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STABILITY PROPERTIES OF CONSTRAINED JUMP-DIFFUSION PROCESSES [1]

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Abstract: We consider a class of jump-diffusion processes, constrained to a polyhedral cone $G \subset I\!R^n$, where the constraint vector field is constant on each face of the boundary. The constraining mechanism corrects for "attempts" of the process to jump outside the domain. Under Lipschitz continuity of the Skorohod map Γ, it is known that there is a cone $\mathcal{C}$ such that the image $\Gamma\phi$ of a deterministic linear trajectory ϕ remains bounded if and only if $\dot{\phi} \in \mathcal{C}$. Denoting the generator of a corresponding unconstrained jump-diffusion by $\mathcal{L}$, we show that a key condition for the process to admit an invariant probability measure is that for $x \in G$, $\mathcal{L}\,\mathrm{id}(x)$ belongs to a compact subset of $\mathcal{C}^o$.

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1 Introduction

In this work we consider stability properties of a class of jump-diffusion processes that are constrained to lie in a convex closed polyhedral cone. Let G be a cone in $\mathbb{R}^n$, given as the intersection $\cap_i G_i$ of the half spaces

$$G_i = \{x \in \mathbb{R}^n : x \cdot n_i \geq 0\}, \quad i = 1, \ldots, N,$$

where n_i, $i = 1, \ldots, N$ are given unit vectors. It is assumed that the origin is a proper vertex of G, in the sense that there exists a closed half space G_0 with $G \cap G_0 = \{0\}$. Equivalently, there exists a unit vector a_0 such that

$$\{x \in G : x \cdot a_0 \leq 1\} \tag{1.1}$$

is compact. Note that, in particular, $N \geq n$. Let $F_i = \partial G \cap \partial G_i$. With each face F_i we associate a unit vector d_i (such that $d_i \cdot n_i > 0$). This vector defines the *direction of constraint* associated with the face F_i. The *constraint vector field* $d(x)$ is defined for $x \in \partial G$ as the set of all unit vectors in the cone generated by $\{d_i, i \in \mathrm{In}(x)\}$, where

$$\mathrm{In}(x) \doteq \{i \in \{1, \ldots, N\} : x \cdot n_i = 0\}.$$

Under further assumptions on (n_i) and (d_i), one can define a *Skorohod map* Γ in the space of right continuous paths with left limits, in a way which is consistent with the constraint vector field d. Namely, Γ maps a path ψ to a path $\phi = \psi + \eta$ taking values in G, so that η is of bounded variation, and, denoting the total variation of η on $[0, s]$ by $|\eta|(s)$, $d\eta(\cdot)/d|\eta|(\cdot) \in d(\phi(\cdot))$. The precise definition of Γ and the conditions assumed are given in Section 2. The constrained jump-diffusion studied in this paper is the second component Z of the pair (X, Z) of processes satisfying

$$X_t = z_0 + \int_0^t \beta(Z_s)ds + \int_0^t a(Z_s)dW_s + \int_{[0,t]\times E} h(\delta(Z_{s-}, z))[N(ds, dz) - q(ds, dz)] \\ + \int_{[0,t]\times E} h'(\delta(Z_{s-}, z))N(ds, dz), \tag{1.2}$$

$$Z = \Gamma(X). \tag{1.3}$$

Here, W and N are the driving m-dimensional Brownian motion and Poisson random measure on $\mathbb{R}_+ \times E$; β, a and δ are (state-dependent) coefficients and h is a truncation function (see Section 2 for definitions and assumptions). For illustration, consider as a special case of (1.2), (1.3), the case where X is a Lévy process with piecewise constant paths and finitely many jumps over finite time intervals. Then $X_t = x + \sum_{s \leq t} \Delta X_s$, where $\Delta X_s = X_s - X_{s-}$. In this case, Z is given as $Z_t = x + \sum_{s \leq t} \Delta Z_s$, where ΔZ_s can be defined recursively in a straightforward way. Namely, if $Z_{s-} + \Delta X_s \in G$, then $\Delta Z_s = \Delta X_s$. Otherwise, $Z_s = Z_{s-} + \Delta X_s + \alpha d$, where $\alpha \in (0, \infty)$, $Z_s \in \partial G$, and $d \in d(Z_s)$. In general, this set of conditions may not have a solution (α, d), or may have multiple solutions. However, the assumptions we put on the map Γ will ensure that this recursion is uniquely solvable, and as a result, that the process Z is well defined.

A related model for which recurrence and transience properties have been studied extensively is that of a semimartingale reflecting Brownian motion (SRBM) in polyhedral cones [3, 8, 11, 12, 13]. Roughly speaking, a SRBM is a constrained version, using a "constraining mechanism" as described above, of a Brownian motion with a drift. In a recent work [1], sufficient conditions for positive recurrence of a constrained diffusion process with a state dependent drift and (uniformly nondegenerate) diffusion coefficients were obtained. Under the assumption of regularity of the map Γ (as in Condition 2.4 below), it was shown that if the drift vector field takes values in the cone $\mathcal{C}$ generated by the vectors $-d_i$, $i = 1, \ldots, N$, and stays away, uniformly, from the boundary of the cone, then the corresponding constrained diffusion process is positive recurrent and admits a unique invariant measure. The technique used there critically relies on certain estimates on the exponential moments of the constrained process. The current work aims at showing that $\mathcal{C}$ plays the role of a stability cone in a much more general setting of constrained jump-diffusions for which only the first moment is assumed to be finite. The natural definition of the drift vector field in the case of a jump-diffusion is $\tilde{\beta} \doteq \mathcal{L}\,\mathrm{id}$, where $\mathcal{L}$ denotes the generator of a related "unconstrained" jump-diffusion (see (2.6)), and id denotes the identity mapping on $I\!R^n$. In the case of a Lévy process with finite mean, the drift is simply $\tilde{\beta}(x) = E_x X_1 - x$ (which is independent of x). Our basic stability assumption is that the range of $\tilde{\beta}$ is contained in $\cup_{k \in I\!N}\, k\mathcal{C}_1$, where $\mathcal{C}_1$ is a compact subset of the interior of $\mathcal{C}$. Under this assumption, our main stability result states (Theorem 2.13): There exists a compact set A such that for any compact $C \subset G$,

$$\sup_{x \in C} E_x \tau_A < \infty, \tag{1.4}$$

where τ_A is the first time Z hits A, and E_x denotes the expectation under which Z starts from x. The proof of this result is based on the construction of a Lyapunov function, and on a careful separate analysis of small and large jumps of the Markov process. As another consequence of the existence of a Lyapunov function we show that Z is bounded in probability. From the Feller property of the process it then follows that it admits at least one invariant measure. Finally, under further suitable communicability conditions (see Conditions 2.18 and 2.20) it follows that the Markov process is positive Harris recurrent and admits a unique invariant measure.

The study of these processes is motivated by problems in stochastic network theory (see [18] for a review). The assumptions we make on the Skorohod map are known to be satisfied by a large class of applications, including single class open queueing networks (see [6], [10]).

For a sampling of stability results on constrained processes with jumps we list [4, 5, 15, 16, 19, 20]. We take an approach similar to that of [8], where the stability properties of SRBM in an orthant are proved by means of constructing a Lyapunov function. At the cost of putting conditions that guarantee strong existence and uniqueness of solutions to the SDE, we are able to treat diffusions with jumps and state-dependent coefficients. One of the key properties of the Lyapunov function f constructed in [8], is that $Df(x) \cdot b \leq -c < 0$ for $x \in G \setminus \{0\}$, where b denotes the constant drift vector of the unconstrained driving Brownian motion. In a state-dependent setting, an analogous condition must hold simultaneously for all b in the range of $\tilde{\beta}$. The construction of the Lyapunov function is therefore much more involved. The basic stability assumption referred to above plays a key role in this construction.

The paper is organized as follows. In Section 2 we present basic definitions, assumptions, statements of main results and their corollaries. Section 3 is devoted to the proof of (1.4), under the assumption that a suitable Lyapunov function exists. We also show in this section that the Markov process is bounded in probability. In Section 4 we present the construction of the Lyapunov function. Since many arguments are similar to those in [8], we have tried to avoid repetition wherever possible. Finally, we have included certain standard arguments in the appendix for the sake of completeness.

The following notation is used in this paper. The boundary relative to $I\!R^n$ of a set $A \subset I\!R^n$ is denoted by ∂A. The convex hull of A is denoted by $\mathrm{conv}(A)$. The cone $\{\sum_{i\in I} \alpha_i v_i : \alpha_i \geq 0, i \in I\}$ generated by $(v_i, i \in I)$, is denoted by $\mathrm{cone}\{v_i, i \in I\}$. The open ball of radius r about x is denoted by $B(x,r)$, and the unit sphere in $I\!R^n$ by S^{n-1}. $D([0,\infty) : I\!R^n)$ denotes the space of functions mapping $[0,\infty)$ to $I\!R^n$ that are right continuous and have limits from the left. We endow $D([0,\infty) : I\!R^n)$ with the usual Skorohod topology. We define $D_A([0,\infty) : I\!R^n) \doteq \{\psi \in D([0,\infty) : I\!R^n) : \psi(0) \in A\}$. For $\eta \in D([0,\infty) : I\!R^n)$, $|\eta|(T)$ denotes the total variation of η on $[0,T]$ with respect to the Euclidean norm on $I\!R^n$. The Borel σ-field on $I\!R^n$ is denoted by $\mathcal{B}(I\!R^n)$ and the space of probability measures on $(I\!R^n, \mathcal{B}(I\!R^n))$ by $\mathcal{P}(I\!R^n)$. Finally, α denotes a positive constant, whose value is unimportant and may change from line to line.

2 Setting and results

Recall from Section 1 the assumptions on the set G and the definition of the vector field d,

$$d(x) \doteq \mathrm{cone}\{d_i, i \in \mathrm{In}(x)\} \cap S^{n-1}.$$

For $x \in \partial G$, define the set $n(x)$ of inward normals to G at x by

$$n(x) \doteq \{\nu : |\nu| = 1, \quad \nu \cdot (x-y) \leq 0, \ \ \forall\, y \in G\}.$$

Let Λ be the collection of all the subsets of $\{1, 2, \ldots, N\}$. We will make the following basic assumption regarding the vectors (d_i, n_i).

Condition 2.1. *For each $\lambda \in \Lambda$, $\lambda \neq \emptyset$, there exists a vector $d^\lambda \in \mathrm{cone}\{d_i, i \in \lambda\}$ with*

$$d^\lambda \cdot n_i > 0 \quad \textit{for all} \ \ i \in \lambda. \tag{2.1}$$

Remark 2.2. An important consequence (cf. [8]) of the above assumption is that for each $\lambda \in \Lambda, \lambda \neq \emptyset$ there exists a vector n^λ such that $n^\lambda \in n(x)$ for all $x \in G$ satisfying $\mathrm{In}(x) = \lambda$ and

$$n^\lambda \cdot d_i > 0 \quad \text{for all} \ \ i \in \lambda. \tag{2.2}$$

Definition 2.3. *Let $\psi \in D_G([0,\infty) : I\!R^n)$ be given. Then $(\phi, \eta) \in D([0,\infty) : I\!R^n) \times D([0,\infty) : I\!R^n)$ solves the Skorohod problem (SP) for ψ with respect to G and d if and only if $\phi(0) = \psi(0)$, and for all $t \in [0,\infty)$ (1) $\phi(t) = \psi(t) + \eta(t)$; (2) $\phi(t) \in G$; (3) $|\eta|(t) < \infty$; (4) $|\eta|(t) = \int_{[0,t]} I_{\{\phi(s)\in\partial G\}} d|\eta|(s)$; (5) There exists (Borel) measurable $\gamma : [0,\infty) \to I\!R^k$ such that $\gamma(t) \in d(\phi(t))$ ($d|\eta|$-almost everywhere) and $\eta(t) = \int_{[0,t]} \gamma(s) d|\eta|(s)$.*

On the domain $D \subset D_G([0,\infty) : I\!R^n)$ on which there is a unique solution to the SP we define the Skorohod map (SM) Γ as $\Gamma(\psi) \doteq \phi$, if $(\phi, \psi - \phi)$ is the unique solution of the SP posed by ψ. We will make the following assumption on the regularity of the SM defined by the data $\{(d_i, n_i); i = 1, 2, \ldots, N\}$.

Condition 2.4. *The SM is well defined on all of $D_G([0,\infty):I\!R^n)$, i.e., $D = D_G([0,\infty):I\!R^n)$ and the SM is Lipschitz continuous in the following sense. There exists a constant $\ell<\infty$ such that for all $\phi_1,\phi_2 \in D_G([0,\infty):I\!R^n)$:*

$$\sup_{0\le t<\infty} |\Gamma(\phi_1)(t)-\Gamma(\phi_2)(t)| \le \ell \sup_{0\le t<\infty} |\phi_1(t)-\phi_2(t)|. \tag{2.3}$$

We will assume without loss of generality that $\ell \ge 1$. We refer the reader to [6, 7, 10] for sufficient conditions for this regularity property to hold.

We now introduce the constrained processes that will be studied in this paper.

Definition 2.5. *Let (X_t) be a Lévy process starting from zero (i.e. $X_0=0$) , with the Lévy measure K on $(I\!R^n,\mathcal{B}(I\!R^n))$. Define a "constrained Lévy process", starting from $z_0\in G$, by the relation*

$$Z \doteq \Gamma(z_0+X).$$

Recall that a Lévy measure K is a measure that satisfies the condition $\int_{I\!R^n}|y|^2\wedge 1K(dy)<\infty$ (see [2], Chapter 1). We will make one additional assumption on K, as follows.

