Electronic Journal of Probability

Vol. 14 (2009), Paper no. 39, pages 1117–1161.

Journal URL
http://www.math.washington.edu/~ejpecp/

The Virgin Island Model

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Abstract

A continuous mass population model with local competition is constructed where every emigrant colonizes an unpopulated island. The population founded by an emigrant is modeled as excursion from zero of an one-dimensional diffusion. With this excursion measure, we construct a process which we call Virgin Island Model. A necessary and sufficient condition for extinction of the total population is obtained for finite initial total mass.

Key words: branching populations, local competition, extinction, survival, excursion measure, Virgin Island Model, Crump-Mode-Jagers process, general branching process.

AMS 2000 Subject Classification: Primary 60K35, 92D25.

Submitted to EJP on October 9, 2008, final version accepted April 22, 2009.

[*]Research supported by the DFG in the Dutch German Bilateral Research Group "Mathematics of Random Spatial Models from Physics and Biology" (FOR 498)

[†]Research supported by EPSRC Grant no GR/T19537/01

1 Introduction

This paper is motivated by an open question on a system of interacting locally regulated diffusions. In [8], a sufficient condition for local extinction is established for such a system. In general, however, there is no criterion available for global extinction, that is, convergence of the total mass process to zero when started in finite total mass.

The method of proof for the local extinction result in [8] is a comparison with a mean field model $(M_t)_{t\geq 0}$ which solves

$$dM_t = \kappa(\mathbf{E}M_t - M_t)dt + h(M_t)dt + \sqrt{2g(M_t)}dB_t \tag{1}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion and where $h, g\colon [0,\infty) \to \mathbb{R}$ are suitable functions satisfying $h(0) = 0 = g(0)$. This mean field model arises as the limit as $N \to \infty$ (see Theorem 1.4 in [19] for the case $h \equiv 0$) of the following system of interacting locally regulated diffusions on N islands with uniform migration

$$\begin{aligned} dX_t^N(i) =& \kappa\Big[\frac{1}{N}\sum_{j=0}^{N-1} X_t^N(j) - X_t^N(i)\Big]\, dt \\ &+ h\big(X_t^N(i)\big)dt + \sqrt{2g\big(X_t^N(i)\big)}\, dB_t(i) \qquad i = 0, \ldots, N-1. \end{aligned} \tag{2}$$

For this convergence, $X_0^N(0), \ldots, X_0^N(N-1)$ may be assumed to be independent and identically distributed with the law of $X_0^N(0)$ being independent of N. The intuition behind the comparison with the mean field model is that if there is competition (modeled through the functions h and g in (2)) among individuals and resources are everywhere the same, then the best strategy for survival of the population is to spread out in space as quickly as possible.

The results of [8] cover translation invariant initial measures and local extinction. For general h and g, not much is known about extinction of the total mass process. Let the solution $(X_t^N)_{t\geq 0}$ of (2) be started in $X_0^N(i) = x\mathbb{1}_{i=0}$, $x \geq 0$. We prove in a forthcoming paper under suitable conditions on the parameters that the total mass $|X_t^N| := \sum_{i=1}^N X_t^N(i)$ converges as $N \to \infty$. In addition, we show in that paper that the limiting process dominates the total mass process of the corresponding system of interacting locally regulated diffusions started in finite total mass. Consequently, a global extinction result for the limiting process would imply a global extinction result for systems of locally regulated diffusions.

In this paper we introduce and study a model which we call *Virgin Island Model* and which is the limiting process of $(X_t^N)_{t\geq 0}$ as $N \to \infty$. Note that in the process $(X_t^N)_{t\geq 0}$ an emigrant moves to a given island with probability $\frac{1}{N}$. This leads to the characteristic property of the Virgin Island Model namely every emigrant moves to an unpopulated island. Our main result is a necessary and sufficient condition (see (28) below) for global extinction for the Virgin Island Model. Moreover, this condition is fairly explicit in terms of the parameters of the model.

Now we define the model. On the 0-th island evolves a diffusion $Y = (Y_t)_{t\geq 0}$ with state space $\mathbb{R}_{\geq 0}$ given by the strong solution of the stochastic differential equation

$$dY_t = -a(Y_t)\, dt + h(Y_t)\, dt + \sqrt{2g(Y_t)}dB_t, \qquad Y_0 = y \geq 0, \tag{3}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion. This diffusion models the total mass of a population and is the diffusion limit of near-critical branching particle processes where both the offspring mean and the offspring variance are regulated by the total population. Later, we will specify conditions on a, h and g so that Y is well-defined. For now, we restrict our attention to the prototype example of a Feller branching diffusion with logistic growth in which $a(y) = \kappa y$, $h(y) = \gamma y(K - y)$ and $g(y) = \beta y$ with $\kappa, \gamma, K, \beta > 0$. Note that zero is a trap for Y, that is, $Y_t = 0$ implies $Y_{t+s} = 0$ for all $s \geq 0$.

Mass emigrates from the 0-th island at rate $a(Y_t)\,dt$ and colonizes unpopulated islands. A new population should evolve as the process $(Y_t)_{t\geq 0}$. Thus, we need the law of excursions of Y from the trap zero. For this, define the set of excursions from zero by

$$U := \big\{\chi \in \mathbf{C}\big((-\infty,\infty),[0,\infty)\big) : T_0 \in (0,\infty],\ \chi_t = 0\ \ \forall\, t \in (-\infty,0] \cup [T_0,\infty)\big\} \tag{4}$$

where $T_y = T_y(\chi) := \inf\{t > 0 : \chi_t = y\}$ is the first hitting time of $y \in [0,\infty)$. The set U is furnished with locally uniform convergence. Throughout the paper, $\mathbf{C}(S_1, S_2)$ and $\mathbf{D}(S_1, S_2)$ denote the set of continuous functions and the set of càdlàg functions, respectively, between two intervals $S_1, S_2 \subset \mathbb{R}$. Furthermore, define

$$\mathbf{D} := \big\{\chi \in \mathbf{D}\big((-\infty,\infty),[0,\infty)\big) : \chi_t = 0 \quad \forall\, t < 0\big\}. \tag{5}$$

The *excursion measure* Q_Y is a σ-finite measure on U. It has been constructed by Pitman and Yor [16] as follows: Under Q_Y, the trajectories come from zero according to an entrance law and then move according to the law of Y. Further characterizations of Q_Y are given in [16], too. For a discussion on the excursion theory of one-dimensional diffusions, see [18]. We will give a definition of Q_Y later.

Next we construct all islands which are colonized from the 0-th island and call these islands the first generation. Then we construct the second generation which is the collection of all islands which have been colonized from islands of the first generation, and so on. Figure 1 illustrates the resulting tree of excursions. For the generation-wise construction, we use a method to index islands which keeps track of which island has been colonized from which island. An island is identified with a triple which indicates its mother island, the time of its colonization and the population size on the island as a function of time. For $\chi \in \mathbf{D}$, let

$$\mathscr{I}_0^\chi := \big\{(\emptyset, 0, \chi)\big\} \tag{6}$$

be a possible 0-th island. For each $n \geq 1$ and $\chi \in \mathbf{D}$, define

$$\mathscr{I}_n^\chi := \big\{\big(\iota_{n-1}, s, \psi\big) : \iota_{n-1} \in \mathscr{I}_{n-1}^\chi, (s,\psi) \in [0,\infty) \times \mathbf{D}\big\} \tag{7}$$

which we will refer to as the set of all possible islands of the n-th generation with fixed 0-th island $(\emptyset, 0, \chi)$. This notation should be read as follows. The island $\iota_n = (\iota_{n-1}, s, \psi) \in \mathscr{I}_n^\chi$ has been colonized from island $\iota_{n-1} \in \mathscr{I}_{n-1}^\chi$ at time s and carries total mass $\psi(t-s)$ at time $t \geq 0$. Notice that there is no mass on an island before the time of its colonization. The island space is defined by

$$\mathscr{I} := \{\emptyset\} \cup \bigcup_{\chi \in \mathbf{D}} \mathscr{I}^\chi \quad \text{where} \quad \mathscr{I}^\chi := \bigcup_{n \geq 0} \mathscr{I}_n^\chi. \tag{8}$$

Denote by $\sigma_\iota := s$ the colonization time of island ι if $\iota = (\iota', s, \psi)$ for some $\iota' \in \mathscr{I}$. Furthermore, let $\{\Pi^\iota : \iota \in \mathscr{I} \setminus \{\emptyset\}\}$ be a set of Poisson point processes on $[0,\infty) \times \mathbf{D}$ with intensity measure

$$\mathbf{E}\big[\Pi^{(\iota,s,\chi)}(dt \otimes d\psi)\big] = a\big(\chi(t-s)\big)\,dt \otimes Q_Y(d\psi) \quad \iota \in \mathscr{I}. \tag{9}$$

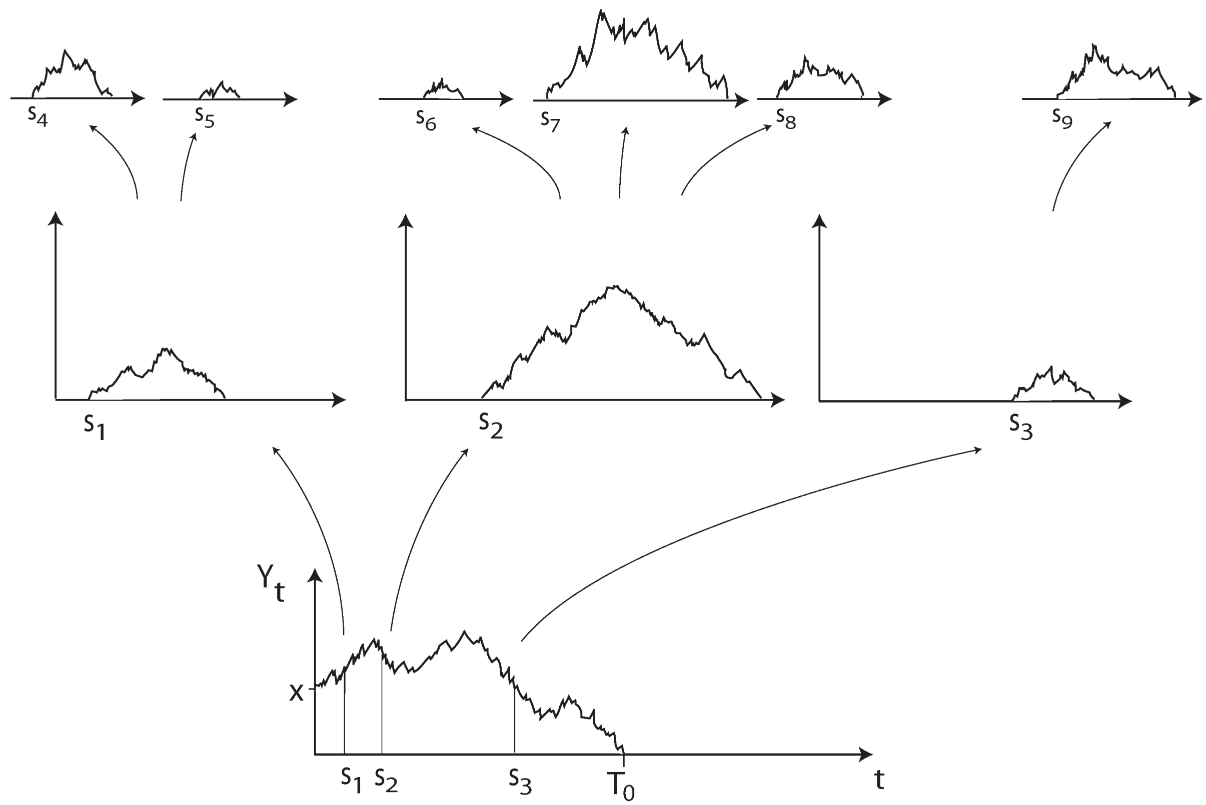

Figure 1: Subtree of the Virgin Island Model. Only offspring islands with a certain excursion height are drawn. Note that infinitely many islands are colonized e.g. between times s_1 and s_2.

For later use, let $\Pi^{\chi} := \Pi^{(\emptyset,0,\chi)}$. We assume that the family $\{\Pi^{\iota} : \iota \in \mathscr{I}^{\chi}\}$ is independent for every $\chi \in \mathbf{D}$.

The Virgin Island Model is defined recursively generation by generation. The 0-th generation only consists of the 0-th island

$$\mathscr{V}^{(0)} := \{(\emptyset, 0, Y)\}. \tag{10}$$

The $(n+1)$-st generation, $n \geq 0$, is the (random) set of all islands which have been colonized from islands of the n-th generation

$$\mathscr{V}^{(n+1)} := \{(\iota_n, s, \psi) \in \mathscr{I} : \iota_n \in \mathscr{V}^{(n)}, \Pi^{\iota_n}(\{(s,\psi)\}) > 0\}. \tag{11}$$

The set of all islands is defined by

$$\mathscr{V} := \bigcup_{n\geq 0} \mathscr{V}^{(n)}. \tag{12}$$

The total mass process of the Virgin Island Model is defined by

$$V_t := \sum_{(\iota,s,\psi)\in\mathscr{V}} \psi(t-s), \quad t \geq 0. \tag{13}$$

Our main interest concerns the behaviour of the law $\mathscr{L}(V_t)$ of V_t as $t \to \infty$.

The following observation is crucial for understanding the behavior of $(V_t)_{t\geq 0}$ as $t \to \infty$. There is an inherent branching structure in the Virgin Island Model. Consider as new "time coordinate" the number of island generations. One offspring island together with all its offspring islands is again a Virgin Island Model but with the path $(Y_t)_{t\geq 0}$ on the 0-th island replaced by an excursion path. Because of this branching structure, the Virgin Island Model is a multi-type Crump-Mode-Jagers branching process (see [10] under "general branching process") if we consider islands as individuals and $[0,\infty)\times\mathbf{D}$ as type space. We recall that a single-type Crump-Mode-Jagers process is a particle process where every particle i gives birth to particles at the time points of a point process ξ_i until its death at time λ_i, and $(\lambda_i,\xi_i)_i$ are independent and identically distributed. The literature on Crump-Mode-Jagers processes assumes that the number of offspring per individual is finite in every finite time interval. In the Virgin Island Model, however, every island has infinitely many offspring islands in a finite time interval because Q_Y is an infinite measure.

The most interesting question about the Virgin Island Model is whether or not the process survives with positive probability as $t\to\infty$. Generally speaking, branching particle processes survive if and only if the expected number of offspring per particle is strictly greater than one, e.g. the Crump-Mode-Jagers process survives if and only if $\mathbf{E}\xi_i[0,\lambda_i]>1$. For the Virgin Island Model, the offspring of an island (ι,s,χ) depends on the emigration intensities $a\big(\chi(t-s)\big)dt$. It is therefore not surprising that the decisive parameter for survival is the expected "sum" over those emigration intensities

$$\int\int_0^\infty a\big(\chi_t\big)\,dt\,Q_Y(d\chi). \tag{14}$$

We denote the expression in (14) as "expected total emigration intensity" of the Virgin Island Model. The observation that (14) is the decisive parameter plus an explicit formula for (14) leads to the following main result. In Theorem 2, we will prove that the Virgin Island Model survives with strictly positive probability if and only if

$$\int_0^\infty \frac{a(y)}{g(y)}\exp\Big(\int_0^y \frac{-a(u)+h(u)}{g(u)}\,du\Big)\,dy > 1. \tag{15}$$

Note that the left-hand side of (15) is equal to $\int_0^\infty a(y)m(dy)$ where $m(dy)$ is the speed measure of the one-dimensional diffusion (3). The method of proof for the extinction result is to study an integral equation (see Lemma 5.3) which the Laplace transform of the total mass V solves. Furthermore, we will show in Lemma 9.8 that the expression in (14) is equal to the left-hand side of (15).

Condition (15) already appeared in [8] as necessary and sufficient condition for existence of a nontrivial invariant measure for the mean field model, see Theorem 1 and Lemma 5.1 in [8]. Thus, the total mass process of the Virgin Island Model dies out if and only if the mean field model (1) dies out. The following duality indicates why the same condition appears in two situations which seem to be fairly different at first view. If $a(x)=\kappa x$, $h(x)=\gamma x(K-x)$ and $g(x)=\beta x$ with $\kappa,\gamma,\beta>0$, that is, in the case of Feller branching diffusions with logistic growth, then model (2) is dual to itself, see Theorem 3 in [8]. If $(X_t^N)_{t\geq 0}$ indeed approximates the Virgin Island Model as $N\to\infty$, then – for this choice of parameters – the total mass process $(V_t)_{t\geq 0}$ is dual to the mean field model. This duality would directly imply that – in the case of Feller branching diffusions with logistic growth – global extinction of the Virgin Island Model is equivalent to local extinction of the mean field model.

An interesting quantity of the Virgin Island process is the area under the path of V. In Theorem 3, we prove that the expectation of this quantity is finite exactly in the subcritical situation in which case we give an expression in terms of a, h and g. In addition, in the critical case and in the supercritical case, we obtain the asymptotic behaviour of the expected area under the path of V up to time t

$$\int_0^t \mathbf{E}^x V_s\, ds \tag{16}$$

as $t \to \infty$ for all $x \geq 0$. More precisely, the order of (16) is $O(t)$ in the critical case. For the supercritical case, let $\alpha > 0$ be the Malthusian parameter defined by

$$\int_0^\infty \left(e^{-\alpha u} \int a(\chi_u) Q_Y(d\chi) \right) du = 1. \tag{17}$$

It turns out that the expression in (16) grows exponentially with rate α as $t \to \infty$.

