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ON THE UNIQUE SOLVABILITY OF SOME NONLINEAR STOCHASTIC PDES

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Abstract: The Cauchy problem for 1-dimensional nonlinear stochastic partial differential equations is studied. The uniqueness and existence of solutions in $\mathcal{H}_p^2(T)$-space are proved.

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1. Introduction

The aim of this paper is to prove the unique solvability of the following one-dimensional nonlinear stochastic partial differential equations(SPDEs):

$$du = [a(t,x,u)u'' + f(t,x)]\,dt + [\sigma^k(t)u' + g^k(t,x)]\,dw^k_t,\ \ u(0,\cdot) = u_0, \tag{1.1}$$

$$du = [a(t,x,u)u'' + f(t,x)]\,dt + g^k(t,x)\,dw^k_t,\ \ u(0,\cdot) = u_0, \tag{1.2}$$

under suitable conditions on a, σ, f, g^k and u_0.

There are not so many works on the solvability of nonlinear SPDEs except some special classes of nonlinear SPDEs.

In [8],[9], Krylov developed an L_p-theory of SPDEs including some nonlinear equations. The main assumption (Assumption 4.6 in [9]) was that the nonlinear terms are strictly subordinated to the *linear* main operators. Our equation (1.1) does not fall into this class because we have a nonlinearity in the main operator.

There are some other interesting classes of nonlinear SPDEs. Among others are semilinear equations and nonlinear equations of monotone type.

Semilinear equations have been extensively studied mostly using semigroup theory. They are evolution equations defined on some Hilbert spaces:

$$du = [Au + F(t,u)]\,dt + B(t,u)\,dW(t), \tag{1.3}$$

where A is the infinitesimal generator of a strongly continuous semigroup $S(t)$ and $W(t)$ is a Hilbert space-valued Wiener process. Under some conditions on nonlinear operators F and B like (local) Lipshitzness or dissipativeness, one obtains the solvability of (1.3). The idea is to convert (1.3) into an integral equation using $S(t)$ and then employ the fixed point type argument. We refer the readers to Da Prato-Zabczyk [7].

The theory for the nonlinear equations of monotone type has been developed by many mathematicians using the variational approach, Bensoussan-Temam [3],[4] (via time discretisation), Pardoux [16],[17], Krylov-Rozovskii [13], Ahmed [1] (via Galerkin approximation), Bensoussan [2] (via splitting up method). One tries to approximate the given nonlinear infinite-dimensional equations by a sequence of "solvable" ones. One then makes use of the monotonicity of the nonlinear operators to pass to the limit.

We note that (1.1) is not included in neither of these two theories.

We also mention recent two papers which consider equations similar to ours. Dalecky and Goncharuk [5] studied abstract quasilinear SPDEs which includes the following equation (in a very simplified form) as an example:

$$\frac{\partial u}{\partial t} = a(t, u(t))u'' + \int_D c(t, x, y, u(t, x))\xi(t, y)\, dy, \tag{1.4}$$

where $\xi(t, y)$ is a space-time white noise and D is a smooth bounded domain. But it is assumed that $a(t, \cdot)$ does not depend on the pointwise value of the solution u, but it is a functional of u. Da Prato and Tubaro [6] considered the following equation as an application of their theory:

$$du = a(t, x, u)u''\, dt + [b(t, x)u' + h(t, x)u]\, dw_t. \tag{1.5}$$

Under rather strong regularity assumptions on b and h, they proved that (1.5) has a unique solution. The key idea was to transform the stochastic PDE (1.5) to a deterministic equation for almost all $\omega \in \Omega$ using the stochastic characteristic method (see [18],[19] and references therein for this method). This allows them to use the Hölder space theory for linear PDEs with nonsmooth coefficients and nonlinear PDEs.

We also apply this transformation technique to reduce (1.1) to (1.2) (if σ^k's also depend on x, this transformation does not work in *our* approach). But note that after this reduction, we still have a *stochastic* PDE. Using the idea in [11], one can further transform (1.2) to a completely deterministic problem in the spirit of [6]. But this process requires a very strong regularity assumption on g. Thus we do not make this transformation, which is the main difference between our work and [6]. Then the Hölder space theory is not available for (1.2). This is the main difficulty in getting the solvability of nonlinear SPDEs like (1.2) in our approach. We first work in some stochastic Sobolev space setting which is available for us by the work of Krylov [8], [9], and then apply the embedding theorems for these spaces to control the Hölder norms of the solution and its derivatives.

Now we briefly describe our method and the organization of this paper.

Our approach heavily depends on Krylov's L_p-theory [8],[9]. Thus in Section 2, we briefly summarize some of the notations, definitions and theorems of this theory. In particular, the solvability of linear equations in $\mathcal{H}_p^n(\tau)$-spaces and the embedding theorems for these spaces enable us to prove in Section 3 the "local existence" of (1.2) by adapting the method of continuity argument for nonlinear PDEs (see §1.1 of [10]) to our stochastic PDE. In Section 4, we first prove the uniqueness of the solution of (1.2). Then we show that the a priori estimate for one-dimensional SPDEs with discontinuous coefficients established in [20] along with the local solvability and the uniqueness yields the main result of this paper, the unique solvability of (1.2). The unique solvability of (1.1) follows from this. In many places, we introduce a sequence of stopping times, which turns out to be a very useful technical tool.

Finally, we remark that we could consider slightly more general equations, for example, we can add lower order terms in (1.1), and get the same result under slightly weaker conditions. But we didn't do this since we tried to present the main idea as free of extra technical details as possible. We also think that one can get the unique solvability of (1.1) with $\sigma^k = \sigma^k(t,x)$ by a perturbation argument under some relatively mild assumption on the regularity of σ^k's. We will make this generalization elsewhere.

2. Notation and Preliminary Results

Here we present some of the notations, definitions and theorems of Krylov [8],[9]. We formulate them in a convenient form for our purpose. We also state an a priori estimate for one-dimensional SPDEs with discontinuous coefficients which is used in Section 4.

Let $\mathbb{R}^1$ be 1-dimensional Euclidean space, T a fixed positive number, $(\Omega, \mathcal{F}, P)$ a complete probability space, $(\{\mathcal{F}_t\}, t \geqslant 0)$ be an increasing filtration of σ-fields $\mathcal{F}_t \subset \mathcal{F}$ containing all P-null subsets of Ω, and $\mathcal{P}$ the predictable σ-field generated by $\{\mathcal{F}_t\}$. Let $\{w_t^k; k = 1, 2, \cdots\}$ be independent one-dimensional $\mathcal{F}_t$-adapted Wiener processes defined on $(\Omega, \mathcal{F}, P)$. For the above standard terminologies, we refer the readers to [12].

Let $\mathcal{D}$ be the set of real-valued Schwartz distributions defined on $C_0^\infty(\mathbb{R}^1)$. For given $p \in [2, \infty)$ and nonnegative real number n, define the space $H_p^n = H_p^n(\mathbb{R}^1)$ (called the space of Bessel potentials or the Sobolev space with fractional derivatives) as the space of all generalized functions u such that $(1-\Delta)^{n/2}u \in L_p = L_p(\mathbb{R}^1)$. For $u \in H_p^n$ and $\phi \in C_0^\infty$ by definition

$$
\begin{aligned}
(u, \phi) &= ((1-\Delta)^{n/2}u, (1-\Delta)^{-n/2}\phi) \\
&= \int_{\mathbb{R}^1} [(1-\Delta)^{n/2}u](x)(1-\Delta)^{-n/2}\phi(x)\, dx.
\end{aligned}
\tag{2.1}
$$

For $u \in H_p^n$ one introduces the norm

$$
\| u \|_{n,p} := \| (1-\Delta)^{n/2}u \|_p,
$$

where $\|\cdot\|_p$ is the norm in L_p. It is known that H^n_p is a Banach space with the norm $\|\cdot\|_{n,p}$ and the set C_0^∞ is dense in H^n_p.

Recall that for integers $n \geqslant 0$ the space H^n_p coincides with the Sobolev space $W^n_p = W^n_p(\mathbb{R}^1)$.

We apply the same definitions to l_2-valued functions g, where l_2 is the set of all real-valued sequences $g = \{g^k; k = 1, 2, \cdots\}$ with the norm defined by $|g|^2_{l_2} := \sum_k |g^k|^2$. Specifically,

$$\| g \|_p := \| \, |g|_{l_2} \, \|_p, \quad \| g \|_{n,p} := \| \, |(1-\Delta)^{n/2} g|_{l_2} \, \|_p \, .$$

Finally, for stopping times $\tau \leqslant T$, we denote $(\!(0,\tau]\!] = \{(\omega, t) : 0 < t \leqslant \tau(\omega)\}$ and

$$\mathbb{H}^n_p(\tau) := L_p((\!(0,\tau]\!], \mathcal{P}; H^n_p), \ \ \mathbb{H}^n_p(\tau, l_2) := L_p((\!(0,\tau]\!], \mathcal{P}; H^n_p(\mathbb{R}^1, l_2)).$$

If $n = 0$, we use $\mathbb{L}$ instead of $\mathbb{H}^0$. The norms in these spaces are defined in an obvious way.

For $n \in \mathbb{R}$ and

$$(f, g) \in \mathcal{F}^n_p(\tau) := \mathbb{H}^n_p(\tau) \times \mathbb{H}^{n+1}_p(\tau, l_2),$$

set

$$\| (f, g) \|_{\mathcal{F}^n_p(\tau)} := \| f \|_{\mathbb{H}^n_p(\tau)} + \| g \|_{\mathbb{H}^{n+1}_p(\tau, l_2)} \, .$$

Every stopping time τ appearing in this paper satisfies $\tau \leqslant T$ a.s.