Condition 2.6. *The Lévy measure K satisfies*

$$\int_{I\!R^n} |y|1_{|y|\ge 1}K(dy)<\infty.$$

The above assumption holds if and only if the Lévy process X_t has finite mean.

We now define the reflected jump-diffusions considered in this work. On a complete filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t),P)$, let an m-dimensional standard Brownian motion W and a Poisson random measure N on $I\!R_+\times E$, with intensity measure $q(dt,dz)=dt\otimes F(dz)$ be given. Here, $(E,\mathcal{E})$ is a Blackwell space and F is a positive σ-finite measure on $(E,\mathcal{E})$. For all practical purposes, $(E,\mathcal{E})$ can be taken to be $(I\!R^n,\mathcal{B}(I\!R^n))$ (see [14]). Let a truncation function $h: I\!R^n\to I\!R^n$ be a continuous bounded function satisfying $h(x)=x$ is a neighborhood of the origin and with compact support. We fix such a function throughout, and denote also $h'(x)\doteq x-h(x)$. The reflected jump-diffusion process (Z_t) is given as the strong solution to the set of equations (1.2), (1.3). The following conditions will be assumed on the coefficients and the intensity measure.

Condition 2.7. *There exists $\theta\in(0,\infty)$ and a measurable function $\rho: E\to[0,\infty)$ such that*

$$\int_E \rho^2(z)F(dz)<\infty,$$

and the following conditions hold.

(i) Lipschitz Condition: For all $y,y'\in I\!R^n$, $z\in E$,

$$|\beta(y)-\beta(y')|+|a(y)-a(y')|\le\theta|y-y'|,$$

$$|h(\delta(y,z))-h(\delta(y',z))|\le\rho(z)|y-y'|,$$

$$|h'(\delta(y,z))-h'(\delta(y',z))|\le\rho^2(z)|y-y'|.$$

(ii) Growth Condition: For all $y \in I\!R^n$, $z \in E$,

$$\frac{|\beta(y)|}{1+|y|} + |a(y)| \leq \theta,$$

$$|h(\delta(y,z))| \leq \rho(z),$$
$$|h'(\delta(y,z))| \leq \rho^2(z) \wedge \rho^4(z).$$

Under the above conditions it can be shown that there is a unique strong solution to (1.2) and (1.3) which is a strong Markov process. I.e., the following result holds.

Theorem 2.8. *Suppose that Conditions 2.4 and 2.7 hold, and that on* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ *we are given processes* (W, N) *as above. Then, for all* $x \in G$ *there exists, on the basis* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, *a unique pair of* $\{\mathcal{F}_t\}$*-adapted processes* $(Z_t, k_t)_{t\geq 0}$ *with paths in* $D([0,\infty) : I\!R^n)$, *and a progressively measurable process* $(\gamma_t)_{t\geq 0}$, *such that the following hold:*

1. $Z_t \in G$, *for all* $t \geq 0$, *a.s.*
2. *For all* $t \geq 0$,
$$\begin{aligned} Z_t &= x + \int_0^t \beta(Z_s)ds + \int_0^t a(Z_s)dW_s + \int_{[0,t]\times E} h(\delta(Z_{s-}, z))[N(ds,dz) - q(ds,dz)] \\ &+ \int_{[0,t]\times E} h'(\delta(Z_{s-}, z))N(ds,dz) + k_t, \end{aligned} \tag{2.4}$$
a.s.
3. *For all* $T \in [0, \infty)$
$$|k|_T < \infty, \quad a.s.$$
4. $$|k|_t = \int_0^t I_{\{Z_s \in \partial G\}} d|k|_s,$$
and $k_t = \int_0^t \gamma_s d|k|_s$ *with* $\gamma_s \in d(Z_s)$ *a.e.* $[d|k|]$.

Furthermore, the pair $(Z_t - k_t, Z_t)$ *is the unique* $\{\mathcal{F}_t\}$*-adapted pair of processes with cadlag paths which satisfies equations (1.2, 1.3) for all* t, *a.s., with the given driving terms* (W, N). *Finally,* (Z_t) *is a strong Markov process on* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$.

The proof of the theorem follows via the usual Picard iteration method on using the Lipschitz property of the SM. We refer the reader to [6] where a similar argument for constrained diffusion processes is presented.

Remark 2.9. Condition 2.7 is a version of the assumptions in [14], Chapter III, where strong existence and uniqueness results for unconstrained jump-diffusion processes are considered. The conditions assumed there are substantially weaker, and can be similarly weakened in the current context as well, via similar arguments.

Remark 2.10. Taking $a(z) \equiv a$, $\beta(z) \equiv \beta$, $(E, \mathcal{E}) \equiv (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, $\delta(y,z) \doteq z$, $\rho(z) \doteq |z| 1_{|z| \le 1} + \sqrt{|z|} 1_{|z| \ge 1}$ and $F(dz) \equiv K(dz)$, we see that a Lévy process satisfying Condition 2.6 is a special case of the process $\{Z_t\}$ in Theorem 2.8.

Here are the main results of this paper. The first result gives sufficient conditions for transience and stability of a reflecting Lévy process. The transience proof is a simple consequence of the law of large numbers, while the stability is treated in a more general framework in the context of a reflected jump-diffusion process. For a Borel set $A \subset G$, let τ_A denote the first time Z hits A. Define

$$\mathcal{C} \doteq \text{cone}\{-d_i, i \in \{1, \ldots, N\}\}. \tag{2.5}$$

Theorem 2.11. *Let X and Z be as in Definition 2.5. Assume that Conditions 2.1, 2.4 and 2.6 hold.*

1. *If $EX_1 \in \mathcal{C}^c$, then there is a constant $\gamma \in G \setminus \{0\}$ such that for all $x \in G$, $Z_t/t \to \gamma$ as $t \to \infty$, P_x-a.s.*
2. *If $EX_1 \in \mathcal{C}^o$, then there is a compact set A such that for all $M \in (0, \infty)$,*

$$\sup_{z \in G, |z| \le M} E_z \tau_A < \infty.$$

Next we consider reflected jump-diffusion processes. If in equation (1.2) X were replaced by Z, and the coefficients a, β and δ were extended to all of $\mathbb{R}^n$, then this equation alone would define a diffusion process with jumps Z, the extended generator of which we denote by $\mathcal{L}$ (see [14], Chapter IX, p. 514 for the form of the extended generator in this setting). Let $\text{id} : \mathbb{R}^n \to \mathbb{R}^n$ denote the identity map, and define

$$\tilde{\beta} \doteq \mathcal{L}\,\text{id} = \beta + \int_E h'(\delta(\cdot, z)) F(dz). \tag{2.6}$$

Note that in view of Condition 2.7, there is a constant $\alpha < \infty$ such that

$$\sup_{x \in \mathbb{R}^n} \left| \int_E h'(\delta(x,z)) F(dz) \right| \le \alpha. \tag{2.7}$$

We use the generator of the "unconstrained" jump-diffusion process only as a motivation to define the vector field $\tilde{\beta}$. Since we only deal with constrained diffusions, we will consider only the restriction of $\tilde{\beta}$ to G, which, with an abuse of notation, we still denote by $\tilde{\beta}$. Of course, $\tilde{\beta}$ can otherwise be defined by the right hand side of (2.6). Our main assumption on $\tilde{\beta}$ is the following.

Condition 2.12. *There exists a compact set $\mathcal{C}_1$ contained in the interior $\mathcal{C}^o$ of $\mathcal{C}$ such that the range of $\tilde{\beta}$ is contained in $\cup_{k \in \mathbb{N}} k\mathcal{C}_1$.*

Here is the main result on the stability of reflected jump-diffusions.

Theorem 2.13. *Let (Z_t) be as in Theorem 2.8. Suppose that Conditions 2.1, 2.4, 2.7 and 2.12 hold. Then there is a compact set A such that for any compact $K \subset G$, $\sup_{z \in K} E_z \tau_A < \infty$.*

Remark 2.14. We will, in fact, obtain a more precise bound, namely $E_z\tau_A \le \alpha|z|+1$, for some constant α independent of $z \in G$.

As an immediate corollary of the above theorem we have the following result.

Corollary 2.15. *Let $p(t,x,dy)$ denote the transition probability function of the Markov process $\{Z_t\}$. Suppose that there is a closed set $S \subset G$ such that $p(t,x,S) = 1$ for all $x \in S$ and $t \in (0,\infty)$. Let the compact set A be as in Theorem 2.13 and suppose that the assumptions of that theorem hold. Then $\sup_{z\in S, |z|\le M} E_z(\tau_{A\cap S}) < \infty$ for all $M \in (0,\infty)$.*

The following result on "boundedness in probability" of the process $\{Z_t\}$ is a consequence of the existence of a suitable Lyapunov function and will be proved in Section 3.

Theorem 2.16. *Let the assumptions of Theorem 2.13 hold. Then for every $M \in (0,\infty)$, the family of probability measures, $\{P_z(Z(t) \in \cdot);\ t \in [0,\infty),\ z \in G \cap B(0,M)\}$ is tight.*

From the above result we have, on using the Feller property of $\{Z_t\}$, the following corollary.

Corollary 2.17. *Suppose the assumptions of Theorem 2.13 hold. Then the Markov process $\{Z_t\}$ admits at least one invariant measure.*

We now impose the following communicability condition on the Markov process $\{Z_t\}$ relative to a set S.

Condition 2.18. *Let S be as in Corollary 2.15 and let ν be a σ-finite measure with support S. Then for all $r \in (0,\infty)$ and $C \in \mathcal{B}(I\!R^n)$ with $\nu(C) > 0$, $\inf_{x\in S,|x|\le r} P_x(Z_1 \in C) > 0$.*

The above assumption is satisfied with $S = G$ and ν as the Lebesgue measure, if the diffusion coefficient a in (2.4) is uniformly non degenerate.

Now we can give the following result on positive Harris recurrence. The proof of the theorem is similar to that of Theorem 2.2 of [1] and thus is omitted.

Theorem 2.19. *Let the assumptions of Theorem 2.13 and Corollary 2.15 hold. Further suppose that Condition 2.18 holds. Then for all closed sets C with $\nu(C) > 0$, and all $M > 0$, we have that $\sup_{z\in S,|z|\le M} I\!E_z(\tau_C) < \infty$.*

Finally, we introduce one more condition which again is satisfied if the diffusion coefficient is uniformly non degenerate and ν is the Lebesgue measure on G.

Condition 2.20. *For some $\lambda \in (0,\infty)$, the probability measure θ on G defined as*

$$\theta(F) \doteq \lambda \int_0^\infty e^{-\lambda t} p(t,x,F)dt, \quad F \in \mathcal{B}(G)$$

is absolutely continuous with respect to ν.

The following theorem is a direct consequence of Theorem 4.2.23, Chapter 1 of [21].

Theorem 2.21. *Let the assumptions in Theorem 2.19 hold. Further suppose that Condition 2.20 holds. Then (Z_t) has a unique invariant probability measure.*

3 Proofs of the main results

We begin with the proof of Theorem 2.11.

Proof of Theorem 2.11: Since part 2 is a special case of Theorem 2.13 (note that Condition 2.7 implies, in the special case of a Lévy process, Condition 2.6), we consider only part 1. Let $\beta = EX_1$ and let $\phi(t) = \beta t$. Then by [3, Lemma 3.1 and Theorem 3.10(2)], $\Gamma(\phi)(t) = \gamma t$, where $\gamma \neq 0$. By the Lipschitz continuity of Γ, $Z_t = \Gamma(\phi)(t) + \lambda_t = \gamma t + \lambda_t$, where

$$|\lambda_t| \leq \ell \sup_{s \leq t} |z_0 + X_s - s\beta|.$$

From the strong law of large numbers $t^{-1}(X_t - t\beta) \to 0$ a.s. Combined with a.s. local boundedness of X, this implies that $\sup_{s \leq t} |z_0 + X_s - s\beta| \to 0$ a.s. Thus $t^{-1}|\lambda_t| \to 0$ a.s., and this proves the result. ■

In the rest of the paper we prove Theorem 2.13 and its consequences. Hence we will assume throughout that Conditions 2.1, 2.4, 2.7 and 2.12 hold. The proof of Theorem 2.13 is based on the existence of a suitable Lyapunov function which is defined as follows.

Definition 3.1. *1. We say that a function $f \in C^2(G \setminus \{0\})$ is a* Lyapunov function for the SP (G, d) with respect to the mean velocity r_0, *if the following conditions hold.*

(a) For all $N \in (0, \infty)$, there exists $M \in (0, \infty)$ such that $(x \in G,\ |x| \geq M)$ implies that $f(x) \geq N$.

(b) For all $\epsilon > 0$ there exists $M \in (0, \infty)$ such that $(x \in G,\ |x| \geq M)$ implies $\|D^2 f(x)\| \leq \epsilon$.

(c) There exists $c \in (0, \infty)$ such that $Df(x) \cdot r_0 \leq -c$, $x \in G \setminus \{0\}$, and $Df(x) \cdot d \leq -c$, $d \in d(x)$, $x \in \partial G \setminus \{0\}$.

(d) There exists $L \in (0, \infty)$ such that $\sup_{x \in G} |Df(x)| \leq L$.