The result of Theorem 3 in the supercritical case suggests that the event that $(V_t)_{t\geq 0}$ grows exponentially with rate α as $t \to \infty$ has positive probability. However, this is not always true. Theorem 7 proves that $e^{-\alpha t} V_t$ converges in distribution to a random variable $W \geq 0$. Furthermore, this variable is not identically zero if and only if

$$\int \left(\int_0^\infty a(\chi_s) e^{-\alpha s}\, ds \right) \log^+ \left(\int_0^\infty a(\chi_s) e^{-\alpha s}\, ds \right) Q_Y(d\chi) < \infty \tag{18}$$

where $\log^+(x) := \max\{0, \log(x)\}$. This $(x \log x)$-criterion is similar to the Kesten-Stigum Theorem (see [14]) for multidimensional Galton-Watson processes. Our proof follows Doney [4] who establishes an $(x \log x)$-criterion for Crump-Mode-Jagers processes.

Our construction introduces as new "time coordinate" the number of island generations. Readers being interested in a construction of the Virgin Island Model in the original time coordinate – for example in a relation between V_t and $(V_s)_{s<t}$ – are referred to Dawson and Li (2003) [3]. In that paper, a superprocess with dependent spatial motion and interactive immigration is constructed as the pathwise unique solution of a stochastic integral equation driven by a Poisson point process whose intensity measure has as one component the excursion measure of the Feller branching diffusion. In a special case (see equation (1.6) in [3] with $x(s, a, t) = a$, $q(Y_s, a) = \kappa Y_s(\mathbb{R})$ and $m(da) = \mathbb{1}_{[0,1]}(a)\, da$), this is just the Virgin Island Model with (3) replaced by a Feller branching diffusion, i.e. $a(y) = \kappa y$, $h(y) = 0$, $g(y) = \beta y$. It would be interesting to know whether existence and uniqueness of such stochastic integral equations still hold if the excursion measure of the Feller branching diffusion is replaced by Q_Y.

Models with competition have been studied by various authors. Mueller and Tribe (1994) [15] and Horridge and Tribe (2004) [7] investigate an one-dimensional SPDE analog of interacting Feller branching diffusions with logistic growth which can also be viewed as KPP equation with branching noise. Bolker and Pacala (1997) [2] propose a branching random walk in which the individual mortality rate is increased by a weighted sum of the entire population. Etheridge (2004) [6] studies two diffusion limits hereof. The "stepping stone version of the Bolker-Pacala model" is a system of interacting Feller branching diffusions with non-local logistic growth. The "superprocess version of the Bolker-Pacala model" is an analog of this in continuous space. Hutzenthaler and Wakolbinger [8], motivated by [6], investigated interacting diffusions with local competition which is an analog of the Virgin Island Model but with mass migrating on $\mathbb{Z}^d$ instead of migration to unpopulated islands.

2 Main results

The following assumption guarantees existence and uniqueness of a strong $[0,\infty)$-valued solution of equation (3), see e.g. Theorem IV.3.1 in [9]. Assumption A2.1 additionally requires that $a(\cdot)$ is essentially linear.

Assumption A2.1. *The three functions $a\colon [0,\infty) \to [0,\infty)$, $h\colon [0,\infty) \to \mathbb{R}$ and $g\colon [0,\infty) \to [0,\infty)$ are locally Lipschitz continuous in $[0,\infty)$ and satisfy $a(0) = h(0) = g(0) = 0$. The function g is strictly positive on $(0,\infty)$. Furthermore, h and $\sqrt{g}$ satisfy the linear growth condition*

$$\limsup_{x\to\infty} \frac{0 \vee h(x) + \sqrt{g(x)}}{x} < \infty \tag{19}$$

where $x \vee y$ denotes the maximum of x and y. In addition, $c_1 \cdot x \le a(x) \le c_2 \cdot x$ holds for all $x \ge 0$ and for some constants $c_1, c_2 \in (0,\infty)$.

The key ingredient in the construction of the Virgin Island Model is the law of excursions of $(Y_t)_{t\ge 0}$ from the boundary zero. Note that under Assumption A2.1, zero is an absorbing boundary for (3), i.e. $Y_t = 0$ implies $Y_{t+s} = 0$ for all $s \ge 0$. As zero is not a regular point, it is not possible to apply the well-established Itô excursion theory. Instead we follow Pitman and Yor [16] and obtain a σ-finite measure $\bar{Q}_Y$ – to be called *excursion measure* – on U (defined in (4)). For this, we additionally assume that $(Y_t)_{t\ge 0}$ hits zero in finite time with positive probability. The following assumption formulates a necessary and sufficient condition for this (see Lemma 15.6.2 in [13]). To formulate the assumption, we define

$$\bar{s}(z) := \exp\Big(-\int_1^z \frac{-a(x) + h(x)}{g(x)}\,dx\Big), \quad \bar{S}(y) := \int_0^y \bar{s}(z)\,dz, \quad z, y > 0. \tag{20}$$

Note that $\bar{S}$ is a scale function, that is,

$$\mathbf{P}^y\big(T_b(Y) < T_c(Y)\big) = \frac{\bar{S}(y) - \bar{S}(c)}{\bar{S}(b) - \bar{S}(c)} \tag{21}$$

holds for all $0 \le c < y < b < \infty$, see Section 15.6 in [13].

Assumption A2.2. *The functions a, g and h satisfy*

$$\int_0^x \bar{S}(y) \frac{1}{g(y)\bar{s}(y)}\,dy < \infty \tag{22}$$

for some $x > 0$.

Note that if Assumption A2.2 is satisfied, then (22) holds for all $x > 0$.

Pitman and Yor [16] construct the excursion measure $\bar{Q}_Y$ in three different ways one being as follows. The set of excursions reaching level $\delta > 0$ has $\bar{Q}_Y$-measure $1/\bar{S}(\delta)$. Conditioned on this event an excursion follows the diffusion $(Y_t)_{t\ge 0}$ conditioned to converge to infinity until this process reaches level δ. From this time on the excursion follows an independent unconditioned process.

We carry out this construction in detail in Section 9. In addition Pitman and Yor [16] describe the excursion measure "in a preliminary way as"

$$\lim_{y\to 0}\frac{1}{\bar{S}(y)}\mathscr{L}^y(Y) \tag{23}$$

where the limit indicates weak convergence of finite measures on $\mathbf{C}\big([0,\infty),[0,\infty)\big)$ away from neighbourhoods of the zero-trajectory. However, they do not give a proof. Having $\bar{Q}_Y$ identified as the limit in (23) will enable us to transfer explicit formulas for $\mathscr{L}(Y)$ to explicit formulas for $\bar{Q}_Y$. We establish the existence of the limit in (23) in Theorem 1 below. For this, let the topology on $\mathbf{C}\big([0,\infty),[0,\infty)\big)$ be given by locally uniform convergence. Furthermore, recall Y from (3), the definition of U from (4) and the definition of $\bar{S}$ from (20).

Theorem 1. *Assume A2.1 and A2.2. Then there exists a σ-finite measure $\bar{Q}_Y$ on U such that*

$$\lim_{y\to 0}\frac{1}{\bar{S}(y)}\mathbf{E}^y F(Y) = \int F(\chi)\bar{Q}_Y(d\chi) \tag{24}$$

for all bounded continuous $F\colon \mathbf{C}\big([0,\infty),[0,\infty)\big)\to\mathbb{R}$ for which there exists an $\varepsilon>0$ such that $F(\chi)=0$ whenever $\sup_{t\geq 0}\chi_t<\varepsilon$.

For our proof of the global extinction result for the Virgin Island Model, we need the scaling function $\bar{S}$ in (24) to behave essentially linearly in a neighbourhood of zero. More precisely, we assume $\bar{S}'(0)$ to exist in $(0,\infty)$. From definition (20) of $\bar{S}$ it is clear that a sufficient condition for this is given by the following assumption.

Assumption A2.3. *The integral $\int_\varepsilon^1 \frac{-a(y)+h(y)}{g(y)}\,dy$ has a limit in $(-\infty,\infty)$ as $\varepsilon\to 0$.*

It follows from dominated convergence and from the local Lipschitz continuity of a and h that Assumption A2.3 holds if $\int_0^1 \frac{y}{g(y)}\,dy$ is finite.

In addition, we assume that the expected total emigration intensity of the Virgin Island Model is finite. Lemma 9.6 shows that, under Assumptions A2.1 and A2.2, an equivalent condition for this is given in Assumption A2.4.

Assumption A2.4. *The functions a, g and h satisfy*

$$\int_x^\infty \frac{a(y)}{g(y)\bar{s}(y)}\,dy<\infty \tag{25}$$

for some and then for all $x>0$.

We mention that if Assumptions A2.1, A2.2 and A2.4 hold, then the process Y hits zero in finite time almost surely (see Lemma 9.5 and Lemma 9.6). Furthermore, we give a generic example for a, h and g namely $a(y)=c_1y$, $h(y)=c_2y^{\kappa_1}-c_3y^{\kappa_2}$, $g(y)=c_4y^{\kappa_3}$ with $c_1,c_2,c_3,c_4>0$. The Assumptions A2.1, A2.2, A2.3 and A2.4 are all satisfied if $\kappa_2>\kappa_1\geq 1$ and if $\kappa_3\in[1,2)$. Assumption A2.2 is not met by $a(y)=\kappa y$, $\kappa>0$, $h(y)=y$ and $g(y)=y^2$ because then $\bar{s}(y)=y^{\kappa-1}$, $\bar{S}(y)=y^\kappa/\kappa$ and condition (22) fails to hold.

Next we formulate the main result of this paper. Theorem 2 proves a nontrivial transition from extinction to survival. For the formulation of this result, we define

$$s(z) := \exp\Big(-\int_0^z \frac{-a(x)+h(x)}{g(x)}\,dx\Big), \quad S(y) := \int_0^y s(z)\,dz, \quad z,y>0, \tag{26}$$

which is well-defined under Assumption A2.3. Note that $\bar{S}(y) = S(y)\bar{S}^{'}(0)$. Define the excursion measure

$$Q_Y := \bar{S}^{'}(0)\bar{Q}_Y \tag{27}$$

and recall the total mass process $(V_t)_{t\geq 0}$ from (13).

Theorem 2. *Assume A2.1, A2.2, A2.3 and A2.4. Then the total mass process $(V_t)_{t\geq 0}$ started in $x>0$ dies out (i.e., converges in probability to zero as $t\to\infty$) if and only if*

$$\int_0^\infty \frac{a(y)}{g(y)s(y)}\,dy \leq 1. \tag{28}$$

If (28) *fails to hold, then V_t converges in distribution as $t\to\infty$ to a random variable V_∞ satisfying*

$$\mathbf{P}^x(V_\infty=0) = 1-\mathbf{P}^x(V_\infty=\infty) = \mathbf{E}^x\exp\Big(-q\int_0^\infty a(Y_s)\,ds\Big) \tag{29}$$

for all $x\geq 0$ and some $q>0$.

Remark 2.1. *The constant $q>0$ is the unique strictly positive fixed-point of a function defined in Lemma 7.1.*

In the critical case, that is, equality in (28), V_t converges to zero in distribution as $t\to\infty$. However, it turns out that the expected area under the graph of V is infinite. In addition, we obtain in Theorem 3 the asymptotic behaviour of the expected area under the graph of V up to time t as $t\to\infty$. For this, define

$$w_a(x) := \int_0^\infty S(x\wedge z)\frac{a(z)}{g(z)s(z)}\,dz,\ x\geq 0, \tag{30}$$

and similarly $w_{id} := w_a$ with $a(z)=z$. If Assumptions A2.1, A2.2, A2.3 and A2.4 hold, then $w_a(x)+w_{id}(x)$ is finite for fixed $x<\infty$; see Lemma 9.6. Furthermore, under Assumptions A2.1, A2.2, A2.3 and A2.4,

$$w_a^{'}(0) = \int_0^\infty \frac{a(z)}{g(z)s(z)}\,dz < \infty \tag{31}$$

by the dominated convergence theorem.

Theorem 3. *Assume A2.1, A2.2, A2.3 and A2.4. If the left-hand side of* (28) *is strictly smaller than one, then the expected area under the path of V is equal to*

$$\mathbf{E}^x\int_0^\infty V_s\,ds = w_{id}(x) + \frac{w_{id}^{'}(0)\,w_a(x)}{1-w_a^{'}(0)} \in (0,\infty) \tag{32}$$

for all $x \geq 0$. Otherwise, the left-hand side of (32) *is infinite. In the critical case, that is, equality in* (28),

$$\frac{1}{t}\int_0^t \mathbf{E}^x V_s\,ds \to \frac{w_{id}'(0)\,w_a(x)}{\int_0^\infty \frac{w_a(y)}{g(y)s(y)}dy} \in [0,\infty) \qquad \text{as } t \to \infty \tag{33}$$

where the right-hand side is interpreted as zero if the denominator is equal to infinity. In the supercritical case, i.e., if (28) *fails to be true, let $\alpha > 0$ be such that*

$$\int_0^\infty e^{-\alpha s}\int a(\chi_s)Q_Y(d\chi)\,ds = 1. \tag{34}$$

Then the order of growth of the expected area under the path of $(V_s)_{s\geq 0}$ up to time t as $t \to \infty$ can be read off from

$$e^{-\alpha t}\int_0^t \mathbf{E}^x V_s\,ds \to \frac{\int_0^\infty e^{-\alpha s}\int \chi_s Q_Y(d\chi)\,ds \cdot \int_0^\infty e^{-\alpha s}\mathbf{E}^x a(Y_s)\,ds}{\int_0^\infty \Big(\alpha s e^{-\alpha s}\int a(\chi_s)Q_Y(d\chi)\Big)ds} \in (0,\infty) \tag{35}$$

for all $x \geq 0$.

The following result is an analog of the Kesten-Stigum Theorem, see [14]. In the supercritical case, $e^{-\alpha t}V_t$ converges to a random variable W as $t \to \infty$. In addition, W is not identically zero if and only if the $(x\log x)$-condition (18) holds. We will prove a more general version hereof in Theorem 7 below. Unfortunately, we do not know of an explicit formula in terms of a, h and g for the left-hand side of (18). Aiming at a condition which is easy to verify, we assume instead of (18) that the second moment $\int(\int_0^\infty a(\chi_s)\,ds)^2 Q(d\chi)$ is finite. In Assumption A2.5, we formulate a condition which is slightly stronger than that, see Lemma 9.8 below.

Assumption A2.5. *The functions a, g and h satisfy*

$$\int_x^\infty a(y)\frac{y + w_a(y)}{g(y)\bar{s}(y)}\,dy < \infty \tag{36}$$

for some and then for all $x > 0$.

Theorem 4. *Assume A2.1, A2.2, A2.3 and A2.5. Suppose that* (28) *fails to be true (supercritical case) and let $\alpha > 0$ be the unique solution of* (34)*. Then*

$$\frac{V_t}{e^{\alpha t}} \xrightarrow{w} W \qquad \text{as } t \to \infty \tag{37}$$

in the weak topology and $\mathbf{P}\{W > 0\} = \mathbf{P}\{V_\infty > 0\}$.

3 Outline

Theorem 1 will be established in Section 9. Note that Section 9 does not depend on the sections 4-8. We will prove the survival and extinction result of Theorem 2 in two steps. In the first step, we obtain a criterion for survival and extinction in terms of Q_Y. More precisely, we prove that the process dies out if and only if the expression in (14) is smaller than or equal to one. In this step, we do not exploit that Q_Y is the excursion measure of Y. In fact, we will prove an analog of Theorem 2 in a more general setting where Q_Y is replaced by some σ-finite measure Q and where the islands are counted with random characteristics. See Section 4 below for the definitions. The analog of Theorem 2 is stated in Theorem 5, see Section 4, and will be proven in Section 7. The key equation for its proof is contained in Lemma 5.1 which formulates the branching structure in the Virgin Island Model. In the second step, we calculate an expression for (14) in terms of a, h and g. This will be done in Lemma 9.8. Theorem 2 is then a corollary of Theorem 5 and of Lemma 9.8, see Section 10. Similarly, a more general version of Theorem 3 is stated in Theorem 6, see Section 4 below. The proofs of Theorem 3 and of Theorem 6 are contained in Section 10 and Section 6, respectively. As mentioned in Section 1, a rescaled version of $(V_t)_{t\geq 0}$ converges in the supercritical case. This convergence is stated in a more general formulation in Theorem 7, see Section 4 below. The proofs of Theorem 4 and of Theorem 7 are contained in Section 10 and in Section 8, respectively.

4 Virgin Island Model counted with random characteristics

In the proof of the extinction result of Theorem 2, we exploit that one offspring island together with all its offspring islands is again a Virgin Island Model but with a typical excursion instead of Y on the 0-th island. For the formulation of this branching property, we need a version of the Virgin Island Model where the population on the 0-th island is governed by Q_Y. More generally, we replace the law $\mathscr{L}(Y)$ of the first island by some measure ν and we replace the excursion measure Q_Y by some measure Q. Given two σ-finite measures ν and Q on the Borel-σ-algebra of $\mathbf{D}$, we define the *Virgin Island Model with initial island measure ν and excursion measure Q* as follows. Define the random sets of islands $\mathscr{V}^{(n),\nu,Q}$, $n \geq 0$, and $\mathscr{V}^{\nu,Q}$ through the definitions (9), (10), (11) and (12) with $\mathscr{L}(Y)$ and Q_Y replaced by ν and Q, respectively. A simple example for ν and Q is $\nu(d\chi) = Q(d\chi) = \mathbf{E}\delta_{t\mapsto \mathbb{1}_{t<L}}(d\chi)$ where $L \geq 0$ is a random variable and δ_ψ is the Dirac measure on the path ψ. Then the Virgin Island Model coincides with a Crump-Mode-Jagers process in which a particle has offspring according to a rate $a(1)$ Poisson process until its death at time L.

