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# ONE-ARM EXPONENT FOR CRITICAL 2D PERCOLATION

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**Abstract** The probability that the cluster of the origin in critical site percolation on the triangular grid has diameter larger than $R$ is proved to decay like $R^{-5/48}$ as $R \to \infty$.

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# 1 Introduction

Critical site percolation on the triangular lattice is obtained by declaring independently each site to be open with probability $p = 1/2$ and otherwise to be closed. Let $0 \leftrightarrow C_R$ denote the event that there exists an open path from the origin to the circle $C_R$ of radius $R$ around the origin. In this paper we prove:

**Theorem 1.1.** *For critical site percolation on the standard triangular grid in the plane,*

$$\mathbf{P}\big[0 \leftrightarrow C_R\big] = R^{-5/48+o(1)}, \qquad R \to \infty\,.$$

This result is based on the recent proof by Stanislav Smirnov [22] that the scaling limit of this percolation process exists and is described by $\mathrm{SLE}_6$, the stochastic Loewner evolution with parameter $\kappa = 6$. This latter result has been conjectured in [20] where $\mathrm{SLE}_\kappa$ was introduced. See [24] for a treatment of other critical exponents for site percolation on the triangular lattice.

The exponent determined in Theorem 1.1 is sometimes called the one-arm exponent. In Appendix B we briefly discuss what the methods of the present paper can say about the monochromatic two-arm or backbone exponent, for which no established conjecture existed. More precisely, we show that this exponent is the highest eigenvalue of a certain differential operator with mixed boundary conditions in a triangle.

Theorem 1.1 has been conjectured in the theoretical physics literature [17, 16, 15, 2], usually in forms involving other critical exponents that imply this one using the scaling relations that have been proved mathematically by Kesten [7]. It is conjectured that the theorem holds for any planar lattice. However, Smirnov's results mentioned above have been established only for site percolation on the triangular lattice, and hence we can only prove our result in this case. In the literature, the exponent calculated in Theorem 1.1 is sometimes denoted by $1/\rho$.

Theorem 1.1 will appear as a corollary of the following similar result about the exponent for the scaling limit. The scaling limit is the process obtained from percolation by letting the mesh of the grid tend to zero (see the next section for more detail).

**Theorem 1.2.** *Let $Q$ denote the union of the clusters meeting the unit circle $\partial\mathbb{U}$ in the scaling limit of critical site percolation on the triangular lattice. There is a constant $c > 0$ such that for all $r \in (0, 1/2)$,*

$$c^{-1}\, r^{5/48} \le \mathbf{P}\big[\mathrm{dist}(Q, 0) < r\big] \le c\, r^{5/48}.$$

This theorem resembles in statement and in proof the determination of the Brownian motion exponents carried out in [13]. Also related to this theorem is the calculation in [21] of the probability of an event for the scaling limit.

We will assume that the reader is familiar with the theory of percolation in the plane (mainly, the Russo-Seymour-Welsh theorem and its consequences), such as appearing in [5, 6]. Also, familiarity with SLE will be assumed. To learn about the basics of SLE the reader is advised to consult the first few sections in [12, 13, 19, 11].

We now turn to discuss the SLE counterpart of Theorem 1.2. Let

$$\lambda = \lambda(\kappa) := \frac{\kappa^2 - 16}{32\kappa}\,. \tag{1.1}$$

**Theorem 1.3.** *For every $\kappa > 4$ with $\kappa \neq 8$, there exists a constant $c > 0$ such the radial* $\mathrm{SLE}_\kappa$ *path $\gamma : [0,\infty) \to \overline{\mathbb{U}}$ satisfies for all $r \in (0,1)$,*

$$c^{-1} r^\lambda \leq \mathbf{P}\big[\gamma[0,T_r] \text{ contains no counterclockwise loop around } 0\big] \leq c\, r^\lambda$$

*where $T_r$ denotes the first time at which $\gamma$ intersects the circle of radius $r$ around the origin.*

The existence of a counterclockwise loop around 0 in $\gamma[0,t]$ means that there are $0 \leq t_1 \leq t_2 \leq t$ such that $\gamma(t_1) = \gamma(t_2)$ and the winding number around 0 of the restriction of $\gamma$ to $[t_1, t_2]$ is 1. The only reason that $\kappa = 8$ is excluded from the theorem is that it has not been proven yet that the SLE path $\gamma$ is a continuous path when $\kappa = 8$; this was proved for all other $\kappa$ in [19]. In the range $\kappa \in [0,4]$ the theorem holds trivially with $\lambda = 0$, because $\gamma$ is a.s. a simple path.

**Acknowlegments.** This paper was planned prior to Smirnov's [22]. At the time, the link with discrete percolation was only conjectural. The conjecture has been established in [22], and therefore Theorems 1.1 and 1.2 can now be stated as unconditional theorems. Therefore, a significant portion of the credit for these results also belongs to S. Smirnov.

We wish to thank Harry Kesten and Michael Solomiak for useful advice.

## 2 The scaling limit exponent

Let $\theta \in [0, 2\pi]$, and let $A_\theta$ be the arc

$$A_\theta := \{e^{is} : s \in [0,\theta]\} \subset \partial\mathbb{U}.$$

Fix some $\delta > 0$, and consider site percolation with parameter $p = 1/2$ on the triangular grid of mesh $\delta$. It is convenient to represent a percolation configuration by coloring the hexagonal faces of the dual grid, black for open, white for closed. Let $\mathfrak{B}_\delta$ denote the union of all the black hexagons, and let $Q_\delta(\theta)$ denote the union of $A_\theta$ with all the connected components of $\mathfrak{B}_\delta \cap \overline{\mathbb{U}}$ which meet $A_\theta$. See Figure 1. Let $Q(\theta)$ denote a random subset of $\overline{\mathbb{U}}$ whose law is the weak limit as $\delta \downarrow 0$ of the law of $Q_\delta(\theta)$. (The law of $Q_\delta(\theta)$ can be thought of as a probability measure on the Hausdorff space of compact subsets of $\overline{\mathbb{U}}$.) By [22, 23], the limit exists, and, moreover, it can be explicitly described via $\mathrm{SLE}_6$, as follows.