Definition 2.1. For a $\mathcal{D}$-valued function $u \in \mathbb{H}^n_p(\tau)$, we write $u \in \mathcal{H}^n_p(\tau)$ if $u'' \in \mathbb{H}^{n-2}_p(\tau)$ and there exists $(f, g) \in \mathcal{F}^{n-2}_p(\tau)$ such that for any $\phi \in C_0^\infty$, with probability 1 the equality

$$(u(t,\cdot), \phi) = (u(0,\cdot), \phi) + \int_0^t (f(s,\cdot), \phi)\, ds + \sum_{k=1}^\infty \int_0^t (g^k(s,\cdot), \phi)\, dw^k_s \tag{2.2}$$

holds for all $t \leqslant \tau$ and $u(0,\cdot) \in L_p(\Omega, \mathcal{F}_0; H^{n-2/p}_p)$. We also define

$$\mathcal{H}^n_{p,0}(\tau) = \mathcal{H}^n_p(\tau) \cap \{u : u(0,\cdot) = 0\},$$

$$\| u \|_{\mathcal{H}^n_p(\tau)} = \| u'' \|_{\mathbb{H}^{n-2}_p(\tau)} + \| (f, g) \|_{\mathcal{F}^{n-2}_p(\tau)} + (E \| u(0,\cdot) \|^p_{H^{n-2/p}_p})^{1/p}. \tag{2.3}$$

Definition 2.2. For $u \in \mathcal{H}^n_p(\tau)$, if (2.2) holds, we write $f = \mathbb{D}u$, $g = \mathbb{S}u$ and we also write

$$u(t) = u(0) + \int_0^t \mathbb{D}u(s)\, ds + \int_0^t \mathbb{S}^k u(s)\, dw^k_s,$$

$$du = f dt + g^k dw^k_t, \ \ t \leqslant \tau.$$

We always understand equation like (1.1) in the sense of Definition 2.2, which means that we look for a function $u \in \mathcal{H}_p^n(\tau)$ such that

$$\mathbb{D}u = a(t,x,u)u'' + f,\ \mathbb{S}^k u = \sigma^k(t)u' + g^k.$$

Lemma 2.3. *The spaces $\mathcal{H}_p^n(\tau)$ and $\mathcal{H}_{p,0}^n(\tau)$ are Banach spaces with norm (2.3). In addition for $u \in \mathcal{H}_p^n(\tau)$*

$$\| u \|_{\mathbb{H}_p^n(\tau)} \leqslant N(T) \| u \|_{\mathcal{H}_p^n(\tau)},$$

$$E \sup_{t \leqslant \tau} \| u(t,\cdot) \|_{n-2,p}^p \leqslant N(T) \| u \|_{\mathcal{H}_p^n(\tau)}^p .$$

Proof. See Theorem 2.7 of [9]. □

Lemma 2.4. *If $u_j \in \mathcal{H}_p^n(T)$ and $\| u_j \|_{\mathcal{H}_p^n(T)} \leqslant K$, where K is a finite constant, then there exists a subsequence j' and a function $u \in \mathcal{H}_p^n(T)$ such that*

(i) $u_{j'}$, $u_{j'}(0,\cdot)$, $\mathbb{D}u_{j'}$, and $\mathbb{S}u_{j'}$ converge weakly to u, $u(0,\cdot)$, $\mathbb{D}u$, and $\mathbb{S}u$ in $\mathbb{H}_p^n(T)$, $L_p(\Omega, H_p^{n-2/p})$, $\mathbb{H}_p^{n-2}(T)$, and $\mathbb{H}_p^{n-1}(T, l_2)$ respectively;

(ii) $\| u \|_{\mathcal{H}_p^n(T)} \leqslant K$;

(iii) for any $\phi \in C_0^\infty$ and any $t \in [0,T]$ we have $(\phi, u_{j'}(t,\cdot)) \rightharpoonup (\phi, u(t,\cdot))$ weakly in $L_p(\Omega)$.

Proof. See Theorem 2.11 of [9]. □

On $[0,\tau]$, consider the following equation

$$du = [a(t,x)u'' + b(t,x)u' + c(t,x)u + f(t,x)]dt + g^k(t,x)dw_t^k, \quad (2.4)$$

where a, b, c, f are real-valued and g is a l_2-valued function defined for $\omega \in \Omega$, $t \geqslant 0$, $x \in \mathbb{R}^1$. We consider this equation in the sense of Definition 2.2.

We make the following assumptions.

Assumption 2.1 (uniform ellipticity). For any $\omega \in \Omega$, $t \geqslant 0$, $x \in \mathbb{R}^1$, we have

$$\lambda \leqslant a(\omega, t, x) \leqslant \Lambda,$$

where λ and Λ are fixed positive constants.

Assumption 2.2 (uniform continuity of a). For any $\varepsilon > 0$, there exists a $\kappa_\varepsilon > 0$ such that

$$|a(t,x) - a(t,y)| < \varepsilon$$

whenever $|x-y| < \kappa_\varepsilon$, $\omega \in \Omega$, $t \geqslant 0$.

Assumption 2.3. a, b, c are $\mathcal{P} \times \mathcal{B}(\mathbb{R}^1)$-measurable functions and for any $\omega \in \Omega$, $t \geqslant 0$, we have $a(t,\cdot), b(t,\cdot), c(t,\cdot) \in C^n(\mathbb{R}^1)$.

$f(t,x)$, $g(t,x)$ are predictable as functions taking values in H_p^n and $H_p^{n+1}(\mathbb{R}^1, l_2)$, respectively.

Assumption 2.4. For any $t \geqslant 0, \omega \in \Omega$,

$$\| a(t,\cdot) \|_{C^n} + \| b(t,\cdot) \|_{C^n} + \| c(t,\cdot) \|_{C^n} \leqslant K, \ (f(\cdot,\cdot), g(\cdot,\cdot)) \in \mathcal{F}^n_p(\tau).$$

Theorem 2.5. *Let Assumptions 2.1-2.4 be satisfied and let*

$$u_0 \in L_p(\Omega, \mathcal{F}_0; H^{n+2-2/p}_p).$$

Then the Cauchy problem for equation (2.4) on $[0,\tau]$ with the initial condition $u(0,\cdot) = u_0$ has a unique solution $u \in \mathcal{H}^{n+2}_p(\tau)$. For this solution, we have

$$\| u \|_{\mathcal{H}^{n+2}_p(\tau)} \leqslant N\{\| (f,g) \|_{\mathcal{F}^n_p(\tau)} + (E \| u_0 \|^p_{n+2-2/p,p})^{1/p}\},$$

where the constant N depends only on $n, \lambda, \Lambda, K, T$ and the function κ_ε.

Proof. See Theorem 4.1 of [9]. □

Theorem 2.6. *(Embedding theorem)*
If $p > 2$, $1/2 > \beta > \alpha > 1/p$, then for any function $u \in \mathcal{H}^n_p(\tau)$, we have $u \in C^{\alpha-1/p}([0,\tau], H^{n-2\beta}_p)$ (a.s.) and for any $t, s \leqslant \tau$,

$$E \| u(t,\cdot) - u(s,\cdot) \|^p_{n-2\beta,p} \leqslant N(\beta,p,T)|t-s|^{\beta p-1} \| u \|^p_{\mathcal{H}^n_p(\tau)},$$

$$E \| u(t,\cdot) \|^p_{C^{\alpha-1/p}([0,\tau],H^{n-2\beta}_p)} \leqslant N(\beta,\alpha,p,T) \| u \|^p_{\mathcal{H}^n_p(\tau)} .$$

Proof. See Theorem 6.2 of [9]. □

As a final preliminary result, we state a theorem recently established by the author [20]. In this theorem, we assume that a is $\mathcal{P} \times \mathcal{B}(\mathbb{R}^1)$-measurable and satisfies $\lambda \leqslant a \leqslant \Lambda$.

Theorem 2.7. *There exists a $p_0 = p_0(\lambda,\Lambda) > 2$ such that if $p \in [2, p_0)$ and if $(f,g) \in \mathcal{F}_p(\tau)$, $u_0 \in L_p(\Omega, \mathcal{F}_0; H^2_p)$, then there exists a unique solution $u \in \mathcal{H}^2_p(\tau)$ of*

$$du = [a(t,x)u'' + f(t,x)]\,dt + g^k(t,x)\,dw^k_t, \ u(0,\cdot) = u_0. \tag{2.5}$$

Moreover, u satisfies

$$\| u \|_{\mathcal{H}^2_p(\tau)} \leqslant N(p,\lambda,\Lambda,T)\{\| (f,g) \|_{\mathcal{F}_p(T)} + (E \| u_0 \|^p_{2,p})^{1/p}\}. \tag{2.6}$$

3. "Local" Solvability and Regularity

We consider the following 1-dimensional nonlinear SPDE:

$$du = [a(t,x,u)u''(t,x) + f(t,x)]\,dt + g^k(t,x)\,dw_t^k,\ \ u(0,\cdot) = u_0. \quad (3.1)$$

We make the following assumptions:

Assumption 3.1 (uniform ellipticity). For any $\omega \in \Omega$, $t \in [0,T]$, $x \in \mathbb{R}^1$, and $u \in \mathbb{R}$, we have

$$\lambda \leqslant a(\omega,t,x,u) \leqslant \Lambda,$$

where λ and Λ are fixed positive constants.

Assumption 3.2. $q \geqslant 2$ and $0 < \nu \leqslant 1$ are numbers satisfying the conditions:

$$1/2 > \beta > \alpha > 1/q \text{ and } \nu > 2\beta + 1/q,$$

for some α and β.

