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Improved Inclusion-Exclusion Identities and Inequalities Based on a Particular Class of Abstract Tubes

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Abstract

Recently, Naiman and Wynn introduced the concept of an abstract tube in order to obtain improved inclusion-exclusion identities and inequalities that involve much fewer terms than their classical counterparts. In this paper, we introduce a particular class of abstract tubes which plays an important role with respect to chromatic polynomials and network reliability. The inclusion-exclusion identities and inequalities associated with this class simultaneously generalize several well-known results such as Whitney's broken circuit theorem, Shier's expression for the reliability of a network as an alternating sum over chains in a semilattice and Narushima's inclusion-exclusion identity for posets. Moreover, we show that under some restrictive assumptions a polynomial time inclusion-exclusion algorithm can be devised, which generalizes an important result of Provan and Ball on network reliability.

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1 Introduction

Inclusion-exclusion identities and inequalities play an important role in many areas of mathematics. For any finite collection of sets $\{A_v\}_{v\in V}$ and any $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the classical inclusion-exclusion inequalities (also known as *Bonferroni inequalities*) state that

$$\chi\left(\bigcup_{v\in V} A_v\right) \geq \sum_{\substack{I\subseteq V,\, I\neq\emptyset \\ |I|\leq n}} (-1)^{|I|-1}\, \chi\left(\bigcap_{i\in I} A_i\right) \qquad (n \text{ even}),$$

$$\chi\left(\bigcup_{v\in V} A_v\right) \leq \sum_{\substack{I\subseteq V,\, I\neq\emptyset \\ |I|\leq n}} (-1)^{|I|-1}\, \chi\left(\bigcap_{i\in I} A_i\right) \qquad (n \text{ odd}),$$

where $\chi(A)$ is used to denote the indicator function of a set A. Note that for $n \geq |V|$ the equals sign holds and thus we have the classical inclusion-exclusion identity (also known as the *sieve formula*). There is no real restriction in using indicator functions rather than measures, since the above inequalities can be integrated with respect to any measure on the σ-algebra generated by $\{A_v\}_{v\in V}$.

The classical inclusion-exclusion inequalities were found in 1936 by Bonferroni [2], and the first implicit use of the inclusion-exclusion identity was probably by N. Bernoulli in 1710; for more historical notes, see Takács [14]. An overview of new developments is given by Galambos and Simonelli [7].

Recently, Naiman and Wynn [10] introduced the notion of an abstract tube in order to improve and generalize the classical inclusion-exclusion inequalities. A similar concept is that of McKee [9], who proposes a graph structure for the same purpose. Related work was done by Tomescu [15], who considered an approach focusing on hypertrees.

Roughly speaking, an abstract tube is a collection of sets together with an appropriate abstract simplicial complex. In the applications of Naiman and Wynn [10] the sets are balls or other geometric objects and the abstract simplicial complex is obtained from a Voronoi decomposition of the underlying space. Here, the sets are finite and the abstract simplicial complex is the broken circuit complex of a graph or the order complex of a partially ordered set.

Our paper is organized as follows: In Section 2, we review the notion of an abstract tube and the main result of Naiman and Wynn [10]. In Section 3, we present our main result, which provides a sufficient condition for a family of sets and an abstract simplicial complex to constitute an abstract tube, and we show that under some restrictive assumptions, a polynomial time inclusion-exclusion algorithm can be devised. In Section 4, applications to chromatic polynomials and network reliability are presented. In the subsection on chromatic polynomials, a link is established between the theory of abstract tubes and the theory of broken circuit complexes, which was initiated by Wilf [17].

2 Abstract tubes

First, we briefly review some terminologies and facts from combinatorial topology. For a detailed exposition and examples clarifying the various concepts, the reader is referred to the fundamental paper of Naiman and Wynn [10].

An *abstract simplicial complex* $\mathfrak{S}$ is a set of non-empty subsets of some finite set such that $I \in \mathfrak{S}$ and $\emptyset \neq J \subset I$ imply $J \in \mathfrak{S}$. The elements of $\mathfrak{S}$ are called *faces* or *simplices* of $\mathfrak{S}$. The dimension of a face is one less than its cardinality. It is well-known that any abstract simplicial complex $\mathfrak{S}$ can be realized as a geometric simplicial complex $\mathfrak{S}'$ in Euclidean space, such that there is a one-to-one correspondence between the faces of $\mathfrak{S}$ and the faces of $\mathfrak{S}'$. A geometric simplicial complex is *contractible* if it can be deformed continuously to a single point. An abstract simplicial complex is *contractible*, if it has a contractible geometric realization. For example, the abstract simplicial complex $\mathfrak{P}^*(V)$ consisting of all non-empty subsets of some finite set V is contractible.

