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Sobolev solution for semilinear PDE with obstacle under monotonicity condition

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Abstract

We prove the existence and uniqueness of Sobolev solution of a semilinear PDE's and PDE's with obstacle under monotonicity condition. Moreover we give the probabilistic interpretation of the solutions in term of Backward SDE and reflected Backward SDE respectively .

Key words: Backward stochastic differential equation, Reflected backward stochastic differential equation, monotonicity condition, Stochastic flow, partial differential equation with obstacle.

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1 Introduction

Our approach is based on Backward Stochastic Differential Equations (in short BSDE's) which were first introduced by Bismut [5] in 1973 as equation for the adjoint process in the stochastic version of Pontryagin maximum principle. Pardoux and Peng [14] generalized the notion in 1990 and were the first to consider general BSDE's and to solve the question of existence and uniqueness in the non-linear case. Since then BSDE's have been widely used in stochastic control and especially in mathematical finance, as any pricing problem by replication can be written in terms of linear BSDEs, or non-linear BSDEs when portfolios constraints are taken into account as in El Karoui, Peng and Quenez [6].

The main motivation to introduce the non-linear BSDE's was to give a probabilistic interpretation (Feynman-Kac's formula) for the solutions of semilinear parabolic PDE's. This result was first obtained by Peng in [16], see also Pardoux and Peng [15] by considering the viscosity and classical solutions of such PDE's. Later, Barles and Lesigne [2] studied the relation between BSDE's and solutions of semi-linear PDE's in Soblev spaces. More recently Bally and Matoussi [3] studied semilinear stochastic PDEs and backward doubly SDE in Sobolev space and their probabilistic method is based on stochastic flow.

The reflected BSDE's was introduced by the five authors El Karoui, Kapoudjian, Pardoux, Peng and Quenez in [7], the setting of those equations is the following: let us consider moreover an adapted stochastic process $L := (L_t)_{t \leqslant T}$ which stands for a barrier. A solution for the reflected BSDE associated with (ξ, g, L) is a triple of adapted stochastic processes $(Y_t, Z_t, K_t)_{t \leqslant T}$ such that

$$\begin{cases} Y_t = \xi + \displaystyle\int_t^T g(s, \omega, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s, \ \forall\, t \in [0, T], \\ Y_t \geqslant L_t \ \text{ and } \ \displaystyle\int_0^T (Y_t - L_t)dK_t = 0. \end{cases}$$

The process K is continuous, increasing and its role is to push upward Y in order to keep it above the barrier L. The requirement $\int_0^T (Y_t - L_t)dK_t = 0$ means that the action of K is made with a minimal energy.

The development of reflected BSDE's (see for example [7], [10], [9]) has been especially motivated by pricing American contingent claim by replication, especially in constrained markets. Actually it has been shown by El Karoui, Pardoux and Quenez [8] that the price of an American contingent claim $(S_t)_{t \leqslant T}$ whose strike is γ in a standard complete financial market is Y_0 where $(Y_t, \pi_t, K_t)_{t \leqslant T}$ is the solution of the following reflected BSDE

$$\begin{cases} -dY_t = b(t, Y_t, \pi_t)dt + dK_t - \pi_t dW_t, \quad Y_T = (S_T - \gamma)^+, \\ Y_t \geqslant (S_t - \gamma)^+ \quad \text{and} \quad \int_0^T (Y_t - (S_t - \gamma)^+)dK_t = 0 \end{cases}$$

for an appropriate choice of the function b. The process π allows to construct a replication strategy and K is a consumption process that could have the buyer of the option. In a standard financial market the function $b(t, \omega, y, z) = r_t y + z\theta_t$ where θ_t is the risk premium and r_t the spot rate to invest or borrow. Now when the market is constrained *i.e.* the interest rates are not the same whether we borrow or invest money then the function $b(t, \omega, y, z) = r_t y + z\theta_t - (R_t - r_t)(y - (z.\sigma_t^{-1}.\mathbf{1}))^-$ where R_t (resp. r_t) is the spot rate to borrow (resp. invest) and σ the volatility.

Partial Differential Equations with obstacles and their connections with optimal control problems have been studied by Bensoussan and Lions [4]. They study such equations in the point of view of variational inequalities. In a recent paper, Bally, Caballero, El Karoui and Fernandez [1] studied the the following semilinear PDE with obstacle

$$(\partial_t + \mathcal{L})u + f(t, x, u, \sigma^* \nabla u) + \nu = 0,\ u \geqslant h,\ u_T = g,$$

where h is the obstacle. The solution of such equation is a pair (u, ν) where u is a function in $\mathbf{L}^2([0,T], \mathcal{H})$ and ν is a positive measure concentrated on the set $\{u = h\}$. The authors proved the uniqueness and existence for the solution to this PDE when the coefficient f is Lipschitz and linear increasing on (y, z), and gave the probabilistic interpretation (Feynman-Kac formula) for u and ∇u by the solution (Y, Z) of the reflected BSDE (in short RBSDE). They prove also the natural relation between Reflected BSDE's and variational inequalities and prove uniqueness of the solution for such variational problem by using the relation between the increasing process K and the measure ν. This is also a point of view in this paper.

On the other hand, Pardoux [13] studied the solution of a BSDE with a coefficient $f(t, \omega, y, z)$, which satisfies only monotonicity, continuous and general increasing conditions on y, and a Lipschitz condition on z, i.e. for some continuous, increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, and real numbers $\mu \in \mathbb{R}$, $k > 0$, $\forall t \in [0,T]$, $\forall y, y' \in \mathbb{R}^n$, $\forall z, z' \in \mathbb{R}^{n \times d}$,

$$\begin{aligned} |f(t,y,0)| &\leqslant |f(t,0,0)| + \varphi(|y|), \text{ a.s.}; \\ \langle y - y', f(t,y,z) - f(t,y',z) \rangle &\leqslant \mu \left|y - y'\right|^2, \text{ a.s.}; \\ \left|f(t,y,z) - f(t,y,z')\right| &\leqslant k \left|z - z'\right|, \text{ a.s..} \end{aligned} \tag{1}$$

In the same paper, he also considered the PDE whose coefficient f satisfies the monotonicity condition (1), proved the existence of a viscosity solution u to this PDE and gave its probabilistic interpretation via the solution of the corresponding BSDE. More recently, Lepeltier, Matoussi and Xu [12] proved the existence and uniqueness of the solution for the reflected BSDE under the monotonicity condition.

In our paper, we study the Sobolev solutions of the PDE and also the PDE with continuous obstacle under the monotonicity condition (1). Using penalization method, we prove the existence of the solution and give the probabilistic interpretation of the solution u and ∇u (resp.$(u, \nabla u, \nu)$) by the solution (Y, Z) of backward SDE (resp. the solution (Y, Z, K) of reflected backward SDE). Furthermore we use equivalence norm results and a stochastic test function to pass from the solution of PDE's to the one of BSDE's in order to get the uniqueness of the solution.

Our paper is organized as following: in section 2, we present the basic assumptions and the definitions of the solutions for PDE and PDE with obstacle, then in section 3, we recall some useful results from [3]. We will prove the main results for PDE and PDE with continuous barrier under monotonicity condition in section 4 and 5 respectively. Finally, we prove an analogue result to Proposition 2.3 in [3] under the monotonicity condition, and we also give a priori estimates for the solution of the reflected BSDE's.

2 Notations and preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and $B = (B_1, B_2, \cdots, B_d)^*$ be a d-dimensional Brownian motion defined on a finite interval $[0,T]$, $0 < T < +\infty$. Denote by $\{\mathcal{F}_s^t; t \leqslant s \leqslant T\}$

the natural filtration generated by the Brownian motion B :

$$\mathcal{F}_s^t = \sigma\{B_s - B_t; t \leqslant r \leqslant s\} \cup \mathcal{F}_0,$$

where $\mathcal{F}_0$ contains all $P-$null sets of $\mathcal{F}$.

We will need the following spaces for studying BSDE or reflected BSDE. For any given $n \in \mathbb{N}$:

- $\mathbf{L}_n^2(\mathcal{F}_s^t)$: the set of n-dimensional $\mathcal{F}_s^t$-measurable random variable ξ, such that $E(|\xi|^2) < +\infty$.
- $\mathbf{H}_{n\times m}^2(t,T)$: the set of $\mathbb{R}^{m\times n}$-valued $\mathcal{F}_s^t$-predictable process ψ on the interval $[t,T]$, such that $E\int_t^T \|\psi(s)\|^2 ds < +\infty$.
- $\mathbf{S}_n^2(t,T)$: the set of n-dimensional $\mathcal{F}_s^t$-progressively measurable process ψ on the interval $[t,T]$, such that $E(\sup_{t\leqslant s\leqslant T} \|\psi(s)\|^2) < +\infty$.
- $\mathbf{A}^2(t,T)$:=$\{K : \Omega \times [t,T] \to \mathbb{R}$, $\mathcal{F}_s^t$–progressively measurable increasing RCLL processes with $K_t = 0$, $E[(K_T)^2] < \infty$ $\}$.

Finally, we shall denote by $\mathcal{P}$ the σ-algebra of predictable sets on $[0,T] \times \Omega$. In the real–valued case, i.e., $n = 1$, these spaces will be simply denoted by $\mathbf{L}^2(\mathcal{F}_s^t)$, $\mathbf{H}^2(t,T)$ and $\mathbf{S}^2(t,T)$, respectively.

For the sake of the Sobolev solution of the PDE, the following notations are needed:

- $C_b^m(\mathbb{R}^d, \mathbb{R}^n)$: the set of C^m-functions $f : \mathbb{R}^d \to \mathbb{R}^n$, whose partial derivatives of order less that or equal to m, are bounded. (The functions themselves need not to be bounded)
- $C_c^{1,m}([0,T] \times \mathbb{R}^d, \mathbb{R}^n)$: the set of continuous functions $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}^n$ with compact support, whose first partial derivative with respect to t and partial derivatives of order less or equal to m with respect to x exist.
- $\rho : \mathbb{R}^d \to \mathbb{R}$, the weight, is a continuous positive function which satisfies $\int_{\mathbb{R}^d} \rho(x)dx < \infty$.
- $\mathbf{L}^2(\mathbb{R}^d, \rho(x)dx)$: the weighted $\mathbf{L}^2$-space with weight function $\rho(x)$, endowed with the norm

$$\|u\|_{\mathbf{L}^2(\mathbb{R}^d,\rho)}^2 = \int_{\mathbb{R}^d} |u(x)|^2 \rho(x)dx$$

We assume:

Assumption 2.1. *$g(\cdot) \in \mathbf{L}^2(\mathbb{R}^d, \rho(x)dx)$.*

Assumption 2.2. *$f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}^n$ is measurable in (t,x,y,z) and*

$$\int_0^T \int_{\mathbb{R}^d} |f(t,x,0,0)|^2 \rho(x)dxdt < \infty.$$

Assumption 2.3. *f satisfies increasing and monotonicity condition on y, for some continuous increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, real numbers $k > 0$, $\mu \in \mathbb{R}$ such that $\forall(t,x,y,y',z,z') \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^{n\times d}$*

(i) $|f(t,x,y,z)| \leqslant |f(t,x,0,z)| + \varphi(|y|)$,

(ii) $|f(t,x,y,z) - f(t,x,y,z')| \leqslant k\,|z-z'|$,

(iii) $\langle y-y', f(t,x,y,z) - f(t,x,y',z)\rangle \leqslant \mu\,|y-y'|^2$,

(iv) $y \to f(t,x,y,z)$ *is continuous.*

For the PDE with obstacle, we consider that f satisfies assumptions 2.2 and 2.3, for $n=1$.

Assumption 2.4. *The obstacle function* $h \in C([0,T]\times\mathbb{R}^d;\mathbb{R})$ *satisfies the following conditions: there exists* $\kappa \in \mathbb{R}$, $\beta > 0$, *such that* $\forall (t,x) \in [0,T]\times\mathbb{R}^d$
(i) $\varphi(e^{\mu t}h^+(t,x)) \in \mathbf{L}^2(\mathbb{R}^d;\rho(x)dx)$,
(ii) $|h(t,x)| \leqslant \kappa(1+|x|^\beta)$,
here h^+ *is the positive part of* h.

Assumption 2.5. $b:[0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ *and* $\sigma:[0,T]\times\mathbb{R}^d \to \mathbb{R}^{d\times d}$ *satisfy*

$$b \in C_b^2(\mathbb{R}^d;\mathbb{R}^d) \quad \text{and} \quad \sigma \in C_b^3(\mathbb{R}^d;\mathbb{R}^{d\times d}).$$

We first study the following PDE

$$\begin{cases} (\partial_t + \mathcal{L})\,u \;+\; F(t,x,u,\nabla u) = 0, \quad \forall\,(t,x) \,\in\, [0,T]\times\mathbb{R}^d \\ u(x,T) = g(x), \quad \forall\, x \,\in\, \mathbb{R}^d \end{cases}$$

where $F:[0,T]\times\mathbb{R}^d\times\mathbb{R}^n\times\mathbb{R}^{n\times d} \to \mathbb{R}$, such that

$$F(t,x,u,p) = f(t,x,u,\sigma^* p)$$

and

$$\mathcal{L} = \sum_{i=1}^d b_i \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^d a_{i,j}\frac{\partial^2}{\partial x_i \partial x_j},$$

$a := \sigma\sigma^*$. Here σ^* is the transposed matrix of σ.
In order to study the weak solution of the PDE, we introduce the following space

$$\mathcal{H} := \{u \in \mathbf{L}^2([0,T]\times\mathbb{R}^d, ds\otimes\rho(x)dx) \;\big|\; \sigma^*\nabla u \in \mathbf{L}^2(([0,T]\times\mathbb{R}^d, ds\otimes\rho(x)dx)\}$$

endowed with the norm

$$\|u\|^2 := \int_{\mathbb{R}^d}\int_0^T [|u(s,x)|^2 + |(\sigma^*\nabla u)(s,x)|^2]\rho(x)dsdx.$$

Definition 2.1. *We say that* $u \in \mathcal{H}$ *is the weak solution of the PDE associated to* (g,f), *if*
(i) $\|u\|^2 < \infty$,
(ii) for every $\phi \in C_c^{1,\infty}([0,T]\times\mathbb{R}^d)$

$$\int_t^T (u_s,\partial_t\phi)ds + (u(t,\cdot),\phi(t,\cdot)) - (g(\cdot),\phi(\cdot,T)) + \int_t^T \mathcal{E}(u_s,\phi_s)ds = \int_t^T (f(s,\cdot,u_s,\sigma^*\nabla u_s),\phi_s)ds. \tag{2}$$

where $(\phi,\psi)=\int_{\mathbb{R}^d}\phi(x)\psi(x)dx$ denotes the scalar product in $\mathbf{L}^2(\mathbb{R}^d,dx)$ and

$$\mathcal{E}(\psi,\phi)=\int_{\mathbb{R}^d}((\sigma^*\nabla\psi)(\sigma^*\nabla\phi)+\phi\nabla((\frac{1}{2}\sigma^*\nabla\sigma+b)\psi))dx$$

is the energy of the system of our PDE which corresponds to the Dirichlet form associated to the operator $\mathcal{L}$ when it is symmetric. Indeed $\mathcal{E}(\psi,\phi)=-(\phi,\mathcal{L}\psi)$.

The probabilistic interpretation of the solution of PDE associated with g,f, which satisfy Assumption 2.1-2.3 was firstly studied by (Pardoux [13]), where the author proved the existence of a viscosity solution to this PDE, and gave its probabilistic interpretation. In section 4, we consider the weak solution to PDE (2) in Sobolev space, and give the proof of the existence and uniqueness of the solution as well as the probabilistic interpretation.

In the second part of this article, we will consider the obstacle problem associated to the PDE (2) with obstacle function h, where we restrict our study in the one dimensional case ($n=1$). Formulaly, The solution u is dominated by h, and verifies the equation in the following sense : $\forall(t,x)\in[0,T]\times\mathbb{R}^d$

$$\begin{aligned}
&\text{(i) } (\partial_t+\mathcal{L})u+F(t,x,u,\nabla u)\leqslant 0, \qquad \text{on} \qquad u(t,x)\geqslant h(t,x),\\
&\text{(ii) } (\partial_t+\mathcal{L})u+F(t,x,u,\nabla u)=0, \qquad \text{on} \qquad u(t,x)>h(t,x),\\
&\text{(iii) } u(x,T)=g(x)\,.
\end{aligned}$$

where $\mathcal{L}=\sum_{i=1}^d b_i\frac{\partial}{\partial x_i}+\frac{1}{2}\sum_{i,j=1}^d a_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}$, $a=\sigma\sigma^*$. In fact, we give the following formulation of the PDE with obstacle.