Furthermore, our results do not only hold for the total mass process (13) but more generally when the islands are counted with random characteristics. This concept is well-known for Crump-Mode-Jagers processes, see Section 6.9 in [10]. Assume that $\phi_\iota = \big(\phi_\iota(t)\big)_{t\in\mathbb{R}}$, $\iota \in \mathscr{I}$, are separable and nonnegative processes with the following properties. It vanishes on the negative half-axis, i.e. $\phi_\iota(t) = 0$ for $t < 0$. Informally speaking our main assumption on ϕ_ι is that it does not depend on the history. Formally we assume that

$$\Big(\phi_{(\iota,s,\chi)}(t)\Big)_{t\in\mathbb{R}} \stackrel{d}{=} \Big(\phi_{(\emptyset,0,\chi)}(t-s)\Big)_{t\in\mathbb{R}} \quad \forall\, \chi \in \mathbf{D}, \iota \in \mathscr{I}, s \geq 0. \tag{38}$$

Furthermore, we assume that the family $\{\phi_\iota, \Pi^\iota : \iota \in \mathscr{I}^\chi\}$ is independent for each $\chi \in \mathbf{D}$ and $(\omega, t, \chi) \mapsto \phi_{(\emptyset,0,\chi)}(t)(\omega)$ is measurable. As a short notation, define $\phi_\chi(t) := \phi(t,\chi) := \phi_{(\emptyset,0,\chi)}(t)$ for $\chi \in \mathbf{D}$. With this, we define

$$V_t^{\phi,\nu,Q} := \sum_{\iota\in\mathscr{V}^{\nu,Q}} \phi_\iota(t-\sigma_\iota), \quad t \geq 0, \tag{39}$$

and say that $\left(V_t^{\phi,\nu,Q}\right)_{t\geq 0}$ is a *Virgin Island process counted with random characteristics* ϕ. Instead of $V_t^{\phi,\delta_\chi,Q}$, we write $V_t^{\phi,\chi,Q}$ for a path $\chi \in \mathbf{D}$ and note that $(\omega,t,\chi) \mapsto V_t^{\phi,\chi,Q}(\omega)$ is measurable. A prominent example for ϕ_χ is the deterministic random variable $\phi_\chi(t) \equiv \chi(t)$. In this case, $V_t^{\nu,Q} := V_t^{\phi,\nu,Q}$ is the total mass of all islands at time t. Notice that $(V_t)_{t\geq 0}$ defined in (13) is a special case hereof, namely $V_t = V_t^{\mathscr{L}(Y),Q_Y}$. Another example for ϕ_χ is $\phi(t,\chi) = \chi(t)\mathbb{1}_{t\leq t_0}$. Then $\mathscr{V}_t^{\phi,\chi,Q}$ is the total mass at time t of all islands which have been colonized in the last t_0 time units. If $\phi(t,\chi) = \int_t^\infty \chi_s\,ds$, then $V_t^{\phi,\chi,Q} = \int_t^\infty V_s^{\chi,Q} ds$.

As in Section 2, we need an assumption which guarantees finiteness of $V_t^{\phi,\nu,Q}$.

Assumption A4.1. *The function* $a\colon [0,\infty) \to [0,\infty)$ *is continuous and there exist* $c_1, c_2 \in (0,\infty)$ *such that* $c_1 x \leq a(x) \leq c_2 x$ *for all* $x \geq 0$. *Furthermore,*

$$\sup_{t\leq T} \int \Big(a(\chi_t) + \mathbf{E}\phi(t,\chi)\Big)\nu(d\chi) + \sup_{t\leq T} \int \Big(a(\chi_t) + \mathbf{E}\phi(t,\chi)\Big)Q(d\chi) < \infty \tag{40}$$

for every $T < \infty$

The analog of Assumption A2.4 in the general setting is the following assumption.

Assumption A4.2. *Both the expected emigration intensity of the* 0*-th island and of subsequent islands are finite:*

$$\int \Big(\int_0^\infty a(\chi_u)\,du\Big)\nu(d\chi) + \int \Big(\int_0^\infty a(\chi_u)\,du\Big)Q(d\chi) < \infty. \tag{41}$$

In Section 2, we assumed that $(Y_t)_{t\geq 0}$ hits zero in finite time with positive probability. See Assumption A2.2 for an equivalent condition. Together with A2.4, this assumption implied almost sure convergence of $(Y_t)_{t\geq 0}$ to zero as $t \to \infty$. In the general setting, we need a similar but somewhat weaker assumption. More precisely, we assume that $\phi(t)$ converges to zero "in distribution" both with respect to ν and with respect to Q.

Assumption A4.3. *The random processes* $\left\{\left(\phi_\chi(t)\right)_{t\geq 0} \colon \chi \in \mathbf{D}\right\}$ *and the measures* Q *and* ν *satisfy*

$$\int \Big(1 - \mathbf{E}e^{-\lambda\phi(t,\chi)}\Big)(\nu + Q)(d\chi) \to 0 \qquad \text{as } t \to \infty \tag{42}$$

for all $\lambda \geq 0$.

Having introduced the necessary assumptions, we now formulate the extinction and survival result of Theorem 2 in the general setting.

Theorem 5. *Let* ν *be a probability measure on* $\mathbf{D}$ *and let* Q *be a measure on* $\mathbf{D}$. *Assume A4.1, A4.2 and A4.3. Then the Virgin Island process* $(V_t^{\phi,\nu,Q})_{t\geq 0}$ *counted with random characteristics* ϕ *with* 0*-th island distribution* ν *and with excursion measure* Q *dies out (i.e., converges to zero in probability) if and only if*

$$\bar{a} := \int \Big(\int_0^\infty a(\chi_u)\,du\Big)Q(d\chi) \leq 1. \tag{43}$$

In case of survival, the process converges weakly as $t \to \infty$ to a probability measure $\mathscr{L}\left(V_\infty^{\phi,\nu,Q}\right)$ with support in $\{0,\infty\}$ which puts mass

$$\int 1-\exp\Big(-q\int_0^\infty a(\chi_s)\,ds\Big)\nu(d\chi) \tag{44}$$

on the point ∞ where $q>0$ is the unique strictly positive fixed-point of

$$z\mapsto \int 1-\exp\Big(-z\int_0^\infty a(\chi_s)\,ds\Big)Q(d\chi),\quad z\geq 0. \tag{45}$$

Remark 4.1. *The assumption on ν to be a probability measure is convenient for the formulation in terms of convergence in probability. For a formulation in the case of a σ-finite measure ν, see the proof of the theorem in Section 7.*

Next we state Theorem 3 in the general setting. For its formulation, define

$$f^\nu(t):=\int \mathbf{E}\phi(t,\chi)\nu(d\chi),\quad t\geq 0, \tag{46}$$

and similarly f^Q with ν replaced by Q.

Theorem 6. *Assume A4.1 and A4.2. If the left-hand side of* (43) *is strictly smaller than one and if both f^ν and f^Q are integrable, then*

$$\int \mathbf{E}\Big[\int_0^\infty V_s^{\phi,\chi,Q}\,ds\Big]\nu(d\chi)=\int_0^\infty f^\nu(s)ds+\frac{\int_0^\infty f^Q(s)\,ds\int\int_0^\infty a(\chi_s)\,ds\nu(d\chi)}{1-\int\Big(\int_0^\infty a(\chi_s)\,ds\Big)Q(d\chi)} \tag{47}$$

which is finite and strictly positive. Otherwise, the left-hand side of (47) *is infinite. If the left-hand side of* (43) *is equal to one and if both f^ν and f^Q are integrable,*

$$\lim_{t\to\infty}\frac{1}{t}\int \mathbf{E}\Big[\int_0^t V_s^{\phi,\chi,Q}\,ds\Big]\nu(d\chi)=\frac{\int_0^\infty f^Q(s)\,ds\cdot\int\int_0^\infty a(\chi_s)\,ds\,\nu(d\chi)}{\int_0^\infty s\int a(\chi_s)Q(d\chi)\,ds}<\infty \tag{48}$$

where the right-hand side is interpreted as zero if the denominator is equal to infinity. In the supercritical case, i.e., if (43) *fails to be true, let $\alpha>0$ be such that*

$$\int_0^\infty\Big(e^{-\alpha s}\int a(\chi_s)\,Q(d\chi)\Big)\,ds=1. \tag{49}$$

Additionally assume that f^Q is continuous a.e. with respect to the Lebesgue measure,

$$\sum_{k=0}^\infty \sup_{k\leq t<k+1}|e^{-\alpha t}f^Q(t)|<\infty \tag{50}$$

and that $e^{-\alpha t}f^\nu(t)\to 0$ as $t\to\infty$. Then the order of convergence of the expected total intensity up to time t can be read off from

$$\lim_{t\to\infty}e^{-\alpha t}\int \mathbf{E}\Big[\int_0^t V_s^{\phi,\chi,Q}\,ds\Big]\nu(d\chi)=\frac{1}{\alpha}\lim_{t\to\infty}e^{-\alpha t}\int \mathbf{E}\big[V_t^{\phi,\chi,Q}\big]\nu(d\chi) \tag{51}$$

and from

$$\lim_{t\to\infty} e^{-\alpha t}\int \mathbf{E}\big[V_t^{\phi,\chi,Q}\big]\nu(d\chi) = \frac{\int_0^\infty e^{-\alpha s} f^Q(s)\,ds\cdot\int_0^\infty e^{-\alpha s}\int a(\chi_s)\nu(d\chi)\,ds}{\int_0^\infty s e^{-\alpha s}\int a(\chi_s)Q(d\chi)\,ds}. \tag{52}$$

For the formulation of the analog of the Kesten-Stigum Theorem, denote by

$$\bar{m} := \frac{\int_0^\infty e^{-\alpha s} f^Q(s)\,ds}{\int_0^\infty s e^{-\alpha s}\int a(\chi_s)Q(d\chi)\,ds} \in (0,\infty) \tag{53}$$

the right-hand side of (52) with ν replaced by Q. Furthermore, define

$$A_\alpha(\chi) := \int_0^\infty a(\chi_s)e^{-\alpha s}\,ds \tag{54}$$

for every path $\chi\in\mathbf{D}$. For our proof of Theorem 7, we additionally assume the following properties of Q.

Assumption A4.4. *The measure Q satisfies*

$$\int\Big(\int_0^T a(\chi_s)\,ds\Big)^2 Q(d\chi) < \infty \tag{55}$$

for every $T<\infty$ and

$$\sup_{t\geq 0}\int\Big[\mathbf{E}\phi_\chi(t)\int_0^t a(\chi_s)\,ds\Big]Q(d\chi)<\infty,\quad \sup_{t\geq 0}\int\mathbf{E}\big(\phi_\chi^2(t)\big)Q(d\chi)<\infty. \tag{56}$$

Theorem 7. *Let ν be a probability measure on $\mathbf{D}$ and let Q be a measure on $\mathbf{D}$. Assume A4.1, A4.2, A4.3 and A4.4. Suppose that $\bar{a}>1$ (supercritical case) and let $\alpha>0$ be the unique solution of* (49)*. Then*

$$\frac{V_t^{\phi,\nu,Q}}{e^{\alpha t}\bar{m}} \xrightarrow{w} W \qquad \text{as } t\to\infty \tag{57}$$

in the weak topology where W is a nonnegative random variable. The variable W is not identically zero if and only if

$$\int A_\alpha(\chi)\log^+\big(A_\alpha(\chi)\big)Q(d\chi)<\infty \tag{58}$$

where $\log^+(x):=\max\{0,\log(x)\}$. If (58) *holds, then*

$$\mathbf{E}W = \int\Big[\int_0^\infty e^{-\alpha s}a(\chi_s)\,ds\Big]\nu(d\chi),\mathbf{P}(W=0) = \int\Big[e^{-q\int_0^\infty a(\chi_s)\,ds}\Big]\nu(d\chi) \tag{59}$$

where $q>0$ is the unique strictly positive fixed-point of (45)*.*

Remark 4.2. *Comparing* (59) *with* (44)*, we see that $\mathbf{P}(W>0)=\mathbf{P}(V_\infty^{\phi,\nu,Q}>0)$. Consequently, the Virgin Island process $\big(V_t^{\phi,\nu,Q}\big)_{t\geq 0}$ conditioned on not converging to zero grows exponentially fast with rate α as $t\to\infty$.*

5 Branching structure

We mentioned in the introduction that there is an inherent branching structure in the Virgin Island Model. One offspring island together with all its offspring islands is again a Virgin Island Model but with a typical excursion instead of Y on the 0-th island. In Lemma 5.1, we formalize this idea. As a corollary thereof, we obtain an integral equation for the modified Laplace transform of the Virgin Island Model in Lemma 5.3 which is the key equation for our proof of the extinction result of Theorem 2. Recall the notation of Section 1 and of Section 4.

Lemma 5.1. *Let $\chi \in \mathbf{D}$. There exists an independent family*

$$\left\{ \left({}^{(s,\psi)}\mathrm{V}_t^{\phi,\chi,Q} \right)_{t\geq 0} : (s,\psi) \in [0,\infty)\times \mathbf{D} \right\} \tag{60}$$

of random variables which is independent of ϕ_χ and of Π^χ such that

$$V_t^{\phi,\chi,Q} = \phi_\chi(t) + \sum_{(s,\psi)\in\Pi^\chi} {}^{(s,\psi)}\mathrm{V}_t^{\phi,\chi,Q} \qquad \forall\ t\geq 0 \tag{61}$$

and such that

$$\left({}^{(s,\psi)}\mathrm{V}_t^{\phi,\chi,Q} \right)_{t\geq 0} \stackrel{d}{=} \left(V_{t-s}^{\phi,\psi,Q} \right)_{t\geq 0} \tag{62}$$

for all $(s,\psi)\in[0,\infty)\times\mathbf{D}$.

Proof. Write $\mathscr{V}^\chi := \mathscr{V}^{\chi,Q}$ and $\mathscr{V}^{(n),\chi} := \mathscr{V}^{(n),\chi,Q}$. Define

$${}^{(s,\psi)}\mathscr{V}^{(1),\chi} := \left\{ \left((\emptyset,0,\chi), s,\psi \right) \right\} \subset \mathscr{I}_1^\chi \text{ and } {}^{(s,\psi)}\mathscr{V}^\chi := \bigcup\nolimits_{n\geq 1} {}^{(s,\psi)}\mathscr{V}^{(n),\chi} \tag{63}$$

for $(s,\psi)\in[0,\infty)\times\mathbf{D}$ where

$${}^{(s,\psi)}\mathscr{V}^{(n+1),\chi} := \left\{ (\iota_n, r,\zeta) \in \mathscr{I}_{n+1}^\chi : \iota_n \in {}^{(s,\psi)}\mathscr{V}^{(n),\chi}, \Pi^{\iota_n}(r,\zeta) > 0 \right\} \tag{64}$$

for $n\geq 1$. Comparing (63) and (64) with (11), we see that

$$\mathscr{V}^{(0),\chi} = \{(\emptyset,0,\chi)\} \text{ and } \mathscr{V}^{(n),\chi} = \bigcup_{(s,\psi)\in\Pi^\chi} {}^{(s,\psi)}\mathscr{V}^{(n),\chi} \quad \forall\ n\geq 1. \tag{65}$$

Define $V_t^{(0),\phi,\chi,Q} = \phi_\chi(t)$ for $t\geq 0$ and for $n\geq 1$

$$V_t^{(n),\phi,\chi,Q} := \sum_{(s,\psi)\in\Pi^\chi} \sum_{\iota\in {}^{(s,\psi)}\mathscr{V}^{(n),\chi}} \phi_\iota(t-\sigma_\iota) =: \sum_{(s,\psi)\in\Pi^\chi} {}^{(s,\psi)}\mathrm{V}_t^{(n),\phi,\chi,Q}. \tag{66}$$

Summing over $n\geq 0$ we obtain for $t\geq 0$

$$V_t^{\phi,\chi,Q} = \phi_\chi(t) + \sum_{(s,\psi)\in\Pi^\chi} \sum_{n\geq 1} {}^{(s,\psi)}\mathrm{V}_t^{(n),\phi,\chi,Q} =: \phi_\chi(t) + \sum_{(s,\psi)\in\Pi^\chi} {}^{(s,\psi)}\mathrm{V}_t^{\phi,\chi,Q}. \tag{67}$$

This is equality (61). Independence of the family (60) follows from independence of $(\Pi^{\iota})_{\iota\in\mathscr{I}^\chi}$ and from independence of $(\phi_\iota)_{\iota\in\mathscr{I}^\chi}$. It remains to prove (62). Because of assumption (38) the random characteristics ϕ_ι only depends on the last part of ι. Therefore

$$\begin{aligned} {}^{(s,\psi)}\mathrm{V}_{\cdot}^{(n),\phi,\chi,Q} &= \sum_{\iota\in{}^{(s,\psi)}\mathscr{V}^{(n),\chi}} \phi_\iota(\cdot-\sigma_\iota) \\ &\stackrel{d}{=} \sum_{\tilde\iota\in\mathscr{V}^{(n-1),\psi,Q}} \phi_{\tilde\iota}(\cdot-(\sigma_{\tilde\iota}+s)) = V_{\cdot-s}^{(n-1),\psi,Q}. \end{aligned} \tag{68}$$

Summing over $n\geq 1$ results in (62) and finishes the proof. □

In order to increase readability, we introduce the following suggestive symbolic abbreviation

$$\mathbf{I}\Big[f\left(V_t^{\phi,\nu,Q}\right)\Big] := \int \mathbf{E}f\left(V_t^{\phi,\chi,Q}\right)\nu(d\chi) \quad t\geq 0, f\in\mathbf{C}\big([0,\infty),[0,\infty)\big). \tag{69}$$

One might want to read this as "expectation" with respect to a non-probability measure. However, (69) is not intended to define an operator.

The following lemma proves that the Virgin Island Model counted with random characteristics as defined in (39) is finite.