Recall from [22] that the limit $\tilde\gamma$ as $\delta \downarrow 0$ of the outer boundary of $Q_\delta(\theta)$ consists of $A_\theta$ concatenated with the path of chordal $\mathrm{SLE}_6$ joining the endpoints of $A_\theta$. By [22, 23], the scaling limit of percolation is conformally invariant. Hence, it is not too hard to verify that $Q(\theta)$ is obtained by "filling in" each component of $\mathbb{U} \setminus \tilde\gamma$ with an independent copy of $Q(2\pi)$. A more precise statement is the following.

**Theorem 2.1 (Smirnov).** *Let $\theta \in (0, 2\pi)$, let $\gamma_{\mathrm{ch}}$ denote the path of chordal* $\mathrm{SLE}_6$ *from 1 to $e^{i\theta}$ in $\mathbb{U}$, and let $\tilde\gamma$ denote the curve obtained by concatenating $A_\theta$, clockwise oriented, with $\gamma_{\mathrm{ch}}$. Let $\mathcal{W}$ denote the collection of all connected components $W$ of $\mathbb{U} \setminus \tilde\gamma$ such that $\tilde\gamma$ has winding number $-1$ about points in $W$. For each $W \in \mathcal{W}$, let $\psi_W : \mathbb{U} \to W$ be a conformal homeomorphism, and let $Q^W$ denote an independent copy of $Q(2\pi)$. Then the distribution of $Q(\theta)$ is the same as the distribution of*

$$\bigcup_{W \in \mathcal{W}} \psi_W(Q^W)\,.$$

*(This union contains $\tilde\gamma$ a.s.)*

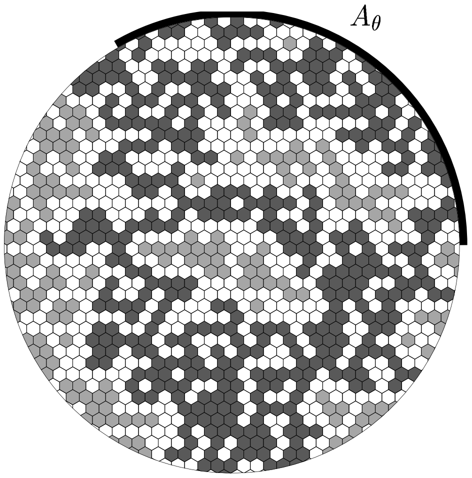


Figure 1: The set $Q_\delta(\theta)$.

Theorem 2.1 follows from the results and methods of [22]. See [23] for further details.

We are interested in the distribution of the distance from $Q(\theta)$ to 0. However, it is more convenient to study the distribution of a very closely related quantity, the *conformal radius.* Let $U(\theta)$ denote the component of 0 in $\mathbb{U} \setminus Q(\theta)$. (It follows from Theorem 2.1 or from the Russo-Seymour-Welsh Theorem that $0 \notin Q(\theta)$ a.s.) Let $\psi = \psi_\theta : U(\theta) \to \mathbb{U}$ be the conformal map, normalized by $\psi(0) = 0$ and $\psi'(0) > 0$. Define the conformal radius $\mathfrak{r}(\theta)$ of $U(\theta)$ about 0 by $\mathfrak{r}(\theta) := 1/\psi'(0)$. A well known consequence of the Koebe 1/4 Theorem and the Schwarz Lemma (see, e.g., [1]) is that

$$\frac{\mathfrak{r}(\theta)}{4} \le \operatorname{dist}\bigl(0, Q(\theta)\bigr) \le \mathfrak{r}(\theta)\,, \tag{2.1}$$

and therefore information about the distribution of $\mathfrak{r}(\theta)$ translates to information about the distribution of $\operatorname{dist}\bigl(0, Q(\theta)\bigr)$. Set

$$h(\theta, t) := \mathbf{P}\bigl[\mathfrak{r}(\theta) \le e^{-t}\bigr]\,. \tag{2.2}$$

Note that for $t > 0$,

$$h(0, t) = \lim_{\theta \downarrow 0} h(\theta, t) = 0\,, \tag{2.3}$$

follows from the Russo-Seymour-Welsh Theorem and the observation that $\mathfrak{r}(\theta)$ tends to 1 if the diameter of $Q(\theta)$ tends to zero.

**Lemma 2.2.** *In the range $\theta \in (0, 2\pi)$, $t > 0$, the function $h$ satisfies the PDE*

$$\frac{\kappa}{2}\,\partial_\theta^2 h(\theta, t) + \cot\bigl(\frac{\theta}{2}\bigr)\,\partial_\theta h(\theta, t) - \partial_t h(\theta, t) = 0\,, \tag{2.4}$$

*with $\kappa = 6$.*

(The reason that we keep $\kappa$ as a parameter is that some of the work done below will also apply to the proof of Theorem 1.3, where $\kappa$ is not necessarily 6.)

