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BSDEs with two reflecting barriers driven by a Brownian and a Poisson noise and related Dynkin game

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Abstract

In this paper we study BSDEs with two reflecting barriers driven by a Brownian motion and an independent Poisson process. We show the existence and uniqueness of *local* and global solutions. As an application we solve the related zero-sum Dynkin game.

Key words: Backward stochastic differential equation ; Poisson measure ; Dynkin game ; Mokobodzki's condition.

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1 Introduction

Let $(\mathcal{F}_t)_{t\leq T}$ be a filtration generated by a Brownian motion $(B_t)_{t\leq T}$ and an independent Poisson measure $\mu(t,\omega,de)$ defined on a probability space $(\Omega,\mathcal{F},P)$. A solution for the backward stochastic differential equation (BSDE for short) with two reflecting barriers associated with a coefficient $f(t,\omega,y,z)$, a terminal value ξ and a lower (resp. an upper) barrier $(L_t)_{t\leq T}$ (resp. $(U_t)_{t\leq T}$) is a quintuple of $\mathcal{F}_t$-predictable processes $(Y_t, Z_t, V_t, K_t^+, K_t^-)_{t\leq T}$ which satisfies:

$$\begin{cases} -dY_t = f(t,Y_t,Z_t,V_t)dt + dK_t^+ - dK_t^- - Z_t dB_t - \int_E V_t(e)\tilde{\mu}(dt,de),\ t\leq T\ ;\ Y_T=\xi \\ L_t \leq Y_t \leq U_t,\ \forall t\leq T \\ K^{\pm} \text{ are continuous non-decreasing and } (Y_t-L_t)dK_t^+ = (Y_t-U_t)dK_t^- = 0\ (K_0^{\pm}=0), \end{cases} \tag{1}$$

where $\tilde{\mu}$ is the compensated measure of μ.

Nonlinear BSDEs have been first introduced by Pardoux and Peng [19], who proved the existence and uniqueness of a solution under suitable hypotheses on the coefficient and the terminal value of the BSDE. Since, these equations have gradually become an important mathematical tool which is encountered in many fields such as finance ([6], [4], [8], [9],...), stochastic games and optimal control ([10], [11], ...), partial differential equations ([1], [18], [20],...).

In the case when the filtration is generated only by a Brownian motion and when we consider just one lower barrier (set $U\equiv+\infty$ and $K^-\equiv 0$ in (1)), the problem of existence and uniqueness of a solution for (1) is considered and solved by El-Karoui et al. in [6]. Their work has been generalized by Cvitanic & Karatzas in [4] where they deal with BSDEs with two reflecting barriers.

BSDEs without reflection (in (1) one should take $L=-\infty$ and $U=+\infty$, thereby $K^{\pm}=0$) driven by a Brownian motion and an independent Poisson measure have been considered first by Tang & Li [17] then by Barles et al. in connection with partial-integral differential equations in [1]. In both papers the authors showed the existence and uniqueness of a solution.

The extension to the case of BSDEs with one reflecting barrier has been established by Hamadène & Ouknine in [13]. The authors showed the existence and uniqueness of the solution when the coefficient f is Lipschitz. Two proofs have been given, the first one is based on the penalization scheme as for the second, it is obtained in using the Snell envelope notion. However both methods make use of a contraction argument since the usual comparison theorem fails to work in the general framework.

So in this work we study BSDEs with two reflecting barriers driven by a Brownian motion and an independent Poisson measure. This is the natural extension of Hamadène & Ouknine's work. However there are at least four motivations for considering this problem. The first one is related to Dynkin zero-sum game. The second is in connection with the real option area since the stopping and starting problem leads to a BSDE with two reflecting barriers (see e.g. [12]). The third one is that our problem can provide solutions for variational inequality problems with two obstacles when the generator is of partial-integral type. Finally we provide a condition easy to check in practice under which the well known Mokobodski's hypothesis is satisfied.

In this paper we begin to show the existence of an adapted and *rcll* (right continuous and left limited) process $Y := (Y_t)_{t\leq T}$ which in a way is a *local solution* for (1). Actually we prove that for any stopping time τ there exist another stopping time $\theta_\tau \geq \tau$ and a quadruple of processes (Z, V, K^+, K^-) which with Y verify (1) on the interval $[\tau, \theta_\tau]$. In addition the process Y reaches the barriers U and L between τ and θ_τ. In the proof of our theorem, the key point is that the predictable projection of a process π whose jumping times are inaccessible is equal to π_-, the process of left limits associated with π.

This result is then applied to deal with the zero-sum Dynkin game associated with L, U, ξ and a process $(g_s)_{s\leq T}$ which stands for the instantaneous payoff. We show that this game is closely related to the notion of *local* solution for (1). Besides we obtain the existence of a saddle-point for the game under conditions out of the scope of the known results on this subject. Finally we give some feature of the value function of the game (see Remark 4.3). Our result can be applied in mathematical finance to deal with American game (or *recallable*) options whose underlying derivatives contain a Poisson part (see [15] for this type of option).

Further we consider the problem of existence and uniqueness of a global solution for (1). When the weak Mokobodski's assumption **[WM]** is satisfied, which roughly speaking turns into the existence of a difference of non-negative supermartingales between L and U, we show existence and uniqueness of the solution. Then we address the issue of the verification of the condition **[WM]**. Actually we prove that under the fully separation of the barriers, *i.e.*, $L_t < U_t$ for any $t \leq T$, the condition **[WM]** holds true.

This paper is organized as follows :

In Section 2, we deal with *local* solutions for (1) while Section 3 is devoted to zero-sum Dynkin games. At the end, in Section 4, we address the problem of existence of a global solution for (1). $\Box$

2 Reflected BSDEs driven by a Brownian motion and an independent Poisson point process

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\leq T})$ be a stochastic basis such that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$, $\mathcal{F}_{t+} = \bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon} = \mathcal{F}_t$, $\forall t < T$, and suppose that the filtration is generated by the following two mutually independent processes :

- a d-dimensional Brownian motion $(B_t)_{t\leq T}$

- a Poisson random measure μ on $I\!R^+ \times E$, where $E := I\!R^l \setminus \{0\}$ is equipped with its Borel fields $\mathcal{E}$, with compensator $\nu(dt, de) = dt\lambda(de)$, such that $\{\tilde{\mu}([0, t] \times A) = (\mu - \nu)([0, t] \times A)\}_{t\leq T}$ is a martingale for every $A \in \mathcal{E}$ satisfying $\lambda(A) < \infty$. The measure λ is assumed to be σ-finite on $(E, \mathcal{E})$ and verifies $\int_E (1 \wedge |e|^2)\lambda(de) < \infty$.

Now let:

- $\mathcal{D}$ be the set of $\mathcal{F}_t$-adapted right continuous with left limits processes $(Y_t)_{t\leq T}$ with values in $I\!R$ and $\mathcal{D}^2 := \{Y \in \mathcal{D}, I\!E[\sup_{t\leq T} |Y_t|^2] < \infty\}$

- $\tilde{\mathcal{P}}$ (resp. $\mathcal{P}$) be the $\mathcal{F}_t$-progressive (resp. predictable) tribe on $\Omega \times [0, T]$

- $\mathcal{H}^{2,k}$ (resp. $\mathcal{H}^k$) be the set of $\tilde{\mathcal{P}}$-measurable processes $Z := (Z_t)_{t\leq T}$ with values in $I\!R^k$ and $dP \otimes dt$-square integrable (resp. P-*a.s.* $Z(\omega) := (Z_t(\omega))_{t\leq T}$ is dt-square integrable)

- $\mathcal{L}^2$ (resp. $\mathcal{L}$) be the set of mappings $V : \Omega \times [0,T] \times E \to I\!R$ which are $\mathcal{P} \otimes \mathcal{E}$-measurable and $I\!E[\int_0^T \|V_s\|^2 ds] < \infty$ (resp. $\int_0^T \|V_s\|^2 ds < \infty$, P-*a.s.*) where $\|v\| := (\int_E |v(e)|^2 \lambda(de))^{\frac{1}{2}}$ for $v : E \to I\!R$

- $\mathcal{C}^2_{ci}$ (resp. $\mathcal{C}_{ci}$) the space of continuous $\mathcal{F}_t$-adapted and non-decreasing processes $(k_t)_{t\le T}$ such that $k_0 = 0$ and $I\!E[k_T^2] < \infty$ (resp. $k_T < \infty$, P-*a.s.*)

- for a stopping time τ, $\mathcal{T}_\tau$ denotes the set of stopping times θ such that $\theta \ge \tau$

- for a given *rcll* process $(w_t)_{t\le T}$, $w_{t-} = \lim_{s \nearrow t} w_s, t \le T$ $(w_{0-} = w_0)$; $w_- := (w_{t-})_{t\le T}$ and $\triangle w := w - w_-$ □

We are now given four objects:

- a terminal value $\xi \in L^2(\Omega, \mathcal{F}_T, P)$

- a map $f : \Omega \times [0,T] \times I\!R^{1+d} \times L^2(E, \mathcal{E}, \lambda; I\!R) \to I\!R$ which with (ω, t, y, z, v) associates $f(\omega, t, y, z, v)$, $\tilde{\mathcal{P}} \otimes \mathcal{B}(I\!R^{1+d}) \otimes \mathcal{B}(L^2(E, \mathcal{E}, \lambda; I\!R))$-measurable and satisfying :

(i) the process $(f(t, 0, 0, 0))_{t\le T}$ belongs to $\mathcal{H}^{2,1}$

(ii) f is uniformly Lipschitz with respect to (y, z, v), *i.e.*, there exists a constant $C \ge 0$ such that for any y, y', z, $z' \in I\!R$ and v, $v' \in L^2(E, \mathcal{E}, \lambda; I\!R)$,

$$P - a.s., \; |f(\omega, t, y, z, v) - f(\omega, t, y', z', v')| \le C(|y - y'| + |z - z'| + \|v - v'\|)$$

- two obstacles $L := (L_t)_{t\le T}$ and $U := (U_t)_{t\le T}$ which are $\mathcal{F}_t-$progressively measurable *rcll*, real valued processes satisfying $L_t \le U_t$, $\forall t \le T$ and $L_T \le \xi \le U_T$, P-a.s.. In addition they belong to $\mathcal{D}^2$, *i.e.*,

$$I\!E[\sup_{0\le t\le T} \{|L_t| + |U_t|\}^2] < \infty.$$

Besides we assume that their jumping times occur only at inaccessible stopping times which roughly speaking means that they are not predictable (see e.g. [2], pp.215 for the accurate definition). If this latter condition is not satisfied and especially if the upper (resp. lower) barrier U (resp. L) has positive (resp. negative) jumps then Y could have predictable jumps and the processes $K^\pm$ would be no longer continuous. Therefore the setting of the problem is not the same as in (1).

Let us now introduce our two barrier reflected BSDE with jumps associated with (f, ξ, L, U). A solution is a 5-uple $(Y, Z, V, K^+, K^-) := (Y_t, Z_t, V_t, K_t^+, K_t^-)_{t\le T}$ of processes with values in $I\!R^{1+d} \times L^2(E, \mathcal{E}, \lambda; I\!R) \times I\!R^+ \times I\!R^+$ such that:

$$\begin{cases} (i) & Y \in \mathcal{D}^2, \; Z \in \mathcal{H}^d, V \in \mathcal{L} \text{ and } K^\pm \in \mathcal{C}_{ci} \\ (ii) & Y_t = \xi + \int_t^T f(s, Y_s, Z_s, V_s) ds + (K_T^+ - K_t^+) - (K_T^- - K_t^-) \\ & \qquad - \int_t^T Z_s dB_s - \int_t^T \int_E V_s(e) \tilde{\mu}(ds, de) \,, \forall t \le T \\ (iii) & \forall t \le T, L_t \le Y_t \le U_t \text{ and } \int_0^T (Y_t - L_t) dK_t^+ = \int_0^T (Y_t - U_t) dK_t^- = 0. \end{cases} \tag{2}$$

Note that equation (2) has not a solution in general. Actually one can take $L = U$ with L not being a semimartingale, then obviously we cannot find a 5-uple which satisfies the relation (ii). □

2.1 BSDEs with one reflecting barrier

To begin with we recall the following result by Hamadène & Ouknine [13] related to reflected BSDEs with one upper barrier (in (2), $L \equiv -\infty$ and $K^+ = 0$) driven by a Brownian motion and an independent Poisson process.

Theorem 2.1 *: There exits a quadruple* $(Y, Z, K, V) := (Y_t, Z_t, K_t, V_t)_{t \le T}$ *of processes with values in* $I\!R^{1+d} \times I\!R^+ \times \mathcal{L}^2$ *which satisfies :*

$$\begin{cases} Y \in \mathcal{D}^2,\ Z \in \mathcal{H}^{2,d},\ K \in \mathcal{C}^2_{ci} \text{ and } V \in \mathcal{L}^2 \\ Y_t = \xi + \int_t^T f(s, Y_s, Z_s, V_s)ds - (K_T - K_t) - \int_t^T Z_s dB_s - \int_t^T \int_U V_s(e)\tilde{\mu}(ds, de),\ t \le T \\ \forall t \le T, Y_t \le U_t \text{ and } \int_0^T (U_t - Y_t)dK_t = 0.\ \square \end{cases} \tag{3}$$

In general we do not have a comparison result for solutions of BSDEs driven by a Brownian motion and an independent Poisson process, reflected or not (see e.g. [1] for a counter-example). However in some specific cases, when the coefficients have some features and especially when they do not depend on the variable v, we actually have comparison.

So let us give another pair (f', ξ') where $f' : (\omega, t, y, z, v) \longmapsto f'(\omega, t, y, z, v) \in I\!R$ and $\xi' \in L^2(\Omega, \mathcal{F}_T, P; I\!R)$. On the other hand, assume there exists a quadruple of processes (Y', Z', V', K') which belongs to $\mathcal{D}^2 \times \mathcal{H}^{2,d} \times \mathcal{L}^2 \times \mathcal{C}^2_{ci}$ and solution for the BSDE with one reflecting upper barrier associated with $(f'(\omega, t, y, z, v), \xi', U)$. Then we have:

Lemma 2.1 *: Assume that :*

(i) f *is independent of* v

(ii) P-a.s. for any $t \le T$, $f(t, Y'_t, Z'_t) \le f'(s, Y'_t, Z'_t, V'_t)$ *and* $\xi \le \xi'$.
Then P-a.s., $\forall t \le T$, $Y_t \le Y'_t$.