Lemma 5.2. *Assume A4.1. Then, for every $T<\infty$,*

$$\sup_{t\leq T}\mathbf{I}\Big[V_t^{\phi,\nu,Q}\Big]<\infty. \tag{70}$$

Furthermore, if

$$\sup_{t\leq T}\int \mathbf{E}\big(\phi_\chi^2(t)\big)+\Big(\int_0^T a(\chi_s)\,ds\Big)^2\,Q(d\chi)<\infty, \tag{71}$$

then there exists a constant $c_T<\infty$ such that

$$\begin{aligned} &\sup_{t\leq T}\mathbf{I}\bigg[\Big(V_t^{\phi,\nu,Q}\Big)^2\bigg] \\ &\leq c_T\Big(1+\sup_{t\leq T}\int \mathbf{E}\big(\phi_\chi^2(t)\big)\,(\nu+Q)(d\chi)+\int\Big(\int_0^T a(\chi_s)ds\Big)^2\,\nu(d\chi)\Big) \end{aligned} \tag{72}$$

for all ν and the right-hand side of (72) *is finite in the special case $\nu=Q$.*

Proof. We exploit the branching property formalized in Lemma 5.1 and apply Gronwall's inequality. Recall $\mathrm{V}^{(n),\chi,Q}$ from the proof of Lemma 5.1. The two equalities (66) and (68) imply

$$\mathbf{I}\Big[V_t^{(0),\phi,\nu,Q}\Big] = \int \mathbf{E}\phi_\chi(t)\,\nu(d\chi)\leq \sup_{s\leq T}\int \mathbf{E}\phi_\chi(s)\,\nu(d\chi) \tag{73}$$

for $t \le T$ and for $n \ge 1$

$$\begin{aligned}
\mathbf{I}\Big[V_t^{(n),\phi,\nu,Q}\Big] &= \int \mathbf{E}\Big[\sum_{(s,\psi)\in\Pi^\chi} \mathbf{E}\big[V_{t-s}^{(n-1),\phi,\psi,Q}\big]\Big]\nu(d\chi) \\
&= \int \Big(\int_0^t \int \mathbf{E}\big[V_{t-s}^{(n-1),\phi,\psi,Q}\big]Q(d\psi)a(\chi_s)ds\Big)\,\nu(d\chi) \\
&\le \sup_{u\le T}\int a(\chi_u)\nu(d\chi)\int_0^t \mathbf{I}\Big[V_s^{(n-1),\phi,Q,Q}\Big]\,ds.
\end{aligned} \tag{74}$$

Using Assumption A4.1 induction on $n \ge 0$ shows that all expressions in (73) and in (74) are finite in the case $\nu = Q$. Summing (74) over $n \le n_0$ we obtain

$$\sum_{n=0}^{n_0}\mathbf{I}\Big[V_t^{(n),\phi,\nu,Q}\Big] \le \int \mathbf{E}\phi_\chi(u)\nu(d\chi) + \int_0^t \sum_{n=0}^{n_0}\mathbf{I}\Big[V_s^{(n),\phi,Q,Q}\Big]\int a(\chi_{t-s})\nu(d\chi)\,ds \tag{75}$$

for $t \le T$. In the special case $\nu = Q$ Gronwall's inequality implies

$$\sum_{n=0}^{n_0}\mathbf{I}\Big[V_t^{(n),\phi,Q,Q}\Big] \le \sup_{u\le T}\int \mathbf{E}\phi_\chi(u)Q(d\chi)\cdot\exp\Big(t\sup_{u\le T}\int a(\chi_u)Q(d\chi)\Big). \tag{76}$$

Summing (74) over $n \le n_0$, inserting (76) into (74) and letting $n_0 \to \infty$ we see that (70) follows from Assumption A4.1.

For the proof of (72), note that (75) with $\nu = \delta_\chi$ and (70) imply

$$\int \Big(\mathbf{E}V_t^{\phi,\chi,Q}\Big)^2 Q(d\chi) \le \int 2\Big(\mathbf{E}\phi_\chi(t)\Big)^2 + \tilde{c}_T\Big(\int_0^T a(\chi_s)ds\Big)^2 Q(d\chi) < \infty \tag{77}$$

for some $\tilde{c}_T < \infty$. In addition the two equalities (66) and (68) together with independence imply

$$\int \mathrm{Var}\,\big(V_t^{(0),\phi,\chi,Q}\big)\nu(d\chi) = \int \mathrm{Var}\,\big(\phi_\chi(t)\big)\nu(d\chi) \tag{78}$$

for $t \ge 0$ and for $n \ge 1$

$$\begin{aligned}
&\int \mathrm{Var}\,\big(V_t^{(n),\phi,\chi,Q}\big)\nu(d\chi) \\
&= \int \mathbf{E}\Big(\sum_{(s,\psi)\in\Pi^\chi}\mathrm{Var}\,\big(V_{t-s}^{(n-1),\phi,\psi,Q}\big)\Big)\,\nu(d\chi) \\
&= \int\int_0^t \Big(a(\chi_s)\int \mathrm{Var}\,\big(V_{t-s}^{(n-1),\phi,\psi,Q}\big)Q(d\psi)\Big)ds\,\nu(d\chi) \\
&\le \int_0^t\int \mathrm{Var}\,\big(V_s^{(n-1),\phi,\psi,Q}\big)Q(d\psi)ds\cdot\sup_{u\le T}\int a(\chi_u)\,\nu(d\chi).
\end{aligned} \tag{79}$$

In the special case $\nu = Q$ induction on $n \geq 0$ together with (71) shows that all involved expressions are finite. A similar estimate as in (79) leads to

$$\begin{aligned}
&\int \mathbf{E}\Big[\Big(\sum_{n=0}^{n_0} V_t^{(n),\phi,\chi,Q}\Big)^2\Big]\nu(d\chi) - \int \Big(\mathbf{E}\sum_{n=0}^{n_0} V_t^{(n),\phi,\chi,Q}\Big)^2 \nu(d\chi) \\
&= \int \operatorname{Var}\big(\phi_\chi(t)\big) + \mathbf{E}\Big(\sum_{(s,\psi)\in\Pi^\chi} \operatorname{Var}\big(\sum_{n=1}^{n_0} V_{t-s}^{(n-1),\phi,\psi,Q}\big)\Big)\nu(d\chi) \\
&= \int \operatorname{Var}\big(\phi_\chi(t)\big) + \int_0^t \Big(a(\chi_s)\int \operatorname{Var}\big(\sum_{n=0}^{n_0-1} V_{t-s}^{(n),\phi,\psi,Q}\big)Q(d\psi)\Big)ds\,\nu(d\chi) \\
&\leq \int \mathbf{E}\Big(\phi_\chi^2(t)\Big)\nu(d\chi) + \int_0^t \int \mathbf{E}\Big[\Big(\sum_{n=0}^{n_0} V_s^{(n),\phi,\psi,Q}\Big)^2\Big]Q(d\psi)ds \cdot \sup_{u\leq T}\int a(\chi_u)\nu(d\chi).
\end{aligned}$$

In the special case $\nu = Q$ Gronwall's inequality together with (77) leads to

$$\begin{aligned}
&\int \mathbf{E}\Big[\Big(\sum_{n=0}^{n_0} V_t^{(n),\phi,\chi,Q}\Big)^2\Big]Q(d\chi) \\
&\leq \Big(\int 3\mathbf{E}\Big(\phi_\chi^2(t)\Big) + \tilde{c}_T\Big(\int_0^T a(\chi_s)\,ds\Big)^2 Q(d\chi)\Big)\exp\Big(\sup_{u\leq T}\int a(\chi_u)Q(d\chi)T\Big)
\end{aligned} \tag{80}$$

which is finite by Assumption A4.1 and assumption (71). Inserting (80) into (79) and letting $n_0 \to \infty$ finishes the proof. □

In the following lemma, we establish an integral equation for the modified Laplace transform of the Virgin Island Model. Recall the definition of $V_t^{\phi,\nu,Q}$ from (39).

Lemma 5.3. *Assume A4.1. The modified Laplace transform* $\mathbf{I}\big[1 - e^{-\lambda V_t^{\phi,\nu,Q}}\big]$ *of the Virgin Island Model counted with random characteristics* ϕ *satisfies*

$$\begin{aligned}
&\mathbf{I}\Big[1 - e^{-\lambda V_t^{\phi,\nu,Q}}\Big] \\
&= \int \mathbf{E}\Big[1 - \exp\Big(-\lambda\phi_\chi(t) - \int_0^\infty \mathbf{I}\big[1 - e^{-\lambda V_{t-s}^{\phi,Q,Q}}\big]a(\chi_s)\,ds\Big)\Big]\nu(d\chi)
\end{aligned} \tag{81}$$

for all $\lambda, t \geq 0$.

Proof. Fix $\lambda, t \geq 0$. Applying Lemma 5.1,

$$\begin{aligned}
&\mathbf{I}\Big[1 - e^{-\lambda V_t^{\phi,\nu,Q}}\Big] \\
&= \int \Big[1 - \mathbf{E}\big(e^{-\lambda\phi_\chi(t)}\big)\cdot\mathbf{E}\Big(\prod_{(s,\psi)\in\Pi^\chi} \mathbf{E}e^{-\lambda V_{t-s}^{\phi,\psi,Q}}\Big)\Big]\nu(d\chi) \\
&= \int \Big[1 - \mathbf{E}\big(e^{-\lambda\phi_\chi(t)}\big)\cdot\exp\Big(-\int_0^\infty \int 1 - \mathbf{E}e^{-\lambda V_{t-s}^{\phi,\psi,Q}}Q(d\psi)a(\chi_s)\,ds\Big)\Big]\nu(d\chi) \\
&= \int \mathbf{E}\Big[1 - \exp\Big(-\lambda\phi_\chi(t) - \int_0^\infty \mathbf{I}\big[1 - e^{-\lambda V_{t-s}^{\phi,Q,Q}}\big]a(\chi_s)\,ds\Big)\Big]\nu(d\chi).
\end{aligned}$$

This proves the assertion. □

6 Proof of Theorem 6

Recall the definition of $(V_t^{\phi,\nu,Q})_{t\geq 0}$ from (39), f^ν from (46) and the notation $\mathbf{I}$ from (69). We begin with the supercritical case and let $\alpha > 0$ be the Malthusian parameter which is the unique solution of (49). Define

$$m^\nu(t) := \mathbf{I}\Big[V_t^{\phi,\nu,Q}\Big] \qquad \mu^\nu(ds) := \int a(\chi_s)\,\nu(d\chi)\,ds \tag{82}$$

for $t \geq 0$. In this notation, equation (74) with ν replaced by Q reads as

$$e^{-\alpha t} m^Q(t) = e^{-\alpha t} f^Q(t) + \int_0^t e^{-\alpha(t-s)} m^Q(t-s) e^{-\alpha s} \mu^Q(ds). \tag{83}$$

This is a renewal equation for $e^{-\alpha t} m^Q(t)$. By definition of α, $e^{-\alpha s}\mu^Q(ds)$ is a probability measure. From Lemma 5.2 we know that m^Q is bounded on finite intervals. By assumption, f^Q is continuous Lebesgue-a.e. and satisfies (50). Hence, we may apply standard renewal theory (e.g. Theorem 5.2.6 of [10]) and obtain

$$\lim_{t\to\infty} e^{-\alpha t} m^Q(t) = \frac{\int_0^\infty e^{-\alpha s} f^Q(s)\,ds}{\int_0^\infty s e^{-\alpha s}\mu^Q(ds)} < \infty. \tag{84}$$

Multiply equation (74) by $e^{-\alpha t}$, recall $e^{-\alpha t} f^\nu(t) \to 0$ as $t \to \infty$ and apply the dominated convergence theorem together with A4.2 to obtain

$$\lim_{t\to\infty} e^{-\alpha t} m^\nu(t) = \int_0^\infty e^{-\alpha s} \lim_{t\to\infty} e^{-\alpha(t-s)} m^Q(t-s) \mu^\nu(ds). \tag{85}$$

Insert (84) to obtain equation (52). An immediate consequence of the existence of the limit on the left-hand side of (85) is equation (51)

$$e^{-\alpha t}\int_0^t m^\nu(s)\,ds = \int_0^\infty e^{-\alpha s} \cdot e^{-\alpha(t-s)} m^\nu(t-s)\,ds \xrightarrow{t\to\infty} \frac{1}{\alpha} \cdot \lim_{t\to\infty} e^{-\alpha t} m^\nu(t) \tag{86}$$

where we used the dominated convergence theorem.

Next we consider the subcritical and the critical case. Define

$$\bar{x}^\nu(t) := \int_0^t \mathbf{I}\Big[V_s^{\phi,\nu,Q}\Big]\,ds, \quad t \geq 0. \tag{87}$$

In this notation, equation (74) integrated over $[0, t]$ reads as

$$\bar{x}^\nu(t) = \int_0^t f^\nu(s)\,ds + \int_0^t \bar{x}^Q(t-u)\mu^\nu(du), \quad t \geq 0. \tag{88}$$

In the subcritical case, f^Q and f^ν are integrable. Theorem 5.2.9 in [10] applied to (88) with ν replaced by Q implies

$$\lim_{t\to\infty} \bar{x}^Q(t) = \frac{\int_0^\infty f^Q(s)ds}{1 - \mu^Q\big([0,\infty)\big)}. \tag{89}$$

Letting $t \to \infty$ in (88), dominated convergence and $\mu^{\nu}\big([0,\infty)\big) < \infty$ imply

$$\lim_{t\to\infty} \bar{x}^{\nu}(t) = \int_0^{\infty} f^{\nu}(s)ds + \int_0^{\infty} \lim_{t\to\infty} \bar{x}^{Q}(t-u)\mu^{\nu}(du). \tag{90}$$

Inserting (89) results in (47). In the critical case, similar arguments lead to

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t}\bar{x}^{\nu}(t) \\
&= \lim_{t\to\infty} \frac{1}{t}\int_0^{t} f^{\nu}(s)ds + \int_0^{\infty} \lim_{t\to\infty} \frac{t-u}{t} \lim_{t\to\infty} \frac{1}{t-u}\bar{x}^{Q}(t-u)\mu^{\nu}(du) \\
&= \frac{\int_0^{\infty} f^{Q}(s)\,ds}{\int_0^{\infty} u\mu^{Q}(du)}\mu^{\nu}\big([0,\infty)\big).
\end{aligned} \tag{91}$$

The last equality follows from (88) with ν replaced by Q and Corollary 5.2.14 of [10] with $c := \int_0^{\infty} f^{Q}(s)\,ds$, $n := 0$ and $\theta := \int_0^{\infty} u\mu^{Q}(du)$. Note that the assumption $\theta < \infty$ of this corollary is not necessary for this conclusion. $\square$

7 Extinction and survival in the Virgin Island Model. Proof of Theorem 5

Recall the definition of $(V_t^{\phi,\nu,Q})_{t\geq 0}$ from (39) and the notation $\mathbf{I}$ from (69). As we pointed out in Section 2, the expected total emigration intensity of the Virgin Island Model plays an important role. The following lemma provides us with some properties of the modified Laplace transform of the total emigration intensity. These properties are crucial for our proof of Theorem 5.

Lemma 7.1. *Assume A4.2. Then the function*

$$k(z) := \int 1 - \exp\Big(-z\int_0^{\infty} a\big(\chi_s\big)\,ds\Big)Q(d\chi), \quad z \geq 0, \tag{92}$$

is concave with at most two fixed-points. Zero is the only fixed-point if and only if

$$k'(0) = \int\int_0^{\infty} a\big(\chi_s\big)\,ds\,Q(d\chi) \leq 1. \tag{93}$$

Denote by q the maximal fixed-point. Then we have for all $z \geq 0$:

$$z \leq k(z) \implies z \leq q \tag{94}$$

$$z \geq k(z) \wedge z > 0 \implies z \geq q. \tag{95}$$

Proof. If $\int_0^{\infty} a\big(\chi_s\big)\,ds = 0$ for Q-a.a. χ, then $k \equiv 0$ and zero is the only fixed-point. For the rest of the proof, we assume w.l.o.g. that $\int\big(\int_0^{\infty} a(\chi_s)\,ds\big)Q(d\chi) > 0$.

The function k has finite values because of $1-e^{-c} \leq c$, $c \geq 0$, and Assumption A4.2. Concavity of k is inherited from the concavity of $x \mapsto 1-e^{-xc}$, $c \geq 0$. Using dominated convergence together with Assumption A4.2, we see that

$$\frac{k(z)}{z} = \int \frac{1-\exp\left(-z\int_0^\infty a(\chi_s)\,ds\right)}{z} Q(d\chi) \xrightarrow{z\to\infty} 0. \tag{96}$$

In addition, dominated convergence together with Assumption A4.2 implies

$$k^{'}(z) = \int \left[\int_0^\infty a(\chi_s)\,ds \exp\left(-z\int_0^\infty a(\chi_s)\,ds\right)\right] Q(d\chi) \quad z \geq 0. \tag{97}$$

Hence, k is strictly concave. Thus, k has a fixed-point which is not zero if and only if $k^{'}(0) > 1$. The implications (94) and (95) follow from the strict concavity of k. □

The method of proof (cf. Section 6.5 in [10]) of the extinction result for a Crump-Mode-Jagers process $(J_t)_{t\geq 0}$ is to study an equation for $(\mathbf{E}e^{-\lambda J_t})_{t\geq 0,\lambda\geq 0}$. The Laplace transform $(\mathbf{E}e^{-\lambda J_t})_{\lambda\geq 0}$ converges monotonically to $\mathbf{P}(J_t = 0)$ as $\lambda \to \infty$, $t \geq 0$. Furthermore, $\mathbf{P}(J_t = 0) = \mathbf{P}(\exists s \leq t : J_s = 0)$ converges monotonically to the extinction probability $\mathbf{P}(\exists s \geq 0 : J_s = 0)$ as $t \to \infty$. Taking monotone limits in the equation for $(\mathbf{E}e^{-\lambda J_t})_{t\geq 0,\lambda\geq 0}$ results in an equation for the extinction probability. In our situation, there is an equation for the modified Laplace transform $(L_t(\lambda))_{t>0,\lambda>0}$ as defined in (98) below. However, the monotone limit of $L_t(\lambda)$ as $\lambda \to \infty$ might be infinite. Thus, it is not clear how to transfer the above method of proof. The following proof of Theorem 2 directly establishes the convergence of the modified Laplace transform.