Definition 2.2. *We say that (u,ν) is the weak solution of the PDE with obstacle associated to (g,f,h), if*
(i) $\|u\|^2<\infty$, $u\geqslant h$, and $u(T,x)=g(x)$.
(ii) ν is a positive Radon measure such that $\int_0^T\int_{\mathbb{R}^d}\rho(x)d\nu(t,x)<\infty$,
(iii) for every $\phi\in C_c^{1,\infty}([0,T]\times\mathbb{R}^d)$

$$\begin{aligned}
&\int_t^T(u_s,\partial_s\phi)ds+(u(t,\cdot),\phi(t,\cdot))-(g(\cdot),\phi(\cdot,T))+\int_t^T\mathcal{E}(u_s,\phi_s)ds \qquad (3)\\
=\ &\int_t^T(f(s,\cdot,u_s,\sigma^*\nabla u_s),\phi_s)ds+\int_t^T\int_{\mathbb{R}^d}\phi(s,x)1_{\{u=h\}}d\nu(x,s).
\end{aligned}$$

3 Stochastic flow and random test functions

Let $(X_s^{t,x})_{t\leqslant s\leqslant T}$ be the solution of

$$\begin{cases}dX_s^{t,x}=b(s,X_s^{t,x})ds+\sigma(s,X_s^{t,x})dB_s,\\ X_t^{t,x}=x,\end{cases}$$

where $b:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$ and $\sigma:[0,T]\times\mathbb{R}^d\to\mathbb{R}^{d\times d}$ satisfy Assumption 2.5.

So $\{X_s^{t,x}, x \in \mathbb{R}^d, t \leqslant s \leqslant T\}$ is the stochastic flow associated to the diffuse $\{X_s^{t,x}\}$ and denote by $\{\widehat{X}_s^{t,x}, t \leqslant s \leqslant T\}$ the inverse flow. It is known that $x \to \widehat{X}_s^{t,x}$ is differentiable (Ikeda and Watanabe [**?**]). We denote by $J(X_s^{t,x})$ the determinant of the Jacobian matrix of $\widehat{X}_s^{t,x}$, which is positive, and $J(X_t^{t,x}) = 1$.

For $\phi \in C_c^\infty(\mathbb{R}^d)$ we define a process $\phi_t : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}$ by

$$\phi_t(s,x) := \phi(\widehat{X}_s^{t,x}) J(\widehat{X}_s^{t,x}).$$

Following Kunita (See [11]), we know that for $v \in \mathbf{L}^2(\mathbb{R}^d)$, the composition of v with the stochastic flow is

$$(v \circ X_s^{t,\cdot}, \phi) := (v, \phi_t(s,\cdot)).$$

Indeed, by a change of variable, we have

$$(v \circ X_s^{t,\cdot}, \phi) = \int_{\mathbb{R}^d} v(y)\phi(\widehat{X}_s^{t,y}) J(\widehat{X}_s^{t,y}) dy = \int_{\mathbb{R}^d} v(X_s^{t,x})\phi(x) dx.$$

The main idea in Bally and Matoussi [3] and Bally et al. [1], is to use ϕ_t as a test function in (2) and (3). The problem is that $s \to \phi_t(s,x)$ is not differentiable so that $\int_t^T (u_s, \partial_s \phi) ds$ has no sense. However $\phi_t(s,x)$ is a semimartingale and they proved the following semimartingale decomposition of $\phi_t(s,x)$:

Lemma 3.1. *For every function $\phi \in \mathbf{C}_c^2(\mathbb{R}^d)$,*

$$\phi_t(s,x) = \phi(x) - \sum_{j=1}^d \int_t^s \left(\sum_{i=1}^d \frac{\partial}{\partial x_i} \left(\sigma_{ij}(x)\phi_t(r,x) \right) \right) dB_r^j + \int_t^s \mathcal{L}^* \phi_t(r,x) dr, \tag{4}$$

where $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$. So

$$d\phi_t(r,x) = -\sum_{j=1}^d \left(\sum_{i=1}^d \frac{\partial}{\partial x_i} \left(\sigma_{ij}(x)\phi_t(r,x) \right) \right) dB_r^j + \mathcal{L}^* \phi_t(r,x) dr, \tag{5}$$

Then in (2) we may replace $\partial_s \phi ds$ by the Itô stochastic integral with respect to $d\phi_t(s,x)$, and have the following proposition which allows us to use ϕ_t as a test function. The proof will be given in the appendix.

Proposition 3.1. *Assume that assumptions 2.1, 2.2 and 2.3 hold. Let $u \in \mathcal{H}$ be a weak solution of PDE (2), then for $s \in [t,T]$ and $\phi \in C_c^2(\mathbb{R}^d)$,*

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_s^T u(r,x) d\phi_t(r,x) dx - (g(\cdot), \phi_t(T,\cdot)) + (u(s,\cdot), \phi_t(s,\cdot)) - \int_s^T \mathcal{E}(u(r,\cdot), \phi_t(r,\cdot)) dr \\ = & \int_{\mathbb{R}^d} \int_s^T f(r,x,u(r,x), \sigma^* \nabla u(r,x)) \phi_t(r,x) dr dx. \ \ a.s. \end{aligned} \tag{6}$$

Remark 3.1. *Here $\phi_t(r,x)$ is $\mathbb{R}$-valued. We consider that in (6), the equality holds for each component of u.*

We need the result of equivalence of norms, which play important roles in existence proof for PDE under monotonic conditions. The equivalence of functional norm and stochastic norm is first proved by Barles and Lesigne [2] for $\rho = 1$. In Bally and Matoussi [3] proved the same result for weighted integrable function by using probabilistic method. Let ρ be a weighted function, we take $\rho(x) := \exp(F(x))$, where $F : \mathbb{R}^d \to \mathbb{R}$ is a continuous function. Moreover, we assume that there exists a constant $R > 0$, such that for $|x| > R$, $F \in C_b^2(\mathbb{R}^d, \mathbb{R})$. For instant, we can take $\rho(x) = (1 + |x|)^{-q}$ or $\rho(x) = \exp \alpha\, |x|$, with $q > d + 1$, $\alpha \in \mathbb{R}$.

Proposition 3.2. *Suppose that assumption 2.5 hold, then there exists two constants k_1, $k_2 > 0$, such that for every $t \leqslant s \leqslant T$ and $\phi \in \mathbf{L}^1(\mathbb{R}^d, \rho(x)dx)$, we have*

$$k_2 \int_{\mathbb{R}^d} |\phi(x)|\, \rho(x)dx \leqslant \int_{\mathbb{R}^d} E(\left|\phi(X_s^{t,x})\right|)\rho(x)dx \leqslant k_1 \int_{\mathbb{R}^d} |\phi(x)|\, \rho(x)dx, \tag{7}$$

Moreover, for every $\psi \in \mathbf{L}^1([0,T] \times \mathbb{R}^d, dt \otimes \rho(x)dx)$

$$\begin{aligned} k_2 \int_{\mathbb{R}^d} \int_t^T |\psi(s,x)|\, \rho(x)dsdx \quad &\leqslant \quad \int_{\mathbb{R}^d} \int_t^T E(\left|\psi(s, X_s^{t,x})\right|)\rho(x)dsdx \\ &\leqslant \quad k_1 \int_{\mathbb{R}^d} \int_t^T |\psi(s,x)|\, \rho(x)dsdx, \end{aligned} \tag{8}$$

where the constants k_1, k_2 depend only on T, ρ and the bounds of the first (resp. first and second) derivatives of b (resp. σ).

This proposition is easy to get from the follwing Lemma, see Lemma 5.1 in Bally and Matoussi [3].

Lemma 3.2. *There exist two constants $c_1 > 0$ and $c_2 > 0$ such that $\forall x \in \mathbb{R}^d$, $0 \leqslant t \leqslant T$*

$$c_1 \leqslant E\left(\frac{\rho(t, \widehat{X}_t^{0,x}) J(\widehat{X}_t^{0,x})}{\rho(x)}\right) \leqslant c_2.$$

4 Sobolev's Solutions for PDE's under monotonicity condition

In this section we shall study the solution of the PDE whose coefficient f satisfies the monotonicity condition. For this sake, we introduce the BSDE associated with (g, f): for $t \leqslant s \leqslant T$,

$$Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_s^{t,x} dB_s. \tag{9}$$

Thanks to the equivalence of the norms result (3.2), we know that $g(X_T^{t,x})$ and $f(s, X_s^{t,x}, 0, 0)$ make sense in the BSDE (9). Moreover we have

$$g(X_T^{t,x}) \in \mathbf{L}_n^2(\mathcal{F}_T) \text{ and } f(., X_.^{t,x}, 0, 0) \in \mathbf{H}_n^2(0,T).$$

It follows from the results from Pardoux [13] that for each (t,x), there exists a unique pair $(Y^{t,x}, Z^{t,x}) \in \mathbf{S}^2(t,T) \times \mathbf{H}_{n\times d}^2(t,T)$ of $\{\mathcal{F}_s^t\}$ progressively measurable processes, which solves this BSDE(g, f).

The main result of this section is

Theorem 4.1. *Suppose that assumptions 2.1-2.3 and 2.4 hold. Then there exists a unique weak solution $u \in \mathcal{H}$ of the PDE (2). Moreover we have the probabilistic interpretation of the solution:*

$$u(t,x) = Y_t^{t,x}, \quad (\sigma^* \nabla u)(t,x) = Z_t^{t,x}, \quad dt \otimes dx - a.e. \tag{10}$$

and moreover $Y_s^{t,x} = u(s, X_s^{t,x})$, $Z_s^{t,x} = (\sigma^* \nabla u)(s, X_s^{t,x})$, $dt \otimes dP \otimes dx$*-a.e.* $\forall s \in [t,T]$.

Proof: We start to prove the existence result.

a) Existence : We prove the existence in three steps. By integration by parts formula, we know that u solves (2) if and only if

$$\widehat{u}(t,x) = e^{\mu t} u(t,x)$$

is a solution of the PDE$(\widehat{g}, \widehat{f})$, where

$$\widehat{g}(x) = e^{\mu T} g(x) \ \text{ and } \ \widehat{f}(t,x,y,z) = e^{\mu t} f(t,x,e^{-\mu t} y, e^{-\mu t} z) - \mu y. \tag{11}$$

Then the coefficient $\widehat{f}$ satisfies the assumption 2.3 as f, except that 2.3-(iii) is replaced by

$$(y - y')(f(t,x,y,z) - f(t,x,y',z)) \leqslant 0. \tag{12}$$

In the first two steps, we consider the case where f does not depend on ∇u, and write $f(t,x,y)$ for $f(t,x,y,v(t,x))$, where v is in $\mathbf{L}^2([0,T] \times \mathbb{R}^d, dt \otimes \rho(x)dx)$.

We assume first that $f(t,x,y)$ satisfies the following assumption 2.3**'**: $\forall (t,x,y,y') \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n$,

(i) $|f(t,x,y)| \leqslant |f(t,x,0)| + \varphi(|y|)$,

(ii) $\langle y - y', f(t,x,y) - f(t,x,y') \rangle \leqslant 0$,

(iii) $y \to f(t,x,y)$ is continuous, $\forall (t,x) \in [0,T] \times \mathbb{R}^d$.

Step 1 : *Suppose that $g(x)$, $f(t,x,0)$ are uniformly bounded*, i.e. there exists a constant C, such that

$$|g(x)| + \sup_{0 \leqslant t \leqslant T} |f(t,x,0)| \leqslant C \tag{13}$$

where C as a constant which can be changed line by line.

Define $f_n(t,y) := (\theta_n * f(t,\cdot))(y)$ where $\theta_n : \mathbb{R}^n \to \mathbb{R}_+$ is a sequence of smooth functions with compact support, which approximate the Dirac distribution at 0, and satisfy $\int \theta_n(z)dz = 1$. Let $\{(Y_s^{n,t,x}, Z_s^{n,t,x}),\ t \leqslant s \leqslant T\}$ be the solution of BSDE associated to $(g(X_T^{t,x}), f_n)$, namely,

$$Y_s^{n,t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{n,t,x}) dr - \int_s^T Z_r^{n,t,x} dB_r, \text{ P-a.s..} \tag{14}$$

Then for each $n \in \mathbb{N}$, we have

$$\left| Y_s^{n,t,x} \right| \leqslant e^T C,$$

and

$$\left|f_n(s,X_s^{t,x},Y_s^{n,t,x})\right|^2 \leqslant 2\left|f_n(s,X_s^{t,x},0)\right|^2 + 2\psi^2(e^{\frac{T}{2}}\sqrt{C})$$

where $\psi(r) := \sup_n \sup_{|y|\leqslant r} \int_{\mathbb{R}^n} \varphi(|y|)\theta_n(y-z)dz$. So there exists a constant $C > 0$, s.t.

$$\sup_n \int_{\mathbb{R}^d} E\int_t^T (\left|Y_s^{n,t,x}\right|^2 + \left|f_n(s,X_s^{t,x},Y_s^{n,t,x})\right|^2 + \left|Z_s^{n,t,x}\right|^2)\rho(x)dsdx \leqslant C. \tag{15}$$

Then let $n\to\infty$ on the both sides of (14), we get that the limit $(Y_s^{t,x}, Z_s^{t,x})$ of $(Y_s^{n,t,x}, Z_s^{n,t,x})$, satisfies

$$Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r,X_r^{t,x},Y_r^{t,x})dr - \int_s^T Z_r^{t,x}dB_r,\ \text{P-a.s..} \tag{16}$$

Moreover we obtain from the estimate (15) that

$$\int_{\mathbb{R}^d}\int_t^T E(\left|Y_s^{t,x}\right|^2 + \left|Z_s^{t,x}\right|^2)\rho(x)dsdx < \infty. \tag{17}$$

Notice that $(Y_t^{t,x}, Z_t^{t,x})$ are $\mathcal{F}_t^t$ measurable, which implies they are deterministic. Define $u(t,x) := Y_t^{t,x}$, and $v(t,x) := Z_t^{t,x}$. By the flow property of $X_r^{s,x}$ and by the uniqueness of the solution of the BSDE (16), we have that $Y_s^{t,x} = u(s,X_s^{t,x})$ and $Z_s^{t,x} = v(s,X_s^{t,x})$.
The terminal condition g and $f(.,.,0,0)$ are not continuous in t and x, and assumed to belong in a suitable weighted L^2 space, so the solution u and for instance v are not in general continuous, and are only defined a.e. in $[0,T]\times\mathbb{R}^d$. So in order to give meaning to the expression $u(s,X_s^{t,x})$ (resp. $v(s,X_s^{t,x})$), and following Bally and Matoussi [3], we apply a regularization procedure on the final condition g and the coefficient f. Actually, according to Pardoux and Peng ([15], Theorem 3.2), if the coefficient (g,f) are smooth, then the PDE (2) admits a unique classical solution $u\in C^{1,2}([0,T]\times\mathbb{R}^d)$. Therefore the approximated expression $u(s,X_s^{t,x})$ (resp. $v(s,X_s^{t,x})$) has a meaning and then pass to the limit in L^2 spaces like us in Bally and Matoussi [3].
Now, the equivalence of norm result (8) and estimate (17) follow that $u,v\in\mathbf{L}^2([0,T]\times\mathbb{R}^d, dt\otimes\rho(x)dx)$. Finally, let $F(r,x) = f(r,X_r^{t,x},Y_r^{t,x})$, we know that $F(s,x)\in\mathbf{L}^2([0,T]\times\mathbb{R}^d, dt\otimes\rho(x)dx)$, in view of

$$\begin{aligned}\int_{\mathbb{R}^d}\int_t^T \left|F(s,x)\right|^2\rho(x)dsdx &\leqslant \frac{1}{k_2}\int_{\mathbb{R}^d}\int_t^T E\left|F(s,X_s^{t,x})\right|^2\rho(x)dsdx\\ &= \frac{1}{k_2}\int_{\mathbb{R}^d}\int_t^T E\left|f(s,X_s^{t,x},Y_s^{t,x})\right|^2\rho(x)dsdx < \infty.\end{aligned}$$

So that from theorem 2.1 in [3], we get that $v=\sigma^*\nabla u$ and that $u\in\mathcal{H}$ solves the PDE associated to (g,f) under the bounded assumption.

