Electronic Journal of Probability

Vol. 13 (2008), Paper no. 67, pages 2031–2068.

Journal URL
http://www.math.washington.edu/~ejpecp/

Delay equations driven by rough paths

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Abstract

In this article, we illustrate the flexibility of the algebraic integration formalism introduced in *M. Gubinelli (2004), Controlling Rough Paths, J. Funct. Anal.* **216***, 86-140,* by establishing an existence and uniqueness result for delay equations driven by rough paths. We then apply our results to the case where the driving path is a fractional Brownian motion with Hurst parameter $H > \frac{1}{3}$.

Key words: rough paths theory; delay equation; fractional Brownian motion; Malliavin calculus.

AMS 2000 Subject Classification: Primary 60H05, 60H07, 60G15.

Submitted to EJP on November 16, 2007, final version accepted November 5, 2008.

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1 Introduction

In the last years, great efforts have been made to develop a stochastic calculus for fractional Brownian motion. The first results gave a rigorous theory for the stochastic integration with respect to fractional Brownian motion and established a corresponding Itô formula, see e.g. [1; 2; 3; 6; 18]. Thereafter, stochastic differential equations driven by fractional Brownian motion have been considered. Here different approaches can be used depending on the dimension of the equation and the Hurst parameter of the driving fractional Brownian motion. In the one-dimensional case [17], existence and uniqueness of the solution can be derived by a regularization procedure introduced in [21]. The case of a multi-dimensional driving fractional Brownian motion can be treated by means of fractional calculus tools, see e.g. [19; 23] or by means of the Young integral [13], when the Hurst coefficient satisfies $H > \frac{1}{2}$. However, only the rough paths theory [13; 12] and its application to fractional Brownian motion [5] allow to solve fractional SDEs in any dimension for a Hurst parameter $H > \frac{1}{4}$. The original rough paths theory developed by T. Lyons relies on deeply involved algebraical and analytical tools. Therefore some alternative methods [8; 9] have been developed recently, trying to catch the essential results of [12] with less theoretical apparatus.

Since it is based on some rather simple algebraic considerations and an extension of Young's integral, the method given in [9], which we call *algebraic integration* in the sequel, has been especially attractive to us. Indeed, we think that the basic properties of fractional differential systems can be studied in a natural and nice way using algebraic integration. (See also [16], where this approach is used to study the law of the solution of a fractional SDE.) In the present article, we will illustrate the flexibility of the algebraic integration formalism by studying fractional equations *with delay*. More specifically, we will consider the following equation:

$$\begin{cases} X_t = \xi_0 + \int_0^t b(X_s, X_{s-r_1}, \ldots, X_{s-r_k}) ds \\ \qquad\qquad\qquad + \int_0^t \sigma(X_s, X_{s-r_1}, \ldots, X_{s-r_k}) dB_s, \quad t \in [0,T], \\ X_t = \xi_t, \qquad t \in [-r_k, 0]. \end{cases} \tag{1}$$

Here the discrete delays satisfy $0 < r_1 < \ldots < r_k < \infty$, the functions $\sigma : \mathbb{R}^{n(k+1)} \to \mathbb{R}^{n,d}$, $b : \mathbb{R}^{n(k+1)} \to \mathbb{R}^n$ are regular, B is a d-dimensional fractional Brownian motion with Hurst parameter $H > \frac{1}{3}$ and the initial condition ξ is a $\mathbb{R}^n$-valued weakly controlled process based on B, see Definition 2.5. (In particular ξ can be any smooth deterministic function from $[-r_k, 0]$ to $\mathbb{R}^n$.) We also use, here and in the sequel, the following notation: we write $\mathbb{R}^{n(k+1)}$ for the set $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$ ($k+1$ times), while $\mathbb{R}^{n,d}$ stands for the set of $n \times d$ matrices with real entries.

The stochastic integral in equation (1) is a generalized Stratonovich integral, which will be explained in detail in Section 2. Actually, in equations like (1), the drift term $\int_0^t b(X_s, X_{s-r_1}, \ldots, X_{s-r_k}) ds$ is usually harmless, but causes some cumbersome notations. Thus, for sake of simplicity, we will rather deal in the sequel with delay equations of the type

$$\begin{cases} X_t = \xi_0 + \int_0^t \sigma(X_s, X_{s-r_1}, \ldots, X_{s-r_k}) dB_s, \quad t \in [0,T], \\ X_t = \xi_t, \qquad t \in [-r_k, 0]. \end{cases} \tag{2}$$

Our main result will be as follows:

Theorem 1.1. *Let $\sigma \in C_b^3(\mathbb{R}^{n(k+1)};\mathbb{R}^{n,d})$, let B be a d-dimensional fractional Brownian motion with Hurst parameter $H > \frac{1}{3}$ and let ξ be a $\mathbb{R}^n$-valued weakly controlled process based on B. Then equation (2) admits a unique solution on $[0,T]$ in the class of weakly controlled processes (see Definition 2.5).*

Stochastic delay equations driven by standard Brownian motion have been studied extensively (see e.g. [15] and [14] for an overview) and are used in many applications. However, delay equations driven by fractional Brownian motion have been only considered so far in [7], where the *one-dimensional* equation

$$\begin{cases} X_t = \xi_0 + \int_0^t \sigma(X_{s-r})dB_s + \int_0^t b(X_s)ds, & t \in [0,T], \\ X_t = \xi_t, & t \in [-r,0], \end{cases} \tag{3}$$

is studied for $H > \frac{1}{2}$. Observe that (3) is a particular case of equation (2).

To solve equation (2), one requires two main ingredients in the algebraic integration setting. First of all, a natural class of paths, in which the equation can be solved. Here, this will be the paths whose increments are controlled by the increments of B. Namely, writing $(\delta z)_{st} = z_t - z_s$ for the increments of an arbitrary function z, a stochastic differential equation driven by B should be solved in the class of paths, whose increments can be decomposed into

$$z_t - z_s = \zeta_s(B_t - B_s) + \rho_{st} \quad \text{for} \quad 0 \le s < t \le T,$$

with ζ belonging to $\mathscr{C}_1^\gamma$ and ρ belonging to $\mathscr{C}_2^{2\gamma}$ for a given $\frac{1}{3} < \gamma < H$. (Here, $\mathscr{C}_i^\mu$ denotes a space of μ-Hölder continuous functions of i variables, see Section 2.) This class of functions will be called the class of *weakly controlled paths* in the sequel.

To solve fractional differential equations *without* delay, the second main tool would be to define the integral of a weakly controlled path with respect to fractional Brownian motion and to show that the resulting process is still a weakly controlled path. To define the integral of a weakly controlled path, a double iterated integral of fractional Brownian motion, called the *Lévy area*, will be required. Once the stability of the class of weakly controlled paths under integration is established, the differential equation is solved by an appropriate fixed point argument.

To solve fractional delay equations, we will have to modify this procedure. More specifically, we need a second class of paths, the class of *delayed controlled paths*, whose increments can be written as

$$z_t - z_s = \zeta_s^{(0)}(B_t - B_s) + \sum_{i=1}^k \zeta_s^{(i)}(B_{t-r_i} - B_{s-r_i}) + \rho_{st} \quad \text{for} \quad 0 \le s < t \le T,$$

where, as above, $\zeta^{(i)}$ belongs to $\mathscr{C}_1^\gamma$ for $i = 0,\ldots,k$, and ρ belongs to $\mathscr{C}_2^{2\gamma}$ for a given $\frac{1}{3} < \gamma < H$. (Note that a classical weakly controlled path is a delayed controlled path with $\zeta^{(i)} = 0$ for $i = 1,\ldots,k$.) For such a delayed controlled path we will then define its integral with respect to fractional Brownian motion. We emphasize the fact that the integral of a delayed controlled path is actually a classical weakly controlled path and satisfies a stability property.

To define this integral we have to introduce a *delayed Lévy area* $\mathbf{B^2}(v)$ of B for $v \in [-r_k, 0]$. This process with values in the space of matrices $\mathbb{R}^{d,d}$ will also be defined as an iterated integral: for

$1 \le i, j \le d$ and $0 \le s < t \le T$ we set

$$\mathbf{B}^{\mathbf{2}}_{st}(v)(i,j) = \int_s^t dB^i_u \int_{s+v}^{u+v} dB^j_w = \int_s^t (B^j_{u+v} - B^j_{s+v}) d^\circ B^i_u, \tag{4}$$

where the integral on the right hand side is a Russo-Vallois integral [21]. Finally, the fractional delay equation (2) will be solved by a fixed point argument.

Here, we would like to stress that we prefer to use the regularization procedure introduced by Russo and Vallois [21] instead of the direct limits of sums approach for the definition of the delayed Lévy area. Malliavin calculus allows us then to verify without too much effort that (4) properly defines a delayed Lévy area for B, see Section 5, while working with limits of sums as e.g. in Coutin and Qian [5] would certainly require more work. Moreover, we would like to mention that we restrict here to a Hurst parameter $H > 1/3$ for sake of conciseness and simplicity. Indeed, the structures needed in order to solve the delay equation (2) for $1/4 < H < 1/3$ are more cumbersome, and include for instance the notions of doubly delayed controlled paths and of volume elements based on B. These problems will be addressed in the forthcoming paper [22].

This article is structured as follows: Throughout the remainder of this article, we consider the general delay equation

$$\begin{cases} dy_t &= \sigma(y_t, y_{t-r_1}, \dots, y_{t-r_k}) dx_t, & t \in [0,T], \\ y_t &= \xi_t, & t \in [-r_k, 0], \end{cases} \tag{5}$$

where x is γ-Hölder continuous function with $\gamma > \frac{1}{3}$ and ξ is a weakly controlled path based on x. In Section 2 we recall some basic facts of the algebraic integration and in particular the definition of a classical weakly controlled path, while in Section 3 we introduce the class of delayed controlled paths and the integral of a delayed controlled path with respect to its controlling rough path. Using the stability of the integral, we show the existence of a unique solution of equation (5) in the class of classical weakly controlled paths under the assumption of the existence of a delayed Lévy area. Finally, in Section 4 we specialize our results to delay equations driven by a fractional Brownian motion with Hurst parameter $H > \frac{1}{3}$.

2 Algebraic integration and rough paths equations

Before we consider equation (5), we recall the strategy introduced in [9] in order to solve an equation without delay, i.e.,

$$dy_t = \sigma(y_t) dx_t, \quad t \in [0,T], \qquad y_0 = \alpha \in \mathbb{R}^n, \tag{6}$$

where x is a $\mathbb{R}^d$-valued γ-Hölder continuous function with $\gamma > \frac{1}{3}$.

2.1 Increments

Here we present the basic algebraic structures, which will allow us to define a pathwise integral with respect to irregular functions. For real numbers $0 \le a \le b \le T < \infty$, a vector space V and an integer $k \geq 1$ we denote by $\mathscr{C}_k([a,b];V)$ the set of functions $g : [a,b]^k \to V$ such that $g_{t_1\cdots t_k} = 0$ whenever $t_i = t_{i+1}$ for some $1 \leq i \leq k-1$. Such a function will be called a $(k-1)$-*increment*, and we will set $\mathscr{C}_*([a,b];V) = \cup_{k\geq 1}\mathscr{C}_k([a,b];V)$. An important operator for our purposes is given by

$$\delta : \mathscr{C}_k([a,b];V) \to \mathscr{C}_{k+1}([a,b];V), \qquad (\delta g)_{t_1\cdots t_{k+1}} = \sum_{i=1}^{k+1}(-1)^{k-i} g_{t_1\cdots \hat{t}_i \cdots t_{k+1}}, \tag{7}$$

where $\hat{t}_i$ means that this argument is omitted. A fundamental property of δ is that $\delta\delta = 0$, where $\delta\delta$ is considered as an operator from $\mathscr{C}_k([a,b];V)$ to $\mathscr{C}_{k+2}([a,b];V)$. We will denote $\mathscr{Z}\mathscr{C}_k([a,b];V) = \mathscr{C}_k([a,b];V) \cap \mathrm{Ker}\delta$ and $\mathscr{B}\mathscr{C}_k([a,b];V) = \mathscr{C}_k([a,b];V) \cap \mathrm{Im}\delta$.

Some simple examples of actions of δ are as follows: For $g \in \mathscr{C}_1([a,b];V)$, $h \in \mathscr{C}_2([a,b];V)$ and $f \in \mathscr{C}_3([a,b];V)$ we have

$$(\delta g)_{st} = g_t - g_s, \quad (\delta h)_{sut} = h_{st} - h_{su} - h_{ut} \quad \text{and} \quad (\delta f)_{suvt} = f_{uvt} - f_{svt} + f_{sut} - f_{suv}$$

for any $s,u,v,t \in [a,b]$. Furthermore, it is easily checked that $\mathscr{Z}\mathscr{C}_{k+1}([a,b];V) = \mathscr{B}\mathscr{C}_k([a,b];\,V)$ for any $k \geq 1$. In particular, the following property holds:

Lemma 2.1. *Let $k \geq 1$ and $h \in \mathscr{Z}\mathscr{C}_{k+1}([a,b];V)$. Then there exists a (non unique) $f \in \mathscr{C}_k([a,b];V)$ such that $h = \delta f$.*

Observe that Lemma 2.1 implies in particular that all elements $h \in \mathscr{C}_2([a,b];V)$ with $\delta h = 0$ can be written as $h = \delta f$ for some $f \in \mathscr{C}_1([a,b];V)$. Thus we have a heuristic interpretation of $\delta|_{\mathscr{C}_2([a,b];V)}$: it measures how much a given 1-increment differs from being an exact increment of a function, i.e., a finite difference.

Our further discussion will mainly rely on k-increments with $k \leq 2$. For simplicity of the exposition, we will assume that $V = \mathbb{R}^m$ in what follows, although V could be in fact any Banach space. We measure the size of the increments by Hölder norms, which are defined in the following way: for $f \in \mathscr{C}_2([a,b];V)$ let

$$\|f\|_\mu = \sup_{s,t\in[a,b]} \frac{|f_{st}|}{|t-s|^\mu}$$

and

$$\mathscr{C}_2^\mu([a,b];V) = \left\{ f \in \mathscr{C}_2([a,b];V);\, \|f\|_\mu < \infty \right\}.$$

Obviously, the usual Hölder spaces $\mathscr{C}_1^\mu([a,b];V)$ are determined in the following way: for a continuous function $g \in \mathscr{C}_1([a,b];V)$ set

$$\|g\|_\mu = \|\delta g\|_\mu,$$

and we will say that $g \in \mathscr{C}_1^\mu([a,b];V)$ iff $\|g\|_\mu$ is finite. Note that $\|\cdot\|_\mu$ is only a semi-norm on $\mathscr{C}_1([a,b];V)$, but we will work in general on spaces of the type

$$\mathscr{C}_{1,\alpha}^\mu([a,b];V) = \left\{ g : [a,b] \to V;\, g_a = \alpha,\, \|g\|_\mu < \infty \right\}$$

for a given $\alpha \in V$, on which $\|g\|_\mu$ is a norm.

For $h \in \mathscr{C}_3([a,b];V)$ we define in the same way

$$\|h\|_{\gamma,\rho} = \sup_{s,u,t\in[a,b]} \frac{|h_{sut}|}{|u-s|^{\gamma}|t-u|^{\rho}}, \tag{8}$$

$$\|h\|_{\mu} = \inf\left\{ \sum_i \|h_i\|_{\rho_i,\mu-\rho_i}; (\rho_i,h_i)_{i\in\mathbb{N}} \text{ with } h_i \in \mathscr{C}_3([a,b];V), \sum_i h_i = h, 0<\rho_i<\mu \right\}$$

and

$$\mathscr{C}_3^{\mu}([a,b];V) = \left\{ h \in \mathscr{C}_3([a,b];V);\ \|h\|_{\mu} < \infty \right\}.$$

Eventually, let $\mathscr{C}_3^{1+}([a,b];V) = \cup_{\mu>1}\mathscr{C}_3^{\mu}([a,b];V)$ and note that the same kind of norms can be considered on the spaces $\mathscr{Z}\mathscr{C}_3([a;b];V)$, leading to the definition of the spaces $\mathscr{Z}\mathscr{C}_3^{\mu}([a;b];V)$ and $\mathscr{Z}\mathscr{C}_3^{1+}([a,b];V)$.

The crucial point in this algebraic approach to the integration of irregular paths is that the operator δ can be inverted under mild smoothness assumptions. This inverse is called Λ. The proof of the following proposition may be found in [9], and in a simpler form in [10].

Proposition 2.2. *There exists a unique linear map $\Lambda : \mathscr{Z}\mathscr{C}_3^{1+}([a,b];V) \to \mathscr{C}_2^{1+}([a,b];V)$ such that*

$$\delta\Lambda = Id_{\mathscr{Z}\mathscr{C}_3^{1+}([a,b];V)} \quad \text{and} \quad \Lambda\delta = Id_{\mathscr{C}_2^{1+}([a,b];V)}.$$

In other words, for any $h \in \mathscr{C}_3^{1+}([a,b];V)$ such that $\delta h = 0$, there exists a unique $g = \Lambda(h) \in \mathscr{C}_2^{1+}([a,b];V)$ such that $\delta g = h$. Furthermore, for any $\mu > 1$, the map Λ is continuous from $\mathscr{Z}\mathscr{C}_3^{\mu}([a,b];V)$ to $\mathscr{C}_2^{\mu}([a,b];V)$ and we have

$$\|\Lambda h\|_{\mu} \le \frac{1}{2^{\mu}-2}\|h\|_{\mu}, \qquad h \in \mathscr{Z}\mathscr{C}_3^{\mu}([a,b];V). \tag{9}$$

This mapping Λ allows to construct a generalized Young integral:

Corollary 2.3. *For any 1-increment $g \in \mathscr{C}_2([a,b];V)$ such that $\delta g \in \mathscr{C}_3^{1+}([a,b];V)$ set $\delta f = (Id - \Lambda\delta)g$. Then*

$$(\delta f)_{st} = \lim_{|\Pi_{st}|\to 0} \sum_{i=0}^{n} g_{t_i\, t_{i+1}}$$

for $a \le s < t \le b$, where the limit is taken over any partition $\Pi_{st} = \{t_0 = s, \ldots, t_n = t\}$ of $[s,t]$, whose mesh tends to zero. Thus, the 1-increment δf is the indefinite integral of the 1-increment g.

We also need some product rules for the operator δ. For this recall the following convention: for $g \in \mathscr{C}_n([a,b];\mathbb{R}^{l,d})$ and $h \in \mathscr{C}_m([a,b];\mathbb{R}^{d,p})$ let gh be the element of $\mathscr{C}_{n+m-1}([a,b];\mathbb{R}^{l,p})$ defined by

$$(gh)_{t_1,\ldots,t_{m+n-1}} = g_{t_1,\ldots,t_n} h_{t_n,\ldots,t_{m+n-1}} \tag{10}$$

for $t_1,\ldots,t_{m+n-1} \in [a,b]$.