Proof: Let $X = (X_t)_{t \le T}$ be a *rcll* semi-martingale, then using Tanaka's formula with the function $(x^+)^2 = (\max\{x, 0\})^2$ reads:

$$(X_t^+)^2 = (X_T^+)^2 - 2\int_t^T X_{s-}^+ dX_s - \int_t^T 1_{[X_s > 0]} d < X^c, X^c >_s - \sum_{t < s \le T} \{(X_s^+)^2 - (X_{s-}^+)^2 - 2X_{s-}^+ \Delta X_s\}.$$

But the function $x \in I\!R \mapsto (x^+)^2$ is convex then $\{(X_s^+)^2 - (X_{s-}^+)^2 - 2X_{s-}^+ \Delta X_s\} \ge 0$. It implies that

$$(X_t^+)^2 + \int_t^T 1_{[X_s > 0]} d < X^c, X^c >_s \le (X_T^+)^2 - 2\int_t^T X_{s-}^+ dX_s.$$

Now using this formula with $Y - Y'$ yields:

$$((Y_t - Y'_t)^+)^2 + \int_t^T 1_{[Y_s - Y'_s > 0]} |Z_s - Z'_s|^2 ds \le -2\int_t^T (Y_{s-} - Y'_{s-})^+ d(Y_s - Y'_s).$$

But $\int_t^T (Y_{s-} - Y'_{s-})^+ d(K_s - K'_s) = \int_t^T (Y_s - Y'_s)^+ d(K_s - K'_s)$ and as usual this last term is non-negative since $(Y_s - Y'_s)^+ dK'_s = 0$ (indeed when $Y_t > Y'_t$ we cannot have $Y'_t = U_t$). Besides

$f(s,Y'_s,Z'_s) \leq f'(s,Y'_s,Z'_s,V'_s)$ and f is Lipschitz continuous then there exist bounded and $\mathcal{F}_t$-adapted processes $(a_s)_{s\leq T}$ and $(b_s)_{s\leq T}$ such that:

$$f(s,Y_s,Z_s) = f(s,Y'_s,Z'_s) + a_s(Y_s - Y'_s) + b_s(Z_s - Z'_s).$$

Therefore we have:

$$\begin{array}{l}((Y_t - Y'_t)^+)^2 + \int_t^T 1_{[Y_s - Y'_s > 0]}|Z_s - Z'_s|^2 ds \leq 2\int_t^T (Y_{s-} - Y'_{s-})^+\{a_s(Y_s - Y'_s) + b_s(Z_s - Z'_s)\}ds \\ \qquad -2\int_t^T (Y'_{s-} - Y_{s-})^+\{(Z_s - Z'_s)dB_s + \int_E V_s(e)\tilde{\mu}(ds,de)\}.\end{array}$$

Taking now expectation, using in an appropriate way the inequality $|a.b| \leq \epsilon|a|^2 + \epsilon^{-1}|b|^2$ ($\epsilon > 0$), and Gronwall's one we obtain $I\!\!E[((Y_t - Y'_t)^+)^2] = 0$ for any $t \leq T$. The result follows thoroughly since Y and Y' are *rcll*. $\Box$

2.2 Local solutions of BSDEs with two reflecting barriers

Throughout this part we assume that the function f does not depend on v. The main reason is, as pointed out in Lemma 2.1, in that case we can use comparison in order to deduce results for the two reflecting barrier BSDE associated with (f,ξ,L,U). Actually we have the following result related to the existence of *local* solutions for (2):

Theorem 2.2 *: There exists a unique $\mathcal{F}_t$-optional process $Y := (Y_t)_{t\leq T}$ such that:*

(i) $Y_T = \xi$ and P-a.s. for any $t \leq T$, $L_t \leq Y_t \leq U_t$

(ii) for any $\mathcal{F}_t$-stopping time τ there exists a quintuple $(\theta_\tau, Z^\tau, V^\tau, K^{\tau,+}, K^{\tau,-})$ which belongs to $\mathcal{T}_\tau \times \mathcal{H}^{2,d} \times \mathcal{L}^2 \times \mathcal{C}^2_{ci} \times \mathcal{C}^2_{ci}$ such that: P-a.s.,

$$E(f,\xi,L,U): \left\{\begin{array}{l} \forall t \in [\tau,\theta_\tau],\ Y_t = Y_{\theta_\tau} + \int_t^{\theta_\tau} f(s,Y_s,Z^\tau_s)ds + \int_t^{\theta_\tau} d(K^{\tau,+}_s - K^{\tau,-}_s) \\ \qquad\qquad\qquad -\int_t^{\theta_\tau} Z^\tau_s dB_s - \int_t^{\theta_\tau}\int_E V^\tau_s(e)\tilde{\mu}(ds,de), \\ \int_\tau^{\theta_\tau}(U_s - Y_s)dK^{\tau,-}_s = \int_\tau^{\theta_\tau}(Y_s - L_s)dK^{\tau,+}_s = 0 \end{array}\right. \tag{4}$$

(iii) if we set $\nu_\tau := \inf\{s \geq \tau, Y_s = U_s\} \wedge T$ and $\sigma_\tau := \inf\{s \geq \tau, Y_s = L_s\} \wedge T$ then $\nu_\tau \vee \sigma_\tau \leq \theta_\tau$, $Y_{\nu_\tau} = U_{\nu_\tau}$ on $[\nu_\tau < T]$ and $Y_{\sigma_\tau} = L_{\sigma_\tau}$ on $[\sigma_\tau < T]$.

Hereafter we call the process $Y := (Y_t)_{t\leq T}$ a local solution for the BSDE with two reflecting barriers associated with (f,ξ,L,U) which we denote $E(f,\xi,L,U)$.

Proof: Let us show uniqueness. Let Y and Y' be two $\mathcal{F}_t$-optional processes which satisfy $(i)-(iii)$. Then for any stopping time τ we have :

$$\begin{aligned} Y_{\sigma_\tau\wedge\nu'_\tau} &= Y_{\sigma_\tau}1_{[\sigma_\tau\leq\nu'_\tau]} + Y_{\nu'_\tau}1_{[\nu'_\tau<\sigma_\tau]} \\ &= Y_{\sigma_\tau}1_{[\sigma_\tau\leq\nu'_\tau]\cap[\sigma_\tau<T]} + Y_{\sigma_\tau}1_{[\sigma_\tau=\nu'_\tau=T]} + Y_{\nu'_\tau}1_{[\nu'_\tau<\sigma_\tau]} \\ &\leq L_{\sigma_\tau}1_{[\sigma_\tau\leq\nu'_\tau]\cap[\sigma_\tau<T]} + \xi 1_{[\sigma_\tau=\nu'_\tau=T]} + U_{\nu'_\tau}1_{[\nu'_\tau<\sigma_\tau]} \\ &= L_{\sigma_\tau}1_{[\sigma_\tau\leq\nu'_\tau]\cap[\sigma_\tau<T]} + \xi 1_{[\sigma_\tau=\nu'_\tau=T]} + Y'_{\nu'_\tau}1_{[\nu'_\tau<\sigma_\tau]} \\ &\leq Y'_{\sigma_\tau\wedge\nu'_\tau}. \end{aligned}$$

Besides P-$a.s.$ for any $t \in [\tau, \sigma_\tau \wedge \nu'_\tau]$ we have:

$$Y_t = Y_{\sigma_\tau \wedge \nu'_\tau} + \int_t^{\sigma_\tau \wedge \nu'_\tau} f(s, Y_s, Z_s^\tau) ds - \int_t^{\sigma_\tau \wedge \nu'_\tau} dK_s^{\tau,-} - \int_t^{\sigma_\tau \wedge \nu'_\tau} \{Z_s^\tau dB_s + \int_E V_s^\tau(e) \tilde{\mu}(ds, de)\}$$

and

$$Y'_t = Y'_{\sigma_\tau \wedge \nu'_\tau} + \int_t^{\sigma_\tau \wedge \nu'_\tau} f(s, Y'_s, Z_s'^\tau) ds + \int_t^{\sigma_\tau \wedge \nu'_\tau} dK_s'^{\tau,+} - \int_t^{\sigma_\tau \wedge \nu'_\tau} \{Z_s'^\tau dB_s + \int_E V_s'^\tau(e) \tilde{\mu}(ds, de)\}$$

since for $t \in [\tau, \sigma_\tau \wedge \nu'_\tau]$, $dK_t^{\tau,+} = 0$ and $dK_t'^{\tau,-} = 0$. Now arguing as in the proof of Lemma 2.1, for any $t \leq T$ we obtain:

$$\begin{aligned}
((Y_{(t\vee\tau)\wedge(\sigma_\tau\wedge\nu'_\tau)} - Y'_{(t\vee\tau)\wedge(\sigma_\tau\wedge\nu'_\tau)})^+)^2 &+ \int_{(t\vee\tau)\wedge(\sigma_\tau\wedge\nu'_\tau)}^{(\sigma_\tau\wedge\nu'_\tau)} 1_{[Y_s - Y'_s > 0]} |Z_s^\tau - Z_s'^\tau|^2 ds \\
&\leq -2 \int_{(t\vee\tau)\wedge(\sigma_\tau\wedge\nu'_\tau)}^{(\sigma_\tau\wedge\nu'_\tau)} (Y_s - Y'_s)^+ d(Y_s - Y'_s) \\
&\leq 2 \int_{(t\vee\tau)\wedge(\sigma_\tau\wedge\nu'_\tau)}^{(\sigma_\tau\wedge\nu'_\tau)} (Y_{s-} - Y'_{s-})^+ \{(f(s, Y_s, Z_s^\tau) - f(s, Y'_s, Z_s'^\tau)) ds - (dK_s^{\tau,-} + dK_s'^{\tau,+})\} \\
&\qquad + M_{(\sigma_\tau\wedge\nu'_\tau)} - M_{(t\vee\tau)\wedge(\sigma_\tau\wedge\nu'_\tau)}, \text{ where } M \text{ is a martingale.}
\end{aligned}$$

But $\int_{(t\vee\tau)\wedge(\sigma_\tau\wedge\nu'_\tau)}^{(\sigma_\tau\wedge\nu'_\tau)} (Y_s - Y'_s)^+ (dK_s^{\tau,-} + dK_s'^{\tau,+}) \geq 0$ then by the same lines as in the proof of Lemma 2.1, we get P-$a.s.$ for any $t \leq T$, $Y_{(t\vee\tau)\wedge(\sigma_\tau\wedge\nu'_\tau)} \leq Y'_{(t\vee\tau)\wedge(\sigma_\tau\wedge\nu'_\tau)}$. Taking now $t = 0$ to obtain $Y_\tau \leq Y'_\tau$. However in a symmetric way we have also $Y'_\tau \leq Y_\tau$ and then $Y'_\tau = Y_\tau$. Finally the optional section theorem ([2], pp.220) implies that $Y \equiv Y'$.

The proof of existence of Y will be obtained after Lemmas 2.2 & 2.4 below. So to begin with, for $n \geq 0$, let $(Y^n, Z^n, K^{n,-}, V^n)$ be the solution of the following reflected BSDE with just one upper barrier U (which exists according to Theorem 2.1):

$$\begin{cases}
(i)\ Y^n \in \mathcal{D}^2,\ Z^n \in \mathcal{H}^{2,d},\ K^{n,-} \in \mathcal{C}^2_{ci} \text{ and } V^n \in \mathcal{L}^2 \\
(ii)\ Y_t^n = \xi + \int_t^T f(s, Y_s^n, Z_s^n) ds + \int_t^T d(K_s^{n,+} - K_s^{n,-}) \\
\qquad\qquad - \int_t^T Z_s^n dB_s - \int_t^T \int_E V_s^n(e) \tilde{\mu}(ds, de),\ \forall\, t \leq T \\
(iii)\ Y^n \leq U \text{ and } \int_0^T (U_s - Y_s^n) dK_s^{n,-} = 0\ ;\ K_t^{n,+} := n \int_0^t (L_s - Y_s^n)^+ ds.
\end{cases} \tag{5}$$

Using comparison, we have for any $n \geq 0$, P-a.s., and for all $t \leq T$, $Y_t^n \leq Y_t^{n+1} \leq U_t$. Then for any $t \leq T$ let us set $Y_t := \lim_{n\to\infty} Y_t^n$. Therefore Y is an $\mathcal{F}_t$-optional process since Y^n is so and obviously we have P-$a.s.$, $\forall\, t \leq T$, $Y_t \leq U_t$. Besides, if we denote ${}^pX$ the predictable projection of a process X then for $t \leq T$, ${}^pY_t = \lim_{n\to\infty} {}^pY_t^n = \lim_{n\to\infty} Y_{t-}^n$. Indeed since the jumping times of Y^n are the same as the ones of $\int_0^t\int_E V_s^n(e)\tilde{\mu}(de, ds)$, then they are inaccessible and ${}^pY^n = Y_-^n$ ([3], pp.113).

Now for a stopping time τ, let $\nu_\tau^n := \inf\{s \geq \tau,\ Y_s^n = U_s\} \wedge T$. Since $Y^n \leq Y^{n+1}$ then the sequence $(\nu_\tau^n)_{n\geq 0}$ is non-increasing and converges to the stopping time $\delta_\tau := \lim_{n\to\infty} \nu_\tau^n$.

Lemma 2.2 : *The following properties hold true :*

(i) for any stopping time τ, $Y_{\delta_\tau} = U_{\delta_\tau} 1_{[\delta_\tau < T]} + \xi 1_{[\delta_\tau = T]}$.

