field of vortices

Intuitively, we want to describe a collection of infinitely many independent Brownian vortex filaments, uniformly distributed in space, with intensity-thickness-length parameters (U, ℓ, T) distributed according to a measure γ. The total vorticity of the fluid is the sum of the vorticity of the single filaments, so, by linearity of the relation vorticity-velocity, the total velocity field will be the sum of the velocity fields of the single filaments.

The rigorous description requires some care, so we split it into a number of steps.

2.1 Underlying Poisson random measure

Let Ξ be the metric space

$$\Xi = \{(U, \ell, T, X) \in \mathbb{R}_+^3 \times C([0,1]; \mathbb{R}^3) : 0 < \ell \leq \sqrt{T} \leq 1\}$$

with its Borel σ-field $\mathcal{B}(\Xi)$. Let $(\Omega, \mathcal{A}, P)$ be a probability space, with expectation denoted by E, and let μ_ω, $\omega \in \Omega$, be a Poisson random measure on $\mathcal{B}(\Xi)$, with intensity ν (a σ-finite measure on $\mathcal{B}(\Xi)$) given by

$$d\nu(U, \ell, T, X) = d\gamma(U, \ell, T) d\mathcal{W}(X).$$

for γ a σ-finite measure on the Borel sets of $\{(U, \ell, T) \in \mathbb{R}^3_+ : 0 < \ell \leq \sqrt{T} \leq 1\}$ and $d\mathcal{W}(X)$ the σ-finite measure defined by

$$\int_{C([0,1];\mathbb{R}^3)} \psi(X) d\mathcal{W}(X) = \int_{\mathbb{R}^3} \left[\int_{C([0,1];\mathbb{R}^3)} \psi(X) d\mathcal{W}_{x_0}(X) \right] dx_0$$

for any integrable test function $\psi : C([0,1]; \mathbb{R}^3) \to \mathbb{R}$ where $d\mathcal{W}_{x_0}(X)$ is the Wiener measure on $C([0,1], \mathbb{R}^3)$ starting at x_0 and dx_0 is the Lebesgue measure on $\mathbb{R}^3$. Heuristically the measure $\mathcal{W}$ describe a Brownian path starting from an uniformly distributed point in all space. The assumptions on γ will be specified at due time.

The random measure μ_ω is uniquely determined by its characteristic function

$$E \exp\left(i \int_\Xi \varphi(\xi) \mu(d\xi) \right) = \exp\left(- \int_\Xi (e^{\varphi(\xi)} - 1) \nu(d\xi) \right)$$

for any bounded measurable function φ on Ξ with support in a set of finite ν-measure.

In particular, for example, the first two moments of μ read

$$E \int_\Xi \varphi(\xi) \mu(d\xi) = \int_\Xi \varphi(\xi) \nu(d\xi)$$

and

$$E \left[\int_\Xi \varphi(\xi) \mu(d\xi) \right]^2 = \left[\int_\Xi \varphi(\xi) \nu(d\xi) \right]^2 + \int_\Xi \varphi^2(\xi) \nu(d\xi).$$

We have to deal also with moments of order p; some useful formulae will be now given.

2.2 Moments of the Poisson Random Field

Let $\varphi : \Xi \to \mathbb{R}$ be a measurable function. We shall say it is μ-integrable if it is μ_ω-integrable for P-a.e. $\omega \in \Omega$. In such a case, by approximation by bounded measurable compact support functions, one can show that the mapping $\omega \mapsto \mu_\omega(\varphi)$ is measurable.

If $\varphi' : \Xi \to \mathbb{R}$ is a measurable function with $\nu(\varphi \neq \varphi') = 0$, since $P(\mu(\varphi \neq \varphi') = 0) = 1$, we have that φ' is μ-integrable and $P(\mu(\varphi) = \mu(\varphi')) = 1$. Therefore the concept of μ-integrability and the random variable $\mu(\varphi)$ depend only on the equivalence class of φ.

Let φ be a measurable function on Ξ, possibly defined only ν-a.s. We shall say that it is μ-integrable if some of its measurable extensions to the whole Ξ is μ-integrable. Neither the condition of μ-integrability nor the equivalence class of $\mu(\varphi)$ depend on the extension, by the previous observations. We need all these remarks in the sequel when we deal with φ given by stochastic integrals.

Lemma 2 *Let φ be a measurable function on Ξ (possibly defined only ν-a.s.), such that $\nu(\varphi^p) < \infty$ for some even integer number p. Then φ is μ-integrable, $\mu(\varphi) \in L^p(\Omega)$, and*

$$E\left[\mu(\varphi)^p\right] \le ep^p \nu(\varphi^p).$$

If in addition we have $\nu(\varphi^k) = 0$ for every odd $k < p$, then

$$E\left[\mu(\varphi)^p\right] \ge \nu(\varphi^p).$$

Proof. Assume for a moment that φ is bounded measurable and with support in a set of finite ν-measure, so that the qualitative parts of the statement are obviously true. Using the moment generating function

$$Ee^{\lambda\mu(\varphi)} = e^{\nu(e^{\lambda\varphi}-1)}$$

we obtain

$$\begin{aligned}
\sum_{p=0}^{\infty} \frac{\lambda^p}{p!} E[\mu(\varphi)^p] &= \sum_{n=0}^{\infty} \frac{1}{n!} \left[\nu(e^{\lambda\varphi} - 1)\right]^n \\
&= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \nu(\varphi^k)\right]^n \\
&= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k_1,\dots,k_n \ge 1} \frac{\lambda^{k_1+\cdots+k_n}}{k_1!\cdots k_n!} \nu(\varphi^{k_1})\cdots\nu(\varphi^{k_n}) \\
&= 1 + \sum_{p=1}^{\infty} \frac{\lambda^p}{p!} \sum_{n=1}^{p} \sum_{\substack{k_1,\dots,k_n \ge 1 \\ k_1+\cdots+k_n = p}} \frac{p!}{n!k_1!\cdots k_n!} \nu(\varphi^{k_1})\cdots\nu(\varphi^{k_n})
\end{aligned}$$

Hence we have an equation for the moments:

$$E[\mu(\varphi)^p] = \sum_{n=1}^{p} \sum_{\substack{k_1,\dots,k_n \ge 1 \\ k_1+\cdots+k_n = p}} \frac{p!}{n!k_1!\cdots k_n!} \nu(\varphi^{k_1})\cdots\nu(\varphi^{k_n}) \tag{7}$$

Since $\nu(|\varphi|^k) \le [\nu(|\varphi|^p)]^{k/p}$ for $k \le p$ we have

$$\begin{aligned}
E[|\mu(\varphi)|^p] &\le E[\mu(|\varphi|)^p] \\
&\le \sum_{n=1}^{p} \sum_{\substack{k_1,\dots,k_n \ge 1 \\ k_1+\cdots+k_n = p}} \frac{p!}{n!k_1!\cdots k_n!} \nu(|\varphi|^p) \\
&\le \nu(|\varphi|^p) \sum_{n=1}^{p} \sum_{\substack{k_1,\dots,k_n \ge 0 \\ k_1+\cdots+k_n = p}} \frac{p!}{n!k_1!\cdots k_n!} \\
&= \nu(|\varphi|^p) \sum_{n=1}^{p} \frac{n^p}{n!} \le ep^p \nu(|\varphi|^p).
\end{aligned}$$

This proves the first inequality of the lemma. A posteriori, we may use it to prove the qualitative part of the first statement, by a simple approximation procedure for general measurable φ.

For the lower bound, from the assumption that $\nu(\varphi^k) = 0$ for odd $k < p$, in the sum (7) we have contributions only when all k_i, $i = 1, \dots, n$ are even. Then, neglecting many terms, we have

$$E[\mu(\varphi)^p] \geq \nu(\varphi^p).$$

The proof is complete.

2.3 Velocity field

Let ρ be a radially symmetric measurable bounded function ρ on $\mathbb{R}^3$ with compact support in the ball $B(0,1)$, and let K_ℓ be defined as in section 1.1. Then we have that

$$K_1 \text{ is Lipschitz continuous.} \tag{8}$$

Indeed by an explicit computation it is possible to show that

$$K_1(x) = -2Q(|x|)\frac{x}{|x|^3}$$

with

$$Q(r) := \int_{B(0,r)} \rho(x)dx$$

(since ρ has support in $B(0,1)$ we have $Q(r) = Q$ if $r \geq 1$). Then

$$\nabla K_1(x) = -2Q'(|x|)\frac{x \otimes x}{|x|^4} - 2Q(|x|)\left[\frac{1}{|x|^3} - 3\frac{x \otimes x}{|x|^5}\right]$$

with $Q'(r) = dQ(r)/dr$ and we can bound

$$|\nabla K_1(x)| \leq C\|\rho\|_\infty$$

since

$$Q(r) \leq C\|\rho\|_\infty r^3, \qquad Q'(r) \leq C\|\rho\|_\infty r^2.$$

For any $x, x_0 \in \mathbb{R}^3$, $\ell, T > 0$, the random variable $X \mapsto \int_0^T K_\ell(x - X_t) \wedge dX_t$ is defined $\mathcal{W}_{x_0}$-a.s. on $C([0,1], \mathbb{R}^3)$. We also have, given $x \in \mathbb{R}^3$, $\ell, T > 0$, that $X \mapsto \int_0^T K_\ell(x - X_t) \wedge dX_t$ is a well defined measurable function, defined $\mathcal{W}$-a.s. on $C([0,1], \mathbb{R}^3)$. More globally, writing $\xi = (U, \ell, T, X)$ for shortness, for any $x \in \mathbb{R}^3$ we may consider the measurable $\mathbb{R}^3$-valued function

$$\xi \mapsto u^{\xi}_{\text{single}}(x) := \frac{U}{\ell^2}\int_0^T K_\ell(x - X_t) \wedge dX_t$$

defined ν-a.s. on Ξ. In plain words, this is the velocity field at point x of a filament specified by ξ.

Again in plain words, given $\omega \in \Omega$, the point measure μ_ω specifies the parameters and locations of infinitely many filaments: formally

$$\mu = \sum_{\alpha \in \mathbb{N}} \delta_{\xi^\alpha} \tag{9}$$

for a sequence of i.i.d. random points $\{\xi^\alpha\}$ distributed in Ξ according to ν (this fact is not rigorous since ν is only σ-finite, but it has a rigorous version by localization explained below). Since the total velocity at a given point $x \in \mathbb{R}^3$ should be the sum of the contributions from each single filament, i.e. in heuristic terms

$$u(x) = \sum_{\alpha \in \mathbb{N}} u_{\text{single}}^{\xi^\alpha}(x) \tag{10}$$

we see that, in the rigorous language of μ, we should write

$$u(x) = \int_\Xi u_{\text{single}}^{\xi}(x)\, \mu(d\xi) = \mu\left(u_{\text{single}}^{\cdot}(x)\right). \tag{11}$$

If we show that $\nu\left(\left|u_{\text{single}}^{\cdot}(x)\right|^p\right) < \infty$ for some even $p \geq 2$, then from lemma 2, $u_{\text{single}}^{\cdot}(x)$ is μ-integrable and the random variable

$$\omega \mapsto u(x,\omega) := \mu_\omega\left(u_{\text{single}}^{\cdot}(x)\right)$$

is well defined.