Assumption 3.3. For any $\omega \in \Omega$, a is Hölder continuous in t, continuously differentiable in x and twice continuously differentiable in u. Moreover,

$$\| a(\cdot,x,u) \|_{C^{\nu-2\beta-1/q}} + \| a(t,\cdot,u) \|_{C^1} + \| a(t,x,\cdot) \|_{C^2} \leqslant K_1,$$

where K_1 is independent of ω, t, x, u.

Assumption 3.4. Fix $p \in (2,p_0)$, where p_0 is from Theorem 2.7.

The initial condition u_0 is a function in $L_p(\Omega,\mathcal{F}_0;H_p^2) \cap L_q(\Omega,\mathcal{F};H_q^{2+\nu})$ and the functions $f(t,x)$, $g(t,x)$ are predictable as functions taking values in $L_p \cap H_q^\nu$ and $H_p^1(\mathbb{R}^1,l_2) \cap H_q^{1+\nu}(\mathbb{R}^1,l_2)$, respectively. Moreover,

$$\max(\| (f,g) \|_{\mathcal{F}_p(T)}, \| (f,g) \|_{\mathcal{F}_q^\nu(T)}) \leqslant K_2.$$

Theorem 3.1. *("Local existence" for (3.1))*

Let Assumptions 3.1-3.4 be satisfied. Then there exists a stopping time $\tau \leqslant T$ and $u \in \mathcal{H}_p^2(\tau)$ such that $E\,\tau > 0$ and u is a solution of (3.1) in $[0,\tau]$.

Proof. For simplicity of presentation, we assume that $u_0 = 0$. The general case is treated in a similar way.

Step 1 Consider

$$du = [\mu u'' + (1-\mu)a(t,x,u)u'' + f]dt + g^k dw_t^k,\ u(0) = 0. \quad (3.2)$$

If $\mu = 1$, there exists a unique solution $u_1 \in \mathcal{H}_p^2(T) \cap \mathcal{H}_q^{2+\nu}(T)$ by Theorem 2.5. Moreover by Theorem 2.5, 2.6 and Assumption 3.3,

$$\begin{aligned} E \| u_1(t,\cdot) \|^q_{C^{\alpha-\frac{1}{q}}([0,T],H_q^{2+\nu-2\beta})} &\leqslant N(\lambda,\Lambda,T) \| u_1 \|^q_{\mathcal{H}_q^{2+\nu}(T)} \\ &\leqslant N(\lambda,\Lambda,T) \| (f,g) \|^q_{\mathcal{F}_q^\nu(T)} \leqslant N(\lambda,\Lambda,T,K_2). \end{aligned} \quad (3.3)$$

Let M be a positive number. By (3.3), Assumption 3.2 and Sobolev embedding $H_q^{2+\nu-2\beta} \subset C^{2+\nu-2\beta-\frac{1}{q}}$, there exists a positive constant γ and a stopping time $\tau_1 \leqslant T$ such that

$$\| u_1(\cdot,\cdot) \|_{C^{\gamma,2+2\gamma}([0,\tau_1]\times\mathbb{R}^1)} \leqslant M.$$

Notice that we can take τ_1 as close to T as we wish by taking larger M. Define

$\Phi(\delta) := \{w \in \mathcal{H}^2_{p,0}(\tau_1) : \| (w-u_1)(\omega,\cdot,\cdot) \|_{C^{\gamma,2+2\gamma}([0,\tau_1]\times\mathbb{R}^1)} < \delta \text{ a.s.}\}$ and then consider the following problem with respect to function v with zero initial condition, where w is a given function from $\Phi(\delta)$ and μ is a number in $[0,1]$.

$$\begin{aligned} dv = {} & [\mu v'' + (1-\mu)\{a(t,x,u_1)v'' + a_u(t,x,u_1)u_1''v\} \\ & + (1-\mu)\{a(t,x,w)w'' - a(t,x,u_1)w'' - a_u(t,x,u_1)u_1''w\} + f]\,dt \\ & + g^k\,dw_t^k. \end{aligned} \tag{3.4}$$

If we define $\widetilde{a}(t,x) := \mu + (1-\mu)\,a(t,x,u_1(t,x))$, $\widetilde{c}(t,x) := (1-\mu)\,a_u(t,x,u_1(t,x))\,u_1''(t,x)$, $\widetilde{f} := (1-\mu)\{a(t,x,w)w'' - a(t,x,u_1)w'' - a_u(t,x,u_1)u_1''w\} + f$ and $\widetilde{g^k} := g^k$, then, it is easy to check that $\widetilde{a}$, $\widetilde{c}$, $\widetilde{f}$ and $\widetilde{g^k}$ satisfy Assumption 2.1-2.4 with $n=0$. For example, we show that $a_u(\cdot,\cdot,u_1)u_1''w \in \mathbb{L}_p(\tau_1)$:

By Lemma 4.2(i) of [9], it suffices to check that $u_1'' \in \mathbb{L}_p(\tau_1)$ and $a_u(\cdot,\cdot,u_1)\,w \in C^0$. Now by our assumption, both inclusions are obvious.

Thus, by Theorem 2.5, (3.4) or

$$dv = [\widetilde{a}(t,x)v'' + \widetilde{c}(t,x)v + \widetilde{f}(t,x)]\,dt + \widetilde{g^k}(t,x)\,dw_t^k$$

has a unique solution $v \in \mathcal{H}^2_{p,0}(\tau_1)$ in $[0,\tau_1]$.

In this way, we can define an operator

$$\Psi_\mu : \Phi(\delta) \to \mathcal{H}^2_{p,0}(\tau_1), \;\; \Psi_\mu(w) = v.$$

We claim that for some $\delta > 0$ and all μ which are sufficiently close to 1, Ψ_μ is a contraction mapping from $\Phi(\delta)$ into $\Phi(\delta)$ in the metric $\sup_{\omega\in\Omega} \| (w-\widetilde{w})(\omega,\cdot,\cdot) \|_{C^{\gamma,2+2\gamma}([0,\tau_1]\times\mathbb{R}^1)}$.

Now note that

$$\begin{aligned} d(v-u_1) = {} & [\mu(v-u_1)'' + (1-\mu)\,a(t,x,u_1)\,(v-u_1)'' \\ & + (1-\mu)a_u(t,x,u_1)u_1''(v-u_1) + (1-\mu)\{a(t,x,w)w'' - a(t,x,u_1)w'' \\ & - a_u(t,x,u_1)u_1''w + a_u(t,x,u_1)u_1''u_1\} + (1-\mu)\{a(t,x,u_1)u_1'' - u_1''\}]\,dt. \end{aligned} \tag{3.5}$$

For fixed ω, (3.5) is a uniformly parabolic equation in $[0,\tau_1(\omega)]\times\mathbb{R}^1$. By Assumption 3.2 and the definition of $\Phi(\delta)$, $a(t,x,u_1)$, $a_u(u_1)$, $a(t,x,w)$, w'', u_1'' and their products which appear in (3.5) are Hölder continuous in (t,x). Thus by Theorem 5.1 of [15] p320, $v-u_1\in C^{1+\gamma,2+2\gamma}([0,\tau_1(\omega)]\times\mathbb{R}^1)$ and for a constant $N=N(\lambda,\Lambda,T,M,K_1)$,

$$\begin{aligned}&\| v-u_1\|_{C^{\gamma,2+2\gamma}([0,\tau_1]\times\mathbb{R}^1)}\\&\leqslant N\| a(\cdot,\cdot,w)w''-a(\cdot,\cdot,u_1)w''-a_u(\cdot,\cdot,u_1)u_1''w\\&+a_u(\cdot,\cdot,u_1)u_1''u_1\|_{C^{\gamma,2\gamma}([0,\tau_1]\times\mathbb{R}^1)}\\&+N(1-\mu)\| a(\cdot,\cdot,u_1)u_1''-u_1''\|_{C^{\gamma,2\gamma}([0,\tau_1]\times\mathbb{R}^1)}\,.\end{aligned}\tag{3.6}$$

Now we show that

$$\begin{aligned}&\| a(\cdot,\cdot,w)w''-a(\cdot,\cdot,u_1)w''-a_u(\cdot,\cdot,u_1)u_1''w\\&+a_u(\cdot,\cdot,u_1)u_1''u_1\|_{C^{\gamma,2\gamma}([0,\tau_1]\times\mathbb{R}^1)}\leqslant\chi(\delta)\delta,\end{aligned}$$

where $\chi(\delta)$ depends only on $\delta,K_1,\lambda,\Lambda,T$ and $\chi(\delta)\searrow 0$ as $\delta\searrow 0$. Indeed, by the fundamental theorem of calculus,

$a(t,x,w)w''-a(t,x,u_1)w''-a_u(t,x,u_1)u_1''w+a_u(t,x,u_1)u_1''u_1$
$=a(t,x,w)w''-a(t,x,u_1)u_1''-a(t,x,u_1)(w-u_1)''-a_u(t,x,u_1)u_1''(w-u_1)=\{\int_0^1 a(t,x,\theta w+(1-\theta)u_1)\,d\theta\}(w-u_1)''+\{\int_0^1 a_u(t,x,\theta w+(1-\theta)u_1)\,(\theta w''+(1-\theta)u_1'')\,d\theta\}(w-u_1)-a(t,x,u_1)(w-u_1)''-a_u(t,x,u_1)u_1''(w-u_1)$.

We can further transform the above equation to get
$[\int_0^1\{a(t,x,\theta w+(1-\theta)u_1)-a(t,x,u_1)\}\,d\theta]\,(w-u_1)''+[\int_0^1\{a_u(t,x,\theta w+(1-\theta)u_1)\,(\theta w''+(1-\theta)u_1'')-a_u(t,x,u_1)u_1''\}\,d\theta](w-u_1)$.