Proposition 2.4. *It holds:*

(i) *Let* $g \in \mathscr{C}_1([a,b];\mathbb{R}^{l,d})$ *and* $h \in \mathscr{C}_1([a,b];\mathbb{R}^d)$*. Then* $gh \in \mathscr{C}_1([a,b];\mathbb{R}^l)$ *and*

$$\delta(gh) = \delta g\, h + g\, \delta h.$$

(ii) *Let* $g \in \mathscr{C}_1([a,b];\mathbb{R}^{l,d})$ *and* $h \in \mathscr{C}_2([a,b];\mathbb{R}^d)$*. Then* $gh \in \mathscr{C}_2([a,b];\mathbb{R}^l)$ *and*

$$\delta(gh) = -\delta g\, h + g\, \delta h.$$

(iii) *Let* $g \in \mathscr{C}_2([a,b];\mathbb{R}^{l,d})$ *and* $h \in \mathscr{C}_1([a,b];\mathbb{R}^d)$*. Then* $gh \in \mathscr{C}_2([a,b];\mathbb{R}^l)$ *and*

$$\delta(gh) = \delta g\, h + g\, \delta h.$$

(iv) *Let* $g \in \mathscr{C}_2([a,b];\mathbb{R}^{l,d})$ *and* $h \in \mathscr{C}_2([a,b];\mathbb{R}^{d,p})$*. Then* $gh \in \mathscr{C}_3([a,b];\mathbb{R}^{l,p})$ *and*

$$\delta(gh) = -\delta g\, h + g\, \delta h.$$

2.2 Classical weakly controlled paths (CCP)

In the remainder of this article, we will use both the notations $\int_s^t f\,dg$ or $\mathscr{J}_{st}(f\,dg)$ for the integral of a function f with respect to a given function g on the interval $[s,t]$. Moreover, we set $\|f\|_\infty = \sup_{x\in\mathbb{R}^{d,l}} |f(x)|$ for a function $f : \mathbb{R}^{d,l} \to \mathbb{R}^{m,n}$ and also $\|g\|_\infty = \sup_{t\in[a,b]} |g_t|$ for a path $g \in \mathscr{C}_1^\kappa([a,b];V)$. To simplify the notation we will write $\mathscr{C}_k^\gamma$ instead of $\mathscr{C}_k^\gamma([a,b];V)$, if $[a,b]$ and V are obvious from the context.

Before we consider the technical details, we will make some heuristic considerations about the properties that the solution of equation (6) should have. Set $\hat{\sigma}_t = \sigma\left(y_t\right)$, and suppose that y is a solution of (6), which satisfies $y \in \mathscr{C}_1^\kappa$ for a given $\frac{1}{3} < \kappa < \gamma$. Then the integral form of our equation can be written as

$$y_t = \alpha + \int_0^t \hat{\sigma}_u dx_u, \qquad t \in [0,T]. \tag{11}$$

Our approach to generalized integrals induces us to work with increments of the form $(\delta y)_{st} = y_t - y_s$ instead of (11). It is immediate that one can decompose the increments of (11) into

$$(\delta y)_{st} = \int_s^t \hat{\sigma}_u dx_u = \hat{\sigma}_s (\delta x)_{st} + \rho_{st} \quad \text{with} \quad \rho_{st} = \int_s^t (\hat{\sigma}_u - \hat{\sigma}_s) dx_u.$$

We thus have obtained a decomposition of y of the form $\delta y = \hat{\sigma}\delta x + \rho$. Let us see, still at a heuristic level, which regularity we can expect for $\hat{\sigma}$ and ρ: If σ is bounded and continuously differentiable, we have that $\hat{\sigma}$ is bounded and

$$|\hat{\sigma}_t - \hat{\sigma}_s| \le \|\sigma'\|_\infty \|y\|_\kappa |t-s|^\kappa,$$

where $\|y\|_\kappa$ denotes the κ-Hölder norm of y. Hence $\hat{\sigma}$ belongs to $\mathscr{C}_1^\kappa$ and is bounded. As far as ρ is concerned, it should inherit both the regularities of $\delta\hat{\sigma}$ and x, provided that the integral

$\int_s^t(\hat{\sigma}_u-\hat{\sigma}_s)dx_u=\int_s^t(\delta\hat{\sigma})_{su}dx_u$ is well defined. Thus, one should expect that $\rho\in\mathscr{C}_2^{2\kappa}$. In summary, we have found that a solution δy of equation (11) should be decomposable into

$$\delta y=\hat{\sigma}\delta x+\rho \quad \text{with} \quad \hat{\sigma}\in\mathscr{C}_1^{\kappa} \text{ bounded and } \rho\in\mathscr{C}_2^{2\kappa}. \tag{12}$$

This is precisely the structure we will demand for a possible solution of equation (6) respectively its integral form (11):

Definition 2.5. *Let $a\leq b\leq T$ and let z be a path in $\mathscr{C}_1^{\kappa}([a,b];\mathbb{R}^n)$ with $\kappa\leq\gamma$ and $2\kappa+\gamma>1$. We say that z is a classical weakly controlled path based on x, if $z_a=\alpha\in\mathbb{R}^n$ and $\delta z\in\mathscr{C}_2^{\kappa}([a,b];\mathbb{R}^n)$ can be decomposed into*

$$\delta z=\zeta\delta x+r, \quad i.e. \quad (\delta z)_{st}=\zeta_s(\delta x)_{st}+\rho_{st}, \quad s,t\in[a,b], \tag{13}$$

with $\zeta\in\mathscr{C}_1^{\kappa}([a,b];\mathbb{R}^{n,d})$ and $\rho\in\mathscr{C}_2^{2\kappa}([a,b];\mathbb{R}^n)$.
The space of classical weakly controlled paths on $[a,b]$ will be denoted by $\mathscr{Q}_{\kappa,\alpha}([a,b];\mathbb{R}^n)$, and a path $z\in\mathscr{Q}_{\kappa,\alpha}([a,b];\mathbb{R}^n)$ should be considered in fact as a couple (z,ζ).
The norm on $\mathscr{Q}_{\kappa,\alpha}([a,b];\mathbb{R}^n)$ is given by

$$\mathscr{N}[z;\mathscr{Q}_{\kappa,\alpha}([a,b];\mathbb{R}^n)]=\|\delta z\|_{\kappa}+\|\rho\|_{2\kappa}+\|\zeta\|_{\infty}+\|\delta\zeta\|_{\kappa}.$$

Note that in the above definition α corresponds to a given initial condition and ρ can be understood as a regular part. Moreover, observe that a can be negative.

Now we can sketch the strategy used in [9], in order to solve equation (6):

(a) Verify the stability of $\mathscr{Q}_{\kappa,\alpha}([a,b];\mathbb{R}^n)$ under a smooth map $\varphi:\mathbb{R}^n\to\mathbb{R}^{n,d}$.

(b) Define rigorously the integral $\int z_u dx_u=\mathscr{J}(zdx)$ for a classical weakly controlled path z and compute its decomposition (13).

(c) Solve equation (6) in the space $\mathscr{Q}_{\kappa,\alpha}([a,b];\mathbb{R}^n)$ by a fixed point argument.

Actually, for the second point one has to impose a priori the following hypothesis on the driving rough path, which is a standard assumption in the rough paths theory:

Hypothesis 2.6. *The $\mathbb{R}^d$-valued γ-Hölder path x admits a Lévy area, i.e. a process $\mathbf{x^2}=\mathscr{J}(dxdx)\in\mathscr{C}_2^{2\gamma}([0,T];\mathbb{R}^{d,d})$, which satisfies $\delta\mathbf{x^2}=\delta x\otimes\delta x$, that is*

$$\left[(\delta\mathbf{x^2})_{sut}\right](i,j)=(\delta x^i)_{su}(\delta x^j)_{ut}, \quad \text{for all} \quad s,u,t\in[0,T],\ i,j\in\{1,\ldots,d\}.$$

Then, using the strategy sketched above, the following result is obtained in [9]:

Theorem 2.7. *Let x be a process satisfying Hypothesis 2.6 and let $\sigma\in C^2(\mathbb{R}^n;\mathbb{R}^{n,d})$ be bounded together with its derivatives. Then we have:*

1. *Equation (6) admits a unique solution y in $\mathscr{Q}_{\kappa,\alpha}([0,T];\mathbb{R}^n)$ for any $\kappa<\gamma$ such that $2\kappa+\gamma>1$.*

2. *The mapping $(\alpha,x,\mathbf{x^2})\mapsto y$ is continuous from $\mathbb{R}^n\times\mathscr{C}_1^{\gamma}([0,T];\mathbb{R}^d)\times\mathscr{C}_2^{2\gamma}([0,T];\mathbb{R}^{d,d})$ to $\mathscr{Q}_{\kappa,\alpha}([0,T];\mathbb{R}^n)$, in a sense which is detailed in [9, Proposition 8].*

3 The delay equation

In this section, we make a first step towards the solution of the delay equation

$$\begin{cases} dy_t = \sigma(y_t, y_{t-r_1}, \ldots, y_{t-r_k})dx_t, & t \in [0,T], \\ y_t = \xi_t, & t \in [-r_k, 0], \end{cases} \tag{14}$$

where x is a $\mathbb{R}^d$-valued γ-Hölder continuous function with $\gamma > \frac{1}{3}$, the function $\sigma \in C^3(\mathbb{R}^{n(k+1)}; \mathbb{R}^{n,d})$ is bounded together with its derivatives, ξ is a $\mathbb{R}^n$-valued weakly controlled path based on x, and $0 < r_1 < \ldots < r_k < \infty$. For convenience, we set $r_0 = 0$ and, moreover, we will use the notation

$$\mathfrak{s}(y)_t = (y_{t-r_1}, \ldots, y_{t-r_k}), \qquad t \in [0,T]. \tag{15}$$

3.1 Delayed controlled paths

As in the previous section, we will first make some heuristic considerations about the properties of a solution: set $\hat{\sigma}_t = \sigma(y_t, \mathfrak{s}(y)_t)$ and suppose that y is a solution of (14) with $y \in \mathscr{C}_1^\kappa$ for a given $\frac{1}{3} < \kappa < \gamma$. Then we can write the integral form of our equation as

$$(\delta y)_{st} = \int_s^t \hat{\sigma}_u dx_u = \hat{\sigma}_s (\delta x)_{st} + \rho_{st} \quad \text{with} \quad \rho_{st} = \int_s^t (\hat{\sigma}_u - \hat{\sigma}_s) dx_u.$$

Thus, we have again obtained a decomposition of y of the form $\delta y = \hat{\sigma}\delta x + \rho$. Moreover, it follows (still at a heuristic level) that $\hat{\sigma}$ is bounded and satisfies

$$|\hat{\sigma}_t - \hat{\sigma}_s| \le \|\sigma'\|_\infty \sum_{i=0}^k |y_{t-r_i} - y_{s-r_i}| \le (k+1)\|\sigma'\|_\infty \|y\|_\kappa |t-s|^\kappa.$$

Thus, with the notation of Section 2.1, we have that $\hat{\sigma}$ belongs to $\mathscr{C}_1^\kappa$ and is bounded. The term ρ should again inherit both the regularities of $\delta\hat{\sigma}$ and x. Thus, one should have that $\rho \in \mathscr{C}_2^{2\kappa}$. In conclusion, the increment δy should be decomposable into

$$\delta y = \hat{\sigma}\delta x + \rho \quad \text{with} \quad \hat{\sigma} \in \mathscr{C}_1^\kappa \text{ bounded and } \rho \in \mathscr{C}_2^{2\kappa}. \tag{16}$$

This is again the structure we will ask for a possible solution to (14). However, this decomposition does not take into account that equation (14) is actually a *delay* equation. To define the integral $\int_s^t \hat{\sigma}_u dx_u$, we have to enlarge the class of functions we will work with, and hence we will define a *delayed controlled path* (hereafter DCP in short).

Definition 3.1. *Let $0 \le a \le b \le T$ and $z \in \mathscr{C}_1^\kappa([a,b]; \mathbb{R}^n)$ with $\frac{1}{3} < \kappa \le \gamma$. We say that z is a delayed controlled path based on x, if $z_a = \alpha$ belongs to $\mathbb{R}^n$ and if $\delta z \in \mathscr{C}_2^\kappa([a,b]; \mathbb{R}^n)$ can be decomposed into*

$$(\delta z)_{st} = \sum_{i=0}^k \zeta_s^{(i)} (\delta x)_{s-r_i, t-r_i} + \rho_{st} \quad \text{for} \quad s,t \in [a,b], \tag{17}$$

where $\rho \in \mathscr{C}_2^{2\kappa}([a,b]; \mathbb{R}^n)$ and $\zeta^{(i)} \in \mathscr{C}_1^\kappa([a,b]; \mathbb{R}^{n,d})$ for $i = 0, \ldots, k$.
The space of delayed controlled paths on $[a,b]$ will be denoted by $\mathscr{D}_{\kappa,\alpha}([a,b]; \mathbb{R}^n)$, and a path $z \in$

$\mathcal{D}_{\kappa,\alpha}([a,b];\mathbb{R}^n)$ should be considered in fact as a $(k+2)$-tuple $(z,\zeta^{(0)},\dots,\zeta^{(k)})$. The norm on $\mathcal{D}_{\kappa,\alpha}([a,b];\mathbb{R}^n)$ is given by

$$\mathcal{N}[z;\mathcal{D}_{\kappa,\alpha}([a,b];\mathbb{R}^n)] = \|\delta z\|_\kappa + \|\rho\|_{2\kappa} + \sum_{i=0}^{k}\|\zeta^{(i)}\|_\infty + \sum_{i=0}^{k}\|\delta\zeta^{(i)}\|_\kappa.$$

Now we can sketch our strategy to solve the delay equation:

1. Consider the map T_σ defined on $\mathcal{Q}_{\kappa,\alpha}([a,b];\mathbb{R}^n)\times\mathcal{Q}_{\kappa,\tilde\alpha}([a-r_k,b-r_1];\mathbb{R}^n)$ by
$$(T_\sigma(z,\tilde z))_t = \sigma(z_t,\mathfrak{s}(\tilde z)_t), \qquad t\in[a,b], \tag{18}$$
where we recall that the notation $\mathfrak{s}(\tilde z)$ has been introduced at (15). We will show that T_σ maps $\mathcal{Q}_{\kappa,\alpha}([a,b];\mathbb{R}^n)\times\mathcal{Q}_{\kappa,\tilde\alpha}([a-r_k,b-r_1];\mathbb{R}^n)$ smoothly onto a space of the form $\mathcal{D}_{\kappa,\hat\alpha}([a,b];\mathbb{R}^{n,d})$.

2. Define rigorously the integral $\int z_u dx_u = \mathcal{J}(zdx)$ for a delayed controlled path $z\in\mathcal{D}_{\kappa,\hat\alpha}([a,b];\mathbb{R}^{n,d})$, show that $\mathcal{J}(zdx)$ belongs to $\mathcal{Q}_{\kappa,\alpha}([a,b];\mathbb{R}^d)$, and compute its decomposition (13). Let us point out the following important fact: T_σ creates "delay", that is $T_\sigma(z,\tilde z)\in\mathcal{D}_{\kappa,\hat\alpha}([a,b];\mathbb{R}^{n,d})$, while $\mathcal{J}$ creates "advance", that is $\mathcal{J}(zdx)\in\mathcal{Q}_{\kappa,\alpha}([a,b];\mathbb{R}^n)$.

3. By combining the first two points, we will solve equation (14) by a fixed point argument on the intervals $[0,r_1],[r_1,2r_1],\dots$.

3.2 Action of the map T on controlled paths

The major part of this section will be devoted to the following two stability results:

Proposition 3.2. *Let $0\le a\le b\le T$, let $\alpha,\tilde\alpha$ be two initial conditions in $\mathbb{R}^n$ and let $\varphi\in C^3(\mathbb{R}^{n(k+1)};\mathbb{R}^l)$ be bounded with bounded derivatives. Define T_φ on $\mathcal{Q}_{\kappa,\alpha}([a;b];\mathbb{R}^n)\times\mathcal{Q}_{\kappa,\tilde\alpha}([a-r_k;b-r_1];\mathbb{R}^n)$ by $T_\varphi(z,\tilde z)=\hat z$, with*

$$\hat z_t = \varphi(z_t,\mathfrak{s}(\tilde z)_t), \qquad t\in[a,b].$$

Then, setting $\hat\alpha=\varphi(\alpha,\mathfrak{s}(\tilde z_a))=\varphi(\alpha,\tilde z_{a-r_1},\dots,\tilde z_{a-r_{k-1}},\tilde\alpha)$, we have $T_\varphi(z,\tilde z)\in\mathcal{D}_{\kappa,\hat\alpha}([a;b];\ \mathbb{R}^l)$ and it admits a decomposition of the form

$$(\delta\hat z)_{st} = \hat\zeta_s\,(\delta x)_{st} + \sum_{i=1}^{k}\hat\zeta_s^{(i)}\,(\delta x)_{s-r_i,t-r_i} + \hat\rho_{st}, \qquad s,t\in[a,b], \tag{19}$$

where $\hat\zeta,\hat\zeta^{(i)}$ are the $\mathbb{R}^{l,d}$-valued paths defined by

$$\hat\zeta_s = \left(\frac{\partial\varphi}{\partial x_{1,0}}(z_s,\mathfrak{s}(\tilde z)_s),\dots,\frac{\partial\varphi}{\partial x_{n,0}}(z_s,\mathfrak{s}(\tilde z)_s)\right)\zeta_s, \qquad s\in[a,b],$$

and

$$\hat\zeta_s^{(i)} = \left(\frac{\partial\varphi}{\partial x_{1,i}}(z_s,\mathfrak{s}(\tilde z)_s),\dots,\frac{\partial\varphi}{\partial x_{n,i}}(z_s,\mathfrak{s}(\tilde z)_s)\right)\tilde\zeta_{s-r_i}, \qquad s\in[a,b],$$

for $i = 1, \ldots, k$. Moreover, the following estimate holds:

$$\begin{aligned}
&\mathcal{N}[\hat{z}; \mathcal{D}_{\kappa,\hat{\alpha}}([a;b];\mathbb{R}^l)] \\
&\quad \le c_{\varphi,T}\left(1+\mathcal{N}^2[z;\mathcal{Q}_{\kappa,\alpha}([a,b];\mathbb{R}^n)]+\mathcal{N}^2[\tilde{z};\mathcal{Q}_{\kappa,\tilde{\alpha}}([a-r_k,b-r_1];\mathbb{R}^n)]\right),
\end{aligned} \tag{20}$$

where the constant $c_{\varphi,T}$ depends only φ and T.

Proof. Fix $s,t \in [a,b]$ and set

$$\psi_s^{(i)} = \left(\frac{\partial \varphi}{\partial x_{1,i}}(z_s, \mathfrak{s}(\tilde{z})_s), \ldots, \frac{\partial \varphi}{\partial x_{n,i}}(z_s, \mathfrak{s}(\tilde{z})_s)\right).$$

for $i = 0, \ldots, k$. It is readily checked that

$$\begin{aligned}
(\delta\hat{z})_{st} &= \varphi(z_{t-r_0}, \tilde{z}_{t-r_1}, \ldots, \tilde{z}_{t-r_k}) - \varphi(z_{s-r_0}, \tilde{z}_{s-r_1}, \ldots, \tilde{z}_{s-r_k}) \\
&= \psi_s^{(0)}\zeta_s(\delta x)_{st} + \sum_{i=1}^{k}\psi_s^{(i)}\tilde{\zeta}_{s-r_i}(\delta x)_{s-r_i,t-r_i} + \hat{\rho}_{st}^{(1)} + \hat{\rho}_{st}^{(2)},
\end{aligned}$$

where

$$\begin{aligned}
\hat{\rho}_{st}^{(1)} &= \psi_s^{(0)}\rho_{st} + \sum_{i=1}^{k}\psi_s^{(i)}\tilde{\rho}_{s-r_i,t-r_i}, \\
\hat{\rho}_{st}^{(2)} &= \varphi(z_{t-r_0}, \tilde{z}_{t-r_1}, \ldots, \tilde{z}_{t-r_k}) - \varphi(z_{s-r_0}, \tilde{z}_{s-r_1}, \ldots, \tilde{z}_{s-r_k}) \\
&\qquad - \psi_s^{(0)}(\delta z)_{st} - \sum_{i=1}^{k}\psi_s^{(i)}(\delta\tilde{z})_{s-r_i,t-r_i}.
\end{aligned}$$

(i) We first have to show that $\hat{\rho}^{(1)}, \hat{\rho}^{(2)} \in \mathcal{C}_2^{2\kappa}([a,b];\mathbb{R}^l)$. For the second remainder term Taylor's formula yields

$$|\hat{\rho}_{st}^{(2)}| \le \frac{1}{2}\|\varphi''\|_\infty\Big(|(\delta z)_{st}|^2 + \sum_{i=1}^{k}|(\delta\tilde{z})_{s-r_i,t-r_i}|^2\Big),$$

and hence clearly, thanks to some straightforward bounds in the spaces $\mathcal{Q}_{\kappa,\alpha}$, we have

$$\frac{|\hat{\rho}_{st}^{(2)}|}{|t-s|^{2\kappa}} \le \frac{1}{2}\|\varphi''\|_\infty\Big(\mathcal{N}^2[z;\mathcal{Q}_{\kappa,\alpha}([a,b];\mathbb{R}^n)] + \sum_{i=1}^{k}\mathcal{N}^2[\tilde{z};\mathcal{Q}_{\kappa,\alpha}([a-r_i,b-r_i];\mathbb{R}^n)]\Big). \tag{21}$$

The first term can also be bounded easily: it can be checked that

$$\frac{|\hat{\rho}_{st}^{(1)}|}{|t-s|^{2\kappa}} \le \|\varphi'\|_\infty\left(\mathcal{N}\left[\rho;\mathcal{C}_2^{2\kappa}([a,b];\mathbb{R}^n)\right] + \sum_{i=1}^{k}\mathcal{N}\left[\tilde{\rho},\mathcal{C}_2^{2\kappa}([a-r_i,b-r_i];\mathbb{R}^n)\right]\right). \tag{22}$$

Putting together the last two inequalities, we have shown that decomposition (19) holds, that is

$$(\delta\hat{z})_{st} = \psi_s^{(0)}\zeta_s(\delta x)_{st} + \sum_{i=1}^{k}\psi_s^{(i)}\tilde{\zeta}_{s-r_i}^{(i)}(\delta x)_{s-r_i,t-r_i} + \hat{\rho}_{st}$$

with $\hat{\rho}_{st} = \hat{\rho}^{(1)}_{st} + \hat{\rho}^{(2)}_{st} \in \mathscr{C}_2^{2\kappa}([a,b];\mathbb{R}^l)$.