2. We say that a function $f \in C^2(G \setminus \{0\})$ is a Lyapunov function for the SP (G, d) with respect to the set (of mean velocities) $\tilde{R} \subset I\!R^n$, *if it is a Lyapunov function for the SP (G, d) with respect to the mean velocity r_0, for any $r_0 \in \tilde{R}$, and if in item (c) above, the constant c does not depend on $r_0 \in \tilde{R}$.*

Remark 3.2. (a) If f is a Lyapunov function for the SP (G, r) with respect to a certain set $\tilde{R}$, then Df is Lipschitz continuous on $\{x \in G : |x| \geq M\}$ with parameter ϵ, where $\epsilon > 0$ can be taken arbitrarily small by letting M be large. This implies a useful consequence of the second part of item (c) in Definition 3.1 as follows: There exist $M_0 \in (0, \infty)$, $\delta_0 \in (0, 1)$ such that $Df(x) \cdot d \leq -c/2$, whenever $d \in d(y)$, $|y - x| \leq \delta_0$, $y \in \partial G$, $|x| \geq M_0$.

(b) If f is a Lyapunov function for (G, d) with respect to a set $\tilde{R}$, then it is automatically a Lyapunov function for (G, d) with respect to $\cup_{k \in I\!N} k\tilde{R}$.

We say that a function f is radially linear on G if $f(sx) = sf(x)$ for all $s \in (0, \infty)$ and $x \in G$.

The following result is key to the proof of Theorem 2.13 and will be proved in Section 4.

Theorem 3.3. *Let the assumptions of Theorem 2.13 hold. Then there exists a Lyapunov function f for the SP (G, d) with respect to the set $\mathcal{C}_1$, where $\mathcal{C}_1$ is as in Condition 2.12. Furthermore, f is radially linear on G.*

We now turn to the proof of Theorem 2.13. Write (2.4) as

$$Z_t = z_0 + \int_0^t \tilde{\beta}(Z_s)ds + \int_0^t a(Z_s)dW_s + M_t^{(1)} + M_t^{(2)} + k_t,$$

where

$$M_t^{(1)} = \int_{[0,t]\times E} h(\delta(Z_{s-}, z))[N(ds,dz) - q(ds,dz)],$$
$$M_t^{(2)} = \int_{[0,t]\times E} h'(\delta(Z_{s-}, z))[N(ds,dz) - q(ds,dz)]$$

and $\tilde{\beta}(\cdot)$ is as in (2.6). Note that the term that has been subtracted and added is finite (e.g., by (2.7)). Let also

$$U_t = \int_0^t \tilde{\beta}(Z_s)ds,$$

and

$$M_t = \int_0^t a(Z_s)dW_s + M_t^{(1)} + M_t^{(2)}.$$

Then

$$Z_t = z_0 + U_t + M_t + k_t. \tag{3.1}$$

Let f be as in Theorem 3.3. From Condition 2.12, it follows (see also Remark 3.2(b)) that

$$Df(x)\cdot \dot{U}_t \le -c, \quad x \in G\setminus\{0\}, \quad t \ge 0, \tag{3.2}$$

where c is as in Definition 3.1. For any $\kappa \in (0,\infty)$ and compact set $A \subset I\!R^n$, define the sequences $(\tilde{\sigma}_n)$, (σ_n) of stopping times as $\tilde{\sigma}_0 = 0$,

$$\tilde{\sigma}_n = \tilde{\sigma}_n(\kappa) \doteq \inf\{t > \tilde{\sigma}_{n-1} : |X_t - X_{\tilde{\sigma}_{n-1}}| \ge \kappa\},$$
$$\sigma_n = \sigma_n(\kappa, A) \doteq \tilde{\sigma}_n \wedge \tau_A.$$

Let also

$$\tilde{n}(t) = \inf\{n : \tilde{\sigma}_n \ge t\},$$
$$\bar{n}(t) = \bar{n}(t,A) = \inf\{n : \sigma_n \ge \tau_A \wedge t\},$$

where the infimum over an empty set is ∞. Note that $\tilde{n}(t\wedge\tau_A) = \bar{n}(t)$, a.s. The following are the main lemmas used in the proof of Theorem 2.13.

Lemma 3.4. *$\{M_t^{(1)}\}$ and $\{\int_0^t a(Z_s)dW_s\}$ are square integrable martingales and $\{M_t^{(2)}\}$ is a martingale.*

Lemma 3.5. *There exists a constant $c_1 = c_1(\kappa) \in (1,\infty)$ such that for any bounded stopping time τ, $E\tilde{n}(\tau) \le c_1(E\tau + 1)$.*

For $s \in [0,\infty)$, and a cadlag process $\{Y_t\}$, we write $Y_s - Y_{s-}$ as ΔY_s.

Lemma 3.6. *There is a $b_0 \in (0,\infty)$ and a function $\bar{\alpha} : [0,\infty) \mapsto [0,\infty)$ with $\bar{\alpha}(b) \to 0$ as $b \to \infty$ such that for any bounded stopping time τ, and $b > b_0$,*

$$E\sum_{s\le\tau} |\Delta X_s| 1_{|\Delta X_s|>b} \le \bar{\alpha}(b)E\tau.$$

Proof of Theorem 2.13: Let f be as in Theorem 3.3 and let a_0 be as in (1.1). For any M, the level set $\{x \in G : f(x) \leq M\}$ is compact. If $\tilde{M} = \tilde{M}(M) = \max\{|x| : f(x) \leq M\}$, then the set

$$A = A(M) = \{x \in G : x \cdot a_0 \leq \tilde{M}\} \tag{3.3}$$

contains the level set, and is compact. In addition, $G \setminus A$ is convex. Definition 3.1.1(a),(b) implies that there is a function $\epsilon_0(\tilde{M})$ such that $\|D^2 f(x)\| \leq \epsilon_0(\tilde{M})$ for $x \in G \setminus A(M)$, and where $\tilde{M} = \tilde{M}(M)$ is as above, and $\epsilon_0(\tilde{M}) \to 0$ as $\tilde{M} \to \infty$. The notation $\epsilon_0(\tilde{M})$ and $\tilde{M}(M)$ is used in what follows.

Write $x_n = X_{\sigma_n}$, $x_{n-} = X_{\sigma_n -}$, where $\{X_t\}$ is as in (1.3). Define similarly k_n, k_{n-}, z_n, z_{n-}, u_n, u_{n-}, m_n, m_{n-} for the processes k, Z, U and M, respectively. Let κ be so small that $2\ell\kappa \leq \delta_0/2$, where δ_0 is as in Remark 3.2(a). κ will be fixed throughout. The proof will be based on establishing a bound on $Ef(z_m) = f(z_0) + \sum_1^m E[f(z_n) - f(z_{n-1})]$. According to (3.1), one has

$$\begin{aligned} z_n - z_{n-1} &= x_n - x_{n-1} + k_n - k_{n-1} \\ &= u_n - u_{n-1} + m_n - m_{n-1} + k_n - k_{n-1}. \end{aligned}$$

We consider two cases.

Case 1: $|x_n - x_{n-1}| \leq 2\kappa$.

Consider the linear interpolation z^θ defined for $\theta \in [0,1]$ as

$$z^\theta = z_{n-1} + \theta(z_n - z_{n-1}).$$

Then

$$f(z_n) - f(z_{n-1}) = \int_0^1 Df(z^\theta) d\theta \cdot (z_n - z_{n-1}). \tag{3.4}$$

By the Lipschitz continuity of the SM, $z^\theta \in B_{2\ell\kappa}(z^0) \subset B_{\delta_0/2}(z^0)$ for $\theta \in [0,1]$. Also note that for $s \in [\sigma_{n-1}, \sigma_n]$,

$$\gamma_s \in \bigcup_{x \in B_{2\ell\kappa}(z^0)} d(x) \subset \bigcup_{x \in B_{\delta_0/2}(z^0)} d(x), \quad [d|k|] \ a.s. \tag{3.5}$$

Let M be so large that $\tilde{M} \geq M_0 + 1$, where M_0 is as in Remark 3.2(a). Then any $x \in A = A(M)$ satisfies $|x| \geq M_0 + 1$. By convexity of A we therefore have for $n \leq \bar{n}(\infty)$ that $|z^\theta| \geq \tilde{M} \geq M_0 + 1$, and we get from (3.5) that for $\theta \in [0,1]$,

$$\begin{aligned} Df(z^\theta) \cdot [k_n - k_{n-1}] &= Df(z^\theta) \int_{\sigma_{n-1}}^{\sigma_n} \gamma_s d|k|_s \\ &\leq -\frac{c}{2}(|k|_n - |k|_{n-1}) \leq 0. \end{aligned} \tag{3.6}$$

Now let $\epsilon \doteq \epsilon_0(\tilde{M} - \ell b)$. By (3.2), $Df(z^0) \cdot (u_n - u_{n-1}) \leq -c(\sigma_n - \sigma_{n-1})$. Therefore, from part (b) of Definition 3.1, and (3.4), (3.6), we have

$$\begin{aligned} f(z_n) - f(z_{n-1}) &\leq Df(z_{n-1})(z_n - z_{n-1}) + \int_0^1 |Df(z^\theta) - Df(z^0)| d\theta \, |z_n - z_{n-1}| \\ &\leq Df(z_{n-1})(z_n - z_{n-1}) + (\epsilon)(2\ell\kappa)(2\ell\kappa) \\ &\leq -c(\sigma_n - \sigma_{n-1}) + Df(z_{n-1})(m_n - m_{n-1}) + 4(\ell)^2 \epsilon \kappa^2. \end{aligned} \tag{3.7}$$

Case 2: $|x_n - x_{n-1}| > 2\kappa$.

The argument applied in Case 1 gives an analogue of (3.7) in the form

$$f(z_{n-}) - f(z_{n-1}) \le -c(\sigma_n - \sigma_{n-1}) + Df(z_{n-1})(m_{n-} - m_{n-1}) + 4\ell\epsilon\kappa^2. \tag{3.8}$$

Next we provide a bound on $f(z_n) - f(z_{n-})$. Let

$$\hat{z}^\theta = z_{n-} + \theta(z_n - z_{n-}), \quad \theta \in [0,1].$$

Note that $k_n - k_{n-} \in d(z_n)$, by Definition 2.3, and therefore $Df(z_n)\cdot(k_n - k_{n-}) \le 0$. Also, recall that $|Df| \le L$. Let b_0 be as in Lemma 3.6 and $b > b_0$ be arbitrary. Then if $|x_n - x_{n-}| \le b$, then for all $\theta \in [0,1]$, $|\hat{z}^\theta - \hat{z}^0| \le \ell b$, and therefore the bound $\|D^2 f(z^\theta)\| \le \epsilon \doteq \epsilon_0(\tilde{M}(M) - \ell b)$ holds. Thus

$$\begin{aligned}
f(z_n) - f(z_{n-}) &= \left(\int_0^1 Df(\hat{z}^\theta)d\theta\right)\cdot[(x_n - x_{n-}) + (k_n - k_{n-})] \\
&= Df(z_{n-})\cdot(x_n - x_{n-}) + \int_0^1 (Df(\hat{z}^\theta) - Df(z_{n-}))d\theta\cdot(x_n - x_{n-}) \\
&\quad + Df(z_n)\cdot(k_n - k_{n-}) + \int_0^1 (Df(\hat{z}^\theta) - Df(z_n))d\theta\cdot(k_n - k_{n-}) \\
&\le Df(z_{n-})\cdot(x_n - x_{n-}) + 2\epsilon\ell^2 b^2 + 4\ell L|x_n - x_{n-}|1_{|x_n - x_{n-}|>b} \\
&= Df(z_{n-})\cdot(m_n - m_{n-}) + 2\epsilon\ell^2 b^2 + 4\ell L|x_n - x_{n-}|1_{|x_n - x_{n-}|>b}.
\end{aligned} \tag{3.9}$$

Let $J_q(t)$ denote the set $\{n \le \bar{n}(t)\wedge q : |x_n - x_{n-1}| > 2\kappa\}$. Combining (3.7), (3.8) and (3.9) we get

$$\begin{aligned}
\sum_{n\le\bar{n}(t)\wedge q} f(z_n) - f(z_{n-1}) &\le -c\sigma_{\overline{n}(t)\wedge q} + \sum_{n\le\bar{n}(t)\wedge q} Df(z_{n-1})(m_n - m_{n-1}) + 4\ell\epsilon\kappa^2\overline{n}(t)\wedge q \\
&+ \sum_{n\in J_q(t)} (Df(z_{n-}) - Df(z_{n-1}))\cdot(m_n - m_{n-}) \\
&+ \sum_{n\in J_q(t)} (2\epsilon\ell^2 b^2 + 4\ell L|x_n - x_{n-}|1_{|x_n - x_{n-}|>b}).
\end{aligned}$$

Since $m_n - m_{n-} = x_n - x_{n-}$, the following inequality holds

$$\sum_{J_q(t)} (Df(z_{n-}) - Df(z_{n-1}))\cdot(m_n - m_{n-}) \le \sum_{J_q(t)} (2\epsilon\ell^2 b^2 + 4\ell L|x_n - x_{n-}|1_{|x_n - x_{n-}|>b}).$$

Writing $\tilde{M}_t = \sum_{n\le\bar{n}(t)\wedge q} Df(z_{n-1})(m_n - m_{n-1})$, we get

$$\begin{aligned}
f(z_{\overline{n}(t)\wedge q}) - f(z_0) &= \sum_{n\le\bar{n}(t)\wedge q} (f(z_n) - f(z_{n-1})) \\
&\le -c(\sigma_{\overline{n}(t)\wedge q}) + \tilde{M}_t + 4\ell\epsilon\kappa^2\bar{n}(t) \\
&+ 2\epsilon\ell^2 b^2|J_q(t)| + \sum_{n\in J_q(t)} 4\ell L|x_n - x_{n-}|1_{|x_n - x_{n-}|>b}.
\end{aligned} \tag{3.10}$$

From Lemma 3.4, $E(\tilde{M}_t) = 0$. Using Lemma 3.5

$$E(\bar{n}(t)) = E(\tilde{n}(t \wedge \tau_A) \leq c_1(E(t \wedge \tau_A) + 1).$$

Observing that

$$\sum_{n \in J_q(t)} |x_n - x_{n-}| 1_{|x_n - x_{n-}| > b} \leq \sum_{s \leq t \wedge \tau_A} |\Delta X_s| 1_{|\Delta X_s| > b},$$

we have from Lemma 3.6 that the expectation of the term on the left side above is bounded by $\bar{\alpha}(b) E(\tau_A \wedge t)$. Combining these observations, we have that

$$\begin{aligned} Ef(z_{\overline{n}(t) \wedge q}) - f(z_0) &\leq -cE(\sigma_{\overline{n}(t) \wedge q}) + \epsilon(4\ell\kappa^2 + 2\ell^2 b^2)(c_1(E(\tau_A \wedge t) + 1) \\ &+ 4\ell L \bar{\alpha}(b) E(\tau_A \wedge t). \end{aligned}$$

Let b be so large that $4\ell L \bar{\alpha}(b) \leq c/3$. Recalling the definition of ϵ, let M be so large, thus ϵ so small, that $\epsilon(4\ell\kappa^2 + 2\ell^2 b^2) c_1 \leq c/3$. Then

$$-f(z_0) - c/3 \leq -cE(\sigma_{\overline{n}(t) \wedge q}) + \frac{2c}{3} E(\tau_A \wedge t),$$

Taking $q \to \infty$, and recalling that $\sigma_{\overline{n}(t)} \geq \tau_A \wedge t$, we see that $E(\tau_A \wedge t) \leq 3f(z_0)/c + 1$. Finally, taking $t \to \infty$, we get for each z_0, $E_{z_0} \tau_A \leq 3f(z_0)/c + 1$. Note that κ, c_1, c, L do not depend on z_0, nor do the choices of $M, \epsilon(M), b, \bar{\alpha}(b)$. The result follows. ■

We now present the proof of Theorem 2.16.