In some proof below we will use the occupation measure of the Brownian motion in the interval $[0,T]$, which is defined, for every Borel set B of $\mathbb{R}^3$ as

$$B \mapsto L_B^T := \int_0^T 1_{X_t \in B} dt.$$

Lemma 3 *Given $x \in \mathbb{R}^3$ and $p > 0$, there exist $C_p > 0$ such that for every $\ell^2 \leq T \leq 1$we have*

$$\mathcal{W}\left[W(x)^{p/2}\right] \leq C_p \ell^p \ell T$$

where

$$W(x) = \int_0^T |K_\ell(x - X_t)|^2\, dt.$$

PROOF. Let us bound K by a multi-scale argument. This is necessary only in the long-range case (see the introduction). If $|y| \leq \ell$ we can bound $|K_\ell(y)| \leq C\ell$. Indeed if $|w| \leq 1$, $|K(w)| \leq C$ for some constant C and then if $|y| < \ell$ we have $|K_\ell(y)| = \ell |K_1(y/\ell)| \leq C\ell$.

Next, given $\Lambda > \ell$ and an integer N, consider a sequence $\{\ell_i\}_{i=0,\dots,N}$ of scales, with $\ell = \ell_0 < \ell_1 < \dots < \ell_N = \Lambda$. Then, for $i = 1, \dots, N$, if $\ell_{i-1} < |y| < \ell_i$, by the explicit formula for $K_\ell(y)$ we have $|K_\ell(y)| \le C\ell(\ell/|y|)^2$. Therefore

$$|K_\ell(y)| \le C\ell \left(\frac{\ell}{\ell_{i-1}}\right)^2 1_{\ell_{i-1}<|y|\le\ell_i}$$

If $|y| > \Lambda$ we simply bound $K^e_\ell(y) \le C\ell(\ell/\Lambda)^2$.
Summing up,

$$\begin{aligned}|K_\ell(y)|^2 &= |K_\ell(y)|^2 \left(1_{|y|\le\ell} + \sum_{i=1}^{N} 1_{\ell_{i-1}<|y|<\ell_i} + 1_{|y|>\Lambda}\right)\\ &\le C\ell^2 1_{|y|\le\ell} + C\sum_{i=1}^{N} \ell^2 \left(\frac{\ell}{\ell_{i-1}}\right)^4 1_{\ell_{i-1}<|y|<\ell_i} + C\ell^2 \left(\frac{\ell}{\Lambda}\right)^4 1_{|y|>\Lambda}\end{aligned}$$

which implies the following bound for $W(x)$:

$$W(x) \le C\ell^2 L^T_{B(x,\ell)} + C\sum_{i=1}^{N} \ell^2 \left(\frac{\ell}{\ell_{i-1}}\right)^4 L^T_{B(x,\ell_i)\setminus B(x,\ell_{i-1})} + C\ell^2 \left(\frac{\ell}{\Lambda}\right)^4 L^T_{B(x,\Lambda)^c}$$

where L^T_B has been defined above. By the additivity of $B \mapsto L^T_B$, the sum appearing in this equation can be rewritten as

$$\begin{aligned}\sum_{i=1}^{N} \left(\frac{\ell}{\ell_{i-1}}\right)^4 L^T_{B(x,\ell_i)\setminus B(x,\ell_{i-1})} &= \sum_{i=1}^{N-1} \left[\left(\frac{\ell}{\ell_{i-1}}\right)^4 - \left(\frac{\ell}{\ell_i}\right)^4\right] L^T_{B(x,\ell_i)}\\ &\quad + \left(\frac{\ell}{\ell_{N-1}}\right)^4 L^T_{B(x,\Lambda)} - L^T_{B(x,\ell)}\\ &\le \sum_{i=1}^{N-1} \left[\left(\frac{\ell}{\ell_{i-1}}\right)^4 - \left(\frac{\ell}{\ell_i}\right)^4\right] L^T_{B(x,\ell_i)} + \left(\frac{\ell}{\ell_{N-1}}\right)^4 L^T_{B(x,\Lambda)}.\end{aligned}$$

Assume that $\ell_i/\ell_{i-1} \le 2$ uniformly in $i = 1, \dots, N$. Then

$$\begin{aligned}&\sum_{i=1}^{N} \left(\frac{\ell}{\ell_{i-1}}\right)^4 L^T_{B(x,\ell_i)\setminus B(x,\ell_{i-1})}\\ &\le C\sum_{i=1}^{N-1} \left[\left(\frac{\ell}{\ell_{i-1}}\right)^2 - \left(\frac{\ell}{\ell_i}\right)^2\right] \left(\frac{\ell}{\ell_i}\right)^2 L^T_{B(x,\ell_i)} + C\left(\frac{\ell}{\Lambda}\right)^4 L^T_{B(x,\Lambda)}.\end{aligned}$$

Notice now that

$$\sum_{i=1}^{N-1}\left[\left(\frac{\ell}{\ell_{i-1}}\right)^2-\left(\frac{\ell}{\ell_i}\right)^2\right]=\left(\frac{\ell}{\ell}\right)^2-\left(\frac{\ell}{\Lambda}\right)^2\leq 1$$

so that, by Cauchy-Schwartz and Jensen inequalities we have

$$\begin{aligned}&\left[\sum_{i=1}^{N}\left(\frac{\ell}{\ell_{i-1}}\right)^4 L^T_{B(x,\ell_i)\setminus B(x,\ell_{i-1})}\right]^{p/2}\\&\leq C_p\sum_{i=1}^{N-1}\left[\left(\frac{\ell}{\ell_{i-1}}\right)^2-\left(\frac{\ell}{\ell_i}\right)^2\right]\left(\frac{\ell}{\ell_i}\right)^p(L^T_{B(x,\ell_i)})^{p/2}+C_p\left(\frac{\ell}{\Lambda}\right)^{2p}\left(L^T_{B(x,\Lambda)}\right)^{p/2}.\end{aligned}$$

An upper bound for $W(x)^{p/2}$ is then obtained as

$$\begin{aligned}W(x)^{p/2}\leq{}& C_p\ell^p(L^T_{B(x,\ell)})^{p/2}\\&+C_p\ell^p\left(\frac{\ell}{\Lambda}\right)^{2p}\left(L^T_{B(x,\Lambda)}\right)^{p/2}+C_p\ell^p\left(\frac{\ell}{\Lambda}\right)^{2p}\left(L^T_{B(x,\Lambda)^c}\right)^{p/2}\\&+C\ell^p\sum_{i=1}^{N-1}\left[\left(\frac{\ell}{\ell_{i-1}}\right)^2-\left(\frac{\ell}{\ell_i}\right)^2\right]\left(\frac{\ell}{\ell_i}\right)^p(L^T_{B(x,\ell_i)})^{p/2}\\\leq{}& C_p\ell^p(L^T_{B(x,\ell)})^{p/2}+C_p\ell^p\left(\frac{\ell}{\Lambda}\right)^{2p}T^{p/2}\\&+C_p\ell^p\sum_{i=1}^{N-1}\left[\left(\frac{\ell}{\ell_{i-1}}\right)^2-\left(\frac{\ell}{\ell_i}\right)^2\right]\left(\frac{\ell}{\ell_i}\right)^p(L^T_{B(x,\ell_i)})^{p/2}\end{aligned}$$

where we have used again Cauchy-Schwartz inequality.

We use now lemma 14 with $\alpha=1$. For a given $\lambda\in(0,1)$, we take both ε and ℓ equal to λ in (24) and (25), and get

$$\mathcal{W}\left[\left(L^T_{B(x,\lambda)}\right)^{p/2}\right]\leq C_p(\lambda\wedge\sqrt{T})^p\lambda(\lambda\vee\sqrt{T})^2.$$

Then we obtain

$$\begin{aligned}&\mathcal{W}\left[W(x)^{p/2}\right]\\&\leq C_p\ell^{2p}\ell T+C_p\ell^p\left(\frac{\ell}{\Lambda}\right)^{2p}T^{p/2}\\&+C_p\ell^p\sum_{i=1}^{N-1}\left[\left(\frac{\ell}{\ell_{i-1}}\right)^2-\left(\frac{\ell}{\ell_i}\right)^2\right]\left(\frac{\ell}{\ell_i}\right)^p(\ell_i\wedge\sqrt{T})^p\ell_i(\ell_i\vee\sqrt{T})^2\end{aligned}$$

and taking the limit as the partition gets finer:

$$
\begin{aligned}
&\mathcal{W}\left[W(x)^{p/2}\right] \\
&\leq C_p \ell^{2p} \ell T + C_p \ell^p \left(\frac{\ell}{\Lambda}\right)^{2p} T^{p/2} \\
&+ C_p \ell^p \int_\ell^\Lambda \left(\frac{\ell}{u}\right)^p (u \wedge \sqrt{T})^p (u \vee \sqrt{T})^2 u d\left[-\left(\frac{\ell}{u}\right)^2\right]
\end{aligned}
$$

The integral can then be computed as

$$
\begin{aligned}
&\int_\ell^\Lambda \left(\frac{\ell}{u}\right)^p (u \wedge \sqrt{T})^p (u \vee \sqrt{T})^2 u \frac{\ell^2}{u^3} du \\
&\quad = \ell^p \ell^2 T \int_\ell^{\sqrt{T}} \frac{du}{u^2} + T^{p/2} \ell^{p+2} \int_{\sqrt{T}}^\Lambda u^{-p} du \\
&\quad = \ell^p \ell^2 T [\ell^{-1} - \sqrt{T}^{-1}] + (p-1)^{-1} T^{p/2} \ell^{p+2} [T^{(1-p)/2} - \Lambda^{(1-p)}] \\
&\quad \leq \ell^p \ell T + (p-1)^{-1} \ell^p \frac{\ell}{\sqrt{T}} \ell T
\end{aligned}
$$

Using the fact that $\ell \leq \sqrt{T}$ and letting $\Lambda \to \infty$ we finally obtain the claim. $\square$

Remark 3 *The multiscale argument above can be rewritten in continuum variables from the very beggining by means of the following identity: if $f : [0, \infty) \to \mathbb{R}$ is of class C^1 and has a suitable decay at infinity, then*

$$\int_0^T f(|x - X_t|)\, dt = -\int_0^\infty f'(r)\, L^T_{B(x,r)} dr.$$

This identity can be applied to $W(x)$. The proof along these lines is not essentially shorter and perhaps it is more obscure, thus we have choosen the discrete multiscale argument which has a neat geometrical interpretation.