Step 2 : *We assume $g\in\mathbf{L}^2(\mathbb{R}^d,\rho(x)dx)$, f satisfies the assumption 2.3' and $f(t,x,0)\in$* $\mathbf{L}^2([0,T]\times\mathbb{R}^d, dt\otimes\rho(x)dx)$. We approximate g and f by bounded functions as follows :

$$\begin{aligned} g_n(x) &= \Pi_n(g(x)), \\ f_n(t,x,y) &= f(t,x,y) - f(t,x,0) + \Pi_n(f(t,x,0)), \end{aligned} \tag{18}$$

where

$$\Pi_n(y) := \frac{\min(n,|y|)}{|y|}y.$$

Clearly, the pair (g_n, f_n) satisfies the assumption (13) of step 1, and

$$\begin{aligned} g_n &\to g \text{ in } \mathbf{L}^2(\mathbb{R}^d, \rho(x)dx), \\ f_n(t,x,0) &\to f(t,x,0) \text{ in } \mathbf{L}^2([0,T]\times\mathbb{R}^d, dt\otimes\rho(x)dx). \end{aligned} \tag{19}$$

Denote $(Y_s^{n,t,x}, Z_s^{n,t,x}) \in \mathbf{S}_n^2(t,T)\times\mathbf{H}_{n\times d}^2(t,T)$ the solution of the BSDE(ξ_n, f_n), where $\xi_n = g_n(X_T^{t,x})$, i.e.

$$Y_s^{n,t,x} = g_n(X_T^{t,x}) + \int_s^T f_n(r, X_r^{t,x}, Y_r^{n,t,x})dr - \int_s^T Z_r^{n,t,x}dB_r.$$

Then from the results in step 1, $u_n(t,x) = Y_t^{n,t,x}$ and $u_n(t,x)\in\mathcal{H}$, is the weak solution of the PDE(g_n, f_n), with

$$Y_s^{n,t,x} = u_n(s, X_s^{t,x}), Z_s^{n,t,x} = (\sigma^*\nabla u_n)(s, X_s^{t,x}), \text{a.s.} \tag{20}$$

For $m,n\in\mathbb{N}$, applying Itô's formula to $\left|Y_s^{m,t,x} - Y_s^{n,t,x}\right|^2$, we get

$$\begin{aligned} E\left|Y_s^{m,t,x} - Y_s^{n,t,x}\right|^2 + E\int_s^T \left|Z_r^{m,t,x} - Z_r^{n,t,x}\right|^2 dr &\leqslant E\left|g_m(X_T^{t,x}) - g_n(X_T^{t,x})\right|^2 \\ &+ E\int_s^T \left|Y_r^{m,t,x} - Y_r^{n,t,x}\right|^2 dr + E\int_s^T \left|f_m(r, X_r^{t,x}, 0) - f_n(r, X_r^{t,x}, 0)\right|^2 dr. \end{aligned} \tag{21}$$

From the equivalence of the norms (7) and (8), it follows

$$\begin{aligned} &\int_{\mathbb{R}^d} E\left|Y_s^{m,t,x} - Y_s^{n,t,x}\right|^2 \rho(x)dx \leqslant \int_{\mathbb{R}^d} E\left|g_m(X_T^{t,x}) - g_n(X_T^{t,x})\right|^2 \rho(x)dx \\ &\quad + \int_{\mathbb{R}^d} E\int_s^T \left|Y_r^{m,t,x} - Y_r^{n,t,x}\right|^2 dr\rho(x)dx + \int_{\mathbb{R}^d} E\int_s^T \left|f_m(r, X_r^{t,x}, 0) - f_n(r, X_r^{t,x}, 0)\right|^2 dr\rho(x)dx \\ &\quad \leqslant \int_{\mathbb{R}^d} E\int_s^T \left|Y_r^{m,t,x} - Y_r^{n,t,x}\right|^2 dr\rho(x)dx + k_1\int_{\mathbb{R}^d} E\left|g_m(x) - g_n(x)\right|^2 \rho(x)dx \\ &\qquad + k_1\int_{\mathbb{R}^d}\int_t^T \left|f_m(r,x,0) - f_n(r,x,0)\right|^2 \rho(x)drdx, \end{aligned}$$

and by Gronwall's inequality and (19), we get as $m,n\to\infty$

$$\sup_{t\leqslant s\leqslant T}\int_{\mathbb{R}^d} E\left|Y_s^{m,t,x} - Y_s^{n,t,x}\right|^2 \rho(x)dx \to 0.$$

It follows immediately as $m,n\to\infty$

$$\int_{\mathbb{R}^d} E\int_s^T \left|Y_r^{m,t,x} - Y_r^{n,t,x}\right|^2 \rho(x)drdx + \int_{\mathbb{R}^d} E\int_s^T \left|Z_r^{m,t,x} - Z_r^{n,t,x}\right|^2 \rho(x)drdx \to 0.$$

Using again the equivalence of the norms (8), we get:

$$\begin{aligned} &\int_t^T\int_{\mathbb{R}^d} \left|u_m(s,x) - u_n(s,x)\right|^2 + \left|\sigma^*\nabla u_m(s,x) - \sigma^*\nabla u_n(s,x)\right|^2 \rho(x)dxds \\ \leqslant\ & \frac{1}{k_2}\int_t^T\int_{\mathbb{R}^d} E(\left|u_m(s, X_s^{t,x}) - u_n(s, X_s^{t,x})\right|^2 + \left|\sigma^*\nabla u_m(s, X_s^{t,x}) - \sigma^*\nabla u_n(s, X_s^{t,x})\right|^2)\rho(x)dsdx \\ =\ & \frac{1}{k_2}\int_t^T\int_{\mathbb{R}^d} E(\left|Y_s^{m,t,x} - Y_s^{n,t,x}\right|^2 + \left|Z_s^{m,t,x} - Z_s^{n,t,x}\right|^2)\rho(x)dsdx \to 0. \end{aligned}$$

as $m,n\to\infty$, i.e. $\{u_n\}$ is Cauchy sequence in $\mathcal{H}$. Denote its limit as u, so $u\in\mathcal{H}$, and satisfies for every $\phi\in C_c^{1,\infty}([0,T]\times\mathbb{R}^d)$,

$$\int_t^T (u_s,\partial_t\phi)ds+(u(t,\cdot),\phi(t,\cdot))-(g(\cdot),\phi(\cdot,T))+\int_t^T \mathcal{E}(u_s,\phi_s)ds=\int_t^T (f(s,\cdot,u_s),\phi_s)ds. \quad (22)$$

On the other hand, $(Y_\cdot^{n,t,x},Z_\cdot^{n,t,x})$ converges to $(Y_\cdot^{t,x},Z_\cdot^{t,x})$ in $\mathbf{S}_n^2(0,T)\times\mathbf{H}_{n\times d}^2(0,T)$, which is the solution of the BSDE with parameters $(g(X_T^{t,x}),f)$; by the equivalence of the norms, we deduce that

$$Y_s^{t,x}=u(s,X_s^{t,x}),Z_s^{t,x}=\sigma^*\nabla u(s,X_s^{t,x}),\ \text{a.s.}\ \forall s\in[t,T],$$

specially $Y_t^{t,x}=u(t,x)$, $Z_t^{t,x}=\sigma^*\nabla u(t,x)$.

Now, it's easy to the generalize the result to the case when f satisfies assumption 2.2 .

Step 3: *In this step, we consider the case where f depends on ∇u.* Assume that g, f satisfy the assumptions 2.1 - 2.3, with assumption 2.3-(iii) replaced by (12). From the result in step 2, for any given $n\times d$-matrix-valued function $v\in\mathbf{L}^2([0,T]\times\mathbb{R}^d,dt\otimes\rho(x)dx)$, $f(t,x,u,v(t,x))$ satisfies the assumptions in step 2. So the PDE$(g,f(t,x,u,v(t,x)))$ admits a unique solution $u\in\mathcal{H}$ satisfying (i) and (ii) in the definition 2.1.

Set $V_s^{t,x}=v(s,X_s^{t,x})$, then $V_s^{t,x}\in\mathbf{H}_{n\times d}^2(0,T)$ in view of the equivalence of the norms. We consider the following BSDE with solution $(Y_\cdot^{t,x},Z_\cdot^{t,x})$

$$Y_s^{t,x}=g(X_T^{t,x})+\int_s^T f(s,X_s^{t,x},Y_s^{t,x},V_s^{t,x})ds-\int_s^T Z_s^{t,x}dB_s,$$

then $Y_s^{t,x}=u(s,X_s^{t,x})$, $Z_s^{t,x}=\sigma^*\nabla u(s,X_s^{t,x})$, a.s. $\forall s\in[t,T]$.

Now we can construct a mapping Ψ from $\mathcal{H}$ into itself. For any $\overline{u}\in\mathcal{H}$, $u=\Psi(\overline{u})$ is the weak solution of the PDE with parameters $g(x)$ and $f(t,x,u,\sigma^*\nabla\overline{u})$.

Symmetrically we introduce a mapping Φ from $\mathbf{H}_n^2(t,T)\times\mathbf{H}_{n\times d}^2(t,T)$ into itself. For any $(U^{t,x},V^{t,x})\in\mathbf{H}_n^2(t,T)\times\mathbf{H}_{n\times d}^2(t,T)$, $(Y^{t,x},Z^{t,x})=\Phi(U^{t,x},V^{t,x})$ is the solution of the BSDE with parameters $g(X_T^{t,x})$ and $f(s,X_s^{t,x},Y_s^{t,x},V_s^{t,x})$. Set $V_s^{t,x}=\sigma^*\nabla\overline{u}(s,X_s^{t,x})$, then $Y_s^{t,x}=u(s,X_s^{t,x})$, $Z_s^{t,x}=\sigma^*\nabla u(s,X_s^{t,x})$, a.s.a.e..

Let $\overline{u}_1$, $\overline{u}_2\in\mathcal{H}$, and $u_1=\Psi(\overline{u}_1)$, $u_2=\Psi(\overline{u}_2)$, we consider the difference $\triangle u:=u_1-u_2$, $\triangle\overline{u}:=\overline{u}_1-\overline{u}_2$. Set $V_s^{t,x,1}:=\sigma^*\nabla\overline{u}_1(s,X_s^{t,x})$, $V_s^{t,x,2}:=\sigma^*\nabla\overline{u}_2(s,X_s^{t,x})$. We denote by $(Y^{t,x,1},Z^{t,x,1})$(resp. $(Y^{t,x,2},Z^{t,x,2})$) the solution of the BSDE with parameters $g(X_T^{t,x})$ and $f(s,X_s^{t,x},Y_s^{t,x},V_s^{t,x,1})$ (resp. $f(s,X_s^{t,x},Y_s^{t,x},V_s^{t,x,2})$); then for a.e. $\forall s\in[t,T]$,

$$\begin{aligned}Y_s^{t,x,1}&=u_1(s,X_s^{t,x}),Z_s^{t,x,1}=\sigma^*\nabla u_1(s,X_s^{t,x}),\\ Y_s^{t,x,2}&=u_2(s,X_s^{t,x}),Z_s^{t,x,2}=\sigma^*\nabla u_2(s,X_s^{t,x}),\end{aligned}$$

Denote $\triangle Y_s^{t,x}:=Y_s^{t,x,1}-Y_s^{t,x,2}$, $\triangle Z_s^{t,x}:=Z_s^{t,x,1}-Z_s^{t,x,2}$, $\triangle V_s^{t,x}:=V_s^{t,x,1}-V_s^{t,x,2}$. By Itô's formula applied to $e^{\gamma t}E\left|\triangle Y_s^{t,x}\right|^2$, for some α and $\gamma\in\mathbb{R}$, we have

$$e^{\gamma t}E\left|\triangle Y_s^{t,x}\right|^2+E\int_s^T e^{\gamma s}(\gamma\left|\triangle Y_r^{t,x}\right|^2+\left|\triangle Z_r^{t,x}\right|^2)dr\leqslant E\int_s^T e^{\gamma s}(\frac{k^2}{\alpha}\left|\triangle Y_r^{t,x}\right|^2+\alpha\left|\triangle V_r^{t,x}\right|^2)dr,$$

Using the equivalence of the norms, we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_t^T e^{\gamma s}(\gamma\left|\triangle u(s,x)\right|^2+\left|\sigma^*\nabla(\triangle u)(s,x)\right|^2)\rho(x)dsdx\\
\leqslant\ & \frac{1}{k_2}\int_{\mathbb{R}^d}\int_t^T e^{\gamma s}E(\gamma\left|\triangle Y_r^{t,x}\right|^2+\left|\triangle Z_r^{t,x}\right|^2)\rho(x)drdx\\
\leqslant\ & \frac{1}{k_2}\int_{\mathbb{R}^d}\int_s^T e^{\gamma s}E(\frac{k^2}{\alpha}\left|\triangle Y_r^{t,x}\right|^2+\alpha\left|\triangle V_r^{t,x}\right|^2)\rho(x)drdx\\
\leqslant\ & \frac{k_1}{k_2}\int_{\mathbb{R}^d}\int_s^T e^{\gamma s}(\frac{k^2}{\alpha}\left|\triangle u(s,x)\right|^2+\alpha\left|\sigma^*\nabla(\triangle \overline{u})(s,x)\right|^2)\rho(x)dsdx.
\end{aligned}$$

Set $\alpha=\frac{k_2}{2k_1}$, $\gamma=1+\frac{2k_1^2}{k_2^2}k^2$, then we get

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_t^T e^{\gamma s}(\left|\triangle u(s,x)\right|^2+\left|\sigma^*\nabla(\triangle u)(s,x)\right|^2)\rho(x)dsdx\\
\leqslant\ & \frac{1}{2}\int_{\mathbb{R}^d}\int_t^T e^{\gamma s}\left|\sigma^*\nabla(\triangle \overline{u})(s,x)\right|^2\rho(x)dsdx,\\
\leqslant\ & \frac{1}{2}\int_{\mathbb{R}^d}\int_t^T e^{\gamma s}(\left|\triangle \overline{u}(s,x)\right|^2+\left|\sigma^*\nabla(\triangle \overline{u})(s,x)\right|^2)\rho(x)dsdx.
\end{aligned}$$

Consequently, Ψ is a strict contraction on $\mathcal{H}$ equipped with the norm

$$\|u\|_\gamma^2:=\int_{\mathbb{R}^d}\int_t^T e^{\gamma s}(\left|u(s,x)\right|^2+\left|\sigma^*\nabla u(s,x)\right|^2)\rho(x)dsdx.$$

So Ψ has fixed point $u\in\mathcal{H}$ which is the solution of the PDE (2) associated to (g,f). Moreover, for $t\leqslant s\leqslant T$,

$$Y_s^{t,x}=u(s,X_s^{t,x}),Z_s^{t,x}=\sigma^*\nabla u(s,X_s^{t,x}),\ \text{.a.e.}$$

and specially $Y_t^{t,x}=u(t,x),Z_t^{t,x}=\sigma^*\nabla u(t,x)$, a.e.

b) Uniqueness : Let u^1 and $u^2\in\mathcal{H}$ be two solutions of the PDE(g,f). From Proposition 3.1, for $\phi\in C_c^2(\mathbb{R}^d)$ and $i=1,2$

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_s^T u^i(r,x)d\phi_t(r,x)dx+(u^i(s,\cdot),\phi_t(s,\cdot))-(g(\cdot),\phi_t(\cdot,T))-\int_s^T\mathcal{E}(u^i(r,\cdot),\phi_t(r,\cdot))dr\\
=\ & \int_s^T\int_{\mathbb{R}^d}\phi_t(r,x)f(r,x,u^i(r,x),\sigma^*\nabla u^i(r,x))drdx. \qquad (23)
\end{aligned}$$