After this transformation one can easily see that our claim follows from Assumption 3.2 and the definition of $\Phi(\delta)$.

Thus, by first taking δ sufficiently small and then choosing μ close to 1 according to δ , we can make

$$\| v-u_1\|_{C^{\gamma,2+2\gamma}([0,\tau_1]\times\mathbb{R}^1)}<\delta.$$

This shows that $\Psi_\mu:\Phi(\delta)\to\Phi(\delta)$. Now we proceed to show the contraction.

If we let $v=\Psi_\mu w$, $\widetilde{v}=\Psi_\mu\widetilde{w}$ for $w,\widetilde{w}\in\Phi(\delta)$,

$$\begin{aligned}&d(v-\widetilde{v})=[\mu(v-\widetilde{v})''+(1-\mu)a(t,x,u_1)(v-\widetilde{v})''\\&+(1-\mu)a_u(t,x,u_1)(v-\widetilde{v})+(1-\mu)\{a(t,x,w)w''-a(t,x,\widetilde{w})\widetilde{w}''\\&-a(t,x,u_1)(w''-\widetilde{w}'')-a_u(t,x,u_1)u_1''(w-\widetilde{w})\}]\,dt.\end{aligned}\tag{3.7}$$

By the same argument as above, one can show that

$$\| v-\widetilde{v}\|_{C^{\gamma,2+2\gamma}([0,\tau_1]\times\mathbb{R}^1)}\leqslant N\chi(\delta)\| w-\widetilde{w}\|_{C^{\gamma,2+2\gamma}([0,\tau_1]\times\mathbb{R}^1)}\,.$$

Now by reducing δ and also μ according to the new δ if necessary, we can make $\Psi_\mu : \Phi(\delta) \to \Phi(\delta)$ and $N\chi(\delta) < 1/2$. We fix δ and corresponding μ. We denote such a μ by μ_0.

We have just finished the proof of the claim. It implies that there exists a $u_{\mu_0} \in \Phi(\delta)$ such that $\Psi_{\mu_0}(u_{\mu_0}) = u_{\mu_0}$ (this statement requires a proof because we didn't show that Ψ_{μ_0} is a contraction in the metric of $\mathcal{H}^2_p(\tau_1)$). Indeed, if we define a sequence $u^n_{\mu_0} \in \Phi(\delta)$ inductively by $u^0_{\mu_0} = u_1$ and $u^{n+1}_{\mu_0} = \Psi_{\mu_0}(u^n_{\mu_0})$ for $n \geqslant 0$, then $\{u^n_{\mu_0}\}$ is a Cauchy sequence in the metric $\sup_{\omega\in\Omega} \| u^n_{\mu_0} - u^m_{\mu_0} \|_{C^{\gamma,2+2\gamma}([0,\tau_1]\times\mathbb{R}^1)}$. By Arzela-Ascoli theorem, there exists a function $u_{\mu_0} = u_{\mu_0}(\omega,t,x)$ such that $u^n_{\mu_0}(\omega,\cdot,\cdot) \to u_{\mu_0}(\omega,\cdot,\cdot)$ in say, $C^{0,2}$ and $\| u_{\mu_0} \|_{C^{\gamma,2+2\gamma}([0,\tau_1]\times\mathbb{R}^1)} < \infty$.

For any $\phi \in C^\infty_0$, $(\omega,t) \in (\!(0,\tau]\!]$,

$$
\begin{aligned}
&\int_{\mathbb{R}^1} u^{n+1}_{\mu_0}(t,x)\phi(x)\,dx \\
&= \int_0^t \int_{\mathbb{R}^1} [\mu_0 (u^{n+1}_{\mu_0})''(s,x)\phi(x) + (1-\mu_0)a(s,x,u_1(s,x))(u^{n+1}_{\mu_0})''(s,x)\phi(x) \\
&+ (1-\mu_0)a_u(s,x,u_1(s,x))\,u_1''(s,x)u^{n+1}_{\mu_0}(s,x)\phi(x) \\
&+ (1-\mu_0)\{a(s,x,u^n_{\mu_0})(u^n_{\mu_0})'' - a(s,x,u_1)(u^n_{\mu_0})'' - a_u(s,x,u_1)u_1''u^n_{\mu_0}\}\phi(x) \\
&+ f(s,x)\phi(x)]\,dxds + \int_0^t \int_{\mathbb{R}^1} g^k(s,x)\phi(x)\,dxdw^k_s.
\end{aligned}
\tag{3.8}
$$

Now by passing to the limit in this equation, we get after some cancellations,

$$
\begin{aligned}
&\int_{\mathbb{R}^1} u_{\mu_0}(t,x)\phi(x)\,dx \\
&= \int_0^t \int_{\mathbb{R}^1} \{\mu_0 u''_{\mu_0}(s,x)\phi(x) + (1-\mu_0)a(s,x,u_{\mu_0})(s,x))\,u''_{\mu_0}(s,x)\phi(x) \\
&+ f(s,x)\phi(x)]\,dxds + \int_0^t \int_{\mathbb{R}^1} g^k(s,x)\phi(x)\,dxdw^k_s.
\end{aligned}
\tag{3.9}
$$

Therefore, u_{μ_0} is a solution of

$$
du = [\mu_0\, u'' + (1-\mu_0)a(t,x,u_{\mu_0})u'' + f]\,dt + g^k\,dw^k_t \tag{3.10}
$$

in $[0,\tau_1]$ in the sense of distribution (see Definition 3.6 of [9]). But since $\mu_0 + (1-\mu_0)a(t,x,u_{\mu_0})$ satisfies Assumption 2.1-2.4 with $n = 0$, there exists a unique $\mathcal{H}^2_p(\tau_1)$-solution $\widetilde{u}_{\mu_0}$ of the above equation.

Then $u_{\mu_0} - \widetilde{u}_{\mu_0}$ satisfies

$$
d(u_{\mu_0} - \widetilde{u}_{\mu_0}) = [\mu_0\,(u_{\mu_0} - \widetilde{u}_{\mu_0})'' + (1-\mu_0)a(t,x,u_{\mu_0})(u_{\mu_0} - \widetilde{u}_{\mu_0})'']\,dt,
$$

$$(u_{\mu_0} - \widetilde{u}_{\mu_0})(0) = 0$$

in the sense of distribution. But since for fixed ω, $\widetilde{u}_{\mu_0}$ is Hölder continuous in (t,x) by the embedding theorem, $u_{\mu_0} - \widetilde{u}_{\mu_0}$ is bounded. Then by the uniqueness of parabolic PDE, $u_{\mu_0} = \widetilde{u}_{\mu_0}$ a.s. and u_{μ_0} is indeed a $\mathcal{H}^2_p(\tau_1)$-solution of (3.10), which is equivalent to $\Psi_{\mu_0}(u_{\mu_0}) = u_{\mu_0}$. This finishes Step 1.

Step 2 Now we take u_{μ_0} instead of u_1 and proceed as before. That is, as a next step, we want to show that we can solve (3.2) for $\mu \in [2\mu_0 - 1, \mu_0]$ starting from μ_0. To do this, we have to make sure that all the constants we claimed "under control" (e.g. M) are independent of this inductive step. Since we don't know $\| u_{\mu_0} \|_{C^{\gamma,2+2\gamma}([0,\tau_1]\times\mathbb{R}^1)} \leqslant M$ (we fixed M in Step 1), we are forced to choose another stopping time $\tau_2 \leqslant \tau_1$ such that

$$\| u_{\mu_0}(\cdot,\cdot) \|_{C^{\gamma,2+2\gamma}([0,\tau_2]\times\mathbb{R}^1)} \leqslant M. \tag{3.11}$$

By the construction in Step 1, $u_{\mu_0} \in \Phi(\delta)$ and

$$\| u_{\mu_0}(\cdot,\cdot) \|_{C^{\gamma,2+2\gamma}([0,\tau_1]\times\mathbb{R}^1)} \leqslant M + \delta \text{ a.s.}$$

So we see that we can indeed choose a *nonzero* stopping time $\tau_2 \leqslant \tau_1$ such that (3.11) is satisfied.

We define

$$\Phi(\delta) := \{ w \in \mathcal{H}^2_{p,0}(\tau_2) : \| (w - u_{\mu_0})(\omega,\cdot,\cdot) \|_{C^{\gamma,2+2\gamma}([0,\tau_2]\times\mathbb{R}^1)} < \delta \; a.s. \}.$$

Then everything in Step 1 goes through with almost no change. After getting τ_2 and $u_{2\mu_0-1}$, if $2\mu_0 - 1 > 0$, we repeat the argument to get τ_3 and $u_{3\mu_0-2}$. In this way, we arrive at $\mu = 0$ in a finite number of steps. Now the theorem is proved.

□

Theorem 3.2. *Suppose Assumptions 3.1-3.4 are satisfied. Let $u \in \mathcal{H}^2_p(\tau)$ be a solution of (3.1) in $[0,\tau]$. Then the following hold:*

(i) There exists an increasing sequence of subsets Ω_m of Ω such that $\Omega_m \uparrow \Omega$.

(ii) For each m, there exists a $v_m \in \mathcal{H}^2_p(\tau) \cap \mathcal{H}^{2+\nu}_q(\tau)$ such that $I_{\Omega_m} v_m = I_{\Omega_m} u$ in $\mathcal{H}^2_p(\tau)$.

(iii) v_m satisfies

$$\max(\| v_m \|_{\mathcal{H}^2_p(\tau)}, \| v_m \|_{\mathcal{H}^{2+\nu}_q(\tau)}) \leqslant N(m),$$

where $N(m) = N(m, \lambda, \Lambda, T, p, K_1, K_2)$.