(ii) Now we have to consider the "density" functions

$$\hat{\zeta}_s = \psi_s^{(0)}\zeta_s, \quad \hat{\zeta}_s^{(i)} = \psi_s^{(i)}\tilde{\zeta}_{s-r_i}, \qquad s \in [a,b].$$

Clearly $\hat{\zeta}, \hat{\zeta}^{(i)}$ are bounded on $[a,b]$, because the functions $\psi^{(i)}$ are bounded (due to the boundedness of φ') and because ζ, $\tilde{\zeta}^{(i)}$ are also bounded. In particular, it holds

$$\sup_{s\in[a,b]} |\hat{\zeta}_s| \leq \|\varphi'\|_\infty \sup_{s\in[a,b]} |\zeta_s|, \qquad \sup_{s\in[a,b]} |\hat{\zeta}_s^{(i)}| \leq \|\varphi'\|_\infty \sup_{s\in[a,b]} |\tilde{\zeta}_{s-r_i}| \tag{23}$$

for $i = 1,\ldots,k$. Moreover, for $i = 1,\ldots,k$, we have

$$\begin{aligned}
&|\hat{\zeta}^{(i)}_{s_1} - \hat{\zeta}^{(i)}_{s_2}| \\
&\quad \leq |(\psi^{(i)}_{s_1} - \psi^{(i)}_{s_2})\tilde{\zeta}_{s_1-r_i}| + |(\tilde{\zeta}_{s_1-r_i} - \tilde{\zeta}_{s_2-r_i})\psi^{(i)}_{s_2}| \\
&\quad \leq \|\varphi''\|_\infty |z_{s_1} - z_{s_2}| \sup_{s\in[a,b]} |\tilde{\zeta}_{s-r_i}| + \|\varphi''\|_\infty \sum_{j=1}^{k} |\tilde{z}_{s_1-r_j} - \tilde{z}_{s_2-r_j}| \sup_{s\in[a,b]} |\tilde{\zeta}_{s-r_i}| \\
&\qquad + \|\psi^{(i)}\|_\infty |\tilde{\zeta}_{s_1-r_i} - \tilde{\zeta}_{s_2-r_i}| \\
&\quad \leq \|\varphi''\|_\infty\, \mathcal{N}[z; \mathscr{C}_1^\kappa([a,b];\mathbb{R}^n)] \sup_{s\in[a,b]} |\tilde{\zeta}_{s-r_i}|\, |s_2 - s_1|^\kappa \\
&\qquad + \|\varphi''\|_\infty \sum_{j=1}^{k} \mathcal{N}[\tilde{z}; \mathscr{C}_1^\kappa([a-r_j, b-r_j];\mathbb{R}^n)] \sup_{s\in[a,b]} |\tilde{\zeta}_{s-r_i}|\, |s_2 - s_1|^\kappa \\
&\qquad + \|\psi^{(i)}\|_\infty\, \mathcal{N}[\tilde{\zeta}; \mathscr{C}_1^\kappa([a-r_i, b-r_i];\mathbb{R}^{n,d})]\, |s_2 - s_1|^\kappa.
\end{aligned} \tag{24}$$

Similarly, we obtain

$$\begin{aligned}
|\hat{\zeta}_{s_1} - \hat{\zeta}_{s_2}| \quad \leq \quad & \|\varphi''\|_\infty\, \mathcal{N}[z; \mathscr{C}_1^\kappa([a,b];\mathbb{R}^n)] \sup_{s\in[a,b]} |\zeta_s|\, |s_2 - s_1|^\kappa \\
& + \|\varphi''\|_\infty \sum_{j=1}^{k} \mathcal{N}[\tilde{z}; \mathscr{C}_1^\kappa([a-r_j, b-r_j];\mathbb{R}^n)] \sup_{s\in[a,b]} |\zeta_s|\, |s_2 - s_1|^\kappa \\
& + \|\psi^{(0)}\|_\infty\, \mathcal{N}[\zeta; \mathscr{C}_1^\kappa([a,b];\mathbb{R}^{n,d})]\, |s_2 - s_1|^\kappa.
\end{aligned} \tag{25}$$

Hence, the densities satisfy the conditions of Definition 3.1.

(iii) Finally, combining the estimates (21) – (25) yields the estimate (20), which ends the proof. □

We thus have shown that the map T_φ is quadratically bounded in z and $\tilde{z}$. Moreover, for fixed $\tilde{z}$ the map $T_\varphi(\cdot,\tilde{z}) : \mathcal{Q}_{\kappa,\alpha}([a;b];\mathbb{R}^n) \to \mathcal{D}_{\kappa,\hat{\alpha}}([a;b];\mathbb{R}^l)$ is locally Lipschitz continuous:

Proposition 3.3. *Let the notations and assumptions of Proposition 3.2 prevail. Let $0 \leq a \leq b \leq T$, let $z^{(1)}, z^{(2)} \in \mathcal{Q}_{\kappa,\alpha}([a,b];\mathbb{R}^n)$ and let $\tilde{z} \in \mathcal{Q}_{\kappa,\tilde{\alpha}}([a-r_k, b-r_1];\mathbb{R}^n)$. Then,*

$$\begin{aligned}
&\mathcal{N}[T_\varphi(z^{(1)},\tilde{z}) - T_\varphi(z^{(2)},\tilde{z}); \mathcal{D}_{\kappa,0}([a;b];\mathbb{R}^l)] \\
&\qquad\qquad \leq c_{\varphi,T} \left(1 + C(z^{(1)}, z^{(2)}, \tilde{z})\right)^2 \mathcal{N}[z^{(1)} - z^{(2)}; \mathcal{Q}_{\kappa,0}([a,b];\mathbb{R}^n)],
\end{aligned} \tag{26}$$

where

$$C(z^{(1)},z^{(2)},\tilde{z}) = \mathcal{N}[\tilde{z};\mathcal{Q}_{\kappa,\tilde{\alpha}}([a-r_k,b-r_1];\mathbb{R}^n)] \\ +\mathcal{N}[z^{(1)};\mathcal{Q}_{\kappa,\alpha}([a,b];\mathbb{R}^n)] + \mathcal{N}[z^{(2)};\mathcal{Q}_{\kappa,\alpha}([a,b];\mathbb{R}^n)] \tag{27}$$

and the constant $c_{\varphi,T}$ depends only on φ and T.

Proof. Denote $\hat{z}^{(j)} = T_\varphi(z^{(j)},\tilde{z})$ for $j=1,2$. By Proposition 3.2 we have

$$\left(\delta\hat{z}^{(j)}\right)_{st} = \hat{\zeta}_s^{(j)}(\delta x)_{st} + \sum_{i=1}^k \hat{\zeta}_s^{(i,j)}(\delta x)_{s-r_i,t-r_i} + \hat{\rho}_{st}^{(j)}, \qquad s,t\in[a,b],$$

with

$$\hat{\zeta}_s^{(j)} = \psi_s^{(0,j)}\zeta_s^{(j)}, \qquad \hat{\zeta}_s^{(i,j)} = \psi_s^{(i,j)}\tilde{\zeta}_{s-r_i}, \qquad s\in[a,b],$$

where

$$\psi_s^{(i,j)} \quad = \quad \left(\frac{\partial\varphi}{\partial x_{1,i}}(z_s^{(j)},\mathfrak{s}(\tilde{z})_s),\ldots,\frac{\partial\varphi}{\partial x_{n,i}}(z_s^{(j)},\mathfrak{s}(\tilde{z})_s)\right), \qquad s\in[a,b],$$

for $i=0,\ldots,k$, $j=1,2$. Furthermore, it holds $\hat{\rho}_{st}^{(j)} = \hat{\rho}_{st}^{(1,j)} + \hat{\rho}_{st}^{(2,j)}$, where

$$\hat{\rho}_{st}^{(1,j)} = \psi_s^{(0,j)}\rho_{st}^{(j)} + \sum_{i=1}^k \psi_s^{(i,j)}\tilde{\rho}_{s-r_i,t-r_i},$$
$$\hat{\rho}_{st}^{(2,j)} = \varphi(z_{t-r_0}^{(j)},\tilde{z}_{t-r_1},\ldots,\tilde{z}_{t-r_k}) - \varphi(z_{s-r_0}^{(j)},\tilde{z}_{s-r_1},\ldots,\tilde{z}_{s-r_k}) \\ - \psi_s^{(0,j)}(\delta z^{(j)})_{st} - \sum_{i=1}^k \psi_s^{(i,j)}(\delta\tilde{z})_{s-r_i,t-r_i}.$$

Thus, we obtain for $\hat{z} = \hat{z}^{(1)} - \hat{z}^{(2)}$ the decomposition

$$(\delta\hat{z})_{st} = \sum_{i=0}^k \hat{\zeta}_s^{(i)}(\delta x)_{s-r_i,t-r_i} + \hat{\rho}_{st}$$

with $\hat{\zeta}_s^{(0)} = \psi_s^{(0,1)}\zeta_s^{(1)} - \psi_s^{(0,2)}\zeta_s^{(2)}$, the paths $\hat{\zeta}^{(i)}$ are defined by $\hat{\zeta}_s^{(i)} = (\psi_s^{(i,1)} - \psi_s^{(i,2)})\tilde{\zeta}_{s-r_i}$ for $i=1,\ldots,k$, and $\hat{\rho}_{st} = \hat{\rho}_{st}^{(1)} - \hat{\rho}_{st}^{(2)}$.

In the following we will denote constants (which depend only on T and φ) by c, regardless of their value. For convenience, we will also use the short notations $\mathcal{N}[\tilde{z}]$, $\mathcal{N}[z^{(1)}]$, $\mathcal{N}[z^{(2)}]$ and $\mathcal{N}[z^{(1)}-z^{(2)}]$ instead of the corresponding quantities in (26) and (27).

(i) We first control the supremum of the density functions $\hat{\zeta}^{(i)}$, $i=0,\ldots,k$. For $i=0$ we can write

$$\hat{\zeta}_s^{(0)} = \psi_s^{(0,1)}(\zeta_s^{(1)} - \zeta_s^{(2)}) + (\psi_s^{(0,1)} - \psi_s^{(0,2)})\zeta_s^{(2)}$$

and thus it follows

$$|\hat{\zeta}_s^{(0)}| \le \|\varphi'\|_\infty|\zeta_s^{(1)} - \zeta_s^{(2)}| + \|\varphi''\|_\infty|z_s^{(1)} - z_s^{(2)}||\zeta_s^{(2)}| \\ \le c\left(1+\mathcal{N}[z^{(2)}]\right)\mathcal{N}[z^{(1)}-z^{(2)}]. \tag{28}$$

Similarly, we get

$$|\hat{\zeta}_s^{(i)}| \le c\,\mathcal{N}[\tilde{z}]\,\mathcal{N}[z^{(1)} - z^{(2)}]. \tag{29}$$

(ii) Now, consider the increments of the density functions. Here, the key is to expand the expression $\psi_s^{(i,1)} - \psi_s^{(i,2)}$ for $i = 0, \ldots, k$. For this define

$$u_s(r) = r(z_s^{(1)} - z_s^{(2)}) + z_s^{(2)}, \qquad r \in [0,1], \quad s \in [a,b].$$

We have

$$\begin{aligned}\frac{\partial \varphi}{\partial x_{p,i}}(z_s^{(1)}, \mathfrak{s}(\tilde{z})_s) - \frac{\partial \varphi}{\partial x_{p,i}}(z_s^{(2)}, \mathfrak{s}(\tilde{z})_s) &= \frac{\partial \varphi}{\partial x_{p,i}}(u_s(1), \mathfrak{s}(\tilde{z})_s) - \frac{\partial \varphi}{\partial x_{p,i}}(u_s(0), \mathfrak{s}(\tilde{z})_s) \\ &= \theta_s^{(p,i)}(z_s^{(1)} - z_s^{(2)}),\end{aligned}$$

where

$$\theta_s^{(p,i)} = \int_0^1 \left(\frac{\partial^2 \varphi}{\partial x_{1,0} \partial x_{p,i}}(u_s(r), \mathfrak{s}(\tilde{z})_s), \ldots, \frac{\partial^2 \varphi}{\partial x_{n,0} \partial x_{p,i}}(u_s(r), \mathfrak{s}(\tilde{z})_s) \right) dr.$$

Hence it follows

$$\psi_s^{(i,1)} - \psi_s^{(i,2)} = \left(\theta_s^{(1,i)}(z_s^{(1)} - z_s^{(2)}), \ldots, \theta_s^{(n,i)}(z_s^{(1)} - z_s^{(2)}) \right). \tag{30}$$

Note that $\theta^{(p,i)}$ is clearly bounded and, under the assumption $\varphi \in C_b^3$, it moreover satisfies

$$|\theta_t^{(p,i)} - \theta_s^{(p,i)}| \le c\, \left(\mathcal{N}[z^{(1)}] + \mathcal{N}[z^{(2)}] + \mathcal{N}[\tilde{z}] \right) |t-s|^\kappa. \tag{31}$$

For $i = 0$ we can now write

$$\begin{aligned}\hat{\zeta}_t^{(0)} - \hat{\zeta}_s^{(0)} = {} & (\psi_t^{(0,1)} - \psi_s^{(0,1)})(\zeta_s^{(1)} - \zeta_s^{(2)}) + \psi_t^{(0,1)}((\zeta_t^{(1)} - \zeta_t^{(2)}) - (\zeta_s^{(1)} - \zeta_s^{(2)})) \\ & + (\psi_s^{(0,1)} - \psi_s^{(0,2)})(\zeta_t^{(2)} - \zeta_s^{(2)}) + \zeta_t^{(2)}((\psi_t^{(0,1)} - \psi_t^{(0,2)}) - (\psi_s^{(0,1)} - \psi_s^{(0,2)})).\end{aligned}$$

It follows

$$\begin{aligned}|\hat{\zeta}_t^{(0)} - \hat{\zeta}_s^{(0)}| \le {} & c\left(\mathcal{N}[z^{(1)}] + \mathcal{N}[\tilde{z}] \right) |t-s|^\kappa\, \mathcal{N}[z^{(1)} - z^{(2)}] + c\,\mathcal{N}[z^{(1)} - z^{(2)}]\,|t-s|^\kappa \\ & + c\,\mathcal{N}[z^{(1)} - z^{(2)}]\,\mathcal{N}[z^{(2)}]\,|t-s|^\kappa \\ & + \mathcal{N}[z^{(2)}]|(\psi_t^{(0,1)} - \psi_t^{(0,2)}) - (\psi_s^{(0,1)} - \psi_s^{(0,2)})|.\end{aligned} \tag{32}$$

Using (30) and (31) we obtain

$$\begin{aligned}&|(\psi_t^{(0,1)} - \psi_t^{(0,2)}) - (\psi_s^{(0,1)} - \psi_s^{(0,2)})| \\ &\qquad \le c\left(1 + \mathcal{N}[z^{(1)}] + \mathcal{N}[z^{(2)}] + \mathcal{N}[\tilde{z}] \right) \mathcal{N}[z^{(1)} - z^{(2)}]\,|t-s|^\kappa.\end{aligned} \tag{33}$$

Combining (32) and (33) yields

$$|\hat{\zeta}_t^{(0)} - \hat{\zeta}_s^{(0)}| \le c\left(1 + \mathcal{N}[z^{(1)}] + \mathcal{N}[z^{(2)}] + \mathcal{N}[\tilde{z}] \right)^2 \mathcal{N}[z^{(1)} - z^{(2)}]\,|t-s|^\kappa. \tag{34}$$

By similar calculations we also have

$$|\hat{\zeta}_t^{(i)} - \hat{\zeta}_s^{(i)}| \le c \left(1 + \mathcal{N}[z^{(1)}] + \mathcal{N}[z^{(2)}] + \mathcal{N}[\tilde{z}]\right)^2 \mathcal{N}[z^{(1)} - z^{(2)}]\, |t-s|^{\kappa} \tag{35}$$

for $i = 1, \dots, k$.