The following definition and proposition are due to NAIMAN and WYNN [10].

Definition 2.1. An *abstract tube* is a pair $(\mathcal{A}, \mathfrak{S})$ consisting of a finite collection of sets $\mathcal{A} = \{A_v\}_{v \in V}$ and an abstract simplicial complex $\mathfrak{S} \subseteq \mathfrak{P}^*(V)$ such that for any $\omega \in \bigcup_{v \in V} A_v$ the abstract simplicial complex

$$\mathfrak{S}(\omega) \;:=\; \left\{ I \in \mathfrak{S} : \omega \in \bigcap_{i \in I} A_i \right\}$$

is contractible.

Proposition 2.2. *Let $(\{A_v\}_{v \in V}, \mathfrak{S})$ be an abstract tube, $n \in \mathbb{N}_0$. Then,*

$$\chi\left(\bigcup_{v \in V} A_v \right) \;\geq\; \sum_{\substack{I \in \mathfrak{S} \\ |I| \leq n}} (-1)^{|I|-1} \, \chi\left(\bigcap_{i \in I} A_i \right) \qquad \textit{(n even)}\,,$$

$$\chi\left(\bigcup_{v \in V} A_v \right) \;\leq\; \sum_{\substack{I \in \mathfrak{S} \\ |I| \leq n}} (-1)^{|I|-1} \, \chi\left(\bigcap_{i \in I} A_i \right) \qquad \textit{(n odd)}\,.$$

Since $(\{A_v\}_{v \in V}, \mathfrak{P}^*(V))$ is an abstract tube for any finite collection of sets $\{A_v\}_{v \in V}$, the classical inclusion-exclusion inequalities are a special case of the NAIMAN-WYNN inequalities. Moreover, NAIMAN and WYNN [10] proved that their inequalities are at least as sharp as their classical counterparts, although in many important cases their computational effort is significantly smaller.

3 A particular class of abstract tubes

Definition 3.1. Let $\{A_v\}_{v \in V}$ be a finite collection of sets whose index set V is equipped with a linear ordering relation. A non-empty set $X \subseteq V$ is *irrelevant* if

$$\bigcap_{x \in X} A_x \;\subseteq \bigcup_{x > \max X} A_x \,.$$

For any set $\mathfrak{X}$ of subsets of V define

$$\mathfrak{I}(V, \mathfrak{X}) \;:=\; \{ I \subseteq V \,:\, I \neq \emptyset \text{ and } I \not\supseteq X \text{ for any } X \in \mathfrak{X} \}\,.$$

From the definition it follows that $\mathfrak{I}(V,\mathfrak{X})$ is an abstract simplicial complex, and evidently, $\mathfrak{I}(V,\mathfrak{X}) = \mathfrak{P}^*(V)$ if and only if $\mathfrak{X} = \emptyset$.

As an example, consider the sets $A_1 = \{a,b\}$, $A_2 = \{b,c\}$ and $A_3 = \{b,d\}$. Then, $A_1 \cap A_2 \subseteq A_3$, $A_1 \cap A_3 \subseteq A_2$ and $A_2 \cap A_3 \subseteq A_1$. Since only the first inclusion is of the required form, $\{1,2\}$ is the only irrelevant subset of $\{1,2,3\}$. Moreover, the definition gives $\mathfrak{I}(\{1,2,3\},\{\{1,2\}\}) = \{\{1\},\{2\},\{3\},\{1,3\},\{2,3\}\}$.