Proof of Theorem 5. Recall q from Lemma 7.1. In the first step, we will prove

$$L_t := L_t(\lambda) := \mathbf{I}\left[1 - e^{-\lambda V_t^{\phi,Q,Q}}\right] \to q \qquad (\text{as } t \to \infty) \tag{98}$$

for all $\lambda > 0$. Set $L_t(0) := 0$. It follows from Lemma 5.2 that $(L_t)_{t\leq T}$ is bounded for every finite T. Lemma 5.3 with ν replaced by Q provides us with the fundamental equation

$$L_t = \int \mathbf{E}\left[1 - \exp\left(-\lambda\phi_\chi(t) - \int_0^\infty a(\chi_s) L_{t-s}\,ds\right)\right] Q(d\chi) \qquad \forall\ t \geq 0. \tag{99}$$

Based on (99), the idea for the proof of (98) is as follows. The term $\lambda\phi_\chi(t)$ vanishes as $t \to \infty$. If L_t converges to some limit, then the limit has to be a fixed-point of the function

$$k(z) = \int \left[1 - \exp\left(-z\int_0^\infty a(\chi_s)\,ds\right)\right] Q(d\chi). \tag{100}$$

By Lemma 7.1, this function is (typically strictly) concave. Therefore, it has exactly one attracting fixed-point. Furthermore, this fact forces L_t to converge as $t \to \infty$.

We will need finiteness of $L_\infty := \limsup_{t\to\infty} L_t$. Looking for a contradiction, we assume $L_\infty = \infty$. Then there exists a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \to \infty$ such that $L_{t_n} \le \sup_{t\le t_n} L_t \le L_{t_n}+1$. We estimate

$$\begin{aligned}
L_{t_n} &\le \int \Big[1-\mathbf{E}\exp\Big(-\lambda\phi_\chi(t_n) - \int_0^\infty a(\chi_s)\sup_{r\le t_n} L_r\,ds\Big)\Big]Q(d\chi)\\
&\le k(L_{t_n}+1) + \int \exp\Big(-\int_0^\infty a(\chi_s)L_{t_n}\,ds\Big)\Big(1-\mathbf{E}e^{-\lambda\phi_\chi(t_n)}\Big)Q(d\chi)\\
&\le k(L_{t_n}+1) + \int \Big(1-\mathbf{E}e^{-\lambda\phi_\chi(t_n)}\Big)Q(d\chi).
\end{aligned} \tag{101}$$

The last summand converges to zero by Assumption A4.3 and is therefore bounded by some constant c. Inequality (101) leads to the contradiction

$$1 \le \lim_{n\to\infty}\frac{k(L_{t_n}+1)}{L_{t_n}} + \lim_{n\to\infty}\frac{c}{L_{t_n}} = 0. \tag{102}$$

The last equation is a consequence of (96) and the assumption $L_\infty = \infty$. Next we prove $L_\infty \le q$ using boundedness of $(L_t)_{t\ge 0}$. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence such that $\lim_{n\to\infty} L_{t_n} = L_\infty < \infty$. Then a calculation as in (101) results in

$$\begin{aligned}
\lim_{n\to\infty} L_{t_n} \le \limsup_{n\to\infty}&\int \Big[1-\exp\Big(-\int_0^\infty a(\chi_s)\sup_{t\ge t_n} L_{t-s}\,ds\Big)\Big]Q(d\chi)\\
&+\limsup_{n\to\infty}\int\Big(1-\mathbf{E}e^{-\lambda\phi_\chi(t_n)}\Big)Q(d\chi).
\end{aligned} \tag{103}$$

The last summand is equal to zero by Assumption A4.3. The first summand on the right-hand side of (103) is dominated by

$$\Big(\sup_{t>0} L_t\Big)\int\Big(\int_0^\infty a(\chi_s)\,ds\Big)Q(d\chi) < \infty \tag{104}$$

which is finite by boundedness of $(L_t)_{t\ge 0}$ and by Assumption A4.2. Applying dominated convergence, we conclude that L_∞ is bounded by

$$L_\infty \le \int\Big[1-\exp\Big(-\int_0^\infty a(\chi_s)\limsup_{t\to\infty} L_{t-s}\,ds\Big)\Big]Q(d\chi) = k(L_\infty). \tag{105}$$

Thus, Lemma 7.1 implies $\limsup_{t\to\infty} L_t \le q$.

Assume $q > 0$ and suppose that $m := \liminf_{t\to\infty} L_t = 0$. Let $(t_n)_{n\in\mathbb{N}}$ be such that $0 < L_{t_n} \ge \inf_{1\le t\le t_n} L_t \ge cL_{t_n} \to 0$ as $n\to\infty$ and $t_n+1 \le t_{n+1}\to\infty$. By Lemma 7.1, there is an n_0 and a $c<1$ such that $c\int\int_0^{t_{n_0}} a(\chi_s)\,dsQ(d\chi) > 1$. We estimate

$$\begin{aligned}
L_{t_n} &\ge \int\Big[1-\exp\Big(-\int_0^{t_n-1} a(\chi_s)\inf_{1\le t\le t_n} L_t\,ds\Big)\Big]Q(d\chi)\\
&\ge \int\Big[1-\exp\Big(-c\int_0^{t_{n_0}} a(\chi_s)L_{t_n}\,ds\Big)\Big]Q(d\chi) \qquad \forall\, n > n_0.
\end{aligned} \tag{106}$$

Using dominated convergence, the assumption $m = 0$ results in the contradiction

$$
\begin{aligned}
1 &\geq \lim_{n\to\infty} \frac{1}{L_{t_n}} \int \Big[1 - \exp\Big(-cL_{t_n} \int_0^{t_{n_0}} a(\chi_s)\, ds\Big)\Big] Q(d\chi) \\
&= c \int \Big(\int_0^{t_{n_0}} a(\chi_s)\, ds\Big) Q(d\chi) > 1.
\end{aligned} \tag{107}
$$

In order to prove $m \geq q$, let $(t_n)_{n\in\mathbb{N}}$ be such that $\lim_{n\to\infty} L_{t_n} = m > 0$. An estimate as above together with dominated convergence yields

$$
\begin{aligned}
m = \lim_{n\to\infty} L_{t_n} &\geq \lim_{n\to\infty} \int \Big[1 - \exp\Big(-\int_0^{t_n} a(\chi_s) \inf_{t\geq t_n} L_{t-s}\, ds\Big)\Big] Q(d\chi) \\
&= \int \Big[1 - \exp\Big(-\int_0^{\infty} a(\chi_s) \liminf_{t\to\infty} L_t\, ds\Big)\Big] Q(d\chi) = k(m).
\end{aligned} \tag{108}
$$

Therefore, Lemma 7.1 implies $\liminf_{t\to\infty} L_t = m \geq q$, which yields (98).

Finally, we finish the proof of Theorem 5. Applying Lemma 5.3, we see that

$$
\begin{aligned}
&\Big|\mathbf{I}\Big[1 - e^{-\lambda V_t^{\phi,\nu,Q}}\Big] - \int \Big[1 - \exp\Big(-q \int_0^\infty a(\chi_s)\, ds\Big)\Big] \nu(d\chi)\Big| \\
&\leq \int \exp\Big(-\int_0^\infty L_{t-s} a(\chi_s)\, ds\Big) \mathbf{E}\Big[1 - e^{-\lambda \phi_\chi(t)}\Big] \nu(d\chi) \\
&\quad + \Big|\int \Big[1 - \exp\Big(-\int_0^\infty L_{t-s} a(\chi_s)\, ds\Big)\Big] \nu(d\chi) \\
&\quad - \int \Big[1 - \exp\Big(-q \int_0^\infty a(\chi_s)\, ds\Big)\Big] \nu(d\chi)\Big|.
\end{aligned} \tag{109}
$$

The first summand on the right-hand side of (109) converges to zero as $t \to \infty$ by Assumption A4.3. By the first step (98), $L_t \to q$ as $t \to \infty$. Hence, by the dominated convergence theorem and Assumption A4.2, the left-hand side of (109) converges to zero as $t \to \infty$. As ν is a probability measure by assumption, we conclude

$$
\lim_{t\to\infty} \mathbf{E} e^{-\lambda V_t^{\phi,\nu,Q}} = \int \exp\Big(-q \int_0^\infty a(\chi)\, ds\Big) \nu(d\chi) \qquad \forall\, \lambda \geq 0. \tag{110}
$$

This implies Theorem 5 as the Laplace transform is convergence determining, see e.g. Lemma 2.1 in [5]. □

8 The supercritical Virgin Island Model. Proof of Theorem 7

Our proof of Theorem 7 follows the proof of Doney (1972) [4] for supercritical Crump-Mode-Jagers processes. Some changes are necessary because the recursive equation (99) differs from the respective recursive equation in [4]. Parts of our proof are analogous to the proof in [4] which we nevertheless include here for the reason of completeness. Lemma 8.9 and Lemma 8.10 below contain the essential part of the proof of Theorem 7. For these two lemmas, we will need auxiliary lemmas which we now provide.

We assume throughout this section that a solution $\alpha \in \mathbb{R}$ of equation (34) exists. Note that this is implied by A4.2 and $Q\big(\int_0^\infty a(\chi_s)\,ds > 0\big) > 0$. Recall the definition of μ^Q from (82).

8.1 Preliminaries

For $\lambda \geq 0$, define

$$H_\alpha(\psi)(\lambda) := \int \Big[1 - \exp\Big(-\int_0^\infty a\big(\chi_s\big)\psi(\lambda e^{-\alpha s})\,ds\Big)\Big]Q(d\chi) \tag{111}$$

for $\psi \in \mathbf{D}$.

Lemma 8.1. *The operator H_α is contracting in the sense that*

$$\Big|H_\alpha(\psi_1)(\lambda) - H_\alpha(\psi_2)(\lambda)\Big| \leq \int_0^\infty \Big|\psi_1(\lambda e^{-\alpha s}) - \psi_2(\lambda e^{-\alpha s})\Big|\mu^Q(ds) \tag{112}$$

for all $\psi_1, \psi_2 \in \mathbf{D}$.

Proof. The lemma follows immediately from $|e^{-x} - e^{-y}| \leq |x - y|$ and from the definition (82) of μ^Q. ◻

Lemma 8.2. *The operator H_α is nondecreasing in the sense that*

$$H_\alpha(\psi_1)(\lambda) \leq H_\alpha(\psi_2)(\lambda) \tag{113}$$

for all $\lambda \geq 0$ if $\psi_1(\lambda) \leq \psi_2(\lambda)$ for all $\lambda \geq 0$.

Proof. The lemma follows from $1 - e^{-cx}$ being increasing in x for every $c > 0$. ◻

For every measurable function $\psi\colon \mathbb{R} \times [0,\infty) \to [0,\infty)$, define

$$\bar{H}_\alpha(\psi)(t,\lambda) := \int \Big[f\Big(\int_0^\infty a\big(\chi_s\big)\psi(t-s,\lambda e^{-\alpha s})ds\Big)\Big]Q(d\chi). \tag{114}$$

for $\lambda \geq 0$ and $t \in \mathbb{R}$ where $f(x) := x - 1 + e^{-x} \geq 0$, $x \geq 0$. If $\tilde{\psi}\colon [0,\infty) \to [0,\infty)$ is a function of one variable, then we set $\bar{H}_\alpha(\tilde{\psi})(\lambda) := \bar{H}_\alpha(\psi)(1,\lambda)$ where $\psi(t,\lambda) := \tilde{\psi}(\lambda)$ for $\lambda \geq 0$, $t \in \mathbb{R}$.

Lemma 8.3. *The operator $\bar{H}_\alpha$ is nondecreasing in the sense that*

$$\bar{H}_\alpha(\psi_1)(t,\lambda) \leq \bar{H}_\alpha(\psi_2)(t,\lambda) \tag{115}$$

for all $\lambda \geq 0$ and $t \in \mathbb{R}$ if $\psi_1(t,\lambda) \leq \psi_2(t,\lambda)$ for all $\lambda \geq 0$, $t \in \mathbb{R}$.

Proof. The assertion follows from the basic fact that f is nondecreasing. ◻

Lemma 8.4. *Assume A4.2. Let $id : \lambda \mapsto \lambda$ be the identity map. The function*

$$\eta(\lambda) := 1 - \frac{1}{\lambda} H_\alpha(id)(\lambda) = \frac{1}{\lambda} \bar{H}_\alpha(id)(\lambda), \quad \lambda > 0, \tag{116}$$

is nonnegative and nondecreasing. Furthermore, $\eta(0+) = 0$.

Proof. Recall the definition of $A_\alpha(\chi)$ from (54). By equation (114), we have $\lambda\eta(\lambda) = \int f(\lambda A_\alpha)\, dQ$. Thus, η is nonnegative. Furthermore, $\eta(0+) = 0$ follows from the dominated convergence theorem and Assumption A4.2. Let $x, y > 0$. Then

$$\eta(x+y) - \eta(x) = \int \frac{xA_\alpha f\left((x+y)A_\alpha\right) - (x+y)A_\alpha f\left(xA_\alpha\right)}{x(x+y)A_\alpha} dQ \geq 0. \tag{117}$$

The inequality follows from $\tilde{x} f(\tilde{x} + \tilde{y}) - (\tilde{x} + \tilde{y}) f(\tilde{x}) \geq 0$ for all $\tilde{x}, \tilde{y} \geq 0$. □

The following lemma, due to Athreya [1], translates the $(x \log x)$-condition (58) into an integrability condition on η . For completeness, we include its proof.

Lemma 8.5. *Assume A4.2. Let η be the function defined in* (116). *Then*

$$\int_{0+} \frac{1}{\lambda} \eta(\lambda)\, d\lambda < \infty \text{ and } \sum_{n=1}^{\infty} \eta(cr^n) < \infty \tag{118}$$

for some and then all $c > 0$, $r < 1$ if and only if the $(x \log x)$-condition (58) *holds.*

Proof. By monotonicity of η (see Lemma 8.4), the two quantities in (118) are finite or infinite at the same time. Fix $c > 0$. Using Fubini's theorem and the substitution $v := \lambda A_\alpha$, we obtain

$$\begin{aligned} \int_0^c \frac{1}{\lambda} \eta(\lambda)\, d\lambda &= \int \left[\int_0^c \left[\frac{\lambda A_\alpha - 1 + e^{-\lambda A_\alpha}}{(\lambda A_\alpha)^2} \right] \left(A_\alpha\right)^2 d\lambda \right] dQ \\ &= \int \left[A_\alpha \int_0^{cA_\alpha} \frac{v - 1 + e^{-v}}{v^2}\, dv \right] dQ. \end{aligned} \tag{119}$$

It is a basic fact that $\int_0^T \frac{1}{v^2}(v - 1 + e^{-v}) dv \sim \log T$ as $T \to \infty$. □

8.2 The limiting equation

In the following two lemmas, we consider uniqueness and existence of a function Ψ which satisfies:

$$\begin{aligned} &\text{(a)} \quad \left|\Psi(\lambda_1) - \Psi(\lambda_2)\right| \leq |\lambda_1 - \lambda_2| \text{ for } \lambda_1, \lambda_2 \geq 0,\ \Psi(0) = 0 \\ &\text{(b)} \quad \Psi(\lambda) = H_\alpha(\Psi)(\lambda) \\ &\text{(c)} \quad \frac{\Psi(\lambda)}{\lambda} \to 1 \qquad \text{as } \lambda \to 0 \\ &\text{(d)} \quad 0 \leq \Psi(\lambda_1) \leq \Psi(\lambda_2) \leq \lambda_2 \quad \forall\, 0 \leq \lambda_1 \leq \lambda_2 \text{ and } \lim_{\lambda \to \infty} \Psi(\lambda) = q \end{aligned} \tag{120}$$

where $q \geq 0$ is as in Lemma 7.1. Notice that the zero function does not satisfy (120)(c). First, we prove uniqueness.