(ii) P-a.s. for any $t \le T$, $U_t \ge Y_t \ge L_t$

(iii) for any stopping time τ, *there is a triple* $(\tilde{Z}^\tau, \tilde{V}^\tau, \tilde{K}^{\tau,+}) \in \mathcal{H}^{2,d} \times \mathcal{L}^2 \times \mathcal{C}^2_{ci}$ *such that: P-a.s.,*

$$\begin{cases} \forall t \in [\tau, \delta_\tau],\ Y_t = Y_{\delta_\tau} + \int_t^{\delta_\tau} f(s, Y_s, \tilde{Z}^\tau_s) ds + \int_t^{\delta_\tau} d\tilde{K}^{\tau,+}_s \\ \qquad\qquad - \int_t^{\delta_\tau} \tilde{Z}^\tau_s dB_s - \int_t^{\delta_\tau} \int_E \tilde{V}^\tau_s(e) \tilde{\mu}(ds, de), \\ \int_\tau^{\delta_\tau} (Y_s - L_s) d\tilde{K}^{\tau,+}_s = 0. \end{cases} \tag{6}$$

Proof: From equation (5) we have: P-*a.s.*, $\forall t \in [\tau, \nu^n_\tau]$

$$\begin{cases} Y^n_t = Y^n_{\nu^n_\tau} + \int_t^{\nu^n_\tau} f(s, Y^n_s, Z^n_s) ds + \int_t^{\nu^n_\tau} n(L_s - Y^n_s)^+ ds \\ \qquad - \int_t^{\nu^n_\tau} Z^n_s dB_s - \int_t^{\nu^n_\tau} \int_E V^n_s(e) \tilde{\mu}(ds, de) \end{cases} \tag{7}$$

since the process $K^{n,-}$ increases only when Y^n reaches the barrier U. Now for any $n \ge 0$, $Y^0 \le Y^n \le U$ then $\sup_{n \ge 0} I\!E[\sup_{t \le T} |Y^n_t|^2] < \infty$ since Y^0 and U belong to $\mathcal{D}^2$. Next using Itô's formula with $(Y^n)^2$ we get : for any $t \le T$,

$$\begin{aligned} (Y^n_{(t\vee\tau)\wedge\nu^n_\tau})^2 &+ \int_{(t\vee\tau)\wedge\nu^n_\tau}^{\nu^n_\tau} |Z^n_s|^2 ds + \sum_{(t\vee\tau)\wedge\nu^n_\tau < s \le \nu^n_\tau} (\Delta_s Y^n)^2 = (Y^n_{\nu^n_\tau})^2 + 2\int_{(t\vee\tau)\wedge\nu^n_\tau}^{\nu^n_\tau} Y^n_{s-} f(s, Y^n_s, Z^n_s) ds \\ &+2\int_{(t\vee\tau)\wedge\tau^n_\tau}^{\nu^n_\tau} Y^n_{s-} n(L_s - Y^n_s)^+ ds - 2\int_{(t\vee\tau)\wedge\nu^n_\tau}^{\nu^n_\tau} Y^n_{s-}\{Z^n_s dB_s + \int_E V^n_s(e)\tilde{\mu}(ds, de)\} \\ &\le (Y^n_{\nu^n_\tau})^2 + 2\int_{(t\vee\tau)\wedge\nu^n_\tau}^{\nu^n_\tau} Y^n_{s-} f(s, Y^n_s, Z^n_s) ds + \epsilon^{-1} \sup_{s \le T} |Y^n_s|^2 + \\ &\quad \epsilon\{\int_{(t\vee\tau)\wedge\nu^n_\tau}^{\nu^n_\tau} n(L_s - Y^n_s)^+ ds\}^2 - 2\int_{(t\vee\tau)\wedge\nu^n_\tau}^{\nu^n_\tau} Y^n_{s-}\{Z^n_s dB_s + \int_E V^n_s(e)\tilde{\mu}(ds, de)\} \end{aligned} \tag{8}$$

for any $\epsilon > 0$. But (7) implies the existence of a constat $C \ge 0$ such that for any $t \le T$ we have:

$$\begin{aligned} (\int_{(t\vee\tau)\wedge\nu^n_\tau}^{\nu^n_\tau} n(L_s - Y^n_s)^+ ds)^2 &\le C\{(Y^n_{(t\vee\tau)\wedge\nu^n_\tau})^2 + (Y^n_{\nu^n_\tau})^2 + (\int_{(t\vee\tau)\wedge\nu^n_\tau}^{\nu^n_\tau} f(s, Y^n_s, Z^n_s) ds)^2 \\ &+ (\int_{(t\vee\tau)\wedge\nu^n_\tau}^{\nu^n_\tau} Z^n_s dB_s - \int_{(t\vee\tau)\wedge\nu^n_\tau}^{\nu^n_\tau} \int_E V^n_s(e)\tilde{\mu}(ds, de))^2\}. \end{aligned} \tag{9}$$

Taking now expectation in both hand-sides, making use of the Burkholder-Davis-Gundy inequality (see e.g. [3], pp.304) and the estimate $\sup_{n\ge 0} I\!E[\sup_{t \le T} |Y^n_t|^2] < \infty$, and finally taking into account the fact that f is Lipschitz yield: $\forall t \le T$,

$$I\!E[(\int_{(t\vee\tau)\wedge\nu^n_\tau}^{\nu^n_\tau} n(L_s - Y^n_s)^+ ds)^2] \le C_1(1 + I\!E[\int_\tau^{\nu^n_\tau} \{|Z^n_s|^2 + \|V^n_s\|^2\} ds]) \tag{10}$$

for some constant C_1. Next taking expectation in (8), plug (10) in (8), and using the inequality $\forall \delta > 0$, $|Y^n_{s-} f(s, Y^n_s, Z^n_s)| \le \delta C_f |Z^n_s|^2 + C_f \delta^{-1} \sup_{s \le T}(|U_s| \vee |Y^0_s|)^2 + |f(s, 0, 0)| \sup_{s \le T}(|U_s| \vee |Y^0_s|)$

(C_f is the Lipschitz constant of f), we obtain by taking $t = 0$ and after an appropriate choice of ϵ and δ,

$$\forall n \geq 0,\ I\!\!E[\int_\tau^{\nu_\tau^n} (|Z_s^n|^2 + \|V_s^n\|^2)ds] \leq C_2,$$

for some constant C_2 independent of n, since $I\!\!E[\sum_{\tau<s\leq\nu_\tau^n}(\Delta_s Y^n)^2] = I\!\!E[\int_\tau^{\nu_\tau^n} \|V_s^n\|^2 ds]$. Henceforth there exists also a constant C such that for any $n \geq 0$,

$$I\!\!E[\int_\tau^{\nu_\tau^n} |f(s, Y_s^n, Z_s^n)|^2 ds + \{\int_\tau^{\nu_\tau^n} n(L_s - Y_s^n)^+ ds\}^2] \leq C. \tag{11}$$

But equation (7) implies that

$$I\!\!E[Y_{\delta_\tau}^n 1_{[\delta_\tau<T]}] \geq I\!\!E[Y_{\nu_\tau^n}^n 1_{[\delta_\tau<T]}] - \sup_{n\geq 0}\{I\!\!E[\int_\tau^{\nu_\tau^n} |f(s, Y_s^n, Z_s^n)|^2 ds]\}^{\frac{1}{2}} \sqrt{I\!\!E[\nu_\tau^n - \delta_\tau]}$$

since $n(Y_s^n - L_s)^+ \geq 0$. Taking now the limit as $n \to \infty$ we obtain $I\!\!E[Y_{\delta_\tau} 1_{[\delta_\tau<T]}] \geq I\!\!E[U_{\delta_\tau} 1_{[\delta_\tau<T]}]$ which implies that $Y_{\delta_\tau} = U_{\delta_\tau} 1_{[\delta_\tau<T]} + \xi 1_{[\delta_\tau=T]}$ since $Y \leq U$.

Let us now show that $Y \geq L$. For this let us consider the following BSDE: P-a.s., $\forall t \in [\tau, \delta_\tau]$

$$\left\{ \begin{array}{l} \bar{Y}_t^n = Y_{\delta_\tau}^n + \int_t^{\delta_\tau} f(s, Y_s^n, Z_s^n)ds + \int_t^{\delta_\tau} n(L_s - \bar{Y}_s^n)ds \\ \qquad\qquad - \int_t^{\delta_\tau} \bar{Z}_s^n dB_s - \int_t^{\delta_\tau}\int_E \bar{V}_s^n(e)\tilde{\mu}(ds, de). \end{array} \right. \tag{12}$$

Once again by comparison with (7) we have, $\forall n \geq 0$, $\bar{Y}_\tau^n \leq Y_\tau^n$. But

$$\bar{Y}_\tau^n = I\!\!E[Y_{\delta_\tau}^n e^{-n(\delta_\tau-\tau)} + \int_\tau^{\delta_\tau} e^{-n(s-\tau)}\{f(s, Y_s^n, Z_s^n) + nL_s\}ds|\mathcal{F}_\tau]$$

and then $(\bar{Y}_\tau^n)_{n\geq 0}$ converges to $Y_{\delta_\tau} 1_{[\delta_\tau=\tau]} + L_\tau 1_{[\delta_\tau>\tau]}$, P-a.s. Actually in $L^1(dP)$, $Y_{\delta_\tau}^n e^{-n(\delta_\tau-\tau)} \to Y_\tau 1_{[\delta_\tau=\tau]}$, $\int_\tau^{\delta_\tau} e^{-n(s-\tau)} f(s, Y_s^n, Z_s^n)ds \to 0$ through (11) and finally $\int_\tau^{\delta_\tau} e^{-n(s-\tau)} nL_s ds \to L_\tau 1_{[\delta_\tau>\tau]}$ since L is *rcll*. Therefore we have $Y_\tau = \lim_n Y_\tau^n \geq \lim_n \bar{Y}_\tau^n = Y_{\delta_\tau} 1_{[\delta_\tau=\tau]} + L_\tau 1_{[\delta_\tau>\tau]} \geq L_\tau$. As τ is a whatever stopping time then the optional section theorem (see e.g. [2], pp.220) implies that P-a.s., $\forall t \leq T,\ \ Y_t \geq L_t$.

Finally it remains to show *(iii)*. Let $(\tilde{Y}^\tau, \tilde{Z}^\tau, \tilde{V}^\tau, \tilde{K}^{\tau,+}) \in \mathcal{D}^2 \times \mathcal{H}^{2,d} \times \mathcal{L}^2 \times \mathcal{C}_{ci}^2$ solution of the following reflected BSDE: P-*a.s.*, $\forall t \in [\tau, \delta_\tau]$,

$$\left\{ \begin{array}{l} \tilde{Y}_t^\tau = Y_{\delta_\tau} + \int_t^{\delta_\tau} f(s, \tilde{Y}_s^\tau, \tilde{Z}_s^\tau)ds + \int_t^{\delta_\tau} d\tilde{K}_s^{\tau,+} \\ \qquad\qquad - \int_t^{\delta_\tau} \tilde{Z}_s^\tau dB_s - \int_t^{\delta_\tau}\int_E \tilde{V}_s^\tau(e)\tilde{\mu}(ds, de); \\ \int_\tau^{\delta_\tau} (\tilde{Y}_s^\tau - L_s)d\tilde{K}_s^{\tau,+} = 0 \text{ and } \forall t \in [\tau, \delta_\tau], \quad \tilde{Y}_t^\tau \geq L_t. \end{array} \right. \tag{13}$$

Write Itô's formula for the *rcll* process $|\tilde{Y}_t^\tau - Y_t^n|^2$ with $t \in [\tau, \delta_\tau]$, taking expectation and finally let $n \to \infty$ to obtain that P-*a.s.*, $\forall t \in [\tau, \delta_\tau]$, $\tilde{Y}_t^\tau = Y_t$. Actually let $C = C_f$, the Lipschitz constant of f, and let σ be a stopping time such that $\sigma \in [\tau, \delta_\tau]$, then :

$$\begin{aligned}
&I\!\!E[|\tilde{Y}^\tau_\sigma - Y^n_\sigma|^2 e^{2(C+C^2)\sigma} + \textstyle\sum_{\sigma<s\le\delta_\tau} e^{2(C+C^2)s}(\Delta_s(\tilde{Y}^\tau_s - Y^n_s))^2] = I\!\!E[|Y_{\delta_\tau} - Y^n_{\delta_\tau}|^2 e^{2(C+C^2)\delta_\tau}]\\
&\quad +2I\!\!E[\int_\sigma^{\delta_\tau} [f(s,\tilde{Y}^\tau_s,\tilde{Z}^\tau_s) - f(s,Y^n_s,Z^n_s) - (C+C^2)(\tilde{Y}^\tau_s - Y^n_s)](\tilde{Y}^\tau_s - Y^n_s)e^{2(C+C^2)s}ds]\\
&\quad -I\!\!E[\int_\sigma^{\delta_\tau} |\tilde{Z}^\tau_s - Z^n_s|^2 e^{2(C+C^2)s}ds] + 2I\!\!E[\int_\sigma^{\delta_\tau} e^{2(C+C^2)s}(\tilde{Y}^\tau_s - Y^n_s)d(\tilde{K}^{\tau,+}_s - K^{n,+}_s)]\\
&\le I\!\!E[|Y_{\delta_\tau} - Y^n_{\delta_\tau}|^2 e^{2(C+C^2)\delta_\tau}] + 2I\!\!E[\int_\sigma^{\delta_\tau} e^{2(C+C^2)s}(L_s - Y^n_s)d\tilde{K}^{\tau,+}_s].
\end{aligned}$$

Taking now the limit as $n \to \infty$ to obtain

$$I\!\!E[|\tilde{Y}^\tau_\sigma - Y_\sigma|^2 e^{2(C+C^2)\sigma}] \le 2I\!\!E[\int_\sigma^{\delta_\tau} e^{2(C+C^2)s}(L_s - Y_s)d\tilde{K}^{\tau,+}_s] \le 0$$

and the proof is complete. $\Box$

We now give the following technical result.

Lemma 2.3 *: Let κ be an inaccessible stopping time. Let $(\theta^n)_{n\ge0}$ be a non-decreasing sequence of stopping times uniformly bounded by T and let us set $\theta := \sup_{n\ge0}\theta^n$. Then $P(\cap_{n\ge0}[\theta^n < \theta] \cap [\theta = \kappa]) = 0$.*

Proof: Let K^p_t be the predictable dual projection of $K_t := 1_{[t\ge\kappa]}$, which is continuous since for all predictable stopping time τ we have $\triangle K^p_\tau = I\!\!E[\triangle K_\tau|\mathcal{F}_{\tau-}]$ (see e.g. [3], pp.149-150), then $\triangle K^p_\tau = I\!\!E[1_{[\tau=\kappa]}/\mathcal{F}_{\tau-}] = 0$. But the process $(1_{]\theta^n,\theta]}(s))_{s\le T}$ is predictable, then we have

$$P([\theta^n < \kappa \le \theta]) = I\!\!E[K_\theta] - I\!\!E[K_{\theta^n}] = I\!\!E[K^p_\theta] - I\!\!E[K^p_{\theta^n}] \to 0 \text{ as } n \to \infty.$$

Finally to obtain the result it enough to remark that:

$$P(\cap_{n\ge0}[\theta^n < \theta] \cap [\theta = \kappa]) = \lim_{n\to\infty} P([\theta^n < \theta] \cap [\theta = \kappa]) \le \lim_{n\to\infty} P([\theta^n < \kappa \le \theta]) = 0.\Box$$

Now let $\theta^n_\tau := \inf\{s \ge \delta_\tau, Y^n_s \le L_s\} \wedge T$. Since $Y^n \le Y^{n+1}$ then the sequence of stopping times $(\theta^n_\tau)_{n\ge0}$ is non-decreasing and converges to $\theta_\tau := \lim_{n\to\infty}\theta^n_\tau$.