By (4), we get

$$\begin{aligned}
\int_{\mathbb{R}^d}\int_s^T u^i d\phi_t(r,x)dx\ =\ & \int_s^T(\int_{\mathbb{R}^d}(\sigma^*\nabla u^i)(r,x)\phi_t(r,x)dx)dB_r\\
&+\int_s^T\int_{\mathbb{R}^d}\left((\sigma^*\nabla u^i)(\sigma^*\nabla\phi_r)+\phi\nabla((\frac{1}{2}\sigma^*\nabla\sigma+b)u_r^i)\right)dxdr.
\end{aligned}$$

We substitute this in (23), and get

$$\begin{aligned}\int_{\mathbb{R}^d} u^i(s,x)\phi_t(s,x)dx &= (g(\cdot),\phi_t(\cdot,T)) - \int_s^T \int_{\mathbb{R}^d} (\sigma^*\nabla u^i)(r,x)\phi_t(r,x)dxdB_r \\ &\quad + \int_s^T \int_{\mathbb{R}^d} \phi_t(r,x) f(r,x,u^i(r,x),\sigma^*\nabla u^i(r,x))drdx.\end{aligned}$$

Then by the change of variable $y = \widehat{X}_r^{t,x}$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} u^i(s,X_s^{t,y})\phi(y)dy &= \int_{\mathbb{R}^d} g(X_T^{t,y})\phi(y)dy + \int_s^T \int_{\mathbb{R}^d} \phi(y) f(s,X_s^{t,y},u^i(s,X_s^{t,y}),\sigma^*\nabla u^i(s,X_s^{t,y}))dyds \\ &\quad - \int_s^T \int_{\mathbb{R}^d} (\sigma^*\nabla u^i)(r,X_r^{t,y})\phi(y)dydB_r.\end{aligned}$$

Since ϕ is arbitrary, we can prove this result for $\rho(y)dy$ almost every y. So $(u^i(s,X_s^{t,y}),(\sigma^*\nabla u^i)(s,X_s^{t,y}))$ solves the BSDE$(g(X_T^{t,y}),f)$, i.e. $\rho(y)dy$ a.s., we have

$$u^i(s,X_s^{t,y}) = g(X_T^{t,y}) + \int_s^T f(s,X_s^{t,y},u^i(s,X_s^{t,y}),\sigma^*\nabla u^i(s,X_s^{t,y}))ds - \int_s^T (\sigma^*\nabla u^i)(r,X_r^{t,y})dB_r.$$

Then by the uniqueness of the BSDE, we know $u^1(s,X_s^{t,y}) = u^2(s,X_s^{t,x})$ and $(\sigma^*\nabla u^1)(s,X_s^{t,y}) = (\sigma^*\nabla u^2)(s,X_s^{t,y})$. Taking $s=t$ we deduce that $u^1(t,y) = u^2(t,y)$, $dt \otimes dy$-a.s. $\square$

5 Sobolev's solution for PDE with obstacle under monotonicity condition

In this section we study the PDE with obstacle associated with (g,f,h), which satisfy the assumptions 2.1-2.4 for $n=1$. We will prove the existence and uniqueness of a weak solution to the obstacle problem. We will restrict our study to the case when φ is polynomial increasing in y, i.e.

Assumption 5.1. *We assume that for some $\kappa_1 \in \mathbb{R}$, $\beta_1 > 0, \forall y \in \mathbb{R}$,*

$$|\varphi(y)| \leqslant \kappa_1(1+|y|^{\beta_1}).$$

For the sake of PDE with obstacle, we introduce the reflected BSDE associated with (g,f,h), like in El Karoui et al. [7]:

$$\begin{cases} Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})dr + K_T^{t,x} - K_t^{t,x} - \int_s^T Z_s^{t,x}dB_s,\ P\text{-}a.s\ \forall s \in [t,T] \\ Y_s^{t,x} \geqslant L_s^{t,x},\ P\text{-}a.s \\ \int_t^T (Y_s^{t,x} - L_s^{t,x})dK_s^{t,x} = 0,\ P\text{-}a.s. \end{cases} \tag{24}$$

where $L_s^{t,x} = h(s, X_s^{t,x})$ is a continuous process. Moreover following Lepeltier et al [12], we shall need to estimate

$$\begin{aligned} E[\sup_{t \leqslant s \leqslant T} \varphi^2(e^{\mu t}(L_s^{t,x})^+)] &= E[\sup_{t \leqslant s \leqslant T} \varphi^2(e^{\mu t} h(s, X_s^{t,x})^+)] \\ &\leqslant Ce^{2\beta_1 \mu T} E[\sup_{t \leqslant s \leqslant T} (1 + \left|X_s^{t,x}\right|^{2\beta_1 \beta})] \\ &\leqslant C(1 + |x|^{2\beta_1 \beta}), \end{aligned}$$

where C is a constant which can be changed line by line. By assumption 2.4-(ii), with same techniques we get for $x \in \mathbb{R}$, $E[\sup_{t \leqslant s \leqslant T} \varphi^2((L_s^{t,x})^+)] < +\infty$. Thanks to the assumption 2.1 and 2.2, by the equivalence of norms 7 and 8, we have

$$g(X_T^{t,x}) \in \mathbf{L}^2(\mathcal{F}_T) \text{ and } f(s, X_s^{t,x}, 0, 0) \in \mathbf{H}^2(0, T).$$

By the existence and uniqueness theorem for the RBSDE in [12], for each (t, x), there exists a unique triple $(Y^{t,x}, Z^{t,x}, K^{t,x}) \in \mathbf{S}^2(t, T) \times \mathbf{H}_d^2(t, T) \times \mathbf{A}^2(t, T)$ of $\{\mathcal{F}_s^t\}$ progressively measurable processes, which is the solution of the reflected BSDE with parameters $(g(X_T^{t,x}), f(s, X_s^{t,x}, y, z), h(s, X_s^{t,x}))$We shall give the probabilistic interpretation for the solution of PDE with obstacle (3).

The main result of this section is

Theorem 5.1. *Assume that assumptions 2.1-2.5 hold and $\rho(x) = (1 + |x|)^{-p}$ with $p \geqslant \gamma$ where $\gamma = \beta_1 \beta + \beta + d + 1$. There exists a pair (u, ν), which is the solution of the PDE with obstacle (3) associated to (g, f, h) i.e. (u, ν) satisfies Definition 2.2-(i) -(iii). Moreover the solution is given by: $u(t, x) = Y_t^{t,x}$, a.e. where $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})_{t \leqslant s \leqslant T}$ is the solution of RBSDE (24), and*

$$Y_s^{t,x} = u(s, X_s^{t,x}), Z_s^{t,x} = (\sigma^* \nabla u)(s, X_s^{t,x}). \tag{25}$$

Moreover, we have for every measurable bounded and positive functions ϕ and ψ,

$$\int_{\mathbb{R}^d} \int_t^T \phi(s, \widehat{X}_s^{t,x}) J(\widehat{X}_s^{t,x}) \psi(s, x) 1_{\{u=h\}}(s, x) d\nu(s, x) = \int_{\mathbb{R}^d} \int_t^T \phi(s, x) \psi(s, X_s^{t,x}) dK_s^{t,x}, \text{ a.s..} \tag{26}$$

If $(\overline{u}, \overline{\nu})$ is another solution of the PDE (3) such that $\overline{\nu}$ satisfies (26) with some $\overline{K}$ instead of K, where $\overline{K}$ is a continuous process in $\mathbf{A}_{\mathcal{F}}^2(t, T)$, then $\overline{u} = u$ and $\overline{\nu} = \nu$.

Remark 5.1. *The expression (26) gives us the probabilistic interpretation (Feymamn-Kac's formula) for the measure ν via the increasing process $K^{t,x}$ of the RBSDE. This formula was first introduced in Bally et al. [1], where the authors prove (26) when f is Lipschitz on y and z uniformly in (t, ω). Here we generalize their result to the case when f is monotonic in y and Lipschitz in z.*

Proof. As in the proof of theorem 4.1 in section 4, we first notice that (u, ν) solves (3) if and only if

$$(\widehat{u}(t, x), d\widehat{\nu}(t, x)) = (e^{\mu t} u(t, x), e^{\mu t} d\nu(t, x))$$

is the solution of the PDE with obstacle $(\widehat{g}, \widehat{f}, \widehat{h})$, where $\widehat{g}$, $\widehat{f}$ are defined as in (12) with

$$\widehat{h}(t, x) = e^{\mu t} h(t, x).$$

Then the coefficient $\widehat{f}$ satisfies the same assumptions in assumption 2.3 with (iii) replaced by (12), which means that f is decreasing on y in the 1-dimensional case. The obstacle $\widehat{h}$ still satisfies assumption 2.4, for $\mu = 0$. In the following we will use (g, f, h) instead of $(\widehat{g}, \widehat{f}, \widehat{h})$, and suppose that (g, f, h) satisfies assumption 2.1, 2.2, 2.4, 2.5 and 2.3 with (iii) replaced by (12).

a) Existence : *The existence of a solution will be proved in 4 steps. From step 1 to step 3, we suppose that f does not depend on ∇u, satisfies assumption 2.3' for $n = 1$, and $f(t, x, 0) \in \mathbf{L}^2([0,T] \times \mathbb{R}^d, dt \otimes \rho(x)dx)$. In the step 4, we study the case when f depend on ∇u.*

Step 1 : *Suppose $g(x)$, $f(t, x, 0)$, $h^+(t, x)$ uniformly bounded* i.e. that there exists a constant C such that

$$|g(x)| + \sup_{0 \leqslant t \leqslant T} |f(t, x, 0)| + \sup_{0 \leqslant t \leqslant T} h^+(t, x) \leqslant C.$$

We will use the penalization method. For $n \in \mathbb{N}$, we consider for all $s \in [t, T]$,

$$Y_s^{n,t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{n,t,x})dr + n\int_s^T (Y_r^{n,t,x} - h(r, X_r^{t,x}))^- dr - \int_s^T Z_r^{n,t,x} dB_r.$$

From Theorem 4.1 in section 3, we know that $u_n(t, x) := Y_t^{n,t,x}$, is solution of the PDE(g, f_n), where $f_n(t, x, y, x) = f(t, x, y, z) + n(y - h(t, x))^-$, i.e. for every $\phi \in C_c^{1,\infty}([0, T] \times \mathbb{R}^d)$

$$\begin{aligned}\int_t^T (u_s^n, \partial_s \phi)ds + (u^n(t, \cdot), \phi(t, \cdot)) - (g(\cdot), \phi(\cdot, T)) + \int_t^T \mathcal{E}(u_s^n, \phi_s)ds \\ = \int_t^T (f(s, \cdot, u_s^n), \phi_s)ds + n\int_t^T ((u^n - h)^-(s, \cdot), \phi_s)ds.\end{aligned}$$

Moreover

$$Y_s^{n,t,x} = u_n(s, X_s^{t,x}), Z_s^{n,t,x} = \sigma^* \nabla u_n(s, X_s^{t,x}), \tag{27}$$

Set $K_s^{n,t,x} = n\int_t^s (Y_r^{n,t,x} - h(r, X_r^{t,x}))^- dr$. Then by (27), we have that $K_s^{n,t,x} = n\int_t^s (u_n - h)^-(r, X_r^{t,x})dr$.

Following the estimates and convergence results for $(Y^{n,t,x}, Z^{n,t,x})$ in the step 1 of the proof of Theorem 2.2 in [12], for m, $n \in \mathbb{N}$, we have,as $m, n \to \infty$

$$E\int_t^T \left|Y_s^{n,t,x} - Y_s^{m,t,x}\right|^2 ds + E\int_t^T \left|Z_s^{n,t,x} - Z_s^{m,t,x}\right|^2 ds + E\sup_{t \leqslant s \leqslant T} \left|K_s^{n,t,x} - K_s^{m,t,x}\right|^2 \to 0,$$

and

$$\sup_n E\int_0^T (\left|Y_s^{n,t,x}\right|^2 + \left|Z_s^{n,t,x}\right|^2 + (K_T^{n,t,x})^2) \leqslant C.$$

By the equivalence of the norms (8), we get

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_t^T \rho(x)(|u_n(s, x) - u_m(s, x)|^2 + |\sigma^*\nabla u_n(s, x) - \sigma^*\nabla u_m(s, x)|^2)dsdx \\ \leqslant\ &\frac{1}{k_2}\int_{\mathbb{R}^d} \rho(x)E\int_t^T (\left|Y_s^{n,t,x} - Y_s^{m,t,x}\right|^2 + \left|Z_s^{n,t,x} - Z_s^{m,t,x}\right|^2)dsdx \to 0.\end{aligned}$$

Thus (u_n) is a Cauchy sequence in $\mathcal{H}$, and the limit $u = \lim_{n\to\infty} u_n$ belongs to $\mathcal{H}$.

Denote $\nu_n(dt,dx) = n(u_n-h)^-(t,x)dtdx$ and $\pi_n(dt,dx) = \rho(x)\nu_n(dt,dx)$, then by (7)

$$\begin{aligned}\pi_n([0,T]\times\mathbb{R}^d) &= \int_{\mathbb{R}^d}\int_0^T \rho(x)\nu_n(dt,dx) = \int_{\mathbb{R}^d}\int_0^T \rho(x)n(u_n-h)^-(t,x)dtdx \\ &\leqslant \frac{1}{k_2}\int_{\mathbb{R}^d}\rho(x)E\left|K_T^{n,0,x}\right|dx \leqslant C\int_{\mathbb{R}^d}\rho(x)dx < \infty.\end{aligned}$$

It follows that

$$\sup_n \pi_n([0,T]\times\mathbb{R}^d) < \infty. \tag{28}$$

In the same way like in the existence proof step 2 of theorem 14 in [1], we can prove that $\pi_n([0,T]\times\mathbb{R}^d)$ is bounded and then π_n is tight. So we may pass to a subsequence and get $\pi_n \to \pi$ where π is a positive measure. Define $\nu = \rho^{-1}\pi$; ν is a positive measure such that $\int_0^T\int_{\mathbb{R}^d}\rho(x)d\nu(t,x) < \infty$, and so we have for $\phi \in C_c^{1,\infty}([0,T]\times\mathbb{R}^d)$ with compact support in x,

$$\int_{\mathbb{R}^d}\int_t^T \phi d\nu_n = \int_{\mathbb{R}^d}\int_t^T \frac{\phi}{\rho}d\pi_n \to \int_{\mathbb{R}^d}\int_t^T \frac{\phi}{\rho}d\pi = \int_{\mathbb{R}^d}\int_t^T \phi d\nu.$$

Now passing to the limit in the PDE(g,f_n), we check that (u,ν) satisfies the PDE with obstacle (g,f,h), i.e. for every $\phi \in C_c^{1,\infty}([0,T]\times\mathbb{R}^d)$, we have

$$\begin{aligned}&\int_t^T (u_s,\partial_s\phi)ds + (u(t,\cdot),\phi(t,\cdot)) - (g(\cdot),\phi(\cdot,T)) + \int_t^T \mathcal{E}(u_s,\phi_s)ds \\ = &\int_t^T (f(s,\cdot,u_s),\phi_s)ds + \int_t^T\int_{\mathbb{R}^d}\phi(s,x)1_{\{u=h\}}(s,x)d\nu(x,s).\end{aligned} \tag{29}$$

The last is to prove that ν satisfies the probabilistic interpretation (26). Since $K^{n,t,x}$ converges to $K^{t,x}$ uniformly in t, the measure $dK^{n,t,x} \to dK^{t,x}$ weakly in probability.