Corollary 1 *Assume*

$$\gamma(U^p \ell T) < \infty$$

for some even integer $p \geq 2$. Then, for any $x \in \mathbb{R}^3$, we have

$$\nu\left(\left|u^{\cdot}_{single}(x)\right|^p\right) < \infty$$

and the random variable

$$\omega \mapsto u(x, \omega) := \mu_\omega\left(u^{\cdot}_{single}(x)\right)$$

has finite p-moment:

$$E\left[|u(x)|^p\right] < \infty.$$

Proof. We have

$$\nu\left(\left|\dot{u_{\text{single}}}(x)\right|^p\right) = \int \mathcal{W}_{x_0}(\left|\dot{u_{\text{single}}}(x)\right|^p)d\gamma(U,\ell,T)dx_0.$$

By Burkholder-Davis-Gundy inequality, there is $C_p > 0$ such that

$$\mathcal{W}_{x_0}(\left|\dot{u_{\text{single}}}(x)\right|^p) \leq C_p \frac{U^p}{\ell^{2p}} \mathcal{W}_{x_0}\left[W(x)^{p/2}\right].$$

Hence

$$\begin{aligned}\mathcal{W}(\left|\dot{u_{\text{single}}}(x)\right|^p) &\leq C_p \frac{U^p}{\ell^{2p}} \mathcal{W}\left[W(x)^{p/2}\right] \\ &\leq C'_p \frac{U^p}{\ell^{2p}} \ell^{2p}\ell T = C'_p U^p \ell T.\end{aligned}$$

Therefore $\nu\left(\left|\dot{u_{\text{single}}}(x)\right|^p\right) < \infty$ by the assumption $\gamma(U^p\ell T) < \infty$. The other claims are a consequence of lemma 2. $\square$

Lemma 4 *Under the previous assumptions, the law of $u(x,\cdot)$ is independent of x and is invariant also under rotations:*

$$Ru(x,\cdot) \stackrel{\mathcal{L}}{=} u(Rx,\cdot)$$

for every rotation matrix R.

Proof. With the usual notation $\xi = (U,\ell,T,X)$ we have

$$u^{\xi}_{\text{single}}(x) = u^{(U,\ell,T,X)}_{\text{single}}(x) = u^{(U,\ell,T,X-x)}_{\text{single}}(0) = u^{\tau_x\xi}_{\text{single}}(0)$$

where $\tau_x(U,\ell,T,X) = (U,\ell,T,X-x)$. The map τ_x is a measurable transformation of Ξ into itself. One can see that ν is τ_x-invariant; we omit the details, but we just notice that ν is not a finite measure, so the invariance means

$$\int_{\Xi} \varphi(\tau_x\xi)\,\nu(d\xi) = \int_{\Xi} \varphi(\xi)\,\nu(d\xi)$$

for every $\varphi \in L^1(\Xi,\nu)$. From this invariance it follows that the law of the random measure μ is the same as the law of the random measure $\tau_x\mu$. Therefore

$$\begin{aligned}\mu\left[u^{\xi}_{\text{single}}(x)\right] &= \mu\left[u^{\tau_x\xi}_{\text{single}}(0)\right] \\ &= (\tau_x\mu)\left[u^{\xi}_{\text{single}}(0)\right] \stackrel{\mathcal{L}}{=} \mu\left[u^{\xi}_{\text{single}}(0)\right].\end{aligned}$$

This proves the first claim.

If R is a rotation, from the explicit form of K_ℓ it is easy to see that

$$RK_\ell(y) = K_\ell(Ry)$$

hence

$$\begin{aligned} Ru^{\xi}_{\text{single}}(x) &= \frac{U}{\ell^2}\int_0^T RK_\ell(x - X_t) \wedge dRX_t \\ &= \frac{U}{\ell^2}\int_0^T K_\ell(Rx - RX_t) \wedge dRX_t \\ &= u^{R\xi}_{\text{single}}(Rx) \end{aligned}$$

where we have set $R(U, \ell, T, X) = (U, \ell, T, RX)$. Again $R\nu = \nu$, $R\mu \overset{\mathcal{L}}{=} \mu$, so the end of the proof is the same as above. $\square$

We say that a random field $u(x, \cdot)$ is *homogeneous* if its law is independent of x and *isotropic* if its law is invariant under rotations.

Corollary 2 *Assume*

$$\gamma(U^p \ell T) < \infty$$

for every $p > 1$. Then $\{u(x, \cdot); x \in \mathbb{R}^3\}$ is an isotropic homogeneous random field, with finite moments of all orders.

This corollary is sufficient to introduce the structure function and state the main results of this paper. However, it is natural to ask whether the random field $\{u(x, \cdot); x \in \mathbb{R}^3\}$ has a continuous modification. Having in mind Kolmogorov regularity theorem, we need good estimates of $E\left[|u(x) - u(y)|^p\right]$. They are as difficult as the careful estimates we shall perform in the next section to understand the scaling of the structure function. Therefore we anticipate the result without proof. It is a direct consequence of Theorem 1.

Proposition 1 *Assume*

$$\gamma(U^p \ell T) < \infty$$

for every $p > 1$. Then, for every even integer p there is a constant $C_p > 0$ such that

$$E\left[|u(x) - u(y)|^p\right] \le C_p \gamma\left[U^p \left(\frac{\ell \wedge |x - y|}{\ell}\right)^p \ell T\right].$$

Consequently, if the measure γ has the property that for some even integer p and real number $\alpha > 3$ there is a constant $C'_p > 0$ such that

$$\gamma\left[U^p \left(\frac{\ell \wedge \varepsilon}{\ell}\right)^p \ell T\right] \le C'_p \varepsilon^\alpha \text{ for any } \varepsilon \in (0, 1), \tag{12}$$

then the random field $u(x)$ has a continuous modification.

Remark 4 *A sufficient condition for (12) is: there are $\alpha > 3$ and $\beta > 0$ such that for every sufficiently large even integer p there is a constant $C_p > 0$ such that*

$$\gamma\left[U^p \ell T \cdot 1_{\ell \leq \varepsilon^{1-\beta}}\right] \leq C_p \varepsilon^\alpha \text{ for any } \varepsilon \in (0,1).$$

Indeed,

$$\begin{aligned}
&\gamma\left[U^p\left(\frac{\ell\wedge\varepsilon}{\ell}\right)^p \ell T\right] \\
&= \gamma\left[U^p\left(\frac{\ell\wedge\varepsilon}{\ell}\right)^p \ell T \cdot 1_{\ell\leq\varepsilon^{1-\beta}}\right] + \gamma\left[U^p\left(\frac{\ell\wedge\varepsilon}{\ell}\right)^p \ell T \cdot 1_{\varepsilon^{1-\beta}\leq\ell}\right] \\
&\leq \gamma\left[U^p \ell T \cdot 1_{\ell\leq\varepsilon^{1-\beta}}\right] + \gamma\left[U^p \varepsilon^{\beta p} \ell T \cdot 1_{\varepsilon^{1-\beta}\leq\ell}\right] \\
&\leq C_p\varepsilon^\alpha + \varepsilon^{\beta p}\gamma\left[U^p \ell T\right]
\end{aligned}$$

so we have (12) with a suitable choice of p. This happens in particular in the multifractal example of section 3.2, remark 7.

The model presented here has a further symmetry which is not physically correct. This symmetry, described in the next lemma, implies that the odd moments of the longitudinal structure function vanish, in contradiction both with experiments and certain rigorous results derived from the Navier-Stokes equation (see [9]). The same drawback is present in other statistical models of vortex structures [1].

Beyond the rigorous formulation, the following property says that the random field $u^{\cdot}_{\text{single}}$ has the same law as $-u^{\cdot}_{\text{single}}$. We cannot use the concept of law since ξ does not live on a probability space.

Lemma 5 *Under the hypotesis of Corollary 1, given $x_1, ..., x_n \in \mathbb{R}^3$, the measurable vector*

$$U_n(\xi) := \left(u^\xi_{single}(x_1), ..., u^\xi_{single}(x_n)\right)$$

has the property

$$\int \varphi(U_n(\xi))\, d\nu(\xi) = \int \varphi(-U_n(\xi))\, d\nu(\xi)$$

for every $\varphi = \varphi(u_1, ..., u_n) : \mathbb{R}^{3n} \to \mathbb{R}$ with a polynomial bound in its variables..

Proof. With the notation $\widetilde{X}_t = X_{T-t}$, we have

$$\begin{aligned}
-u^\xi_{\text{single}}(x_k) &= -\frac{U}{\ell^2}\int_0^T K_\ell(x_k - X_t) \wedge \circ dX_t \\
&= \frac{U}{\ell^2}\int_0^T K_\ell(x_k - \widetilde{X}_s) \wedge \circ d\widetilde{X}_s \\
&= u^{S\xi}_{\text{single}}(x_k)
\end{aligned}$$

where $S(U,\ell,T,X) = (U,\ell,T,\widetilde{X})$. Since $S\nu = \nu$, we have the result (using the integrability of Corollary 1). □

Lemma 6 *If p is an odd positive integer, then*

$$\nu\left[\left\langle u^{\cdot}_{single}\left(y\right)-u^{\cdot}_{single}\left(x\right),y-x\right\rangle^{p}\right]=0$$

for every $x,y\in\mathbb{R}^{3}$.

PROOF. It is sufficient to apply the lemma to the function

$$\varphi\left(u_{1},u_{2}\right):=\left\langle u_{1}-u_{2},y-x\right\rangle^{p}$$

and the points $x_1 = y$, $x_2 = x$, with the observation that

$$\varphi\left(-u_{1},-u_{2}\right)=-\varphi\left(u_{1},u_{2}\right).$$

□

2.4 Localization

At the technical level, we do not need to localize the σ-finite measures of the present work. However, we give a few remarks on localization to help the intuitive interpretation of the model. Essentially we are going to introduce rigorous analogs of the heuristic expressions (9) and (10) written at the beginning of the previous section. The problem there was that the law of ξ^{α} should be ν, which is only a σ-finite measure. For this reason one has to localize ν and μ.