**Proof.** We assume that $\gamma_{\rm ch}$ and $Q(\theta)$ are coupled as in Theorem 2.1. Let $T_{\rm ch}=T_{\rm ch}(\theta)$ be the first time $t$ such that $\gamma_{\rm ch}[0,t]$ disconnects $0$ from $e^{i\theta}$ in $\overline{\mathbb{U}}$. Let $\gamma_{\rm ra}$ denote the path of radial SLE$_6$ from $1$ to $0$ in $\overline{\mathbb{U}}$, and let $T=T_{\rm ra}(\theta)$ denote the first time $t$ such that $\gamma_{\rm ra}[0,t]$ disconnects $0$ from $e^{i\theta}$ in $\overline{\mathbb{U}}$. From [13, Theorem 4.1] we know that up to time reparameterization, the restriction of $\gamma_{\rm ra}$ to $[0,T]$ has the same distribution as the restriction of $\gamma_{\rm ch}$ to $[0,T_{\rm ch}]$. We therefore assume that indeed $\gamma_{\rm ra}$ restricted to $[0,T]$ is a reparameterization of $\gamma_{\rm ch}$ restricted to $[0,T_{\rm ch}]$. Let $U_t$ be the connected component of $\mathbb{U}\setminus\gamma_{\rm ra}[0,t]$ which contains $0$, and let

$$g_t : U_t \to \mathbb{U}$$

be the conformal map, normalized by $g_t(0)=0$, $g_t'(0)>0$. By the definition of radial SLE$_6$, the maps $g_t$ satisfy

$$\partial_t g_t(z) = -g_t(z)\,\frac{g_t(z)+e^{i\sqrt{\kappa}B_t}}{g_t(z)-e^{i\sqrt{\kappa}B_t}}\,, \qquad g_0(z)=z\,, \tag{2.5}$$

where $\kappa=6$ and $B_t$ is Brownian motion on $\mathbb{R}$ with $B_0=0$. Differentiating (2.5) with respect to $z$ at $z=0$ gives

$$g_t'(0)=e^t\,. \tag{2.6}$$

At time $T$, $\gamma_{\rm ra}$ separates $0$ from $e^{i\theta}$. Let $W'$ be the connected component of $\mathbb{U}\setminus\gamma_{\rm ra}[0,T]$ which contains $0$. We distinguish between two possibilities. Either the boundary of $W'$ on $\gamma_{\rm ra}$ is on the right hand side of $\gamma_{\rm ra}$, in which case set $\nu:=1$, or on the left hand side, and then set $\nu:=-1$. In the notation of Theorem 2.1, if $\nu=-1$, then $W'\notin\mathcal{W}$. Hence, by (2.6) and the coupling of $\gamma_{\rm ra}$ with $Q(\theta)$,

$$\mathfrak{r}(\theta)=e^{-T}\,, \qquad \text{if } \nu=-1\,. \tag{2.7}$$

We also want to understand the distribution of $\mathfrak{r}(\theta)$ given $\gamma_{\rm ra}$ and $\nu=1$. In that case, $W'\in\mathcal{W}$. In Theorem 2.1, we may take the map $\psi_{W'}$ to satisfy $\psi_{W'}(0)=0$. Let $r'$ denote the conformal radius about $0$ of $\mathbb{U}\setminus Q^{W'}$, where $Q^{W'}$ is as in Theorem 2.1. Then $r'$ has the same distribution as $\mathfrak{r}(2\pi)$ and is independent from $\bigcup_{s>0}\mathcal{F}_s$, where $\mathcal{F}_s$ denotes the $\sigma$-field generated by $(B_t : t\le s)$. Moreover, by the chain rule for the derivative at zero,

$$\mathfrak{r}(\theta)=r'\,e^{-T}\,, \qquad \text{if } \nu=1\,. \tag{2.8}$$

To make use of the relations (2.7) and (2.8), we have to understand the relation between $\nu$ and $\mathcal{F}_T$. Recall that $g_t\big(\gamma_{\rm ra}(t)\big)=\exp(i\sqrt{\kappa}B_t)$. Set for $t<T$,

$$Y_t = Y_t^\theta := -i\log g_t(e^{i\theta}) - \sqrt{\kappa}\,B_t\,, \tag{2.9}$$

with $Y_0^\theta=\theta$ and the log branch chosen continuously. That is, $Y_t$ is the length of the arc on $\partial\mathbb{U}$ which corresponds under $g_t^{-1}$ to the union of $A_\theta$ with the right hand side $\gamma_{\rm rhs}[0,t]$ of $\gamma_{\rm ra}[0,t]$. Suppose, for a moment that $\nu=1$. That means that the boundary of $W'$ is contained in $\gamma_{\rm rhs}$ and $A_\theta$. Consequently, as $t\uparrow T$, the harmonic measure from $0$ of $A_\theta\cup\gamma_{\rm rhs}$ tends to $1$. By conformal invariance of harmonic measure, this means that $Y_t\to 2\pi$ as $t\uparrow T$ on the event $\nu=1$. Similarly, we have $Y_t\to 0$ as $t\uparrow T$ on the event $\nu=-1$. Set $Y_T:=\lim_{t\uparrow T}Y_t$. By (2.7) and (2.8), we now have

$$\mathbf{P}\Big[\mathfrak{r}(\theta)\le e^{-t}\;\Big|\;\mathcal{F}_T\Big] = 1_{\{Y_T=2\pi\}}\,\mathbf{P}\big[r'\le e^{T-t}\mid\mathcal{F}_T\big] + 1_{\{Y_T=0,\,t\le T\}}\,.$$

Taking expectation gives

$$h(\theta,t) = \mathbf{E}\big[h(Y_T^\theta, t-T)\big]\,, \tag{2.10}$$

where we use the fact that $h(\theta,t)=1$ for $t\le 0$ and $\theta\in[0,2\pi]$. From (2.5) and (2.9) we get

$$dY_t = \cot(Y_t/2)\,dt - \sqrt{\kappa}\,dB_t\,. \tag{2.11}$$

By (2.10), for every constant $s>0$ the process $t\mapsto h(Y_t^\theta, s-t)$ is a local martingale on $t<\min\{T,s\}$. The theory of diffusion processes and (2.10), (2.11) imply that $h(\theta,t)$ is smooth in the range $\theta\in(0,2\pi)$, $t>0$. By Itô's formula at time $t=0$

$$dh(Y_t, s-t) = \Big(\frac{\kappa}{2}\,\partial_\theta^2 h(\theta,s) + \cot\frac{\theta}{2}\,\partial_\theta h(\theta,s) - \partial_s h(\theta,s)\Big)dt - \sqrt{\kappa}\,\partial_\theta h(\theta,s)\,dB_t.$$

Since $h(Y_t,s-t)$ is a local martingale, the $dt$ term must vanish, and we obtain (2.4). $\square$

In order to derive boundary conditions, it turns out that it is more convenient to work with the following modified version of $h$:

$$\tilde h(\theta,t) := \int_0^1 h(\theta, t+s)\,ds\,.$$

Since, $h$ is smooth, $\tilde h$ also satisfies the PDE (2.4).