Lemma 3.2. *Let V be a finite set and $\mathfrak{X}$ a set of subsets of V. Then $\mathfrak{I}(V,\mathfrak{X})$ is contractible or $\mathfrak{X}$ is a covering of V.*

Proof. Suppose that $\mathfrak{X}$ is not a covering of V. Fix some $v \in V \setminus \bigcup \mathfrak{X}$. It is obvious that v is contained in every maximal face of $\mathfrak{I}(V,\mathfrak{X})$. Therefore, each realization of $\mathfrak{I}(V,\mathfrak{X})$ is star-shaped with respect to v and hence contractible. □

We now state our main result. Recall that every abstract tube corresponds to an inclusion-exclusion identity and a series of inclusion-exclusion inequalities. Although the identity and the inequalities corresponding to our main result are new, we do not mention them explicitly, since they can easily be read from Proposition 2.2. Thus, our main result reads as follows:

Theorem 3.3. *Let $\{A_v\}_{v\in V}$ be a finite collection of sets whose index set V is equipped with a linear ordering relation, and let $\mathfrak{X}$ be any set of irrelevant subsets of V. Then $(\{A_v\}_{v\in V}, \mathfrak{I}(V,\mathfrak{X}))$ is an abstract tube.*

Proof. We prove the theorem by contradiction. Assume that there is some $\omega \in \bigcup_{v\in V} A_v$ such that $\mathfrak{I}(V,\mathfrak{X})(\omega)$ is not contractible. Obviously, $\mathfrak{I}(V,\mathfrak{X})(\omega) = \mathfrak{I}(V_\omega, \mathfrak{X} \cap \mathfrak{P}^*(V_\omega))$, where $V_\omega := \{v \in V : \omega \in A_v\}$. From this, the assumption and Lemma 3.2 we conclude that $\mathfrak{X} \cap \mathfrak{P}^*(V_\omega)$ is a covering of V_ω. Hence, there is some $X \in \mathfrak{X}$ such that $X \subseteq V_\omega$ and $\max X = \max V_\omega$. It follows that

$$\omega \in \bigcap_{v\in V_\omega} A_v \subseteq \bigcap_{v\in X} A_v \subseteq \bigcup_{v>\max X} A_v \subseteq \bigcup_{v\notin V_\omega} A_v$$

and hence, $\omega \in A_v$ for some $v \notin V_\omega$, which contradicts the definition of V_ω. □

The identity corresponding to the particular case where any $X \in \mathfrak{X}$ satisfies $\bigcap_{x\in X} A_x \subseteq A_{x^*}$ for some $x^* > \max X$ has already been applied to network reliability problems in [3]. Note that in Theorem 3.3 it is not required that $\mathfrak{X}$ consists of *all* irrelevant subsets of V.

The identity corresponding to the next corollary is due to NARUSHIMA [11]. The corresponding inequalities are new.

Corollary 3.4. *Let $\{A_v\}_{v\in V}$ be a finite collection of sets whose index set V is equipped with a partial ordering relation such that for any $x,y \in V$, $A_x \cap A_y \subseteq A_z$ for some upper bound z of x and y, and let $\mathfrak{C}(V)$ denote the set of non-empty chains of V. Then $(\{A_v\}_{v\in V}, \mathfrak{C}(V))$ is an abstract tube.*

Proof. The corollary follows from Theorem 3.3 by defining $\mathfrak{X}$ as the set of all unordered pairs of incomparable elements of V and then considering an arbitrary linear extension of the partial ordering relation on V. □

The assumptions of Corollary 3.4 entrain that V contains a unique maximal element, and obviously they are satisfied if V is a $\vee$-semilattice and $A_x \cap A_y \subseteq A_{x\vee y}$ for any $x,y \in V$. In the original version of NARUSHIMA's result [11], it is

required that for any $x, y \in V$, $A_x \cap A_y \subseteq A_z$ for some *minimal* upper bound z of x and y. Note that in Corollary 3.4 the minimality of z is not required.

Corollary 3.5. *Let $\{A_v\}_{v\in V}$ satisfy the assumptions of Corollary 3.4. In addition, let μ be a probability measure on the σ-algebra generated by $\{A_v\}_{v\in V}$ such that $\mu(\bigcap_{v\in V} A_v) > 0$ and $\mu(A_{i_k} \mid A_{i_{k-1}} \cap \cdots \cap A_{i_1}) = \mu(A_{i_k} \mid A_{i_{k-1}})$ for any chain $i_1 < \cdots < i_k$ in V where $k > 1$. Then,*

$$\mu\left(\bigcup_{v\in V} A_v\right) = \sum_{v\in V} \Lambda(v)\,,$$

where Λ is defined by the following recursive scheme:

$$\Lambda(v) := \mu(A_v) - \sum_{w<v} \Lambda(w)\,\mu(A_v|A_w)\,. \tag{1}$$

Proof. By Corollary 3.4 and Proposition 2.2 it suffices to prove that

$$\Lambda(v) = \sum_{\substack{I\in\mathfrak{C}(V)\\ \max I = v}} (-1)^{|I|-1}\,\mu\left(\bigcap_{i\in I} A_i\right)\,,$$