Fix two continuous functions ϕ, ψ : $[0,T]\times\mathbb{R}^d \to \mathbb{R}^+$ which have compact support in x and a continuous function with compact support $\theta : \mathbb{R}^d \to \mathbb{R}^+$, we have

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_t^T \phi(s,\widehat{X}_s^{t,x})J(\widehat{X}_s^{t,x})\psi(s,x)\theta(x)d\nu(s,x) \\ = &\lim_{n\to\infty}\int_{\mathbb{R}^d}\int_t^T \phi(s,\widehat{X}_s^{t,x})J(\widehat{X}_s^{t,x})\psi(s,x)\theta(x)n(u_n-h)^-(t,x)dtdx \\ = &\lim_{n\to\infty}\int_{\mathbb{R}^d}\int_t^T \phi(s,x)\psi(s,X_s^{t,x})\theta(X_s^{t,x})n(u_n-h)^-(t,X_s^{t,x})dtdx \\ = &\lim_{n\to\infty}\int_{\mathbb{R}^d}\int_t^T \phi(s,x)\psi(s,X_s^{t,x})\theta(X_s^{t,x})dK_s^{n,t,x}dx \\ = &\int_{\mathbb{R}^d}\int_t^T \phi(s,x)\psi(s,X_s^{t,x})\theta(X_s^{t,x})dK_s^{t,x}dx.\end{aligned}$$

We take $\theta = \theta_R$ to be the regularization of the indicator function of the ball of radius R and pass to the limit with $R \to \infty$, it follows that

$$\int_{\mathbb{R}^d}\int_t^T \phi(s,\widehat{X}_s^{t,x})J(\widehat{X}_s^{t,x})\psi(s,x)d\nu(s,x) = \int_{\mathbb{R}^d}\int_t^T \phi(s,x)\psi(s,X_s^{t,x})dK_s^{t,x}dx. \tag{30}$$

Since $(Y_s^{n,t,x}, Z_s^{n,t,x}, K_s^{n,t,x})$ converges to $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})$ as $n \to \infty$ in $\mathbf{S}^2(t,T)\times\mathbf{H}^2(t,T) \times \mathbf{A}^2(t,T)$, and $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})$ is the solution of RBSDE$(g(X_T^{t,x}), f, h)$, then we have

$$\int_t^T (Y_s^{t,x} - L_s^{t,x}) dK_s^{t,x} = \int_t^T (u-h)(t, X_s^{t,x}) dK_s^{t,x} = 0, \text{a.s.}$$

it follows that $dK_s^{t,x} = 1_{\{u=h\}}(s, X_s^{t,x}) dK_s^{t,x}$. In (30), setting $\psi = 1_{\{u=h\}}$ yields

$$\int_{\mathbb{R}^d}\int_t^T \phi(s, \widehat{X}_s^{t,x}) J(\widehat{X}_s^{t,x}) 1_{\{u=h\}}(s,x) d\nu(s,x) = \int_{\mathbb{R}^d}\int_t^T \phi(s, \widehat{X}_s^{t,x}) J(\widehat{X}_s^{t,x}) d\nu(s,x), \text{ a.s.}$$

Note that the family of functions $A(\omega) = \{(s,x) \to \phi(s, \widehat{X}_s^{t,x}) : \phi \in C_c^\infty\}$ is an algebra which separates the points (because $x \to \widehat{X}_s^{t,x}$ is a bijection). Given a compact set G, $A(\omega)$ is dense in $C([0,T] \times G)$. It follows that $J(\widehat{X}_s^{t,x}) 1_{\{u=h\}}(s,x) d\nu(s,x) = J(\widehat{X}_s^{t,x}) d\nu(s,x)$ for almost every ω. While $J(\widehat{X}_s^{t,x}) > 0$ for almost every ω, we get $d\nu(s,x) = 1_{\{u=h\}}(s,x) d\nu(s,x)$, and (26) follows.

Then we get easily that $Y_s^{t,x} = u(s, X_s^{t,x})$ and $Z_s^{t,x} = \sigma^* \nabla u(s, X_s^{t,x})$, in view of the convergence results for $(Y_s^{n,t,x}, Z_s^{n,t,x})$ and the equivalence of the norms. So $u(s, X_s^{t,x}) = Y_s^{t,x} \geqslant h(t,x)$. Specially for $s = t$, we have $u(t,x) \geqslant h(t,x)$

Step 2 : *As in the proof of the RBSDE in Theorem 2.2 in [12], step 2, we relax the bounded condition on the barrier h in step 1, and prove the existence of the solution under assumption 2.4.*

Similarly to step 2 in the proof of theorem 2.2 in [12], after some transformation, we know that it is sufficient to prove the existence of the solution for the PDE with obstacle (g,f,h), where (g,f,h) satisfies

$$g(x), f(t,x,0) \leqslant 0.$$

Let $h(t,x)$ satisfy assumption 2.4 for $\mu = 0$, i.e. $\forall (t,x) \in [0,T] \times, \mathbb{R}^d$

$$\varphi(h(t,x)^+) \in \mathbf{L}^2(\mathbb{R}^d; \rho(x)dx),$$

and

$$|h(t,x)| \leqslant \kappa(1 + |x|^\beta).$$

Set

$$h_n(t,x) = h(t,x) \wedge n,$$

then the function $h_n(t,x)$ are continuous, $\sup_{0 \leqslant t \leqslant T} h_n^+(t,x) \leqslant n$, and $h_n(s, X_s^{t,x}) \to h(s, X_s^{t,x})$ in $\mathbf{S}^2_{\mathcal{F}}(t,T)$, in view of Dini's theorem and dominated convergence theorem.

We consider the PDE with obstacle associated with (g,f,h_n). By the results of step 1, there exists (u_n, ν_n), which is the solution of the PDE with obstacle associated to (g,f,h_n), where $u_n \in \mathcal{H}$ and ν_n is a positive measure such that $\int_0^T \int_{\mathbb{R}^d} \rho(x) d\nu_n(t,x) < \infty$. Moreover

$$\begin{cases} Y_s^{n,t,x} = u_n(s, X_s^{t,x}), Z_s^{n,t,x} = \sigma^* \nabla u_n(s, X_s^{t,x}), \\ \displaystyle\int_{\mathbb{R}^d}\int_t^T \phi(s, \widehat{X}_s^{t,x}) J(\widehat{X}_s^{t,x}) \psi(s,x) 1_{\{u_n = h_n\}}(s,x) d\nu_n(s,x) = \int_{\mathbb{R}^d}\int_t^T \phi(s,x) \psi(s, X_s^{t,x}) dK_s^{n,t,x} dx, \end{cases} \tag{31}$$

Here $(Y^{n,t,x}, Z^{n,t,x}, K^{n,t,x})$ is the solution of the RBSDE$(g(X_T^{t,x}), f, h_n)$. Thanks to proposition 6.1 in Appendix, and the bounded assumption of g and f, we know that

$$
\begin{aligned}
& E\,\Big[\int_t^T \big(\big|Y_s^{n,t,x}\big|^2 + \big|Z_s^{n,t,x}\big|^2\,\big)ds + (K_T^{n,0,x})^2\Big] \\
& \leqslant C\,\big(1 + E\,\big[\varphi^2(\sup_{0\,\leqslant\,t\,\leqslant\,T} h^+(t, X_t^{0,x})\,) + \sup_{0\,\leqslant\,t\,\leqslant\,T}(h^+(t, X_t^{0,x}))^2\big]\big) \\
& \leqslant C(1 + |x|^{2\beta_1\beta} + |x|^{2\beta}).
\end{aligned}
\tag{32}
$$

By the Lemma 2.3 in [12], $Y_s^{n,t,x} \to Y_s^{t,x}$ in $\mathbf{S}^2(0,T)$, $Z_s^{n,t,x} \to Z_s^{t,x}$ in $\mathbf{H}_d^2(0,T)$ and $K_s^{n,t,x} \to K_s^{t,x}$ in $\mathbf{A}^2(0,T)$, as $n \to \infty$. Moreover $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})$ is the solution of RBSDE$(g(X_T^{t,x}), f, h)$.

By the convergence result of $(Y_s^{n,t,x}, Z_s^{n,t,x})$ and the equivalence of the norms (8), we get

$$
\begin{aligned}
& \int_{\mathbb{R}^d} \rho(x) \int_t^T (|u_n(t,x) - u_m(t,x)|^2 + |\sigma^* \nabla u_n(s,x) - \sigma^* \nabla u_m(s,x)|^2) ds dx \\
\leqslant \quad & \frac{1}{k_2} \int_{\mathbb{R}^d} \rho(x) E \int_t^T (\big|Y_s^{n,t,x} - Y_s^{m,t,x}\big|^2 + \big|Z_s^{n,t,x} - Z_s^{m,t,x}\big|^2) ds dx \to 0.
\end{aligned}
$$

So $\{u_n\}$ is a Cauchy sequence in $\mathcal{H}$, and admits a limit $u \in \mathcal{H}$. Moreover $Y_s^{t,x} = u(s, X_s^{t,x})$, $Z_s^{t,x} = \sigma^* \nabla u(s, X_s^{t,x})$. In particular $u(t,x) = Y_t^{t,x} \geqslant h(t,x)$.

Set $\pi_n = \rho \nu_n$, like in step 1, we first need to prove that $\pi_n([0,T] \times \mathbb{R}^d)$ is uniformly bounded. In (31), let $\phi = \rho$, $\psi = 1$, then we have

$$
\int_{\mathbb{R}^d} \int_0^T \rho(\widehat{X}_s^{0,x}) J(\widehat{X}_s^{0,x}) d\nu_n(s,x) = \int_{\mathbb{R}^d} \int_0^T \rho(x) dK_s^{n,0,x} dx.
$$

Recall Lemma 3.2: there exist two constants $c_1 > 0$ and $c_2 > 0$ such that $\forall x \in \mathbb{R}^d$, $0 \leqslant t \leqslant T$

$$
c_1 \leqslant E \left(\frac{\rho(t, \widehat{X}_t^{0,x}) J(\widehat{X}_t^{0,x})}{\rho(x)} \right) \leqslant c_2.
$$

Applying Hölder's inequality and Schwartz's inequality, we have

$$\begin{aligned}
& \pi_n([0,T]\times\mathbb{R}^d) \\
= & \int_{\mathbb{R}^d}\int_0^T \rho(x)\nu_n(dt,dx) \\
= & \int_{\mathbb{R}^d}\int_0^T \frac{\rho^{\frac12}(x)}{\rho^{\frac12}(t,\widehat{X}_t^{0,x})J^{\frac12}(\widehat{X}_t^{0,x})}\rho^{\frac12}(x)\rho^{\frac12}(t,\widehat{X}_t^{0,x})J^{\frac12}(\widehat{X}_t^{0,x})\nu_n(dt,dx) \\
\leqslant & E[\left(\int_{\mathbb{R}^d}\int_0^T \frac{\rho(x)}{\rho(t,\widehat{X}_t^{0,x})J(\widehat{X}_t^{0,x})}\rho(x)\nu_n(dt,dx)\right)^{\frac12}\left(\int_{\mathbb{R}^d}\int_0^T \rho(t,\widehat{X}_t^{0,x})J(\widehat{X}_t^{0,x})\nu_n(dt,dx)\right)^{\frac12}] \\
\leqslant & \left(E\int_{\mathbb{R}^d}\int_0^T \frac{\rho(x)}{\rho(t,\widehat{X}_t^{0,x})J(\widehat{X}_t^{0,x})}\rho(x)\nu_n(dt,dx)\right)^{\frac12}\left(E\int_{\mathbb{R}^d}\int_0^T \rho(t,\widehat{X}_t^{0,x})J(\widehat{X}_t^{0,x})\nu_n(dt,dx)\right)^{\frac12} \\
= & \left(\int_{\mathbb{R}^d}\int_0^T E\left(\frac{\rho(x)}{\rho(t,\widehat{X}_t^{0,x})J(\widehat{X}_t^{0,x})}\right)\rho(x)\nu_n(dt,dx)\right)^{\frac12}\left(\int_{\mathbb{R}^d}E\int_0^T dK_t^{n,0,x}\rho(x)dx\right)^{\frac12} \\
\leqslant & \left(\frac{1}{c_1}\int_{\mathbb{R}^d}\int_0^T \rho(x)\nu_n(dt,dx)\right)^{\frac12}\left(\int_{\mathbb{R}^d}\rho(x)E[K_T^{n,0,x}]dx\right)^{\frac12}.
\end{aligned}$$

So by (32) and (7), we get

$$\begin{aligned}
\sup_n \pi_n([0,T]\times\mathbb{R}^d) &\leqslant C\int_{\mathbb{R}^d}\rho(x)E[K_T^{n,0,x}]dx \\
&\leqslant C\int_{\mathbb{R}^d}\rho(x)(1+|x|^{\beta_1\beta}+|x|^{\beta})dx<\infty.
\end{aligned}\tag{33}$$

Using the same arguments as in step 1, we deduce that π_n is tight. So we may pass to a subsequence and get $\pi_n \to \pi$ where π is a positive measure.

Define $\nu = \rho^{-1}\pi$, then ν is a positive measure such that $\int_0^T\int_{\mathbb{R}^d}\rho(x)d\nu(t,x) < \infty$. Then for $\phi \in C([0,T]\times\mathbb{R}^d)$ with compact support in x, we have as $n\to\infty$,

$$\int_t^T\int \phi d\nu_n = \int_t^T\int\frac{\phi}{\rho}d\pi_n \to \int_t^T\int\frac{\phi}{\rho}d\pi = \int_t^T\int\phi d\nu.$$

Now passing to the limit in the PDE(g,f,h_n), we check that (u,ν) satisfies the PDE with obstacle associated to (g,f,h), i.e. for every $\phi\in C_c^{1,\infty}([0,T]\times\mathbb{R}^d)$

$$\begin{aligned}
&\int_t^T(u_s,\partial_s\phi)ds + (u(t,\cdot),\phi(t,\cdot)) - (g(\cdot),\phi(\cdot,T)) + \int_t^T\mathcal{E}(u_s,\phi_s)ds \\
= & \int_t^T(f(s,\cdot,u_s),\phi_s)ds + \int_t^T\int_{\mathbb{R}^d}\phi(s,x)1_{\{u=h\}}d\nu(x,s).
\end{aligned}\tag{34}$$

Then we will check if the probabilistic interpretation (26) still holds. Fix two continuous functions $\phi,\psi : [0,T]\times\mathbb{R}^d\to\mathbb{R}^+$ which have compact support in x. With the convergence result of

$K^{n,t,x}$, which implies $dK^{n,t,x} \to dK^{t,x}$ weakly in probability, in the same way as step 1, passing to the limit in (31) we have

$$\int_{\mathbb{R}^d}\int_t^T \phi(s,\widehat{X}_s^{t,x})J(\widehat{X}_s^{t,x})\psi(s,x)d\nu(s,x) = \int_{\mathbb{R}^d}\int_t^T \phi(s,x)\psi(s,X_s^{t,x})dK_s^{t,x}dx$$

Since $(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x})$ is the solution of RBSDE$(g(X_T^{t,x}), f, h)$, then by the integral condition, we deduce the $dK_s^{t,x} = 1_{\{u=h\}}(s, X_s^{t,x})dK_s^{t,x}$. In (35), setting $\psi = 1_{\{u=h\}}$ yields

$$\int_{\mathbb{R}^d}\int_t^T \phi(s,\widehat{X}_s^{t,x})J(\widehat{X}_s^{t,x})1_{\{u=h\}}(s,x)d\nu(s,x) = \int_{\mathbb{R}^d}\int_t^T \phi(s,\widehat{X}_s^{t,x})J(\widehat{X}_s^{t,x})d\nu(s,x).$$

With the same arguments, we get that $d\nu(s,x) = 1_{\{u=h\}}(s,x)d\nu(s,x)$, and (26)holds for ν and K.

Step 3 : *Now we will relax the bounded condition on $g(x)$ and $f(t,x,0)$.* Then for $m,n \in \mathbb{N}$, let

$$\begin{aligned} g_{m,n}(x) &= (g(x)\wedge n)\vee(-m), \\ f_{m,n}(t,x,y) &= f(t,x,y) - f(t,x,0) + (f(t,x,0)\wedge n)\vee(-m). \end{aligned}$$

So $g_{m,n}(x)$ and $f_{m,n}(t,x,0)$ are bounded and for fixed $m \in \mathbb{N}$, as $n \to \infty$, we have

$$\begin{aligned} g_{m,n}(x) &\to g_m(x) \text{ in } \mathbf{L}^2(\mathbb{R}^d, \rho(x)dx), \\ f_{m,n}(t,x,0) &\to f_m(t,x,0) \text{ in } \mathbf{L}^2([0,T]\times\mathbb{R}^d, dt\otimes\rho(x)dx), \end{aligned}$$

where

$$\begin{aligned} g_m(x) &= g(x)\vee(-m), \\ f_m(t,x,y) &= f(t,x,y) - f(t,x,0) + f(t,x,0)\vee(-m). \end{aligned}$$

Then as $m \to \infty$, we have

$$\begin{aligned} g_m(x) &\to g(x) \text{ in } \mathbf{L}^2(\mathbb{R}^d, \rho(x)dx), \\ f_m(t,x,0) &\to f(t,x,0) \text{ in } \mathbf{L}^2([0,T]\times\mathbb{R}^d, dt\otimes\rho(x)dx), \end{aligned}$$

in view of assumption 2.1 and $f(t,x,0) \in \mathbf{L}^2([0,T]\times\mathbb{R}^d, dt\otimes\rho(x)dx)$.