Proof of Theorem 2.16. The proof is adapted from [17], pages 146-147. Since the Lyapunov function f satisfies $f(z) \to \infty$ as $|z| \to \infty$, it suffices to show that: For all $\delta > 0$ and $L_0 \in (0, \infty)$ there exists an η such that

$$\inf_{x \in G, |x| \leq L_0} P_x(f(Z(t)) \leq \eta) \geq 1 - \delta. \tag{3.11}$$

Let A be as in the proof of Theorem 2.13. Fix $\lambda > \tilde{M}$ and define $A_\lambda \doteq \{x \in G : x \cdot a_0 \geq \lambda\}$. Let $\overline{\lambda} \doteq \sup\{|x| : x \in A_\lambda^c\}$ and set $\rho \doteq \sup\{f(x) : x \in G \cap B(0,1)\}$. Recalling the radial property of the Lyapunov function we have that for all $x \neq 0$; $f(x) \leq \rho|x|$.

Now we define a sequence of stopping times $\{\tau_n\}$ as follows. Set $\tau_0 = 0$. Define

$$\tau_{2n+1} \doteq \inf\{t > \tau_{2n} : Z(t) \in A\}; \quad n \in I\!N_0$$

and

$$\tau_{2n+2} \doteq \inf\{t > \tau_{2n+1} : Z(t) \in A_\lambda\}; \quad n \in I\!N_0.$$

Without loss of generality, we assume that $\tau_n < \infty$ with probability 1 for all n. From Remark 2.14 we have that, for all $n \in I\!N_0$,

$$\begin{aligned} E(\tau_{2n+1} - \tau_{2n} \mid \mathcal{F}_{\tau_{2n}}) &\leq c|Z_{\tau_{2n}}| + 1 \\ &\leq c(|\Delta Z_{\tau_{2n}}| + \bar{\lambda}) + 1 \\ &\leq \alpha|\Delta Z_{\tau_{2n}}| + \alpha. \end{aligned} \tag{3.12}$$

Next observe that, for all $\eta > \overline{\lambda}\rho$:

$$f(z(t)) \leq \eta, \quad t \in [\tau_{2n+1}, \tau_{2n+2}), n \in I\!N_0. \tag{3.13}$$

Now we claim that there is a constant α such that for all $\eta > 0$ and $x \in G$

$$P_x(\sup_{0 \le t < \tau_1} f(Z(t)) \ge \eta) \le \alpha \frac{f(x)+1}{\eta} \tag{3.14}$$

and for all $n \in I\!N$

$$P(\sup_{\tau_{2n} \le t < \tau_{2n+1}} f(Z(t)) \ge \eta \mid \mathcal{F}_{\tau_{2n}}) \le \alpha \frac{|\Delta Z_{\tau_{2n}}|+1}{\eta}. \tag{3.15}$$

We only show (3.15), since the proof of (3.14) is similar. By arguing as in the proof of Theorem 2.13 (see (3.10)), we have that

$$\sup_{\tau_{2n} \le t < \tau_{2n+1}} f(Z(t)) \le f(Z_{\tau_{2n}}) + L\ell\kappa + \sup_{1 \le k \le \overline{n}(\tau_{2n+1})} \sum_{1 \le j \le k} (f(z_j) - f(z_{j-1})) \tag{3.16}$$

where $\{\sigma_j\}$ and $\overline{n}(\cdot)$ are defined as in the displays below (3.2) with $\tilde{\sigma}_0 \doteq \tau_{2n}$ (rather than 0) and τ_A replaced by τ_{2n+1}. Given a stopping time τ, denote the conditional expectation and conditional probability with respect to the σ-field $\mathcal{F}_\tau$ by $I\!E_\tau$ and $I\!P_\tau$ respectively. Then, we have via arguments as in Theorem 2.13 that

$$\begin{aligned} & I\!E_{\tau_{2n}}(\sup_{1 \le k \le \overline{n}(\tau_{2n+1})} \sum_{1 \le j \le k} (f(z_j) - f(z_{j-1}))) \\ & \le I\!E_{\tau_{2n}}(\sup_{1 \le k \le \overline{n}(\tau_{2n+1})} |\sum_{1 \le j \le k} Df(z_{j-1})(m_j - m_{j-1})|) \\ & \quad + \alpha(I\!E_{\tau_{2n}}(\tau_{2n+1} - \tau_{2n}) + 1). \end{aligned}$$

Doob's inequality yields that

$$I\!E_{\tau_{2n}}(\sup_{1 \le k \le \overline{n}(\tau_{2n+1})} |\sum_{1 \le j \le k} Df(z_{j-1})(m_j - m_{j-1})|) \le \alpha(I\!E_{\tau_{2n}}(\tau_{2n+1} - \tau_{2n}) + 1).$$

Combining the above observations with (3.12) we have that

$$I\!E_{\tau_{2n}}(\sup_{1 \le n \le \overline{n}(\tau_{2n+1})} \sum_{1 \le j \le n} (f(z_j) - f(z_{j-1}))) \le \alpha(|\Delta Z_{\tau_{2n}}| + 1).$$

Combining this with (3.16) we have (3.15).

Following [17] we can choose an integer k_δ and, for each t, an integer valued random variable $j(t,\delta)$ such that $\tau_{j(t,\delta)}$ are stopping times and

$$P(\tau_{j(t,\delta)} \le t \le \tau_{j(t,\delta)+k_\delta}) \ge 1 - \delta/2.$$

Now define $J_i \doteq [\tau_{j(t,\delta)+i-1}, \tau_{j(t,\delta)+i})$ and fix $\eta > \overline{\lambda}\rho$. Let τ' be the hitting time of the set A_λ by Z_t. Then

$$\begin{aligned} P_x(f(Z(t)) \ge \eta) & \le \frac{\delta}{2} + \sum_{i=1}^{k_\delta} P_x(\sup_{s \in J_i} f(Z(s)) \ge \eta) \\ & \le \frac{\delta}{2} + \sum_{i=1}^{k_\delta} E_x(I\!E_{\tau_{j(t,\delta)}}(I\!P_{\tau_{j(t,\delta)+i-1}}(\sup_{s \in J_i} f(Z(s)) \ge \eta))) \end{aligned}$$

$$
\begin{aligned}
+ & \quad P_x(\sup_{0\le s\le\tau_1} f(X(s)) \ge \eta) \\
\le & \quad \frac{\delta}{2} + \frac{\alpha}{\eta}\sum_{i=1}^{k_\delta} E_x(I\!\!E_{\tau_{j(t,\delta)}}(|\Delta Z_{\tau_{j(t,\delta)+i-1}}| + 1)) \\
+ & \quad \frac{\alpha(f(x)+1)}{\eta} \\
\le & \quad \frac{\delta}{2} + \frac{\alpha(f(x)+1+k_\delta+bk_\delta)}{\eta} \\
+ & \quad \frac{\alpha}{\eta} E_x(I\!\!E_{\tau_{j(t,\delta)}}(\sum_{i=1}^{k_\delta} |\Delta Z_{\tau_{j(t,\delta)+i-1}}| 1_{|\Delta Z_{\tau_{j(t,\delta)+i-1}}|>b})), \qquad (3.17)
\end{aligned}
$$

where the third inequality above is the consequence of (3.13), (3.15) and (3.14) and in the fourth inequality $b \in (1,\infty)$ is arbitrary. Next note that

$$
\begin{aligned}
& E_x(I\!\!E_{\tau_{j(t,\delta)}}(\sum_{i=1}^{k_\delta} |\Delta Z_{\tau_{j(t,\delta)+i-1}}| 1_{|\Delta Z_{\tau_{j(t,\delta)+i-1}}|>b})) \\
\le & \quad E_x(I\!\!E_{\tau_{j(t,\delta)}}(\sum_{s\in[\tau_{j(t,\delta)},\tau_{j(t,\delta)+k_\delta+1})} |\Delta Z_s| 1_{|\Delta Z_s|>b})) \\
\le & \quad E_x(I\!\!E_{\tau_{j(t,\delta)}}(\int_{[\tau_{j(t,\delta)},\tau_{j(t,\delta)+k_\delta+1})\times E} h'(\delta(Z_{s-},z)) 1_{|h'(\delta(Z_{s-},z))|>b} F(dz)ds)) \\
\le & \quad (k_\delta+1)\bar{\alpha}(b),
\end{aligned}
$$

where $\bar{\alpha}(b) \doteq \int_E \rho^2(z) 1_{\rho^2(z)>b} F(dz)$.
Using the above observation in (3.17) we have that

$$
P_x(f(Z(t)) \ge \eta) \le \frac{\delta}{2} + \frac{\alpha(f(x)+1+k_\delta+bk_\delta)}{\eta} + \frac{\alpha}{\eta}(k_\delta+1)\bar{\alpha}(b).
$$

The result now follows on taking η suitably large. ■

We now give the proofs of the lemmas.

Proof of Lemma 3.4: Since $a(\cdot)$ is a bounded function we have that $\int_0^t a(Z_s)dW_s$ is a square integrable martingale. In order to show that $M_t^{(2)}$ is a martingale, it suffices to show, in view of Theorem II.1.8 of [14] that for all $T \in [0,\infty)$,

$$
\int_{[0,T]\times E} E|h'(\delta(Z_{s-},z))| q(ds,dz) < \infty.
$$

(The cited theorem states a local martingale property, however the proof there shows the above stronger assertion.) The inequality follows on observing that from Condition 2.7 the above expression is bounded by $T\int_E \rho^2(z)F(dz) < \infty$. Finally, in view of Theorem II.1.33 of [14], to show that $M_t^{(1)}$ is a square integrable martingale, it suffices to show that

$$
\sup_{s\in[0,T]} \int_E E|h(\delta(Z_{s-},z))|^2 F(dz) < \infty.
$$

The last inequality follows, once more from Condition 2.7. ■

Proof of Lemma 3.5: Recall that $X_t = z_0 + \int_0^t \tilde{\beta}(Z_s)ds + \int_0^t a(Z_s)dW_s + M_t^{(1)} + M_t^{(2)}$, where $M_t^{(i)}$ are martingales. Since $M_t^{(1)}$ is a square integrable martingale, by Doob's inequality we have

$$\begin{aligned} E\sup_{s\le\epsilon}|M_s^{(1)}|^2 &\le 4E|M_\epsilon^{(1)}|^2 \\ &= 4E\int_{[0,\epsilon]\times E}|h(\delta(Z_{s-},z))|^2F(dz)ds \\ &\le 4\epsilon\int_E \rho^2(z)F(dz). \end{aligned}$$

Also observe that

$$\begin{aligned} E|M_\epsilon^{(2)}| &\le E|\int_{[0,\epsilon\times E} h'(\delta(Z_{s-},z))N(ds,dz)| + E|\int_{[0,\epsilon]\times E} h'(\delta(Z_{s-},z))q(ds,dz)| \\ &\le 2E\int_{[0,\epsilon]\times E}|h'(\delta(Z_{s-},z))|q(ds,dz) \\ &\le 2\epsilon\int_E \rho^2(z)F(dz), \end{aligned}$$

where the second inequality is a consequence of Theorem II.1.8 of [14] and the last inequality follows from Condition 2.7. Using the linear growth of $\tilde{\beta}$ and the Lipschitz property of Γ, the above moment bounds show that

$$E\sup_{s\le\epsilon}|X_s - z_0| \le \alpha\sqrt{\epsilon} + \alpha\int_0^\epsilon E\sup_{s\le\theta}|X_s - z_0|d\theta.$$

Hence, by Gronwall's inequality, for every $\delta > 0$ there is $\epsilon > 0$ such that $E(\sup_{0\le s\le\epsilon}|X_s - z_0|) \le \delta$. By choosing $\epsilon \in (0,1)$ small enough one can obtain

$$P(\tilde{\sigma}_1 \le \epsilon) = P(\sup_{s\le\epsilon}|X_s - X_0| \ge \kappa) \le 1/2.$$