Lemma 8.6. *Assume A4.2 and $\alpha > 0$. If Ψ_1 and Ψ_2 satisfy* (120)*, then $\Psi_1 = \Psi_2$.*

Proof. Notice that $\Psi_1(0) = \Psi_2(0)$. Define $\Lambda(\lambda) := \frac{1}{\lambda}|\Psi_1(\lambda) - \Psi_2(\lambda)|$ for $\lambda > 0$ and note that $\Lambda(0+) = 0$ by (120)(c). From Lemma 8.1, we obtain for $\lambda > 0$

$$\Lambda(\lambda) \leq \frac{1}{\lambda}\int_0^\infty \left|\Psi_1(\lambda e^{-\alpha s}) - \Psi_2(\lambda e^{-\alpha s})\right|\mu^Q(ds) = \int_0^\infty \Lambda(\lambda e^{-\alpha s})\mu_\alpha^Q(ds) \tag{121}$$

where $\mu_\alpha^Q(ds) := e^{-\alpha s}\mu^Q(ds)$ is a probability measure because α solves equation (49). Let R_i, $i \geq 1$, be independent variables with distribution μ_α^Q and note that $\mathbf{E}R_1 < \infty$. We may assume that $\mathbf{E}R_1 > 0$ because $\mu^Q([0,\infty)) = 0$ implies $\Psi_i = H_\alpha(\Psi_i) = 0$ for $i = 1, 2$. Iterating inequality (121), we arrive at

$$\Lambda(\lambda) \leq \mathbf{E}\Lambda\left(\lambda e^{-\alpha R_1}\right) \leq \mathbf{E}\Lambda\left(\lambda e^{-\alpha(R_1+\ldots+R_n)}\right) \longrightarrow \Lambda(0+) = 0 \quad \text{as } n \to \infty. \tag{122}$$

The convergence in (122) follows from the weak law of large numbers. □

Lemma 8.7. *Assume A4.2 and $\alpha > 0$. There exists a solution Ψ of* (120) *if and only if the* $(x\log x)$*-condition* (58) *holds.*

Proof. Assume that (58) holds. Define $\Psi_0(\lambda) := \lambda$, $\Psi_{n+1}(\lambda) := H_\alpha(\Psi_n)(\lambda)$ for $\lambda \geq 0$ and $\Lambda_{n+1}(\lambda) := \frac{1}{\lambda}\left|\Psi_{n+1}(\lambda) - \Psi_n(\lambda)\right|$ for $\lambda > 0$ and $n \geq 0$. Recall μ_α^Q and $(R_i)_{i\in\mathbb{N}}$ from the proof of Lemma 8.6. Note that $\mathbf{E}R_1 > 0$ because of $\alpha > 0$. Arguments as in the proof of Lemma 8.6 imply

$$\Lambda_{n+1}(\lambda) \leq \mathbf{E}\Lambda_n\left(\lambda e^{-\alpha R_1}\right) \leq \mathbf{E}\Lambda_1\left(\lambda e^{-\alpha S_n}\right) \tag{123}$$

where $S_n := R_1 + \ldots + R_n$ for $n \geq 0$. Since $\eta \geq 0$ by Lemma 8.4 and

$$\Psi_0(\lambda) - \Psi_1(\lambda) = \lambda - H_\alpha(id)(\lambda) = \lambda\eta(\lambda), \tag{124}$$

we see that $\eta = \Lambda_1$. In addition, we conclude from $\eta \geq 0$ that $\Psi_1(\lambda) \leq \Psi_0(\lambda) = \lambda$. By Lemma 8.2, this implies inductively $\Psi_n(\lambda) \leq \lambda$ for $n \geq 0$, $\lambda \geq 0$. Let $\Lambda(\lambda) := \sum_{n\geq 1}\Lambda_n(\lambda)$. We need to prove that $\Lambda(\lambda) < \infty$. Clearly $0 < \mathbf{E}e^{-R_1} < 1$, so we can choose $\varepsilon > 0$ with $e^\varepsilon\mathbf{E}e^{-R_1} < 1$. Then

$$\sum_{n=0}^\infty \mathbf{P}\left(S_n \leq n\varepsilon\right) \leq \sum_{n=0}^\infty e^{n\varepsilon}\mathbf{E}e^{-S_n} = \sum_{n=0}^\infty \left(e^\varepsilon\mathbf{E}e^{-R_1}\right)^n < \infty. \tag{125}$$

Define $\bar{\eta}(\lambda) := \sup_{0<u\leq\lambda}\eta(u)$. It follows from (123), (125), Lemma 8.4 and Lemma 8.5 that for all $\lambda > 0$

$$\Lambda(\lambda) \leq \sum_{n=0}^\infty \mathbf{E}\eta\left(\lambda e^{-\alpha S_n}\right) \leq \bar{\eta}(\lambda)\sum_{n=0}^\infty \mathbf{P}(S_n \leq n\varepsilon) + \sum_{n=0}^\infty \eta(\lambda e^{-n\alpha\varepsilon}) < \infty. \tag{126}$$

Thus, $(\Psi_n(\lambda))_{n\geq 0}$ is a Cauchy sequence in $[0,\lambda]$. Hence, we conclude the existence of a function Ψ such that $\Psi(\lambda) = \lim_{n\to\infty}\Psi_n(\lambda)$ for every $\lambda \geq 0$. By the dominated convergence theorem, Ψ satisfies (120)(b). To check (120)(a), we prove that Ψ_n is Lipschitz continuous with constant one. The induction step follows from Lemma 8.1

$$\begin{aligned} \left|\Psi_{n+1}(\lambda_1) - \Psi_{n+1}(\lambda_2)\right| &\leq \int_0^\infty \left|\Psi_n(\lambda_1 e^{-\alpha s}) - \Psi_n(\lambda_2 e^{-\alpha s})\right|\mu^Q(ds) \\ &\leq \left|\lambda_1 - \lambda_2\right|\int_0^\infty e^{-\alpha s}\mu^Q(ds) = \left|\lambda_1 - \lambda_2\right|. \end{aligned} \tag{127}$$

In order to check (120)(c), note that since $\eta(0+) = 0$, it follows from (126) that $\Lambda(0+) = 0$. Thus,

$$\Big|\frac{\Psi(\lambda)}{\lambda} - 1\Big| \le \limsup_{n\to\infty} \frac{1}{\lambda}\Big|\Psi_n(\lambda) - \lambda\Big| \le \Lambda(\lambda) \longrightarrow 0 \qquad \text{as } \lambda \to 0, \tag{128}$$

as required. Finally, monotonicity of Ψ_n and $\Psi_n(\lambda) \le \lambda$ for all $n \in \mathbb{N}$ imply monotonicity of Ψ and $\Psi(\lambda) \le \lambda$, respectively. The last claim of (120)(d), namely $\Psi(\infty) = q$, follows from letting $\lambda \to \infty$ in $\Psi(\lambda) = H_\alpha(\Psi)(\lambda)$, monotonicity of Ψ and from Lemma 7.1 together with $\Psi(\infty) > 0$.

For the "only if"-part of the lemma, suppose that there exists a solution Ψ of (120). Write $\tilde{g}(\lambda) := \frac{\Psi(\lambda)}{\lambda}$ for $\lambda > 0$. Since $\tilde{g} \ge 0$ and $\tilde{g}(0+) = 1$, there exist constants $c_1, c_2, c_3 > 0$ such that $c_2 \le \tilde{g}(\lambda) \le c_3$ for all $\lambda \in (0, c_1]$. Using (120)(b), $\Psi(\lambda) \ge \lambda c_2$ for $\lambda \in (0, c_1]$ and Lemma 8.3, we obtain for $\lambda \in (0, c_1]$

$$\begin{aligned}\tilde{g}(\lambda) &= \frac{H_\alpha(\Psi)(\lambda)}{\lambda} = \frac{1}{\lambda}\int_0^\infty \Psi(\lambda e^{-\alpha s})\mu^Q(ds) - \frac{1}{\lambda}\bar{H}_\alpha(\Psi)(\lambda) \\ &\le \int_0^\infty \tilde{g}\left(\lambda e^{-\alpha s}\right)\mu_\alpha^Q(ds) - \frac{1}{\lambda}\bar{H}_\alpha(c_2\cdot)(\lambda) = \mathbf{E}\tilde{g}\left(\lambda e^{-\alpha R_1}\right) - c_2\eta(c_2\lambda).\end{aligned} \tag{129}$$

Let t_0 be such that $0 < c_4 := \mu_\alpha^Q\left([0, t_0]\right) < 1$ and write $\tilde{g}^*(\lambda) := \sup_{u\le\lambda} \tilde{g}(u)$. Then

$$\tilde{g}^*(\lambda) \le c_4\tilde{g}^*(\lambda) + (1 - c_4)\tilde{g}^*\left(\lambda e^{-\alpha t_0}\right) - c_2\eta(c_2\lambda) \tag{130}$$

which we rewrite as $\tilde{g}^*(\lambda) \le \tilde{g}^*(\tau\lambda) - c_5\eta(c_2\lambda)$ where $\tau := e^{-\alpha t_0}$ and $c_5 := \frac{c_2}{1-c_4}$. Iterating this inequality results in $\tilde{g}^*(\lambda) \le \tilde{g}^*(\lambda\tau^{n+1}) - c_5\sum_{k=0}^n \eta(c_2\lambda\tau^k)$ for $n \ge 0$. Since $\tilde{g}^*$ is bounded on $(0, c_1]$ this implies that $\sum_{k=0}^\infty \eta(c_2\lambda\tau^k) < \infty$. Therefore, by Lemma 8.5, the $(x\log x)$-condition holds. □

8.3 Convergence

Recall $\bar{m}$, $\mathbf{I}$, m^Q and L_t from (53), (69), (82) and (98), respectively. As before, let $\mu_\alpha^Q(ds) := e^{-\alpha s}\mu^Q(ds)$. Define

$$D(\lambda, t) := \frac{m^Q(t)}{e^{\alpha t}\bar{m}} - \frac{1}{\lambda}L_t\Big(\frac{\lambda}{e^{\alpha t}\bar{m}}\Big), \tag{131}$$

$D_T(\lambda) := \sup_{s\le T} |D(\lambda, s)|$ and $D_\infty(\lambda) := \lim_{T\to\infty} D_T(\lambda)$ for $\lambda > 0$ and $t, T \ge 0$. The following two lemmas follow Lemma 5.1 and Lemma 5.2, respectively, in [4].

Lemma 8.8. *Assume A4.1, A4.2, A4.4 and $\alpha > 0$. If the $(x\log x)$-condition* (58) *holds, then $D_\infty(\lambda) \to 0$ as $\lambda \to 0$.*

Proof. Inserting the definitions (82) and (98) of m^Q and L_t, respectively, into (131), we see that

$$D(\lambda, t) = \frac{1}{\lambda}\mathbf{I}\Big[f\Big(\frac{\lambda V_t^{\phi,Q,Q}}{e^{\alpha t}\bar{m}}\Big)\Big] \ge 0 \qquad \lambda, t > 0 \tag{132}$$

is nonnegative where $f(x) := x - 1 + e^{-x}$, $x \geq 0$. Insert the recursive equations (83) and (99) for m^Q and $(L_t)_{t\geq 0}$, respectively, into (131) to obtain

$$\begin{aligned}
D(\lambda, t) &= \int \frac{\mathbf{E}\phi(t,\chi)}{e^{\alpha t}\bar{m}} Q(d\chi) + \int_0^t \frac{m^Q(t-s)}{e^{\alpha(t-s)}\bar{m}} \mu_\alpha^Q(ds) \\
&\quad - \frac{1}{\lambda}\int \Big[1 - \mathbf{E}\exp\Big(-\lambda\frac{\phi_\chi(t)}{e^{\alpha t}\bar{m}} - \int_0^t a(\chi_s) L_{t-s}\big(\frac{\lambda}{e^{\alpha t}\bar{m}}\big)\, ds\Big)\Big] Q(d\chi) \\
&= \int_0^t \Big[\frac{m^Q(t-s)}{e^{\alpha(t-s)}\bar{m}} - \frac{1}{\lambda e^{-\alpha s}} L_{t-s}\Big(\frac{\lambda e^{-\alpha s}}{e^{\alpha(t-s)}\bar{m}}\Big)\Big]\mu_\alpha^Q(ds) \\
&\quad + \frac{1}{\lambda}\int_0^t L_{t-s}\Big(\frac{\lambda e^{-\alpha s}}{e^{\alpha(t-s)}\bar{m}}\Big)\mu^Q(ds) \\
&\qquad - \frac{1}{\lambda}\int \Big[1 - \exp\Big(-\int_0^t a(\chi_s) L_{t-s}\big(\frac{\lambda e^{-\alpha s}}{e^{\alpha(t-s)}\bar{m}}\big)\, ds\Big)\Big] Q(d\chi) \\
&\quad + \int \mathbf{E}\Big[\frac{1 - \exp\big(-\lambda\frac{\phi_\chi(t)}{e^{\alpha t}\bar{m}}\big)}{\lambda}\Big]\Big[1 - \exp\Big(-\int_0^t a(\chi_s) L_{t-s}\big(\frac{\lambda}{e^{\alpha t}\bar{m}}\big)\, ds\Big)\Big] Q(d\chi) \\
&\quad + \frac{1}{\lambda}\int \mathbf{E}\Big[\frac{\lambda\phi(t,\chi)}{e^{\alpha t}\bar{m}} - 1 + \exp\Big(-\frac{\lambda\phi(t,\chi)}{e^{\alpha t}\bar{m}}\Big)\Big] Q(d\chi) \\
&=: \int_0^t D(\lambda e^{-\alpha s}, t-s)\mu_\alpha^Q(ds) + \frac{1}{\lambda}\bar{H}_\alpha\Big((t,\lambda) \mapsto L_t\big(\frac{\lambda}{e^{\alpha t}\bar{m}}\big)\Big) + T_1 + T_2
\end{aligned} \tag{133}$$

where T_1 and T_2 are suitably defined. Inequality (132) implies

$$L_t\Big(\frac{\lambda}{e^{\alpha t}\bar{m}}\Big) \leq \lambda\frac{m^Q(t)}{e^{\alpha t}\bar{m}} \leq \lambda c_1 \qquad t, \lambda \geq 0 \tag{134}$$

where c_1 is a finite constant. The last inequality is a consequence of Theorem 6, equation (52), with ν replaced by Q. Lemma 8.3 together with (134) implies

$$\frac{1}{\lambda}\bar{H}_\alpha\Big((t,\lambda) \mapsto L_t\big(\frac{\lambda}{e^{\alpha t}\bar{m}}\big)\Big) \leq \frac{1}{\lambda}\bar{H}_\alpha\Big(c_1 \cdot id\Big) = c_1\eta(\lambda c_1). \tag{135}$$

Using $1 - e^{-x} \leq x$, inequality (134) and $x - 1 + e^{-x} \leq \frac{1}{2}x^2$, $x \geq 0$, we see that the expressions T_1 and T_2 are bounded above by

$$\begin{aligned}
T_1 &\leq \int \Big[\frac{\mathbf{E}\phi_\chi(t)}{e^{\alpha t}\bar{m}} \int_0^t a(\chi_s)\lambda e^{-\alpha s} c_1 ds\Big] Q(d\chi) \leq c_2\lambda \\
T_2 &\leq \frac{\lambda}{2}\int \mathbf{E}\Big(\frac{\phi_\chi(t)}{e^{\alpha t}\bar{m}}\Big)^2 Q(d\chi) \leq c_3\lambda
\end{aligned} \tag{136}$$

for all $\lambda, t > 0$ where c_2, c_3 are finite constants which are independent of $t > 0$ and $\lambda > 0$. Such constants exist by Assumption A4.4. Taking supremum over $t \in [0, T]$ in (133) and inserting (135) and (136) results in

$$D_T(\lambda) \leq \int_0^T D_T(\lambda e^{-\alpha s})\mu_\alpha^Q(ds) + c_1\eta(\lambda c_1) + c_2\lambda + c_3\lambda \qquad \forall\, \lambda > 0. \tag{137}$$

Choose $t_0 > 0$ such that $c_4 := \mu_\alpha^Q([0, t_0]) \in (0, 1)$. Then, by monotonicity of D_T,

$$D_T(\lambda) \le c_4 D_T(\lambda) + (1 - c_4) D_T(\lambda e^{-\alpha t_0}) + c_1 \eta(\lambda c_1) + c_2 \lambda + c_3 \lambda \tag{138}$$

for all $\lambda > 0$. Hence, D_T is bounded by

$$D_T(\lambda) \le D_T(\lambda e^{-\alpha t_0}) + c_5 \eta(c_1 \lambda) + c_6 \lambda \qquad \forall\, \lambda > 0 \tag{139}$$

where $c_5 := \frac{c_1}{1-c_4}$ and $c_6 := \frac{c_2+c_3}{1-c_4}$. Iterate this inequality to obtain

$$\begin{aligned} D_T(\lambda) &\le D_T(\lambda e^{-\alpha t_0 n}) + \sum_{k=0}^{n-1} \Big(c_5 \eta(c_1 \lambda e^{-\alpha t_0 k}) + c_6 \lambda e^{-\alpha t_0 k} \Big) \\ &\xrightarrow{n\to\infty} D_T(0+) + \sum_{k=0}^{\infty} \Big(c_5 \eta(c_1 \lambda e^{-\alpha t_0 k}) + c_6 \lambda e^{-\alpha t_0 k} \Big). \end{aligned} \tag{140}$$

Now we need to prove $D_T(0+) = 0$. Looking at (132) and using $f(x) \le x^2/2$, we see that $D_T(\lambda)$ is bounded by $\frac{\lambda}{2\bar{m}^2} \sup_{t \le T} \mathbf{I}\big[\big(V_t^{\phi,Q,Q}\big)^2\big]$. This is finite because of inequality (72) with $\nu = Q$ together with A4.4. Therefore $D_T(0+) = 0$. Letting $T \to \infty$ in (140), we obtain

$$D_\infty(\lambda) \le c_5 \sum_{k=0}^{\infty} \eta\big(\lambda c_1 e^{-\alpha t_0 k}\big) + \lambda c_6 \sum_{k=0}^{\infty} e^{-\alpha t_0 k} \qquad \forall\, \lambda > 0. \tag{141}$$

The right-hand side is finite by Lemma 8.5. By Lemma 8.4, we know that $\eta(0+) = 0$ and that η is nondecreasing. Letting $\lambda \to 0$ in (141) and using the dominated convergence theorem implies $D_\infty(\lambda) \to 0$ as $\lambda \to 0$. □

Lemma 8.9. *Assume A4.1, A4.2, A4.3, A4.4 and $\alpha > 0$ If the $(x \log x)$-condition* (58) *holds, then*

$$L_t\Big(\frac{\lambda}{\bar{m} e^{\alpha t}}\Big) \to \Psi(\lambda) \qquad \text{as } t \to \infty \tag{142}$$

for every $\lambda \ge 0$ where Ψ is the unique solution of (120).