Now we consider the PDE with obstacle associated to $(g_{m,n}, f_{m,n}, h)$. By step 2, there exists a $(u_{m,n}, \nu_{m,n})$ which is the solution of the PDE with obstacle associated to $(g_{m,n}, f_{m,n}, h)$. In particular the representation formulas (25) and (26) are satisfied. Denote by $(Y^{m,n,t,x}, Z^{m,n,t,x}, K^{m,n,t,x})$ the solution of the RBSDE $(g_{m,n}(X_T^{t,x}), f_{m,n}, h)$.

Recall the convergence results in step 3 of theorem 2.2 in [12], we know that for fixed $m \in \mathbb{N}$, as $n \to \infty$, $(Y_s^{m,n,t,x}, Z_s^{m,n,t,x}, K_s^{m,n,t,x}) \to (Y_s^{m,t,x}, Z_s^{m,t,x}, K_s^{m,t,x})$ in $\mathbf{S}^2(0,T)\times\mathbf{H}_d^2(0,T)\times\mathbf{A}^2(0,T)$, and that $(Y_s^{m,t,x}, Z_s^{m,t,x}, K_s^{m,t,x})$ is the solution of RBSDE$(g_m(X_T^{t,x}), f_m, h)$.

By Itô's formula, we have for $n,p \in \mathbb{N}$,

$$\begin{aligned} & E\int_t^T (\left|Y_s^{m,n,t,x} - Y_s^{m,p,t,x}\right|^2 + \left|Z_s^{m,n,t,x} - Z_s^{m,p,t,x}\right|^2)ds \\ \leqslant \quad & CE\left|g_{m,n}(X_T^{t,x}) - g_{m,p}(X_T^{t,x})\right|^2 + CE\int_t^T \left|f_{m,n}(s,X_s^{t,x},0) - f_{m,p}(s,X_s^{t,x},0)\right|^2 ds, \end{aligned}$$

so by the equivalence of the norms (7) and (8), it follows that as $n \to \infty$,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_t^T \rho(x)(|u_{m,n}(t,x)-u_{m,p}(t,x)|^2+|\sigma^*\nabla u_{m,n}(s,x)-\sigma^*\nabla u_{m,p}(s,x)|^2)dsdx \\
\leqslant\quad & \frac{Ck_1}{k_2}\int_{\mathbb{R}^d}\rho(x)\left|g_{m,n}(x)-g_{m,p}(x)\right|^2dx+\frac{Ck_1}{k_2}\int_{\mathbb{R}^d}\int_t^T\rho(x)\left|f_{m,n}(s,x,0)-f_{m,p}(s,x,0)\right|^2dsdx \to 0.
\end{aligned}$$

i.e. for each fixed $m \in \mathbb{N}$, $\{u_{m,n}\}$ is a Cauchy sequence in $\mathcal{H}$, and admits a limit $u_m \in \mathcal{H}$. Moreover $Y_s^{m,t,x} = u_m(s,X_s^{t,x}), Z_s^{m,t,x} = \sigma^*\nabla u_m(s,X_s^{t,x})$, a.s., in particular $u_m(t,x) = Y_t^{m,t,x} \geqslant h(t,x)$.

Then we find the measure ν_m by the sequence $\{\nu_{m,n}\}$. Set $\pi_{m,n} = \rho\nu_{m,n}$, by proposition 6.1 in Appendix, we have for each $m,n \in \mathbb{N}$, $0 \leqslant t \leqslant T$

$$\begin{aligned}
E(\left|K_T^{m,n,t,x}\right|^2) \quad &\leqslant\quad CE[g_{m,n}^2(X_T^{t,x})+\int_0^T f_{m,n}^2(s,X_s^{t,x},0,0)ds+\varphi^2(\sup_{t\leqslant s\leqslant T}(h^+(s,X_s^{t,x}))) \\
&\qquad +\sup_{t\leqslant s\leqslant T}(h^+(s,X_s^{t,x}))^2+1+\varphi^2(2T)] \\
&\leqslant\quad CE[g(X_T^{t,x})^2+\int_0^T f^2(s,X_s^{t,x},0,0)ds+\varphi^2(\sup_{0\leqslant s\leqslant T}(h^+(s,X_s^{t,x}))) \\
&\qquad +\sup_{0\leqslant s\leqslant T}(h^+(s,X_s^{t,x}))^2+1+\varphi^2(2T)] \\
&\leqslant\quad C(1+|x|^{2\beta_1\beta}+|x|^{2\beta}). \qquad (35)
\end{aligned}$$

By the same way as in step 2, we deduce that for each fixed $m \in \mathbb{N}$, $\pi_{m,n}$ is tight, we may pass to a subsequence and get $\pi_{m,n} \to \pi_m$ where π_m is a positive measure. If we define $\nu_m = \rho^{-1}\pi_m$, then ν_m is a positive measure such that $\int_0^T\int_{\mathbb{R}^d}\rho(x)d\nu_m(t,x) < \infty$. So we have for all $\phi \in C([0,T]\times\mathbb{R}^d)$ with compact support in x,

$$\int_t^T\int\phi d\nu_{m,n} = \int_t^T\int\frac{\phi}{\rho}d\pi_{m,n} \to \int_t^T\int\frac{\phi}{\rho}d\pi_m = \int_t^T\int\phi d\nu_m.$$

Now for each fixed $m \in \mathbb{N}$, let $n \to \infty$, in the PDE$(g_{m,n}, f_{m,n}, h)$, we check that (u_m, ν_m) satisfies the PDE with obstacle associated to (g_m, f_m, h), and by the weak convergence result of $dK^{m,n,t,x}$, we have easily that the probabilistic interpretation (26) holds for ν_m and K^m.

Then let $m \to \infty$, by the convergence results in step 4 of theorem 2.2 in [12], we apply the same method as before. We deduce that $\lim_{m\to\infty} u_m = u$ is in $\mathcal{H}$ and $Y_s^{t,x} = u(s,X_s^{t,x}), Z_s^{t,x} = \sigma^*\nabla u(s,X_s^{t,x})$, a.s., where $(Y^{t,x}, Z^{t,x}, K^{t,x})$ is the solution of the RBSDE(g,f,h), in particular, setting $s=t$, $u_(t,x) = Y_t^{t,x} \geqslant h(t,x)$.

From (35), it follows that

$$E[(K_T^{m,t,x})^2] \leqslant C(1+|x|^{2\beta\beta_1}+|x|^{2\beta}).$$

By the same arguments, we can find the measure ν by the sequence $\{\nu_m\}$, which satisfies that for all ϕ and ψ with compact support,

$$\int_{\mathbb{R}^d}\int_t^T\phi(s,\widehat{X}_s^{t,x})J(\widehat{X}_s^{t,x})\psi(s,x)1_{\{u=h\}}(s,x)d\nu(s,x) = \int_{\mathbb{R}^d}\int_t^T\phi(s,x)\psi(s,X_s^{t,x})dK_s^{t,x}dx.$$

Finally we find a solution (u,ν) to the PDE with obstacle (g,f,h), when f does not depend on ∇u. So for every $\phi \in C_c^{1,\infty}([0,T]\times\mathbb{R}^d)$

$$\begin{aligned} & \int_t^T (u_s,\partial_s\phi)ds + (u(t,\cdot),\phi(t,\cdot)) - (g(\cdot),\phi(\cdot,T)) + \int_t^T \mathcal{E}(u_s,\phi_s)ds \\ = \;& \int_t^T (f(s,\cdot,u_s),\phi_s)ds + \int_t^T \int_{\mathbb{R}^d} \phi(s,x)1_{\{u=h\}}d\nu(x,s). \end{aligned} \tag{36}$$

Step 4 : *Finally we study the case when f depends on ∇u, and satisfies a Lipschitz condition on ∇u.* We construct a mapping Ψ from $\mathcal{H}$ into itself. For some $\overline{u}\in\mathcal{H}$, define

$$u = \Psi(\overline{u}),$$

where (u,ν) is a weak solution of the PDE with obstacle $(g, f(t,x,u,\sigma\nabla\overline{u}),h)$. Then by this mapping, we denote a sequence $\{u_n\}$ in $\mathcal{H}$, beginning with a function $v^0 \in \mathbf{L}^2([0,T]\times\mathbb{R}^d, dt\otimes\rho(x)dx)$. Since $f(t,x,u,v^0(t,x))$ satisfies the assumptions of step 3, the PDE$(g,f(t,x,u,v^0(t,x)),h)$ admits a solution $(u_1,v_1)\in\mathcal{H}$. For $n\in\mathbb{N}$, set $u_n(t,x) = \Psi(u_{n-1}(t,x))$.

Symmetrically we introduce a mapping Φ from $\mathbf{H}^2(t,T)\times\mathbf{H}^2_d(t,T)$ into itself. For $V^{t,x,0} = v^0(s,X_s^{t,x}))$, then $V_s^{t,x}\in\mathbf{H}^2_d(t,T)$ in view of the equivalence of the norms. Set

$$(Y^{t,x,n},Z^{t,x,n}) = \Phi(Y^{t,x,n-1},Z^{t,x,n-1}),$$

where $(Y^{t,x,n},Z^{t,x,n},K^{t,x,n})$is the solution of the RBSDE with parameters $g(X_T^{t,x})$, $f(s,X_s^{t,x},Y_s^{t,x},Z_s^{t,x,n-1})$ and $h(s,X_s^{t,x})$.Then $Y_s^{t,x,n} = u_n(s,X_s^{t,x})$, $Z_s^{t,x,n} = \sigma^*\nabla u_n(s,X_s^{t,x})$ a.s. and

$$\int_{\mathbb{R}^d}\int_t^T \phi(s,\widehat{X}_s^{t,x})J(\widehat{X}_s^{t,x})\psi(s,x)1_{\{u=h\}}(s,x)d\nu_n(s,x) = \int_{\mathbb{R}^d}\int_t^T \phi(s,x)\psi(s,X_s^{t,x})dK_s^{t,x,n}dx.$$

Set $\widetilde{u}_n(t,x) := u_n(t,x) - u_{n-1}(t,x)$. To deal with the difference $\widetilde{u}_n$, we need the difference of the corresponding BSDE, denote $\widetilde{Y}_s^{t,x,n} := Y_s^{t,x,n} - Y_s^{t,x,n-1}$, $\widetilde{Z}_s^{t,x,n} := Z_s^{t,x,n} - Z_s^{t,x,n-1}$, $\widetilde{K}_s^{t,x,n} := K_s^{t,x,n} - K_s^{t,x,n-1}$. It follows from Itô's formula, for some α, $\gamma\in\mathbb{R}$,

$$\begin{aligned} e^{\gamma t}E\left|\widetilde{Y}_s^{t,x,n}\right|^2 \quad &+\quad E\int_s^T e^{\gamma r}(\gamma\left|\widetilde{Y}_r^{t,x,n}\right|^2 + \left|\widetilde{Z}_r^{t,x,n}\right|^2)dr \\ &\leqslant\quad E\int_s^T e^{\gamma r}(\frac{k^2}{\alpha}\left|\widetilde{Y}_r^{t,x,n}\right|^2 + \alpha\left|\widetilde{Z}_r^{t,x,n-1}\right|^2)dr, \end{aligned}$$

since

$$\begin{aligned} & \int_s^T e^{\gamma r}\widetilde{Y}_r^{t,x,n}d\widetilde{K}_r^{t,x,n} \\ = \;& \int_s^T e^{\gamma r}(Y_s^{t,x,n} - h(r,X_r^{t,x}))dK^{t,x,n} + \int_s^T e^{\gamma r}(Y_s^{t,x,n-1} - h(r,X_r^{t,x}))dK^{t,x,n-1} \\ & -\int_s^T e^{\gamma r}(Y_s^{t,x,n} - h(r,X_r^{t,x}))dK^{t,x,n-1} + \int_s^T e^{\gamma r}(Y_s^{t,x,n-1} - h(r,X_r^{t,x}))dK^{t,x,n} \\ \leqslant \;& 0. \end{aligned}$$

then by the equivalence of the norms, for $\gamma = 1 + \frac{2k_1^2}{k_2^2}k^2$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_t^T e^{\gamma s}(|\widetilde{u}_n(s,x)|^2 + |\sigma^*\nabla(\widetilde{u}_n)(s,x)|^2)\rho(x)dsdx \\
\leqslant\ & (\frac{1}{2})^{n-1}\int_{\mathbb{R}^d}\int_t^T e^{\gamma s}(|\widetilde{u}_2(s,x)|^2 + |\sigma^*\nabla(\widetilde{u}_2)(s,x)|^2)\rho(x)dsdx \\
\leqslant\ & (\frac{1}{2})^{n-1}(\|u_1(s,x)\|_\gamma^2 + \|u_2(s,x)\|_\gamma^2).
\end{aligned}$$

where $\|u\|_\gamma^2 := \int_{\mathbb{R}^d}\int_t^T e^{\gamma s}(|u(s,x)|^2 + |\sigma^*\nabla u(s,x)|^2)\rho(x)dsdx$, which is equivalent to the norm $\|\cdot\|$ of $\mathcal{H}$. So $\{u_n\}$ is a Cauchy sequence in $\mathcal{H}$, it admits a limit u in $\mathcal{H}$, which is the solution to the PDE with obstacle (2). Then consider $\sigma^*\nabla u$ as a known function by the result of step 3, we know that there exists a positive measure ν such that $\int_0^T\int_{\mathbb{R}^d}\rho(x)d\nu(t,x) < \infty$, and for every $\phi \in C_c^{1,\infty}([0,T]\times\mathbb{R}^d)$,

$$\begin{aligned}
&\int_t^T (u_s, \partial_s\phi)ds + (u(t,\cdot),\phi(t,\cdot)) - (g(\cdot),\phi(\cdot,T)) + \int_t^T \mathcal{E}(u_s,\phi_s)ds \\
=\ & \int_t^T (f(s,\cdot,u_s,\sigma^*\nabla u_s),\phi_s)ds + \int_t^T\int_{\mathbb{R}^d}\phi(s,x)1_{\{u=h\}}d\nu(x,s). \qquad (37)
\end{aligned}$$

Moreover, for $t \leqslant s \leqslant T$,

$$Y_s^{t,x} = u(s,X_s^{t,x}),\ Z_s^{t,x} = \sigma^*\nabla u(s,X_s^{t,x}),\ \text{a.s.a.e.},$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_t^T \phi(s,\widehat{X}_s^{t,x})J(\widehat{X}_s^{t,x})\psi(s,x)1_{\{u=h\}}(s,x)d\nu(s,x) \\
=\ & \int_{\mathbb{R}^d}\int_t^T \phi(s,x)\psi(s,X_s^{t,x})dK_s^{t,x}.
\end{aligned}$$

b) Uniqueness : Set $(\overline{u},\overline{\nu})$ to be another solution of the PDE with obstacle (3) associated to (g,f,h); with $\overline{\nu}$ verifies (26) for an increasing process $\overline{K}$. We fix $\phi:\mathbb{R}^d\to\mathbb{R}$, a smooth function in $C_c^2(\mathbb{R}^d)$ with compact support and denote $\phi_t(s,x) = \phi(\widehat{X}_s^{t,x})J(\widehat{X}_s^{t,x})$. From proposition 3.1, one may use $\phi_t(s,x)$ as a test function in the PDE(g,f,h) with $\partial_s\phi(s,x)ds$ replaced by a stochastic integral with respect to the semimartingale $\phi_t(s,x)$. Then we get, for $t \leqslant s \leqslant T$

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_s^T \overline{u}(r,x)d\phi_t(r,x)dx + (\overline{u}(s,\cdot),\phi_t(s,\cdot)) - (g(\cdot),\phi_t(\cdot,T)) + \int_s^T \mathcal{E}(\overline{u}_r,\phi_r)dr \qquad (38) \\
=\ & \int_s^T\int_{\mathbb{R}^d} f(r,x,\overline{u}(r,x),\sigma^*\nabla\overline{u}(r,x))\phi_t(r,\cdot)dr + \int_s^T\int_{\mathbf{R}^d}\phi_t(r,x)1_{\{\overline{u}=h\}}d\overline{\nu}(x,r).
\end{aligned}$$