where again $\mathfrak{C}(V)$ denotes the set of non-empty chains of V. We proceed by induction on the height of v. If the height of v is zero, that is, if v is minimal, then $\Lambda(v) = \mu(A_v)$ and the statement is proven. For any non-minimal $v \in V$ the recursive definition and the induction hypothesis give

$$\begin{aligned}
\Lambda(v) &= \mu(A_v) - \sum_{w<v} \sum_{\substack{I\in\mathfrak{C}(V)\\ \max I = w}} (-1)^{|I|-1}\,\mu\left(\bigcap_{i\in I} A_i\right)\,\mu(A_v|A_w)\\
&= \mu(A_v) - \sum_{w<v} \sum_{\substack{I\in\mathfrak{C}(V)\\ \max I = w}} (-1)^{|I|-1}\,\mu\left(\bigcap_{i\in I} A_i\right)\,\mu\left(A_v \,\middle|\, \bigcap_{i\in I} A_i\right)\\
&= \mu(A_v) + \sum_{w<v} \sum_{\substack{I\in\mathfrak{C}(V)\\ \max I = w}} (-1)^{|I\cup\{v\}|-1}\,\mu\left(\bigcap_{i\in I\cup\{v\}} A_i\right)\\
&= \mu(A_v) + \sum_{\substack{I\in\mathfrak{C}(V)\\ \max I = v\\ I\neq\{v\}}} (-1)^{|I|-1}\,\mu\left(\bigcap_{i\in I} A_i\right)\,. \qquad \square
\end{aligned}$$

Remark. By the technique of dynamic programming, the recursive scheme gives an algorithm whose space complexity is $O(|V|)$ and whose time complexity is $O(|V|^2 \times T)$ where T is the time needed to compute $\mu(A_v)$ and $\mu(A_v|A_w)$.

We remark that there are several other algorithms combining the principle of inclusion-exclusion with dynamic programming; see e.g., Karp [8].

4 Applications

4.1 Chromatic polynomials

We consider finite undirected graphs without loops or multiple edges. For any graph G and any $\lambda \in \mathbb{N}$, a *λ-coloring* of G is a mapping of the vertex-set of G

into $\{1,\ldots,\lambda\}$; it is *proper,* if no edge is monochromatic, that is, if no edge joins two vertices having the same color. Let $P_G(\lambda)$ denote the number of proper λ-colorings of G. By BIRKHOFF [1], $P_G(\lambda)$ is a polynomial in λ of degree $v(G)$, where $v(G)$ denotes the number of vertices of G. The polynomial $P_G(\lambda)$ is the so-called *chromatic polynomial* of G.

Now, let the edge-set of G be equipped with a linear ordering relation. Due to WHITNEY [16], a *broken circuit* of G is obtained from the edge-set of a cycle in G by removing the maximum edge of the cycle. The *broken circuit complex* of G, which was initiated by WILF [17], is the abstract simplicial complex

$$\mathfrak{K}(G) \;:=\; \mathfrak{I}(E(G),\mathfrak{B}(G))\,,$$

where $E(G)$ denotes the set of edges and $\mathfrak{B}(G)$ the set of broken circuits of G. For each edge e of G let A_e denote the set of all λ-colorings of G such that e is monochromatic. By virtue of Theorem 3.3 it is easily verified that $(\{A_e\}_{e\in E(G)},\mathfrak{K}(G))$ is an abstract tube. Integrating the corresponding identity with respect to the counting measure $|\cdot|$ we straightforwardly obtain

$$P_G(\lambda) \;=\; \lambda^{v(G)} - \left|\bigcup_{e\in E(G)} A_e\right| \;=\; \lambda^{v(G)} + \sum_{k=1}^{v(G)} (-1)^k \sum_{\substack{I\in\mathfrak{K}(G)\\ |I|=k}} \left|\bigcap_{i\in I} A_i\right| .$$