**Lemma 2.3.** *For every fixed $t>0$, the one sided $\theta$-derivative of $\tilde h$ at $2\pi$ is zero; that is,*

$$\lim_{\theta\uparrow 2\pi} \frac{\tilde h(2\pi,t)-\tilde h(\theta,t)}{2\pi-\theta} = 0\,. \tag{2.12}$$

**Proof.** Let $\epsilon>0$ be very small and set $\theta=2\pi-\epsilon$. Let $\delta>0$ be smaller than $\epsilon$, and let $\mathcal{Z}(r,\epsilon)$ denote the event that there is a connected component of $\mathfrak{B}_\delta\cap\overline{\mathbb{U}}$ which does not intersect $A_\theta$ but does intersect the two circles of radii $\epsilon$ and $r$ about the point 1.

We claim that there are constants $c,\alpha>0$ such that

$$\mathbf{P}\big[\mathcal{Z}(r,\epsilon)\big] \le c(\epsilon/r)^{1+\alpha}\,, \tag{2.13}$$

provided $0<\delta<\epsilon<r<2$. This well known result is also used in Smirnov's arguments. As we could not track down an explicit proof of this statement in the literature and for the reader's convenience, we include a proof of this fact in the appendix.

Let $Q' := Q(2\pi)\setminus Q(\theta)$. By letting $\delta\downarrow 0$, it follows from (2.13) that

$$\mathbf{P}\big[\operatorname{diam} Q' \ge r\big] \le c\,(\epsilon/r)^{1+\alpha}\,. \tag{2.14}$$

It is easy to verify that there is a constant $c_1>0$ such that for every connected compact $K\subset\overline{\mathbb{U}}$ which intersects $\partial\mathbb{U}$, the harmonic measure $\mu(\mathbb{U},K,0)$ in $\mathbb{U}$ of $K$ from 0 satisfies

$$c_1^{-1}\operatorname{diam} K \le \mu(\mathbb{U},K,0) \le c_1 \operatorname{diam} K\,. \tag{2.15}$$

Moreover, if $\mathfrak{r}(K)$ denotes the conformal radius of $\mathbb{U}\setminus K$, then

$$\min\Big\{-\log\mathfrak{r}(K), 1\Big\} \le c_2\big(\operatorname{diam} K\big)^2\,, \tag{2.16}$$

where $c_2$ is some constant. To justify this, note that $\mathfrak{r}(K)$ is monotone non-increasing in $K$ and hence $\mathfrak{r}(K) \geq \mathfrak{r}(B)$, where $B$ is the smallest disk with $\partial B$ orthogonal to $\partial \mathbb{U}$ which contains $K$. For such a $B$, one can calculate $\mathfrak{r}(B)$ explicitly, since the normalized conformal map from $\mathbb{U} \setminus B$ to $\mathbb{U}$ is conjugate to the map $z \to \sqrt{z}$ by Möbius transformations. Now

$$\begin{aligned}
\min\Big\{\log \mathfrak{r}(\theta) - \log \mathfrak{r}(2\pi), 1\Big\} &= \min\Big\{-\log \mathfrak{r}\big(\psi_\theta^{-1}(Q')\big), 1\Big\} \\
&\leq c_2 \big(\operatorname{diam} \psi_\theta^{-1}(Q')\big)^2 \\
&\leq c_2\, c_1^2\, \mu(\mathbb{U}, \psi_\theta^{-1}(Q'), 0)^2 \\
&= c_2\, c_1^2\, \mu(\mathbb{U} \setminus Q(\theta), Q', 0)^2 \\
&\leq c_2\, c_1^2\, \mu(\mathbb{U}, Q', 0)^2 \leq c_2\, c_1^4\, (\operatorname{diam} Q')^2 .
\end{aligned}$$

(The equality is due to conformal invariance of harmonic measure.) Combining this with (2.14) gives

$$\mathbf{P}\big[\log \mathfrak{r}(\theta) - \log \mathfrak{r}(2\pi) \geq s\big] \leq \min\big\{1, c_3\, (\epsilon/\sqrt{s})^{1+\alpha}\big\}, \qquad s \leq 1 .$$

Hence,

$$\tilde{h}(2\pi, t) - \tilde{h}(\theta, t) \leq \int_0^1 \mathbf{P}\big[\log \mathfrak{r}(\theta) - \log \mathfrak{r}(2\pi) \geq s\big]\, ds = o(\epsilon) ,$$

and the lemma is established. □

**Proof of Theorem 1.2.** We are going to give a non-probabilistic proof based on the PDE and boundary conditions that we derived (but other justifications are also possible). Let $\Lambda$ denote the differential operator on the left hand side of (2.4), and set $S = (0, 2\pi) \times (0, \infty)$. Set

$$q := \frac{\kappa - 4}{\kappa}, \qquad H(\theta, t) := \big(\sin(\theta/4)\big)^q\, e^{-\lambda t} ,$$

where $\lambda$ is defined by (1.1). The function $\tilde{h}$ satisfies $\Lambda \tilde{h} = 0$ in $S$, the Neumann boundary condition (2.12) the Dirichlet condition $\tilde{h}(0, t) = 0$ for $t > 0$, $0 \leq \tilde{h}(\theta, t) \leq 1$ and $\tilde{h}(\theta, 0) > 0$ for $\theta \in (0, 2\pi]$. Note that $H$ also has all these properties. We now use the Maximum Principle to show that there are positive constants $c, c'$ such that

$$\forall t \geq 1\,,\ \forall \theta \in [0, 2\pi]\,, \qquad c\, H(\theta, t) \leq \tilde{h}(\theta, t) \leq c'\, H(\theta, t) . \tag{2.17}$$

This is a bit tricky because of the singularities of (2.4) at the boundary, but otherwise quite standard.