Proof. By Theorem 2.7,

$$\| u \|_{\mathcal{H}^2_p(\tau)} \leqslant N(\lambda, \Lambda, T, p, K_2).$$

Then, by Theorem 2.6, u satisfies

$$E \parallel u(t,\cdot) \parallel^p_{C^{\alpha-1/p}([0,\tau],H^{2-2\beta}_p)} \leqslant N(\lambda,\Lambda,T,p,K_2,\alpha,\beta).$$

Since $2-2\beta>1$ and $1/p<1/2$, $H^{2-2\beta}_p \subset C^{1/2}$ by the Sobolev embedding theorem. Thus, we conclude that

$$E \parallel u(t,\cdot) \parallel^p_{C^{\alpha-1/p}([0,\tau],C^{1/2})} \leqslant N(\lambda,\Lambda,T,p,K_2,\alpha,\beta). \tag{3.12}$$

Now by the Chebychev inequality and (3.12),

$$\begin{aligned} &P\{\omega\in\Omega : \parallel u(t,\cdot) \parallel^p_{C^{\alpha-1/p}([0,\tau],C^{1/2})} > k^p\} \\ &\leqslant \frac{1}{k^p} E \parallel u(t,\cdot) \parallel^p_{C^{\alpha-1/p}([0,\tau],C^{1/2})} \leqslant \frac{N}{k^p}. \end{aligned}$$

Therefore, if we define

$$\widetilde{\Omega_k} := \{\omega\in\Omega : \parallel u(t,\cdot) \parallel_{C^{\alpha-1/p}([0,\tau],C^{1/2})} \leqslant k\},$$

then $\widetilde{\Omega_k} \uparrow \Omega$ as $k\to\infty$.

We consider the following equation for v:

$$dv = [a(t,x,I_{\widetilde{\Omega_k}}u)v'' + f]\,dt + g^k\,dw^k_t,\ \ v(0)=u_0. \tag{3.13}$$

It is easy to see that one can apply Theorem 2.5 (n=0) to the above equation and we obtain a unique solution $\widetilde{v_k} \in \mathcal{H}^2_p(\tau)\cap\mathcal{H}^2_q(\tau)$. Moreover

$$\parallel \widetilde{v_k} \parallel_{\mathcal{H}^2_q(\tau)} \leqslant N(k,\lambda,\Lambda,T,K_1,K_2). \tag{3.14}$$

Now we claim that $\widetilde{v_k}(\omega,\cdot,\cdot) = u(\omega,\cdot,\cdot)$ for almost all $\omega\in\widetilde{\Omega_k}$. Indeed, $\widetilde{v_k} - u \in \mathcal{H}^2_{p,0}(\tau)$ satisfies

$$\begin{aligned} d(\widetilde{v_k}-u) &= [a(t,x,I_{\widetilde{\Omega_k}}u)\widetilde{v_k}'' - a(t,x,u)u'']dt \\ &= I_{\widetilde{\Omega_k}} a(t,x,u)(\widetilde{v_k}-u)''dt + I_{\widetilde{\Omega_k^c}}[a(t,x,0)\widetilde{v_k}'' - a(t,x,u)u'']dt. \end{aligned}$$

For fixed $\omega\in\widetilde{\Omega_k}$, $\widetilde{v_k}-u \in H^{1,2}_p(\tau(\omega))$ is a generalized solution of a uniformly elliptic parabolic PDE with Hölder continuous coefficient. Since the initial condition is zero, our claim follows from the uniqueness of the solution.

We apply Theorem 2.6 once more to $\widetilde{v_k}$. Then by (3.14), we get

$$E \parallel \widetilde{v_k}(t,\cdot) \parallel^q_{C^{\alpha-\frac{1}{q}}}([0,\tau],H^{2-2\beta}_q) \leqslant N(k,\lambda,\Lambda,T,K_1,K_2).$$

But since $H^{2-2\beta}_q \subset C^{2-2\beta-\frac{1}{q}}$ and $2-2\beta-\frac{1}{q}>1$ by Assumption 3.2, we have for some $\gamma>0$

$$E \parallel \widetilde{v_k}(t,\cdot) \parallel^q_{C^\gamma([0,\tau],C^{1+\gamma})} \leqslant N(k,\lambda,\Lambda,T,K_1,K_2). \tag{3.15}$$

By Chebychev inequality and (3.15),

$$P\{\omega \in \Omega : \| \widetilde{v_k}(t,\cdot) \|^q_{C^\gamma([0,\tau],C^{1+\gamma})} > l^q\} \leqslant \frac{N(k)}{l^q}.$$

We define

$$\widetilde{\Omega_{k,l}} := \{\omega \in \Omega : \| \widetilde{v_k}(t,\cdot) \|_{C^\gamma([0,\tau],C^{1+\gamma})} \leqslant l\}.$$

Recall that $\widetilde{v_k}(\omega,\cdot,\cdot) = u(\omega,\cdot,\cdot)$ for almost all $\omega \in \widetilde{\Omega_k}$. Thus,

$$\begin{aligned}
&P\{\omega \in \Omega : \| u(t,\cdot) \|_{C^\gamma([0,\tau],C^{1+\gamma})} \leqslant l\} \\
&\geqslant P\{\omega \in \Omega : \widetilde{v_k}(\omega,\cdot,\cdot) = u(\omega,\cdot,\cdot) \text{ and } \| \tilde{v_k}(t,\cdot) \|_{C^\gamma([0,\tau],C^{1+\gamma})} \leqslant l\} \\
&\geqslant 1 - P(\widetilde{\Omega_k}^c) - P(\widetilde{\Omega_{k,l}}^c) \geqslant 1 - \frac{N}{k^p} - \frac{N(k)}{l^q}.
\end{aligned}$$

Note that we can make $\frac{N}{k^p} + \frac{N(k)}{l^q}$ as small as we like by taking large enough k and l. We first take large enough k and fix. Then choose sufficiently large l according to k. ¿From the above construction, we see that if we define

$$\Omega_m := \{\omega \in \Omega : \| u(t,\cdot) \|_{C^\gamma([0,\tau(\omega)],C^{1+\gamma})} \leqslant l(m)\},$$

for some increasing sequence $l(m)$, then $\Omega_m \uparrow \Omega$.

Now consider the following equation for v:

$$dv = [a(t,x,I_{\Omega_m} u)v'' + f]\,dt + g^k\,dw^k_t, \ v(0) = u_0. \tag{3.16}$$

¿From the definition of Ω_m, we see that the above equation satisfies the assumptions of Theorem 2.5 with $n = 0$ and $n = \nu$. Thus, there exists a unique solution $v_m \in \mathcal{H}^2_p(\tau) \cap \mathcal{H}^{2+\nu}_q(\tau)$ of (3.16) and v_m satisfies

$$\max(\| v_m \|_{\mathcal{H}^2_p(\tau)}, \| v_m \|_{\mathcal{H}^{2+\nu}_q(\tau)}) \leqslant N(m,\lambda,\Lambda,p,T,K_1,K_2).$$

One can also show that $v_m(\omega,\cdot,\cdot) = u(\omega,\cdot,\cdot)$ for almost all $\omega \in \Omega_m$ by arguing as before. The theorem is proved. □

4. Uniqueness and "Global" Solvability

Throughout this section, Assumptions 3.1-3.4 are in force.

Lemma 4.1. *Let u_1 and u_2 be two solutions of*

$$du = [a(t,x,u)u'' + f]\,dt + g^k\,dw^k_t, \ u(0) = u_0 \tag{4.1}$$

in $\mathcal{H}^2_p(\tau)$ for some stopping time τ. Suppose also that there are two stopping times $\tau_1, \tau_2 \leqslant \tau$ such that $u_1 = u_2$ as functions in $\mathcal{H}^2_p(\tau_1)$ and $\mathcal{H}^2_p(\tau_2)$. Then $u_1 = u_2$ as functions in $\mathcal{H}^2_p(\tau_1 \vee \tau_2)$.

Proof. This is obvious. □

Theorem 4.2. *(Uniqueness)*
Let u_1, u_2 be two solutions of (4.1) in $\mathcal{H}_p^2(\tau)$ for some stopping time τ. Then $u_1 = u_2$ in $\mathcal{H}_p^2(\tau)$.

Proof. It suffices to show that there exists a sequence of stopping times τ_n such that $\tau_n \uparrow \tau$ a.s. and $u_1 = u_2$ as functions in $\mathcal{H}_p^2(\tau_n)$.

Define

$$\sigma_n(\omega) := \inf\{0 \leqslant t \leqslant \tau(\omega) : \max_{i=1,2} \| u_i(\cdot,\cdot) \|_{C^{\alpha-1/p,1/2}([0,t]\times\mathbb{R}^1)} \geqslant n\}.$$

Since by Theorem 2.6,

$$E \| u_i(s,\cdot) \|^p_{C^{\alpha-1/p}([0,\tau],H_p^{2-2\beta})} \leqslant N(\beta,\alpha,p,T) \| u_i \|^p_{\mathcal{H}_p^2(\tau)},$$

and $2-2\beta-1/p > 1/2$, $\sigma_n \uparrow \tau$ a.s. as $n \to \infty$.

Then, for a.e. $(\omega,s) \in (\!(0,\sigma_n]\!]$, $a(s,x,u_i) = a(s,x,u_i(\omega,s,x)) =: \widetilde{a}(\omega,s,x)$ satisfies Assumptions 2.1-2.4 with $n=0$. By applying Theorem 2.5, we conclude that $u_i \in \mathcal{H}_q^2(\sigma_n)$ for each n.