Since each face of $\mathfrak{K}(G)$ is cycle-free, we can apply Lemma 3.1 of [5], which states that $\left|\bigcap_{i\in I} A_i\right| = \lambda^{v(G)-|I|}$ whenever I is cycle-free. Thus, we have

$$P_G(\lambda) \;=\; \sum_{k=0}^{v(G)} (-1)^k a_k(G)\lambda^{v(G)-k}\,, \tag{2}$$

where $a_0(G)=1$ and $a_k(G)$, $k>0$, counts the faces of cardinality k (dimension $k-1$) in the broken circuit complex of G. Identity (2) is known as WHITNEY's broken circuit theorem [16]. The corresponding inequalities

$$\begin{aligned} P_G(\lambda) &\;\le\; \sum_{k=0}^{n} (-1)^k a_k(G)\lambda^{v(G)-k} \qquad (n \text{ even})\,,\\ P_G(\lambda) &\;\ge\; \sum_{k=0}^{n} (-1)^k a_k(G)\lambda^{v(G)-k} \qquad (n \text{ odd})\,, \end{aligned}$$

are new. An inductive proof of these inequalities will appear in [6].

4.2 Network reliability

We consider a probabilistic directed or undirected network G whose nodes are perfectly reliable and whose edges fail randomly and independently with known probabilities. For distinguished nodes s and t of G we are interested in identities and inequalities for the two-terminal reliability $\mathrm{Rel}_{st}(G)$ which is the probability that s and t are connected by a path of operating edges. (The all-terminal reliability is treated separately; the interested reader is referred to [4].)

A key role in calculating $\mathrm{Rel}_{st}(G)$ is played by the s,t-paths and s,t-cutsets of G: An *s,t-path* of G is a minimal set of edges connecting s and t, and an

s,t-*cutset* of G is a minimal set of edges disconnecting s and t. An s,t-path of G *operates* if all of its edges operate; an s,t-cutset of G *fails* if all of its edges fail. If the underlying probability measure is denoted by μ, then

$$\mathrm{Rel}_{st}(G) \;=\; \mu\left(\bigcup_{P\in\mathcal{P}}\{P \text{ operates}\}\right) \;=\; 1-\mu\left(\bigcup_{C\in\mathcal{C}}\{C \text{ fails}\}\right),$$

where $\mathcal{P}$ and $\mathcal{C}$ denote the set of s,t-paths and s,t-cutsets of G, respectively. In order to apply the results of Section 3, we adopt the partial ordering relations imposed by Shier [13, Section 6.1]: For any s,t-cutsets C and D of G define

$$C \preceq D \quad :\Leftrightarrow \quad N(C)\subseteq N(D),$$

where $N(C)$ is the set of nodes reachable from s after removing C. If G is a planar network, then for any s,t-paths P and Q of G define

$$P \preceq Q \quad :\Leftrightarrow \quad P \text{ lies below } Q.$$

In both cases a lattice structure is induced where the supremum and infimum of any two sets is included by the union of these two sets. Hence, by Corollary 3.4, $\big(\{\{C \text{ fails}\}\}_{C\in\mathcal{C}}, \mathfrak{C}(\mathcal{C})\big)$ and $\big(\{\{P \text{ operates}\}\}_{P\in\mathcal{P}}, \mathfrak{C}(\mathcal{P})\big)$ are abstract tubes. By integrating the corresponding identities and inequalities with respect to μ, results for $\mathrm{Rel}_{st}(G)$ are obtained. The identities for $\mathrm{Rel}_{st}(G)$ are due to Shier [13, Section 6.3], whereupon the inequalities are new. If each edge e fails with probability $q_e = 1-p_e$, then these inequalities can be written as

$$\sum_{\substack{\mathcal{I}\in\mathfrak{C}(\mathcal{P})\\ |\mathcal{I}|\le n}} (-1)^{|\mathcal{I}|-1}\prod_{e\,\in\bigcup\mathcal{I}} p_e \;\le\; \mathrm{Rel}_{st}(G) \;\le\; 1-\sum_{\substack{\mathcal{I}\in\mathfrak{C}(\mathcal{C})\\ |\mathcal{I}|\le n}} (-1)^{|\mathcal{I}|-1}\prod_{e\,\in\bigcup\mathcal{I}} q_e \quad (n \text{ even}),$$

$$\sum_{\substack{\mathcal{I}\in\mathfrak{C}(\mathcal{P})\\ |\mathcal{I}|\le n}} (-1)^{|\mathcal{I}|-1}\prod_{e\,\in\bigcup\mathcal{I}} p_e \;\ge\; \mathrm{Rel}_{st}(G) \;\ge\; 1-\sum_{\substack{\mathcal{I}\in\mathfrak{C}(\mathcal{C})\\ |\mathcal{I}|\le n}} (-1)^{|\mathcal{I}|-1}\prod_{e\,\in\bigcup\mathcal{I}} q_e \quad (n \text{ odd}),$$