Suppose that a function $G$ is defined and continuous on $S^* := \overline{S} \setminus \{(0,0)\}$, non-negative in some neighborhood of $(0,0)$ in $S^*$, $G \geq 0$ on $t = 0$, $G = 0$ on $\theta = 0$, $\partial_\theta G = 0$ on $\theta = 2\pi$ and $\Lambda(G) = 0$ in $S$. Then, we claim that $G \geq 0$ on $S^*$.

Let $\epsilon > 0$ and set $F := G + \epsilon\, \theta^{q/2} + \epsilon$. Assume that $F < 0$ somewhere in $S$. The function $F$ is continuous on $S^*$, positive in a neighborhood of $(0,0)$ in $S^*$ and $F \geq \epsilon$ when $t = 0$ or $\theta = 0$. Continuity then implies that there is a pair $(\theta_0, t_0) \in S^*$ such that $F(\theta_0, t_0) \leq 0$ and $t_0$ is minimal among all such points in $S^*$. Clearly, $t_0 > 0$ and $\theta_0 > 0$. Observe also that $\theta_0 \neq 2\pi$, because $\partial_\theta F > 0$ at $\theta = 2\pi$. Hence, $(\theta_0, t_0) \in S$. A computation using the explicit value of $q$ shows that $\Lambda(F) = \epsilon \Lambda(\theta^{q/2}) < 0$ in $S$. By the minimality of $t_0$, we have

$$\partial_\theta F(\theta_0, t_0) = 0\,, \qquad \partial_\theta^2 F(\theta_0, t_0) \geq 0\,, \qquad \partial_t F(\theta_0, t_0) \leq 0 .$$

But the definition of $\Lambda$ shows that this contradicts $\Lambda F < 0$. Therefore $F \geq 0$ on $S^*$. Since $\epsilon > 0$ was arbitrary, this proves that $G \geq 0$.

Now, note that $\Lambda(2 - 2t - \theta^2) < 0$ for $\theta \in (0, \pi)$. Let

$$F_1 := c_1 \, H - \tilde{h} + 2 - 2t - \theta^2 \,, \qquad F_2 := c_2 \, \tilde{h} - H + 2 - 2t - \theta^2 \,,$$

where the constants $c_1, c_2$ are chosen so that $F_1 > 0$ and $F_2 > 0$ on $\{\pi\} \times [0,1]$ and on $[0,\pi] \times \{0\}$. The above argument applied to the functions $F_j$ with $S$ replaced by $(0,\pi) \times (0,1)$, shows that $c_1 \, H(\theta,1) - \tilde{h}(\theta,1) \geq F_1(\theta,1) \geq 0$ and $c_2 \, \tilde{h}(\theta,1) - H(\theta,1) \geq F_2(\theta,1) \geq 0$ for $\theta \in [0,\pi]$. By changing the constants $c_1$ and $c_2$ if necessary, we may make sure that this same inequalities hold for all $\theta \in [0, 2\pi]$. Yet another application of the same Maximum Principle argument, this time in the range $[0,2\pi] \times [1,\infty)$ proves that $c_1 \, H - \tilde{h} \geq 0$ and $c_2 \, \tilde{h} - H \geq 0$ for $t \geq 1$, thereby establishing (2.17).

The theorem now follows from (2.17), the definition of $\tilde{h}$ and (2.1). □

# 3 The discrete exponent

We now show that the discrete exponent and the continuous exponent we derived are the same. The proof is quite direct. However, for exponents related to crossings of several colors (that is, black and white or occupied and vacant), the analogous equivalence is not as easy. See [24] for a treatment of this more involved situation.

**Proof of Theorem 1.1.** Let $u(r)$ denote the probability that $Q = Q(2\pi)$ intersects the circle $r\partial\mathbb{U}$, and let $\mathcal{C}(R_1, R_2)$ denote the event that in site percolation with parameter $p = 1/2$ on the standard triangular grid, there is an open cluster crossing the annulus with radii $R_1$ and $R_2$ about 0. Let $\mathcal{A}(R)$ denote the event that there is an open path separating the circles of radii $R$ and $2R$ about 0. By the definition of the scaling limit it follows that for all fixed $r \in (0,1)$ there is some $s_0 = s_0(r)$ such that

$$\forall R \geq s_0, \qquad u(r/2)/2 \leq \mathbf{P}\big[\mathcal{C}(rR, R)\big] \leq 2\, u(2r)\,. \tag{3.1}$$

Let $R$ be large, $r > 0$ small, set $\tilde{r} = 2r$, and let $N$ be the minimal integer satisfying $s_0(r)\, \tilde{r}^{-N} > R$. Note that the event $\{0 \leftrightarrow C_R\}$ that 0 is connected to the circle $C_R$ satisfies

$$\{0 \leftrightarrow C_R\} \supset \{0 \leftrightarrow C_{2s_0}\} \cap \bigcap_{j=0}^{N-1} \big(\mathcal{C}(s_0\, \tilde{r}^{-j}, 2\, s_0\, \tilde{r}^{-j-1}) \cap \mathcal{A}(s_0\, \tilde{r}^{-j})\big)\,.$$