(iii) Now, we have to control the remainder term $\hat{\rho}$. For this we decompose $\hat{\rho}$ as

$$\hat{\rho}_{st} = \rho_{st}^{(1)} + \rho_{st}^{(2)},$$

where

$$\begin{aligned}
\rho_{st}^{(1)} &= \psi_s^{(0,1)} \rho_{st}^{(1)} - \psi_s^{(0,2)} \rho_{st}^{(2)} + \sum_{i=1}^{k} \left(\psi_s^{(i,1)} - \psi_s^{(i,2)}\right) \tilde{\rho}_{s-r_i, t-r_i},\\
\rho_{st}^{(2)} &= \left(\varphi(z_{t-r_0}^{(1)}, \tilde{z}_{t-r_1}, \dots, \tilde{z}_{t-r_k}) - \varphi(z_{s-r_0}^{(1)}, \tilde{z}_{s-r_1}, \dots, \tilde{z}_{s-r_k})\right)\\
&\quad - \left(\varphi(z_{t-r_0}^{(2)}, \tilde{z}_{t-r_1}, \dots, \tilde{z}_{t-r_k}) - \varphi(z_{s-r_0}^{(2)}, \tilde{z}_{s-r_1}, \dots, \tilde{z}_{s-r_k})\right)\\
&\quad - \left(\psi_s^{(0,1)} (\delta z^{(1)})_{st} - \psi_s^{(0,2)} (\delta z^{(2)})_{st}\right) - \sum_{i=1}^{k} \left(\psi_s^{(i,1)} - \psi_s^{(i,2)}\right) (\delta \tilde{z})_{s-r_i, t-r_i}.
\end{aligned}$$

We consider first $\rho^{(1)}$: for this term, some straightforward calculations yield

$$|\rho_{st}^{(1)}| \le c(1 + \mathcal{N}[z^{(1)}] + \mathcal{N}[z^{(2)}] + \mathcal{N}[\tilde{z}])\, \mathcal{N}[z^{(1)} - z^{(2)}]\, |t-s|^{2\kappa}. \tag{36}$$

Now consider $\rho^{(2)}$. The mean value theorem yields

$$\begin{aligned}
\rho_{st}^{(2)} &= \left(\bar{\psi}_s^{(0,1)} - \psi_s^{(0,1)}\right) (\delta z^{(1)})_{st} - \left(\bar{\psi}_s^{(0,2)} - \psi_s^{(0,2)}\right) (\delta z^{(2)})_{st}\\
&\qquad + \sum_{i=1}^{k} \left(\left(\bar{\psi}_s^{(i,1)} - \bar{\psi}_s^{(i,2)}\right) - \left(\psi_s^{(i,1)} - \psi_s^{(i,2)}\right)\right) (\delta \tilde{z})_{s-r_i, t-r_i}\\
&= \left(\bar{\psi}_s^{(0,1)} - \psi_s^{(0,1)}\right) (\delta(z^{(1)} - z^{(2)}))_{st}\\
&\qquad + \left(\left(\bar{\psi}_s^{(0,1)} - \bar{\psi}_s^{(0,2)}\right) - \left(\psi_s^{(0,1)} - \psi_s^{(0,2)}\right)\right) (\delta z^{(2)})_{st}\\
&\qquad + \sum_{i=1}^{k} \left(\left(\bar{\psi}_s^{(i,1)} - \bar{\psi}_s^{(i,2)}\right) - \left(\psi_s^{(i,1)} - \psi_s^{(i,2)}\right)\right) (\delta \tilde{z})_{s-r_i, t-r_i}\\
&\triangleq Q_1 + Q_2 + Q_3,
\end{aligned}$$

with

$$\begin{aligned}
\bar{\psi}_s^{(i,j)} &= \int_0^1 \left(\frac{\partial \varphi}{\partial x_{1,i}}(v_s^{(j)}(r)), \dots, \frac{\partial \varphi}{\partial x_{n,i}}(v_s^{(j)}(r))\right) dr,\\
v_s^{(j)}(r) &= \left(z_s^{(j)} + r(z_t^{(j)} - z_s^{(j)}), \tilde{z}_{s-r_1} + r(\tilde{z}_{t-r_1} - \tilde{z}_{s-r_1}), \dots, \tilde{z}_{s-r_k} + r(\tilde{z}_{t-r_k} - \tilde{z}_{s-r_k})\right).
\end{aligned}$$

We shall now bound Q_1, Q_2 and Q_3 separately: it is readily checked that

$$|\bar{\psi}_s^{(i,j)} - \psi_s^{(i,j)}| \le c \left(1 + \mathcal{N}[z^{(1)}] + \mathcal{N}[z^{(2)}] + \mathcal{N}[\tilde{z}]\right) |t-s|^{\kappa},$$

and thus we obtain

$$Q_1 \le c\left(1+\mathcal{N}[z^{(1)}]+\mathcal{N}[z^{(2)}]+\mathcal{N}[\tilde{z}]\right)\mathcal{N}[z^{(1)}-z^{(2)}]\,|t-s|^{2\kappa}. \tag{37}$$

In order to estimate Q_2 and Q_3, recall that by (30) in part *(ii)* we have

$$\psi_s^{(i,1)}-\psi_s^{(i,2)}=\left(\theta_s^{(1,i)}(z_s^{(1)}-z_s^{(2)}),\dots,\theta_s^{(n,i)}(z_s^{(1)}-z_s^{(2)})\right), \tag{38}$$

where

$$\theta_s^{(p,i)}=\int_0^1\left(\frac{\partial^2\varphi}{\partial x_{1,0}\partial x_{p,i}}(u_s(r'),\mathfrak{s}(\tilde{z})_s),\dots,\frac{\partial^2\varphi}{\partial x_{n,0}\partial x_{p,i}}(u_s(r'),\mathfrak{s}(\tilde{z})_s)\right)dr',$$
$$u_s(r')=z_s^{(2)}+r'(z_s^{(1)}-z_s^{(2)}).$$

Similarly, we also obtain that

$$\bar{\psi}_s^{(i,1)}-\bar{\psi}_s^{(i,2)}=\left(\bar{\theta}_s^{(1,i)}(z_s^{(1)}-z_s^{(2)}),\dots,\bar{\theta}_s^{(n,i)}(z_s^{(1)}-z_s^{(2)})\right)+q_{st}^{(i)} \tag{39}$$

with

$$\begin{aligned}\bar{\theta}_s^{(p,i)} &= \int_0^1\int_0^1\left(\frac{\partial^2\varphi}{\partial x_{1,0}\partial x_{p,i}}(\bar{u}_s(r,r')),\dots,\frac{\partial^2\varphi}{\partial x_{n,0}\partial x_{p,i}}(\bar{u}_s(r,r'))\right)dr\,dr',\\ \bar{u}_s(r,r') &= v_s^{(2)}(r)+r'\left(v_s^{(1)}(r)-v_s^{(2)}(r)\right)\end{aligned}$$

and

$$|q_{st}^{(i)}|\le c\,\mathcal{N}[z^{(1)}-z^{(2)}]\,|t-s|^{\kappa}.$$

Now, using (38) and (39) we can write

$$\begin{aligned}&\left(\bar{\psi}_s^{(i,1)}-\bar{\psi}_s^{(i,2)}\right)-\left(\psi_s^{(i,1)}-\psi_s^{(i,2)}\right)\\ &\quad=\left((\bar{\theta}_s^{(1,i)}-\theta_s^{(1,i)})(z^{(1)}-z^{(2)}),\dots,(\bar{\theta}_s^{(n,i)}-\theta_s^{(n,i)})(z^{(1)}-z^{(2)})\right)+q_{st}^{(i)}\end{aligned}$$

for any $i=0,\dots,k$. Since moreover

$$\begin{aligned}&\bar{u}_s(r,r')-(u_s(r'),\mathfrak{s}(\tilde{z})_s)\\ &\quad=r\left((z_t^{(2)}-z_s^{(2)})+r'(z_t^{(1)}-z_s^{(1)}-(z_t^{(2)}-z_s^{(2)})),\tilde{z}_{t-r_1}-\tilde{z}_{s-r_1},\dots,\tilde{z}_{t-r_k}-\tilde{z}_{s-r_k}\right),\end{aligned}$$

another Taylor expansion yields

$$|\bar{\theta}_s^{(p,i)}-\theta_s^{(p,i)}|\le c\left(\mathcal{N}[z^{(1)}]+\mathcal{N}[z^{(2)}]+\mathcal{N}[\tilde{z}]\right)|t-s|^{\kappa}.$$

Hence, we obtain

$$\begin{aligned}&\left|\left(\bar{\psi}_s^{(i,1)}-\bar{\psi}_s^{(i,2)}\right)-\left(\psi_s^{(i,1)}-\psi_s^{(i,2)}\right)\right|\\ &\qquad\le c\left(\mathcal{N}[z^{(1)}]+\mathcal{N}[z^{(2)}]+\mathcal{N}[\tilde{z}]\right)\mathcal{N}[z^{(1)}-z^{(2)}]\,|t-s|^{\kappa},\end{aligned} \tag{40}$$

from which suitable bounds for Q_2 and Q_3 are easily deduced. Thus it follows by (37) and (40) that

$$|\rho_{st}^{(2)}|\le c\left(1+\mathcal{N}[z^{(1)}]+\mathcal{N}[z^{(2)}]+\mathcal{N}[\tilde{z}]\right)^2\mathcal{N}[z^{(1)}-z^{(2)}]\,|t-s|^{2\kappa}.$$

Combining this estimate with (36) we finally have

$$|\rho_{st}|\le c\left(1+\mathcal{N}[z^{(1)}]+\mathcal{N}[z^{(2)}]+\mathcal{N}[\tilde{z}]\right)^2\mathcal{N}[z^{(1)}-z^{(2)}]\,|t-s|^{2\kappa}. \tag{41}$$

(iv) The assertion follows now from (28), (29), (34), (35) and (41).

□

3.3 Integration of delayed controlled paths (DCP)

The aim of this section is to define the integral $\mathcal{J}(m^* dx)$, where m is a delayed controlled path $m \in \mathcal{D}_{\kappa,\hat{\alpha}}([a,b];\mathbb{R}^d)$. Here we denote by A^* the transposition of a vector or matrix A and by $A_1 \cdot A_2$ the inner product of two vectors or two matrices A_1 and A_2, namely, if $A_1, A_2 \in \mathbb{R}^{n,d}$, then $A_1 \cdot A_2 = \mathrm{Tr}(A_1 A_2^*)$. We will also write $\mathcal{Q}_{\kappa,\alpha}$. We will also write $\mathcal{Q}_{\kappa,\alpha}$ (resp. $\mathcal{D}_{\kappa,\alpha}$) instead of $\mathcal{Q}_{\kappa,\alpha}([a,b];V)$ (resp. $\mathcal{D}_{\kappa,\alpha}([a,b];V)$) if there is no risk of confusion about $[a,b]$ and V.

Note that if the increments of m can be expressed like in (17), m^* admits the decomposition

$$(\delta m^*)_{st} = \sum_{i=0}^{k} (\delta x)^*_{s-r_i,t-r_i} \zeta_s^{(i)*} + \rho^*_{st}, \tag{42}$$

where $\rho^* \in \mathcal{C}_2^{2\kappa}([a,b];\mathbb{R}^{1,d})$ and the densities $\zeta^{(i)}$, $i = 0,\dots,k$, satisfy the conditions of Definition 3.1.

To illustrate the structure of the integral of a DCP, we first assume that the paths $x, \zeta^{(i)}$ and ρ are smooth, and we express $\mathcal{J}(m^* dx)$ in terms of the operators δ and Λ. In this case, $\mathcal{J}(m^* dx)$ is well defined, and we have

$$\int_s^t m_u^* dx_u = m_s^*(x_t - x_s) + \int_s^t (m_u^* - m_s^*) dx_u$$

for $a \le s \le t \le b$, or in other words

$$\mathcal{J}(m^*\, dx) = m^*\, \delta x + \mathcal{J}(\delta m^*\, dx). \tag{43}$$

Now consider the term $\mathcal{J}(\delta m^*\, dx)$: Using the decomposition (42) we obtain

$$\mathcal{J}(\delta m^*\, dx) = \int_s^t \left(\sum_{i=0}^{k} (\delta x)^*_{s-r_i,u-r_i} \zeta_s^{(i)*} + \rho^*_{su} \right) dx_u = A_{st} + \mathcal{J}_{st}(\rho^*\, dx) \tag{44}$$

with

$$A_{st} = \sum_{i=0}^{k} \int_s^t (\delta x)^*_{s-r_i,u-r_i} \zeta_s^{(i)*} dx_u.$$

Since, for the moment, we are dealing with smooth paths, the density $\zeta^{(i)}$ can be taken out of the integral above, and we have

$$A_{st} = \sum_{i=0}^{k} \zeta_s^{(i)} \cdot \mathbf{x}^{\mathbf{2}}_{st}(-r_i),$$

with the $d \times d$ matrix $\mathbf{x}^{\mathbf{2}}_{st}(v)$ defined by

$$\mathbf{x}^{\mathbf{2}}_{st}(v) = \left(\int_s^t \left(\int_{s+v}^{u+v} dx_w \right) dx_u^{(1)}, \dots, \int_s^t \left(\int_{s+v}^{u+v} dx_w \right) dx_u^{(d)} \right), \qquad 0 \le s \le t \le T,$$

for $v \in \{-r_k, \dots, -r_0\}$. Indeed, we can write

$$\begin{aligned} \int_s^t (\delta x)^*_{s-r_i,u-r_i} \zeta_s^{(i)*} dx_u &= \int_s^t \zeta_s^{(i)} \cdot [(\delta x)_{s-r_i,u-r_i} \otimes dx_u] \\ &= \zeta_s^{(i)} \cdot \int_s^t (\delta x)_{s-r_i,u-r_i} \otimes dx_u = \zeta_s^{(i)} \cdot \mathbf{x}^{\mathbf{2}}_{st}(-r_i). \end{aligned}$$

Inserting the expression of A_{st} into (43) and (44) we obtain

$$\mathscr{I}_{st}(m^*\,dx) = m_s^*(\delta x)_{st} + \sum_{i=0}^{k} \zeta_s^{(i)} \cdot \mathbf{x}_{st}^{\mathbf{2}}(-r_i) + \mathscr{I}_{st}(\rho^*\,dx) \tag{45}$$

for $a \le s \le t \le b$.

Let us now consider the Lévy area term $\mathbf{x}_{st}^{\mathbf{2}}(-r_i)$. If x is a smooth path, it is readily checked that

$$(\delta \mathbf{x}^{\mathbf{2}}(-r_i))_{sut} = \mathbf{x}_{st}^{\mathbf{2}}(-r_i) - \mathbf{x}_{su}^{\mathbf{2}}(-r_i) - \mathbf{x}_{ut}^{\mathbf{2}}(-r_i) = (\delta x)_{s-r_i,u-r_i} \otimes (\delta x)_{ut},$$

for any $i = 0, \ldots, k$. This decomposition of $\delta \mathbf{x}^{\mathbf{2}}(-r_i)$ into a product of increments is the fundamental algebraic property we will use to extend the above integral to non-smooth paths. Hence, we will need the following assumption:

Hypothesis 3.4. *The path x is a $\mathbb{R}^d$-valued γ-Hölder continuous function with $\gamma > \frac{1}{3}$ and admits a delayed Lévy area, i.e., for all $v \in \{-r_k, \ldots, -r_0\}$, there exists a path $\mathbf{x}^{\mathbf{2}}(v) \in \mathscr{C}_2^{2\gamma}([0,T];\mathbb{R}^{d,d})$, which satisfies*

$$\delta \mathbf{x}^{\mathbf{2}}(v) = \delta x(v) \otimes \delta x, \tag{46}$$

that is

$$\left[(\delta \mathbf{x}^{\mathbf{2}}(v))_{sut}\right](i,j) = (\delta x^i)_{s+v,u+v}(\delta x^j)_{ut} \qquad \text{for all} \qquad s,u,t \in [0,T], \quad i,j \in \{1,\ldots,d\}.$$

In the above formulae, we have set $x(v)$ for the shifted path $x(v)_s = x_{s+v}$.

To finish the analysis of the smooth case it remains to find a suitable expression for $\mathscr{I}(\rho^*\,dx)$. For this, we write (45) as

$$\mathscr{I}_{st}(\rho^*\,dx) = \mathscr{I}_{st}(m^*\,dx) - m_s^*(\delta x)_{st} - \sum_{i=0}^{k} \zeta_s^{(i)} \cdot \mathbf{x}_{st}^{\mathbf{2}}(-r_i) \tag{47}$$

and we apply δ to both sides of the above equation. For smooth paths m and x we have

$$\delta(\mathscr{I}(m^*\,dx)) = 0, \qquad\qquad \delta(m^*\,\delta x) = -\delta m^*\,\delta x,$$

by Proposition 2.4. Hence, applying these relations to the right hand side of (47), using the decomposition (42) and again Proposition 2.4, we obtain

$$\begin{aligned}
&[\delta(\mathscr{I}(\rho^*\,dx))]_{sut} \\
&= (\delta m^*)_{su}(\delta x)_{ut} + \sum_{i=0}^{k}(\delta \zeta^{(i)})_{su} \cdot \mathbf{x}_{st}^{\mathbf{2}}(-r_i) - \sum_{i=0}^{k} \zeta_s^{(i)} \cdot (\delta \mathbf{x}^{\mathbf{2}}(-r_i))_{sut} \\
&= \sum_{i=0}^{k}(\delta x)^*_{s-r_i,u-r_i}\zeta_s^{(i)*}(\delta x)_{ut} + \rho_{su}^*(\delta x)_{ut} \\
&\qquad\qquad + \sum_{i=0}^{k}(\delta \zeta^{(i)})_{su} \cdot \mathbf{x}_{st}^{\mathbf{2}}(-r_i) - \sum_{i=0}^{k} \zeta_s^{(i)} \cdot [(\delta x)_{s-r_i,t-r_i} \otimes (\delta x)_{ut}] \\
&= \rho_{su}^*(\delta x)_{ut} + \sum_{i=0}^{k}(\delta \zeta^{(i)})_{su} \cdot \mathbf{x}_{st}^{\mathbf{2}}(-r_i).
\end{aligned}$$

In summary, we have derived the representation

$$\delta(\mathscr{J}(\rho^* \, dx)) = \rho^* \, \delta x + \sum_{i=0}^{k} \delta \zeta^{(i)} \cdot \mathbf{x}^2(-r_i),$$

for two regular paths m and x.
If $m, x, \zeta^{(i)}$, $i = 0, \ldots, k$ and $\mathbf{x}^2$ are smooth enough, we have $\delta(\mathscr{J}(\rho^* \, dx)) \in \mathscr{Z}\mathscr{C}_3^{1+}$ and thus belongs to the domain of Λ due to Proposition 2.2. (Recall that $\delta\delta = 0$.) Hence, it follows

$$\mathscr{J}(\rho^* \, dx) = \Lambda \left(\rho^* \, \delta x + \sum_{i=0}^{k} \delta \zeta^{(i)} \cdot \mathbf{x}^2(-r_i) \right),$$

and inserting this identity into (45) we end up with

$$\mathscr{J}(m^* \, dx) = m^* \delta x + \sum_{i=0}^{k} \zeta^{(i)} \cdot \mathbf{x}^2(-r_i) + \Lambda \left(\rho^* \delta x + \sum_{i=0}^{k} \delta \zeta^{(i)} \cdot \mathbf{x}^2(-r_i) \right). \tag{48}$$

The expression above can be generalized to the non-smooth case, since $\mathscr{J}(m^* \, dx)$ has been expressed only in terms of increments of m and x. Consequently, we will use (48) as the definition for our extended integral.

Proposition 3.5. *For fixed $\frac{1}{3} < \kappa < \gamma$, let x be a path satisfying Hypothesis 3.4. Furthermore, let $m \in \mathscr{D}_{\kappa,\hat{\alpha}}([a,b]; \mathbb{R}^d)$ such that the increments of m are given by (17). Define z by $z_a = \alpha$ with $\alpha \in \mathbb{R}$ and*

$$(\delta z)_{st} = m_s^*(\delta x)_{st} + \sum_{i=0}^{k} \zeta_s^{(i)} \cdot \mathbf{x}_{st}^2(-r_i) + \Lambda_{st} \left(\rho^* \delta x + \sum_{i=0}^{k} \delta \zeta^{(i)} \cdot \mathbf{x}^2(-r_i) \right) \tag{49}$$

for $a \le s \le t \le b$. Finally, set

$$\mathscr{J}(m^* \, dx) = \delta z. \tag{50}$$

Then:

1. *$\mathscr{J}(m^* \, dx)$ coincides with the usual Riemann integral, whenever m and x are smooth functions.*
2. *z is well-defined as an element of $\mathscr{Q}_{\kappa,\alpha}([a,b]; \mathbb{R})$ with decomposition $\delta z = m^* \delta x + \hat{\rho}$, where $\hat{\rho} \in \mathscr{C}_2^{2\kappa}([a,b]; \mathbb{R})$ is given by*

$$\hat{\rho} = \sum_{i=0}^{k} \zeta^{(i)} \cdot \mathbf{x}^2(-r_i) + \Lambda \left(\rho^* \delta x + \sum_{i=0}^{k} \delta \zeta^{(i)} \cdot \mathbf{x}^2(-r_i) \right).$$

3. *The semi-norm of z can be estimated as*

$$\mathscr{N}[z; \mathscr{Q}_{\kappa,\alpha}([a,b]; \mathbb{R})] \le \|m\|_\infty + c_{int}(b-a)^{\gamma-\kappa}(|\hat{\alpha}| + \mathscr{N}[m; \mathscr{D}_{\kappa,\hat{\alpha}}([a,b]; \mathbb{R}^d)]) \tag{51}$$

   where

$$c_{int} = c_{\kappa,\gamma,T} \left(\|x\|_\gamma + \sum_{i=0}^{k} \|\mathbf{x}^2(-r_i)\|_{2\gamma} \right)$$

   with the constant $c_{\kappa,\gamma,T}$ depending only on κ, γ and T. In particular,

$$\|z\|_\kappa \le c_{int}(b-a)^{\gamma-\kappa}(|\hat{\alpha}| + \mathscr{N}[m; \mathscr{D}_{\kappa,\hat{\alpha}}([a,b]; \mathbb{R}^d)]). \tag{52}$$

4. It holds

$$\mathscr{I}_{st}(m^* \, dx) = \lim_{|\Pi_{st}| \to 0} \sum_{i=0}^{N} \left[m^*_{t_i} (\delta x)_{t_i \, t_{i+1}} + \sum_{j=0}^{k} \zeta^{(j)}_{t_i} \cdot \mathbf{x}^{\mathbf{2}}_{t_i \, t_{i+1}}(-r_j) \right] \tag{53}$$

for any $a \le s < t \le b$, where the limit is taken over all partitions $\Pi_{st} = \{s = t_0, \dots, t_N = t\}$ of $[s, t]$, as the mesh of the partition goes to zero.