Given $A\in\mathcal{B}\left(\Xi\right)$ with $0<\nu\left(A\right)<\infty$, consider the measure μ_A defined as the restriction of μ to A:

$$\mu_{A}\left(B\right)=\mu\left(A\cap B\right)$$

for any $B\in\mathcal{B}\left(\Xi\right)$. It can be written (the equality is in law, or a.s. over a possibly enlarged probability space) as the sum of independent random atoms each distributed according to the probability measure $B\in\mathcal{B}\left(A\right)\mapsto\widetilde{\nu}_{A}\left(B\right):=\nu(B|A)$:

$$\mu_{A}(d\xi)=\sum_{\alpha=1}^{N_{A}}\delta_{\xi^{\alpha}}(d\xi)$$

where N_A is a Poisson random variable with intensity $\nu(A)$ and the family of random variables $\{\xi^{\alpha}\}_{\alpha\in\mathbb{N}}$ is independent and identically distributed according to $\widetilde{\nu}_{A}$. Moreover if $\{A_i\}_{i\in\mathbb{N}}$ is a family of mutually disjoint subsets of Ξ then the r.v.s $\{\mu_{A_i}\}_{i\in\mathbb{N}}$ are independent.

Sets A as above with a physical significance are the following ones. Given $0<\eta<1$ and $R>0$, let

$$A_{\eta,R}=\{(U,T,\ell,X)\in\Xi:\ell>\eta,|x_{0}|\leq R\}.$$

In a fluid model we meet these sets if we consider only vortexes up to some scale η (it could be the Kolmogorov dissipation scale) and roughly confined in a ball of radius R. If we assume that the measure γ satisfies $0 < \gamma(\ell > \eta) < \infty$ for each $\eta > 0$, then $0 < \nu(A_{\eta,R}) < \infty$, and for the measure $\mu_{\eta,R} := \mu_{A_{\eta,R}}$ we have the representation

$$\mu_{\eta,R}(d\xi) = \sum_{\alpha=1}^{N_{\eta,R}} \delta_{\xi^\alpha}(d\xi)$$

where $N_{\eta,R}$ is $\mathcal{P}(\nu(A_{\eta,R}))$ and $\{\xi^\alpha\}_{\alpha\in\mathbb{N}}$ are i.i.d. with law $\widetilde{\nu}_{\eta,R} := \widetilde{\nu}_{A_{\eta,R}}$.

For any $x \in \mathbb{R}^3$ we may consider $\xi \mapsto u^{\xi}_{\text{single}}(x)$ as a random variable in $\mathbb{R}^3$, defined $\widetilde{\nu}_{\eta,R}$-a.s. on $A_{\eta,R}$. Moreover we may consider the r.v. $u_{\eta,R}(x)$ on $(\Omega, \mathcal{A}, P)$ defined as

$$u_{\eta,R}(x) := \int_{A_{\eta,R}} u^{\xi}_{\text{single}}(x)\,\mu(d\xi) = \int_{\Xi} u^{\xi}_{\text{single}}(x)\,\mu_{\eta,R}(d\xi). \tag{13}$$

It is the velocity field at point x, generated by the vortex filaments in $A_{\eta,R} \in \mathcal{B}(\Xi)$. In this case we have the representation

$$u_{\eta,R}(x) = \sum_{\alpha=1}^{N_{\eta,R}} u^{\xi^\alpha}_{\text{single}}(x) = \sum_{\alpha=1}^{N_{\eta,R}} \frac{U^\alpha}{(\ell^\alpha)^2} \int_0^{T^\alpha} K_\ell(x - X^\alpha_t) \wedge dX^\alpha_t$$

where the quadruples $\xi^\alpha = (U^\alpha, \ell^\alpha, T^\alpha, X^\alpha)$ are distributed according to $\widetilde{\nu}_{\eta,R}$ and are independent. If $\omega \mapsto \xi(\omega)$ is any one of such quadruples, the random variable

$$\omega \mapsto u^{\xi(\omega)}_{\text{single}}(x)$$

is well defined, since the law of ξ is $\widetilde{\nu}_{\eta,R}$ and the random variable $\xi \mapsto u^{\xi}_{\text{single}}(x)$ is well defined $\widetilde{\nu}_{\eta,R}$-a.s. on $A_{\eta,R}$. Therefore $u_{\eta,R}(x)$ is a well defined random variable on $(\Omega, \mathcal{A}, P)$. We have noticed this in contrast to the fact that the definition of $u(x)$ required difficult estimates, because of the contribution of infinitely many filaments.

Given the Poisson random field, by localization we have constructed the velocity fields $u_{\eta,R}(x)$ that have a reasonable intuitive interpretation. Connections between $u_{\eta,R}(x)$ and $u(x)$ can be established rigorously at various levels. We limit ourselves to the following example of statement.

Lemma 7 *Assume*

$$\gamma(U^p \ell T) < \infty$$

for any $p > 0$. Then, at any $x \in \mathbb{R}^3$,

$$\lim_{(\eta, R^{-1}) \to (0^+, 0^+)} E\left[|u_{\eta,R}(x) - u(x)|^p\right] = 0.$$

Proof. Since

$$u_{\eta,R}(x) - u(x) = \int_{\Xi} \left(1_{A_{\eta,R}} - 1\right) u^{\xi}_{\text{single}}(x)\, \mu(d\xi)$$

we have

$$\begin{aligned} E\left[|u_{\eta,R}(x) - u(x)|^p\right] &\le C_p \nu\left[\left|\left(1_{A_{\eta,R}} - 1\right) u^{\cdot}_{\text{single}}(x)\right|^p\right] \\ &= C_p \nu\left[\left(1_{A_{\eta,R}} - 1\right) \left|u^{\cdot}_{\text{single}}(x)\right|^p\right] \end{aligned}$$

Notice that we do not have $\nu\left(A^c_{\eta,R}\right) \to 0$, in general, so the argument to prove the lemma must take into account the properties of the r.v. $u^{\cdot}_{\text{single}}(x)$. We have (by Burkholder-Davis-Gundy inequality)

$$\begin{aligned} &\nu\left[\left(1_{A_{\eta,R}} - 1\right) \left|u^{\cdot}_{\text{single}}(x)\right|^p\right] \\ &= \int_{|x_0|\ge R} dx_0 \int_{l\ge\eta} \mathcal{W}_{x_0}(\left|u^{\cdot}_{\text{single}}(x)\right|^p) d\gamma(U,\ell,T) \\ &\le C_p \int_{|x_0|\ge R} dx_0 \int_{l\ge\eta} \frac{U^p}{\ell^{2p}} \mathcal{W}_{x_0}\left[W(x)^{p/2}\right] d\gamma(U,\ell,T) \\ &\le C_p \int \frac{U^p}{\ell^{2p}} d\gamma(U,\ell,T) \int_{|x_0|\ge R} \mathcal{W}_{x_0}\left[W(x)^{p/2}\right] dx_0 \end{aligned}$$

where $W(x)$ has been defined in lemma 3. Let us show that

$$\int_{|x_0|\ge R} \mathcal{W}_{x_0}\left[W(x)^{p/2}\right] dx_0 \le C_p \ell^{2p} \ell T \cdot \theta(R) \tag{14}$$

where $\theta(R) \to 0$ as $R \to \infty$. The proof will be complete after this result. Recall from the proof of lemma 3 that we have

$$\begin{aligned} W(x)^{p/2} &\le C_p \ell^p (L^T_{B(x,\ell)})^{p/2} + C_p \ell^p \left(\frac{\ell}{\Lambda}\right)^{2p} T^{p/2} \\ &\quad + C_p \ell^p \sum_{i=1}^{N-1} \left[\left(\frac{\ell}{\ell_{i-1}}\right)^2 - \left(\frac{\ell}{\ell_i}\right)^2\right] \left(\frac{\ell}{\ell_i}\right)^p (L^T_{B(x,\ell_i)})^{p/2}. \end{aligned}$$

From lemma 15 we have

$$\begin{aligned} &\int_{|x_0|\ge R} \mathcal{W}_{x_0}\left[\left(L^T_{B(x,\lambda)}\right)^{p/2}\right] dx_0 \\ &\le C_p (\lambda \wedge \sqrt{T})^p \lambda (\lambda \vee \sqrt{T})^2 \exp\left(-\frac{R - (|x| + \lambda)}{\sqrt{T}}\right). \end{aligned}$$

Hence

$$\begin{aligned}
&\int_{|x_0|\ \geq R} \mathcal{W}_{x_0}\left[W(x)^{p/2}\right] dx_0 \\
&\leq C_p \ell^{2p} \ell T \exp\left(-\frac{R-(|x|+\ell)}{\sqrt{T}}\right) + C_p \ell^p \left(\frac{\ell}{\Lambda}\right)^{2p} T^{p/2} \\
&+ C_p \ell^p \sum_{i=1}^{N-1} \left[\left(\frac{\ell}{\ell_{i-1}}\right)^2 - \left(\frac{\ell}{\ell_i}\right)^2\right] \left(\frac{\ell}{\ell_i}\right)^p \\
&\times (\ell_i \wedge \sqrt{T})^p \ell_i (\ell_i \vee \sqrt{T})^2 \exp\left(-\frac{R-(|x|+\ell_i)}{\sqrt{T}}\right).
\end{aligned}$$

Repeating the arguments of lemma 3 we arrive at

$$\begin{aligned}
&\int_{|x_0|\ \geq R} \mathcal{W}_{x_0}\left[W(x)^{p/2}\right] dx_0 \\
&\leq C_p \ell^{2p} \ell T \exp\left(-\frac{R-(|x|+\Lambda)}{\sqrt{T}}\right) + C_p \ell^p \left(\frac{\ell}{\Lambda}\right)^{2p} T^{p/2}.
\end{aligned}$$

Since T and ℓ are smaller than one, and $p \geq 2$, we also have

$$\begin{aligned}
&\int_{|x_0|\ \geq R} \mathcal{W}_{x_0}\left[W(x)^{p/2}\right] dx_0 \\
&\leq C_p \ell^{2p} \ell T \left(\exp\left(-R-(|x|+\Lambda)\right) + \Lambda^{-2p}\right).
\end{aligned}$$

With the choice $\Lambda = R/2$ we prove (14). The proof is complete. □

3 The structure function

Given the random velocity field $u(x,\cdot)$ constructed above, under the assumption of Corollary 2, a quantity of major interest in the theory of turbulence is the *longitudinal structure function* defined for every integer p and $\varepsilon > 0$ as

$$S_p^{\|}(\varepsilon) = E[\langle e, u(x+\varepsilon e) - u(x)\rangle]^p \tag{15}$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean scalar product in $\mathbb{R}^3$, $e \in \mathbb{R}^3$ is a unit vector and $x \in \mathbb{R}^3$, and E, we recall, is the expectation on $(\Omega, \mathcal{A}, P)$.