By the Harris/Fortuin-Kasteleyn-Ginibre (FKG) inequality, we therefore have

$$\mathbf{P}[0 \leftrightarrow C_R] \geq \mathbf{P}[0 \leftrightarrow C_{2s_0}] \prod_{j=0}^{N-1} \mathbf{P}\big[\mathcal{C}(s_0\, \tilde{r}^{-j}, 2\, s_0\, \tilde{r}^{-j-1})\big] \prod_{j=0}^{N-1} \mathbf{P}\big[\mathcal{A}(s_0\, \tilde{r}^{-j})\big]\,.$$

By the Russo-Seymour-Welsh Theorem (RSW), there is a constant $c_1 > 0$ such that $\mathbf{P}\big[\mathcal{A}(R)\big] \geq c_1$ for every $R$. Applying this and (3.1) in the above, gives

$$\mathbf{P}[0 \leftrightarrow C_R] \geq \mathbf{P}[0 \leftrightarrow C_{2s_0}]\, c_1^N \,\big(u(r/2)/2\big)^N\,.$$

By Theorem 1.2, $\log u(r)/\log r \to 5/48$ as $r \downarrow 0$. Therefore,

$$\liminf_{R\to\infty} \frac{\log \mathbf{P}[0 \leftrightarrow C_R]}{\log R} \geq \frac{\log(c_1/2) + \log u(r/2)}{-\log r} \to -5/48\,,$$

as $r \downarrow 0$. This proves the required lower bound on $\mathbf{P}[0 \leftrightarrow C_R]$. The proof for the upper bound is similar (using independence on disjoint sets, in place of FKG), and is left to the reader. □

# 4 Counterclockwise loops for $\mathrm{SLE}_\kappa$

**Proof of Theorem 1.3.** Let $\kappa > 4$, $\kappa \neq 8$. Let $\gamma$ denote the path of radial $\mathrm{SLE}_\kappa$ started from 1, and for $\theta \in [0, 2\pi]$ let $\gamma_\theta^t$ denote the concatenation of the counterclockwise oriented arc $\partial\mathbb{U} \setminus A_\theta$ with $\gamma[0, t]$. Let $\mathcal{E}(\theta, t)$ denote the event that $\gamma_\theta^t$ does not contain a counterclockwise loop around 0, and set

$$\hat{h}(\theta, t) := \mathbf{P}\big[\mathcal{E}(\theta, t)\big].$$

We know that a.s. for all $t > 0$, $\theta > 0$ the path $\gamma[0, t]$ intersects $A_\theta$. (This follows by comparing with a Bessel process. See, e.g., [19].) Hence, $\lim_{\theta\downarrow 0} \hat{h}(\theta, t) = 0$ for all $t > 0$.

Let $K_t$ denote the SLE hull (i.e., $\gamma[0, t]$ union with the set of points in $\overline{\mathbb{U}}$ separated from 0 by $\gamma[0, t]$), and let $U_t := \mathbb{U} \setminus K_t$, which is the component of $\mathbb{U} \setminus \gamma[0, t]$ containing 0. Let

$$I_t := \big\{\theta \in (0, 2\pi) : e^{i\theta} \notin K_t\big\}\,.$$

Let $g_t$ denote the conformal map $g_t : U_t \to \mathbb{U}$ normalized by $g_t(0) = 0$ and $g_t'(0) > 0$.

Let $T(\theta)$ denote the first time such that $e^{i\theta} \in K_t$, and let $\hat{T} := \sup_\theta T(\theta)$. Then $\hat{T}$ is also the first time $t$ such that $\gamma[0, t]$ contains a loop around 0, and is the first time such that $I_t = \emptyset$. Define $Y_t^\theta$ for $t < T(\theta)$ as in (2.9). A moment's thought shows that (2.10) holds with $\hat{h}$ in place of $h$. Therefore, $\hat{h}$ satisfies (2.4) for $\theta \in (0, 2\pi)$ and $t > 0$.

We now need to verify that $\hat{h}$ also satisfies the Neumann boundary condition (2.12). It is immediate that there is a constant $c_1 > 0$ such that

$$\forall t > 0,\ \forall \theta \in [\pi/2, 3\pi/2], \qquad \big|\partial_\theta \hat{h}(\theta, t)\big| \leq c_1\,. \tag{4.1}$$

(This is a Harnack type inequality.) For $\theta \in [0, 2\pi]$ and $t \in [T(\theta), \hat{T})$ *define*,

$$Y_t^\theta := \begin{cases} \inf\{Y_t^\alpha : \alpha \in I_t\} & \theta \leq \inf I_t\,, \\ \sup\{Y_t^\alpha : \alpha \in I_t\} & \theta \geq \sup I_t\,. \end{cases}$$

For fixed $\theta$, the process $Y_t^\theta$ is a process satisfying (2.11) with instantaneous reflection in the boundary (see, e.g., [18] for a detailed treatment of such processes) stopped at the time $\hat{T}$. Clearly, for all $t \in [0, \hat{T})$ the set of $\theta \in [0, 2\pi]$ for which $\mathcal{E}(\theta, t)$ holds is $\big\{\theta \in [0, 2\pi] : \inf_{s\leq t} Y_s^\theta > 0\big\}$. Also, for all $t \geq \hat{T}$ the set of $\theta \in [0, 2\pi]$ for which $\mathcal{E}(\theta, t)$ holds is either $\emptyset$ or $\big\{\theta \in [0, 2\pi] : \inf_{s<\hat{T}} Y_s^\theta > 0\big\}$.