Proof. (1) The first of our claims is a direct consequence of the derivation of equation (48).

(2) Set $c_x = \|x\|_\gamma + \sum_{i=0}^k \|\mathbf{x}^{\mathbf{2}}(-r_i)\|_{2\gamma}$. Now we show that equation (49) defines a classical weakly controlled path. Actually, the term $m^* \, \delta x$ is trivially of the desired form for an element of $\mathscr{Q}_{\kappa,\alpha}$. So consider the term $h^{(1)}_{st} = \sum_{i=0}^k \zeta^{(i)}_s \cdot \mathbf{x}^{\mathbf{2}}_{st}(-r_i)$ for $a \le s \le t \le b$. We have

$$\begin{aligned} |h^{(1)}_{st}| &\le \sum_{i=0}^{k} \|\zeta^{(i)}\|_\infty |\mathbf{x}^{\mathbf{2}}_{st}(-r_i)| \le \Big(\sum_{i=0}^{k} \|\zeta^{(i)}\|_\infty \Big) c_x |t-s|^{2\gamma} \\ &\le \Big(\sum_{i=0}^{k} \|\zeta^{(i)}\|_\infty \Big) c_x (b-a)^{2(\gamma-\kappa)} |t-s|^{2\kappa}. \end{aligned}$$

Thus

$$\|h^{(1)}\|_{2\kappa} \le c_x (b-a)^{2(\gamma-\kappa)} \mathscr{N}[m; \mathscr{D}_{\kappa,\hat{\alpha}}([a,b]; \mathbb{R}^d)].$$

The term

$$h^{(2)} = \rho^* \delta x + \sum_{i=0}^{k} \delta \zeta^{(i)} \cdot \mathbf{x}^{\mathbf{2}}(-r_i)$$

satisfies $\delta h^{(2)} = 0$. Indeed, recall that the increments of m are given by (17). Hence we have

$$\rho = \delta m - \sum_{i=0}^{k} \zeta^{(i)} \delta x(-r_i)$$

and

$$\begin{aligned} h^{(2)} &= \delta m^* \delta x - \sum_{i=0}^{k} \delta x(-r_i)^* \zeta^{(i)*} \delta x + \sum_{i=0}^{k} \delta \zeta^{(i)} \cdot \mathbf{x}^{\mathbf{2}}(-r_i) \\ &= \delta m^* \delta x - \sum_{i=0}^{k} \zeta^{(i)} \cdot \delta \mathbf{x}^{\mathbf{2}}(-r_i) + \sum_{i=0}^{k} \delta \zeta^{(i)} \cdot \mathbf{x}^{\mathbf{2}}(-r_i) \end{aligned}$$

using Hypothesis 3.4 for the Lévy area. But Proposition 2.4 gives

$$\delta(\zeta^{(i)} \cdot \mathbf{x}^{\mathbf{2}}(-r_i)) = -\delta \zeta^{(i)} \cdot \mathbf{x}^{\mathbf{2}}(-r_i) + \zeta^{(i)} \cdot \delta \mathbf{x}^{\mathbf{2}}(-r_i).$$

Hence it follows

$$h^{(2)} = \delta m^* \delta x - \sum_{i=0}^{k} \delta(\zeta^{(i)} \cdot \mathbf{x}^{\mathbf{2}}(-r_i))$$

and thus

$$\delta h^{(2)} = 0$$

using again Proposition 2.4 and $\delta\delta = 0$.

Moreover, recalling the notation (8), it holds

$$\|\rho^* \delta x\|_{2\kappa,\kappa} \le c_x (b-a)^{\gamma-\kappa} \|\rho\|_{2\kappa}$$

and

$$\|\sum_{i=0}^{k} \delta\zeta^{(i)} \cdot \mathbf{x}^{\mathbf{2}}(-r_i)\|_{\kappa,2\kappa} \le c_x (b-a)^{2(\gamma-\kappa)} \sum_{i=0}^{k} \|\zeta^{(i)}\|_{\kappa}.$$

Since $\gamma > \kappa > \frac{1}{3}$ and $\delta h^{(2)} = 0$, we have $h^{(2)} \in \mathrm{Dom}(\Lambda)$ and

$$\|h^{(2)}\|_{3\kappa} \le c_x (1 + T^{\gamma-\kappa})(b-a)^{\gamma-\kappa} \mathcal{N}[m; \mathcal{D}_{\kappa,\hat{\alpha}}([a,b];\mathbb{R}^d)].$$

By Proposition 2.2 it follows

$$\|\Lambda(h^{(2)})\|_{3\kappa} \le \frac{1}{2^{3\kappa}-2} \|h^{(2)}\|_{3\kappa}$$

and we finally obtain

$$\|h^{(1)} + \Lambda(h^{(2)})\|_{2\kappa} \le c_x \frac{2^{3\kappa}-1}{2^{3\kappa}-2} (1 + T^{\gamma-\kappa})(b-a)^{\gamma-\kappa} \mathcal{N}[m; \mathcal{D}_{\kappa,\hat{\alpha}}([a,b];\mathbb{R}^d)]. \tag{54}$$

Thus we have proved that $\hat{\rho} \in \mathcal{C}_2^{2\kappa}([a,b];\mathbb{R})$ and hence that $z \in \mathcal{Q}_{\kappa,\alpha}([a,b];\mathbb{R})$.

(3) Because of $(\delta z)_{st} = m_s^*(\delta x)_{st} + \hat{\rho}_{st}$ and $m \in \mathcal{D}_{\kappa,\hat{\alpha}}([a,b];\mathbb{R}^d)$ the estimates (51) and (52) now follow from (54).

(4) By Proposition 2.4 *(ii)* and the decomposition (17) we have that

$$\delta(m^* \delta x)_{sut} = -(\delta m^*)_{su} (\delta x)_{ut} = -\rho_{su}^* (\delta x)_{ut} - \sum_{i=0}^{k} (\delta x)^*_{s-r_i,u-r_i} \zeta_u^{(i)*} (\delta x)_{ut}.$$

Thus, applying again Proposition 2.4 *(ii)*, and recalling Hypothesis 3.4 for the Lévy area, we obtain that

$$\delta\left(m^* \,\delta x + \sum_{i=0}^{k} \zeta^{(i)} \cdot \mathbf{x}^{\mathbf{2}}(-r_i) \right) = -\left[\rho^* \,\delta x + \sum_{i=0}^{k} \delta\zeta^{(i)} \cdot \mathbf{x}^{\mathbf{2}}(-r_i) \right].$$

Hence, equation (49) can also be written as

$$\mathcal{J}(m^* \,dx) = [\mathrm{Id} - \Lambda\delta] \left(m^* \,\delta x + \sum_{i=0}^{k} \zeta^{(i)} \cdot \mathbf{x}^{\mathbf{2}}(-r_i) \right), \tag{55}$$

and a direct application of Corollary 2.3 yields (53), which ends our proof.

□

Recall that the notation A^* stands for the transpose of a matrix A. Moreover, in the sequel, we will denote by c_{norm} a constant, which depends only on the chosen norm of $\mathbb{R}^{n,d}$. Then, for a matrix-valued delayed controlled path $m \in \mathcal{D}_{\kappa,\hat{\alpha}}([a,b];\mathbb{R}^{n,d})$, the integral $\mathcal{J}(m\,dx)$ will be defined by

$$\mathcal{J}(m\,dx) = \begin{pmatrix} \mathcal{J}(m^{(1)*}dx) \\ \vdots \\ \mathcal{J}(m^{(n)*}dx) \end{pmatrix},$$

where $m^{(i)} \in \mathcal{D}_{\kappa,\hat{\alpha}}([a,b];\mathbb{R}^d)$ for $i = 1,\dots,n$ and we have set $m = (m^{(1)},\dots,m^{(n)})^*$. Now we have by (51) that

$$\begin{aligned} &\mathcal{N}[\mathcal{J}(m\,dx); \mathcal{Q}_{\kappa,\alpha}([a,b];\mathbb{R}^n)] \\ &\qquad \le c_{norm}\left(\|m\|_\infty + c_{int}(b-a)^{\gamma-\kappa}(|\hat{\alpha}| + \mathcal{N}[m; \mathcal{D}_{\kappa,\hat{\alpha}}([a,b];\mathbb{R}^{n,d})])\right). \end{aligned} \tag{56}$$

For two paths $m^{(1)}, m^{(2)} \in \mathcal{D}_{\kappa,\hat{\alpha}}([a,b];\mathbb{R}^{n,d})$ we obtain the following estimate for the difference of $z^{(1)} = \mathcal{J}(m^{(1)}\,dx)$ and $z^{(2)} = \mathcal{J}(m^{(2)}\,dx)$: Since $m_a^{(1)} - m_a^{(2)} = 0$, we have

$$\begin{aligned} &\mathcal{N}[z^{(1)} - z^{(2)}; \mathcal{Q}_{\kappa,0}([a,b];\mathbb{R}^n)] \\ &\qquad \le 2c_{norm}\,c_{int}(b-a)^{\gamma-\kappa}\mathcal{N}\left[m^{(1)} - m^{(2)}; \mathcal{D}_{\kappa,0}([a,b];\mathbb{R}^{n,d})\right]. \end{aligned} \tag{57}$$

4 Solution to the delay equation

With the preparations of the last section, we can now solve the equation

$$\left\{ \begin{array}{rcll} dy_t &=& \sigma(y_t, y_{t-r_1}, \dots, y_{t-r_k})dx_t, & t \in [0,T], \\ y_t &=& \xi_t, & t \in [-r_k, 0], \end{array} \right. \tag{58}$$

in the class of classical weakly controlled paths. For this, it will be crucial to use mappings of the type

$$\Gamma : \mathcal{Q}_{\kappa,\alpha}([a,b];\mathbb{R}^n) \times \mathcal{Q}_{\kappa,\tilde{\alpha}}([a-r_k, b-r_1];\mathbb{R}^n) \to \mathcal{Q}_{\kappa,\alpha}([a,b];\mathbb{R}^n)$$

for $0 \le a \le b \le T$, which are defined by $(z,\tilde{z}) \mapsto \hat{z}$, where $\hat{z}_0 = \alpha$ and $\delta\hat{z}$ given by $\delta\hat{z} = \mathcal{J}(T_\sigma(z,\tilde{z})\,dx)$, with T_σ as in Proposition 3.2. The first part of the current section will be devoted to the study of this map. By (56) we have that

$$\begin{aligned} &\mathcal{N}[\mathcal{J}(T_\sigma(z,\tilde{z})\,dx); \mathcal{Q}_{\kappa,\alpha}([a,b];\mathbb{R}^n)] \\ &\quad \le c_{norm}\left(\|\sigma\|_\infty + c_{int}(b-a)^{\gamma-\kappa}(|\hat{\alpha}| + \mathcal{N}[T_\sigma(z,\tilde{z}); \mathcal{D}_{\kappa,\hat{\alpha}}([a,b];\mathbb{R}^{n,d})])\right) \\ &\quad \le c_{norm}\left(\|\sigma\|_\infty(1 + c_{int}T^{\gamma-\kappa}) + c_{int}(b-a)^{\gamma-\kappa}\mathcal{N}[T_\sigma(z,\tilde{z}); \mathcal{D}_{\kappa,\hat{\alpha}}([a,b];\mathbb{R}^{n,d})]\right) \end{aligned}$$

due to $|\hat{\alpha}| = |\sigma(z_a, \mathfrak{s}(\tilde{z})_a)| \le \|\sigma\|_\infty$. Since

$$T_\sigma(z,\tilde{z}) = \begin{pmatrix} T_{\sigma^{(1)*}}(z,\tilde{z}) \\ \vdots \\ T_{\sigma^{(n)*}}(z,\tilde{z}) \end{pmatrix},$$

where $\sigma^{(i)} \in C_b^3(\mathbb{R}^{n(k+1)};\mathbb{R}^d)$ for $i=1,\dots,n$ and $\sigma=(\sigma^{(1)},\dots,\sigma^{(n)})^*$, it follows by (20) that

$$\begin{aligned}&\mathcal{N}[T_\sigma(z,\tilde{z});\mathcal{D}_{\kappa,\hat{\alpha}}([a;b];\mathbb{R}^{n,d})]\\&\qquad\le c_{\sigma,T}\left(1+\mathcal{N}^2[z;\mathcal{Q}_{\kappa,\alpha}([a,b];\mathbb{R}^n)]+\mathcal{N}^2[\tilde{z};\mathcal{Q}_{\kappa,\tilde{\alpha}}([a-r_k,b-r_1];\mathbb{R}^n)]\right).\end{aligned}$$

Combining these two estimates we obtain

$$\begin{aligned}&\mathcal{N}\left[\Gamma(z,\tilde{z});\mathcal{Q}_{\kappa,\alpha}([a;b];\mathbb{R}^n)\right]\\&\quad\le c_{growth}\left(1+\mathcal{N}^2[\tilde{z};\mathcal{Q}_{\kappa,\tilde{\alpha}}([a-r_k,b-r_1];\mathbb{R}^n)]\right)\left(1+(b-a)^{\gamma-\kappa}\mathcal{N}^2[z;\mathcal{Q}_{\kappa,\alpha}([a,b];\mathbb{R}^n)]\right),\end{aligned}\tag{59}$$

where the constant c_{growth} depends only on c_{int}, c_{norm}, σ, κ, γ and T. Thus the semi-norm of the mapping Γ is quadratically bounded in terms of the semi-norms of z and $\tilde{z}$.

Now let $z^{(1)},z^{(2)}\in\mathcal{Q}_{\kappa,\alpha}([a,b];\mathbb{R}^n)$ and $\tilde{z}\in\mathcal{Q}_{\kappa,\tilde{\alpha}}([a-r_k,b-r_1];\mathbb{R}^n)$. Then, by (57) we have

$$\begin{aligned}&\mathcal{N}[\Gamma(z^{(1)},\tilde{z})-\Gamma(z^{(2)},\tilde{z});\mathcal{Q}_{\kappa,0}([a,b];\mathbb{R}^n)]\\&\qquad\le 2c_{norm}\,c_{int}(b-a)^{\gamma-\kappa}\mathcal{N}[T_\sigma(z^{(1)},\tilde{z})-T_\sigma(z^{(2)},\tilde{z});\mathcal{D}_{\kappa,0}([a,b];\mathbb{R}^{n,d})].\end{aligned}\tag{60}$$

Applying Proposition 3.3, i.e. inequality (26), to the right hand side of the above equation we obtain that

$$\begin{aligned}&\mathcal{N}[\Gamma(z^{(1)},\tilde{z})-\Gamma(z^{(2)},\tilde{z});\mathcal{Q}_{\kappa,0}([a,b];\mathbb{R}^n)]\\&\qquad\le c_{lip}\left(1+C(z^{(1)},z^{(2)},\tilde{z})\right)^2\mathcal{N}[z^{(1)}-z^{(2)};\mathcal{Q}_{\kappa,0}([a,b];\mathbb{R}^n)]\,(b-a)^{\gamma-\kappa},\end{aligned}\tag{61}$$

with a constant c_{lip} depending only on c_{int}, c_{norm}, σ, κ, γ and T, and moreover

$$\begin{aligned}C(z^{(1)},z^{(2)},\tilde{z})&=\mathcal{N}[\tilde{z};\mathcal{Q}_{\kappa,\tilde{\alpha}}([a-r_k,b-r_1];\mathbb{R}^n)]\\&\quad+\mathcal{N}[z^{(1)};\mathcal{Q}_{\kappa,\alpha}([a,b];\mathbb{R}^n)]+\mathcal{N}[z^{(2)};\mathcal{Q}_{\kappa,\alpha}([a,b];\mathbb{R}^n)].\end{aligned}$$

Thus, for fixed $\tilde{z}$ the mappings $\Gamma(\cdot,\tilde{z})$ are locally Lipschitz continuous with respect to the semi-norm $\mathcal{N}[\cdot;\mathcal{Q}_{\kappa,0}([a,b];\mathbb{R}^n)]$.

We also need the following Lemma, which can be shown by straightforward calculations:

Lemma 4.1. *Let $c,\alpha\ge 0$, $\tau\in[0,T]$ and define*

$$\mathcal{A}_\tau^{c,\alpha}=\{u\in\mathbb{R}_+^*:\ c(1+\tau^\alpha u^2)\le u\}.$$

Set also $\tau^*=(8c^2)^{-1/\alpha}$. Then we have $\mathcal{A}_{\tau^*}^{c,\alpha}\ne\emptyset$ and $\sup\{u\in\mathbb{R}_+^*;\,u\in\mathcal{A}_{\tau^*}^{c,\alpha}\}\le(4+2\sqrt{2})c$.

Now we can state and prove our main result. Recall that we use the notations $r=r_k$ and $r_0=0$.

Theorem 4.2. *Let x be a path satisfying Hypothesis 3.4, let $\xi\in\mathcal{Q}_{\kappa,\alpha}([-r,0];\mathbb{R}^n)$ and let $\sigma\in C_b^3(\mathbb{R}^{n(k+1)};\mathbb{R}^{n,d})$. Then we have:*

1. *Equation (58) admits a unique solution y in $\mathcal{Q}_{\kappa,\xi_0}([0,T];\mathbb{R}^n)$ for any $\frac{1}{3}<\kappa<\gamma$ and any $T>0$.*

2. *Let* $F : \mathscr{Q}_{\kappa,\alpha}([-r,0];\mathbb{R}^n) \times \mathscr{C}_1^\gamma([0,T];\mathbb{R}^d) \times \left(\mathscr{C}_2^{2\gamma}([0,T];\mathbb{R}^{d,d})\right)^{k+1} \to \mathscr{C}_1^\kappa([0,T];\mathbb{R}^n)$ *be the mapping defined by*
$$F\left(\xi, x, \mathbf{x}^2(0), \mathbf{x}^2(-r_1), \ldots, \mathbf{x}^2(-r_k)\right) = y,$$
where y *is the unique solution of equation (58). This mapping is locally Lipschitz continuous in the following sense: Let* $\tilde{x}$ *be another driving rough path with corresponding delayed Lévy area* $\tilde{\mathbf{x}}^2(-v)$, $v \in \{-r_k, \ldots, -r_0\}$, *and* $\tilde{\xi}$ *another initial condition. Moreover denote by* $\tilde{y}$ *the unique solution of the corresponding delay equation. Then, for every* $N > 0$, *there exists a constant* $K_N > 0$ *such that*
$$\|y - \tilde{y}\|_{\kappa,\infty} \le K_N \left(\|x - \tilde{x}\|_{\gamma,\infty} + \sum_{i=0}^{k} \|\mathbf{x}^2(-r_i) - \tilde{\mathbf{x}}^2(-r_i)\|_{2\gamma} + \|\xi - \tilde{\xi}\|_{\kappa,\infty} \right)$$
holds for all tuples $(\xi, x, \mathbf{x}^2(-r_0), \ldots, \mathbf{x}^2(-r_k)), (\tilde{\xi}, \tilde{x}, \tilde{\mathbf{x}}^2(-r_0), \ldots, \tilde{\mathbf{x}}^2(-r_k))$ *with*
$$\begin{aligned} &\|\xi\|_\infty + \mathscr{N}[\xi, \mathscr{Q}_{\kappa,\alpha}([-r,0];\mathbb{R}^n)] + \|\tilde{\xi}\|_\infty + \mathscr{N}[\tilde{\xi}, \mathscr{Q}_{\kappa,\tilde{\alpha}}([-r,0];\mathbb{R}^n)] \\ &\quad + \|x\|_{\gamma,\infty} + \sum_{i=0}^{k} \|\mathbf{x}^2(-r_i)\|_{2\gamma} + \|\tilde{x}\|_{\gamma,\infty} + \sum_{i=0}^{k} \|\tilde{\mathbf{x}}^2(-r_i)\|_{2\gamma} \le N, \end{aligned}$$
where $\|f\|_{\mu,\infty} = \|f\|_\infty + \|\delta f\|_\mu$ *denotes the usual Hölder norm of a path* f.