We again use the embedding theorem and repeat the above argument:

Since $H_q^{2-2\beta} \subset C^{2-2\beta-1/q}$,

$$E \| u_i(s,\cdot) \|^q_{C^{\alpha-1/q}([0,\sigma_n],C^{2-2\beta-\frac{1}{q}})} \leqslant N(\beta,\alpha,p,T) \| u_i \|^q_{\mathcal{H}_q^2(\sigma_n)}.$$

This implies that if we define

$$\sigma_{nk}(\omega) := \inf\{0 \leqslant t \leqslant \sigma_n(\omega) : \max_{i=1,2} \| u_i(\cdot,\cdot) \|_{C^{\alpha-1/p,2-2\beta-1/q}([0,t]\times\mathbb{R}^1)} \geqslant k\},$$

then $\sigma_{nk} \uparrow \sigma_n$ a.s. as $k \to \infty$. For $(\omega,s) \in (\!(0,\sigma_{nk}]\!]$, $\widetilde{a}$ now satisfies Assumptions 2.1-2.4 with $n=\nu$. Thus $u_i \in \mathcal{H}_q^{2+\nu}(\sigma_{nk})$ for each n,k.

We repeat the whole process one more time. If we define

$$\sigma_{nkl}(\omega) := \inf\{0 \leqslant t \leqslant \sigma_{nk}(\omega) : \max_{i=1,2} \| u_i(\cdot,\cdot) \|_{C^{\alpha-\frac{1}{q},2+\nu-2\beta-\frac{1}{q}}([0,t]\times\mathbb{R}^1)} \geqslant l\},$$

then $\sigma_{nkl} \uparrow \sigma_{nk}$ a.s. as $l \to \infty$. Recall that $2+\nu-2\beta-1/q > 2$. For each n,k,l, the function u_1-u_2 satisfies the following equation in $\mathcal{H}_{q,0}^{2+\nu}(\sigma_{nkl})$:

$$\begin{aligned} d(u_1-u_2) &= [a(t,x,u_1)u_1'' - a(t,x,u_2)u_2'']\,dt \\ &= [\int_0^1 \frac{d}{d\theta}\{a(t,x,\theta u_1 + (1-\theta)u_2)\,(\theta u_1'' + (1-\theta)u_2'')\}\,d\theta]\,dt. \end{aligned} \tag{4.2}$$

The last term is equal to $[\{\int_0^1 a(t,x,\theta u_1+(1-\theta)u_2)\,d\theta\}(u_1-u_2)'' + \{\int_0^1 a_u(\theta u_1+(1-\theta)u_2)(\theta u_1''+(1-\theta)u_2'')\,d\theta\}(u_1-u_2)]\,dt$.

But by the definition of σ_{nkl},

$\int_0^1 a(t,x,\theta u_1+(1-\theta)u_2)\,d\theta$ and $\int_0^1 a_u(\theta u_1+(1-\theta)u_2)(\theta u_1''+(1-\theta)u_2'')\,d\theta$ satisfy Assumptions 2.1-2.4 for $n=0$. Hence by the unique solvability, $u_1 = u_2$ as functions in $\mathcal{H}^2_q(\sigma_{nkl})$ and so as functions in $\mathcal{H}^2_p(\sigma_{nkl})$.

Now we define a new sequence of stopping times as follows:

$$\tau_n := \sup_{i \leqslant n} \eta_i,$$

where $\{\eta_i\}$ is a rearrangement of $\{\sigma_{nkl}\}$. Clearly, $\tau_n \uparrow \tau$ and by Lemma 4.1, $u_1 = u_2$ as functions in $\mathcal{H}^2_p(\tau_n)$. The theorem is proved. □

Lemma 4.3. *Suppose that there are two stopping times τ_1, τ_2 such that $u_1 \in \mathcal{H}^2_p(\tau_1)$ and $u_2 \in \mathcal{H}^2_p(\tau_2)$ are solutions of (4.1) in $[0,\tau_1]$ and $[0,\tau_2]$, respectively.*

Then there exist a solution $u \in \mathcal{H}^2_p(\tau_1 \vee \tau_2)$ of (4.1) in $[0, \tau_1 \vee \tau_2]$.

Proof. We define

$$u(\omega,t,x) = \begin{cases} u_1(\omega,t,x) & \text{if } t \leqslant \tau_1(\omega) \\ u_2(\omega,t,x) & \text{if } t \leqslant \tau_2(\omega). \end{cases}$$

By Theorem 4.2, $u_1 = u_2 \in \mathcal{H}^2_p(\tau_1 \wedge \tau_2)$, so u is well-defined. It is clear that u is a solution in the sense of distribution. Thus it suffices to show that $u \in \mathcal{H}^2_p(\tau_1 \vee \tau_2)$. For that, we check:

$E \int_0^{\tau_1 \vee \tau_2} \| u(t,\cdot) \|^p_{2,p} \; dt = E(\int_0^{\tau_1} \| u(t,\cdot) \|^p_{2,p} \; dt)\, I_{\tau_1 \geqslant \tau_2}$
$+E(\int_0^{\tau_2} \| u(t,\cdot) \|^p_{2,p} \; dt)\, I_{\tau_1 < \tau_2} \leqslant \| u_1 \|^p_{\mathbb{H}^2_p(\tau_1)} + \| u_2 \|^p_{\mathbb{H}^2_p(\tau_2)}$.

Thus, $u \in \mathbb{H}^2_p(\tau_1 \vee \tau_2)$. Similarly, one can show that $(a(t,x,u)u'' + f, g) \in \mathcal{F}_p(\tau_1 \vee \tau_2)$. The lemma is proved. □

We present the main theorem of this paper.

Theorem 4.4. *(Unique solvability)*

There exists a unique solution $u \in \mathcal{H}^2_p(T)$ of (3.1) and u satisfies

$$\| u \|_{\mathcal{H}^2_p(T)} \leqslant N \, \| (f,g) \|_{\mathcal{F}_p(T)} + N(E \, \| u_0 \|^p_{2,p})^{1/p},$$

where $N = N(\lambda, \Lambda, p, T)$.

Proof. We only need to prove the existence. The uniqueness and the estimate follow from Theorem 4.2 and Theorem 2.7.

Step 1 Let $\Pi :=$ {nonzero stopping time τ : there exists a solution in $\mathcal{H}^2_p(\tau)$ of (3.1) in $[0,\tau]$}. Then by Theorem 3.1, Π is nonempty. Observe that if $\tau_1, \tau_2 \in \Pi$, then $\tau_1 \vee \tau_2 \in \Pi$.

Now let $\bar{\tau} := \bigvee_{\tau \in \Pi} \tau$. We claim that $\bar{\tau} \in \Pi$:

First we show that $\bar{\tau}$ is a stopping time. Let $r := \sup_{\tau \in \Pi}\{E\,\tau\}$. Since $0 \leqslant \tau \leqslant T$, r is a finite number. And there exists a sequence $\tau_n \in \Pi$ such that $E\,\tau_n \to r$ as $n \to \infty$. By the above observation, without loss of generality, we may assume that $\tau_n \uparrow$. Define $\tau' := \lim_{n\to\infty} \tau_n$. $\tau' \leqslant T$

is a stopping time and by the Monotone convergence theorem $E\,\tau' = r$. Clearly $\tau' \leqslant \bar\tau$ a.s. But $\tau' \geqslant \bar\tau$ a.s., for otherwise there exists a $\tau \in \Pi$ such that $E\,\tau > E\,\tau' = r$. This is a contradiction to the definition of supremum. Thus, $\tau' = \bar\tau$ a.s. and $\bar\tau$ is a stopping time.

Now we identify τ' and $\bar\tau$. If we let u_n be the corresponding solution to τ_n, then by the uniqueness,

$$u_n = u_m \text{ in } \mathcal{H}^2_p(\tau_n \wedge \tau_m) = \mathcal{H}^2_p(\tau_{n\wedge m}). \tag{4.3}$$

We extend u_n such that $u_n = 0$ between τ_n and $\bar\tau$. Then by Theorem 2.7,

$$\| u_n \|_{\mathcal{H}^2_p(\tau_n)} = \| u_n \|_{\mathcal{H}^2_p(\bar\tau)} \leqslant N,$$

where N is a constant independent of n. By Lemma 2.4, there exists a subsequence n_k and $u \in \mathcal{H}^2_p(\bar\tau)$ such that $u_{n_k} \rightharpoonup u$ weakly in $\mathcal{H}^2_p(\bar\tau)$. Combining this with (4.3), we get

$$u = u_n \text{ in } \mathcal{H}^2_p(\tau_n). \tag{4.4}$$

On the other hand, for any $\phi \in C_0^\infty$, $t \in [0,\bar\tau]$ and $m \in \mathbb{N}$, with probability one,

$$\begin{aligned}(u_{n_k}((t-1/m)_+ \wedge \tau_{n_k}, \cdot), \phi) &= (u_0, \phi)\\ &+ \int_0^{(t-1/m)_+\wedge\tau_{n_k}} (a(s,x,u_{n_k}) u''_{n_k}(s,\cdot), \phi)\, ds\\ &+ \int_0^{(t-1/m)_+\wedge\tau_{n_k}} (f(s,\cdot),\phi)\, ds + \int_0^{(t-1/m)_+\wedge\tau_{n_k}} (g^k(s,\cdot),\phi)\, dw^k_s.\end{aligned} \tag{4.5}$$

First using (4.4), we replace u_{n_k} in (4.5) by u and then let $k \to \infty$. Since $\tau_{n_k} \to \bar\tau$ and u is Hölder continuous in (t,x) by Theorem 2.6, we get

$$\begin{aligned}(u((t-1/m)_+ \wedge \bar\tau, \cdot), \phi) &= (u_0,\phi) + \int_0^{(t-1/m)_+\wedge\bar\tau} (a(s,x,u) u''(s,\cdot), \phi)\, ds\\ &+ \int_0^{(t-1/m)_+\wedge\bar\tau} (f(s,\cdot),\phi)\, ds + \int_0^{(t-1/m)_+\wedge\bar\tau} (g^k(s,\cdot),\phi)\, dw^k_s.\end{aligned} \tag{4.6}$$

Now we let $m \to \infty$ in (4.6) and conclude that u is a solution of (3.1) in $[0,\bar\tau]$ in the sense of distribution. Since we already know that $u \in \mathcal{H}^2_p(\bar\tau)$, $\bar\tau \in \Pi$. This proves the claim.