Proof. The case $\lambda = 0$ is trivial. For $\lambda > 0, t \ge 0$, define

$$J(\lambda, t) := \frac{1}{\lambda} \Big(L_t\big(\frac{\lambda}{\bar{m} e^{\alpha t}}\big) - \Psi(\lambda) \Big). \tag{143}$$

Furthermore, let $J_T(\lambda) := \sup_{t \ge T} |J(\lambda, t)|$ and $J_\infty(\lambda) := \lim_{T\to\infty} J_T(\lambda)$ for $\lambda > 0$. We will prove $J_\infty(\lambda) = 0$ for $\lambda > 0$. By Theorem 6 and (120)(c),

$$\big|J(\lambda, t) + D(\lambda, t)\big| \le \Big|\frac{m^Q(t)}{\bar{m} e^{\alpha t}} - 1\Big| + \Big|\frac{\Psi(\lambda)}{\lambda} - 1\Big| \xrightarrow{t\to\infty} \Big|\frac{\Psi(\lambda)}{\lambda} - 1\Big| \xrightarrow{\lambda\to 0} 0. \tag{144}$$

Hence, $J_\infty(0+) = 0$ by Lemma 8.8. Using (99) and (120)(b), we estimate

$$\begin{aligned}
&\Big|\lambda J(\lambda, 2t) - \int 1 - \mathbf{E}\exp\Big(-\tfrac{\lambda}{\bar{m}e^{\alpha 2t}}\phi_\chi(2t) - \int_0^t a(\chi_s) L_{2t-s}\big(\tfrac{\lambda}{\bar{m}e^{\alpha 2t}}\big) ds\Big) Q(d\chi) \\
&\quad + \int 1 - \exp\Big(-\int_0^t a(\chi_s)\Psi(\lambda e^{-\alpha s}) ds\Big) Q(d\chi)\Big| \\
&= \Big|\int \mathbf{E}\exp\Big(-\tfrac{\lambda}{\bar{m}e^{\alpha 2t}}\phi_\chi(2t)\Big)\Big\{\exp\Big(-\int_0^t a(\chi_s) L_{2t-s}\big(\tfrac{\lambda}{\bar{m}e^{\alpha 2t}}\big) ds\Big) \\
&\qquad\qquad - \exp\Big(-\int_0^{2t} a(\chi_s) L_{2t-s}\big(\tfrac{\lambda}{\bar{m}e^{\alpha 2t}}\big) ds\Big)\Big\} Q(d\chi) \qquad (145) \\
&\quad - \int \exp\Big(-\int_0^t a(\chi_s)\Psi(\lambda e^{-\alpha s}) ds\Big) + \exp\Big(-\int_0^\infty a(\chi_s)\Psi(\lambda e^{-\alpha s}) ds\Big) Q(d\chi)\Big| \\
&\le \int \int_t^\infty a(\chi_s)\Big\{\sup_{u \ge 0} L_u\big(\tfrac{\lambda}{\bar{m}}\big) + \Psi(\lambda e^{-\alpha s})\Big\} ds\, Q(d\chi) \\
&\le c \int \int_t^\infty a(\chi_s) ds\, Q(d\chi)
\end{aligned}$$

for a suitable constant c. The last inequality uses boundedness of $(L_t)_{t \ge 0}$, see the proof of Theorem 5, and of Ψ. By Assumption A4.2, the right-hand side of (145) converges to zero as $t \to \infty$. Fix $\lambda > 0$ and let $(t_n)_{n \ge 1}$ be such that $\lim_{n\to\infty} |J(\lambda, 2t_n)| = J_\infty(\lambda)$. With this, we get

$$\begin{aligned}
&\Big|\int 1 - \mathbf{E}\exp\Big(-\tfrac{\lambda}{\bar{m}e^{\alpha 2t_n}}\phi_\chi(2t_n) - \int_0^{t_n} a(\chi_s) L_{2t_n-s}\big(\tfrac{\lambda}{\bar{m}e^{\alpha 2t_n}}\big) ds\Big) Q(d\chi) \\
&\quad - \int 1 - \exp\Big(-\int_0^{t_n} a(\chi_s)\Psi(\lambda e^{-\alpha s}) ds\Big) Q(d\chi)\Big| \\
&\le \int 1 - \mathbf{E}\exp\Big(-\tfrac{\lambda}{\bar{m}}\phi_\chi(2t_n)\Big) Q(d\chi) \\
&\quad + \int\int_0^{t_n} a(\chi_s)\Big| L_{2t_n-s}\big(\tfrac{\lambda e^{-\alpha s}}{\bar{m}e^{\alpha(2t_n-s)}}\big) - \Psi(\lambda e^{-\alpha s})\Big| ds\, Q(d\chi) \qquad (146) \\
&\le \int 1 - \mathbf{E}\exp\big(-\tfrac{\lambda}{\bar{m}}\phi_\chi(2t_n)\big) Q(d\chi) + \lambda \int_0^{t_n} J_{t_n}(\lambda e^{-\alpha s}) \mu_\alpha^Q(ds) \\
&\xrightarrow{n\to\infty} \lambda \int_0^\infty J_\infty(\lambda e^{-\alpha s}) \mu_\alpha^Q(ds).
\end{aligned}$$

The convergence in (146) follows from A4.3 and from the dominated convergence theorem together with Assumption A4.2. Recall $(R_i)_{i \ge 1}$ from the proof of Lemma 8.6. Putting (145) and (146) together, we arrive at

$$\begin{aligned}
J_\infty(\lambda) &= \lim_{n\to\infty} |J(\lambda, 2t_n)| \le \int_0^\infty J_\infty\big(\lambda e^{-\alpha s}\big) \mu_\alpha^Q(ds) \qquad (147) \\
&= \mathbf{E} J_\infty\big(\lambda e^{-\alpha R_1}\big) \le \ldots \le \mathbf{E} J_\infty\big(\lambda e^{-\alpha(R_1 + \ldots + R_m)}\big) \xrightarrow{m\to\infty} J_\infty(0+) = 0.
\end{aligned}$$

This finishes the proof. $\square$

If the $(x \log x)$-condition fails to hold, then the rescaled supercritical Virgin Island Model converges to zero. The proof of this assertion follows Kaplan [12].

Lemma 8.10. *Assume A4.1, A4.2, A4.3 and $\alpha > 0$. If the $(x \log x)$-condition (58) fails to hold, then*

$$L_t\Big(\frac{\lambda}{\bar{m}e^{\alpha t}}\Big) \longrightarrow 0 \qquad \text{as } t \to \infty \tag{148}$$

for every $\lambda \geq 0$

Proof. Define $K(\lambda, t) := \frac{1}{\lambda} L_t(\lambda e^{-\alpha t})$ for $\lambda > 0$ and $K(0, t) := \mathbf{I}\big(V_t^{\phi,Q,Q}\big)e^{-\alpha t}$. It suffices to prove

$$K_\infty(\lambda) := \lim_{T\to\infty} K_T(\lambda) := \lim_{T\to\infty} \sup_{t\geq T} K(\lambda, t) = 0. \tag{149}$$

Assume that $K_\infty(\lambda_0) =: \delta > 0$ for some $\lambda_0 > 0$. We will prove that the $(x \log x)$-condition (58) holds. An elementary calculation shows that $\lambda \mapsto \frac{1}{\lambda}(1 - e^{-\lambda})$ is decreasing. Thus, both $K(\lambda, t)$ and $K_\infty(\lambda)$ are decreasing in λ. Furthermore, by Theorem 6,

$$\delta \leq K_\infty(\lambda) \leq \sup_{t\geq 0} K(\lambda, t) \leq \sup_{t\geq 0} \frac{\mathbf{E}V_t^{\phi,Q,Q}}{e^{\alpha t}} =: C < \infty \qquad \forall\, \lambda \leq \lambda_0. \tag{150}$$

Fix $t_0 > 0$, $\lambda \leq \lambda_0$ and let $t \geq 2t_0$. Inserting the recursive equation (99),

$$\begin{aligned}
&\lambda K(\lambda, t) = L_t(\lambda e^{-\alpha t})\\
&= \int \Big[1 - \mathbf{E}\exp\big(-\frac{\lambda}{e^{\alpha t}}\phi_\chi(t)\big)\exp\Big(-\int_0^t a(\chi_s)L_{t-s}(\lambda e^{-\alpha t})ds\Big)\Big]Q(d\chi)\\
&\leq \sup_{u\geq t_0} \int \Big(1 - \mathbf{E}e^{-\lambda\phi_\chi(u)}\Big)Q(d\chi)\\
&\qquad + \int \Big(1 - \exp\Big(-\int_0^{t-t_0} a(\chi_s)L_{t-s}(\lambda e^{-\alpha s}e^{-\alpha(t-s)})ds\Big)\Big)Q(d\chi)\\
&\qquad + \int \Big(1 - \exp\Big(-\int_{t-t_0}^{t} a(\chi_s)L_{t-s}(\lambda e^{-\alpha s}e^{-\alpha(t-s)})ds\Big)\Big)Q(d\chi)\\
&=: T_1 + T_2 + T_3.
\end{aligned} \tag{151}$$

By Assumption A4.3, the first term converges to zero uniformly in $t \geq 2t_0$ as $t_0 \to \infty$. For the third term, we use inequality (150) to obtain

$$\begin{aligned}
T_3 &= \int \Big(1 - \exp\big(-\int_{t-t_0}^{t} a(\chi_s)K(\lambda e^{-\alpha s}, t-s)\lambda e^{-\alpha s}ds\big)\Big)Q(d\chi)\\
&\leq \int \int_{t_0}^{\infty} a(\chi_s)C\lambda e^{-\alpha s}\, ds Q(d\chi).
\end{aligned} \tag{152}$$

The right-hand side converges to zero uniformly in $t \geq 2t_0$ as $t_0 \to \infty$ by Assumption A4.2. The second term is bounded above by

$$T_2 \leq \int \Big(1 - \exp\big(-\int_0^\infty a(\chi_s)K_{t_0}(\lambda e^{-\alpha s})\lambda e^{-\alpha s}ds\big)\Big)Q(d\chi). \tag{153}$$

Recall $(R_i)_{i\geq 1}$ from the proof of Lemma 8.6. Define $S_0 = 0$ and $S_n := R_1 + \ldots + R_n$, $n \geq 1$. Taking supremum over $t \geq 2t_0$ in (151) and letting $t_0 \to \infty$, we arrive at

$$\begin{aligned}
&K_\infty(\lambda) \leq \frac{1}{\lambda}\int \Big(1 - \exp\big(-\int_0^\infty a(\chi_s)K_\infty(\lambda e^{-\alpha s})\lambda e^{-\alpha s}ds\big)\Big)Q(d\chi)\\
&= \int_0^\infty K_\infty(\lambda e^{-\alpha s})e^{-\alpha s}\mu^Q(ds) - \frac{1}{\lambda}\bar{H}_\alpha\Big(\tilde{\lambda} \mapsto K_\infty(\tilde{\lambda})\tilde{\lambda}\Big)(\lambda)\\
&\leq \mathbf{E}\Big[K_\infty(\lambda e^{-\alpha R_1})\Big] - \frac{1}{\lambda}\bar{H}_\alpha(\tilde{\lambda} \mapsto \delta\tilde{\lambda})(\lambda) = \mathbf{E}\Big[K_\infty(\lambda e^{-\alpha R_1})\Big] - \delta\eta(\delta\lambda)\\
&\leq \cdots \leq \mathbf{E}\Big[K_\infty(\lambda e^{-\alpha S_n})\Big] - \delta\sum_{k=0}^{n-1}\mathbf{E}\eta(\delta\lambda e^{-\alpha S_k})
\end{aligned} \tag{154}$$

for all $n \geq 0$. The second inequality follows from $\delta \leq K_\infty(\tilde{\lambda})$ for $\tilde{\lambda} \leq \lambda_0$ and Lemma 8.3. Boundedness of K_∞ on $(0, \lambda_0]$, see (150), implies

$$\sum_{k=0}^{\infty}\eta(\delta\lambda e^{-\alpha S_k}) < \infty \qquad \text{a.s..} \tag{155}$$

By the law of large numbers, we know that $S_k \leq k(\mathbf{E}R_1 + \varepsilon)$ for large k a.s. Hence,

$$\sum_{k=0}^{\infty}\eta(\delta\lambda r^k) < \infty \tag{156}$$

where $r = e^{-\alpha(\mathbf{E}R_1+\varepsilon)} \in (0,1)$. Therefore, the $(x\log x)$-condition (58) holds by Lemma 8.5. This finishes the proof. □

Proof of Theorem 7. Assume that the $(x\log x)$-condition (58) holds. Insert (142) into (81) and use Assumption A4.3 to obtain

$$\mathbf{E}\Big[\exp\Big(-\frac{\lambda V_t^{\phi,\nu,Q}}{\bar{m}e^{\alpha t}}\Big)\Big] \xrightarrow{t\to\infty} \int \Big[\exp\Big(-\int_0^\infty \Psi(\lambda e^{-\alpha s})a(\chi_s)\,ds\Big)\Big]\nu(d\chi) \tag{157}$$

for $\lambda \geq 0$. For this, we applied the dominated convergence theorem together with Assumption A4.2. Denote the right-hand side of (157) by $\tilde{\Psi}(\lambda)$ and note that $\tilde{\Psi}$ is continuous and satisfies $\tilde{\Psi}(0+) = 1$. A standard result, e.g. Lemma 2.1 in [5], provides us with the existence of a random variable $W \geq 0$ such that $\mathbf{E}e^{-\lambda W} = \tilde{\Psi}(\lambda)$ for all $\lambda \geq 0$. This proves the weak convergence (37) as the Laplace transform is convergence determining. Note that

$$\mathbf{P}(W = 0) = \tilde{\Psi}(\infty) = \int \Big[\exp\Big(-\int_0^\infty \Psi(\infty)a(\chi_s)\,ds\Big)\Big]\nu(d\chi) \tag{158}$$

by the dominated convergence theorem. Furthermore,

$$\mathbf{E}W = \lim_{\lambda\to 0}\frac{1-\tilde{\Psi}(\lambda)}{\lambda} = \int \Big[\int_0^\infty e^{-\alpha s}a(\chi_s)ds\Big]\,\nu(d\chi). \tag{159}$$

If the $(x\log x)$-condition fails to hold, then $\mathbf{E}\big[1-\exp\big(-\lambda V_t^{\phi,\nu,Q}/e^{\alpha t}\big)\big] \to 0$ as $t\to\infty$ follows by inserting (148) into (81) together with A4.3. □

9 Excursions from a trap of one-dimensional diffusions. Proof of Theorem 1

Recall the Assumptions A2.1, A2.2, A2.3, A2.4 and A2.5 from Section 2. The process $(Y_t)_{t\geq 0}$, the excursion set U and the scale function $\bar{S}$ have been defined in (3), in (4) and in (20), respectively. The stopping time T_ε has been introduced shortly after (4).

In this section, we define the excursion measure $\bar{Q}_Y$ and prove the convergence result of Theorem 1. We follow Pitman and Yor [16] in the construction of the excursion measure. Under Assumptions A2.1 and A2.2, zero is an absorbing point for Y. Thus, we cannot simply start in zero and wait until the process returns to zero. Informally speaking, we instead condition the process to converge to infinity. One way to achieve this is by Doob's h-transformation. Note that $\big(\bar{S}(Y_{t\wedge T_\varepsilon})\big)_{t\geq 0}$ is a bounded martingale for every $\varepsilon>0$, see Section V.28 in [17]. In particular,

$$\mathbf{E}^y\big[\bar{S}(Y_{t\wedge T_\varepsilon})\big] = \bar{S}(y) \tag{160}$$

for every $y<\varepsilon$. For $\varepsilon>0$, consider the diffusion $(Y_t^{\uparrow,\varepsilon})_{t\geq 0}$ on $[0,\infty)$ – to be called the ↑-diffusion stopped at time T_ε – defined by the semigroup $(T_t^\varepsilon)_{t\geq 0}$ where

$$T_t^\varepsilon f(y) := \frac{1}{\bar{S}(y)}\mathbf{E}^y\big[\bar{S}(Y_{t\wedge T_\varepsilon})f(Y_{t\wedge T_\varepsilon})\big], \quad y>0, t\geq 0, f\in\mathbf{C}_b\big([0,\infty),\mathbb{R}\big). \tag{161}$$

The sequence of processes $\big((Y_t^{\uparrow,\varepsilon})_{t\geq 0},\varepsilon>0\big)$ is consistent in the sense that

$$\mathscr{L}^y\left(Y^{\uparrow,\varepsilon+\delta}_{\cdot\wedge T_\varepsilon}\right) = \mathscr{L}^y\left(Y^{\uparrow,\varepsilon}_{\cdot}\right) \tag{162}$$

for all $0\leq y\leq\varepsilon$ and $\delta>0$. Therefore, we may define a process $Y^\uparrow = (Y_t^\uparrow)_{0\leq t\leq T_\infty}$ which coincides with $(Y_t^{\uparrow,\varepsilon})_{t\geq 0}$ until time T_ε for every $\varepsilon>0$. Note that the ↑-diffusion possibly explodes in finite time.

The following important observation of Williams has been quoted by Pitman and Yor [16]. Because we assume that zero is an exit boundary for $(Y_t)_{t\geq 0}$, zero is an entrance boundary but not an exit boundary for the ↑-diffusion. More precisely, the ↑-diffusion started at its entrance boundary zero and run up to the last time it hits a level $y>0$ is described by Theorem 2.5 of Williams [20] as the time reversal back from T_0 of the ↓-diffusion started at y, where the ↓-diffusion is the process $(Y_t)_{t\geq 0}$ conditioned on $T_0<\infty$. Hence, the process $\big(Y_t^\uparrow\big)_{t\geq 0}$ may be started in zero but takes strictly positive values at positive times.