Proof. The proof of Theorem 4.2 is obtained by means of a fixed point argument, based on the map Γ defined above.

1) Existence and uniqueness. Without loss of generality assume that $T = N r_1$. We will construct the solution of equation (58) by induction over the intervals $[0, r_1]$, $[0, 2r_1]$, …, $[0, N r_1]$, where we recall that r_1 is the smallest delay in (58).

(i) We will first show that equation (58) has a solution on the interval $[0, r_1]$. For this define the mapping
$$\Gamma_{[0,\tau_1]} : \mathscr{Q}_{\kappa,\xi_0}([0,\tau_1];\mathbb{R}^n) \to \mathscr{Q}_{\kappa,\xi_0}([0,\tau_1];\mathbb{R}^n)$$
given by $\hat{z} = \Gamma_{[0,\tau_1]}(z)$, where
$$(\delta \hat{z})_{st} = \mathscr{J}_{st}(T_\sigma(z,\xi)dx)$$
for $0 \le s \le t \le \tau_1$. Moreover, set
$$\tilde{\tau}_1 = (8c_1^2)^{-1/(\gamma-\kappa)} \wedge r_1,$$
where
$$c_1 = c_{growth}\left(1 + \mathscr{N}^2[\xi; \mathscr{Q}_{\kappa,\alpha}([-r,0];\mathbb{R}^n)]\right).$$
Clearly, if z is a fixed point of the map $\Gamma_{[0,\tau_1]}$, then z solves equation (58) on the interval $[0,\tau_1]$. We shall thus prove that such a fixed point exists. First, due to (59) we have the estimate
$$\mathscr{N}\left[\Gamma_{[0,\tau_1]}(z); \mathscr{Q}_{\kappa,\xi_0}([0,\tau_1];\mathbb{R}^n)\right] \le c_1\left(1 + \tau_1{}^{\gamma-\kappa}\mathscr{N}^2[z; \mathscr{Q}_{\kappa,\xi_0}([0,\tau_1];\mathbb{R}^n)]\right).$$

Thanks to our choice of $\tilde{\tau}_1$ and Lemma 4.1 we can now choose $M_1 \in \mathscr{A}_{\tilde{\tau}_1}^{c_1,\gamma-\kappa}$ accordingly and obtain that the ball

$$B_1^{(1)} = \left\{z \in \mathcal{Q}_{\kappa,\xi_0}([0,\tilde{\tau}_1];\mathbb{R}^n);\ \mathcal{N}[z;\mathcal{Q}_{\kappa,\xi_0}([0,\tilde{\tau}_1];\mathbb{R}^n)] \leq M_1\right\}$$

is left invariant under $\Gamma_{[0,\tilde{\tau}_1]}$. Moreover, by equation (61) we now have

$$\mathcal{N}[\Gamma_{[0,\tilde{\tau}_1]}(y^{(1)}) - \Gamma_{[0,\tilde{\tau}_1]}(y^{(2)});\mathcal{Q}_{\kappa,0}([0,\tilde{\tau}_1];\mathbb{R}^n)] \leq \tilde{c}_1 \tilde{\tau}_1^{\gamma-\kappa}\, \mathcal{N}[y^{(1)} - y^{(2)};\mathcal{Q}_{\kappa,0}([0,\tilde{\tau}_1];\mathbb{R}^n)]$$

for $y^{(1)}, y^{(2)} \in B_1^{(1)}$ where

$$\tilde{c}_1 = c_{lip}\left(1 + 2M_1 + \mathcal{N}[\xi;\mathcal{Q}_{\kappa,\alpha}([-r,0];\mathbb{R}^n)]\right)^2.$$

Choosing τ_1^* such that

$$0 < \tau_1^* \leq \left(2\tilde{c}_1\right)^{-1/(\gamma-\kappa)} \wedge \tilde{\tau}_1$$

and

$$N_1 \tau_1^* = r_1$$

for an integer $N_1 \in \mathbb{N}$ yields that the mapping $\Gamma_{[0,\tau_1^*]}$ is a contraction on $B_1^{(1)}$. Thus, the Banach theorem implies that $\Gamma_{[0,\tau_1^*]}$ has a unique fixed point, which leads to a unique solution $z^{(1)}$ of equation (58) on the interval $[0,\tau_1^*]$.

If $\tau_1^* = r_1$, the first step of the proof is finished. Otherwise, define the mapping

$$\Gamma_{[\tau_1^*,2\tau_1^*]} : \mathcal{Q}_{\kappa,z^{(1)}_{\tau_1^*}}([\tau_1^*,2\tau_1^*];\mathbb{R}^n) \to \mathcal{Q}_{\kappa,z^{(1)}_{\tau_1^*}}([\tau_1^*,2\tau_1^*];\mathbb{R}^n)$$

given by $\hat{z} = \Gamma_{[\tau_1^*,2\tau_1^*]}(z)$, where

$$(\delta\hat{z})_{st} = \mathcal{J}_{st}(T_\sigma(z,\xi)dx)$$

for $\tau_1^* \leq s \leq t \leq 2\tau_1^*$. Since $2\tau_1^* \leq r_1$, it still holds

$$\mathcal{N}[\Gamma_{[\tau_1^*,2\tau_1^*]}(z);\mathcal{Q}_{\kappa,z^{(1)}_{\tau_1^*}}([\tau_1^*,2\tau_1^*];\mathbb{R}^n)] \leq c_1\left(1 + \tau_1^{*\gamma-\kappa}\mathcal{N}^2[z;\mathcal{Q}_{\kappa,z^{(1)}_{\tau_1^*}}([\tau_1^*,2\tau_1^*];\mathbb{R}^n)]\right)$$

and thus the ball

$$B_2^{(1)} = \left\{z \in \mathcal{Q}_{\kappa,z^{(1)}_{\tau_1^*}}([\tau_1^*,2\tau_1^*];\mathbb{R}^n);\ \mathcal{N}[z;\mathcal{Q}_{\kappa,z^{(1)}_{\tau_1^*}}([\tau_1^*,2\tau_1^*];\mathbb{R}^n)] \leq M_1\right\},$$

where M_1 is as in the previous step, is left invariant under $\Gamma_{[\tau_1^*,2\tau_1^*]}$. Hence, we obtain as above that

$$\mathcal{N}[\Gamma_{[\tau_1^*,2\tau_1^*]}(y^{(1)}) - \Gamma_{[\tau_1^*,2\tau_1^*]}(y^{(2)});\mathcal{Q}_{\kappa,0}([\tau_1^*,2\tau_1^*];\mathbb{R}^n)] \leq \frac{1}{2}\mathcal{N}[y^{(1)} - y^{(2)};\mathcal{Q}_{\kappa,0}([\tau_1^*,2\tau_1^*];\mathbb{R}^n)]$$

for $y^{(1)}, y^{(2)} \in B_2^{(1)}$ and again the Banach fixed point theorem gives the existence of a unique solution $z^{(2)}$ of equation (58) on the interval $[\tau_1^*,2\tau_1^*]$.

Repeating this step as often as necessary, which is possible since the estimates on the norms of the mappings $\Gamma_{[j\tau_1^*,(j+1)\tau_1^*]}$, $j = 0,\ldots,N_1-1$, are as above, i.e. the constants c_1 and $\tilde{c}_1$ do not change, we obtain that $z = \sum_{j=1}^{N_1} z^{(j)}\,\mathbf{1}_{I_j}$, where $I_j = [(j-1)\tau_1^*, j\tau_1^*]$, is the unique solution to the equation (58) on the interval $[0,r_1]$.

Now, it remains to verify that $z = \sum_{j=1}^{N_1} z^{(j)} \mathbf{1}_{I_j}$ is in fact a CCP. First note that by construction z is continuous on $[0, r_1]$ and moreover that z is a CCP on the subintervals I_j with decomposition

$$(\delta z)_{st} = \zeta_s^{(j)} (\delta x)_{st} + \rho_{st}^{(j)}, \qquad s, t \in I_j,$$

for $s \leq t$, where

$$\zeta_s^{(j)} = \sigma(z_s, \mathfrak{s}(z)_s), \qquad s \in I_j.$$

Clearly, we have

$$(\delta z)_{st} = \sum_{j=j_s}^{j_t} (\delta z^{(j)})_{s \vee t_j, t \wedge t_{j+1}}$$

for $0 \leq s \leq t \leq r_1$, where $t_j = (j-1)\tau_1$ and $j_s, j_t \in \{1, \ldots, N_1\}$ are such that

$$t_{j_s} \leq s < t_{j_s+1} < \ldots < t_{j_t} < t \leq t_{j_t+1}.$$

Setting

$$\zeta_s = \sum_{j=1}^{N_1} \zeta_s^{(j)} \mathbf{1}_{I_j}(s), \qquad s \in [0, r_1],$$

and

$$\rho_{st} = \sum_{j=j_s}^{j_t} (\zeta_{s \vee t_j}^{(j)} - \zeta_s^{(j_s)})(\delta x)_{s \vee t_j, t \wedge t_{j+1}} + \sum_{j=j_s}^{j_t} \rho_{s \vee t_j, t \wedge t_{j+1}}^{(j)}$$

we obtain

$$(\delta z)_{st} = \zeta_s (\delta x)_{st} + \rho_{st}$$

for $0 \leq s \leq t \leq r_1$.

Now, it follows easily by the subadditivity of the Hölder norms that

$$\sup_{s,t \in [0,r_1]} \frac{|(\delta z)_{st}|}{|s-t|^\kappa} \leq \sum_{j=1}^{N_1} \sup_{s,t \in I_j} \frac{|(\delta z^{(j)})_{st}|}{|s-t|^\kappa}$$

and

$$\sup_{t \in [0,r_1]} |\zeta_t| = \sup_{j=1,\ldots,N_1} \sup_{t \in I_j} |\zeta_t^{(j)}|, \qquad \sup_{s,t \in [0,r_1]} \frac{|(\delta \zeta)_{st}|}{|s-t|^\kappa} \leq \sum_{j=1}^{N_1} \sup_{s,t \in I_j} \frac{|(\delta \zeta^{(j)})_{st}|}{|s-t|^\kappa}.$$

Furthermore, we obtain

$$\sup_{s,t \in [0,r_1]} \frac{|\rho_{st}|}{|s-t|^{2\kappa}} \leq \sum_{j=1}^{N_1} \sup_{s,t \in I_j} \frac{|(\rho^{(j)})_{st}|}{|s-t|^{2\kappa}} + N_1 \sup_{s,t \in [0,r_1]} \frac{|(\delta x)_{st}|}{|s-t|^\kappa} \sum_{j=1}^{N_1} \sup_{s,t \in I_j} \frac{|(\delta \zeta^{(j)})_{st}|}{|s-t|^\kappa}.$$

Thus, we have in fact that $z \in \mathcal{Q}_{\kappa,\xi_0}([0, r_1]; \mathbb{R}^n)$.

(ii) Let $l = 1, \ldots, N-1$ and assume that $\tilde{z} \in \mathcal{Q}_{\kappa,\xi_0}([0, l r_1]; \mathbb{R}^n)$ is the solution of the delay equation (58) on the interval $[0, l r_1]$. To show the existence of a unique solution on the interval $[l r_1, (l+1) r_1]$ define the mapping

$$\Gamma_{[l r_1, l r_1 + \tau_l]} : \mathcal{Q}_{\kappa, \tilde{z}_{l r_1}}([l r_1, l r_1 + \tau_l]; \mathbb{R}^n) \to \mathcal{Q}_{\kappa, \tilde{z}_{l r_1}}([l r_1, l r_1 + \tau_l]; \mathbb{R}^n)$$

given by $\hat{z} = \Gamma_{[lr_1, lr_1+\tau_l]}(z)$, where

$$(\delta \hat{z})_{st} = \mathcal{J}_{st}(T_\sigma(z, \mathfrak{s}(\tilde{z}))dx)$$

for $lr_1 \leq s \leq t \leq lr_1 + \tau_l$. Moreover, set

$$\begin{aligned} c_{l+1} &= c_{growth}\left(1 + \mathcal{N}^2[\tilde{z}; \mathcal{Q}_{\kappa,\tilde{z}_{lr_1-r_k}}([lr_1 - r_k, lr_1]; \mathbb{R}^n)]\right), \\ \tilde{\tau}_{l+1} &= (8c_{l+1}^2)^{-1/(\gamma-\kappa)} \wedge r_1, \end{aligned}$$

for which Lemma 4.1 leads to a corresponding choice of M_{l+1} and $B_1^{(l+1)}$. It remains to define

$$\tilde{c}_{l+1} = c_{lip}\left(1 + 2M_{l+1} + \mathcal{N}[\tilde{z}; \mathcal{Q}_{\kappa,\tilde{z}_{lr_1-r_k}}([lr_1 - r_k, lr_1]; \mathbb{R}^n)]\right)^2$$

and to choose τ_{l+1}^* such that

$$0 < \tau_{l+1}^* \leq \left(2\tilde{c}_{l+1}\right)^{-1/(\gamma-\kappa)} \wedge \tilde{\tau}_{l+1}$$

and

$$N_{l+1}\tau_{l+1}^* = r_1$$

for an integer $N_{l+1} \in \mathbb{N}$.

Proceeding completely analogous to step *(i)* we obtain the existence of a unique path $z \in \mathcal{Q}_{\kappa,\tilde{z}_{lr_1}}([lr_1, (l+1)r_1]; \mathbb{R}^n)$, which solves the delay equation (58) on the interval $[lr_1, (l+1)r_1]$ for a given "initial path" $\tilde{z} \in \mathcal{Q}_{\kappa,\xi_0}([0, lr_1]; \mathbb{R}^n)$. Patching these two paths together, we obtain (using the same arguments as at the end of step *(i)*) a path $z \in \mathcal{Q}_{\kappa,\xi_0}([0, (l+1)r_1]; \mathbb{R}^n)$, which solves equation (58) on the interval $[0, (l+1)r_1]$.

Thus we have shown that there exists a unique path $z \in \mathcal{Q}_{\kappa,\xi_0}([0, T]; \mathbb{R}^n)$, which solves the equation (58). Moreover, by the above construction we obtain the following bound on the norm of this path:

$$\mathcal{N}[z; \mathcal{Q}_{\kappa,\xi_0}([0,T]; \mathbb{R}^n)] \leq f\left(\|\delta x\|_\gamma + \sum_{i=0}^{k} \|\mathbf{x}^2(-r_i)\|_{2\gamma} + \mathcal{N}[\xi; \mathcal{Q}_{\kappa,\alpha}([-r, 0]; \mathbb{R}^n)]\right). \tag{62}$$

Here $f : [0, \infty) \to (0, \infty)$ is a continuous non-decreasing function, which depends only on $\kappa, \gamma, n, d, k, \sigma, T$ and $r_1, \ldots, r_k$.

2) Continuity of the Itô map. For convenience and to simplify the notation, we assume now that $n = 1$. The general case $n \in \mathbb{N}$ can be shown similar, but requires more cumbersome notations. Let $y = F\left(\xi, x, \mathbf{x}^2(0), \mathbf{x}^2(-r_1), \ldots, \mathbf{x}^2(-r_k)\right)$. Since y solves equation (58), we have $(\delta y)_{st} = \mathcal{J}_{st}(\sigma^*(y, \mathfrak{s}(y))dx)$ with $\sigma \in C_b^3(\mathbb{R}^{1(k+1)}, \mathbb{R}^d)$. It follows by the Propositions 3.2 and 3.5 that

$$(\delta y)_{st} = m_s^*(\delta x)_{st} + \sum_{i=0}^{k} \zeta_s^{(i)} \cdot \mathbf{x}_{st}^2(-r_i) + \Lambda_{st}\left(\hat{\rho}^* \delta x + \sum_{i=0}^{k} \delta \zeta^{(i)} \cdot \mathbf{x}^2(-r_i)\right) \tag{63}$$

for $0 \leq s \leq t \leq T$ with

$$m_s = \sigma(y_s, \mathfrak{s}(y)_s), \qquad \psi_s^{(i)} = \frac{\partial \sigma}{\partial x_i}(y_s, \mathfrak{s}(y)_s) \qquad \zeta_s^{(i)} = \psi_s^{(i)} m_{s-r_i}^* \tag{64}$$

for $i = 0, \ldots, k$. Note moreover that $\hat{\rho} = \hat{\rho}^{(1)} + \hat{\rho}^{(2)}$ is given by the relation

$$\hat{\rho}^{(1)}_{st} = \sum_{i=0}^{k} \psi^{(i)}_{s} \rho_{s-r_i, t-r_i}, \tag{65}$$

$$\hat{\rho}^{(2)}_{st} = \sigma(y_t, \mathfrak{s}(y)_t) - \sigma(y_s, \mathfrak{s}(y)_s) - \sum_{i=0}^{k} \psi^{(i)}_{s} (\delta y)_{s-r_i, t-r_i} \tag{66}$$

with ρ in turn given by

$$\rho_{st} = \sum_{i=0}^{k} \zeta^{(i)}_{s} \cdot \mathbf{x}^{\mathbf{2}}_{st}(-r_i) + \Lambda_{st}\left(\hat{\rho}^{*} \delta x + \sum_{i=0}^{k} \delta \zeta^{(i)} \cdot \mathbf{x}^{\mathbf{2}}(-r_i) \right). \tag{67}$$

Now consider (58) with a different initial path $\tilde{\xi}$, driving rough path $\tilde{x}$ and corresponding delayed Lévy area $\tilde{\mathbf{x}}^{\mathbf{2}}(v)$ for $v \in \{-r_k, \ldots, -r_0\}$. If the assumptions of the theorem are satisfied, then also the equation

$$\begin{cases} d\tilde{y}_t = \sigma(\tilde{y}_t, \tilde{y}_{t-r_1}, \ldots, \tilde{y}_{t-r_k})\, d\tilde{x}_t, & t \in [0, T], \\ \tilde{y}_t = \tilde{\xi}_t, & t \in [-r_k, 0] \end{cases}$$

admits a unique solution $\tilde{y} = F(\tilde{\xi}, \tilde{x}, \tilde{\mathbf{x}}^{\mathbf{2}}(0), \tilde{\mathbf{x}}^{\mathbf{2}}(-r_1), \ldots, \tilde{\mathbf{x}}^{\mathbf{2}}(-r_k))$. Clearly we also have in this case

$$(\delta \tilde{y})_{st} = \tilde{m}^{*}_{s} (\delta \tilde{x})_{st} + \sum_{i=0}^{k} \tilde{\zeta}^{(i)}_{s} \cdot \tilde{\mathbf{x}}^{\mathbf{2}}(-r_i)_{st} + \Lambda_{st}\left(\tilde{\hat{\rho}}^{*} \delta \tilde{x} + \sum_{i=0}^{k} \delta \tilde{\zeta}^{(i)} \cdot \tilde{\mathbf{x}}^{\mathbf{2}}(-r_i) \right) \tag{68}$$

for $0 \le s \le t \le T$ with $\tilde{m}$, $\tilde{\psi}^{(i)}$, $\tilde{\zeta}^{(i)}$, $\tilde{\hat{\rho}}$, $\tilde{\rho}$ defined according to (64), (65), (66) and (67).