where $\bigcup\mathcal{I}$ denotes the union of all $I\in\mathcal{I}$. Moreover, $\{\{C \text{ fails}\}\}_{C\in\mathcal{C}}$ and $\{\{P \text{ operates}\}\}_{P\in\mathcal{P}}$ satisfy the assumptions of Corollary 3.5. For the first collection of events, the recursive scheme (1) becomes

$$\Lambda(C) \;=\; \prod_{e\in C} q_e - \sum_{D\prec C}\Lambda(D)\prod_{e\in C\setminus D} q_e \qquad (C\in\mathcal{C}),$$

which is due to Provan and Ball [12], and for the second one it states that

$$\Lambda(P) \;=\; \prod_{e\in P} p_e - \sum_{Q\prec P}\Lambda(Q)\prod_{e\in P\setminus Q} p_e \qquad (P\in\mathcal{P}),$$

which is due to Shier [13, Section 6.2].

Example. Consider the network in Figure 1. We are interested in bounding the two-terminal reliability of the network with respect to the nodes s and t. For simplicity, assume that all edges fail with the same probability $q = 1-p$.

Let's first apply the classical inclusion-exclusion inequalities to the s,t-paths and s,t-cutsets of the network. The corresponding bounds $a_n(p)$ and $b_n(q)$

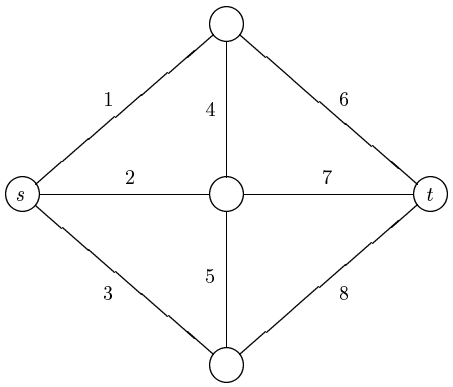

Figure 1: A sample network with terminal nodes s and t.

(which we call *classical bounds*) are shown in Tables 1 and 2, together with the number of sets inspected during the computation of each bound.

Now, let's compute the bounds corresponding to our abstract tubes. The Hasse diagrams for the s,t-paths and s,t-cutsets are shown in Figures 2 and 3. The resulting bounds $a_n^*(p)$ and $b_n^*(q)$ (which we call *improved bounds*) are presented in Tables 3 and 4, together with the number of sets inspected.

Note that in Tables 1 and 3 even and odd values of n correspond to lower and upper bounds, respectively, whereupon in Tables 2 and 4 the correspondence is vice versa. The last bound in each case represents the exact network reliability. Also note that in Tables 3 and 4 the exact network reliability is already reached when $n = 5$ resp. $n = 4$. As expected, the improved bounds employ much fewer sets than the classical bounds. In Figures 4 and 5, some of the bounds are plotted. A numerical comparism of classical and improved bounds is shown in Tables 5 and 6. We observe that both classical and improved bounds based on paths are satisfactory only for small values of p (the less typical case), whereupon those based on cutsets are satisfactory only for small values of q (the more typical case). As expected, the improved bounds beat the classical bounds, and the difference between them grows with p resp. q.

n	$a_n(p)$	# sets
1	$3p^2 + 4p^3 + 2p^4$	9
2	$3p^2 + 4p^3 - 9p^4 - 16p^5 - 9p^6$	45
3	$3p^2 + 4p^3 - 9p^4 - 8p^5 + 34p^6 + 30p^7 + 3p^8$	129
4	$3p^2 + 4p^3 - 9p^4 - 10p^5 + 27p^6 - 50p^7 - 34p^8$	255
5	$3p^2 + 4p^3 - 9p^4 - 10p^5 + 27p^6 - 12p^7 + 54p^8$	381
6	$3p^2 + 4p^3 - 9p^4 - 10p^5 + 27p^6 - 18p^7 - 24p^8$	465
7	$3p^2 + 4p^3 - 9p^4 - 10p^5 + 27p^6 - 18p^7 + 12p^8$	501
8	$3p^2 + 4p^3 - 9p^4 - 10p^5 + 27p^6 - 18p^7 + 3p^8$	510
9	$3p^2 + 4p^3 - 9p^4 - 10p^5 + 27p^6 - 18p^7 + 4p^8$	511