Let $\mathcal{F}^n = \mathcal{F}_{\tilde{\sigma}_n}$. By the strong Markov property of Z on $\mathcal{F}_t$, and by considering the martingales $M^{(i)}_{\tilde{\sigma}_{n-1}+\epsilon} - M^{(i)}_{\tilde{\sigma}_{n-1}}$ in place of $M_\epsilon^{(i)}$, one obtains that for any n, $P(\tilde{\sigma}_n - \tilde{\sigma}_{n-1} > \epsilon|\mathcal{F}^{n-1}) > 1/2$. Let τ be a bounded $(\mathcal{F}_t)$-stopping time. An application of Chebychev's inequality and the observation that since $\epsilon \in (0,1)$ the sets $\{(\tilde{\sigma}_i - \tilde{\sigma}_{i-1}) > \epsilon\}$ and $\{(\tilde{\sigma}_i - \tilde{\sigma}_{i-1}) \wedge 1 > \epsilon\}$ are equal, we have that

$$\frac{\epsilon}{2}E[\tilde{n}(\tau)\wedge k] \le E\sum_{i=1}^{\tilde{n}(\tau)\wedge k} E[(\tilde{\sigma}_i - \tilde{\sigma}_{i-1})\wedge 1|\mathcal{F}^{i-1}]. \tag{3.18}$$

Define

$$S_j \doteq \sum_{i=1}^{j}\left((\tilde{\sigma}_i - \tilde{\sigma}_{i-1})\wedge 1 - E[(\tilde{\sigma}_i - \tilde{\sigma}_{i-1})\wedge 1|\mathcal{F}^{i-1}]\right)$$

Then $(S_j, \mathcal{F}^j)$ is a zero mean martingale. Observing that $\tilde{n}_\tau$ is a stopping time on the filtration $(\mathcal{F}^n)$ we have that for all $k \in I\!N$, $E(S_{\tilde{n}(\tau)\wedge k}) = 0$. Hence from (3.18) it follows that

$$\frac{\epsilon}{2}E[\tilde{n}(\tau)\wedge k] \le E\sum_{i=1}^{\tilde{n}(\tau)\wedge k}[(\tilde{\sigma}_i - \tilde{\sigma}_{i-1})\wedge 1] \le E(\tilde{\sigma}_{\tilde{n}(\tau)-1}) + 1 \le E\tau + 1.$$

Taking $k \uparrow \infty$, the result follows. ■

Proof of Lemma 3.6: Let $b \in (0,\infty)$ be large enough so that $h(x) = 0$ for $|x| \geq \frac{b}{2}$ and $\sup_{x\in I\!R^n} |h(x)| \leq \frac{b}{2}$. Now let $\psi : I\!R^n \to [0,\infty)$ be defined as $\psi(z) \doteq |z| 1_{b\leq |z| \leq b'}$, where $b' \in (b,\infty)$. Clearly, for all $x \in I\!R^n$, $\psi(x) = \psi(h'(x))$. Now from Theorem II.1.8 of [14]

$$\begin{aligned} E\sum_{s\leq\tau} |\Delta X_s| 1_{b\leq|\Delta X_s|\leq b'} &= E\int_{[0,\tau]\times E} |h'(\delta(Z_{s-},z))| 1_{b\leq |h'(\delta(Z_{s-},z))|\leq b'} F(dz)ds \\ &\leq E(\tau)\int_E \rho^2(z) 1_{b\leq\rho^2(z)} F(dz) \\ &\leq \bar{\alpha}(b)E(\tau), \end{aligned}$$

where $\bar{\alpha}(b) \to 0$ as $b \to \infty$. The result now follows upon taking $b' \to \infty$. ■

4 Construction of the Lyapunov function

This section is devoted to the proof of Theorem 3.3. We begin with a stability result on constrained deterministic trajectories which was proved in [1].

Let $\mathcal{C}_1$ be as in Condition 2.12. Let $\delta > 0$ be such that $\mathrm{dist}(x, \partial\mathcal{C}) \geq \delta$ for all $x \in \mathcal{C}_1$. Define

$$V = \{v \in B : \int_0^t |v(s)|ds < \infty, v(t) \in \mathcal{C}_1,\ t \in (0,\infty)\},$$

where B is the set of measurable maps $[0,\infty) \to I\!R^n$. For $x \in G$ let

$$\mathbf{Z}_x = \{\Gamma(x + \int_0^{\cdot} v(s)ds) : v \in V\}.$$

Proposition 4.1. [1] *For any $x \in G$ and $z \in \mathbf{Z}_x$, the following holds:*

$$|z(t)| \leq \frac{\ell^2|x|^2}{\ell|x| + \delta t}, \quad t \in [0,\infty),$$

where ℓ is the finite constant in (2.3).

Using the above result, the following was used in [1] as a Lyapunov function:

$$T(x) \doteq \sup_z \inf\{t \in [0,\infty) : z(t) = 0\}, \tag{4.19}$$

where the supremum is taken over all trajectories $z \in \mathbf{Z}_x$. This function played a key role in the proof of positive recurrence of certain constrained diffusion processes studied in [1]. The proof in [1] uses crucially certain estimates on the exponential moments of the Markov process. Since, in the setting of the current work the Markov process need not even have finite second moment, the techniques of [1] do not apply. However, we will show that by using the ideas from [8] and by suitable smoothing and modifying the hitting time function $T(\cdot)$, one can obtain a Lyapunov function in the sense of Definition 3.1(2) with $\tilde{R}$ there replaced by $\mathcal{C}_1$. Since for $z \in \mathbf{Z}_x$, $z(s) = 0$ implies $z(t) = 0$ for $t > s$, the function $T(\cdot)$ can be rewritten as

$$T(x) \doteq \sup_{\mathbf{Z}_x} \int_0^\infty 1_{(0,\infty)}(|z(s)|)ds.$$

Our first step in the construction is to replace the above indicator function by a smooth function η defined as follows. Let $\eta : I\!R \to [0,1]$ be in $\mathcal{C}^\infty(I\!R)$. Further assume that $\eta(z) = 0$ for all $z \in (-\infty, 1]$, $\eta(z) = 1$ for all $z \in [2, \infty)$ and $\eta'(z) \geq 0$ for all $z \in I\!R$. The next step in constructing a C^2 Lyapunov function is an appropriate modification of this new $T(\cdot)$ function near the boundary and a suitable extension of the function to a neighborhood of G.

For each $\lambda \in \Lambda$, $\lambda \neq \emptyset$, fix a vector d^λ as in Condition 2.1. Define for $\beta, x \in I\!R^n$

$$\begin{aligned} v(\beta, x) &= \beta \quad \text{for } x \in G \\ &= d^{\lambda(x)} \quad \text{for } x \notin G, \end{aligned}$$

where $\lambda(x) = \{i \in \{1, \ldots, N\} : x \cdot n_i \leq 0\}$. Now $\rho \in C^\infty(I\!R^n)$ be such that the support of ρ is contained in $\{x : |x| \leq 1\}$ and $\int_{I\!R^n} \rho(x)dx = 1$. Define for $a > 0$

$$v^a(\beta, x) \doteq \frac{1}{(a|x|)^n} \int_{I\!R^n} \rho(\frac{x-y}{a|x|}) v(\beta, y) dy, \quad x \neq 0.$$

Now let $g : I\!R \to [0,1]$ be a smooth function such that $g(z) = 1$ for $z \in [0, \frac{1}{2}]$ and $g(z) = 0$ for $z \in [1, \infty)$. Define for $i = 1, \ldots, N$, $x \neq 0$ and $\beta \in I\!R^n$

$$v_i^a(\beta, x) = g(\frac{\text{dist}(x, F_i)}{a|x|}) d_i + \left[1 - g(\frac{\text{dist}(x, F_i)}{a|x|})\right] v^a(\beta, x)$$

and

$$v_0^a(\beta, x) = g(\frac{\text{dist}(x, G)}{a|x|}) \beta + \left[1 - g(\frac{\text{dist}(x, G)}{a|x|})\right] v^a(\beta, x).$$

Also set $v_i^a(\beta, 0) = v_0^a(\beta, 0) = \mathbf{0}$, where $\mathbf{0} \doteq (0, \ldots, 0)'_{1 \times N}$. Let

$$K^a(\beta, x) \doteq \text{conv}\{v_i^a(\beta, x); i = 0, 1, \ldots, N\}.$$

Finally, define

$$K^a(x) \doteq \cup_{\beta \in \mathcal{C}_1} K^a(\beta, x), \quad x \in I\!R^n.$$

Now we can define our second modification to the hitting time function. In this modified form the supremum in (4.19) is taken, instead, over all solutions to the differential inclusion $\dot{\phi}(t) \in K^a(\phi(t))$; $\phi(0) = x$. More precisely, for a given $x \in I\!R^n$ let $\phi(\cdot)$ be an absolutely continuous function on $[0, \infty)$ such that

$$\dot{\phi}(t) \in K^a(\phi(t)); \quad \phi(0) = x; \quad t \in [0, \infty).$$

Denote the class of all such $\phi(\cdot)$ (for a given x) by $H^a(x)$. It will be shown in Lemma 4.4 that $H^a(x)$ is nonempty. Our modified form of the Lyapunov function ($V^a(\cdot)$) is defined as follows.

$$V^a(x) \doteq \sup_{\phi \in H^a(x)} \int_0^\infty \eta(|\phi(t)|) dt, \quad x \in I\!R^n.$$

The main step in the proof is the following result. Once this result is proven, parts (a), (b) and (c) of Definition 3.1 used in the statement of Theorem 3.3 follow immediately via one final modification, which consists of further smoothing, radial linearization, and restriction to G, in exactly the form of [8](pages 696-697). Radial linearity of the function thus obtained holds by construction. Finally, part (d) of Definition 3.1 follows immediately from radial linearity and the fact that the function is C^2 on $G \setminus \{0\}$.

Theorem 4.2. *There exist $a_0 \in (0,\infty)$ such that the following hold for all $a \in (0,a_0)$.*

1. *There exists $r \in (0,1)$, not depending on a such that $V^a(x) = 0$ for all $x \in B(0,r)$.*
2. *$V^a(\cdot)$ is locally Lipschitz on $I\!R^n$. In fact, for all $R \in I\!R^n$ there exists $\alpha(R) \in (0,\infty)$ and $\overline{C}(R)$ such that for all $x,y \in I\!R^n$ with $|x| \le R$ and $|x-y| \le \overline{C}(R)$,*
$$|V^a(x) - V^a(y)| \le \alpha(R)|x-y|, \quad \forall a \in (0,a_0).$$
3. *For a.e. $x \in I\!R^n$; $|x| \ge 2$,*
$$\max_{u \in K^a(x)} DV^a(x) \cdot u \le -1.$$
4. *There exists $D \in (0,\infty)$ such that for all $x \in I\!R^n$ with $|x| \ge 2$, $V^a(x) \ge \frac{|x|-2}{D}$.*
5. *There exists $M \in (0,\infty)$ such that*
$$\operatorname{ess\,inf}_{x \in I\!R^n : |x| \ge M} DV^a(x) \cdot \frac{x}{|x|} > 0.$$

In the remaining part of this section we will prove the above theorem. The main idea in the proof is that the stability properties of the trajectories introduced in Proposition 4.1 imply similar properties for the solutions of the differential inclusion $\dot{\phi}(t) \in K^a(\phi(t))$; $\phi(0) = x$ for small enough value of a. More precisely, the following result will be shown.

Proposition 4.3. *There exist $a_0, T \in (0,\infty)$ such that the following hold.*

1. *Whenever $g(\cdot)$ is an absolutely continuous function such that for some $a \in (0,a_0)$,*
$$\dot{g}(t) \in K^a(g(t)) \quad \text{a.e. } t \in [0,\infty); \quad |g(0)| \le 2^m,$$
we have that $g(t) = 0$ for all $t \ge 2^{m+1}T$.
2. *There exist $r \in (0,1)$ such that whenever $\phi(\cdot)$ is an absolutely continuous function and*
$$\dot{\phi}(t) \in K^a(\phi(t)), \quad \text{a.e. } t; \quad |\phi(0)| \le r; \quad a \in (0,a_0),$$
we have that $\sup_{0 \le t < \infty} |\phi(t)| \le 1$.

We now give the proof of Theorem 4.2 assuming that Proposition 4.3 holds.

A straightforward calculation shows that for a fixed $a \in (0,\infty)$, and m, M positive finite numbers, there exists a constant $C(a,m,M) < \infty$ such that
$$\max_{i \in \{0,\dots,N\}} \sup_{\beta \in \mathcal{C}_1} \sup_{x,y : m \le |x|,|y| \le M} |v_i^a(\beta,x) - v_i^a(\beta,y)| \le C(a,m,M)|x-y|. \tag{4.20}$$

Furthermore , it is easy to see that there exists $D \in (0,\infty)$ such that
$$\max_{i \in \{0,\dots,N\}} \sup_{\beta \in \mathcal{C}_1} \sup_{x \in I\!R^n} \sup_{a \in (0,\infty)} |v_i^a(\beta,x)| \le D. \tag{4.21}$$

Proof of Theorem 4.2. Let a_0 and r be as in Proposition 4.3. The choice of r implies that if $|x| \le r$ and if $\phi(\cdot) \in H^a(x)$ then $\phi(t) \in B(0,1)$ for all $t \in [0,\infty)$. This implies that $\eta(|\phi(t)|) = 0$ for all t, thus for such x, $V^a(x) = 0$. This proves part 1. Now we show the local Lipschitz property in 2. Let $x \in I\!\!R^n$ be such that $|x| \le R$. Without loss of generality we can assume that $|x| \ge \frac{r}{2}$ for else local Lipschitz property holds trivially. From Proposition 4.3(1) it follows that we can choose $T_0 < \infty$ such that for any $\phi \in H^a(y)$; $|y| \le R+1$ we have that $\phi(t) = 0$ for all $t \ge T_0$. For an absolutely continuous trajectory $\phi : [0,\infty) \mapsto I\!\!R^n$, define

$$\tau^*(\phi) \doteq \inf\{t \in (0,\infty) : \phi(t) \in B(0, r/2)\}.$$

Now let $\phi \in H^a(x)$ and $x \in B(0,R)$ be such that

$$V^a(x) \le \int_0^{\tau^*(\phi)} \eta(|\phi(t)|)dt + \epsilon.$$

Note that we could replace ∞ by $\tau^*(\phi)$ in the upper limit of the integral on the right, because of Proposition 4.3(2). Let $y \in I\!\!R^n$ be such that, $|y| \le R+1$.