By (5) in Lemma 3.1, we have

$$\begin{aligned}
\int_{\mathbb{R}^d}\int_s^T \overline{u}d_r\phi_t(r,x)dx =\ & \int_s^T(\int_{\mathbb{R}^d}(\sigma^*\nabla\overline{u})(r,x)\phi_t(r,x)dx)dB_r \\
& + \int_s^T\int_{\mathbb{R}^d}\left((\sigma^*\nabla\overline{u})(\sigma^*\nabla\phi_r) + \phi_t\nabla((\frac{1}{2}\sigma^*\nabla\sigma + b)\overline{u})\right)dxdr.
\end{aligned}$$

Substitute this equality in (38), we get

$$\begin{aligned}\int_{\mathbb{R}^d}\overline{u}(s,x)\phi_t(s,x)dx &= (g(\cdot),\phi_t(\cdot,T))-\int_s^T(\int_{\mathbb{R}^d}(\sigma^*\nabla\overline{u})(r,x)\phi_t(r,x)dx)dB_r\\ &\quad+\int_{\mathbb{R}^d}\int_s^T f(r,x,\overline{u}(r,x),\sigma^*\nabla\overline{u}(r,x))\phi_t(s,\cdot)dr+\int_s^T\int_{\mathbf{R}^d}\phi_t(r,x)1_{\{\overline{u}=h\}}d\overline{\nu}(x,r).\end{aligned}$$

Then by changing of variable $y=\widehat{X}_r^{t,x}$ and applying (26) for $\overline{\nu}$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^d}\overline{u}(s,X_s^{t,y})\phi(y)dy\\ =&\int_{\mathbb{R}^d}g(X_T^{t,y})\phi(y)dy+\int_s^T\phi(y)f(s,X_s^{t,y},\overline{u}(s,X_s^{t,y}),\sigma^*\nabla\overline{u}(s,X_s^{t,y})ds\\ &+\int_s^T\int_{\mathbb{R}^d}\phi(y)1_{\{\overline{u}=h\}}(r,X_s^{t,y})d\overline{K}_r^{t,y}dy-\int_s^T(\int_{\mathbb{R}^d}(\sigma^*\nabla\overline{u})(r,X_r^{t,y})\phi(y)dy)dB_r.\end{aligned}$$

Since ϕ is arbitrary, we can prove that for $\rho(y)dy$ almost every y, $(\overline{u}(s,X_s^{t,y}),(\sigma^*\nabla\overline{u})(s,X_s^{t,y}),\widehat{K}_s^{t,x})$ solves the RBSDE$(g(X_T^{t,y}),f,h)$. Here $\widehat{K}_s^{t,x}=\int_t^s 1_{\{\overline{u}=h\}}(r,X_r^{t,y})d\overline{K}_r^{t,y}$. Then by the uniqueness of the solution of the reflected BSDE, we know $\overline{u}(s,X_s^{t,y})=Y_s^{t,y}=u(s,X_s^{t,x})$, $(\sigma^*\nabla\overline{u})(s,X_s^{t,y})=Z_s^{t,y}=(\sigma^*\nabla u)(s,X_s^{t,y})$ and $\widehat{K}_s^{t,y}=K_s^{t,y}$. Taking $s=t$ we deduce that $\overline{u}(t,y)=u(t,y)$, $\rho(y)dy$-a.s. and by the probabilistic interpretation (26), we obtain

$$\int_s^T\int\phi_t(r,x)1_{\{\overline{u}=h\}}(r,x)d\overline{\nu}(x,r)=\int_s^T\int\phi_t(r,x)1_{\{u=h\}}(r,x)d\nu(x,r).$$

So $1_{\{\overline{u}=h\}}(r,x)d\overline{\nu}(x,r)=1_{\{u=h\}}(r,x)d\nu(x,r)$. □

6 Appendix

6.1 Proof of proposition 3.1

First we consider the case when f does not depend on z and satisfies assumption 2.3'. As in step 2 of the proof of theorem 4.1, we approximate g and f as in (18), then $g_n\to g$ in $\mathbf{L}^2(\mathbb{R}^d,\rho(x)dx)$ and $f_n(t,x,0)\to f(t,x,0)$ in $\mathbf{L}^2([0,T]\times\mathbb{R}^d,dt\otimes\rho(x)dx)$, as $n\to\infty$.

Since for each $n\in\mathbb{N}$, $|g_n|\leqslant n$ and $|f_n(t,x,0)|\leqslant n$, by the result of the step 1 of theorem 4.1, the PDE(g_n,f_n) admits the weak solution $u_n\in\mathcal{H}$ and $\sup_{0\leqslant t\leqslant T}|u_n(t,x)|\leqslant C_n$. So we know

$$|f_n(t,x,u_n(t,x))|^2\leqslant|f_n(t,x,0)|^2+\varphi(\sup_{0\leqslant t\leqslant T}|u_n(t,x)|)\leqslant C_n.$$

Set $F_n(t,x):=f_n(t,x,u_n(t,x))$, then $F_n(t,x)\in\mathbf{L}^2([0,T]\times\mathbb{R}^n,dt\otimes\rho(x)dx)$.

From proposition 2.3 in Bally and Matoussi [3], for $\phi\in C_c^2(\mathbb{R}^d)$, we get, for $t\leqslant s\leqslant T$

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_s^T u_n(r,x)d\phi_t(r,x)dx+(u_n(s,\cdot),\phi_t(s,\cdot))-(g_n(\cdot),\phi_t(\cdot,T))+\int_s^T\mathcal{E}(u_n(r,\cdot),\phi_t(r,\cdot))dr\\ &=\int_{\mathbb{R}^d}\int_s^T f(r,x,u_n(r,x))\phi_t(r,x)drdx+\int_{\mathbb{R}^d}\int_s^T(f_n(r,x,0)-f(r,x,0))\phi_t(r,x)drdx.\end{aligned}\tag{39}$$

By step 2, we know that as $n \to \infty$, $u_n \to u$ in $\mathcal{H}$, where u is a weak solution of the PDE(g,f), i.e.

$$\begin{array}{rcl} u_n & \to & u \text{ in } \mathbf{L}^2([0,T]\times\mathbb{R}^d, dt\otimes\rho(x)dx), \\ \sigma^*\nabla u_n & \to & \nabla u \text{ in } \mathbf{L}^2([0,T]\times\mathbb{R}^d, dt\otimes\rho(x)dx). \end{array}$$

Then there exists a function u^* in $\mathbf{L}^2([0,T]\times\mathbb{R}^d, dt\otimes\rho(x)dx)$, such that for a subsequence of $\{u_n\}$, $|u_{n_k}| \leqslant |u^*|$ and $u_{n_k} \to u$, $dt\otimes dx$-a.e. Thanks to assumption 2.3'-(iii), we have that $f(r,x,u_n(r,x)) \to f(r,x,u(r,x))$, $dt\otimes dx$-a.e. Now, for all compact support function $\phi \in C_c^2(\mathbb{R}^d)$, the second term in the right hand side of (39) converge to 0 as $n\to\infty$ and it is not hard to prove by using the dominated convergence theorem the term in the left hand side of (39) converges. Thus, we conclude that $\lim_{n\to\infty}\int_{\mathbb{R}^d}\int_s^T f(r,x,u_n(r,x))\phi_t(r,x)drdx$ exists. Moreover by the monotonocity condition of f and the same arguments as in step 2 of the proof of theorem 4.1, we get for all compact support function $\phi\in C_c^2(\mathbb{R}^d)$

$$\begin{array}{rl} & \displaystyle\int_{\mathbb{R}^d}\int_s^T u(r,x)d\phi_t(r,x)dx + (u(s,\cdot),\phi_t(s,\cdot)) - (g(\cdot),\phi_t(\cdot,T)) + \int_s^T \mathcal{E}(u(r,\cdot),\phi_t(r,\cdot))dr \\ = & \displaystyle\int_{\mathbb{R}^d}\int_s^T f(r,x,u(r,x))\phi_t(r,x)drdx\,. \end{array}$$

Now we consider the case when f depends on ∇u and satisfies the assumption 2.3 with (iii) replaced by (12). Like in the step 3 of the proof of theorem 4.1, we construct a mapping Ψ from $\mathcal{H}$ into itself. Then by this mapping, we define a sequence $\{u_n\}$ in $\mathcal{H}$, beginning with a matrix-valued function $v^0\in\mathbf{L}^2([0,T]\times\mathbb{R}^{n\times d}, dt\otimes\rho(x)dx)$. Since $f(t,x,u,v^0(t,x))$ satisfies the assumptions of step 2, the PDE$(g,f(t,x,u,v^0(t,x)))$ admits a unique solution $u_1\in\mathcal{H}$. For $n\in\mathbb{N}$, denote

$$u_n(t,x) = \Psi(u_{n-1}(t,x)),$$

i.e. u_n is the weak solution of the PDE$(g,f(t,x,u,\sigma^*\nabla u_{n-1}(t,x)))$. Set $\widetilde{u}_n(t,x) := u_n(t,x) - u_{n-1}(t,x)$. In order to estimate the difference, we introduce the corresponding BSDE(g,f_n) for $n=1$, where $f_n(t,x,u) = f(t,x,u,\nabla u_{n-1}(t,x))$. So we have $Y_s^{n,t,x} = u_n(s,X_s^{t,x}), Z_s^{n,t,x} = \sigma\nabla u_n(s,X_s^{t,x})$. Then we apply the Itô's formula to $|\widetilde{Y}^{n,t,x}|^2$, where $\widetilde{Y}_s^{n,t,x} := Y_s^{n,t,x} - Y_s^{n-1,t,x}$. With the equivalence of the norms, similarly as in step 3, for $\gamma = 1+\frac{2k_1^2}{k_2^2}k^2$, we have

$$\begin{array}{rl} & \displaystyle\int_{\mathbb{R}^d}\int_t^T e^{\gamma s}(|\widetilde{u}_n(s,x)|^2 + |\sigma^*\nabla(\widetilde{u}_n)(s,x)|^2)\rho(x)dsdx \\ \leqslant & \displaystyle(\frac{1}{2})^{n-1}\int_{\mathbb{R}^d}\int_t^T e^{\gamma s}(|\widetilde{u}_2(s,x)|^2 + |\sigma^*\nabla(\widetilde{u}_2)(s,x)|^2)\rho(x)dsdx \\ \leqslant & \displaystyle(\frac{1}{2})^{n-1}(\|u_1(s,x)\|_\gamma^2 + \|u_2(s,x)\|_\gamma^2). \end{array}$$

where $\|u\|_\gamma^2 := \int_{\mathbb{R}^d}\int_t^T e^{\gamma s}(|u(s,x)|^2 + |\sigma^*\nabla u(s,x)|^2)\rho(x)dsdx$, which is equivalent to the norm $\|\cdot\|$ of $\mathcal{H}$. So $\{u_n\}$ is a Cauchy sequence in $\mathcal{H}$, it admits a limit u in $\mathcal{H}$, and by the fixed point theorem, u is a solution of the PDE(g,f).

Then for each $n \in \mathbb{N}$, we have for $\phi \in C_c^2(\mathbb{R}^d)$

$$\begin{aligned}
& \int_{\mathbb{R}^d}\int_s^T u_n(r,x)d\phi_t(r,x)dx + (u_n(s,\cdot),\phi_t(s,\cdot)) - (g(\cdot),\phi_t(\cdot,T)) + \int_s^T \mathcal{E}(u_n(r,\cdot),\phi_r(r,\cdot))dr \\
= & \int_{\mathbb{R}^d}\int_s^T f(r,x,u_n(r,x),\sigma^*\nabla u_{n-1}(r,x))\phi_t(r,x)drdx \\
= & \int_{\mathbb{R}^d}\int_s^T f(r,x,u_n(r,x),\sigma^*\nabla u(r,x))\phi_t(r,x)drdx \\
& + \int_{\mathbb{R}^d}\int_s^T [f(r,x,u_n(r,x),\sigma^*\nabla u_{n-1}(r,x)) - f(r,x,u_n(r,x),\sigma^*\nabla u(r,x))]\phi_t(r,x)drdx.
\end{aligned}$$

Noticing that f is Lipschitz in z, we get

$$|f(r,x,u_n(r,x),\sigma^*\nabla u_{n-1}(r,x)) - f(r,x,u_n(r,x),\sigma^*\nabla u(r,x))| \leqslant k\,|\sigma^*\nabla u_{n-1}(r,x) - \sigma^*\nabla u(r,x)|\,.$$

So the last term of the right side converges to 0, since $\{\sigma^*\nabla u_n\}$ converges to $\sigma^*\nabla u$ in $\mathbf{L}^2([0,T]\times\mathbb{R}^d, dt\otimes\rho(x)dx)$. Now we are in the same situation as in the first part of proof, and in the same way, we deduce that the following holds: for $\phi \in C_c^2(\mathbb{R}^d)$

$$\begin{aligned}
& \int_{\mathbb{R}^d}\int_s^T u(r,x)d\phi_t(r,x)dx + (u(s,\cdot),\phi_t(s,\cdot)) - (g(\cdot),\phi_t(\cdot,T)) + \int_s^T \mathcal{E}(u(r,\cdot),\phi_t(r,\cdot))dr \\
= & \int_{\mathbb{R}^d}\int_s^T f(r,x,u(r,x),\sigma^*\nabla u(r,x))\phi_t(r,x)drdx, dt\otimes dx, \text{ a.s..}
\end{aligned}$$

Now if f satisfies assumption 2.3, we know that u is solution of the PDE(g,f) if and only if $\widehat{u} = e^{\mu t}u$ is solution of the PDE$(\widehat{g},\widehat{f})$, where

$$\widehat{g}(x) = e^{\mu T}g(x),\ \widehat{f}(t,x,y,x) = e^{\mu t}f(t,x,e^{-\mu t}y,e^{-\mu t}z) - \mu y,$$

and $\widehat{f}$ satisfies assumption 2.3-(iii) replaced by (12). So we know now: for $\phi \in C_c^2(\mathbb{R}^d)$,

$$\begin{aligned}
& \int_{\mathbb{R}^d}\int_s^T \widehat{u}(r,x)d\phi_t(r,x)dx + (\widehat{u}(s,\cdot),\phi_t(s,\cdot)) - (\widehat{g}(\cdot),\phi_t(\cdot,T)) + \int_s^T \mathcal{E}(\widehat{u}(r,\cdot),\phi_t(r,\cdot))dr \\
= & \int_{\mathbb{R}^d}\int_s^T \widehat{f}(r,x,\widehat{u}(r,x),\nabla\widehat{u}(r,x))\phi_t(r,x)drdx, dt\otimes dx, \text{ a.s..}
\end{aligned}$$

Notice that $d(e^{\mu r}u(r,x)) = \mu e^{\mu r}u(r,x)dr + e^{\mu r}d(u(r,x))$, so by the integration by parts formula (for stochastic process), we get

$$\begin{aligned}
& \int_{\mathbb{R}^d}\int_s^T u(r,x)d\phi_t(r,x)dx \\
= & \int_{\mathbb{R}^d}\int_s^T e^{-\mu r}\widehat{u}(r,x)d\phi_t(r,x)dxdr \\
= & e^{-\mu T}(\widehat{g}(\cdot),\phi_t(\cdot,T)) - e^{-\mu s}(\widehat{u}(s,\cdot),\phi_t(s,\cdot)) + \mu\int_s^T e^{-\mu r}\int_{\mathbb{R}^d}\widehat{u}(r,x)\phi_t(r,x)dxdr \\
& - \int_s^T\int_{\mathbb{R}^d} e^{-\mu r}\phi_t(r,x)[\mathcal{L}\widehat{u}(r,x) + \widehat{f}(r,x,\widehat{u}(r,x),\nabla\widehat{u}(r,x))]drdx.
\end{aligned}$$

Using (11), we get that for $\phi \in C_c^2(\mathbb{R}^d)$,

$$\begin{aligned}\int_{\mathbb{R}^d}\int_s^T u(r,x)d\phi_t(r,x)dx &= (g(\cdot),\phi_t(\cdot,T)) - (u(s,\cdot),\phi_t(s,\cdot)) - \int_s^T\int_{\mathbb{R}^d}\phi_t(r,x)\mathcal{L}u(r,x)drdx \\ &\quad + \int_s^T\int_{\mathbb{R}^d}\phi_t(r,x)f(r,x,u(r,x),\nabla u(r,x))drdx \\ &= (g(\cdot),\phi_t(\cdot,T)) - (u(s,\cdot),\phi_t(s,\cdot)) - \int_s^T \mathcal{E}(u(r,\cdot),\phi_t(r,\cdot))dr \\ &\quad + \int_s^T\int_{\mathbb{R}^d}\phi_t(r,x)f(r,x,u(r,x),\nabla u(r,x))drdx,\end{aligned}$$

and finally, the result follows. □

6.2 Some a priori estimates

In this subsection, we consider the non-markovian Reflected BSDE associated to (ξ, f, L) :

$$\begin{cases} Y_t = \xi + \int_t^T f(t,Y_s,Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s, \\ Y_t \geqslant L_t, \\ \int_0^T (Y_s - L_s)dK_s = 0 \end{cases}$$

under the following assumptions :

(H1) a final condition $\xi \in \mathbf{L}^2(\mathcal{F}_T)$,

(H2) a coefficient $f : \Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$, which is such that for some continuous increasing function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, a real numbers μ and $C > 0$:

(i) $f(\cdot, y, z)$ is progressively measurable, $\forall (y,z) \in \mathbb{R}\times\mathbb{R}^d$;

(ii) $|f(t,y,0)| \leqslant |f(t,0,0)| + \varphi(|y|)$, $\forall (t,y) \in [0,T]\times\mathbb{R}$, a.s.;

(iii) $E\int_0^T |f(t,0,0)|^2\, dt < \infty$;

(iv) $|f(t,y,z) - f(t,y,z')| \leqslant C\,|z - z'|$, $\forall (t,y) \in [0,T]\times\mathbb{R}$, $z, z' \in \mathbb{R}^d$, a.s.