Table 1: Classical bounds based on paths.

n	$b_n(q)$	# sets
1	$1-2q^3-4q^4-2q^5$	9
2	$1-2q^3-4q^4+2q^5+13q^6+10q^7+\ \ q^8$	37
3	$1-2q^3-4q^4+2q^5+13q^6-22q^7-23q^8$	93
4	$1-2q^3-4q^4+2q^5+13q^6-14q^7+39q^8$	163
5	$1-2q^3-4q^4+2q^5+13q^6-14q^7-17q^8$	219
6	$1-2q^3-4q^4+2q^5+13q^6-14q^7+11q^8$	247
7	$1-2q^3-4q^4+2q^5+13q^6-14q^7+\ \ 3q^8$	255
8	$1-2q^3-4q^4+2q^5+13q^6-14q^7+\ \ 4q^8$	256

Table 2: Classical bounds based on cutsets.

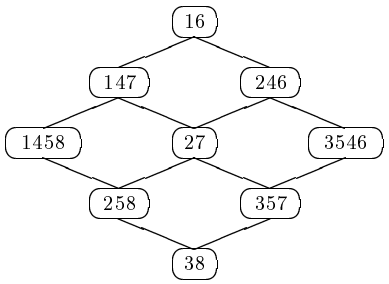

Figure 2: The Hasse diagram of s,t-paths.

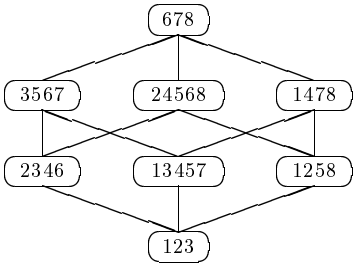

Figure 3: The Hasse diagram of s,t-cutsets.

n	$a_n^*(p)$	# sets
1	$3p^2+4p^3+2p^4$	9
2	$3p^2+4p^3-9p^4-14p^5-\ \ 2p^6$	36
3	$3p^2+4p^3-9p^4-10p^5+27p^6+\ \ 4p^7$	73
4	$3p^2+4p^3-9p^4-10p^5+27p^6-18p^7-2p^8$	97
5–9	$3p^2+4p^3-9p^4-10p^5+27p^6-18p^7+4p^8$	103

Table 3: Improved bounds based on paths.

n	$b_n^*(q)$	# sets
1	$1-2q^3-4q^4-2q^5$	9
2	$1-2q^3-4q^4+2q^5+13q^6+\ 2q^7$	28
3	$1-2q^3-4q^4+2q^5+13q^6-14q^7-2q^8$	46
4–8	$1-2q^3-4q^4+2q^5+13q^6-14q^7+4q^8$	52

Table 4: Improved bounds based on cutsets.

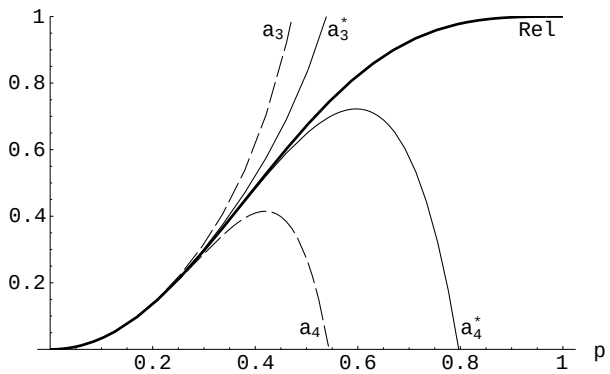

Figure 4: Bounds based on paths.

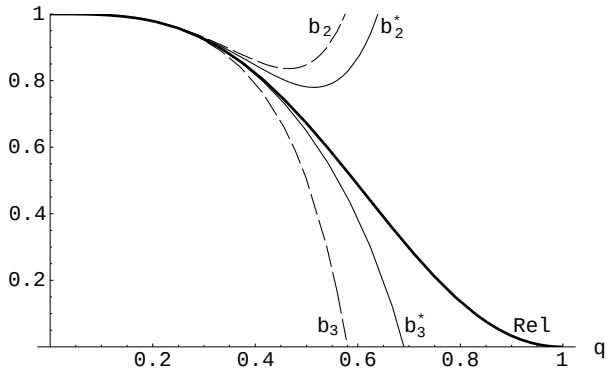

Figure 5: Bounds based on cutsets.

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