Remark 5 *We warn the reader familiar with the literature on statistical fluid mechanics that ε here is not the dissipation energy, but just the spatial scale parameter. In the physical literature, it is commonly denoted by ℓ; however, in our mathematical analysis we need two parameters: the scale parameter of the statistical observation, which we denote by ε, and a parameter internal to the model that describes the thickness of the different vortex filaments, that we denote by ℓ.*

The moments $S_p^{\parallel}(\varepsilon)$ depend only on ε and p, since $u(x,\cdot)$ is homogeneous and isotropic: if $e = R \cdot e_1$ where e_1 is a given unit vector and R is a rotation matrix taking e_1 on e, using that the adjoint of R is R^{-1}, we have

$$\begin{aligned} E\left[\langle u(x+\varepsilon e) - u(x), e\rangle^p\right] &= E\left[\langle u(\varepsilon e) - u(0), e\rangle^p\right] \\ &= E\left[\langle Ru(\varepsilon e_1) - u(0), e\rangle^p\right] \\ &= E\left[\langle u(\varepsilon e_1) - u(0), e_1\rangle^p\right]. \end{aligned}$$

For this reason we do not write explicitly the dependence on x and e.

Let us also recall the *(non-directional) structure function*

$$S_p(\varepsilon) = E\left[|u(x+\varepsilon e) - u(x)|^p\right]$$

which as $S_p^{\parallel}(\varepsilon)$ depends only on ε and p. We obviously have

$$S_p^{\parallel}(\varepsilon) \leq S_p(\varepsilon).$$

We shall see that, for even integers p, they have the same scaling properties. At the technical level, due to the previous inequality, it will be sufficient to estimate carefully $S_p^{\parallel}(\varepsilon)$ from below and $S_p(\varepsilon)$ from above.

3.1 The main result

The quantities $S_p^{\parallel}(\varepsilon)$ and $S_p(\varepsilon)$ describe the statistical behavior of the increments of the velocity field when $\varepsilon \to 0$ and have been extensively investigated, see [9]. Both are expected to have a characteristic power-like behavior of the form (1). Our aim is to prove that, for the model described in the previous section with a suitable choice of γ, (1) holds true in the sense that the limit

$$\zeta_p = \lim_{\varepsilon \to 0} \frac{\log S_p(\varepsilon)}{\log \varepsilon} \tag{16}$$

exists (similarly for $S_p^{\parallel}(\varepsilon)$) and is computable. The following theorem gives us the necessary estimates from above and below, for a rather general measure γ. Then, in the next subsection, we make a choice of γ in order to obtain the classical multifractal scaling behaviour.

Theorem 1 *Assume that*

$$\gamma(U^p \ell T) < \infty$$

for every $p > 1$. Then, for any even integer $p > 1$ there exist two constants $C_p, c_p > 0$ such that

$$c_p \gamma\left[U^p \ell T 1_{\ell < \varepsilon}\right] \leq S_p^{\parallel}(\varepsilon) \leq S_p(\varepsilon) \leq C_p \gamma\left[U^p \left(\frac{\ell \wedge \varepsilon}{\ell}\right)^p \ell T\right] \tag{17}$$

for every $\varepsilon \in (0,1)$.

The proof of this result is long and reported in a separate section.

We would like to give a very rough heuristic that could explain this result. It must be said that we would not believe in this heuristic without the proof, since some steps are too vague (we have devised this heuristic only a posteriori).

What we are going to explain is that

$$\mathcal{W}(|u_{\text{single}}(x+\varepsilon e)-u_{\text{single}}(x)|^p)\sim U^p\left(\frac{\ell\wedge\varepsilon}{\ell}\right)^p\ell T.$$

This is the hard part of the estimate.

Let us discuss separately the case $\varepsilon>\ell$ from the opposite one. When $\varepsilon>\ell$ the vortex structure u_{single} is very thin compared to the length ε of observation of the displacement, thus the difference $u_{\text{single}}(x+\varepsilon e)-u_{\text{single}}(x)$ does not really play a role and the value of $\mathcal{W}(|u_{\text{single}}(x+\varepsilon e)-u_{\text{single}}(x)|^p)$ comes roughly from the separate contributions of $u_{\text{single}}(x+\varepsilon e)$ and $u_{\text{single}}(x)$, which are similar. Let us compute $\mathcal{W}(|u_{\text{single}}(x)|^p)$.

Consider the expression (6) which defines $u_{\text{single}}(x)$. Strictly speaking, consider the short-range case, otherwise there is a correction which makes even more difficult the intuition. Very roughly, $K_\ell(x-X_t)$ behaves like $\ell\cdot 1_{X_t\in B(x,\ell)}$, hence, even more roughly, $u_{\text{single}}(x)$ behaves like

$$u_{\text{single}}(x)\sim\frac{U}{\ell}\int_0^T 1_{X_t\in B(x,\ell)}dX_t.$$

When X_t is a smooth curve, say a straight line (at distances compared to ℓ), then the quantity $\int_0^T 1_{X_t\in B(x,\ell)}dX_t$ is roughly proportional to ℓ if X_t crosses $B(x,\ell)$, while it is zero otherwise. We assume the same result holds true when X_t is a Brownian motion. In addition, X_t crosses $B(x,\ell)$ with a probability proportional to the volume of the Wiener sausage, which is ℓT. Summarizing, we have

$$\int_0^T 1_{X_t\in B(x,\ell)}dX_t\sim\begin{cases}\ell & \text{with probability } \ell T\\ 0 & \text{otherwise}\end{cases}.$$

Therefore $u_{\text{single}}(x)$ takes rougly two values, U with probability ℓT and 0 otherwise. It follows that $\mathcal{W}(|u_{\text{single}}(x)|^p)\sim U^p\ell T$.

Consider now the case $\varepsilon<\ell$. The difference now is important. Since the gradient of K_ℓ is of order one, we have

$$u_{\text{single}}(x+\varepsilon e)-u_{\text{single}}(x)\sim\frac{U}{\ell^2}\varepsilon\int_0^T 1_{X_t\in B(x,\ell)}dX_t.$$

As above we conclude that $u_{\text{single}}(x+\varepsilon e)-u_{\text{single}}(x)$ takes rougly two values, $\varepsilon U/\ell$ with probability ℓT and 0 otherwise. It follows that $\mathcal{W}(|u_{\text{single}}(x)|^p)\sim U^p\left(\frac{\varepsilon}{\ell}\right)^p\ell T$. The intuitive argument is complete.

3.2 Example: the multifractal model

The most elementary idea to introduce a measure γ on the parameters is to take U and T as suitable powers of ℓ, thus prescribing a relation between the thickness ℓ and the length and intensity. That is, a relation of the form

$$d\gamma(U,\ell,T) = \delta_{\ell^h}(dU)\delta_{\ell^a}(dT)\ell^{-b}d\ell.$$

Moreover, we have to prescribe the distribution of ℓ itself, which could again be given by a power law $\ell^{-b}d\ell$. The K41 scaling described below is such an example.

Having in mind multi-scale phenomena related to intermittency, we consider a superposition of the previous scheme. Take a probability measure θ on an interval $I \subset \mathbb{R}_+$ (which measures the relative importance of the scaling exponent $h \in I$). Given two functions $a, b : I \to \mathbb{R}_+$ with $a(h) \leq 2$ (to ensure $\ell^2 \leq T$) consider the measure

$$d\gamma(U,\ell,T) = \int_I \left[\delta_{\ell^h}(dU)\delta_{\ell^{a(h)}}(dT)\ell^{-b(h)}d\ell\right]\theta(dh). \tag{18}$$

Then, according to Theorem 1 , we must evaluate

$$\begin{aligned}\gamma(U^p\ell T 1_{\ell<\varepsilon}) &= \int_{[0,\varepsilon]\times I} \ell^{hp+1+a(h)-b(h)}d\ell\theta(dh)\\ &= \int_I c_{p,h}\varepsilon^{hp+2+a(h)-b(h)}\theta(dh)\end{aligned} \tag{19}$$

while

$$\begin{aligned}\gamma\left[U^p\left(\frac{\varepsilon\wedge\ell}{\ell}\right)^p\ell T\right] &= \int_{[0,\varepsilon]\times I}\ell^{hp+1+a(h)-b(h)}d\ell\theta(dh) + \varepsilon^p\int_{[\varepsilon,+\infty)\times I}\ell^{(h-1)p+1+a(h)-b(h)}d\ell\theta(dh)\\ &= \int_I C_{p,h}\varepsilon^{hp+2+a(h)-b(h)}\theta(dh)\end{aligned}$$

As $\varepsilon \to 0$, by Laplace method, we get

$$\lim_{\varepsilon\to 0}\frac{\log S_p(\varepsilon)}{\log\varepsilon} = \inf_{h\in I}[hp+3-D(h)] = \zeta_p$$

with $D(h) = b(h) - a(h) + 1$. With this choice of γ we have recovered the scaling properties of the *multifractal model* of [11]. See [9] for a review.

Consider the specific choice $\theta(dh) = \delta_{1/3}(dh)$ and $a(1/3) = 2$, $b(1/3) = 4$. We have

$$\zeta_p = [hp+3-D(h)]_{h=1/3} = \frac{p}{3}.$$

This is the Kolmogorov K41 scaling law for 3d turbulence. The choice $a(1/3) = 2$, namely $T = \ell^2$, has the following geometrical meaning: the spatial displacement and the thickness of the structure are comparable (remember that the curves are Brownian), hence their shape is blob-like, as in the classical discussions of "eddies" around K41. The choice $b(1/3) = 4$, namely the measure $\ell^{-4}d\ell$ for the parameter ℓ, corresponds to the idea that the eddies are space-filling: it is easy to see that in a box of unit volume the number of eddies of size larger that ℓ is of order ℓ^{-3}. Finally, the choice $\theta(dh) = \delta_{1/3}(dh)$, namely $U = \ell^{1/3}$, is the key point that produces $\zeta_p = \frac{p}{3}$; one may attempt to justify it by dimensional analysis or other means, but it is essentially one of the issues that should require a better understanding.

Remark 6 *K41 can be obtain from this model also for other choices of the functions $a(h)$ and $b(h)$. For example, we may take $\theta(dh) = \delta_{1/3}(dh)$, $a(1/3) = 0$ and $b(1/3) = 2$. The value choice $\theta(dh) = \delta_{1/3}(dh)$ is again the essential point. The value $a(1/3) = 0$ means $T = 1$, hence the model is made of thin filaments of length comparable to the integral scale, instead of blob-like objects. The value $b(1/3) = 2$, namely the measure $\ell^{-2}d\ell$ for the parameter ℓ, is again space filling in view of the major length of the single vortex structures (recall also that the Hausdorff dimension of Brownian trajectories is 2). From this example we see that in the model described here it is possible to reconcile K41 with a geometry made of thin long vortexes.*

Remark 7 *Assume $\inf I > 0$ and, for instance, $\sup_I D < \infty$. Then $\lim_{p\to\infty} \zeta_p = +\infty$. In particular, $\zeta_p > 3$ for some even integer p. Since, by (19) and Laplace method,*

$$\gamma(U^p \ell T 1_{\ell<\varepsilon^{1-\beta}}) \leq C_{p,\beta}\varepsilon^{\zeta_p(1-\beta)}$$

for any $\beta > 0$, the condition of remark 4 is satisfied. Therefore the velocity field has a continuous modification.