Lemma 2.4 *: We have the following properties:*

(i) P-a.s., for any $t \in [\delta_\tau, \theta_\tau]$, $Y^n_{t\wedge\theta^n_\tau} \to Y_t$ as $n \to \infty$
(ii) P-a.s., $Y_{\theta_\tau} = L_{\theta_\tau}1_{[\theta_\tau<T]} + \xi 1_{[\theta_\tau=T]}$
(iii) for all stopping time τ there is $(\bar{Z}^\tau, \bar{V}^\tau, \bar{K}^{\tau,-}) \in \mathcal{H}^{2,d} \times \mathcal{L}^2 \times \mathcal{C}^2_{ci}$ such that :

$$\begin{cases} Y_t = Y_{\theta_\tau} + \displaystyle\int_t^{\theta_\tau} f(s,Y_s,\bar{Z}^\tau_s)ds - \int_t^{\theta_\tau} d\bar{K}^{\tau,-}_s \\ \qquad\qquad - \displaystyle\int_t^{\theta_\tau} \bar{Z}^\tau_s dB_s - \int_t^{\theta_\tau}\int_E \bar{V}^\tau_s(e)\tilde{\mu}(ds,de), \ \forall t \in [\delta_\tau, \theta_\tau] \\ \displaystyle\int_{\delta_\tau}^{\theta_\tau} (Y_s - U_s)d\bar{K}^{\tau,-}_s = 0. \end{cases} \tag{14}$$

Proof: First let us show (i) and (ii). On the event $\{\delta_\tau \leq t < \theta_\tau\}$ and for n large enough we have $Y^n_{t\wedge\theta^n_\tau} = Y^n_t \to Y_t$ as $n \to \infty$ since on that event for n large we have $\theta^n_\tau > t$. On the other hand (we take $t = \theta_\tau$) $\lim_{n\to\infty} Y^n_{\theta_\tau\wedge\theta^n_\tau} = (\lim_{n\to\infty} Y^n_{\theta^n_\tau})1_{\cap_n[\theta^n_\tau<\theta_\tau]} + Y_{\theta_\tau}1_{\cup_n[\theta^n_\tau=\theta_\tau]}$, provided that $\lim_{n\to\infty} Y^n_{\theta^n_\tau}$ exists on the event $\cap_n[\theta^n_\tau < \theta_\tau]$. But on that event, for all $n \geq j$, $Y^j_{\theta^n_\tau} \leq Y^n_{\theta^n_\tau} \leq L_{\theta^n_\tau}$ since we have also $[\theta^n_\tau < T]$. Therefore taking the limit as n tends to $+\infty$ to get:

$$Y^j_{\theta_\tau-} = Y^j_{\theta_\tau} - \triangle Y^j_{\theta_\tau} \leq \liminf_{n\to\infty} Y^n_{\theta^n_\tau} \leq \limsup_{n\to\infty} Y^n_{\theta^n_\tau} \leq L_{\theta_\tau} - \triangle L_{\theta_\tau} = L_{\theta_\tau-}.$$

Now since the jumping times of Y^j and L are inaccessible and for any inaccessible time κ we have $P(\cap_n[\theta^n_\tau < \theta_\tau]\cap[\theta_\tau = \kappa]) = 0$ (through Lemma 2.3) then $\triangle Y^j_{\theta_\tau} = \triangle L_{\theta_\tau} = 0$ on $\cap_n[\theta^n_\tau < \theta_\tau]$. Taking the limit as $j \to \infty$ to obtain

$$Y_{\theta_\tau} \leq \liminf_{n\to\infty} Y^n_{\theta^n_\tau} \leq \limsup_{n\to\infty} Y^n_{\theta^n_\tau} \leq L_{\theta_\tau} \text{ on } \cap_n [\theta^n_\tau < \theta_\tau].$$

As $Y \geq L$ then on the event $\cap_n[\theta^n_\tau < \theta_\tau]$ we have :

$$Y_{\theta_\tau} = \lim_{n\to\infty} Y^n_{\theta^n_\tau} \text{ and } Y_{\theta_\tau} = L_{\theta_\tau}$$

and (i) follows thoroughly. Let us remark that this equality implies also that on $\cap_n[\theta^n_\tau < \theta_\tau] \cap [\theta_\tau = T]$ we necessarily have $L_T = \xi$.

Now let us show (ii). First we have $Y_{\theta_\tau} = L_{\theta_\tau}$ on the event $\cap_n[\theta^n_\tau < \theta_\tau]$. Next let $\omega \in \cup_n[\theta^n_\tau = \theta_\tau < T]$, then there exists $n(\omega)$ such that $\forall j \geq 0$, $\theta^{n(\omega)+j}_\tau(\omega) = \theta_\tau(\omega)$. It follows that for all $j \geq 0$

$$Y^j_{\theta_\tau(\omega)}(\omega) \leq Y^{n(\omega)+j}_{\theta_\tau(\omega)}(\omega) = Y^{n(\omega)+j}_{\theta^{n(\omega)+j}_\tau(\omega)}(\omega) \leq L_{\theta^{n(\omega)+j}_\tau(\omega)}(\omega) = L_{\theta_\tau}(\omega).$$

Taking the limit as $j \to \infty$ and taking into account that $Y \geq L$ to obtain

$$Y_{\theta_\tau} = L_{\theta_\tau} \text{ on } \cup_n [\theta^n_\tau = \theta_\tau < T].$$

Finally if $\omega \in \cup_n[\theta^n_\tau = \theta_\tau = T]$ then once again there is $n(\omega)$ such that for any $n \geq n(\omega)$, $\theta^n_\tau(\omega) = T$. Therefore for any $n \geq n(\omega)$, $Y^n_{\theta^n_\tau}(\omega) = \xi(\omega) = Y_{\theta_\tau}(\omega)$. Summarizing all of that and taking into account the remark above to obtain:

$$\begin{aligned} Y_{\theta_\tau} &= L_{\theta_\tau}1_{(\cap_n[\theta^n_\tau<\theta_\tau])\cap[\theta_\tau<T]} + \xi 1_{(\cap_n[\theta^n_\tau<\theta_\tau])\cap[\theta_\tau=T]} + L_{\theta_\tau}1_{(\cup_n[\theta^n_\tau=\theta_\tau])\cap[\theta_\tau<T]} + \xi 1_{(\cup_n[\theta^n_\tau=\theta_\tau])\cap[\theta_\tau=T]} \\ &= L_{\theta_\tau}1_{[\theta_\tau<T]} + \xi 1_{[\theta_\tau=T]}, \end{aligned}$$

which is the desired result.

(iii) Let $(\bar{Y}^\tau, \bar{Z}^\tau, \bar{V}^\tau, \bar{K}^{\tau,-}) \in \mathcal{D}^2 \times \mathcal{H}^{2,d} \times \mathcal{L}^2 \times \mathcal{C}^2_{ci}$ such that: P-$a.s.$,

$$\begin{cases} (i)\ \forall t \in [\delta_\tau, \theta_\tau], \bar{Y}^\tau_t = Y_{\theta_\tau} + \int_t^{\theta_\tau} f(s, \bar{Y}^\tau_s, \bar{Z}^\tau_s)ds - \int_t^{\theta_\tau} d\bar{K}^{\tau,-}_s \\ \qquad - \int_t^{\theta_\tau} \bar{Z}^\tau_s dB_s - \int_t^{\theta_\tau}\int_E \bar{V}^\tau_s(e)\tilde{\mu}(ds, de), \\ (ii)\ \int_{\delta_\tau}^{\theta_\tau} (\bar{Y}^\tau_s - U_s)d\bar{K}^{\tau,-}_s = 0 \text{ and } \forall t \in [\delta_\tau, \theta_\tau],\quad \bar{Y}^\tau_t \leq U_t. \end{cases} \tag{15}$$

Write Itô's formula for the process $|\bar{Y}_t^\tau - Y^n_{t\wedge\theta_\tau^n}|^2 e^{2(C+C^2)(t\wedge\theta_\tau^n)}$ with $t\in[\delta_\tau,\theta_\tau]$ (C is the Lipschitz constant of f), taking $t=\sigma$ where σ is a stopping time such that $\delta_\tau\le\sigma\le\theta_\tau$ and finally taking expectation in both hand-sides to obtain:

$$\begin{aligned}
&I\!\!E[|\bar{Y}_\sigma^\tau - Y^n_{\sigma\wedge\theta_\tau^n}|^2 e^{2(C+C^2)(\sigma\wedge\theta_\tau^n)} + \textstyle\sum_{\sigma\wedge\theta_\tau^n<s\le\theta_\tau} e^{2(C+C^2)s\wedge\theta_\tau^n}(\Delta_s\bar{Y}^\tau - \Delta_{s\wedge\theta_\tau^n}Y^n)^2] = \\
&\quad I\!\!E[|Y_{\theta_\tau} - Y^n_{\theta_\tau^n}|^2 e^{2(C+C^2)\theta_\tau^n}] - 2I\!\!E[\int_\sigma^{\theta_\tau} e^{2(C+C^2)(s\wedge\theta_\tau^n)}(\bar{Y}_s^\tau - Y^n_{s\wedge\theta_\tau^n})(d\bar{K}_s^{\tau,-} - 1_{[s<\theta_\tau^n]}dK_s^{n,-})] \\
&\quad -2(C+C^2)I\!\!E[\int_\sigma^{\theta_\tau}(\bar{Y}_s^\tau - Y^n_{s\wedge\theta_\tau^n})^2 e^{2(C+C^2)s\wedge\theta_\tau^n}ds] - I\!\!E[\int_\sigma^{\theta_\tau}|\bar{Z}_s^\tau - 1_{[s<\theta_\tau^n]}Z_s^n|^2 e^{2(C+C^2)(s\wedge\theta_\tau^n)}ds] \\
&\quad +2I\!\!E[\int_\sigma^{\theta_\tau}[f(s,\bar{Y}_s^\tau,\bar{Z}_s^\tau) - 1_{[s<\theta_\tau^n]}f(s,Y_s^n,Z_s^n)](\bar{Y}_s^\tau - Y^n_{s\wedge\theta_\tau^n})e^{2(C+C^2)(s\wedge\theta_\tau^n)}ds] \\
&\le I\!\!E[|Y_{\theta_\tau} - Y^n_{\theta_\tau^n}|^2 e^{2(C+C^2)\theta_\tau^n}] - 2I\!\!E[\int_\sigma^{\theta_\tau} e^{2(C+C^2)\theta_\tau^n}1_{[s>\theta_\tau^n]}(U_s - U_{\theta_\tau^n})d\bar{K}_s^{\tau,-}] \\
&\qquad\qquad +2I\!\!E[\int_\sigma^{\theta_\tau}[1_{[s>\theta_\tau^n]}f(s,\bar{Y}_s^\tau,\bar{Z}_s^\tau)](\bar{Y}_s^\tau - Y^n_{\theta_\tau^n})e^{2(C+C^2)\theta_\tau^n}ds].
\end{aligned}$$

Now taking the limit as $n\to\infty$ we obtain $I\!\!E[|\bar{Y}_\sigma^\tau - Y_\sigma|^2 e^{2(C+C^2)\sigma}]\le 0$. Henceforth $Y_t = \bar{Y}_t^\tau, \forall t\in[\delta_\tau,\theta_\tau]$ and (iii) is proved .□

Let us now construct the processes Z^τ and $K^{\tau,\pm}$ of the theorem and show that Y satisfies the equation of $E(f,\xi,L,U)$.

Let τ be a stopping time and, δ_τ, θ_τ the stopping times constructed as previously. There exist triples of processes $(\tilde{Z}^\tau,\tilde{V}^\tau,\tilde{K}^{\tau,+})$ (resp. $(\bar{Z}^\tau,\bar{V}^\tau,\bar{K}^{\tau,-})$ which belongs to $\mathcal{H}^{2,d}\times\mathcal{L}^2\times\mathcal{C}^2_{ci}$ and which with the process Y satisfies (6) (resp. (14)). So for any $t\le T$ let us set :

$$\begin{array}{l}
Z_t^\tau := \tilde{Z}_t^\tau 1_{[t\le\delta\tau]} + \bar{Z}_t^\tau 1_{[\delta_\tau<t\le\theta_\tau]},\ V_t^\tau := \tilde{V}_t^\tau 1_{[t\le\delta\tau]} + \bar{V}_t^\tau 1_{[\delta_\tau<t\le\theta_\tau]} \\
K_t^{\tau,-} := \bar{K}^{\tau,-}_{(t\vee\delta_\tau)\wedge\theta_\tau} - \bar{K}^{\tau,-}_{\delta_\tau} \text{ and } K_t^{\tau,+} := (\tilde{K}^{\tau,+}_{t\wedge\delta_\tau} - \tilde{K}^{\tau,+}_\tau)1_{[\tau\le t]}.
\end{array}$$

Now since $K_t^{\tau,-}=0$ for $t\le\delta_\tau$ then from (6) and (14) we easily deduce that the 5-uple $(Y_t, Z_t^\tau, V_t^\tau, K_\tau^{\tau,+}, K_t^{\tau,-})_{t\le T}$ satisfies (4).

Finally taking into account the facts that $\theta_\tau\ge\delta_\tau$, $Y_{\theta_\tau}1_{[\theta_\tau<T]} = L_{\theta_\tau}1_{[\theta_\tau<T]}$, $Y_{\delta_\tau}1_{[\delta_\tau<T]} = U_{\delta_\tau}1_{[\delta_\tau<T]}$ we deduce that $\nu_\tau\vee\sigma_\tau\le\theta_\tau$ and $Y_{\nu_\tau}=U_{\nu_\tau}$ on $[\nu_\tau<T]$ and $Y_{\sigma_\tau}=L_{\sigma_\tau}$ on $[\sigma_\tau<T]$. Actually if $\theta_\tau=T$ then $\nu_\tau\vee\sigma_\tau\le\theta_\tau$ and Y is *rcll* in $[\tau,T]$ since it satisfies (4). Therefore if $\nu_\tau<T$ (resp. $\sigma_\tau<T$) then $Y_{\nu_\tau}=U_{\nu_\tau}$ (resp. $Y_{\sigma_\tau}=L_{\sigma_\tau}$). On the other hand assume that $\theta_\tau<T$. Then once again from (4) Y is *rcll* in $[\tau,\theta_\tau]$ and $Y_{\delta_\tau}=U_{\delta_\tau}$, $Y_{\theta_\tau}=L_{\theta_\tau}$. It follows that $\nu_\tau\le\delta_\tau$, $\sigma_\tau\le\theta_\tau$ and $Y_{\nu_\tau}=U_{\nu_\tau}$ and $Y_{\sigma_\tau}=L_{\sigma_\tau}$. Thus the proof of the theorem is complete. □

We now focus on some other regularity properties of the process $Y=(Y_t)_{t\le T}$ constructed in Theorem 2.2.