It will be shown in Lemma 4.5 that there exist measurable functions $q_i : [0,\infty) \to [0,1]$; $i = 0,\ldots,N$ and $\beta : [0,\infty) \to \mathcal{C}_1$ such that $\phi(\cdot)$ solves

$$\begin{aligned} \dot{\phi}(t) &= \sum_{i=0}^{N} q_i(t) v_i^a(\beta(t), \phi(t)), \quad \text{a.e. } t \in [0,\infty) \\ \phi(0) &= x. \end{aligned}$$

Now let $\psi(\cdot)$ be an absolutely continuous function such that for a.e. $t \in [0,\infty)$

$$\begin{aligned} \dot{\psi}(t) &= \sum_{i=0}^{N} q_i(t) v_i^a(\beta(t), \psi(t)), \\ \psi(0) &= y. \end{aligned}$$

Existence of such a $\psi(\cdot)$ will be proved in Lemma 4.4. Since $\psi \in H^a(y)$, we have that $\tau^*(\psi) \le T_0$. We now claim that if y is sufficiently close to x then both $\phi(\tau^*(\phi) \wedge \tau^*(\psi))$ and $\psi(\tau^*(\phi) \wedge \tau^*(\psi))$ are in $B(0,r)$. To see this note that as a consequence of (4.20) and (4.21), for $t \in [0, \tau^*(\phi) \wedge \tau^*(\psi)]$,

$$|\phi(t) - \psi(t)| \le |y - x| + C^* \int_0^t |\phi(s) - \psi(s)| ds,$$

where $C^* \doteq C(a, \frac{r}{2}, R + 1 + T_0 D)$. By an application of Gronwall's inequality we see now that if $|y - x| \le \frac{r}{2} \exp(-C^* T_0) \equiv \overline{C}$ then

$$|\phi(t) - \psi(t)| \le \exp(C^* T_0)|y - x| \le \frac{r}{2} \tag{4.22}$$

for all $t \in [0, \tau^*(\phi) \wedge \tau^*(\psi)]$.

This means that for such y both $\phi(\cdot)$ and $\psi(\cdot)$ are in $B(0,r)$ at time $\tau^*(\phi) \wedge \tau^*(\psi)$. Henceforth we will only consider such y (i.e. $|y - x| \le \overline{C}$). Note next that

$$\begin{aligned} V^a(x) - V^a(y) &\le \int_0^{\tau^*(\phi)\wedge\tau^*(\psi)} (\eta(|\phi(t)|) - \eta(|\psi(t)|)dt + \epsilon. \\ &\le \eta_{lip} T_0 e^{C^* T_0} |y - x| + \epsilon, \end{aligned}$$

where η_{lip} is the Lipschitz constant for $\eta(|\cdot|)$. Sending $\epsilon \to 0$ and using the symmetry of the above calculation we have that

$$|V^a(x) - V^a(y)| \leq \eta_{lip} e^{C^* T_0} |y - x| \tag{4.23}$$

for all $|x| \leq R$ and $|y - x| \leq \overline{C}$. Since $R > 0$ is arbitrary, this proves part 2.

To prove part 3, we will show that at all points x at which $V^a(\cdot)$ is differentiable and $|x| \geq r$

$$\max_{u \in K^a(x)} DV^a(x) \cdot u \leq -\eta(|x|). \tag{4.24}$$

Fix $R \in [2, \infty)$. Now let $r \leq |x| \leq R - 1$ and $u \in K^a(x)$. Then there exist $q_i \in [0,1]$; $i = 0, \ldots, N$ satisfying $\sum_{i=0}^N q_i = 1$ and $\beta \in \mathcal{C}_1$ such that $u = \sum_{i=0}^N q_i v_i^a(\beta, x)$. Define for y such that $|x - y| < \frac{r}{2}$, $u(y) \doteq \sum_{i=0}^N q_i v_i^a(\beta, y)$. In view of (4.20) there exists $\tilde{C} \equiv C(a, r/2, R+1)$ such that

$$|u(x) - u(y)| \leq \tilde{C}|x - y|.$$

Now for a given y such that $|y| \geq \frac{r}{2}$ and $|x - y| < r/2$. define $\phi_y(\cdot)$ to be the absolutely continuous function which satisfies

$$\dot{\phi}_y(t) \in K^a(\phi_y(t)), \;\; \phi_y(0) = y$$

for $t \in [0, \infty)$ and is ϵ-optimal, i.e.

$$V^a(y) \leq \int_0^\infty \eta(|\phi_y(s)|) ds + \epsilon.$$

Let $\phi(\cdot)$ be an absolutely continuous function such that $\phi(\cdot)$ solves:

$$\dot{\phi}(t) = u(\phi(t)); \;\; \phi(0) = x,$$

$t \in [0, \infty)$. The existence of such a ϕ is again assured from Lemma 4.4. Now let $\tau_0 > 0$ be such that for all $t \in [0, \tau_0]$, $|\phi(t) - x| < \frac{\overline{C}}{2}$ and $\tau_0 |u| < \frac{\overline{C}}{2}$. Now set $y \equiv \phi(\tau_0)$. Note that since $\overline{C} < r$, we have that $|x - y| \leq \frac{r}{2}$ and $|y| \geq r/2$. Consider the following modification of the trajectory $\phi(\cdot)$.

$$\begin{aligned} \tilde{\phi}(t) &= \phi(t); \;\; t \in [0, \tau_0] \\ \tilde{\phi}(t) &= \phi_y(t - \tau_0); \;\; t \geq \tau_0 \end{aligned}$$

Note that by construction $\tilde{\phi}(\cdot)$ solves the differential inclusion:

$$\dot{\tilde{\phi}}(t) \in K^a(\tilde{\phi}(t)); \;\; \forall t \in [0, \infty).$$

Now an argument, exactly as on pages 694-695 of [8] shows that

$$\frac{V^a(x + \tau_0 u) - V^a(x)}{\tau_0} \leq -\eta(|x|) + O(\tau_0) - \frac{1}{\tau_0} \int_0^{\tau_0} (\eta(|\tilde{\phi}(s)|) - \eta(|x|)) ds.$$

Taking limit as $\tau_0 \to 0$ we have part 3.

Now we consider part 4. Let $\phi \in H^a(x)$ and let $\tilde{\tau} \doteq \inf\{t : \phi(t) \in B(0,2)\}$. Then we have that

$$\begin{aligned} 2 &\geq |\phi(0)| - |\int_0^{\tilde{\tau}} \dot{\phi}(s)ds| \\ &\geq |x| - \tilde{\tau}D. \end{aligned}$$

Thus $\tilde{\tau} \geq \frac{|x|-2}{D}$. Since $\eta(|x|) = 1$ for $|x| \geq 2$ we have that

$$V^a(x) \geq \int_0^{\tilde{\tau}} \eta(|\phi(s)|)ds \geq \frac{|x|-2}{D}. \tag{4.25}$$

This proves part 4.

Finally, we consider part 5. We will show that there exists $\alpha \in (0,\infty)$ such that for all $x \in I\!R^n$ for which $V^a(x)$ is differentiable, we have that

$$DV^a(x) \cdot \frac{x}{|x|} \geq \frac{1}{\alpha}(1 - \frac{2}{|x|}).$$

This will clearly yield part 5. Without loss of generality assume that $|x| \geq \frac{r}{2}$ since otherwise the inequality holds trivially. In order to show the inequality it suffices to show, in view of part 4, that

$$DV^a(x) \cdot \frac{x}{|x|} \geq \frac{V^a(x)}{|x|}. \tag{4.26}$$

Now the proof of (4.26) is identical to the proof of Proposition 3.7 of [8] on observing that if $\phi \in H^a(x)$ then for $c \in (0,\infty)$, the trajectory $\theta^c(\cdot)$, defined as $\theta^c(t) \doteq (1+c)\phi(\frac{t}{1+c})$, $t > 0$, is in $H^a((1+c)x)$. We omit the details. ■

The proof of Theorem 4.2 used in addition to Proposition 4.3, the following two lemmas. The first lemma is a classical existence and uniqueness result, a sketch of whose proof is provided in the appendix, while the second is a result on measurable selections.

Lemma 4.4. *Let a_0 be as in Proposition 4.3 and $a \in (0, a_0)$ be fixed. Let $q_i(\cdot)$; $i = 0, \ldots, N$ be measurable functions from $[0,\infty) \to [0,1]$ such that $\sum_{i=0}^N q_i(t) = 1$ for all $t \in [0,\infty)$ and $\beta(\cdot)$ be a measurable function from $[0,\infty) \to \mathcal{C}_1$. Let $y \in I\!R^n$ be arbitrary. Then there exists an absolutely continuous function $\phi(\cdot)$ on $[0,\infty)$ such that*

$$\begin{aligned} \dot{\phi}(t) &= \sum_{i=0}^N q_i(t) v_i^a(\beta(t), \phi(t)); \ \text{a.e. } t \in [0,\infty) \\ \phi(0) &= y. \end{aligned} \tag{4.27}$$

Furthermore if $\psi(\cdot)$ is another absolutely continuous function solving (4.27), then $\phi = \psi$.

Lemma 4.5. *Let $a > 0$ be fixed and $\phi(t)$ be an absolutely continuous function on $[0,T]$ such that*

$$\dot{\phi}(t) \in K^a(\phi(t)), \quad \text{a.e. } t \in [0,T].$$

Then there exist measurable functions $q_i : [0,T] \to [0,1]$; $i = 0, \ldots, N$ and $\beta : [0,T] \to \mathcal{C}_1$ such that $\sum_i q_i(t) = 1$ and

$$\dot{\phi}(t) = \sum_{i=0}^N q_i(t) v_i^a(\beta(t), \phi(t)), \quad \text{a.e. } t \in [0,T]. \tag{4.28}$$

Proof: Let B be the subset of $I\!R^n \times I\!R^n \setminus \{0\}$ defined as

$$\{(u,x) \in I\!R^n \times I\!R^n \setminus \{0\} : u = \sum_{i=0}^{N} q_i v_i^a(\beta, x); \;\; q_i \in [0,1]; i = 0, \dots, N, \;\; \beta \in \mathcal{C}_1, \;\; \sum_{i=0}^{N} q_i = 1\}.$$

Let $\mathcal{B}^N$ be the Borel σ-field on $[0,1]^{N+1} \times \mathcal{C}_1$. Define $F : B \mapsto \mathcal{B}^N$ as

$$F(u,x) = \{(q,\beta) : q \in [0,1]^{N+1}; \beta \in \mathcal{C}_1; \sum_{i=0}^{N} q_i v_i^a(\beta, x) = u; \;\; \sum_{i=0}^{N} q_i = 1\}.$$

Note that the map $(x,\beta) \to v_i^a(\beta, x)$ is continuous on $I\!R^n \setminus \{0\} \times \mathcal{C}_1$ for all $i = 0, 1, \dots, N$. This implies that if we have a sequence $(q_k, \beta_k, u_k, x_k) \to (q, \beta, u, x) \in [0,1]^{N+1} \times \mathcal{C}_1 \times I\!R^n \times I\!R^n \setminus \{0\}$ and $(q_k, \beta_k) \in F(u_k, x_k)$ for all k then $(q, \beta) \in F(u, x)$. Thus in view of Corollary 10.3, Appendix of [9] there exists a measurable selection for F, i.e. there exists a measurable map:

$$f : B \to [0,1]^{N+1} \times \mathcal{C}_1$$

such that $f(u,x) \in F(u,x)$ for all $(u,x) \in B$. Choose an arbitrary element $(\hat{q}, \hat{\beta}) \in [0,1]^{N+1} \times \mathcal{C}_1$ such that $\sum_{i=0}^{N} q_i = 1$ and extend f to $\hat{B} \doteq B \cup (\mathbf{0}, 0)$ by setting $f(\mathbf{0}, 0) \doteq (\hat{q}, \hat{\beta})$. Now write $f(\cdot)$ as $(\{f_i(\cdot)\}_{i=0}^{N}, f_*(\cdot))$, i.e. we denote the first $N+1$ coordinates of the vector function f by f_i; $i = 0, \dots, N$, and we denote the $N+2$'th coordinate by f_*. Define

$$q_i(t) \doteq f_i(\dot{\phi}(t), \phi(t)); i = 0, 1, \dots, N; \;\; \text{a.e. } t \in [0, \infty)$$

$$\beta(t) \doteq f_*(\dot{\phi}(t), \phi(t)); \text{a.e. } t \in [0, \infty).$$

Clearly $q(\cdot)$ and $\beta(\cdot)$ are measurable functions and by construction (4.28) holds. ■

Now we turn to the proof of Proposition 4.3. The key idea is to relate the solutions of the differential inclusion $\dot{\phi}(t) \in K^a(\phi(t))$; $\phi(0) = x$, for small enough value of a, with the solutions of the SP for trajectories with velocity in $\mathcal{C}_1$. The following two results are central in that respect. Define for $x \in I\!R^n$

$$\begin{aligned} K(x) \;\; \doteq \;\; & \{v \in I\!R^n : \text{there exists a sequence } (a_k, x_k, v_k)_{k \ge 1} \subset (0,1] \times I\!R^n \times I\!R^n \\ & \text{s.t. } a_k \to 0; \; x_k \to x; \; v_k \to v; \text{ and } v_k \in K^{a_k}(x_k)\}. \end{aligned}$$

We will denote the closure of the convex hull of $K(x)$ by $\overline{K}(x)$. The first result shows that as a approaches 0 the solutions of the differential inclusion converge to a trajectory which also solves a differential inclusion given in terms of $\overline{K}$. The proof is quite similar to the proof of Proposition 3.3 of [8]. We provide a sketch in the appendix.