4 Proof of Theorem 1

Let us introduce some objects related to the structure functions at the level of a single vortex filament. Let e be a given unit vector, the first element of the canonical basis, to fix the ideas. Since $u(x) = \mu\left(u_{\text{single}}^{\cdot}(x)\right)$, then

$$\langle u(\varepsilon e) - u(0), e\rangle = \mu\left[\langle \delta_\varepsilon u_{\text{single}}, e\rangle\right]$$

where

$$\delta_\varepsilon u_{\text{single}} := u_{\text{single}}^{\cdot}(\varepsilon e) - u_{\text{single}}^{\cdot}(0)$$

and similarly

$$|u(\varepsilon e) - u(0)| = \left|\mu\left[\delta_\varepsilon u_{\text{single}}\right]\right|.$$

Of major technical interest will be the quantities, of structure function type,

$$\mathcal{S}_p^e(\varepsilon) = \mathcal{W}(\langle \delta_\varepsilon u_{\text{single}}, e\rangle^p)$$

$$\mathcal{S}_p(\varepsilon) = \mathcal{W}(|\delta_\varepsilon u_{\text{single}}|^p).$$

They depend also on ℓ, T, U.

4.1 Lower bound

As a direct consequence of lemma 6 and lemma 2 we have:

Corollary 3 *If k is an odd positive integer, then*

$$S_k^{\|}(\varepsilon) = 0.$$

Moreover, for any even integer $p > 1$ there is a constant c_p such that

$$\begin{aligned} S_p^{\|}(\varepsilon) &\geq c_p \nu \left[\langle \delta_\varepsilon u_{single}, e \rangle^p \right] \\ &= c_p \gamma \left[\mathcal{S}_p^e(\varepsilon) \right]. \end{aligned}$$

If we prove that

$$\mathcal{S}_p^e(\varepsilon) \geq c'_p U^p \ell T$$

for every $\varepsilon \in (\ell, 1)$ and some constant $c'_p > 0$, then

$$\gamma \left[\mathcal{S}_p^e(\varepsilon) \right] \geq c''_p \gamma \left[U^p \ell T 1_{\ell < \varepsilon} \right]$$

for every $\varepsilon \in (0, 1)$ and some constant $c''_p > 0$, which implies the lower bound of theorem 1. We have

$$\mathcal{S}_p^e(\varepsilon) = \int \mathcal{W}_{x_0} \left[\langle \delta_\varepsilon u_{\text{single}}, e \rangle^p \right] dx_0.$$

Since

$$\langle \delta_\varepsilon u_{\text{single}}, e \rangle = \frac{U}{\ell^2} \int_0^T \langle K_\ell^e(\varepsilon e - X_t) - K_\ell^e(0 - X_t), dX_t \rangle$$

where

$$K_\ell^e(y) = K_\ell(y) \wedge e$$

by Burkholder-Davis-Gundy inequality, there is $c_p > 0$ such that

$$\mathcal{W}_{x_0} \left[\langle \delta_\varepsilon u_{\text{single}}, e \rangle^p \right] \geq c_p \frac{U^p}{\ell^{2p}} \mathcal{W}_{x_0} \left[(W_\varepsilon^e)^{p/2} \right]$$

where

$$W_\varepsilon^e = \int_0^T dt |K_\ell^e(\varepsilon e - X_t) - K_\ell^e(0 - X_t)|^2.$$

Here $K_\ell^e(y) = \langle K_\ell(y), e \rangle$. Therefore

$$\begin{aligned} \mathcal{S}_p^e(\varepsilon) &\geq c_p \frac{U^p}{\ell^{2p}} \int \mathcal{W}_{x_0} \left[(W_\varepsilon^e)^{p/2} \right] dx_0 \\ &= c_p \frac{U^p}{\ell^{2p}} \mathcal{W} \left[(W_\varepsilon^e)^{p/2} \right]. \end{aligned}$$

The proof of the theorem is then complete with the following basic estimate.

Lemma 8 *Given $p > 0$, there exist $c_p > 0$ such that for every $\ell^2 \leq T \leq 1$ and*

$$\varepsilon > \ell$$

we have

$$\mathcal{W}\left[(W_\varepsilon^e)^{p/2}\right] \geq c_p \ell^{2p} \ell T.$$

PROOF. Recall that

$$K_1(x) = -2Q\frac{x}{|x|^3} \quad \text{for } |x| \geq 1.$$

Consider the function

$$\begin{aligned} f(z,\alpha) &= K_1^e(z) - K_1^e(z+\alpha e) \\ &= (K_1(z) - K_1(z+\alpha e)) \wedge e \end{aligned}$$

defined for $z \in \mathbb{R}^3$ and $\alpha \geq 1$. Let $e^\perp$ be any unit vector orthogonal to e. We have

$$\begin{aligned} \left|f\left(e^\perp,\alpha\right)\right| &= 2Q\left|1 - \frac{\left(e^\perp + \alpha e\right) \wedge e}{\left|e^\perp + \alpha e\right|^3}\right| \\ &= 2Q\left|1 - \frac{1}{\left|e^\perp + \alpha e\right|^3}\right| \\ &\geq f\left(e^\perp, 1\right) = C_0 Q \end{aligned}$$

for $C_0 = 2\left(1 - 2^{-\frac{3}{2}}\right)$. Moreover, K_1 is globally Lipschitz, hence there is $L > 0$ such that

$$\begin{aligned} &\left|f\left(e^\perp + w,\alpha\right) - f\left(e^\perp,\alpha\right)\right| \\ &\leq \left|K_1(e^\perp + w) - K_1(e^\perp)\right| + \left|K_1(e^\perp + \alpha e + w) - K_1(e^\perp + \alpha e)\right| \leq L\left|w\right| \end{aligned}$$

for every $w \in \mathbb{R}^3$ and $\alpha \geq 1$. Therefore, there exists a ball $B(e^\perp, a) \subset B(0,2)$ and a constant $C_1 > 0$ such that when $z \in B(e^\perp, a)$ we have

$$|K_1^e(z) - K_1^e(z+\alpha e)| \geq C_1$$

uniformly in $\alpha \geq 1$. Then reintroducing the scaling factor ℓ we obtain that for $y \in B(\ell e^\perp, \ell a)$

$$|K_\ell^e(y) - K_\ell^e(y+\varepsilon e)| > C_1 \ell$$

uniformly in $\ell \in (0,1)$ and $\varepsilon > \ell$. Then we have

$$|K_\ell^e(0 - X_t) - K_\ell^e(\varepsilon e - X_t)| \geq C_1 \ell 1_{X_t \in B(-\ell e^\perp, \ell a)}.$$

Hence, if $\varepsilon > \ell$ we have

$$W_\varepsilon^e = \int_0^T dt |K_\ell^e(0 - X_t) - K_\ell^e(\varepsilon e - X_t)|^2$$
$$\geq C_1 \ell^2 L_{B(-\ell e^\perp, \ell a)}^T.$$

From the lower bound in (24) proved in the next section,

$$\mathcal{W}\left[(W_\varepsilon^e)^{p/2}\right] \geq c_p \ell^p \mathcal{W}\left[\left(L_{B(-\ell e^\perp, \ell a)}^T\right)^{p/2}\right] \geq c'_p \ell^{2p} \ell T.$$

The claim is proved. □

Remark 8 *With a bit more of effort is is also possible to prove the bound*

$$\mathcal{S}_p^e(\varepsilon) \geq c_p U^p \left(\frac{\varepsilon \wedge \ell}{\ell}\right)^p \ell T$$

valid for every $\varepsilon \in (0, 1)$ *(not only for* $\varepsilon > \ell$*). This would be the same as the upper bound, but we do not need it to prove that the behaviors as* $\varepsilon \to 0$ *of the upper and lower bound is the same.*

4.2 Upper bound

Lemma 9 *For every even p there exists a constant* $C_p > 0$ *such that*

$$S_p(\varepsilon) \leq C_p \nu\left[|\delta_\varepsilon u_{single}|^p\right]$$
$$= C_p \gamma\left[\mathcal{S}_p(\varepsilon)\right].$$

PROOF. Let $[\delta_\varepsilon u_{\text{single}}(\xi)]_i$ be the i-th component of $\delta_\varepsilon u_{\text{single}}(\xi)$. We have

$$S_p(\varepsilon) \leq C_p \sum_{i=1}^3 E\left[\left(\mu\left[[\delta_\varepsilon u_{\text{single}}(\xi)]_i\right]\right)^p\right]$$

and thus, by lemma 2,

$$S_p(\varepsilon) \leq C'_p \sum_{i=1}^3 \nu\left[[\delta_\varepsilon u_{\text{single}}(\xi)]_i^p\right]$$

which implies the claim. □

It is then sufficient to prove the bound

$$\mathcal{S}_p(\varepsilon) \leq C_p U^p \left(\frac{\varepsilon \wedge \ell}{\ell}\right)^p \ell T.$$

Again as above, We have

$$\mathcal{S}_p(\varepsilon) = \int \mathcal{W}_{x_0}\left[|\delta_\varepsilon u_{\text{single}}|^p\right] dx_0$$

where, by Burkholder-Davis-Gundy inequality, there is $C_p > 0$ such that

$$\mathcal{W}_{x_0}\left[|\delta_\varepsilon u_{\text{single}}|^p\right] \leq C_p \frac{U^p}{\ell^{2p}} \mathcal{W}_{x_0}\left[W_\varepsilon^{p/2}\right]$$

where

$$W_\varepsilon = \int_0^T dt |K_\ell(\varepsilon e - X_t) - K_\ell(0 - X_t)|^2.$$

Therefore

$$\begin{aligned}\mathcal{S}_p(\varepsilon) &\leq C_p \frac{U^p}{\ell^{2p}} \int \mathcal{W}_{x_0}\left[W_\varepsilon^{p/2}\right] dx_0 \\ &= C_p \frac{U^p}{\ell^{2p}} \mathcal{W}\left[W_\varepsilon^{p/2}\right].\end{aligned}$$

It is then sufficient to prove the bound

$$\mathcal{W}\left[W_\varepsilon^{p/2}\right] \leq C_p \ell^{2p} \left(\frac{\varepsilon \wedge \ell}{\ell}\right)^p \ell T.$$

For $\varepsilon > \ell$ it is not necessary to keep into account the closeness of εe to 0: each term in the difference of W_ε has already the necessary scaling. The hard part of the work has been done in lemma 3 above.