(v) $(y-y')(f(t,y,z) - f(t,y',z)) \leqslant \mu(y-y')^2$, $\forall (t,z) \in [0,T]\times\mathbb{R}^d$, $y, y' \in \mathbb{R}$, a.s.

(vi) $y \to f(t,y,z)$ is continuous, $\forall (t,z) \in [0,T]\times\mathbb{R}^d$, a.s.

(H3) a barrier $(L_t)_{0\,\leqslant\, t\,\leqslant\, T}$, which is a continuous progressively measurable real-valued process, satisfying

$$E[\varphi^2(\sup_{0\,\leqslant\, t\,\leqslant\, T}(e^{\mu t}L_t^+))] < \infty,$$

and $(L_t^+)_{0\,\leqslant\, t\,\leqslant\, T} \in \mathbf{S}^2(0,T)$, $L_T \leqslant \xi$, a.s.

We shall give an a priori estimate of the solution (Y, Z, K) with respect to the terminal condition ξ, the coefficient f and the barrier L. Unlike the Lipshitz case, we have in addition the term $E\varphi^2(\sup_{0\,\leqslant\, t\,\leqslant\, T}(L_t^+))$ and a constant, which only depends on φ, μ, k and T :

Proposition 6.1. *There exists a constant C, which only depends on T, μ and k, such that*

$$E\left[\sup_{0\leqslant t\leqslant T}|Y_t|^2+\int_0^T|Z_s|^2\,ds+|K_T|^2\right]\leqslant CE\left[\xi^2+\int_0^T f^2(t,0,0)dt+\varphi^2(\sup_{0\leqslant t\leqslant T}(L_t^+))\right] + CE[\sup_{0\leqslant t\leqslant T}(L_t^+)^2+1+\varphi^2(2T)].$$

Proof. Applying Itô's formula to $|Y_t|^2$, and taking expectation, then

$$\begin{aligned} E[|Y_t|^2+\int_t^T|Z_s|^2\,ds] &= E[|\xi|^2+2\int_t^T Y_sf(s,Y_s,Z_s)ds+2\int_t^T L_sdK_s \\ &\leqslant E[|\xi|^2+2\int_t^T Y_sf(s,0,0)ds+2\int_t^T(\mu|Y_s|^2+k|Y_s||Z_s|)ds+2\int_t^T L_sdK_s]. \end{aligned}$$

It follows that

$$E[|Y_t|^2+\frac{1}{2}\int_t^T|Z_s|^2\,ds]\leqslant E[|\xi|^2+2\int_t^T f^2(s,0,0)ds+(2\mu+1+2k^2)\int_t^T|Y_s|^2\,ds+2\int_t^T L_sdK_s].$$

Then by Gronwall's inequality, we have

$$E|Y_t|^2\leqslant CE[|\xi|^2+\int_0^T f^2(s,0,0)ds+\int_0^T L_sdK_s], \tag{40}$$

then

$$E\int_0^T|Z_s|^2\,ds\leqslant CE[|\xi|^2+\int_0^T f^2(s,0,0)ds+\int_0^T L_sdK_s], \tag{41}$$

where C is a constant only depends on μ, k and T, in the following this constant can be changed line by line.

Now we estimate K by approximation. By the existence of the solution, theorem 2.2 in Lepeltier et al. [12], we take the process Z as a known process. Without losing generality we write $f(t,y)$ for $f(t,y,Z_t)$, here $f(t,0)=f(t,0,Z_t)$ is a process in $\mathbf{H}^2(0,T)$. Set

$$\begin{aligned} \xi^{m,n} &= (\xi\vee(-n))\wedge m, \\ f^{m,n}(t,y) &= f(t,y)-f(t,0)+(f(t,0)\vee(-n))\wedge m. \end{aligned}$$

For m, $n\in\mathbf{N}$, $\xi^{m,n}$ and $\sup_{0\leqslant t\leqslant T}f^{m,n}(t,0)$ are uniformly bounded. Consider the RBSDE$(\xi^{m,n},f^{m,n},L)$,

$$\begin{aligned} Y_t^{m,n} &= \xi^{m,n}+\int_t^T f^{m,n}(t,Y_s^{m,n})ds+K_T^{m,n}-K_t^{m,n}-\int_t^T Z_s^{m,n}dB_s, \\ Y_t^{m,n} &\geqslant L_t,\int_0^T(Y_s^{m,n}-L_s)dK_s^{m,n}=0. \end{aligned}$$

if we recall the transform in step 2 of the proof of theorem 2.2 in Lepeltier et al. [12], since $\xi^{m,n}$, $f^{m,n}(t,0)\leqslant m$, we know that $(Y_t^{m,n},Z_t^{m,n},K_t^{m,n})$ is the solution of this RBSDE, if and only if $(Y^{m,n\prime},Z^{m,n\prime},K^{m,n\prime})$ is the solution of RBSDE$(\xi^{m,n\prime},f^{m,n\prime},L')$, where

$$(Y_t^{m,n\prime},Z_t^{m,n\prime},K_t^{m,n\prime})=(Y_t^{m,n}+m(t-2(T\vee1)),Z_t^{m,n},K_t^{m,n})$$

and

$$\begin{aligned}
\xi^{m,n\prime} &= \xi^{m,n} + 2mT - m(T \vee 1), \\
f^{m,n\prime}(t,y) &= f^{m,n}(t, y - m(t - 2(T \vee 1))) - m, \\
L'_t &= L_t + m(t - 2(T \vee 1)).
\end{aligned}$$

Without losing generality we set $T \geqslant 1$. Then $\xi^{m,n\prime} \leqslant 0$ and $f^{m,n\prime}(t,0) \leqslant 0$. Since $(Y^{m,n\prime}, Z^{m,n\prime}, K^{m,n\prime})$ is the solution of RBSDE$(\xi^{m,n\prime}, f^{m,n\prime}, L')$, then we have

$$K_T^{m,n\prime} = Y_0^{m,n\prime} - \xi^{m,n\prime} - \int_0^T f^{m,n\prime}(s, Y_s^{m,n\prime}, Z_s)ds + \int_0^T Z_s^{m,n\prime} dB_s,$$

which follows

$$E[(K_T^{m,n\prime})^2] \leqslant 4E[\left|Y_0^{m,n\prime}\right|^2 + \left|\xi^{m,n\prime}\right|^2 + (\int_0^T f^{m,n\prime}(s, Y_s^{m,n\prime})ds)^2 + \int_0^T \left|Z_s^{m,n\prime}\right|^2 ds]. \tag{42}$$

Applying Itô's formula to $|Y^{m,n}|^2$, like (40) and (41), we have

$$E\left|Y_t^{m,n}\right|^2 + E\int_t^T \left|Z_s^{m,n}\right|^2 ds \leqslant CE[|\xi^{m,n}|^2 + \int_t^T (f^{m,n}(s,0))^2 ds + \int_t^T L_s dK_s^{m,n}].$$

So

$$\begin{aligned}
\left|Y_0^{m,n\prime}\right|^2 + \int_0^T \left|Z_s^{m,n\prime}\right|^2 ds &= 2\left|Y_0^{m,n}\right|^2 + 8m^2T^2 + E\int_0^T \left|Z_s^{m,n}\right|^2 ds \\
&\leqslant CE[|\xi^{m,n}|^2 + \int_0^T (f^{m,n}(s,0)ds)^2 + \int_0^T L_s dK_s^{m,n}] + 8m^2T^2.
\end{aligned}$$

For the third term on the right side of (42), from Lemma 2.3 in Lepeltier et al. [12], we remember that

$$(\int_0^T f^{m,n\prime}(s, Y_s^{m,n\prime})ds)^2 \leqslant \max\{(\int_0^T f^{m,n\prime}(s, \widetilde{Y}_s^{m,n})ds)^2, (\int_0^T f^{m,n\prime}(s, \overline{Y}_s^{m,n})ds)^2\}, \tag{43}$$

where $(\widetilde{Y}^{m,n}, \widetilde{Z}^{m,n})$ is the solution the following BSDE

$$\widetilde{Y}_t^{m,n} = \xi^{m,n\prime} + \int_0^T f^{m,n\prime}(s, \widetilde{Y}_s^{m,n})ds - \int_0^T \widetilde{Z}_s^{m,n} dB_s, \tag{44}$$

and

$$\overline{Y}_s^{m,n} = ess \sup_{\tau \in \mathcal{T}_{t,T}} E[(L'_\tau)^+ 1_{\{\tau < T\}} + (\xi^{m,n})^+ 1_{\{\tau = T\}} | \mathcal{F}_t] = ess \sup_{\tau \in \mathcal{T}_{t,T}} E[(L'_\tau)^+ | \mathcal{F}_t].$$

From (44), and proposition 2.2 in Pardoux [13], we have

$$\begin{aligned}
E(\int_0^T f^{m,n\prime}(s, \widetilde{Y}_s^{m,n})ds)^2 &\leqslant CE[\left|\xi^{m,n\prime}\right|^2 + (\int_0^T f^{m,n\prime}(s,0)ds)^2] \\
&\leqslant CE[|\xi^{m,n}|^2 + \int_0^T (f^{m,n}(s,0))^2 ds] + C\varphi^2(2mT) + Cm^2.
\end{aligned}$$

For the other term in (43), with $\sup_{0 \leqslant t \leqslant T} \overline{Y}_s^{m,n} = \sup_{0 \leqslant t \leqslant T}(L'_t)^+$, we get

$$\begin{aligned} E(\int_0^T f^{m,n\prime}(s,\overline{Y}_s^{m,n})ds)^2 &\leqslant E[\int_0^T 2(f^{m,n\prime}(s,0))^2 ds + 2T\varphi^2(\sup_{0 \leqslant t \leqslant T}(L'_t)^+] \\ &\leqslant E[4\int_0^T f^{m,n}(s,0)^2 ds + 2T\varphi^2(\sup_{0 \leqslant t \leqslant T}(L_t)^+)] + 2m^2T + 4T\varphi^2(2mT). \end{aligned}$$

Consequently, we deduce that

$$\begin{aligned} E[(K_T^{m,n})^2] &= E[(K_T^{m,n\prime})^2] \\ &\leqslant CE[|\xi^{m,n}|^2 + \int_0^T (f^{m,n}(s,0))^2 ds + \int_t^T L_s dK_s^{m,n} + \varphi^2(\sup_{0 \leqslant t \leqslant T}(L_t)^+) + m^2 + \varphi^2(2mT)] \\ &\leqslant CE[|\xi|^2 + \int_0^T (f(s,0,Z_s))^2 ds + \varphi^2(\sup_{0 \leqslant t \leqslant T}(L_t)^+) + \sup_{0 \leqslant t \leqslant T}((L_t)^+)^2] + \frac{1}{2}E[(K_T^{m,n})^2] \\ &\quad + C(m^2 + \varphi^2(2mT)). \end{aligned}$$

Moreover using (41) and the fact that f is Lipschitz on z, it follows that

$$\begin{aligned} E[(K_T^{m,n})^2] &\leqslant CE[|\xi|^2 + \int_0^T (f(s,0,0))^2 ds + \varphi^2(\sup_{0 \leqslant t \leqslant T}(L_t)^+) + \sup_{0 \leqslant t \leqslant T}((L_t)^+)^2 \\ &\quad + \int_0^T L_s dK_s] + C(m^2 + \varphi^2(2mT)). \end{aligned} \tag{45}$$

Let $m \to \infty$, then

$$E[|\xi^{m,n} - \xi^n|^2] \to 0, E\int_0^T |f^{m,n}(t,0) - f^n(t,0)|^2 \to 0,$$

where $\xi^n = \xi \vee (-n)$ and $f^n(t,y) = f(t,y) - f(t,0) + f(t,0) \vee (-n)$.

Thanks to the convergence result of step 3 of the proof for theorem 2.2 in [12], we know that $(Y^{m,n}, Z^{m,n}, K^{m,n}) \to (Y^n, Z^n, K^n)$ in $\mathbf{S}^2(0,T) \times \mathbf{H}_d^2(0,T) \times \mathbf{A}^2(0,T)$, where (Y^n, Z^n, K^n) is the soultion of the RBSDE(ξ^n, f^n, L). Moreover $K_T^{m,n} \searrow K_T^n$ in $\mathbf{L}^2(\mathcal{F}_T)$, so we have $K_T^n \leqslant K_T^{1,n}$, which implies for each $n \in \mathbf{N}$,

$$E[(K_T^n)^2] \leqslant E[(K_T^{1,n})^2] \tag{46}$$

Then, letting $n \to \infty$, by the convergence result in step 4, since

$$E[|\xi^n - \xi|^2] \to 0, E\int_0^T |f^n(t,0) - f(t,0)|^2 \to 0,$$

the sequence $(Y^n, Z^n, K^n) \to (Y, Z, K)$ in $\mathbf{S}^2(0,T) \times \mathbf{H}_d^2(0,T) \times \mathbf{A}^2(0,T)$, where (Y, Z, K) is the solution of the RBSDE(ξ, f, L). From (46), and (45) for $m=1$, we get

$$\begin{aligned}
E[(K_T)^2] \quad &\leqslant \quad CE[|\xi|^2 + \int_0^T (f(s,0,0))^2 ds + \varphi^2(\sup_{0\leqslant t\leqslant T}(L_t)^+) + \sup_{0\leqslant t\leqslant T}((L_t)^+)^2 \\
&\quad + \int_0^T L_s dK_s] + C(1+\varphi^2(2T)) \\
&\leqslant \quad CE[|\xi|^2 + \int_0^T (f(s,0,0))^2 ds + \varphi^2(\sup_{0\leqslant t\leqslant T}(L_t)^+) + \sup_{0\leqslant t\leqslant T}((L_t)^+)^2] \\
&\quad + \frac{1}{2}E[(K_T)^2] + C(1+\varphi^2(2T)).
\end{aligned}$$

Then it follows that for each $t \in [0,T]$,

$$\begin{aligned}
E[|Y_t|^2 + \int_0^T |Z_s|^2\, ds + (K_T)^2] \quad &\leqslant \quad CE[|\xi|^2 + \int_0^T (f(s,0,0))^2 ds + \varphi^2(\sup_{0\leqslant t\leqslant T}(L_t)^+) \\
&\quad + \sup_{0\leqslant t\leqslant T}((L_t)^+)^2] + C(1+\varphi^2(2T)).
\end{aligned}$$

Finally we get the result, by applying BDG inequality. □

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