Step 2 Suppose that $\bar\tau \neq T$. Otherwise we are done.

In Step 1, we showed that there exists a solution u in $\mathcal{H}^2_p(\bar\tau)$ of (3.1). Then by Theorem 3.2, there exists a set $\Omega' \subset \Omega$ and a function $v \in \mathcal{H}^2_p(\bar\tau) \cap \mathcal{H}^{2+\nu}_q(\bar\tau)$ such that $P(\Omega') > 1/2$ and $v(\omega,\cdot,\cdot) = u(\omega,\cdot,\cdot)$ as functions in $H^2_p(\bar\tau(\omega))$ for almost all $\omega \in \Omega'$.

By the definition of $\mathcal{H}^2_p(\bar\tau)$ and $\mathcal{H}^{2+\nu}_q(\bar\tau)$,

$$dv = \mathbb{D}v\,dt + \mathbb{S}^k v\,dw^k_t,$$

where $\mathbb{D}v \in \mathbb{L}_p(\bar\tau)\cap\mathbb{H}^\nu_q(\bar\tau)$ and $\mathbb{S}v \in \mathbb{H}^1_p(\bar\tau, l_2)\cap\mathbb{H}^{1+\nu}_q(\bar\tau, l_2)$.

We define $\widetilde{f} := (\mathbb{D}v - \Delta v)I_{t\leqslant\bar\tau}$ and $\widetilde{g}^k := (\mathbb{S}^k v)I_{t\leqslant\bar\tau}$. Notice that $\widetilde{f} \in \mathbb{L}_p(T)\cap\mathbb{H}^\nu_q(T)$, $\widetilde{g} \in \mathbb{H}^1_p(T, l_2)\cap\mathbb{H}^{1+\nu}_q(T, l_2)$ and on $[0,\bar\tau]$ v satisfies

$$dy = (\Delta y + \widetilde{f})\,dt + \widetilde{g}^k\,dw^k_t,\ y(0) = 0. \tag{4.7}$$

By Theorem 2.5, (4.7) has a unique solution $y \in \mathcal{H}^2_p(T)\cap\mathcal{H}^{2+\nu}_q(T)$. The difference $y - v$ satisfies the heat equation on $[0,\bar\tau]$ with zero initial condition, thus it follows that $y(\omega,\cdot,\cdot) = v(\omega,\cdot,\cdot)$ on $[0,\bar\tau]$ a.s.

Now we are ready to solve the following equation "locally":

$$\begin{cases} dz = [a(t+\bar\tau, x, z)z'' + f(t+\bar\tau, x)]\,dt + g^k(t+\bar\tau, x)\,dw^k_{t+\bar\tau} \\ z(0,\cdot) = v(\bar\tau,\cdot). \end{cases} \tag{4.8}$$

As we showed above, since v has an extension as a function in $\mathcal{H}^2_p(T)\cap\mathcal{H}^{2+\nu}_q(T)$, we may assume that $v(\bar\tau,\cdot) \in L_p(\Omega, \mathcal{F}_{\bar\tau}; H^2_p)\cap L_q(\Omega, \mathcal{F}_{\bar\tau}; H^{2+\nu}_q)$. Actually we understand the initial condition exactly in this sense in Krylov's L_p-theory.

We apply Theorem 3.1 to (4.8). It is easy to check that the coefficient and the data satisfy the assumption of Theorem 3.1. The only thing is that we have $w^k_{t+\bar\tau}$ as a driving noise. But the proofs of Theorem 2.5 and Theorem 2.6 go without any change in this situation. Thus, by repeating almost word by word, we can prove the local existence for (4.8). This implies that there exists a stopping time σ such that $E\sigma > 0$ and if we define

$$U := \begin{cases} u & \text{if } t \leqslant \bar\tau(\omega) \\ z(\cdot - \bar\tau, \cdot) & \text{if } \bar\tau(\omega) < t \leqslant (\bar\tau + \sigma)(\omega) \end{cases}$$

then U is a well-defined function in $\mathcal{H}^2_p(\bar\tau + \sigma)$ and U satisfies (3.1) on $[0, \bar\tau + \sigma I_{\Omega'}]$ by the construction and the definition of v and z.

Thus $\bar\tau + \sigma I_{\Omega'} \in \Pi$. But since we may assume that $E\sigma I_{\Omega'} > 0$, $E(\bar\tau + \sigma I_{\Omega'}) > E\bar\tau$, which is a contradiction to the definition of $\bar\tau$. Therefore, $\tau \equiv T$ and the theorem is proved. □

Remark. The choices $\nu = 1$, $q = 4$ satisfy the Assumption 3.2. In this case, one can give a simpler proof for the existence. Indeed, if we let $v := u - \int_0^t g^k(s,\cdot)\,dw^k_s$, then v satisfies for each fixed ω

$$\frac{\partial v}{\partial t} = \widetilde{a}(\cdot,\cdot,v)v + f + \int_0^t \Delta g^k(s,\cdot)\,dw^k_s,$$

which is a deterministic partial differential equation.

Finally, we prove the unique solvability of (1.1).

Theorem 4.5. *Suppose that $\sigma(t) = \{\sigma^k(t), k \geqslant 1\}$ is an l_2-valued function, where σ^k's are $\mathcal{P}$-measurable and $\alpha(t) := \frac{1}{2}\sum_k(\sigma^k)^2(t)$ is Hölder continuous such that $\sup_{\omega\in\Omega} \| \alpha(\cdot) \|_{C^{\nu-2\beta-1/q}([0,T])} \leqslant K_3$ for some constant K_3. Assume also that $\lambda \leqslant a(t,x,u) - \alpha(t) \leqslant \Lambda$ for all $t \in [0,T], x \in \mathbb{R}^1, u \in \mathbb{R}$.*

Then there exists a unique solution $u \in \mathcal{H}_p^2(T)$ of

$$\begin{cases} du = [a(t,x,u)u'' + f(t,x)]\,dt + [\sigma^k(t)u' + g^k(t,x)]\,dw_t^k \\ u(0,\cdot) = u_0. \end{cases} \tag{4.9}$$

and

$$\| u \|_{\mathcal{H}_p^2(T)} \leqslant N \| (f,g) \|_{\mathcal{F}_p(T)} + N(E \| u_0 \|_{2,p}^p)^{1/p},$$

where $N = N(\lambda, \Lambda, p, T, K_3)$.

Proof. Suppose that u is a generalized solution i.e., solution in the sense of distribution (see Definition 3.6 of [9]) of (4.9). We define a process x_t and a function v by

$$x_t := \int_0^t \sigma^k(s)\,dw_s^k, \quad v(t,x) := u(t, x - x_t).$$

Then by the Itô-Wentzell formula (see [14]), we get

$$dv = [a(t,x-x_t,v)v'' - \alpha(t)v'' + f(t,x-x_t) - (g^k)'(t,x-x_t)\sigma^k(t)]\,dt \\ + g^k(t,x-x_t)\,dw_t^k.$$

Let $\widetilde{a}(t,x,v) := a(t,x-x_t,v) - \alpha(t)$, $\widetilde{f}(t,x) := f(t,x-x_t) - (g^k)'(t,x-x_t)\,\sigma^k(t)$, and $\widetilde{g}^k(t,x) := g^k(t,x-x_t)$. Then v satisfies

$$\begin{cases} dv = [\widetilde{a}(t,x,v)v'' + \widetilde{f}(t,x)]\,dt + \widetilde{g}^k(t,x)\,dw_t^k \\ v(0,\cdot) = u_0. \end{cases} \tag{4.10}$$

By Lemma 3.7 of [9], (4.9) and (4.10) are equivalent, that is, (4.9) holds (in the sense of distribution) if and only if (4.10) holds (in the sense of distribution). Thus, it suffices to show that (4.10) has a unique solution in $\mathcal{H}_p^2(T)$. Because of x_t in the definition of $\widetilde{a}$, the Hölder norm of $\widetilde{a}$ with respect to t is not uniform in ω. So we cannot apply Theorem 4.4 directly to (4.10). Therefore, we proceed in the following way.

Recall that $x.$ is Hölder continuous with exponent say $1/4$ (it follows for example from the Kolmogorov test and Burkholder-Davis-Gundy inequality). Let $\Omega^n := \{\omega \in \Omega : \| x.(\omega) \|_{C^{\nu-2\beta-1/q}([0,T])} \leqslant n\}$. Then $\Omega^n \uparrow \Omega$ as $n \to \infty$. Consider

$$\begin{cases} dv^n = [\widetilde{a}^n(t,x,v^n)(v^n)'' + \widetilde{f}(t,x)]\,dt + \widetilde{g}^k(t,x)\,dw_t^k \\ v^n(0,\cdot) = u_0, \end{cases} \tag{4.11}$$

where $\widetilde{a}^n(t,x,v^n) := a(t, x - I_{\Omega^n} x_t, v^n) - \alpha(t)$.