Lemma 4.6. *Consider the sequence $(x_k, a_k, \phi_k(\cdot))_{k \ge 1} \subset I\!R^n \times (0,1] \times \mathcal{C}([0,\infty); I\!R^n)$ such that $x_k \to x$; $a_k \to 0$, and $\phi_k(\cdot) \to \phi(\cdot)$ (uniformly on compacts). Suppose further that each ϕ_k is absolutely continuous and solves the differential inclusion:*

$$\dot{\phi}_k(t) \in K^{a_k}(\phi_k(t)); \text{ a.e. } t \in [0, \infty); \quad \phi_k(0) = x_k.$$

Then $\phi(\cdot)$ is Lipschitz continuous (and thus absolutely continuous) and it solves the differential inclusion:

$$\dot{\phi}(t) \in \overline{K}(\phi(t)); \text{ a.e. } t \in [0, \infty); \quad \phi(0) = x.$$

The proposition below provides the connection between the solutions of the differential inclusion $\dot{\phi}(t) \in \overline{K}(\phi(t))$; $\phi(0) = x$ and certain solutions to the SP. Define

$$\delta_0 \doteq \inf_{(\lambda,i):\lambda\in\Lambda,\, i\in\lambda} d^\lambda \cdot n_i. \tag{4.29}$$

By Condition 2.1, $\delta_0 > 0$.

Proposition 4.7. *Let $\phi : [0,\infty) \to I\!R^n$ be an absolutely continuous function which solves the differential inclusion:*

$$\dot{\phi}(t) \in \overline{K}(\phi(t)), \quad \text{a.e. } t \in [0,\infty).$$

Then there exists a $\tau \in [0, |\phi(0)|/\delta_0)$, a strictly increasing, onto function $\alpha : [0,\infty) \to [0,\infty)$ and a measurable function $\overline{\beta} : [0,\infty) \to \mathcal{C}_1$ such that

$$\psi(t) \doteq \phi(\tau + \alpha(t)) = \Gamma\left(\phi(\tau) + \int_0^{\cdot} \overline{\beta}(u)du\right)(t), \quad t \in (0,\infty).$$

Before presenting the proof of this proposition, we show how the proof of Proposition 4.3 follows. For an absolutely continuous trajectory $\theta : [0,\infty) \mapsto I\!R^n$, define

$$\tau(\theta) \doteq \inf\{t \in (0,\infty) : \theta(t) = 0\}. \tag{4.30}$$

Proof of Proposition 4.3. First we show that there exist $\tilde{a}_0, T \in (0,\infty)$ such that for all $a \in (0,\tilde{a}_0)$ and absolutely continuous $\phi(\cdot)$ on $[0,\infty)$ satisfying

$$\dot{\phi}(t) \in K^a(\phi); \;\; |\phi(0)| \le 1; \;\; \text{a.e. } t \in [0,\infty),$$

we have that

$$\inf_{0\le t<T} |\phi(t)| \le 1/2. \tag{4.31}$$

We argue by contradiction. Suppose that there exists a sequences $\{T_k\}_{k\ge1}$ increasing to ∞, $\{a_k\}_{k\ge1}$ decreasing to 0 and $\{\phi_k(\cdot)\}_{k\ge1}$ such that for all k, $\phi_k(\cdot)$ is absolutely continuous, for a.e. $t \in [0,\infty)$,

$$\dot{\phi}_k(t) \in K^{a_k}(\phi_k(t)); \;\; |\phi_k(0)| \le 1$$

and $\inf_{0\le t<T_k} |\phi_k(t)| > 1/2$. As in Proposition 3.3(i) of [8] we have that $\{\phi_k; k \ge 1\}$ is precompact in $C([0,\infty); I\!R^n)$. Assume without loss of generality that $\phi_k(\cdot)$ converges to $\phi(\cdot)$ uniformly on compacts. Clearly $|\phi(0)| \le 1$ and

$$|\phi(t)| \ge 1/2, \;\; \text{for all } \; t \in [0,\infty). \tag{4.32}$$

From Lemma 4.6 we have that $\phi(\cdot)$ is absolutely continuous and solves the differential inclusion:

$$\dot{\phi}(t) \in \overline{K}(\phi(t)) \quad \text{a.e. } t \in [0,\infty).$$

Therefore from Proposition 4.7 we have that there exists $\tau \in [0, {\delta_0}^{-1}]$, a strictly increasing, onto function $\alpha : [0,\infty) \to [0,\infty)$ and a measurable function $\overline{\beta} : [0,\infty) \to \mathcal{C}_1$ such that for all $t \ge 0$

$$\phi(\tau + \alpha(t)) = \Gamma\left(\phi(\tau) + \int_0^{\cdot} \overline{\beta}(u)du\right)(t).$$

Now applying Proposition 4.1 we have that $\lim_{t\to\infty}\phi(\tau+\alpha(t))=0$. This is a contradiction to (4.32). Hence (4.31) is proven.

Now let ψ be an absolutely continuous function on $[0,\infty)$ satisfying

$$\dot{\psi}(t)\in K^a(\psi);\ \ |\psi(0)|\le 2^k;\ \ \text{a.e. } t\in[0,\infty).$$

Assume without loss of generality that $\psi(0)\neq 0$ and define $\phi(t)\doteq 2^{-k}\psi(2^k t)$. Since $K^a(x)=K^a(\alpha x)$ for all $\alpha>0$ we have that

$$\dot{\phi}(t)\in K^a(\phi);\ \ |\phi(0)|\le 1;\ \ \text{a.e. } t\in[0,\infty).$$

Thus

$$\begin{aligned}\inf_{0\le t\le T2^k}|\psi(t)| &= \inf_{0\le t\le T2^k}2^k|\phi(t2^{-k})|\\ &= 2^k\inf_{0\le t\le T}|\phi(t)|\\ &\le 2^k/2.\end{aligned}$$

Now let $g(\cdot)$ be as in the proposition. Then letting $k=m,(m-1),\ldots,0,-1,\ldots$ we have that $\inf_{0\le t<2^{m+1}T}|g(t)|=0$ Since $K^a(0)=\mathbf{0}$, we have part 1.

Now we consider part 2. Let $(\tilde{a}_0,T)$ be as above. We will show that there exists $a_0\le\tilde{a}_0$ and $r\in(0,1)$ such that the statement 2 in the proposition holds. We will once more argue by contradiction. Suppose that there exist sequences $(a_k,r_k,\phi^k(\cdot))$ such that

$$\dot{\phi}^k(t)\in K^{a_k}(\phi^k(t));\ \ \text{a.e. } t\in[0,\infty),$$

$|\phi^k(0)|\le r_k$, $r_k\to 0$, $a_k\to 0$ and

$$|\phi^k(t_k)|>c \text{ for some } \ t_k\in[0,\infty). \tag{4.33}$$

Assume without loss of generality that $r_k\le 1$ and $a_k\le a_0$ for all $k\ge 1$. From 1 we know that $\tau(\phi^k)\le 2T$ for all $k\ge 1$. Also note that from the uniform Lipschitz property of $(\phi^k(\cdot))$ and noting that $\phi^k(0)\to 0$ as $k\to\infty$ we have that $T^*\doteq\inf_k\tau(\phi^k)>0$, since otherwise $\phi^k(t_k)$ converges to 0 along some subsequence, which contradicts (4.33). So now assume without loss of generality that $\tau(\phi^k)\to T^*$, $t_k\to t^*$ and $\phi^k(\cdot)\to\phi(\cdot)$ uniformly on $[0,T^*]$ as $k\to\infty$. From Lemma 4.6 we have that $\phi(\cdot)$ is absolutely continuous on $[0,T^*]$ and solves the differential inclusion $\dot{\phi}(t)\in\overline{K}(\phi(t))$, a.e. $t\in[0,T^*]$, $\phi(0)=0$. From Proposition 4.7 and Proposition 4.1 we then have that $\phi(t)=0$ for all $t\in[0,T^*]$. But on the other hand, since $\phi^k(t_k)>c$, we have that $\phi(t^*)\ge c$, which is a contradiction. This proves part 2 and hence the proposition. ■

We now prove Proposition 4.7. We will need the following three lemmas. The first lemma characterizes the set $\overline{K}(x)$ and its proof is similar to the proof of Proposition 3.2 of [8] (and is thus omitted). The second lemma says that a solution of the differential inclusion $\dot{\phi}(t)\in\overline{K}(\phi(t))$, enters G after some finite time, and then stays within G. The third lemma gives a representation for a solution to the above differential inclusion.

Lemma 4.8. *For $x \in I\!R^n \setminus \{0\}$.*

$$\begin{aligned}\overline{K}(x) &\subset \operatorname{conv}\{\mathcal{C}_1 \cup \{d_i : i \in \operatorname{In}(x)\}, \quad \text{if } x \in \partial G\\ &\subset \mathcal{C}_1, \quad \text{if } x \in G^0\\ &\subset \operatorname{conv}\{d^\lambda : \lambda \supset \{i : x \cdot n_i < 0\}\} \quad \text{if } x \in G^c.\end{aligned} \tag{4.34}$$

Lemma 4.9. *Let $\phi : [0,\infty) \to I\!R^n$ be an absolutely continuous function which solves the differential inclusion:*

$$\dot{\phi}(t) \in \overline{K}(\phi(t)), \text{a.e. } t \in [0,\infty).$$

Then the following hold.

1. *Let $t \in [0,\infty)$ be so that $\phi(\cdot)$ is differentiable at t and $\phi(t) \notin G$, then*
$$\frac{d}{dt}\left[\min_{i\in\{1,\ldots,N\}} \phi(t)\cdot e_i\right] \geq \delta_0.$$
2. *If $\phi(0) \in G$ then $\phi(t) \in G$ for all $t \in [0,\infty)$.*
3. *If $\phi(0) \notin G$ then there exists $\tau \leq \frac{|\phi(0)|}{\delta_0}$ such that $\phi(t) \in G$ for all $t \geq \tau$.*

Proof: Parts 2 and 3 follow immediately once 1 is proven. We now present the proof of (1). Fix $y \in G^c$. Define $\lambda_1(y) \doteq \{i : y \cdot n_i < 0\}$. Note that whenever $\lambda \supset \lambda_1(y)$, we have from (4.29) that $d^\lambda \cdot n_i \geq \delta_0, \quad \forall i \in \lambda_1(y)$. This yields the implication:

$$v \in \operatorname{conv}\{d^\lambda : \lambda \supset \lambda_1(y)\} \quad \Rightarrow \quad v \cdot n_i \geq \delta_0, \ \forall i \in \lambda_1(y). \tag{4.35}$$

Define

$$\overline{\lambda}(y) \doteq \{i \in \lambda_1(y) : y \cdot n_i = \min_i\{y \cdot n_i\}\}.$$

Now let $t \in [0,\infty)$ be such that $\phi(\cdot)$ is differentiable at t and $\phi(t) \in G^c$. Then by continuity of $\phi(\cdot)$ we can choose $\epsilon > 0$ such that for all $0 \leq h < \epsilon$:

$$\min_{i\in\{1,\ldots,N\}} \phi(t+h)\cdot n_i = \min_{i\in\overline{\lambda}(\phi(t))} \phi(t+h)\cdot n_i.$$

and

$$\lambda_1(\phi(t+h)) \supset \lambda_1(\phi(t)). \tag{4.36}$$

Next observe that

$$\begin{aligned}\frac{d}{dt}\left[\min_{i\in\{1,\ldots,N\}} \phi(t)\cdot n_i\right] &= \lim_{h\to 0}\frac{1}{h}\left(\min_{i\in\{1,\ldots,N\}} \phi(t+h)\cdot n_i - \min_{i\in\{1,\ldots,N\}} \phi(t)\cdot n_i\right)\\ &= \lim_{h\to 0}\frac{1}{h}\min_{i\in\overline{\lambda}(\phi(t))} \left(\phi(t+h)\cdot n_i - \phi(t)\cdot n_i\right)\\ &= \lim_{h\to 0}\min_{i\in\overline{\lambda}(\phi(t))}\frac{1}{h}\int_0^h \dot{\phi}(t+s)\cdot n_i ds\\ &\geq \delta_0,\end{aligned}$$

where the last step follows from (4.35) on observing that in view of Lemma 4.8 and (4.36) for a.e. $s \in [0,h]$

$$\begin{aligned}\dot{\phi}(t+s) &\in \overline{K}(\phi(t+s)) \\ &\subset \operatorname{conv}\{d^\lambda : \lambda \supset \lambda_1(\phi(t+s))\} \\ &\subset \operatorname{conv}\{d^\lambda : \lambda \supset \lambda_1(\phi(t))\}.\end{aligned}$$

This proves the lemma. ■

The following lemma once more uses a result on measurable selections. The proof is quite similar to Lemma 4.5, and a sketch is given in the appendix.