(i) We first analyze the difference between ρ and $\tilde{\rho}$. Here we have

$$\rho_{st} - \tilde{\rho}_{st} = e^{(1)}_{st} + \Lambda_{st}(e^{(2)}) \tag{69}$$

with

$$e^{(1)}_{st} = \sum_{i=0}^{k} \zeta^{(i)}_{s} \cdot (\mathbf{x}^{\mathbf{2}}(-r_i))_{st} - \sum_{i=0}^{k} \tilde{\zeta}^{(i)}_{s} \cdot (\tilde{\mathbf{x}}^{\mathbf{2}}(-r_i))_{st},$$

$$e^{(2)} = \hat{\rho}^{*} \delta x - \tilde{\hat{\rho}}^{*} \delta \tilde{x} + \sum_{i=0}^{k} \delta \zeta^{(i)} \cdot \mathbf{x}^{\mathbf{2}}(-r_i) - \sum_{i=0}^{k} \delta \tilde{\zeta}^{(i)} \cdot \tilde{\mathbf{x}}^{\mathbf{2}}(-r_i).$$

Now set

$$C(y) = \|x\|_{\gamma,\infty} + \sum_{i=0}^{k} \|\mathbf{x}^{\mathbf{2}}(-r_i)\|_{2\gamma} + \mathcal{N}[y; \mathcal{Q}_{\kappa,\xi_0}([0,T]; \mathbb{R})] + \|\xi\|_{\infty} + \mathcal{N}[\xi; \mathcal{Q}_{\kappa,\alpha}([-r,0]; \mathbb{R})],$$

define $C(\tilde{y})$ accordingly for $\tilde{y}$, and let R be the quantity

$$R = \|x - \tilde{x}\|_{\gamma,\infty} + \sum_{i=0}^{k} \|\mathbf{x}^{\mathbf{2}}(-r_i) - \tilde{\mathbf{x}}^{\mathbf{2}}(-r_i)\|_{2\gamma} + \|\xi - \tilde{\xi}\|_{\kappa,\infty}.$$

In the following we will denote constants, which depend only on $\kappa, \gamma, n, d, k, \sigma, T$ and $r_1, \dots, r_k$, by c regardless of their value. Moreover, we will use for quantities, which depend continuously on $c, C(y), C(\tilde{y})$, the short notation C, i.e. we set

$$C := g(c, C(y), C(\tilde{y}))$$

if $g : (0,\infty)^3 \to [0,\infty)$ is a continuous and non-decreasing function.

Fix an interval $[a,b] \subseteq [0,T]$. By straightforward calculations we have

$$|e_{st}^{(1)}| \leq c\,|t-s|^{2\gamma} R + c\,C(\tilde{y})|t-s|^{2\gamma} \sup_{\tau \in [s-r_k,t]} |y_\tau - \tilde{y}_\tau| \tag{70}$$

for $s,t \in [a,b]$. Now, consider the term $e^{(2)}$. We have

$$\begin{aligned}
e_{sut}^{(2)} &= \hat{\rho}_{su}^*(\delta x)_{ut} - \tilde{\hat{\rho}}_{su}^*(\delta \tilde{x})_{ut} + \sum_{i=0}^{k} (\delta \zeta^{(i)})_{su} \cdot \mathbf{x}_{ut}^{\mathbf{2}}(-r_i) - \sum_{i=0}^{k} (\delta \tilde{\zeta}^{(i)})_{su} \cdot \tilde{\mathbf{x}}_{ut}^{\mathbf{2}}(-r_i) \\
&= (\hat{\rho}^* - \tilde{\hat{\rho}}^*)_{su}(\delta x)_{ut} + \tilde{\hat{\rho}}_{su}^*(\delta(x-\tilde{x}))_{ut} \\
&\qquad + \sum_{i=0}^{k} (\delta(\zeta^{(i)} - \tilde{\zeta}^{(i)}))_{su} \cdot \mathbf{x}_{ut}^{\mathbf{2}}(-r_i) - \sum_{i=0}^{k} (\delta \tilde{\zeta}^{(i)})_{su} \cdot (\mathbf{x}_{ut}^{\mathbf{2}}(-r_i) - \tilde{\mathbf{x}}_{ut}^{\mathbf{2}}(-r_i))
\end{aligned}$$

for $s,u,t \in [a,b]$. Clearly, it holds

$$\begin{aligned}
\left|(\hat{\rho}^* - \tilde{\hat{\rho}}^*)_{su}(\delta x)_{ut}\right| &\leq C(y)|t-u|^{\gamma}|s-u|^{2\kappa}\, \mathcal{N}[\hat{\rho} - \tilde{\hat{\rho}}; \mathcal{C}_2^{2\kappa}([a,b];\mathbb{R}^d)], \\
\left|\tilde{\hat{\rho}}_{su}^*(\delta(x-\tilde{x}))_{ut}\right| &\leq |t-u|^{\gamma}|s-u|^{2\kappa}\, \mathcal{N}[\tilde{\hat{\rho}}; \mathcal{C}_2^{2\kappa}([a,b];\mathbb{R}^d)]\, R
\end{aligned}$$

and

$$\begin{aligned}
\left|\sum_{i=0}^{k} (\delta(\zeta^{(i)} - \tilde{\zeta}^{(i)}))_{su} \cdot \mathbf{x}_{ut}^{\mathbf{2}}(-r_i)\right| &\leq c\,C(y)|t-u|^{2\gamma} \sum_{i=0}^{k} \left|(\delta(\zeta^{(i)} - \tilde{\zeta}^{(i)}))_{su}\right|, \\
\left|\sum_{i=0}^{k} (\delta \tilde{\zeta}^{(i)})_{su} \cdot (\mathbf{x}_{ut}^{\mathbf{2}}(-r_i) - \tilde{\mathbf{x}}_{ut}^{\mathbf{2}}(-r_i))\right| &\leq |t-u|^{2\gamma} R \sum_{i=0}^{k} \left|(\delta \tilde{\zeta}^{(i)})_{su}\right|.
\end{aligned}$$

Furthermore, we have

$$\left|(\delta \tilde{\zeta}^{(i)})_{su}\right| \leq C(\tilde{y})|s-u|^{\kappa}$$

for any $i = 0, \dots, k$ and we also obtain by tedious but straightforward computations (similar to the derivation of equation (34)) that

$$\left|\delta(\zeta^{(i)} - \tilde{\zeta}^{(i)})_{su}\right| \leq c(1 + C(y) + C(\tilde{y}))^2 \sup_{\tau_1,\tau_2 \in [s-r_k,u]} |(y_{\tau_1} - \tilde{y}_{\tau_1}) - (y_{\tau_2} - \tilde{y}_{\tau_2})|$$

for any $i = 0, \dots, k$.

Recall that the classical Hölder norm of a path f is defined by

$$\|f\|_{\mu,\infty,[s,t]} = \sup_{\tau \in [s,t]} |f_\tau| + \sup_{\tau_1,\tau_2 \in [s,t]} \frac{|f_{\tau_1} - f_{\tau_2}|}{|\tau_2 - \tau_1|^{\mu}}.$$

Using this notation and combining the previous estimates, we end up with

$$\left|e^{(2)}_{sut}\right| \leq C|t-u|^{\gamma}|s-u|^{2\kappa} R + C|t-u|^{2\gamma}|s-u|^{\kappa}\|y-\tilde{y}\|_{\kappa,\infty,[a-r_k,b]} + C|t-u|^{\gamma}|s-u|^{2\kappa}\,\mathcal{N}[\hat{\rho}-\tilde{\hat{\rho}};\mathcal{C}_2^{2\kappa}([a,b];\mathbb{R}^d)]. \quad (71)$$

Hence $e^{(2)}$ belongs to $\mathrm{Dom}(\Lambda)$ and we obtain by Proposition 2.2 that

$$\|\Lambda(e^{(2)})\|_{3\kappa} \leq C\big(R + \|y-\tilde{y}\|_{\kappa,\infty,[a-r_k,b]} + \mathcal{N}[\hat{\rho}-\tilde{\hat{\rho}};\mathcal{C}_2^{2\kappa}([a,b];\mathbb{R}^d)]\big)(b-a)^{\gamma-\kappa}. \quad (72)$$

Inserting the estimates for $e^{(1)}$ and $\Lambda(e^{(2)})$, i.e. (70) and (72), into the definition (69) of $\rho-\tilde{\rho}$ gives finally

$$\mathcal{N}[\rho-\tilde{\rho};\mathcal{C}_2^{2\kappa}([a,b];\mathbb{R})] \leq C|b-a|^{\gamma-\kappa}\|y-\tilde{y}\|_{\kappa,\infty,[a-r_k,b]} + C|b-a|^{\gamma-\kappa}R + C|b-a|^{\gamma-\kappa}\,\mathcal{N}[\hat{\rho}-\tilde{\hat{\rho}};\mathcal{C}_2^{2\kappa}([a,b];\mathbb{R}^d)],$$

and due to the subadditivity of the Hölder norms we get

$$\mathcal{N}[\rho-\tilde{\rho};\mathcal{C}_2^{2\kappa}([a,b];\mathbb{R})] \leq C|b-a|^{\gamma-\kappa}\,\mathcal{N}[\hat{\rho}-\tilde{\hat{\rho}};\mathcal{C}_2^{2\kappa}([a,b];\mathbb{R}^d)] + C|b-a|^{\gamma-\kappa}\,\|y-\tilde{y}\|_{\kappa,\infty,[a,b]} + C|b-a|^{\gamma-\kappa}\,\|y-\tilde{y}\|_{\kappa,\infty,[a-r_k,a]} + C|b-a|^{\gamma-\kappa}R. \quad (73)$$

(ii) Now we have to consider the difference between $\hat{\rho}$ and $\tilde{\hat{\rho}}$. Recall that $\hat{\rho} = \hat{\rho}^{(1)} + \hat{\rho}^{(2)}$ is given by

$$\hat{\rho}^{(1)}_{st} = \sum_{i=0}^{k} \psi^{(i)}_s \rho_{s-r_i,t-r_i},$$

$$\hat{\rho}^{(2)}_{st} = \sigma(y_t,\mathfrak{s}(y)_t) - \sigma(y_s,\mathfrak{s}(y)_s) - \sum_{i=0}^{k} \psi^{(i)}_s (\delta y)_{s-r_i,t-r_i}$$

and $\tilde{\hat{\rho}}$ accordingly. Analogous to the proof of Proposition 3.3 we obtain

$$\mathcal{N}[\hat{\rho}^{(1)}-\tilde{\hat{\rho}}^{(1)};\mathcal{C}_2^{2\kappa}([a,b];\mathbb{R}^d)] \leq C\mathcal{N}[\rho-\tilde{\rho};\mathcal{C}_2^{2\kappa}([a-r_k,b];\mathbb{R})] + C\|y-\tilde{y}\|_{\kappa,\infty,[a-r_k,b]}$$

and

$$\mathcal{N}[\hat{\rho}^{(2)}-\tilde{\hat{\rho}}^{(2)};\mathcal{C}_2^{2\kappa}([a,b];\mathbb{R}^d)] \leq C\|y-\tilde{y}\|_{\kappa,\infty,[a-r_k,b]}.$$

Inserting this into (73) gives

$$\begin{aligned} &\mathcal{N}[\rho-\tilde{\rho};\mathcal{C}_2^{2\kappa}([a,b];\mathbb{R})] \\ &\quad\leq C|b-a|^{\gamma-\kappa}\,\mathcal{N}[\rho-\tilde{\rho};\mathcal{C}_2^{2\kappa}([a,b];\mathbb{R})] + C|b-a|^{\gamma-\kappa}\,\|y-\tilde{y}\|_{\kappa,\infty,[a,b]} \\ &\quad\quad + C|b-a|^{\gamma-\kappa}\,\mathcal{N}[\rho-\tilde{\rho};\mathcal{C}_2^{2\kappa}([a-r_k,a];\mathbb{R})] + C|b-a|^{\gamma-\kappa}\,\|y-\tilde{y}\|_{\kappa,\infty,[a-r_k,a]} \\ &\quad\quad + C|b-a|^{\gamma-\kappa}R. \end{aligned} \quad (74)$$

(iii) Finally, consider the difference between y and $\tilde{y}$. Completely analogous to step *(i)* we also obtain that

$$\begin{aligned}
&\mathcal{N}[y-\tilde{y};\mathscr{C}_1^\kappa([a,b];\mathbb{R})] \\
&\quad\le C|b-a|^{\gamma-\kappa}\,\mathcal{N}[\rho-\tilde{\rho};\mathscr{C}_2^{2\kappa}([a,b];\mathbb{R})]+C|b-a|^{\gamma-\kappa}\,\|y-\tilde{y}\|_{\kappa,\infty,[a,b]} \\
&\qquad+C|b-a|^{\gamma-\kappa}\,\mathcal{N}[\rho-\tilde{\rho};\mathscr{C}_2^{2\kappa}([a-r_k,a];\mathbb{R})]+C|b-a|^{\gamma-\kappa}\,\|y-\tilde{y}\|_{\kappa,\infty,[a-r_k,a]} \\
&\qquad+C|b-a|^{\gamma-\kappa}R.
\end{aligned}$$

Moreover, since

$$\sup_{\tau\in[a,b]}|y_\tau-\tilde{y}_\tau|\le|y_a-\tilde{y}_a|+(b-a)^\kappa\,\mathcal{N}[y-\tilde{y};\mathscr{C}_1^\kappa([a,b];\mathbb{R})],$$

we also have

$$\begin{aligned}
&\|y-\tilde{y}\|_{\kappa,\infty,[a,b]} \\
&\quad\le C|b-a|^{\gamma-\kappa}\,\mathcal{N}[\rho-\tilde{\rho};\mathscr{C}_2^{2\kappa}([a,b];\mathbb{R})]+C|b-a|^{\gamma-\kappa}\,\|y-\tilde{y}\|_{\kappa,\infty,[a,b]} \\
&\qquad+C|b-a|^{\gamma-\kappa}\,\mathcal{N}[\rho-\tilde{\rho};\mathscr{C}_2^{2\kappa}([a-r_k,a];\mathbb{R})]+C|b-a|^{\gamma-\kappa}\,\|y-\tilde{y}\|_{\kappa,\infty,[a-r_k,a]} \\
&\qquad+|y_a-\tilde{y}_a|+C|b-a|^{\gamma-\kappa}R.
\end{aligned}\tag{75}$$

(Note that we consider here the case $\gamma-\kappa\le\kappa$ without loss of generality. If $\gamma-\kappa>\kappa$, then only $(b-a)^{\gamma-\kappa}$ had to be replaced by $(b-a)^\kappa$ in (75) and in the following arguments.)

(iv) Now set

$$\Delta(a,b)=\mathcal{N}[\rho-\tilde{\rho};\mathscr{C}_2^{2\kappa}([a,b];\mathbb{R})]+\|y-\tilde{y}\|_{\kappa,\infty,[a,b]}.$$

By combining (74) and (75) we finally have that

$$\begin{aligned}
\Delta(a,b)\le{}& C|b-a|^{\gamma-\kappa}\,\Delta(a,b)+C|b-a|^{\gamma-\kappa}\,\Delta((a-r_k)^+,a) \\
&+C|b-a|^{\gamma-\kappa}\,\|\xi-\tilde{\xi}\|_{\kappa,\infty,[-r_k,0]}+|y_a-\tilde{y}_a|+C|b-a|^{\gamma-\kappa}R
\end{aligned}$$

and thus

$$\begin{aligned}
\Delta(a,b)\le{}& C|b-a|^{\gamma-\kappa}\,\Delta(a,b)+C|b-a|^{\gamma-\kappa}\Delta((a-r_k)^+,a) \\
&+|y_a-\tilde{y}_a|+C|b-a|^{\gamma-\kappa}R.
\end{aligned}\tag{76}$$

Now choose $a=0$ and $b_1=(2C)^{-1/(\gamma-\kappa)}$. In this case, we obtain from (76) that

$$\Delta(0,b_1)\le\frac{1}{2}\Delta(0,b_1)+|\xi_0-\tilde{\xi}_0|+\frac{1}{2}R,$$

which yields

$$\Delta(0,b_1)\le R+2|\xi_0-\tilde{\xi}_0|\le 3R.\tag{77}$$

For the next interval $[b_1,2b_1]$, we obtain in turn that

$$\Delta(b_1,2b_1)\le\frac{1}{2}\Delta(b_1,2b_1)+\frac{1}{2}\Delta(0,b_1)+|y_{b_1}-\tilde{y}_{b_1}|+\frac{1}{2}R,$$

and hence

$$\Delta(b_1, 2b_1) \leq \Delta(0, b_1) + 2|y_{b_1} - \tilde{y}_{b_1}| + R \leq 10R,$$

by (77).

Repeating this step $\lfloor T/b_1 \rfloor$-times we obtain that there exists a continuous non-decreasing function $h : (0, \infty) \to [0, \infty)$ such that

$$\Delta(ib_1 \wedge T, (i+1)b_1 \wedge T) \leq h(T/b_1)R$$

for all $i = 0, \ldots, \lfloor T/b_1 \rfloor$. Using the subadditivity of the Hölder norms, the estimate

$$\Delta(0, T) \leq (1 + T/b_1)h(T/b_1)R = \left(1 + T(2C)^{1/(\gamma-\kappa)}\right)h\left(T(2C)^{1/(\gamma-\kappa)}\right)R \tag{78}$$

follows. Now recall that C depends continuously on $c, C(y)$, $C(\tilde{y})$, where c is an arbitrary constant, which depends only on $\kappa, \gamma, n, d, k, \sigma, T$ and $r_1, \ldots, r_k$,

$$C(y) = \|x\|_{\gamma,\infty} + \sum_{i=0}^{k} \|\mathbf{x}^2(-r_i)\|_{2\gamma} + \mathcal{N}[y; \mathcal{Q}_{\kappa,\xi_0}([0,T]; \mathbb{R})] + \|\xi\|_\infty + \mathcal{N}[\xi; \mathcal{Q}_{\kappa,\alpha}([-r, 0]; \mathbb{R})]$$

and $C(\tilde{y})$ is defined accordingly. However, by (62) there exists a continuous and non-decreasing function $f : (0, \infty) \to [0, \infty)$ such that

$$C(y) \leq D + f(D), \qquad C(\tilde{y}) \leq \tilde{D} + f(\tilde{D}),$$

where

$$D = \|x\|_{\gamma,\infty} + \sum_{i=0}^{k} \|\mathbf{x}^2(-r_i)\|_{2\gamma} + \|\xi\|_\infty + \mathcal{N}[\xi; \mathcal{Q}_{\kappa,\alpha}([-r, 0]; \mathbb{R})],$$

and $\tilde{D}$ is again defined accordingly. Thus, we obtain now from (78) that there exists a continuous function $l : [0, \infty) \to [0, \infty)$, which depends only on $\kappa, \gamma, \sigma, n, d, k, T$ and $r_1, \ldots, r_k$, such that

$$\Delta(0, T) \leq l(D + \tilde{D})R.$$

Hence, the assertion follows.

□

5 Application to the fractional Brownian motion

All the previous constructions rely on the specific assumptions we have made on the path x. In this section, we will show how our results can be applied to the fractional Brownian motion.