Proposition 2.1 *: The process Y is rcll. Moreover if $(Y^n)_{n\ge0}$ is the sequence of processes constructed in (5) then $I\!\!E[\sup_{t\le T}|Y_t^n - Y_t|^2]\to 0$ as $n\to\infty$.*

Proof: First let us show that Y is *rcll*. To begin with let us point out that Y is a limit of an increasing sequence $(Y^n)_{n\ge0}$ of *rcll* processes. On the other hand according to Theorem 2.2 there exists an $\mathcal{F}_t$-optional process $\hat{Y}:=(\hat{Y}_t)_{t\le T}$ such that:

(i) $-U \leq \hat{Y} \leq -L$ and $\hat{Y}_T = -\xi$

(ii) for any stopping time τ there exists $(\hat{\theta}_\tau, \hat{Z}^\tau, \hat{V}^\tau, \hat{K}^{\tau,+}, \hat{K}^{\tau,-})$ such that: P-$a.s.$,

$$\begin{cases} \forall t \in [\tau, \hat{\theta}_\tau],\ \hat{Y}_t = \hat{Y}_{\hat{\theta}_\tau} - \int_t^{\hat{\theta}_\tau} f(s, -\hat{Y}_s, -\hat{Z}_s^\tau)ds + \int_t^{\hat{\theta}_\tau} d(\hat{K}_s^{\tau,+} - \hat{K}_s^{\tau,-}) \\ \qquad\qquad\qquad\qquad - \int_t^{\hat{\theta}_\tau} \hat{Z}_s^\tau dB_s - \int_t^{\hat{\theta}_\tau} \int_E \hat{V}_s^\tau(e)\tilde{\mu}(ds, de), \\ \int_\tau^{\hat{\theta}_\tau} (U_s + \hat{Y}_s) d\hat{K}_s^{\tau,+} = \int_\tau^{\hat{\theta}_\tau} (\hat{Y}_s + L_s) d\hat{K}_s^{\tau,-} = 0 \end{cases} \tag{16}$$

(iii) if we set $\hat{\nu}_\tau := \inf\{s \geq \tau, \hat{Y}_s = -L_s\} \wedge T$ and $\hat{\sigma}_\tau := \inf\{s \geq \tau, Y_s = -U_s\} \wedge T$ then $\hat{\nu}_\tau \vee \hat{\sigma}_\tau \leq \hat{\theta}_\tau$, $\hat{Y}_{\tilde{\nu}_\tau} = -L_{\hat{\nu}_\tau}$ on $[\hat{\nu}_\tau < T]$ and $\hat{Y}_{\hat{\sigma}_\tau} = -U_{\hat{\sigma}_\tau}$ on $[\hat{\sigma}_\tau < T]$.

But uniqueness of the process which satisfies $(i) - (iii)$ implies that $-Y = \hat{Y}$ and then Y is also a limit of a decreasing sequence $(\hat{Y}^n)_{n \geq 0}$ of *rcll* processes. Therefore Y is right continuous.

Next for $t \geq 0$ and $n \geq 0$, $Y_t^n \leq Y_t$ then $Y_{t-}^n \leq \liminf_{s \nearrow t} Y_s$ since Y^n is *rcll*. On the other hand $Y_{t-}^n = ({}^pY^n)_t \to {}^pY_t$ as $n \to \infty$. It follows that ${}^pY_t \leq \liminf_{s \nearrow t} Y_s$. Now considering $\hat{Y}^n$ instead of Y^n we obtain that ${}^pY_t \geq \limsup_{s \nearrow t} Y_s$ and then ${}^pY_t = \limsup_{s \nearrow t} Y_s = \liminf_{s \nearrow t} Y_s = Y_{t-}$. Therefore Y is *rcll*. In addition $Y_t^n \nearrow Y_t$ and $Y_{t-}^n \nearrow Y_{t-}$ then a weak version of Dini's Theorem (see e.g. [3], pp.202) implies that $\sup_{t \leq T}(Y_t^n - Y_t)^2 \to 0$ as $n \to \infty$. Finally the dominated convergence theorem yields the desired result .□

3 Connection with Dynkin games

Let us consider a process $g := (g_s)_{s \leq T}$ which belongs to $\mathcal{H}^{2,1}$ and τ a stopping time. The Dynkin game on $[\tau, T]$ associated with (g, ξ, L, U) is a zero-sum game on stopping times where the payoff after τ is given by:

$$\Gamma_\tau(\nu, \sigma) := I\!\!E[\int_\tau^{\nu \wedge \sigma} g_s ds + L_\sigma 1_{[\sigma \leq \nu < T]} + U_\nu 1_{[\nu < \sigma]} + \xi 1_{[\nu = \sigma = T]} | \mathcal{F}_\tau],\ \forall \nu,\ \sigma \in \mathcal{T}_\tau.$$

Dynkin games arise naturally when two agents a_1 and a_2, whose advantages are antagonistic, act on a system up to the time when one of them decides to stop its intervention. In the literature there were many works on Dynkin games (see e.g. [4, 16, 22] and the references therein). In mathematical finance, American game options are typically Dynkin games (see e.g. [5], [9], [15]).

The value function of the Dynkin game on $[\tau, T]$ is an $(\mathcal{F}_t)_{t \leq T}$-adapted process $(Y_t)_{t \in [\tau, T]}$ such that P$-a.s.$,

$$\forall t \in [\tau, T],\ \ Y_t = \text{essinf}_{\nu \in \mathcal{T}_t} \text{esssup}_{\sigma \in \mathcal{T}_t} \Gamma_t(\nu, \sigma) = \text{esssup}_{\sigma \in \mathcal{T}_t} \text{essinf}_{\nu \in \mathcal{T}_t} \Gamma_t(\nu, \sigma).$$

In that case, the random variable Y_τ is just called the value of the game on $[\tau, T]$. Besides a couple of stopping times (ν_τ, σ_τ) which belongs to $\mathcal{T}_\tau \times \mathcal{T}_\tau$ and which satisfies

$$\Gamma_\tau(\nu_\tau, \sigma) \leq \Gamma_\tau(\nu_\tau, \sigma_\tau) \leq \Gamma_\tau(\nu, \sigma_\tau), \quad \forall \nu, \sigma \in \mathcal{T}_\tau$$

is called a *saddle-point* for the Dynkin game on $[\tau, T]$.

Now let $(Y, \theta_\tau, Z^\tau, V^\tau, K^{\tau,+}, K^{\tau,-})$ be a solution of $E(\xi, g, L, U)$. Let ν_τ, σ_τ be the stopping times defined as:

$$\nu_\tau := \inf\{s \geq \tau, Y_s = U_s\} \wedge T \text{ and } \sigma_\tau := \inf\{s \geq \tau, Y_s = L_s\} \wedge T.$$

In the following we show that (ν_τ, σ_τ) is a saddle-point for the game. This result is out of the scope of the known ones on this subject (see e.g. [16] which is the most general paper related to Dynkin games when the strategies are only stopping times) since the process L (resp. U) may have a negative (resp. positive) jump (see Example 3.1 below).

Theorem 3.1 *It holds true that:*
(i) $Y_\tau = \Gamma_\tau(\nu_\tau, \sigma_\tau)$
(ii) $\Gamma_\tau(\nu_\tau, \sigma) \leq Y_\tau \leq \Gamma_\tau(\nu, \sigma_\tau)$ *for any* $\nu, \sigma \in \mathcal{T}_\tau$.
Therefore Y_τ *is the value of the Dynkin game on* $[\tau, T]$ *and* (ν_τ, σ_τ) *is a saddle-point for the game after* τ.

Proof: Since P-*a.s.*, $\max\{\nu_\tau, \sigma_\tau\} \leq \theta_\tau$, then we have:

$$\begin{aligned} Y_\tau = Y_{\nu_\tau\wedge\sigma_\tau} &+ \int_\tau^{\nu_\tau\wedge\sigma_\tau} g(s)ds + (K^{\tau,+}_{\nu_\tau\wedge\sigma_\tau} - K^{\tau,+}_\tau) - (K^{\tau,-}_{\nu_\tau\wedge\sigma_\tau} - K^{\tau,-}_\tau) \\ &- \int_\tau^{\nu_\tau\wedge\sigma_\tau} Z^\tau_s dB_s - \int_\tau^{\nu_\tau\wedge\sigma_\tau}\int_E V^\tau_s(e)\tilde{\mu}(de, ds). \end{aligned} \tag{17}$$

But $\int_\tau^{\theta_\tau}(Y_s - L_s)dK^{\tau,+}_s = \int_\tau^{\theta_\tau}(U_s - Y_s)dK^{\tau,-}_s = 0$ therefore $K^{\tau,+}_{\nu_\tau\wedge\sigma_\tau} - K^{\tau,+}_\tau = 0$ and $K^{\tau,-}_{\nu_\tau\wedge\sigma_\tau} - K^{\tau,-}_\tau = 0$. Besides we have:

$$\begin{aligned} Y_{\nu_\tau\wedge\sigma_\tau} &= Y_{\sigma_\tau}1_{[\sigma_\tau\leq\nu_\tau<T]} + Y_{\nu_\tau}1_{[\nu_\tau<\sigma_\tau]} + \xi 1_{[\nu_\tau=\sigma_\tau=T]} \\ &= L_{\sigma_\tau}1_{[\sigma_\tau\leq\nu_\tau<T]} + U_{\nu_\tau}1_{[\nu_\tau<\sigma_\tau]} + \xi 1_{[\nu_\tau=\sigma_\tau=T]} \end{aligned}$$

since P-*a.s.*, $Y_{\nu_\tau} = U_{\nu_\tau}$ on $[\nu_\tau < T]$ and $Y_{\sigma_\tau} = L_{\sigma_\tau}$ on $[\sigma_\tau < T]$. It follows that

$$Y_\tau = I\!E[\int_\tau^{\nu_\tau\wedge\sigma_\tau} g(s)ds + L_{\sigma_\tau}1_{[\sigma_\tau\leq\nu_\tau<T]} + U_{\nu_\tau}1_{[\nu_\tau<\sigma_\tau]} + \xi 1_{[\nu_\tau=\sigma_\tau=T]}|\mathcal{F}_\tau] = \Gamma_\tau(\nu_\tau, \sigma_\tau)$$

after taking the conditional expectation in (17).

Next let ν be a stopping time of $\mathcal{T}_\tau$. Since $\nu \wedge \sigma_\tau \leq \theta_\tau$ then

$$Y_\tau = Y_{\nu\wedge\sigma_\tau} + \int_\tau^{\nu\wedge\sigma_\tau} g(s)ds + (K^{\tau,+}_{\nu\wedge\sigma_\tau} - K^{\tau,+}_\tau) - (K^{\tau,-}_{\nu\wedge\sigma_\tau} - K^{\tau,-}_\tau) - \int_\tau^{\nu\wedge\sigma_\tau} Z^\tau_s dB_s - \int_\tau^{\nu\wedge\sigma_\tau}\int_E V^\tau_s(e)\tilde{\mu}(de, ds).$$

But $K^{\tau,+}_{\nu\wedge\sigma_\tau} - K^{\tau,+}_\tau = 0$ and $K^{\tau,-}_{\nu\wedge\sigma_\tau} - K^{\tau,-}_\tau \geq 0$ therefore we have :

$$Y_\tau \leq Y_{\nu\wedge\sigma_\tau} + \int_\tau^{\nu\wedge\sigma_\tau} g(s)ds - \int_\tau^{\nu\wedge\sigma_\tau} Z^\tau_s dB_s - \int_\tau^{\nu\wedge\sigma_\tau}\int_E V^\tau_s(e)\tilde{\mu}(de, ds).$$

As

$$\begin{aligned} Y_{\nu\wedge\sigma_\tau} &= Y_{\sigma_\tau}1_{[\sigma_\tau\leq\nu<T]} + Y_\nu 1_{[\nu<\sigma_\tau]} + \xi 1_{[\nu=\sigma_\tau=T]} \\ &\leq L_{\sigma_\tau}1_{[\sigma_\tau\leq\nu<T]} + U_\nu 1_{[\nu<\sigma_\tau]} + \xi 1_{[\nu=\sigma_\tau=T]}, \end{aligned}$$

then, after taking the conditional expectation, we obtain

$$Y_\tau \leq I\!E[\int_\tau^{\nu\wedge\sigma_\tau} g(s)ds + L_{\sigma_\tau}1_{[\sigma_\tau\leq\nu<T]} + U_\nu 1_{[\nu<\sigma_\tau]} + \xi 1_{[\nu=\sigma_\tau=T]}|\mathcal{F}_\tau] = \Gamma_\tau(\nu,\sigma_\tau).$$

In the same way we can show that:

$$Y_\tau \geq I\!E[\int_\tau^{\nu_\tau\wedge\sigma} g(s)ds + L_\sigma 1_{[\sigma\leq\nu_\tau<T]} + U_{\nu_\tau}1_{[\nu_\tau<\sigma]} + \xi 1_{[\nu_\tau=\sigma=T]}|\mathcal{F}_\tau] = \Gamma_\tau(\nu_\tau,\sigma).$$

Thus we have $\Gamma_\tau(\nu_\tau,\sigma) \leq Y_\tau \leq \Gamma_\tau(\nu,\sigma_\tau)$ which implies that:

$$\text{essinf}_{\nu\in\mathcal{T}_\tau}\text{esssup}_{\sigma\in\mathcal{T}_\tau}\Gamma_\tau(\nu,\sigma) \leq Y_\tau \leq \text{esssup}_{\sigma\in\mathcal{T}_\tau}\text{essinf}_{\nu\in\mathcal{T}_\tau}\Gamma_\tau(\nu,\sigma). \quad (18)$$

Therefore we have equalities instead of inequalities since the left-hand side is greater than the right-hand one. It follows that Y_τ is the value of the Dynkin game on $[\tau,T]$. □

Example 3.1 *Assume that* $E = I\!R - \{0\}$, $\nu(dt,de) = dt\frac{1}{2}1_{[-1,1]}(e)de$ *and for any* $t \leq T$, $U_t = |B_t| + \int_0^t\int_E e\mu(dt,de)$, $L_t = \frac{1}{2}|B_t| + \min\{0, \int_0^t\int_E e\mu(dt,de)\}$ *and* $\xi = \frac{U_T+L_T}{2}$. *So the processes U and L have negative and positive jumps since their laws are the same as the ones of* $|B_t| + \sum_{n\geq1} X_n 1_{[T_1+\ldots+T_n\leq t]}$ *and* $\frac{1}{2}|B_t| + \min\{0, \sum_{n\geq1} X_n 1_{[T_1+\ldots+T_n\leq t]}\}$ *respectively, where* $(T_n)_{n\geq1}$ *(resp.* $(X_n)_{n\geq1}$*) is a sequence of i.i.d. random variables whose law is exponential with parameter 1 (resp. uniform on* $[-1,1]$*). We suppose also that they are independent of the Brownian motion.*

Theorem 3.1 implies that the zero-sum Dynkin game associated with (L,U) has a saddle-point since the processes L and U satisfy the requirements. Actually they are square integrable and their jumps occur only in inaccessible stopping times. Now on the ground of the result by Lepeltier & Maingueneau [16] we cannot infer that such a saddle point exists since U (resp. L) has positive (resp. negative) jumps. In [16], the authors show that a saddle-point for the game exists solely if U (resp. L) has only negative (resp. positive) jumps. □

4 Reflected BSDEs with a general coefficient f

Let us recall that for general barriers L and U, equation (2) may not have a solution. Therefore in order to obtain a solution we are led to assume more regularity assumptions, especially on L and U. So in this section we are going to study under which conditions as weak as possible and easy to verify, the BSDE (2) has a solution. To begin with assume that the following hypothesis, called *Mokobodski's condition*, is fulfilled.