Lemma 10 *Given $p > 0$, there exist $C_p > 0$ such that for every $\ell^2 \leq T \leq 1$ and*

$$\varepsilon > \ell$$

we have

$$\mathcal{W}\left[W_\varepsilon^{p/2}\right] \leq C_p \ell^{2p} \ell T.$$

Proof. Since

$$W_\varepsilon \leq 2W(\varepsilon e) + 2W(0)$$

where $W(x) = \int_0^T |K_\ell(x - X_t)|^2 \, dt$, by lemma 3 we have the result. $\square$

For $\varepsilon \leq \ell$ need to extract a power of ε from the estimate of $\mathcal{W}\left[W_\varepsilon^{p/2}\right]$. We essentially repeat the multi-scale argument in the proof of lemma 3, with suitable modifications.

Lemma 11 *As in the previous lemma, when*

$$\varepsilon \leq \ell$$

we have

$$\mathcal{W}\left[W_\varepsilon^{p/2}\right] \leq C_p \varepsilon^p \ell^p \ell T.$$

PROOF. Since now ε is smaller that ℓ we bound $K_\ell(y) - K_\ell(z)$ for $|y - z| \leq \varepsilon$ as

$$|K_\ell(y) - K_\ell(z)| \leq C\varepsilon$$

if $|y| \leq 2\ell$. If $|y| > 2\ell$ then $|z| \geq \ell$ and using the explicit form of the kernel K_ℓ we have the bound

$$|K_\ell(y) - K_\ell(z)| \leq C\varepsilon \left(\frac{\ell}{u}\right)^3 \leq C\varepsilon \left(\frac{\ell}{u}\right)^2$$

where u is the minimum between $|y|$ and $|z|$ and in this case $\ell/u < 1$. Then, given a partition of $[2\ell, \Lambda]$, say $2\ell = \ell_0 < \ell_1 < ... < \ell_N = \Lambda$, as in the proof of lemma 3, we get

$$|K_\ell(y) - K_\ell(z)|^2 = |K_\ell(y) - K_\ell(z)|^2 \left(1_{|y|\leq 2\ell} + \sum_{i=1}^N 1_{\ell_{i-1}<|y|<\ell_i} + 1_{|y|>\Lambda}\right)$$

$$\leq C\varepsilon^2 1_{|y|\leq 2\ell} + C\sum_{i=1}^N \varepsilon^2 \left(\frac{\ell}{\ell_{i-1} - \ell}\right)^4 1_{\ell_{i-1}<|y|<\ell_i} + C\varepsilon^2 \left(\frac{\ell}{\Lambda}\right)^4 1_{|y|>\Lambda}$$

$$\leq C\varepsilon^2 1_{|y|\leq 2\ell} + 2^4 C\sum_{i=1}^N \varepsilon^2 \left(\frac{\ell}{\ell_{i-1}}\right)^4 1_{\ell_{i-1}<|y|<\ell_i} + C\varepsilon^2 \left(\frac{\ell}{\Lambda}\right)^4 1_{|y|>\Lambda}$$

where we have used the fact that $(u - \ell)^{-1} \leq 2/u$ for $u \geq 2\ell$. Then

$$W_\varepsilon \leq C\varepsilon^2 L^T_{B(0,2\ell)} + C\sum_{i=1}^N \varepsilon^2 \left(\frac{\ell}{\ell_{i-1}}\right)^4 L^T_{B(0,\ell_i)\setminus B(x,\ell_{i-1})} + C\varepsilon^2 \left(\frac{\ell}{\Lambda}\right)^4 L^T_{B(x,\Lambda)^c}$$

and arguing as in that proof

$$W_\varepsilon^{p/2} \leq C\varepsilon^p (L^T_{B(0,2\ell)})^{p/2} + C\varepsilon^p \left(\frac{\ell}{\Lambda}\right)^{2p} [L^T_{B(0,\Lambda)}]^{p/2} + C\varepsilon^p \left(\frac{\ell}{\Lambda}\right)^{2p} \left(L^T_{B(x,\Lambda)^c}\right)^{p/2}$$

$$+ C\varepsilon^p \sum_{i=1}^{N-1} \left[\left(\frac{\ell}{\ell_{i-1}}\right)^2 - \left(\frac{\ell}{\ell_i}\right)^2\right] \left(\frac{\ell}{\ell_i}\right)^p (L^T_{B(0,\ell_i)})^{p/2}.$$

Then, from lemma 14 in the form

$$\mathcal{W}\left[\left(L^T_{B(x,\lambda)}\right)^{p/2}\right] \leq C_p(\lambda \wedge \sqrt{T})^p \lambda (\lambda \vee \sqrt{T})^2.$$

and the obvious bound $L_B^T \le T$ for every Borel set B, we have

$$\mathcal{W}\left[W_\varepsilon^{p/2}\right] \le C\varepsilon^p \ell^p \ell T + C\varepsilon^p \left(\frac{\ell}{\Lambda}\right)^{2p} T^{p/2}$$

$$+C\varepsilon^p \sum_{i=1}^{N-1} \left[\left(\frac{\ell}{\ell_{i-1}}\right)^2 - \left(\frac{\ell}{\ell_i}\right)^2\right] \left(\frac{\ell}{\ell_i}\right)^p (\ell_i \wedge \sqrt{T})^p \ell_i (\ell_i \vee \sqrt{T})^2$$

and taking the limit as the partition gets finer:

$$\mathcal{W}\left[W_\varepsilon^{p/2}\right] \le C\varepsilon^p \ell^p \ell T + C\varepsilon^p \left(\frac{\ell}{\Lambda}\right)^{2p} T^{p/2}$$

$$+C\varepsilon^p \int_{2\ell}^{\Lambda} \left(\frac{\ell}{u}\right)^p (u \wedge \sqrt{T})^p (u \vee \sqrt{T})^2 u d\left[-\left(\frac{\ell}{u}\right)^2\right].$$

A direct computation of the integral as in the proof of lemma 3 and the limit as $\Lambda \to \infty$ complete the proof. □

5 Auxiliary results on Brownian occupation measure

In this section we prove the estimates on $\mathcal{W}[|L^T_{B(u,\ell)}|^{p/2}]$ which constitute the technical core of the previous sections. The literature on Brownian occupation measure is wide, so it is possible that results proved here are given somewhere or may be deduced from known results. However, we have not found the uniform estimates we needed, so we prefer to give full self-contained proofs for completeness. Of course, several ideas we use are inspired by the existing literature (in particular, a main source of inspiration has been [15]).

First, notice that $\mathcal{W}[|L^T_{B(u,\ell)}|^{p/2}]$ does not depend on u. But this is not of great help. When $p=2$, $\mathcal{W}[|L^T_{B(u,\ell)}|^{p/2}]$ can be explicitly computed:

$$\begin{aligned}
\mathcal{W}[L^T_{B(u,\ell)}] &= \int_{\mathbb{R}^3} dx_0 \int_0^T dt \int_{B(u,\ell)} dz\; p_t\left(z - x_0\right) \\
&= \int_{\mathbb{R}^3} dx_0 \int_0^T dt \int_{B(u,\ell)} dz\; p_t\left(x_0\right) \\
&= |B(u,\ell)| \int_0^T dt \int_{\mathbb{R}^3} p_t\left(x_0\right) dx_0 = \varpi \ell^3 T
\end{aligned}$$

where ϖ is a geometrical constant and $p_t\left(x\right)$ is the density of the 3D Brownian motion at time t. The estimate of $\mathcal{W}[|L^T_{B(u,\ell)}|^{p/2}]$ for general p requires much more work.

Let

$$\tau_{B(u,\ell)} = \inf\{t \geq 0 : X_t \in B(u,\ell)\}$$

the entrance time in $B(u,\ell)$ for the canonical process. We continue to denote by $\mathcal{W}_{x_0}$ the Wiener measure starting at x_0 and also the mean value with respect to it; similarly for $\mathcal{W}$, the σ-finite measure $d\mathcal{W}_{x_0}dx_0$.

Lemma 12 *For any $p > 0$, $T > 0$, $\ell > 0$, $x_0, u \in \mathbb{R}^3$, it holds that*

$$\begin{aligned}\mathcal{W}_{x_0}[\tau_{B(u,\ell/2)} \leq T/2]\mathcal{W}_0[|L^{T/2}_{B(0,\ell/2)}|^p] &\leq \mathcal{W}_{x_0}[|L^T_{B(u,\ell)}|^p] \\ &\leq \mathcal{W}_{x_0}[\tau_{B(u,\ell)} \leq T]\mathcal{W}_0[|L^T_{B(0,2\ell)}|^p]. \end{aligned} \tag{20}$$

Proof. Let us prove the upper bound. Set, for simplicity, $\tau = \tau_{B(u,\ell)} \wedge T$. When $\tau \leq t < T$ we have

$$X_t \in B(u,\ell) \Rightarrow X_t - X_\tau \in B(0,2\ell)$$

then

$$L^T_{B(u,\ell)} = \int_\tau^T 1_{X_t \in B(u,\ell)} dt \leq \int_\tau^T 1_{\{(X_t - X_\tau) \in B(0,2\ell)\}} dt = \int_0^{T-\tau} 1_{(X_{\tau+t} - X_\tau) \in B(0,2\ell)} dt$$

Then, taking into account that $\tau = T$ implies $L^T_{B(u,\ell)} = 0$,

$$L^T_{B(u,\ell)} \leq 1_{\{\tau < T\}} \int_0^T 1_{(X_{\tau+t} - X_\tau) \in B(0,2\ell)} dt$$

which gives us, using the strong Markov property,

$$\begin{aligned}\mathcal{W}_{x_0}|L^T_{B(u,\ell)}|^p &\leq \mathcal{W}_{x_0}\left[1_{\tau<T}\left(\int_0^T 1_{(X_{\tau+t}-X_\tau)\in B(0,2\ell)} dt\right)^p\right] \\ &= \mathcal{W}_{x_0}[\tau < T]\mathcal{W}_0\left[\left(\int_0^T 1_{X_t \in B(0,2\ell)} dt\right)^p\right] \\ &= \mathcal{W}_{x_0}[\tau_{B(u,\ell)} < T]\mathcal{W}_0\left[(L^T_{B(0,2\ell)})^p\right].\end{aligned}$$

The upper bound is proved.