5.1 Definition

We consider in this section a d-dimensional fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$ defined on the real line, that is a centered Gaussian process

$$B = \left\{B_t = (B_t^1, \ldots, B_t^d);\ t \in \mathbb{R}\right\},$$

where $B^1, \dots, B^d$ are d independent one-dimensional fBms, i.e. each B^i is a centered Gaussian process with continuous sample paths and covariance function

$$R_H(t,s) = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\right) \tag{79}$$

for $i = 1, \dots, d$. The fBm verifies the following two important properties:

$$\text{(scaling)} \quad \text{For any } c > 0,\ B^{(c)} = c^H B_{\cdot/c} \text{ is a fBm,} \tag{80}$$

$$\text{(stationarity)} \quad \text{For any } h \in \mathbb{R},\ B_{\cdot+h} - B_h \text{ is a fBm.} \tag{81}$$

Notice that, for Malliavin calculus purposes, we shall assume in the sequel that B is defined on a complete probability space $(\Omega, \mathscr{F}, P)$, and that $\mathscr{F} = \sigma(B_s;\, s \in \mathbb{R})$.

5.2 Malliavin calculus with respect to fBm

Let us give a few facts about the Gaussian structure of fractional Brownian motion and its Malliavin derivative process, following Section 2 of [18]. Let $\mathscr{E}$ be the set of step-functions on $\mathbb{R}$ with values in $\mathbb{R}^d$. Consider the Hilbert space $\mathscr{H}$ defined as the closure of $\mathscr{E}$ with respect to the scalar product induced by

$$\begin{aligned}&\left\langle (\mathbf{1}_{[t_1,t^1]}, \dots, \mathbf{1}_{[t_d,t^d]}), (\mathbf{1}_{[s_1,s^1]}, \dots, \mathbf{1}_{[s_d,s^d]}) \right\rangle_{\mathscr{H}} \\ &\qquad = \sum_{i=1}^{d} \left(R_H(t^i,s^i) - R_H(t^i,s_i) - R_H(t_i,s^i) + R_H(t_i,s_i)\right),\end{aligned}$$

for any $-\infty < s_i < s^i < +\infty$ and $-\infty < t_i < t^i < +\infty$, and where $R_H(t,s)$ is given by (79). The mapping

$$(\mathbf{1}_{[t_1,t^1]}, \dots, \mathbf{1}_{[t_d,t^d]}) \mapsto \sum_{i=1}^{d} \left(B^i_{t^i} - B^i_{t_i}\right)$$

can be extended to an isometry between $\mathscr{H}$ and the Gaussian space $H_1(B)$ associated with $B = (B^1, \dots, B^d)$. We denote this isometry by $\varphi \mapsto B(\varphi)$. Let $\mathscr{S}$ be the set of smooth cylindrical random variables of the form

$$F = f(B(\varphi_1), \dots, B(\varphi_k)), \qquad \varphi_i \in \mathscr{H}, \quad i = 1, \dots, k,$$

where $f \in C^\infty(\mathbb{R}^{dk}, \mathbb{R})$ is bounded with bounded derivatives. The derivative operator D of a smooth cylindrical random variable of the above form is defined as the $\mathscr{H}$-valued random variable

$$DF = \sum_{i=1}^{k} \frac{\partial f}{\partial x_i}(B(\varphi_1), \dots, B(\varphi_k))\varphi_i.$$

This operator is closable from $L^p(\Omega)$ into $L^p(\Omega; \mathscr{H})$. As usual, $\mathbb{D}^{1,2}$ denotes the closure of the set of smooth random variables with respect to the norm

$$\|F\|_{1,2}^2 \;=\; \mathrm{E}|F|^2 + \mathrm{E}\|DF\|_{\mathscr{H}}^2.$$

In particular, if D^iF denotes the Malliavin derivative of $F \in \mathbb{D}^{1,2}$ with respect to B^i, we have $D^iB_t^j = \delta_{i,j}\mathbf{1}_{[0,t]}$ for $i,j = 1,\dots,d$.

The divergence operator I is the adjoint of the derivative operator. If a random variable $u \in L^2(\Omega;\mathscr{H})$ belongs to $\mathrm{dom}(I)$, the domain of the divergence operator, then $I(u)$ is defined by the duality relationship

$$\mathrm{E}(F\,I(u)) = \mathrm{E}\langle DF, u\rangle_{\mathscr{H}}, \tag{82}$$

for every $F \in \mathbb{D}^{1,2}$. Moreover, let us recall two useful properties verified by D and I:

- If $u \in \mathrm{dom}(I)$ and $F \in \mathbb{D}^{1,2}$ such that $Fu \in L^2(\Omega;\mathscr{H})$, then we have the following integration by parts formula:
$$I(Fu) = FI(u) - \langle DF, u\rangle_{\mathscr{H}}. \tag{83}$$
- If u verifies $\mathrm{E}\|u\|_{\mathscr{H}}^2 + \mathrm{E}\|Du\|_{\mathscr{H}\otimes\mathscr{H}}^2 < \infty$, $D_r u \in \mathrm{dom}(I)$ for all $r \in \mathbb{R}$ and $(I(D_r u))_{r\in\mathbb{R}}$ is an element of $L^2(\Omega;\mathscr{H})$, then
$$D_r I(u) = u_r + I(D_r u). \tag{84}$$

5.3 Delayed Lévy area and fractional Brownian motion

The stochastic integrals we shall use in order to construct our delayed Lévy area are defined, in a natural way, by Russo-Vallois symmetric approximations, that is, for a given process ϕ we have

$$\int_s^t \phi_w\, d^\circ B_w^i \;= L^2 - \lim_{\varepsilon\to 0}(2\varepsilon)^{-1}\int_s^t \phi_w\left(B_{w+\varepsilon}^i - B_{w-\varepsilon}^i\right)dw,$$

provided the limit exists. This pathwise type notion of integral can be related to some stochastic analysis criterions in the following way (for a proof, see [1]):

Theorem 5.1. *Fix $t \ge 0$ and let $\phi \in \mathbb{D}^{1,2}(\mathscr{H})$ be a process such that*

$$\mathrm{Tr}_{[0,t]}D^{B^i}\phi := L^2 - \lim_{\varepsilon\to 0}(2\varepsilon)^{-1}\int_0^t \langle D^{B^i}\phi_u, \mathbf{1}_{[u-\varepsilon,u+\varepsilon]}\rangle_{\mathscr{H}}du$$

exists, and such that, setting $\ell(t,u) = u^{2H-1} + (t-u)^{2H-1}$ for $0 \le u < t$,

$$\int_0^t \mathrm{E}\,\phi_u^2\,\ell(t,u)du + \int_0^t\int_0^t \mathrm{E}\,(D_r^{B^i}\phi_u)^2\,\ell(t,u)dudr < \infty.$$

Then $\int_0^t \phi_w d^\circ B_w^i$ exists and verifies

$$\int_0^t \phi_w d^\circ B_w^i = I^{B^i}(\phi\mathbf{1}_{[0,t]}) + \mathrm{Tr}_{[0,t]}D^{B^i}\phi.$$

With these notations in mind, the main result of this section is the following:

Proposition 5.2. *Let B be a d-dimensional fractional Brownian motion and suppose $H > \frac{1}{3}$. Then almost all sample paths of B satisfy Hypothesis 3.4.*

Proof. A natural candidate to be the delayed Lévy area (provided it is well-defined) is given, for any $v \in \{-r_k, \dots, -r_0\}$, by:

$$\mathbf{B}^{\mathbf{2}}_{st}(v) = \int_s^t d^\circ B_u \otimes \int_{s+v}^{u+v} d^\circ B_r, \text{ i.e. } \mathbf{B}^{\mathbf{2}}_{st}(v)(i,j) = \int_s^t d^\circ B^i_u \int_{s+v}^{u+v} d^\circ B^j_r, \quad i,j \in \{1,\dots,d\}. \tag{85}$$

When $H = \frac{1}{2}$, the desired conclusion is easily obtained, because the Russo-Vallois symmetric integral coincides with the Stratonovich integral.

When $H > \frac{1}{2}$, the Russo-Vallois symmetric integral coincides with the Young integral, which is well defined in this case, and the assertion still follows easily from the properties of Young integrals.

When $\frac{1}{3} < H < \frac{1}{2}$, by Theorem 5.1, we shall obtain (see below) the almost sure existence of the iterated integral (85) for *fixed* integration bounds, but unfortunately not as a process. In order to bypass this difficulty, and rather than trying to prove the uniform version of Theorem 5.1, our strategy actually consists in defining $\mathbf{B}^{\mathbf{2}}(v)$ by (85) only for *rational* times and then to consider its continuous version (which well exists, since $\mathbf{B}^{\mathbf{2}}(v)$ will verify the Kolmogorov criterion).

Now, let us proceed with the proof in the case where $H \in (\frac{1}{3}, \frac{1}{2})$ is fixed. It is a classical fact that $B \in \mathscr{C}^\gamma_1([0,T];\mathbb{R}^d)$ for any $\frac{1}{3} < \gamma < H$. Due to the stationarity property (81) we will work without loss of generality on the interval $[0, t-s]$ instead of $[s,t]$ in the sequel.

1) Case $i = j$. When $v = 0$, it is readily checked that

$$\mathrm{E}|\mathbf{B}^{\mathbf{2}}_{st}(0)(i,i)|^2 = \frac{1}{4}\,\mathrm{E}|B_t - B_s|^4 = \frac{3}{4}|t-s|^{4H}.$$

Let us now consider the case where $v < 0$. For $\phi = (B^i_{\cdot+v} - B^i_v)\mathbf{1}_{[0,t-s]}(\cdot)$, the conditions of Theorem 5.1 are easily verified, hence $\int_0^{t-s} \phi_u d^\circ B^i_u$ exists. Notice moreover that we have $D^{B^i}_r \phi_u = \mathbf{1}_{[v,u+v]}(r)\mathbf{1}_{[0,t-s]}(u)$ and, for $u \in [0, t-s]$ and $\varepsilon \in [0, -v]$ (which is always the case, for a fixed $v < 0$ and ε small enough) it holds

$$\begin{aligned}\langle \mathbf{1}_{[v,u+v]}, \mathbf{1}_{[u-\varepsilon,u+\varepsilon]}\rangle_{\mathscr{H}} &= \frac{1}{2}\left(|v+\varepsilon|^{2H} - |v-\varepsilon|^{2H} + |v-u-\varepsilon|^{2H} - |v-u+\varepsilon|^{2H}\right) \\ &= \frac{1}{2}\left((-v-\varepsilon)^{2H} - (-v+\varepsilon)^{2H} + (-v+u+\varepsilon)^{2H} - (-v+u-\varepsilon)^{2H}\right).\end{aligned}$$

Thus, we obtain

$$\mathrm{Tr}_{[0,t-s]} D^{B^i}\phi = -H(-v)^{2H-1}(t-s) + \frac{1}{2}\left((t-s-v)^{2H} - (-v)^{2H}\right).$$

For $x \geq 0$, it is well-known that $0 \leq ((-v)+x)^{2H} - (-v)^{2H} \leq 2H(-v)^{2H-1}x$. Applying this inequality to the second term of the right hand side of $\mathrm{Tr}_{[0,t-s]} D^{B^i}\phi$ we get

$$\left|\mathrm{Tr}_{[0,t-s]} D^{B^i}\phi\right| \leq H(-v)^{2H-1}(t-s). \tag{86}$$

On the other hand, we have by (84)

$$D^{B_i}_r I^{B_i}(\phi) = \phi_r + I^{B_i}(D^{B_i}_r \phi) = \left(\phi_r + I^{B_i}(\mathbf{1}_{[r-v,+\infty)\cap[0,t-s]})\right)\mathbf{1}_{[0,t-s]}(r). \tag{87}$$

When $-v \geq t-s$, then $[r-v,+\infty)\cap[0,t-s]=\emptyset$ for any $r\in[0,t-s]$. By using (82) we deduce

$$\begin{aligned} \mathrm{E}|I^{B^i}(\phi)|^2 &= \mathrm{E}\|\phi\|^2_{\mathscr{H}} = \mathrm{E}\|B^i_{\cdot+v}-B^i_v\|^2_{\mathscr{H}([0,t-s])} \\ &= \mathrm{E}\|B^i\|^2_{\mathscr{H}([0,t-s])} = (t-s)^{4H}\mathrm{E}\|B^i\|^2_{\mathscr{H}([0,1])}, \end{aligned} \tag{88}$$

where the two last equalities are due to the stationarity (81) and scaling (80) properties of fractional Brownian motion.

When $-v < t-s$, then

$$I^{B_i}(\mathbf{1}_{[r-v,+\infty)\cap[0,t-s]}) = (B^i_{t-s}-B^i_{r-v})\mathbf{1}_{[0,t-s+v]}(r). \tag{89}$$

We deduce

$$\begin{aligned}
&\mathrm{E}|I^{B^i}(\phi)|^2 \\
&= \mathrm{E}\langle DI^{B^i}(\phi),\phi\rangle_{\mathscr{H}} \quad \text{by (82)} \\
&= \mathrm{E}\|\phi\|^2_{\mathscr{H}([0,t-s])} + \mathrm{E}\langle I^{B^i}(\mathbf{1}_{[r-v,\infty)\cap[0,t-s]}),\phi\rangle_{\mathscr{H}([0,t-s])} \quad \text{by (87)} \\
&= \mathrm{E}\|\phi\|^2_{\mathscr{H}([0,t-s])} + \mathrm{E}\langle (B^i_{t-s}-B^i_{\cdot-v})\mathbf{1}_{[0,t-s+v]},\phi\rangle_{\mathscr{H}([0,t-s])} \quad \text{by (89)} \\
&\leq \mathrm{E}\|\phi\|^2_{\mathscr{H}([0,t-s])} + \mathrm{E}\left(\|(B^i_{t-s}-B^i_{\cdot-v})\mathbf{1}_{[0,t-s+v]}\|_{\mathscr{H}([0,t-s])}\,\|\phi\|_{\mathscr{H}([0,t-s])}\right) \\
&\leq \frac{3}{2}\mathrm{E}\|\phi\|^2_{\mathscr{H}([0,t-s])} + \frac{1}{2}\mathrm{E}\|(B^i_{t-s}-B^i_{\cdot-v})\mathbf{1}_{[0,t-s+v]}\|^2_{\mathscr{H}([0,t-s])} \quad \text{because } ab\leq \tfrac{1}{2}(a^2+b^2) \\
&= \frac{3}{2}(t-s)^{4H}\mathrm{E}\|B^i\|^2_{\mathscr{H}([0,1])} + \frac{1}{2}\mathrm{E}\|(B^i_{t-s+v}-B^i)\mathbf{1}_{[0,t-s+v]}\|^2_{\mathscr{H}([0,t-s])} \quad \text{by (80) and (81)} \\
&= \frac{3}{2}(t-s)^{4H}\mathrm{E}\|B^i\|^2_{\mathscr{H}([0,1])} + \frac{1}{2}(t-s+v)^{2H}\mathrm{E}\|(B^i_{t-s+v}-B^i_{(t+s-v)\cdot})\|^2_{\mathscr{H}([0,1])} \\
&= \frac{3}{2}(t-s)^{4H}\mathrm{E}\|B^i\|^2_{\mathscr{H}([0,1])} + \frac{1}{2}(t-s+v)^{4H}\mathrm{E}\|(B^i_1-B^i)\|^2_{\mathscr{H}([0,1])} \quad \text{by (80)} \\
&\leq \frac{1}{2}(t-s)^{4H}\left(3\mathrm{E}\|B^i\|^2_{\mathscr{H}([0,1])} + \mathrm{E}\|(B^i_1-B^i)\|^2_{\mathscr{H}([0,1])}\right).
\end{aligned} \tag{90}$$

Finally, we can summarize (88) and (90) in

$$\mathrm{E}|I^{B^i}(\phi)|^2 \leq c_H|t-s|^{4H},$$

with a constant $c_H>0$, which is in particular independent of v. Putting together this last estimate with inequality (86), we end up with

$$\mathrm{E}|\mathbf{B}^{\mathbf{2}}_{st}(v)(i,i)|^2 \leq c_H(1+|v|^{2H-1})|t-s|^{4H}$$

for any $v\in\{-r_k,\dots,-r_1\}$.

2) Case where $i\neq j$. By stationarity (81), we have for any $v\in\{-r_k,\dots,-r_0\}$ that

$$\left(B^j_{u+v}-B^j_v, B^i_u\right)_{u\in[0,t-s]} \stackrel{\mathscr{L}}{=} \left(B^j_u, B^i_u\right)_{u\in[0,t-s]}.$$

Thus, the delayed Lévy area $\mathbf{B}^{\mathbf{2}}_{0,t-s}(v)(i,j)=\int_0^{t-s}(B^j_{u+v}-B^j_v)d^\circ B^i_u$ for $v<0$ behaves as in the case where $v=0$. But it is a classical result that $\mathbf{B}^{\mathbf{2}}_{0,t-s}(0)$ is well-defined for $H>1/3$ (see e.g. [20]). Moreover, it follows again by the stationarity (81) and the scaling (80) properties that

$$\mathrm{E}|\mathbf{B}^{\mathbf{2}}_{0,t-s}(v)(i,j)|^2 = \mathrm{E}|\mathbf{B}^{\mathbf{2}}_{0,t-s}(0)(i,j)|^2 = |t-s|^{4H}\mathrm{E}|\mathbf{B}^{\mathbf{2}}_{01}(0)(i,j)|^2 \leq c_H|t-s|^{4H}.$$

Immediately, we deduce that

$$\mathrm{E}|\mathbf{B}^{\mathbf{2}}_{st}(v)(i,j)|^2 \le c_H |t-s|^{4H}$$

for any $v \in \{-r_k, \dots, -r_0\}$.

Both in the cases $i = j$ and $i \neq j$ the substitution formula for Russo-Vallois integrals easily yields that $\delta \mathbf{B}^{\mathbf{2}}(v) = \delta B(v) \otimes \delta B$. Furthermore, since $\mathbf{B}^{\mathbf{2}}(v)$ is a process belonging to the second chaos of the fractional Brownian motion B, on which all L^p norms are equivalent for $p > 1$, we get that

$$\mathrm{E}|\mathbf{B}^{\mathbf{2}}_{st}(v)(i,j)|^p \le c_p |t-s|^{2pH} \tag{91}$$

for $i \neq j$ and

$$\mathrm{E}|I^{B_i}(\phi)|^p \le c_p |t-s|^{2pH} \tag{92}$$

when $i = j$. In order to conclude that $\mathbf{B}^{\mathbf{2}}(v) \in \mathscr{C}_2^{2\gamma}(\mathbb{R}^{d\times d})$ for any $\frac{1}{3} < \gamma < H$ and $v \in \{-r_k, \dots, -r_0\}$, let us recall the following inequality from [9]: Let $g \in \mathscr{C}_2(V)$ for a given Banach space V. Then, for any $\kappa > 0$ and $p \ge 1$ we have

$$\|g\|_\kappa \le c \left(U_{\kappa+2/p;p}(g) + \|\delta g\|_\kappa \right) \quad \text{with} \quad U_{\gamma;p}(g) = \left(\int_0^T \int_0^T \frac{|g_{st}|^p}{|t-s|^{\gamma p}} ds dt \right)^{1/p}. \tag{93}$$

By plugging inequality (91)-(92) into (93), by recalling that $\delta \mathbf{B}^{\mathbf{2}}(v) = \delta B(v) \otimes \delta B$ and (86) hold, we obtain that $\mathbf{B}^{\mathbf{2}}(v)(i,j) \in \mathscr{C}_2^{2\gamma}(\mathbb{R}^{d\times d})$ for any $\frac{1}{3} < \gamma < H$ and $i, j = 1, \dots, d$.

□

Acknowledgements. The authors are grateful to the anonymous referees for a careful and thorough reading of this work and also for their valuable suggestions.

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