[M] : There exit two non-negative supermartingales of $\mathcal{D}^2, h := (h_t)_{t\leq T}$ and $h' := (h'_t)_{t\leq T}$ such that $L_t \leq h_t - h'_t \leq U_t,\ \forall t \leq T$.

Then we have:

Proposition 4.1 *: Assume that [**M**] is fulfilled and the mapping* f *does not depend on* (y,z,v), *i.e.,* $f(t,y,z,v) \equiv f(t)$, *then the two barrier reflected BSDE (2) has a solution* (Y,Z,V,K^+,K^-) *in the space* $\mathcal{D}^2 \times \mathcal{H}^{2,d} \times \mathcal{L}^2 \times (\mathcal{C}^2_{ci})^2$. *Furthermore if* (Y',Z',V',K'^+,K'^-) *is another solution for (2) then* $Y = Y'$, $Z = Z'$, $V = V'$ *and* $K^+ - K^- = K'^+ - K'^-$.

Proof: In its main steps, the proof is classical (see e.g. [4] or [11]). First let us recall that a process $A := (A_t)_{t\leq T}$ is called of class [D] if the set of random variables $\{A_\tau, \tau \in \mathcal{T}_0\}$ is uniformly integrable.

Now for a general process X of $\mathcal{D}^2$, let us denote $R(X) := (R(X)_t)_{t\leq T}$ its Snell envelope which is defined by:

$$R(X)_t = esssup_{\tau\in\mathcal{T}_t} I\!\!E[X_\tau|\mathcal{F}_t],\ \forall t \leq T\ ;$$

$R(X)$ is the smallest *rcll* supermartingale of class [D] such that P-*a.s.*, $R(X) \geq X$ (see e.g. [3], pp.431 or [7], pp.126).

Next let us consider the following processes defined by: $\forall t \leq T$,

$$H_t = (h_t + I\!\!E[\xi^-|\mathcal{F}_t])1_{[t<T]} + I\!\!E[\int_t^T f(s)^- ds|\mathcal{F}_t],\ \Theta_t = (h'_t + I\!\!E[\xi^+|\mathcal{F}_t])1_{[t<T]} + I\!\!E[\int_t^T f(s)^+ ds|\mathcal{F}_t],$$
$$\tilde{L}_t = L_t 1_{[t<T]} + \xi 1_{[t=T]} - I\!\!E[\xi + \int_t^T f(s)ds|\mathcal{F}_t] \text{ and } \tilde{U}_t = U_t 1_{[t<T]} + \xi 1_{[t=T]} - I\!\!E[\xi + \int_t^T f(s)ds|\mathcal{F}_t]$$

where $\xi^+ = max(\xi, 0)$, $\xi^- = max(-\xi, 0)$ and the same holds for $f(s)^-$ and $f(s)^+$. Since h and h' are non-negative supermartingales then H and Θ are also non-negative supermartingales which moreover belong to $\mathcal{D}^2$ and verify $H_T = \Theta_T = 0$. On the other hand, through **[M]**, we can easily verify that for any $t \leq T$ we have:

$$\tilde{L}_t \leq H_t - \Theta_t \leq \tilde{U}_t. \tag{19}$$

Next let us consider the sequences $(N_n^\pm)_{n\geq 0}$ of processes defined recursively as follows:

$$N_0^\pm = 0 \text{ and for } n \geq 0, N_{n+1}^+ = R(N_n^- + \tilde{L}) \text{ and } N_{n+1}^- = R(N_n^+ - \tilde{U}).$$

By induction and in using (19) we can easily verify that:

$$\forall n \geq 0,\ 0 \leq N_n^+ \leq N_{n+1}^+ \leq H \text{ and } 0 \leq N_n^- \leq N_{n+1}^- \leq \Theta.$$

It follows that the sequence $(N_n^+)_{n\geq 0}$ (resp. $(N_n^-)_{n\geq 0}$) converges pointwisely to a supermartingale N^+ (resp. N^-) (see e.g. [14], pp.21). In addition N^+ and N^- belong to $\mathcal{D}^2$ and verify (see e.g. [4], pp.2055) :

$$N^+ = R(N^- + \tilde{L}) \text{ and } N^- = R(N^+ - \tilde{U}).$$

Next the Doob-Meyer decompositions of $N^\pm$ yield :

$$\forall t \leq T, N_t^\pm = M_t^\pm - K_t^\pm$$

where $M^\pm$ are *rcll* martingales and $K^\pm$ non-decreasing processes such that $K_0^\pm = 0$. Moreover since $N^\pm \in \mathcal{D}^2$ then $I\!\!E[(K_T^\pm)^2] < \infty$ (see e.g. [3], pp.221). Therefore $M^\pm$ are also elements of $\mathcal{D}^2$ and then there exist processes $Z^\pm \in \mathcal{H}^{2,d}$ and $V^\pm \in \mathcal{L}^2$ such that:

$$\forall t \leq T, M_t^\pm = M_0^\pm + \int_0^t \{Z_s^\pm dB_s + \int_E V_s^\pm(e)\tilde{\mu}(ds, de)\}.$$

Let us now show that $K^{+,d} \equiv K^{-,d}$ where $K^{\pm,d}$ are the purely discontinuous part of $K^\pm$. It is well known from the Snell envelope theory that $K^{\pm,d}$ are predictable and thus if τ is a predictable stopping time we have ([7], pp.131)

$$\{\Delta K_\tau^{+,d} > 0\} \subset \{N_{\tau-}^+ = N_{\tau-}^- + \tilde{L}_{\tau-}\} \cap \{\Delta_\tau N^- < 0\} \cap \{-\Delta_\tau N^+ \leq -\Delta_\tau N^-\}$$

and

$$\{\Delta K_\tau^{-,d} > 0\} \subset \{N_{\tau-}^- = N_{\tau-}^+ - \tilde{U}_{\tau-}\} \cap \{\Delta_\tau N^+ < 0\} \cap \{-\Delta_\tau N^- \le -\Delta_\tau N^+\}.$$

But $\Delta K_\tau^{+,d} = \Delta N_\tau^+$ and $\Delta K_\tau^{-,d} = \Delta N_\tau^+$. Then those inclusions imply that the predictable jumps of N^+ and N^- occur in the same time and they are equal. It follows that $\Delta_\tau K^{+,d} = \Delta_\tau K^{-,d}$ for any predictable stopping time, *i.e.*, $K^{+,d} \equiv K^{-,d}$.

Let us now show that $\int_0^T (N_t^+ - N_t^- - \tilde{L}_t) dK_t^{+,c} = 0$ where $K^{+,c}$ is the continuous part of K^+. Actually for any $t \le T$, $N_t^+ = M_t^+ - K_t^{+,c} - K_t^{+,d}$ then $N_t^+ + K_t^{+,d} = M_t^+ - K_t^{+,c}$. Thereby $(N_t^+ + K_t^{+,d})_{t \le T}$ is a supermartingale which belongs to $\mathcal{D}^2$. More than that, we have also $N^+ + K^{+,d} = R(N^- + K^{+,d} + \tilde{L})$. Indeed $N_t^+ + K^{+,d} \ge N^- + K^{+,d} + \tilde{L}$. Now since $N^+ + K^{+,d}$ is a supermartingale of $\mathcal{D}^2$ then $N^+ + K^{+,d} \ge R(N^- + K^{+,d} + \tilde{L})$. On the other hand, let X be a *rcll* supermartingale of class [D] such that $X \ge N^- + K^{+,d} - \tilde{L}$ then $X - K^{+,d} \ge N^- + \tilde{L}$ which implies that $X - K^{+,d} \ge N^+$ since $X - K^{+,d}$ is a supermartingale of class [D]. Therefore $X \ge N^+ + K^{+,d}$ and then $N^+ + K^{+,d}$ is the smallest supermartingale of class [D] which dominates $N^- + K^{+,d} + \tilde{L}$, *i.e.*, $N^+ + K^{+,d} = R(N^- + K^{+,d} + \tilde{L})$.

Next for any $t \le T$ let $\tau_t = \inf\{s \ge t, K_s^{+,c} > K_t^{+,c}\} \wedge T$. Since $(N^+ + K^{-,d})^p = (M^+ - K^{+,c})^p = M_-^+ - K^{+,c} = N_-^+ + K_-^{-,d}$ then the Snell envelope $N^+ + K^{-,d}$ is regular, therefore τ_t is the largest optimal stopping time after t (see e.g. [7], pp.140). It implies that $(N^+ + K^{-,d})_{\tau_t} = (N^- + K^{-,d} + \tilde{L})_{\tau_t}$, and then for any $s \in [t, \tau_t]$ we have $(N_s^+ - N_s^- - \tilde{L}_s) dK_s^{+,c} = 0$. It follows that for any $s \in [0, T]$ we have $(N_s^+ - N_s^- - \tilde{L}_s) dK_s^{+,c} = 0$, *i.e.*, $\int_0^T (N_s^+ - N_s^- - \tilde{L}_s) dK_s^{+,c} = 0$. In the same way we can show that $\int_0^T (N_s^- - N_s^+ + \tilde{U}_s) dK_s^{+,c} = 0$.

Let us now set:

$$\forall t \le T,\ Y_t = N_t^+ - N_t^- + I\!E[\xi + \int_t^T f(s) ds | \mathcal{F}_t],\ Z_t = Z_t^+ - Z_t^- + \eta_t,\ V_t = V_t^+ - V_t^- + \rho_t$$

where the processes η and ρ are elements of $\mathcal{H}^{2,d}$ and $\mathcal{L}^2$ respectively and verify:

$$I\!E[\xi + \int_0^T f(s) ds | \mathcal{F}_t] = I\!E[\xi + \int_0^T f(s) ds] + \int_0^t \{\eta_s dB_s + \int_E \rho_s(e) \tilde{\mu}(ds, de)\},\ \forall t \le T.$$

Then we can easily check that $(Y, Z, V, K^{+,c}, K^{-,c})$ belongs to $\mathcal{D}^2 \times \mathcal{H}^{2,d} \times \mathcal{L}^2 \times (\mathcal{C}_{ci}^2)^2$ and is a solution for the BSDE with two reflecting barriers associated with $(f(t), \xi, L, U)$.

Let us now deal with the issue of uniqueness. Let $(Y', Z', V', K'^+, K'^-) \in \mathcal{D}^2 \times \mathcal{H}^d \times \mathcal{L} \times (\mathcal{C}_{ci})^2$ be another solution of the reflected BSDE associated with $(f(t), \xi, L, U)$. Since there is a lack of integrability of the processes (Z', V', K'^+, K'^-) we are led to introduce the following stopping times:

$$\forall p \ge 0,\ \alpha_p := \inf\{t \in [0, T], \int_0^t \{|Z_s'|^2 + \|V_s'\|^2\} ds > p\} \wedge T.$$

Then the sequence $(\alpha_p)_{p \ge 0}$ is non-decreasing and converges to T. Moreover it is of stationary type i.e. for any $\omega \in \Omega$, there is $p_0(\omega)$ such that $\alpha_p(\omega) = T$ for $p \ge p_0(\omega)$. Using now Itô's formula with $(Y_{t\wedge\alpha_p}-Y'_{t\wedge\alpha_p})^2$ for $t\le T$, yields:

$$\begin{array}{l}(Y_{t\wedge\alpha_p}-Y'_{t\wedge\alpha_p})^2+\int_{t\wedge\alpha_p}^{\alpha_p}|Z_s-Z'_s|^2ds+\sum_{t\wedge\alpha_p<s\le\alpha_p}(\Delta(Y-Y')_s)^2= \\ \qquad (Y_{\alpha_p}-Y'_{\alpha_p})^2+2\int_{t\wedge\alpha_p}^{\alpha_p}(Y_s-Y'_s)(dK_s^+-dK_s^--dK_s'^++dK_s'^-) \\ \qquad\quad -2\int_{t\wedge\alpha_p}^{\alpha_p}(Y_{s-}-Y'_{s-})\{(Z_s-Z'_s)dB_s+\int_E(V_s(e)-V'_s(e))\tilde{\mu}(ds,de)\}\end{array}$$

But $(Y_s-Y'_s)(dK_s^+-dK_s^--dK_s'^++dK_s'^-)\le 0$ and taking the expectation in the two hand-sides yield:

$$I\!\!E[(Y_{t\wedge\alpha_p}-Y'_{t\wedge\alpha_p})^2+\int_{t\wedge\alpha_p}^{\alpha_p}|Z_s-Z'_s|^2ds+\int_{t\wedge\alpha_p}^{\alpha_p}\int_E|V_s(e)-V'_s(e)|^2ds\lambda(de)]\le I\!\!E[(Y_{\alpha_p}-Y'_{\alpha_p})^2].$$

Using now Fatou's Lemma and Lebesgue dominated convergence theorem w.r.t. p we deduce that $Y=Y'$, $Z=Z'$ and $V=V'$. Thus we have also $K_t^+-K_t^-=K_t'^+-K_t'^-$ for any $t\le T$.

Remark 4.1 *On the uniqueness of the processes K^+ and K^-.*
The process $K^\pm$ can be chosen in a unique way if we moreover require that the measures dK^+ and dK^- are singular. Indeed dK^+-dK^- is a signed measure which has a unique decomposition into $d\lambda^+-d\lambda^-$, i.e.,

$$dK^+-dK^-=d\lambda^+-d\lambda^-$$

where λ^+ and λ^- are non-negative singular measures. Therefore $dK^++d\lambda^-=dK^-+d\lambda^+$ and then $d\lambda^+\le dK^+$ and $d\lambda^-\le dK^-$. Henceforth we have $\lambda^+(ds)=a_sdK_s^+$ and $\lambda^-(ds)=b_sdK_s^-$. It follows that $(Y_t-L_t)\lambda^+(dt)=(Y_t-L_t)a_tdK_t^+=0$. In the same way we have $(U_t-Y_t)\lambda^-(ds)=0$. Whence the claim. Finally let us point out that when $L_t<U_t$ for any $t<T$ then K^+ and K^- are singular and then they are unique.□

We now deal with the BSDE (2) when the mapping f depends also on (y,z,v) and the barriers L and U satisfy the assumption **[M]**.