Let us proceed with the lower bound. Let $\tau' = \tau_{B(u,\ell/2)} \wedge T$. When $\tau' \leq t \leq T$ we have

$$X_t - X_{\tau'} \in B(0,\ell/2) \Rightarrow X_t \in B(u,\ell)$$

then

$$\begin{aligned}
L^T_{B(u,\ell)} &\geq \int_{\tau'}^T 1_{X_t \in B(u,\ell)} dt \geq \int_{\tau'}^T 1_{X_t - X_{\tau'} \in B(0,\ell/2)} dt \\
&\geq \int_0^{T-\tau'} 1_{X_{\tau'+t} - X_{\tau'} \in B(0,\ell/2)} dt \\
&\geq 1_{\tau' \leq T/2} \int_0^{T-\tau'} 1_{X_{\tau'+t} - X_{\tau'} \in B(0,\ell/2)} dt \\
&\geq 1_{\tau' \leq T/2} \int_0^{T/2} 1_{X_{\tau'+t} - X_{\tau'} \in B(0,\ell/2)} dt.
\end{aligned}$$

Then, using again the strong Markov property with respect to the stopping time τ', we obtain

$$\mathcal{W}_{x_0}[|L^T_{B(u,\ell)}|^p] \geq \mathcal{W}_{x_0}[\tau_{B(u,\ell/2)} \leq T/2] \mathcal{W}_0[(L^{T/2}_{B(0,\ell/2)})^p]$$

and the proof is complete. □

Letting $p = 1$ in the previous lemma and using the scale invariance of BM we obtain an upper bounds for $\mathcal{W}_{x_0}[\tau_{B(u,\ell)} \leq T]$ as

$$\mathcal{W}_{x_0}[\tau_{B(u,\ell)} \leq T] = \mathcal{W}_{x_0/\sqrt{2}}[\tau_{B(u/\sqrt{2},\ell/\sqrt{2})} \leq T/2] \leq \frac{\mathcal{W}_{x_0/\sqrt{2}}[L^T_{B(u/\sqrt{2},\sqrt{2}\ell)}]}{\mathcal{W}_0[L^{T/2}_{B(0,\ell/\sqrt{2})}]}$$

and the corresponding lower bound for $\mathcal{W}_{x_0}[\tau_{B(u,\ell/2)} \leq T/2]$:

$$\mathcal{W}_{x_0}[\tau_{B(u,\ell/2)} \leq T/2] = \mathcal{W}_{\sqrt{2}x_0}[\tau_{B(\sqrt{2}u,\ell/\sqrt{2})} \leq T] \geq \frac{\mathcal{W}_{\sqrt{2}x_0}[L^T_{B(\sqrt{2}u,\ell/\sqrt{2})}]}{\mathcal{W}_0[L^T_{B(0,\sqrt{2}\ell)}]}$$

which leads to the following easy corollary:

Corollary 4 *For any $p > 0$, $T > 0$, $\ell > 0$, $x_0, u \in \mathbb{R}^3$, it holds that*

$$\begin{aligned}
\frac{\mathcal{W}_{\sqrt{2}x_0}[L^T_{B(\sqrt{2}u,\ell/\sqrt{2})}]\mathcal{W}_0[|L^{T/2}_{B(0,\ell/2)}|^p]}{\mathcal{W}_0[L^{T/2}_{B(0,\sqrt{2}\ell)}]} &\leq \mathcal{W}_{x_0}[|L^T_{B(u,\ell)}|^p] \\
&\leq \frac{\mathcal{W}_{x_0/\sqrt{2}}[L^T_{B(u/\sqrt{2},\sqrt{2}\ell)}]\mathcal{W}_0[|L^T_{B(0,2\ell)}|^p]}{\mathcal{W}_0[L^T_{B(0,\ell/\sqrt{2})}]} \qquad (21)
\end{aligned}$$

Lemma 13 *Given $\alpha > 0$ and $p > 0$, there exist constants $c, C > 0$ such that the following properties hold true: for every T, ℓ satisfying $T/\ell^2 \geq \alpha$ we have*

$$c\ell^p \leq \mathcal{W}_0 |L^T_{B(0,\ell)}|^{p/2} \leq C\ell^p$$

while if $T/\ell^2 \leq \alpha$

$$cT^{p/2} \leq \mathcal{W}_0 |L^T_{B(0,\ell)}|^{p/2} \leq CT^{p/2}.$$

Proof. Consider first $T/\ell^2 \geq \alpha$. In distribution

$$L^T_{B(0,\ell)} \stackrel{\mathcal{L}}{=} \ell^2 \int_0^{T/\ell^2} 1_{X_t \in B(0,1)} dt \tag{22}$$

and moreover we have that

$$c \leq \mathcal{W}_0 \left[\int_0^{T/\ell^2} 1_{X_t \in B(0,1)} dt \right]^p \leq C \tag{23}$$

uniformly in $T/\ell^2 \geq \alpha > 0$ (the constants depend on α and p). The lower bound is obtained by setting $T/\ell^2 = \alpha$ while the upper bound is given by the fact that

$$\mathcal{W}_0 \left[\int_0^{\infty} 1_{X_t \in B(0,1)} dt \right]^p < \infty$$

for any $p > 0$ (see for instance [13], section 3). From (22) and (23) we get the first claim of the lemma.

Next, if $T/\ell^2 \leq \alpha$, in distribution:

$$L^T_{B(0,\ell)} \stackrel{\mathcal{L}}{=} T \int_0^1 1_{X_t \in B(0,\ell/\sqrt{T})} dt$$

and

$$\mathcal{W}_0 |L^1_{B(0,1/\sqrt{\alpha})}|^p \leq \mathcal{W}_0 |L^1_{B(0,\ell/\sqrt{T})}|^p \leq 1$$

so the second claim is also proved. $\square$

Lemma 14 *Given $p > 0$, $\alpha > 0$, there are constants $c, C > 0$ such that, for $T \geq \alpha\ell^2$*

$$c\ell^p \ell T \leq \mathcal{W}[|L^T_{B(u,\ell)}|^{p/2}] \leq C\ell^p \ell T \tag{24}$$

while for $T \leq \alpha\varepsilon^2$

$$cT^{p/2}\varepsilon^3 \leq \mathcal{W}[|L^T_{B(u,\varepsilon)}|^{p/2}] \leq CT^{p/2}\varepsilon^3. \tag{25}$$

PROOF. Let us prove (24). Using lemma 13, equation (21) becomes

$$c\ell^{p-2}\mathcal{W}_{\sqrt{2}x_0}[L^T_{B(\sqrt{2}u,\ell/\sqrt{2})}] \le \mathcal{W}_{x_0}[|L^T_{B(u,\ell)}|^{p/2}] \le C\ell^{p-2}\mathcal{W}_{x_0/\sqrt{2}}[L^T_{B(u/\sqrt{2},\sqrt{2}\ell)}] \tag{26}$$

for two constants $c, C > 0$ (depending on α and p). Moreover, as we remarked at the beginning of the section,

$$\mathcal{W}[L^T_{B(u,\ell)}] = \varpi\ell^3 T.$$

Using this identity in (26), we get (24).

Now consider eq.(25). Assume $T \le \alpha\varepsilon^2$. Using eq.(21) and lemma 13 we have

$$cT^{\frac{p}{2}-1}\mathcal{W}_{\sqrt{2}x_0}[L^T_{B(\sqrt{2}u,\varepsilon/\sqrt{2})}] \le \mathcal{W}_{x_0}[|L^T_{B(u,\varepsilon)}|^{p/2}] \le CT^{\frac{p}{2}-1}\mathcal{W}_{x_0/\sqrt{2}}[L^T_{B(u/\sqrt{2},\sqrt{2}\varepsilon)}] \tag{27}$$

and (25) is a consequence of the identity

$$\mathcal{W}[L^T_{B(u,\varepsilon)}] = \varpi\varepsilon^3 T.$$

The proof is complete. □

The previous lemma solves the main problem posed at the beginning of the section. We prove also a related result in finite volume that we need for the complementary results of section 2.4. We limit ourselves to the upper bounds, for shortness.

Lemma 15 *Given $p > 0$, $\alpha > 0$, $u \in \mathbb{R}^3$, there is a constant $C > 0$ and a function $\theta(R)$ with $\lim_{R\to\infty}\theta(R) = 0$, such that, for $R > |u| + \ell$ and $T \ge \alpha\ell^2$*

$$\int_{|x_0|\ge R} \mathcal{W}_{x_0}[|L^T_{B(u,\ell)}|^{p/2}]dx_0 \le C\ell^p\ell T \exp\left(-\frac{R-(|u|+\ell)}{\sqrt{T}}\right)$$

while for $T \le \alpha\varepsilon^2$

$$\int_{|x_0|\ge R} \mathcal{W}_{x_0}[|L^T_{B(u,\varepsilon)}|^{p/2}]dx_0 \le CT^{p/2}\varepsilon^3 \exp\left(-\frac{R-(|u|+\varepsilon)}{\sqrt{T}}\right).$$

PROOF. Arguing as in the previous lemma, it is sufficient to prove that

$$\int_{|x_0|\ge R} \mathcal{W}_{x_0}[L^T_{B(u,\ell)}]dx_0 \le C\ell^3 T \exp\left(-\frac{R-(|u|+\ell)}{\sqrt{T}}\right).$$

We have

$$\begin{aligned}\int_{|x_0|\ge R} \mathcal{W}_{x_0}[L^T_{B(u,\ell)}]dx_0 &= \int_{B(0,R)^c} dx_0 \int_0^T dt \int_{B(u,\ell)} dz\, p_t(z-x_0)\\ &= \varpi\ell^3 T \int_0^T \frac{dt}{T}\int_{B(u,\ell)}\frac{dz}{|B(u,\ell)|}\int_{B(0,R)^c} dx_0\, p_t(z-x_0)\\ &\le \varpi\ell^3 T g(R;T,u,\ell)\end{aligned}$$

with (recall that $\ell \leq 1$)

$$g(R;T,u,\ell) = \sup_{t\in(0,T],|z|\leq|u|+\ell} \int_{B(0,R)^c} dx_0 \, p_t(z-x_0) \, .$$

Now (denoting by (W_t) a 3D Brownian motion)

$$\begin{aligned}
g(R;T,u,\ell) &= \sup_{t\in(0,T],|z|\leq|u|+\ell} P(|z-W_t| \geq R) \\
&\leq \sup_{t\in(0,T]} P(|W_t| \geq R - (|u|+\ell)) \\
&= \sup_{t\in(0,T]} P\left(|W_1| \geq \frac{R-(|u|+\ell)}{\sqrt{t}}\right) \\
&\leq P\left(|W_1| \geq \frac{R-(|u|+\ell)}{\sqrt{T}}\right)
\end{aligned}$$

This completes the proof. □

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