Now let $\theta_1$ and $\theta_2$ satisfy $3\pi/2 < \theta_1 < \theta_2 < 2\pi$, and fix some $t_1 > 0$. Let $\tau := \inf\big\{t \in [0, \hat{T}] : Y_t^{\theta_1} = \pi\big\}$, and let $\mathcal{S}$ be the event that $\tau \leq t_1$ and $Y_\tau^{\theta_2} \neq \pi$. Observe that $\mathcal{E}(\theta_1, t_1)$ is equivalent to $\mathcal{E}(\theta_2, t_1)$ on the complement of $\mathcal{S}$. Consequently, the strong Markov property at time $\tau$ gives

$$\hat{h}(\theta_2, t_1) - \hat{h}(\theta_1, t_1) \leq \mathbf{P}[\mathcal{S}]\,\mathbf{E}\big[\hat{h}(Y_\tau^{\theta_2}, t_1 - \tau) - \hat{h}(Y_\tau^{\theta_1}, t_1 - \tau)\ \big|\ \mathcal{S}\big]\,. \tag{4.2}$$

Observe from (2.11) that

$$\partial_t\big(Y_t^{\theta_2} - Y_t^{\theta_1}\big) \le 0\,.$$

Therefore, on $\mathcal{S}$ we have $Y_\tau^{\theta_2} - Y_\tau^{\theta_1} \le \theta_2 - \theta_1$. Hence, by (4.1) combined with (4.2),

$$\hat h(\theta_2, t_1) - \hat h(\theta_1, t_1) \le c_1\,\mathbf{P}[\mathcal{S}]\,(\theta_2 - \theta_1)\,. \tag{4.3}$$

But when $\theta_1 < \theta_2 \le 2\pi$ are both close to $2\pi$, we know that with probability close to 1 there will be some time $t_0 \in (0, t_1)$ such that $Y_{t_0}^{\theta_1} = 2\pi$. In that case also $Y_{t_0}^{\theta_2} = 2\pi$, and hence $Y_t^{\theta_1} = Y_t^{\theta_2}$ for all $t \in [t_0, \hat T)$. In particular, $\mathbf{P}[\mathcal{S}] \to 0$ as $\theta_1 \to 2\pi$. By (4.3) we get the Neumann condition:

$$\lim_{\theta\uparrow 2\pi} \partial_\theta \hat h(\theta, t_1) = 0\,.$$

Now the proof of the theorem is completed exactly as for the corresponding proof of (2.17). □

# A Appendix: an a priori half plane exponent

We now give a brief outline of the proof of the a priori estimate (2.13) on the probability that there exists three disjoint crossings of a half-annulus. This is based on the exponent for the existence of two disjoint crossings.

**Lemma A.1.** *There is a constant $c > 0$ such that for all $R > r > 2$ the probability $f(R, r)$ that there are two disjoint crossings between $C_R$ and $C_r$ within the half-plane $i\overline{\mathbb{H}} = \{z : \operatorname{Re} z \le 0\}$ in site percolation with parameter $p = 1/2$ on the standard triangular grid satisfies*

$$c^{-1}\, r/R \le f(R, r) \le c\, r/R\,.$$

Loosely speaking, this lemma says that the two-arms half plane exponent is 1. Note that the half-plane exponents for $\mathrm{SLE}_6$ were calculated in [12, 14]. They are now valid for critical site percolation on the triangular grid, since $\mathrm{SLE}_6$ describes the scaling limit. However, to show the equivalence, this lemma seems to be needed.

**Proof.** Let $S_R$ be the strip $S_R := \{z \in \mathbb{C} : \operatorname{Re} z \in [-R, 0]\}$, and let $X_R$ be the set of clusters in $S_R$ which join the interval $I_R := [-iR, iR] \subset i\mathbb{R}$ to the line $L_R := -R + i\mathbb{R}$. It follows from RSW that

$$\sup_{R>0} \mathbf{E}\big[|X_R|\big] < \infty\,, \qquad \inf_{R>2} \mathbf{P}\big[|X_R| > 2\big] > 0\,. \tag{A.1}$$

Define the event $\mathcal{D}_R$ that there exists a path of occupied site in $S_R$ joining the origin to $L_R$ and a path of vacant sites in $S_R$ joining the vertex directly below the origin to $L_R$. Note that $\mathcal{D}_R$ is the same as the event that there is some $C \in X_R$ such that 0 is the lowest vertex in $C \cap i\mathbb{R}$. By invariance under vertical translations, since the number of vertices in $I_R$ is of order $R$, (A.1) implies that

$$\forall R > 2, \qquad \mathbf{P}[\mathcal{D}_R] \asymp 1/R\,,$$

where $g \asymp h$ means that $g/h$ is bounded and bounded away from zero.

We now employ the standard "color-flipping" argument (see e.g., [2, 12]). That is, suppose that the origin is connected to $L_R$ in $S_R$. By flipping all the vertices below the topmost crossing from 0 to $L_R$ in $S_R$, it follows that $\mathbf{P}[\mathcal{D}_R]$ is equal to the probability of the event $\mathcal{D}_R^2$ that there are disjoint open paths in $S_R$ connecting the origin and the vertex below the origin to $L_R$. Let $\mathcal{C}_R^2$ be the event that there are two disjoint paths connecting the origin and the vertex below the origin to $C_R$ in $i\overline{\mathbb{H}}$. Clearly, $\mathcal{C}_R^2 \supset \mathcal{D}_R^2$. On the other hand, when $R$ is sufficiently large, by RSW the probability that there are two disjoint left-right crossings of each of the the rectangles $[-R, 0] \times [R/10, R/5]$ and $[-R, 0] \times [-R/5, -R/10]$ is bounded away from zero. Therefore, the FKG inequality implies that there is some constant $c > 0$ such that

$$\mathbf{P}[\mathcal{D}_R^2] \geq c\, \mathbf{P}[\mathcal{C}_{2R}^2]\,.$$

Consequently, we find that $\mathbf{P}[\mathcal{C}_R^2] \asymp R^{-1}$ for $R > 2$. Another application of RSW and FKG as in the proof of Theorem 1.1 shows that

$$\mathbf{P}[\mathcal{C}_R^2] \asymp f(R, r)\, \mathbf{P}[\mathcal{C}_r^2]\,, \qquad \text{when } R > 2r > 4\,.$$

The lemma follows. $\square$

By using RSW yet again combined with the van den Berg-Kesten inequality (BK), it follows that there are $c, \epsilon > 0$ such that the probability that $C_r$ and $C_R$ are connected in $i\overline{\mathbb{H}}$ by two open crossings and one closed crossing is at most $c\,(r/R)^{1+\epsilon}$, when $r > 1$. This immediately implies (2.13), because the existence of a cluster in $\overline{\mathbb{U}}$ intersecting $C_r$ and $C_\epsilon$ but not intersecting $A_\theta$ implies that there are also two disjoint closed crossings from $C_\epsilon$ to $C_r$ in $\overline{\mathbb{U}}$.