Theorem 4.1 *: Assume that the barriers L and U satisfy the assumption **[M]**, then the reflected BSDE (2) associated with $(f(t,y,z,v),\xi,L,U)$ has a solution (Y,Z,V,K^+,K^-) which belongs to $\mathcal{D}^2\times\mathcal{H}^{2,d}\times\mathcal{L}^2\times(\mathcal{C}^2_{ci})^2$. Furthermore if (Y',Z',V',K'^+,K'^-) is another solution for (2) then $Y=Y'$, $Z=Z'$, $V=V'$ and $K^+-K^-=K'^+-K'^-$. The processes $K^\pm$ can be chosen singular and then they are unique.*

Proof: We give a brief proof since it is classical. Let $\mathcal{H}:=\mathcal{D}^2\times\mathcal{H}^{2,d}\times\mathcal{L}^2$ and Φ be the following application:

$$\begin{array}{ll}\Phi: & \mathcal{H}\longrightarrow\mathcal{H} \\ (y,z,v): & \mapsto\Phi(y,z,v)=(\bar{Y},\bar{Z},\bar{V})\end{array}$$

where $(\bar{Y},\bar{Z},\bar{V})$ is the triple for which there exists two other processes $\bar{K}^\pm$ which belong to $\mathcal{C}^2_{ci}$ such that $(\bar{Y},\bar{Z},\bar{V},\bar{K}^+,\bar{K}^-)$ is a solution for the BSDE with two reflecting barriers associated with $(f(t,y_t,z_t,v_t),\xi,L,U)$ (which exists thanks to Proposition 4.1). Now let $\alpha>0$, $(y',v',z')\in$

$\mathcal{H}$ and $(\bar{Y}', \bar{Z}', \bar{V}') = \Phi(y', z', v')$. Using Itô's formula and taking into account that $e^{\alpha s}(\bar{Y}_s - \bar{Y}'_s)d(\bar{K}^+_s - \bar{K}^-_s - \bar{K}'^+_s + \bar{K}'^-_s) \leq 0$ we can show the existence of a constant $\bar{C} < 1$ (see e.g. [11] or [13]) such that:

$$\begin{aligned} I\!E[\int_0^T e^{\alpha s}\{(\bar{Y}_s - \bar{Y}'_s)^2 + |\bar{Z}_s - \bar{Z}'_s|^2 + \int_E |\bar{V}_s(e) - \bar{V}'_s(e)|^2 \lambda(de)\}ds] \\ \leq \bar{C} I\!E[\int_0^T e^{\alpha s}\{|y_s - y'_s|^2 + |z_s - z'_s|^2 + \|v_s - v'_s\|^2\}ds]. \end{aligned}$$

Therefore Φ is a contraction and then has a unique fixed point (Y, Z, V) which actually belongs to $\mathcal{H}$. Moreover there exists $K^{\pm} \in \mathcal{C}^2_{ci}$ ($K^{\pm}_0 = 0$) such that (Y, Z, V, K^+, K^-) is solution for the reflected BSDE associated with (f, ξ, L, U). Finally a word about uniqueness. Let (Y', Z', V', K'^+, K'^-) be another solution for (2). Once more since there is a lack of integrability of the processes (Z', V', K'^+, K'^-), we can argue as the in the proof of uniqueness of Proposition 4.1 to obtain that $Y = Y'$, $Z = Z'$, $V = V'$ and $K^+ - K^- = K'^+ - K'^-$. As shown in Remark 4.1, the processes $K^{\pm}$ can be chosen singular and then they are unique. □

Remark 4.2 *As a by-product of Theorem 4.1, we deduce that assumption [M] is satisfied if and only if the BSDE (2) has a solution (Y, Z, V, K^+, K^-) which belongs to $\mathcal{D}^2 \times \mathcal{H}^{2,d} \times \mathcal{L}^2 \times (\mathcal{C}^2_{ci})^2$. The proof of the reverse inequality is obtained in splitting ξ into ξ^+ and ξ^- and so on.* □

Let us now consider the following condition which is weaker than Mokobodski's one.

[WM]: There exists a sequence $(\gamma_k)_{k \geq 0}$ of stopping times such that:

(i) for any $k \geq 0$, $\gamma_k \leq \gamma_{k+1}$ and $P[\gamma_k < T, \forall k \geq 0] = 0$ ($\gamma_0 = 0$)

(ii) for any $k \geq 0$ there exists a pair (h^k, h'^k) of non-negative supermartingales which belong to $\mathcal{D}^2$ such that:

$$\text{P-}a.s., \forall t \leq \gamma_k,\ L_t \leq h^k_t - h'^k_t \leq U_t.$$

We are going to show that the reflected BSDE (2) has a solution iff **[WM]** is satisfied.

Theorem 4.2 *: The reflected BSDE (2) has a solution iff [**WM**] is satisfied. In addition if $(Y', Z', V', K'^{\pm})$ is another solution then $Y = Y'$, $Z = Z'$, $V = V'$ and $K^+ - K^- = K'^+ - K'^-$.*

Proof: The condition is necessary. Indeed assume that (2) has a solution (Y, Z, V, K^+, K^-). For $k \geq 0$ let us set:

$$\gamma_k := \{s \geq 0, K^+_s + K^-_s \geq k\} \wedge T.$$

Therefore $\gamma_k \leq \gamma_{k+1}$. On the other hand if the event $A = \{\omega, \gamma_k < T, \forall k \geq 0\}$ is such that $P(A) > 0$ then $K^+_T + K^-_T = \infty$ on A which is contradictory. Therefore $P(A) = 0$. Finally for $k \geq 0$ and $t \leq T$ let us set:

$$\begin{aligned} h^k_t &= I\!E[Y^+_{\gamma_k} + \int_t^{\gamma_k} f(s, Y_s, Z_s, V_s)^+ ds + (K^+_{\gamma_k} - K^+_t)|\mathcal{F}_t] \text{ and} \\ h'^k_t &= I\!E[Y^-_{\gamma_k} + \int_t^{\gamma_k} f(s, Y_s, Z_s, V_s)^- ds + (K^-_{\gamma_k} - K^-_t)|\mathcal{F}_t] \end{aligned}$$

then h^k, h'^k belong to $\mathcal{D}^2$ since $I\!\!E[\int_0^{\gamma_k} ds\{|Z_s|^2 + \int_E |V_s(e)|^2\lambda(de)\}] < \infty$ and $\forall t \le \gamma_k$, $L_t \le h_t^k - h_t'^k \le U_t$.

Let us show that the condition is sufficient. It will be divided in two steps.

Step 1: *Assume that the mapping f does not depend on (z,v), i.e., $f(t,y,z,v) \equiv f(t,y)$. Then there exists a 5-uple of processes (Y,Z,V,K^+,K^-) solution of the reflected BSDE (2). In addition for any $k \ge 0$ we have:*

$$I\!\!E[\int_0^{\gamma_k} ds\{|Z_s|^2 + \int_E |V_s(e)|^2\lambda(de)\} + (K_{\gamma_k}^+)^2 + (K_{\gamma_k}^-)^2] < \infty. \tag{20}$$

Let $(Y_t)_{t\le T}$ be the solution of $E(f,\xi,L,U)$ defined in Theorem 2.2. Therefore for any $k \ge 0$, $(Y_{t\wedge\gamma_k})_{t\le T}$ is the solution of $I\!\!E[f1_{[t\le\gamma_k]}, Y_{\gamma_k}, L_{t\wedge\gamma_k}, U_{t\wedge\gamma_k}]$. Now let $(Y^k, Z^k, V^k, K^{k,+}, K^{k,-}) \in \mathcal{D}^2 \times \mathcal{H}^{2,d} \times \mathcal{L}^2 \times (\mathcal{C}_{ci}^2)^2$ be the solution of the BSDE associated with $(f1_{[t\le\gamma_k]}, Y_{\gamma_k}, L_{t\wedge\gamma_k}, U_{t\wedge\gamma_k})$ which exists according to Proposition 4.1. Henceforth Y^k is also the solution of $I\!\!E[f1_{[t\le\gamma_k]}, Y_{\gamma_k}, L_{t\wedge\gamma_k}, U_{t\wedge\gamma_k}]$. Now uniqueness implies that for any $k \ge 0$ and $t \le T$ we have $Y_{t\wedge\gamma_k} = Y_{t\wedge\gamma_k}^k$ and then $Y_{t\wedge\gamma_k}^{k+1} = Y_{t\wedge\gamma_k}^k$, $\forall t \le T$. It implies that for any $k \ge 0$ and $t \le T$, we have:

$$Y_{t\wedge\gamma_k} = Y_{\gamma_k} + \int_{t\wedge\gamma_k}^{\gamma_k} f(s,Y_s)ds + (K_{\gamma_k}^{k,+} - K_{t\wedge\gamma_k}^{k,+}) - (K_{\gamma_k}^{k,-} - K_{t\wedge\gamma_k}^{k,-}) - \int_{t\wedge\gamma_k}^{\gamma_k} Z_s^k dB_s - \int_{t\wedge\gamma_k}^{\gamma_k}\int_E V_s^k(e)\tilde{\mu}(ds,de). \tag{21}$$

On the other hand, through uniqueness we get: for any $t \le T$,

$Z_t^{k+1}1_{[t\le\gamma_k]} = Z_t^k 1_{[t\le\gamma_k]}$, $V_t^{k+1}1_{[t\le\gamma_k]} = V_t^k 1_{[t\le\gamma_k]}$ and $(K^{k+1,+} - K^{k+1,-})_{t\wedge\gamma_k} = (K^{k,+} - K^{k,-})_{t\wedge\gamma_k}$.

Now let us set for $t \le T$,

$$K_t^{\pm}(\omega) = \begin{cases} K_t^{0,\pm}(\omega) \text{ if } t \in [0,\gamma_0(\omega)] \\ K_{\gamma_{k-1}}^{\pm}(\omega) + (K_t^{k,\pm} - K_{\gamma_{k-1}}^{k,\pm})) \text{ if } t \in]\gamma_{k-1}(\omega), \gamma_k(\omega)] \end{cases}$$
$$Z_t(\omega) = Z_t^0(\omega)1_{[0,\gamma_0(\omega)]}(t) + \textstyle\sum_{k\ge1} Z_t^k(\omega)1_{]\gamma_{k-1}(\omega),\gamma_k(\omega)]}(t) \text{ and}$$
$$V_t(\omega) = V_t^0(\omega)1_{[0,\gamma_0(\omega)]}(t) + \textstyle\sum_{k\ge1} V_t^k(\omega)1_{]\gamma_{k-1}(\omega),\gamma_k(\omega)]}(t).$$

Therefore $(Y,Z,V,K^{\pm}) \in \mathcal{D}^2 \times \mathcal{H}^d \times \mathcal{L} \times \mathcal{C}_{ci}$ through the properties satisfied by Z^k, V^k and $K^{k,\pm}$ and since the sequence of stopping times $(\gamma_k)_{k\ge0}$ is of stationary type. In addition for any $t \le T$, $L_t \le Y_t \le U_t$. Next let us show that $(Y_s - L_s)dK_s^+ = 0$. Let ω be fixed. There exists $k_0(\omega)$ such that $\gamma_{k_0}(\omega) = T$. Then

$$\int_0^T (Y_s(\omega) - L_s(\omega))dK_s^+(\omega) = \sum_{k=1,k_0(\omega)} \int_{\gamma_{k-1}}^{\gamma_k} (Y_s - L_s)dK_s^+(\omega) = \sum_{k=1,k_0(\omega)} \int_{\gamma_{k-1}}^{\gamma_k} (Y_s - L_s)dK_s^{k,+}(\omega) = 0.$$

In the same way we have $(U_s - Y_s)dK_s^- = 0$. Finally let us show that the processes (Y,Z,V,K^+,K^-) verify the equation of (2). For $t = T$ the equation is obviously satisfied. Now let $t < T$. From (21) and the definitions of $K^{\pm}$ and Z, for any $k \ge 0$ we have :

$$Y_{t\wedge\gamma_k}(\omega) = Y_{\gamma_k}(\omega) + \int_{t\wedge\gamma_k}^{\gamma_k} f(s,Y_s)ds + (K_{\gamma_k}^+ - K_{t\wedge\gamma_k}^+) - (K_{\gamma_k}^- - K_{t\wedge\gamma_k}^-) -$$
$$\int_{t\wedge\gamma_k}^{\gamma_k} Z_s dB_s - \int_{t\wedge\gamma_k}^{\gamma_k}\int_E V_s(e)\tilde{\mu}(ds,de).$$

As the sequence $(\gamma_k)_{k\geq 0}$ is of stationary type then for k great enough we have $\gamma_k(\omega) = T$. Therefore

$$Y_t(\omega) = \xi(\omega) + \int_t^T f(s, Y_s)(\omega)ds + (K_T^+ - K_t^+)(\omega) - (K_T^- - K_t^-)(\omega) - \int_t^T Z_s dB_s(\omega) - \int_t^T \int_E V_s(e)\tilde{\mu}(ds, de)(\omega).$$

Finally, by construction, we have (20). On the other hand if $(Y', Z', V', K^{',+}, K^{',-})$ is another solution then applying Itô's formula for $(Y_{t\wedge\gamma_k} - Y'_{t\wedge\gamma_k})^2$ taking the expectation and the limit as $k \to \infty$ we obtain $Y = Y'$, $Z = Z'$, $V = V'$ and $K^+ - K^- = K'^+ - K'^-$ $\square$

Step 2: *The general case of the coefficient* f, *i.e, when it depends on* (z, v).

Let us consider the following scheme: $(Z^0, V^0) = (0, 0)$ and for all $j \geq 1$,

$$\begin{cases} Y_t^j = \xi + \int_t^T 1_{[s\leq\gamma_j]} f(s, Y_s^j, Z_s^{j-1}, V_s^{j-1})ds + \int_t^T d(K_s^{j,+} - K_s^{j,-}) \\ \qquad\qquad - \int_t^T Z_s^j dB_s - \int_t^T \int_E V_s^j(e)\tilde{\mu}(ds, de), \quad \forall t \in [0, T] \\ \forall t \leq T,\ L_t \leq Y_t^j \leq U_t \text{ and } (U_t - Y_t^j)dK_t^{-,j} = (L_t - Y_t^j)dK_t^{+,j} = 0,\ P - a.s. \\ \forall k \geq 0,\ \mathbb{E}[\int_0^{\gamma^k} (|Z_s^j|^2 + \|V_s^j\|^2)ds] < \infty. \end{cases} \tag{22}$$

In this definition the indicator $1_{[.\leq\gamma_j]}$ is in place in order to have a coefficient which belongs to $\mathcal{H}^{2,1}$ through (20).