Now we check Assumptions 3.1, 3.3 and 3.4 for $\widetilde{a}^n$, $\widetilde{f}$ and $\widetilde{g}$:

(i) $\lambda \leqslant \widetilde{a}^n(t,x,u) \leqslant \Lambda$,
(ii) $\| \widetilde{a}^n(\cdot,x,u) \|_{C^{\nu-2\beta-1/q}} + \| \widetilde{a}^n(t,\cdot,u) \|_{C^1} + \| \widetilde{a}^n(t,x,\cdot) \|_{C^2} \leqslant \widetilde{K_1}$ for $\widetilde{K_1}$ independent of ω, t, x, u,
(iii) $\max(\| (\widetilde{f},\widetilde{g}) \|_{\mathcal{F}_p(T)}, \| (\widetilde{f},\widetilde{g}) \|_{\mathcal{F}_q^\nu(T)}) \leqslant \widetilde{K_2}$.

(i) and (iii) are immediate from our assumptions and definitions. For (ii), we only need to show that $\| \widetilde{a}^n(\cdot,x,u) \|_{C^{\nu-2\beta-1/q}} \leqslant K$ for some constant K independent of ω, t, x, u ($\delta := \nu - 2\beta - 1/q$):

$$\begin{aligned}
&\sup_{\omega,x,u} \| \widetilde{a}^n(\cdot,x,u) \|_{C^\delta} \leqslant \sup_{\omega,x,u} \| a(\cdot, x - I_{\Omega^n} x_\cdot, u) \|_{C^\delta} + \sup_{\omega} \| \alpha(\cdot) \|_{C^\delta} \\
&\leqslant \sup_{\omega,x,u} \| a(\cdot,x,u) \|_{C^\delta} + \sup_{\omega,x,u} \| a(t,\cdot,u) \|_{C^1} \sup_{\omega\in\Omega^n} \| x_\cdot \|_{C^\delta} \\
&+ \sup_{\omega} \| \alpha(\cdot) \|_{C^\delta} \leqslant K_1 + nK_1 + K_3 =: K
\end{aligned}$$

Thus we can apply Theorem 4.4 and obtain a unique solution v^n of (4.11) and

$$\| v^n \|_{\mathcal{H}_p^2(T)} \leqslant N \| (\widetilde{f},\widetilde{g}) \|_{\mathcal{F}_p(T)} + N(E \| u_0 \|_{2,p}^p)^{1/p}, \tag{4.12}$$

where $N = N(\lambda, \Lambda, p, T)$ is independent of n.

Now we claim that if $n \leqslant m$, then

$$v^n(\omega,\cdot,\cdot) = v^m(\omega,\cdot,\cdot)$$

for almost all $\omega \in \Omega^n = \Omega^n \cap \Omega^m$. Suppose the claim is true. Then we get the unique solvability of (4.10). Indeed, by (4.12) and Lemma 2.4, (after passing to a subsequence) v^n converges weakly to some $\bar{v} \in \mathcal{H}_p^2(T)$. But from our claim, $\bar{v}(\omega,\cdot,\cdot) = v^n(\omega,\cdot,\cdot)$ for almost all $\omega \in \Omega^n$. Then we can pass to the limit in (4.11) by an argument similar to the one given at the end of Step 1 of the proof of Theorem 4.4 and we conclude that $\bar{v}$ is a solution of (4.10). The uniqueness follows from Theorem 4.2 (for the proof of this theorem we do not need the Hölder continuity of the coefficient in t).

Now it remains to prove the claim. For an arbitrary n, by Theorem 3.2, there exist sequences $\Omega_k^n \subset \Omega$ and $v_k^n \in \mathcal{H}_p^2(T) \cap \mathcal{H}_q^{2+\nu}(T)$ such that $\Omega_k^n \uparrow \Omega$ and $v_k^n(\omega,\cdot,\cdot) = v^n(\omega,\cdot,\cdot)$ in $H_p^2(T)$ for almost all $\omega \in \Omega_k^n$.

By the definition of a solution,

$$\begin{aligned}
(v^n(t,\cdot),\phi) &= (u_0,\phi) + \int_0^t (\widetilde{a}^n(s,\cdot,v^n)(v^n)'',\phi)\,ds \\
&+ \int_0^t (\widetilde{f}(s,\cdot),\phi)\,ds + \int_0^t (\widetilde{g}^k(s,\cdot),\phi)\,dw_s^k
\end{aligned} \tag{4.13}$$

holds for all $\phi \in C_0^\infty$ and $t \leqslant T$ a.s. (and the same for v^m).

We fix $\omega \in \Omega^n$ such that (4.13) holds for both v^n and v^m. Note that such ω's have the full probability of Ω^n. Then for this ω, take sufficiently large k such that $\omega \in \Omega_k^n \cap \Omega_k^m$. Then from the equation (4.13) for v^n and v^m,

$$\begin{aligned}
&((v^n - v^m)(\omega, t, \cdot), \phi) = ((v_k^n - v_k^m)(\omega, t, \cdot), \phi) \\
&= \int_0^t (\widetilde{a}(\omega, s, \cdot, v^n)(v^n)'' - \widetilde{a}(\omega, s, \cdot, v^m)(v^m)'', \phi)\, ds \\
&= \int_0^t (\widetilde{a}(\omega, s, \cdot, v_k^n)(v_k^n)'' - \widetilde{a}(\omega, s, \cdot, v_k^m)(v_k^m)'', \phi)\, ds.
\end{aligned}$$

But $\widetilde{a}(\omega, s, x, v_k^n)(v_k^n)'' - \widetilde{a}(\omega, s, x, v_k^m)(v_k^m)'' = [\int_0^1 \widetilde{a}(\omega, s, x, \theta v_k^n + (1-\theta)v_k^m)\, d\theta](v_k^n - v_k^m)'' + [\int_0^1 \widetilde{a}_u(\omega, s, x, \theta v_k^n + (1-\theta)v_k^m)(\theta(v_k^n)'' + (1-\theta)(v_k^m)'')\, d\theta](v_k^n - v_k^m)$. Since v_k^n and v_k^m are in $\mathcal{H}_q^{2+\nu}(T)$, the coefficients of $(v_k^n - v_k^m)''$ and $(v_k^n - v_k^m)$ are Hölder continuous in t and x a.s. Thus, from the uniqueness of solution of parabolic PDE, $v^n(\omega, \cdot, \cdot) = v_k^n(\omega, \cdot, \cdot) = v_k^m(\omega, \cdot, \cdot) = v^m(\omega, \cdot, \cdot)$. The claim is proved. □

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References

[1] N.U. Ahmed, *An existence theorem for stochastic nonlinear evolution equations on Banach space*, Stoch. Anal. Appl., 10(4) (1992), 379-385.

[2] A. Bensoussan, *Some existence results for stochastic partial differential equations*, in Stochastic Partial Differential Equations and Applications, G. Da Prato and L. Tubaro, eds., Pitman Res. Notes Math. Ser., 268 (1992), 37-53.

[3] A. Bensoussan, R. Temam, *Equations aux dérivées partielles stochastiques non linéaires (1)*, Israel J. Math., 11, No. 1 (1972), 95-129.

[4] A. Bensoussan, R. Temam, *Equations stochastiques du type Navier-Stokes*, J. Funct. Anal., 13, No. 2 (1973), 195-222.

[5] Yu.L. Dalecky, N.Yu. Goncharuk, *On a quasilinear stochastic differential equation of parabolic type*, Stoch. Anal. Appl., 12(1) (1994), 103-129.

[6] G. Da Prato, L. Tubaro, *Fully nonlinear stochastic partial differential equations*, SIAM J. Math. Anal. Vol.27, No. 1 (1996), 40-55.

[7] G. Da Prato, J. Zabczyk, *Stochastic equations in infinite dimensions*, Cambridge University Press, New York, NY, 1992.

[8] N.V.Krylov, *On L_p-theory of stochastic partial differential equations in the whole space*, SIAM J. Math. Anal. Vol.27, No.2 (1996), 313-340. 1996.

[9] N.V. Krylov, *An analytic approach to SPDEs*, in Stochastic Partial Differential Equations, Six Perspectives, Mathematical Surveys and Monographs, AMS, submitted.

[10] N.V. Krylov, *Nonlinear elliptic and parabolic equations of the second order*, D.Reidel, Dordrect, 1987.

[11] N.V. Krylov, *On explicit formula for solutions of evolutionary SPDEs (a kind of introduction to the theory)*, Lecture Notes in Control and Inform. Sci., Vol. 176 (1992), 153-164.

[12] N.V. Krylov, *Introduction to the theory of diffusion processes*, Amer. Math. Soc., Providence, RI, 1995.

[13] N.V. Krylov, B.L. Rozovskii, *Stochastic evolution equations*, J. Sov. Math., 16 (1981), 1233-1277.

[14] N.V. Krylov, B.L. Rozovskii, *Stochastic partial differential equations and diffusion processes*, Russian Math. Surveys, 37 (1982), 81-105.

[15] O.A. Ladyzenskaya, V.V. Solonnikov, N.N. Ural'tseva *Linear and quasilinear equations of parabolic type*, Transl AMS, Vol. 23, Am. Math. Soc. Providence, RI, 1968.

[16] E. Pardoux, *Equations aux derivées partielles stochastiques nonlinéaires monotones*, Thèse, Université Paris XI, 1975

[17] E. Pardoux, *Stochastic partial differential equations and filtering of diffusion processes*, Stochastics, Vol.3 (1979), 127-167.

[18] B.L.Rozovskii *Stochastic evolution systems*, Kluwer, Dordrecht, 1990.

[19] L. Tubaro, *Some results on stochastic partial differential equations by the stochastic characteristics method*, Stoch. Anal. Appl., 6(2) (1988), 217-230.

[20] H. Yoo, *L_p-estimates for stochastic PDEs with discontinuous coefficients*, to appear in Stoch. Anal. Appl.