By looking at the right-most vertices of clusters of diameter greater than $R$, an argument similar to the proof of Lemma A.1 can be used to prove directly that the three-arms exponent in a half-plane is 2. An analogous argument [8, Lemma 5] shows that the multicolor five-arms exponent in the plane is equal to 2, which then implies that the multicolor six arms exponent is strictly larger than 2. (This is also an instrumental a priori estimate in [23].) Similarly, the Brownian intersection exponent $\xi(2, 1) = 2$ can be easily determined [9], and a direct proof also works for some exponents associated to loop-erased random walks (e.g., [10]). However, these exponents that take the value 1 or 2 are exceptional. For the determination of fractional exponents, such as, for instance, the one-arm exponent in the present paper, such direct arguments do not work.

## B The monochromatic 2-arm exponent

This informal appendix will discuss the monochromatic 2-arm exponent (sometimes also called "backbone exponent"); that is, the exponent describing the decay of the probability that there are two disjoint open crossings from $C_r$ to $C_R$. So far, despite some attempts, there is no established prediction in the theoretical physics community for the value of this exponent, as far as we are aware. Simulations [4] show that its numerical value is $.3568 \pm .0008$.

The color switching argument used above shows that for half-plane exponents, the required colors of the arms do not matter. For full plane exponents, this argument can only show that every two $k$-arm exponents are the same, if both colors are present in both [2]. For example, the plane exponent for the existence of open, closed, open, closed, open crossings, in clockwise order, is the same as the exponent for the closed, open, open, open, open exponent. Thus, for each $k = 2, 3, \ldots$, there are two types of full-plane exponents, the monochromatic $k$-arm exponent, and the multichromatic $k$-arm exponent. The multichromatic exponents with $k \geq 2$ can all be worked out from [13] combined with [22] (see [24]). We now show (ommiting many details) that the monochromatic 2-arm exponent is in fact also the leading eigenvalue of a differential operator. So far, we have not been able to solve explicitely the PDE problem and to give an explicit formula for the value of the exponent, but this might perhaps be doable by someone more proficient in this field.

Like in the one-arm case, we have to work with a slightly more general problem. Let $\alpha, \beta > 0$ satisfy $\alpha + \beta < 2\pi$, and let $A$ and $A'$ be two consecutive arcs on $\partial\mathbb{U}$ with lengths $\alpha$ and $\beta$, respectively, and set $A'' = \partial\mathbb{U} \setminus (A \cup A')$. Let $\gamma = 2\pi - \alpha - \beta$ denote the length of $A''$ and let $Q$ be the scaling limit of the set of points $p \in \overline{\mathbb{U}}$ which are connected to $A''$ by two paths lying in the union of $A'$ with the set of black hexagons, such that the two paths do not share any hexagon except for the hexagon containing $p$. Let $G(\alpha, \gamma, t)$ be the probability that the conformal radius of $\mathbb{U} \setminus Q$ about 0 is at most $e^{-t}$. Using arguments as above, it is easy to check, by starting an $\mathrm{SLE}_6$ path from the common endpoint of $A$ and $A'$, that $G(\alpha, \gamma, t)$ satisfies the PDE

$$3\,\partial_\alpha^2 G + \cot\big(\frac{\alpha}{2}\big)\partial_\alpha G + \Big(\cot\big(\frac{\alpha+\gamma}{2}\big) - \cot\big(\frac{\alpha}{2}\big)\Big)\partial_\gamma G - \partial_t G = 0.$$

The Dirichlet condition

$$G = 0\,, \qquad \text{when } \gamma = 0$$

is immediate. Similarly, one can check that for all $\gamma \neq 0$,

$$G(0, \gamma) = G(0, 2\pi).$$

Finally, using arguments as for the one-arm exponent, one can show that the boundary condition on $\alpha + \gamma = 2\pi$ is

$$\partial_\gamma G = 0.$$

Hence, the monochromatic two-arm exponent is a number (and in fact the unique number) $\lambda > 0$ such that there is a non-negative function $G_1(\alpha, \gamma)$, not identically zero, such that $e^{-\lambda t} G_1(\alpha, \gamma)$ solves the above PDE and has the corresponding boundary behaviour (one can fix the multiplicative constant by setting $G_1(0, 2\pi) = 1$). We have been unable to guess an explicit solution to this boundary value problem. Some insight might be gained by noting that when $\alpha$ and $\gamma$ go down to zero, the solution should behave like Cardy's [3] formula in $\alpha/(\alpha + \gamma)$, which is a hypergeometric function.

One can rewrite the PDE using other variables. In terms of the function $G_2(\alpha, \beta) := G_1(\alpha, 2\pi - \alpha - \beta)$, one looks for the non-negative eigenfunctions of the more symmetric operator

$$3(\partial_\alpha^2 - 2\partial_\alpha\partial_\beta + \partial_\beta^2) + \cot\big(\frac{\alpha}{2}\big)\partial_\alpha - \cot\big(\frac{\beta}{2}\big)\partial_\beta$$

with boundary conditions (for $\alpha, \beta > 0$, $\alpha + \beta < 2\pi$),

$$G_2(\alpha, 2\pi - \alpha) = 0,\ \ G_2(0, \beta) = 1 \text{ and } \partial_\beta G_2(\alpha, 0) = 0.$$

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