Let $i, j, k \in N$ such that $k \leq i \leq j$. Using Itô's formula we obtain: for any $\alpha \in \mathbb{R}$ and $t \leq T$,

$$\begin{aligned} e^{\alpha(t\wedge\gamma_k)}(Y^i_{t\wedge\gamma_k} - Y^j_{t\wedge\gamma_k})^2 + \int_{t\wedge\gamma_k}^{\gamma_k} e^{\alpha s}|Z_s^i - Z_s^j|^2 ds + \alpha \int_{t\wedge\gamma_k}^{\gamma_k} e^{\alpha s}|Y_s^i - Y_s^j|^2 ds + \sum_{t\wedge\gamma_k < s \leq \gamma_k} e^{\alpha s}\Delta_s(Y^i - Y^j)^2 \\ \leq e^{\alpha\gamma_k}(Y^i_{\gamma_k} - Y^j_{\gamma_k})^2 + 2\int_{t\wedge\gamma_k}^{\gamma_k} e^{\alpha s}(Y_s^i - Y_s^j)(f(s, Y_s^i, Z_s^{i-1}, V_s^{i-1}) - f(s, Y_s^j, Z_s^{j-1}, V_s^{j-1}))ds \\ -2\int_{t\wedge\gamma_k}^{\gamma_k} e^{\alpha s}(Y^i_{s-} - Y^j_{s-})\{(Z_s^i - Z_s^j)dB_s + \int_E (V_s^i(e) - V_s^j(e))\tilde{\mu}(ds, de)\}. \end{aligned} \tag{23}$$

Therefore for α great enough, in taking the expectation in each hand-side we obtain:

$$\begin{aligned} \mathbb{E}[\int_0^{\gamma_k} e^{\alpha s}|Z_s^i - Z_s^j|^2 ds \quad + \int_0^{\gamma_k} e^{\alpha s}\|V_s^i - V_s^j\|^2 ds] \leq \mathbb{E}[e^{\alpha\gamma_k}(Y^i_{\gamma_k} - Y^j_{\gamma_k})^2] \\ + C\epsilon\mathbb{E}[\int_0^{\gamma_k} e^{\alpha s}|Z_s^{i-1} - Z_s^{j-1}|^2 ds + \int_0^{\gamma_k} e^{\alpha s}\|V_s^{i-1} - V_s^{j-1}\|^2 ds] \end{aligned}$$

where $C = C_f$, the Lipschitz constant of f, and ϵ another constant which can be chosen small enough since α is great enough. Thereby we choose it such that $\epsilon C < \frac{1}{2}$. But for any $k \geq 0$,

$$\mathbb{E}[e^{\alpha\gamma_k}(Y^i_{\gamma_k} - Y^j_{\gamma_k})^2] = \mathbb{E}[e^{\alpha\gamma_k}(Y^i_{\gamma_k} - Y^j_{\gamma_k})^2 1_{[\gamma_k < T]}] \leq e^{\alpha T}\mathbb{E}[(U_{\gamma_k} - L_{\gamma_k})^2 1_{[\gamma_k < T]}] := v_k$$

Now for any $i, j \geq k$ we have

$$\Gamma^k_{i,j} := \mathbb{E}[\int_0^{\gamma_k} e^{\alpha s}|Z_s^i - Z_s^j|^2 ds + \int_0^{\gamma_k} e^{\alpha s}\|V_s^i - V_s^j\|^2 ds] \leq v_k + C\epsilon\Gamma^k_{i-1,j-1} \tag{24}$$

which implies that, when finite, for any $k \geq 0$, $\limsup_{i,j\to\infty} \Gamma^k_{i,j} \leq \frac{v_k}{1-\epsilon C}$. But for any $k \leq k'$, $\gamma_k \leq \gamma_{k'}$ and then $\limsup_{i,j\to\infty} \Gamma^k_{i,j} \leq \limsup_{i,j\to\infty} \Gamma^{k'}_{i,j} \leq \frac{v_{k'}}{1-\epsilon C}$. Now let $k' \to \infty$ to obtain that for any $k \geq 0$, $\limsup_{i,j\to\infty} \Gamma^k_{i,j} = 0$.

Let us show that $\limsup_{i,j\to\infty} \Gamma^k_{i,j}$ is finite. By induction we have,

$$\Gamma^k_{i,j} \leq (1 + ... + (C\epsilon)^{i-k})v_k + (C\epsilon)^{i-k}\Gamma^k_{k,j-i+k}. \tag{25}$$

On the other hand, the inequality (24) implies for any $p \geq k$,

$$\Gamma^k_{k,p} \leq v_k + 2C\epsilon\Gamma^k_{k,p-1} + 2C\epsilon\Upsilon^k \tag{26}$$

where $\Upsilon^k := I\!\!E[\int_0^{\gamma^k} e^{\alpha_s}(|Z^k_s - Z^{k-1}_s|^2 + \|V^k_s - V^{k-1}_s\|^2)ds]$. Therefore (25) and (26) imply that $\limsup_{i,j\to\infty} \Gamma^k_{i,j} < \infty$.

Going back now to (23), taking the supremum and using the Burkholder-Davis-Gundy inequality ([3], pp.304) we obtain: for any $k \geq 0$,

$$\limsup_{i,j\to\infty} I\!\!E[\sup_{t\leq T} e^{(\alpha t\wedge\gamma_k)}|Y^i_{t\wedge\gamma_k} - Y^j_{t\wedge\gamma_k}|^2] = 0.$$

But for any $k \geq 0$, $\gamma_k \leq \gamma_{k+1}$ then there exists a triple of process $(Y, Z, V) \in \mathcal{D}^2 \times \mathcal{H}^d \times \mathcal{L}$ such that:

$$\lim_{j\to\infty} I\!\!E[\sup_{s\leq\gamma^k} |Y^j_s - Y_s|^2 + I\!\!E\int_0^{\gamma^k} (|Z^j_s - Z_s|^2 + \|V^j_s - V_s\|^2)ds] = 0.$$

Actually it is enough to choose Y, Z and V such that for any $k \geq 0$,

$$Y_{t\wedge\gamma_k}(\omega) = \lim_{j\to\infty} Y^j_{t\wedge\gamma_k}(\omega), Z_{t\wedge\gamma_k}(\omega) = \lim_{j\to\infty} Z^j_{t\wedge\gamma_k}(\omega) \text{ and } V_{t\wedge\gamma_k}(\omega) = \lim_{j\to\infty} V^j_{t\wedge\gamma_k}(\omega).$$

Now for $i \geq 0$, let $(\overline{Y}^i, \overline{Z}^i, \overline{V}^i, \overline{K}^{i,+}, \overline{K}^{i,-})$ be the solution of the following reflected BSDE:

$$\begin{cases} \overline{Y}^i_t = Y_{\gamma^i} + \int_t^{\gamma^i} f(s, Y_s, Z_s, V_s)ds - \int_t^{\gamma^i} \overline{Z}^i_s dB_s - \int_t^{\gamma^i}\int_E \overline{V}^i_s(e)\tilde{\mu}(ds, de) \\ \qquad\qquad + \int_t^{\gamma^i} d(\overline{K}^{i,+}_s - \overline{K}^{i,-}_s) \quad \forall t \in [0, \gamma^i] \\ L_t \leq \overline{Y}^i_t \leq U_t, \quad \forall t \in [0, \gamma^i] \text{ and } \int_0^{\gamma^i} (\overline{Y}^i_s - L_s)d\overline{K}^{i,+} = \int_0^{\gamma^i} (\overline{Y}^i_s - U_s)d\overline{K}^{i,-} = 0. \end{cases}$$

According to Theorem 4.1 this solution exists. Writing Itô's formula for the process $|\overline{Y}^i_t - Y^j_t|^2$ with $t \leq \gamma^i$ and $j \geq i$ and let $j \longrightarrow \infty$, we obtain for all i:

$$\overline{Y}^i = Y, \;\; \overline{Z}^i = Z \;\text{ and }\; \overline{V}^i = V \;\text{ on } [0, \gamma^i].$$

Next let $K^{\pm}_t := \sum_{i=1}^{\infty} \int_0^t 1_{]\gamma^{i-1};\gamma^i]}(s)d\overline{K}^{i,\pm}_s$, $t \leq T$, then for any i and $t \in [0, \gamma^i]$ we have :

$$\begin{cases} Y_t = Y_{\gamma^i} + \int_t^{\gamma^i} f(s, Y_s, Z_s, V_s)ds - \int_t^{\gamma^i} Z_s dB_s - \int_t^{\gamma^i}\int_E V_s(e)\tilde{\mu}(ds, de) + \int_t^{\gamma^i} d(K^+_s - K^-_s), \\ L_t \leq Y_t \leq U_t \text{ and } (Y_t - L_t)dK^+_t = (Y_t - U_t)dK^-_t = 0. \end{cases}$$

As the sequence $(\gamma^i)_{i\geq 0}$ is stationary then we have: for any $t \leq T$,

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s, V_s)ds - \int_t^T Z_s dB_s - \int_t^T \int_E V_s(e)\tilde{\mu}(ds, de) + \int_t^T d(K_s^+ - K_s^-), \\ L_t \leq Y_t \leq U_t \text{ and } (Y_t - L_t)dK_t^+ = (Y_t - U_t)dK_t^- = 0. \end{cases}$$

The proof of existence is now complete. Let us focus on uniqueness. If $(Y', Z', V', K^{'+}, K^{'-}) \in \mathcal{D}^2 \times \mathcal{H}^d \times \mathcal{L} \times \mathcal{C}_{ci} \times \mathcal{C}_{ci}$ is another solution for the BSDE (2) then $Y = Y'$, $Z = Z'$, $V = V'$ and $dK^+ - dK^- = dK^{'+} - dK^{'-}$. Actually let us consider the following stationary stopping time:

$$\beta_k := \inf\{s \geq 0 : \int_0^s (|Z_r'|^2 + \|V_r'\|^2)dr \geq k\} \wedge \gamma_k.$$

Then using Itô's formula, there exists a constant C such that for all $k \geq 0$,

$$\begin{aligned} I\!\!E[\sup_{t\leq\beta_k} |Y_t - Y_t'|^2 + \int_0^{\beta_k} (|Z_s - Z_s'|^2 + \|V_s - V_s'\|^2)ds] &\leq C I\!\!E[|Y_{\beta_k} - Y'_{\beta_k}|^2] \\ &\leq C I\!\!E[1_{[\beta_k<T]}(U_{\beta_k} - L_{\beta_k})^2]. \end{aligned}$$

Let $k \to \infty$ to have $(Y, Z, V) = (Y', Z', V')$ and finally $dK^+ - dK^- = dK^{'+} - dK^{'-}$. As pointed out in Remark 4.1. the processes $K^\pm$ are unique if we require they are singular $\Box$

The problem is now to find conditions, easy to check in practice, under which the assumption **[WM]** is satisfied. In the sequel of this section we will focus on with this issue.

Theorem 4.3 *If for any $t \in [0, T]$, $L_t < U_t$ then **[WM]** is satisfied.*

Proof: For any $\tau \in \mathcal{T}_0$, let $(Y, \theta_\tau, Z^\tau, V^\tau, K^{\tau,+}, K^{\tau,-})$ be the 6-uple defined in Theorem 2.2 with $\xi = \frac{L_T+U_T}{2}$ and $f \equiv 0$. Therefore $(Y, Z^\tau, V^\tau, K^{\tau,+}, K^{\tau,-})$ belongs to $\mathcal{D}^2 \times \mathcal{T}_\tau \times \mathcal{H}^{2,d} \times \mathcal{L}^2 \times \mathcal{C}^2_{ci} \times \mathcal{C}^2_{ci}$ and

$$\begin{cases} Y_T = \frac{L_T+U_T}{2} \\ Y_t = Y_{\theta_\tau} + \int_t^{\theta_\tau} d(K_s^{\tau,+} - K_s^{\tau,-}) - \int_t^{\theta_\tau} Z_s^\tau dB_s - \int_t^{\theta_\tau} \int_E V_s^\tau(e)\tilde{\mu}(ds, de), \ \forall t \in [\tau, \theta_\tau] \\ \int_\tau^{\theta_\tau} (U_s - Y_s)dK_s^{\tau,-} = \int_\tau^{\theta_\tau} (Y_s - L_s)dK_s^{\tau,+} = 0. \end{cases}$$

Now let us set $\gamma^0 = 0$, $\gamma^{k+1} = \theta_{\gamma^k}$, $k \geq 0$ and $\gamma := \lim_{k\to\infty} \gamma_k(\omega)$. First let us point out that through Lemma 2.4 for any $k \geq 0$ we have $Y_{\gamma_k} 1_{[\gamma_k<T]} = L_{\gamma_k} 1_{[\gamma_k<T]}$. On the other hand for $t \leq T$, let us set

$$Z_t := \sum_{k=0}^{\infty} 1_{]\gamma^k,\gamma^{k+1}]}(t) Z_t^{\gamma^{k+1}}, V_t := \sum_{k=0}^{\infty} 1_{]\gamma^k,\gamma^{k+1}]}(t) V_t^{\gamma^{k+1}} \text{ and } K_t^\pm := \sum_{k=0}^{\infty} \int_0^t 1_{]\gamma^k,\gamma^{k+1}]}(s) dK_s^{\gamma^{k+1},\pm}.$$

Then for any $k \geq 0$ and $t \leq \gamma^k$ we have,

$$Y_t = Y_{\gamma^k} + \int_t^{\gamma^k} d(K_s^+ - K_s^-) - \int_t^{\gamma^k} Z_s dB_s - \int_t^{\gamma^k} \int_E V_s(e)\tilde{\mu}(ds, de)$$

which implies that for any $t \leq \gamma_k$

$$Y_{t\wedge\gamma_k} = I\!E[Y^+_{\gamma_k} + (K^+_{\gamma_k} - K^+_{t\wedge\gamma_k})|\mathcal{F}_{t\wedge\gamma_k}] - I\!E[Y^-_{\gamma_k} + (K^-_{\gamma_k} - K^-_{t\wedge\gamma_k})|\mathcal{F}_{t\wedge\gamma_k}] = h^k_t - h'^k_t.$$

It remains to show $P\{\gamma^k < T, \forall k \geq 0\} = 0$. To do so let us set $A = \{\omega, \gamma^k(\omega) < T, \forall k \geq 0\}$ and $\delta^k := \inf\{s \geq \gamma^k : Y_s = U_s\} \wedge T$. Then through Theorem 2.2-(iii), for any $\omega \in A$ and $k \geq 0$, $Y_{\delta^k}(\omega) = U_{\delta^k}(\omega)$ and $\gamma^k(\omega) \leq \delta^k(\omega) \leq \gamma^{k+1}(\omega)$ then $\delta^k(\omega) \to \gamma(\omega)$ as $k \to \infty$. Now since $L < U$ then for any $k \geq 0$, we have $\gamma_k(\omega)(\omega) < \gamma_{k+1}(\omega) < \gamma(\omega)$ and $\delta^k(\omega) < \delta^{k+1}(\omega) < \gamma(\omega)$. Therefore for $\omega \in A$,

$$U_{\gamma-} = \lim_{k\to\infty} U_{\delta^k}(\omega) = \lim_{k\to\infty} Y_{\delta^k}(\omega) = \lim_{k\to\infty} Y_{\gamma^k}(\omega) = \lim_{k\to\infty} L_{\gamma^k}(\omega) = L_{\gamma-}.$$

But the jumping times of L and U are inaccessible then through Lemma 2.3, $\triangle L_\gamma = \triangle U_\gamma = 0$ on A, *i.e.*, $L_\gamma(\omega) = U_\gamma(\omega)$. As $L < U$ then we have $P(A) = 0$. $\Box$

Remark 4.3 *As a by-product of Theorems 4.3 & 4.2, in combination with Theorem 3.1, the value function of a Dynkin game is a semi-martingale if $L_t < U_t$, for any $t \leq T$.* $\Box$

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