Lemma 4.10. *Let $\phi : [0,\infty) \to \mathbb{R}^n$ be an absolutely continuous function such that $\phi(0) \in G$ and ϕ solves the differential inclusion:*

$$\dot{\phi}(t) \in \overline{K}(\phi(t)); \quad \text{a.e. } t.$$

Then there exist measurable functions $q_i : [0,\infty) \to [0,1]$; $i = 0,1,\ldots,N$, satisfying the equality $\sum_{i=0}^N q_i(t) = 1$, and measurable map $\beta_0 : [0,\infty) \to \mathcal{C}_1$ such that for a.e. $t \in [0,\infty)$

$$\dot{\phi}(t) = \sum_{i \in \mathrm{In}(\phi(t))} q_i(t) d_i + q_0(t)\beta_0(t).$$

Proof of Proposition 4.7: From Lemma 4.9 we know that $\phi(t) \in G$ for a.e. $t > \tau$. From Lemma 4.10 it follows that there exist measurable functions $q_i : [0,\infty) \to [0,1]$; $i = 0,1,\ldots,N$ and $\beta_0 : [0,\infty) \to \mathcal{C}_1$ such that for a.e. $t \geq \tau$

$$\dot{\phi}(t) = \sum_{i \in \mathrm{In}(\phi(t))} q_i(t) d_i + q_0(t)\beta_0(t).$$

Let $\{n^\lambda\}_{\lambda \in \Lambda}$ be as in Remark 2.2. Define $\delta_* \doteq \inf_{\lambda \in \Lambda; i \in \lambda} n^\lambda \cdot d_i$. Also let $\gamma \doteq \sup_{\beta \in \mathcal{C}_1} |\beta|$. We now claim that for a.e. $t \in [\tau, \infty)$

$$q_0(t) \geq \frac{\delta_*}{\delta_* + \gamma}. \tag{4.37}$$

Let $\lambda \in \Lambda \setminus \{0\}$ be arbitrary. Define

$$F^\lambda \doteq \{x \in \mathbb{R}^n : \mathrm{In}(x) \supset \lambda\}.$$

Since $\phi(\cdot)$ is absolutely continuous and F^λ is a linear subspace of $\mathbb{R}^n$ we have that for a.e. t whenever $\phi(t) \in F^\lambda$ we have that $\dot{\phi}(t) \in F^\lambda$. Thus for a.e. t, $I_{\{\phi(t) \in F^\lambda\}}\dot{\phi}(t) \cdot n^\lambda = 0$. Now observe that for a.e $t \geq \tau$ such that $\phi(t) \in F^\lambda$:

$$\begin{aligned}0 &= n^\lambda \cdot \dot{\phi}(t) \\ &= n^\lambda \cdot (\dot{\phi}(t) - q_0(t)\beta_0(t)) + n^\lambda \cdot q_0(t)\beta_0(t) \\ &= n^\lambda \cdot \sum_{i \in \lambda} q_i(t) d_i + q_0(t) n^\lambda \cdot \beta_0(t) \\ &\geq \delta_* \sum_{i \in \lambda} q_i(t) - \gamma q_0(t).\end{aligned}$$

This proves (4.37) for a.e. $t \geq \tau$ such that $\phi(t) \in F^{\lambda}$. Also the claim holds trivially if $\phi(t) \in G^0$ since then $q_0(t) \equiv 1$. Now letting λ run over all the subsets of Λ we have the claim. Next define the strictly increasing function $a : [0,\infty) \to [0,\infty)$ as

$$a(t) \doteq \int_0^t q_0(\tau + s)ds; \quad t \in [0,\infty).$$

Also set $\alpha(t) \doteq a^{-1}(t)$. Finally we show that $\psi(\cdot) \doteq \phi(\tau + \alpha(\cdot))$ solves the SP for

$$x(\cdot) \doteq x(0) + \int_0^{\cdot} \overline{\beta}(s)ds,$$

where $x(0) = \phi(\tau)$ and $\overline{\beta}(t) \doteq \beta_0(\tau + \alpha(t))$, $t \in [0,\infty)$. To see this we only need to observe that for a.e. $t \geq 0$

$$\begin{aligned}
\dot{\psi}(t) &= \frac{\dot{\phi}(\tau + \alpha(t))}{q_0(\tau + \alpha(t))} \\
&= \beta_0(\tau + \alpha(t)) + \sum_{i \in \mathrm{In}(\psi(t))} \frac{q_i(\tau + \alpha(t))}{q_0(\tau + \alpha(t))} d_i.
\end{aligned}$$

This proves the lemma. ■

5 Appendix

Proof of Lemma 4.4: The proof is via Picard iteration method. Define $\phi^{(0)}(\cdot) \equiv y$ on $[0,T]$. For $k \geq 1$, define for $t \in [0,T]$

$$\phi^{(k)}(t) \doteq y + \sum_{i=0}^{N} \int_0^t q_i(s) v_i^a(\beta(s), \phi^{(k-1)}(s))ds.$$

Note that the boundedness of $q_i(\cdot)$ and $v_i^a(\beta(\cdot), \cdot)$ assures that $\phi^{(k)}(\cdot)$ is an equicontinuous family (in fact uniformly Lipschitz continuous) which is pointwise bounded on $[0,T]$ for all $T < \infty$. Thus there exists a subsequential (uniform) limit $\phi(\cdot)$. Clearly $\phi(\cdot)$ is Lipschitz continuous and thus absolutely continuous. Note that the map $(x,\beta) \to v_i^a(\beta, x)$ is continuous on $I\!R^n \setminus \{0\} \times \mathcal{C}_1$. Therefore we have that as $k \to \infty$, $v_i^a(\beta(t), \phi^{(k-1)}(t)) \to v_i^a(\beta(t), \phi(t))$ for all $t \in [0, \tau(\phi))$, where $\tau(\phi)$ is as defined in (4.30). Now a straightforward application of the dominated convergence theorem shows that $\phi(\cdot)$ solves (4.27) on $[0,\tau(\phi))$ and hence, since $v_i^a(\cdot, 0) = \mathbf{0}$, on $[0,\infty)$. Now let $\phi(\cdot)$ and $\psi(\cdot)$ be two solutions to (4.27). We will show that

$$\phi(t) = \psi(t), \forall\, t \in [0, \tau(\phi) \wedge \tau(\psi)). \tag{5.38}$$

Fix $\epsilon > 0$. Then there exists $m > 0$ such that $\min(|\phi(t)|, |\psi(t)|) > m$ on $[0, \tau(\phi) \wedge \tau(\psi) - \epsilon)$. Also let $M \doteq \sup_{0 \leq t \leq \tau(\phi)} \max\{|\phi(t)|, |\psi(t)|\}$. Then from (4.20) we have that

$$|\phi(t) - \psi(t)| \leq C(a,m,M) \int_0^t |\phi(s) - \psi(s)|ds,$$

for all $t \in [0, \tau(\phi) \wedge \tau(\psi) - \epsilon)$. An application of Gronwall's inequality shows that ϕ and ψ are equal on $[0, \tau(\phi) \wedge \tau(\psi) - \epsilon)$. Since $\epsilon > 0$ is arbitrary, we have (5.38). This also implies that $\tau(\phi) = \tau(\psi)$ and since both trajectories stay at 0 once they hit 0, we have the desired uniqueness on $[0, \infty)$. ■

Proof of Lemma 4.6. The Lipschitz continuity of ϕ follows immediately on observing that for $0 \leq s \leq t < \infty$ and $k \geq 1$ $|\phi_k(t) - \phi_k(s)| \leq D|t - s|$. We will show that for all $T \in [0, \infty)$, $\dot{\phi}(t) \in \overline{K}(\phi(t))$; a.e. $t \in [0, T]$. Fix $T \in [0, \infty)$. Define a sequence of probability measures on $\Omega_0 \doteq I\!\!R^n \times I\!\!R^n \times [0, T]$ as follows. For $f \in C_b(I\!\!R^n \times I\!\!R^n \times [0, T])$ define

$$\int_{\Omega_0} f(x, y, t) d\mu_k(x, y, t) \doteq \frac{1}{T} \int_0^T f(\phi_k(s), \dot{\phi}_k(s), s) ds.$$

Since

$$\sup_{k \geq 1, s \in [0,T]} |\phi_k(s)| \leq \sup_{k \geq 1} |x_k| + DT \doteq \overline{C} < \infty \tag{5.39}$$

and $|\dot{\phi}_k(s)| \leq D$ a.e. s, we have that $\{\mu_k\}_{k \geq 1}$ is a tight family of probability measures. Without loss of generality assume that μ_k converges weakly to μ. The sequence $\{\mu_k\}$ gives the following useful representation for $\{\phi_k\}$:

$$\phi_k(t) = x_k + \int_0^t \int_{I\!\!R^n \times I\!\!R^n} y d\mu_k(x, y, s).$$

Taking limits in the above equality we have

$$\phi(t) = x + \int_0^t \int_{I\!\!R^n \times I\!\!R^n} y d\mu(x, y, s). \tag{5.40}$$

Next note that the marginal distribution of μ_k in the time variable is the normalized Lebesgue measure on $[0, T]$ for every k and thus μ also has the same marginal distribution. Therefore there exists $\tilde{\mu}(s, \cdot)$, a regular conditional probability distribution, such that for $f \in C_b(I\!\!R^n \times I\!\!R^n \times [0, T])$

$$\int_{\Omega_0} f(x, y, t) d\mu(x, y, t) = \frac{1}{T} \int_0^T \left(\int_{I\!\!R^n \times I\!\!R^n} f(x, y, t) \tilde{\mu}(t, dx, dy) \right) dt. \tag{5.41}$$

Thus from (5.40) we have that

$$\phi(t) = x + \int_0^t \left(\int_{I\!\!R^n \times I\!\!R^n} y \tilde{\mu}(s, dx, dy) \right) ds.$$

This shows that for a.e. $t \in [0, T]$,

$$\dot{\phi}(t) = \int_{I\!\!R^n \times I\!\!R^n} y \tilde{\mu}(t, dx, dy). \tag{5.42}$$

Using the upper-semi continuity of the set $K(x)$, it follows as in [8] (see pages 687-689) that the support of $\tilde{\mu}(t, dx, dy)$ is contained in $\{\phi(t)\} \times \overline{K}(\phi(t))$. Thus we have from (5.42), on noting that $K(x) \subset \overline{K}(x)$ and $\overline{K}(x)$ is a closed convex set, that $\dot{\phi}(t) \in \overline{K}(\phi(t))$. This proves the lemma. ■

Proof of Lemma 4.10. For $\lambda \in \Lambda$ define

$$B^\lambda \doteq \{u \in I\!\!R^n : u = \sum_{i\in\lambda} q_i d_i + q_0\beta; \ \sum_{i\in\lambda} q_i + q_0 = 1; \ \ q_i \geq 0; \ \ \beta \in \mathcal{C}_1\}.$$

Denote the class of Borel subsets of $[0,1]^{|\lambda|+1} \times \mathcal{C}_1$ by $\mathcal{B}^{|\lambda|}$. Define the set-valued map $F^\lambda : B^\lambda \to \mathcal{B}^{|\lambda|}$ as follows. For $u \in B^\lambda$

$$F^\lambda(u) \doteq \{(q,\beta) : q \equiv (q_i)_{i\in\lambda\cup\{0\}} \in [0,1]^{|\lambda|+1}; \beta \in \mathcal{C}_1; \ \sum_{i\in\lambda} q_i + q_0 = 1; \text{and} \sum_{i\in\lambda} q_i d_i + q_0\beta = u\}.$$

We would like to show that there exists a measurable selection for F^λ, i.e. there exists a measurable map:

$$f^\lambda : B^\lambda \to [0,1]^{|\lambda|+1} \times \mathcal{C}_1$$

such that for all $u \in B^\lambda$, $f^\lambda(u) \in F^\lambda(u)$. In order to show this it will suffice to show (in view of Corollary 10.3, Appendix, [9]) that if $(q_k,\beta_k) \in F^\lambda(u_k)$ and $u_k \to u$ then the sequence $(q_k,\beta_k)_{k\geq 1}$ has a limit point in $F^\lambda(u)$. But this is an immediate consequence of the compactness of $[0,1]^{|\lambda|+1} \times \mathcal{C}_1$. Now fix such a measurable selection for every $\lambda \in \Lambda$. Set

$$f^\lambda(\cdot) \equiv ((f_i^\lambda(\cdot))_{i\in\lambda\cup\{0\}}, f_{vel}^\lambda),$$

where $f_i^\lambda : B^\lambda \to [0,1]$ for $i \in \lambda \cup \{0\}$ and $f_{vel}^\lambda : B^\lambda \to \mathcal{C}_1$ are the coordinate maps defined in the obvious way. Let $\phi(\cdot)$ be as in the statement of the theorem. Define for all t for which $\text{In}(\phi(t)) = \lambda$ and $\dot\phi(t) \in \overline{K}(\phi(t))$,

$$q_0(t) \doteq f_0^\lambda(\dot\phi(t)); \ q_i(t) \doteq f_i^\lambda(\dot\phi(t)); \ i \in \lambda; \ q_i(t) \doteq 0; \ \ i \notin \lambda \text{ and } \beta_0(t) \doteq f_{vel}^\lambda(\dot\phi(t)) \ .$$

Thus letting λ vary over all the subsets of Λ we have a.e. defined measurable functions $(q_i(\cdot))_{i=0,1,\ldots,N}$, $\beta(\cdot)$ as required in the statement of the lemma. ■

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