Pitman and Yor [16] define the excursion measure $\bar{Q}_Y$ as follows. Under

$$\bar{Q}_Y(\boldsymbol{.}|T_\varepsilon < T_0), \tag{163}$$

that is, conditional on "excursions reach level ε", an excursion follows the $\uparrow$-diffusion until time T_ε and then follows the dynamics of $(Y_t)_{t\geq 0}$. In addition, $\bar{Q}_Y\big(T_\varepsilon < T_0\big) = \frac{1}{\bar{S}(\varepsilon)}$. With this in mind, define a process $\hat{Y}^\varepsilon := \big(\hat{Y}^\varepsilon_t\big)_{t\geq 0}$ which satisfies

$$\mathscr{L}^y\big((\hat{Y}^\varepsilon_{t\wedge T_\varepsilon})_{t\geq 0}\big) \;=\; \mathscr{L}^y\big((Y^{\uparrow,\varepsilon}_t)_{t\geq 0}\big) \tag{164}$$

$$\mathscr{L}^y\big((\hat{Y}^\varepsilon_{T_\varepsilon+t})_{t\geq 0}\big) \;=\; \mathscr{L}^\varepsilon\big((Y_t)_{t\geq 0}\big) \tag{165}$$

for $y \geq 0$. In addition, $(\hat{Y}^\varepsilon_t, t \leq T_\varepsilon)$ and $(\hat{Y}^\varepsilon_t, t \geq T_\varepsilon)$ are independent. Define the excursion measure $\bar{Q}_Y$ on U by

$$\mathbb{1}_{T_\varepsilon<T_0}\bar{Q}_Y(d\chi) := \frac{1}{\bar{S}(\varepsilon)}\mathbf{P}^0\big(\hat{Y}^\varepsilon \in d\chi\big), \quad \varepsilon > 0. \tag{166}$$

This is well-defined if

$$\mathbb{1}_{T_{\varepsilon+\delta}<T_0}\frac{1}{\bar{S}(\varepsilon)}\mathbf{P}^0\big(\hat{Y}^\varepsilon \in d\chi\big) = \frac{1}{\bar{S}(\varepsilon+\delta)}\mathbf{P}^0\big(\hat{Y}^{\varepsilon+\delta} \in d\chi\big) \tag{167}$$

holds for all $\varepsilon, \delta > 0$. The critical part here is the path between T_ε and $T_{\varepsilon+\delta}$. Therefore, (167) follows from

$$\begin{aligned}&\frac{1}{\bar{S}(\varepsilon)}\mathbf{E}^\varepsilon\big[F(Y)\mathbb{1}_{T_{\varepsilon+\delta}<T_0}\big] = \frac{1}{\bar{S}(\varepsilon+\delta)}\mathbf{E}^\varepsilon\big[F(Y)|T_{\varepsilon+\delta} < T_0\big]\\ &= \frac{1}{\bar{S}(\varepsilon+\delta)}\mathbf{E}^\varepsilon\big[F(\hat{Y}^{\varepsilon+\delta})\big] = \frac{1}{\bar{S}(\varepsilon+\delta)}\mathbf{E}^0\big[F(\hat{Y}^{\varepsilon+\delta}_{T_\varepsilon+\boldsymbol{.}})\big].\end{aligned} \tag{168}$$

The first equality is equation (21) with $c = 0$, $y = \varepsilon$ and $b = \varepsilon + \delta$. The last equality is the strong Markov property of $Y^{\uparrow,\varepsilon+\delta}$. The last but one equality is the following lemma.

Lemma 9.1. *Assume A2.1 and A2.2. Let* $0 < y < \varepsilon$. *Then*

$$\mathscr{L}^y\big(\,Y\,|\,T_\varepsilon < T_0\big) = \mathscr{L}^y\big(\hat{Y}^\varepsilon\big). \tag{169}$$

Proof. We begin with the proof of independence of $(\hat{Y}^\varepsilon_t, t \leq T_\varepsilon)$ and of $(\hat{Y}^\varepsilon_t, t \geq T_\varepsilon)$. Let F and G be two bounded continuous functions on the path space. Denote by $\mathscr{F}_{T_\varepsilon}$ the σ-algebra generated by $(Y_t)_{t\leq T_\varepsilon}$. Then

$$\begin{aligned}&\mathbf{E}^y\big[F\big(Y_{T_\varepsilon\wedge\boldsymbol{.}}\big)G\big(Y_{T_\varepsilon+\boldsymbol{.}}\big)|T_\varepsilon < T_0\big]\\ &= \mathbf{E}^y\Big[F\big(Y_{T_\varepsilon\wedge\boldsymbol{.}}\big)\mathbf{E}^y\big[G\big(Y_{T_\varepsilon+\boldsymbol{.}}\big)|\mathscr{F}_{T_\varepsilon}\big]|T_\varepsilon < T_0\Big]\\ &= \mathbf{E}^y\big[F\big(Y_{T_\varepsilon\wedge\boldsymbol{.}}\big)|T_\varepsilon < T_0\big]\mathbf{E}^\varepsilon\big[G\big(Y_{\boldsymbol{.}}\big)\big].\end{aligned} \tag{170}$$

The last equality is the strong Markov property of Y. Choosing $F \equiv 1$ in (170) proves that the left-hand side of (169) satisfies (165). In addition, equation (170) proves the desired independence. For the proof of

$$\mathbf{P}^y\big((Y^{\uparrow,\varepsilon}_t)_{t\geq 0}\big) = \mathbf{P}^y\big((Y_{t\wedge T_\varepsilon})_{t\geq 0}|T_\varepsilon < T_0\big), \tag{171}$$

we repeatedly apply the semigroup (161) of $(Y_t^{\uparrow,\varepsilon})_{t\geq 0}$ to obtain

$$\mathbf{E}^y\Big[\prod_{i=1}^n f_i\big(Y_{t_i}^{\uparrow,\varepsilon}\big)\Big]=\frac{1}{\bar{S}(y)}\mathbf{E}^y\Big[\bar{S}(Y_{t_n\wedge T_\varepsilon})\prod_{i=1}^n f_i\big(Y_{t_i\wedge T_\varepsilon}\big)\Big] \tag{172}$$

for bounded, continuous functions $f_1,...,f_n$ and time points $0\leq t_1<...<t_n$. By equation (21) with $c=0$,

$$\bar{S}(Y_{t_n\wedge T_\varepsilon})=\bar{S}(\varepsilon)\mathbf{P}^{Y_{t_n\wedge T_\varepsilon}}\big[T_\varepsilon<T_0\big]=\bar{S}(\varepsilon)\mathbf{E}^y\big[\mathbb{1}_{T_\varepsilon<T_0}|\mathscr{F}_{t_n\wedge T_\varepsilon}\big] \tag{173}$$

$\mathbf{P}^y$–almost surely where $\mathscr{F}_{t_n\wedge T_\varepsilon}$ is the σ-algebra generated by $(Y_s)_{s\leq t_n\wedge T_\varepsilon}$. Insert this identity in the right-hand side of (172) to obtain

$$\mathbf{E}^y\Big[\prod_{i=1}^n f_i\big(Y_{t_i}^{\uparrow,\varepsilon}\big)\Big]=\frac{1}{\mathbf{P}^y\big(T_\varepsilon<T_0\big)}\mathbf{E}^y\Big[\mathbb{1}_{T_\varepsilon<T_0}\prod_{i=1}^n f_i\big(Y_{t_i\wedge T_\varepsilon}\big)\Big]. \tag{174}$$

This proves (171) because finite-dimensional distributions determine the law of a process. □

Now we prove convergence to the excursion measure $\bar{Q}_Y$.

Proof of Theorem 1. Let $F:\mathbf{C}\big([0,\infty),[0,\infty)\big)\to\mathbb{R}$ be a bounded continuous function for which there exists an $\varepsilon>0$ such that $F(\chi)\mathbb{1}_{T_0<T_\varepsilon}=0$ for every path χ. Let $0<y<\varepsilon$. By Lemma 9.1, we obtain

$$\begin{aligned}\frac{1}{\bar{S}(y)}\mathbf{E}^yF(Y)&=\frac{1}{\bar{S}(\varepsilon)\mathbf{P}^y(T_\varepsilon<T_0)}\mathbf{E}^y\big[F(Y)\mathbb{1}_{T_\varepsilon<T_0}\big]\\&=\frac{1}{\bar{S}(\varepsilon)}\mathbf{E}^yF(\hat{Y}^\varepsilon)=\frac{1}{\bar{S}(\varepsilon)}\mathbf{E}^0F(\hat{Y}^\varepsilon_{T_y+\cdot}).\end{aligned} \tag{175}$$

The last equality is the strong Markov property of the $\uparrow$-diffusion. The random time T_y converges to zero almost surely as $y\to 0$. Another observation we need is that every continuous path $(\chi_t)_{t\geq 0}$ is uniformly continuous on any compact set $[0,T]$. Hence, the sequence of paths $\big((\chi_{T_y+t})_{t\geq 0},y>0\big)$ converges locally uniformly to the path $\big(\chi_t\big)_{t\geq 0}$ almost surely as $y\to 0$. Therefore, the dominated convergence theorem implies

$$\lim_{y\to 0}\mathbf{E}^0F(\hat{Y}^\varepsilon_{T_y+\cdot})=\mathbf{E}^0\lim_{y\to 0}F(\hat{Y}^\varepsilon_{T_y+\cdot})=\mathbf{E}^0F(\hat{Y}^\varepsilon_\cdot). \tag{176}$$

Putting (175) and (176) together, we arrive at

$$\lim_{y\to 0}\frac{1}{\bar{S}(y)}\mathbf{E}^yF(Y)=\frac{1}{\bar{S}(\varepsilon)}\mathbf{E}^0F(\hat{Y}^\varepsilon)=\int F(\chi)\bar{Q}_Y(d\chi), \tag{177}$$

which proves the theorem. □

We will employ Lemma 9.1 to calculate explicit expressions for some functionals of $\bar{Q}_Y$. For example, we will prove in Lemma 9.8 together with Lemma 9.6 that

$$\int\Big(\int_0^\infty a(\chi_s)\,ds\Big)\bar{Q}_Y(d\chi)=\int_0^\infty\frac{a(z)}{g(z)\bar{s}(z)}\,dz \tag{178}$$

provided that Assumptions A2.1, A2.2 and A2.4 hold. Equation (178) shows that condition (43) and condition (28) are equivalent. The following lemmas prepare for the proof of (178).

Lemma 9.2. *Assume A2.1 and A2.2. Let $f \in \mathbf{C}\big([0,\infty),[0,\infty)\big)$ have compact support in $(0,\infty)$. Furthermore, let the continuous function $\psi\colon [0,\infty) \to \mathbb{R}$ be nonnegative and nondecreasing. Then*

$$\frac{1}{\bar{S}(y)}\mathbf{E}^y\left[\Big(\int_0^{T_b}\psi(s)f(Y_s)\,ds\Big)^m\right] \xrightarrow{y\to 0} \int\left[\Big(\int_0^{T_b}\psi(s)f(\chi_s)\,ds\Big)^m\right]\bar{Q}_Y(d\chi) \tag{179}$$

for every $b \le \infty$ and $m \in \mathbb{N}_{\ge 0}$.

Proof. W.l.o.g. assume $m \ge 1$. Let $\varepsilon > 0$ be such that $\varepsilon < \inf\operatorname{supp} f$ and let $y < \varepsilon$. Using Lemma 9.1, we see that the left-hand side of (179) is equal to

$$\begin{aligned}
&\frac{1}{\bar{S}(y)}\mathbf{E}^y\left[\Big(\int_0^{T_b}\psi(s)f(Y_s)\,ds\Big)^m \mathbb{1}_{T_\varepsilon<T_0}\right] = \frac{1}{\bar{S}(\varepsilon)}\mathbf{E}^y\left[\Big(\int_0^{T_b}\psi(s)f(\hat{Y}_s^\varepsilon)\,ds\Big)^m\right]\\
&= \frac{1}{\bar{S}(\varepsilon)}\mathbf{E}^0\left[\Big(\int_{T_y}^{T_b}\psi(s-T_y)f(\hat{Y}_s^\varepsilon)\,ds\Big)^m\right] \xrightarrow{y\to 0} \int\Big(\int_0^{T_b}\psi(s)f(\chi_s)\,ds\Big)^m\bar{Q}_Y(d\chi).
\end{aligned}$$

The second equality is the strong Markov property of $Y^{\uparrow,\varepsilon}$ and the change of variable $s \mapsto s - T_y$. For the convergence, we applied the monotone convergence theorem. □

The explicit formula on the right-hand side of (178) originates in the explicit formula (180) below, which we recall from the literature.

Lemma 9.3. *Assume A2.1 and A2.2. If $f \in \mathbf{C}_b[0,\infty)$ or $f \in \mathbf{C}\big([0,\infty),[0,\infty)\big)$, then*

$$\mathbf{E}^y\Big(\int_0^{T_0\wedge T_b} f(Y_s)\,ds\Big) = \int_0^b \Big(f(z)\frac{\bar{S}(b)-\bar{S}(y\vee z)}{\bar{S}(b)}\frac{\bar{S}(y\wedge z)}{g(z)\bar{s}(z)}\Big)\,dz \tag{180}$$

for all $0 \le y \le b < \infty$.

Proof. See e.g. Section 15.3 of Karlin and Taylor [13]. □

Let $(\tilde{Y}_t)_{t\ge 0}$ be a Markov process with càdlàg sample paths and state space E which is a Polish space. For an open set $O \subset E$, denote by τ the first exit time of $(\tilde{Y}_t)_{t\ge 0}$ from the set O. Notice that τ is a stopping time. For $m \in \mathbb{N}_0$, define

$$w_m(y) := \mathbf{E}^y\left[\Big(\int_0^\tau f(\tilde{Y}_s)\,ds\Big)^m\right],\ y \in E, m \in \mathbb{N}_0, \tag{181}$$

for a given function $f \in \mathbf{C}\big(O,[0,\infty)\big)$. In the following lemma, we derive an expression for w_2 for which Lemma 9.3 is applicable.

Lemma 9.4. *Let $(\tilde{Y}_t)_{t\ge 0}$ be a time-homogeneous Markov process with càdlàg sample paths and state space E which is a Polish space. Let w_m be as in* (181) *with an open set $O \subset E$ and with a function $f \in \mathbf{C}\big(O,[0,\infty)\big)$. Then*

$$\mathbf{E}^y\Big(\int_0^\tau sf(\tilde{Y}_s)\,ds\Big) = \mathbf{E}^y\Big(\int_0^\tau w_1(\tilde{Y}_s)\,ds\Big) \tag{182}$$

$$\mathbf{E}^y\left[\Big(\int_0^\tau f(\tilde{Y}_s)\,ds\Big)^2\right] = \mathbf{E}^y\Big(\int_0^\tau 2f(\tilde{Y}_s)w_1(\tilde{Y}_s)\,ds\Big) \tag{183}$$

for all $y \in E$.

Proof. Let $y \in E$ be fixed. For the proof of (182), we apply Fubini to obtain

$$\begin{aligned}\mathbf{E}^y\Big(\int_0^\tau \int_0^s dr\, f(\tilde{Y}_s)\, ds\Big) &= \mathbf{E}^y\Big(\int_0^\tau \int_r^\tau f(\tilde{Y}_s)\, ds\, dr\Big) \\ &= \int_0^\infty \mathbf{E}^y\Big(\mathbb{1}_{r<\tau}\int_0^\infty \mathbb{1}_{s+r<\tau} f(\tilde{Y}_{s+r}) ds\Big)\, dr.\end{aligned} \tag{184}$$

The last equality follows from Fubini and a change of variables. The stopping time τ can be expressed as $\tau = F\big((\tilde{Y}_u)_{u\geq 0}\big)$ with a suitable path functional F. Furthermore, τ satisfies

$$\{r < \tau\} \cap \{s + r < \tau\} = \{r < \tau\} \cap \{s < F\big((\tilde{Y}_{u+r})_{u\geq 0}\big)\} \tag{185}$$

for $r, s \geq 0$. Therefore, the right-hand side of (184) is equal to

$$\begin{aligned}&\int_0^\infty \mathbf{E}^y\Big(\mathbb{1}_{r<\tau}\int_0^\infty \mathbb{1}_{s<F\left((\tilde{Y}_{u+r})_{u\geq 0}\right)} f(\tilde{Y}_{s+r})\, ds\Big)\, dr \\ &= \int_0^\infty \mathbf{E}^y\Big(\mathbb{1}_{r<\tau}\mathbf{E}^{\tilde{Y}_r}\Big[\int_0^\infty \mathbb{1}_{s<\tau} f(\tilde{Y}_s)\, ds\Big]\Big)\, dr = \mathbf{E}^y\Big(\int_0^\tau w_1(\tilde{Y}_r)\, dr\Big).\end{aligned} \tag{186}$$

The last but one equality is the Markov property of $(\tilde{Y}_t)_{t\geq 0}$. This proves (182). For the proof of (183), break the symmetry in the square of $w_2(y)$ to see that $w_2(y)$ is equal to

$$\begin{aligned}&\mathbf{E}^y\Big(2\int_0^\tau \Big[f(\tilde{Y}_r)\int_r^\tau f(\tilde{Y}_s)\, ds\Big] dr\Big) \\ &= 2\int_0^\infty \mathbf{E}^y\Big(\mathbb{1}_{r<\tau} f(\tilde{Y}_r)\mathbf{E}^{\tilde{Y}_r}\Big[\int_0^\tau f(\tilde{Y}_s)\, ds\Big]\Big)\, dr = \mathbf{E}^y\Big(\int_0^\tau 2f(\tilde{Y}_s) w_1(\tilde{Y}_s)\, ds\Big).\end{aligned} \tag{187}$$

This finishes the proof. ☐

We will need that $(Y_t)_{t\geq 0}$ dies out in finite time. The following lemma gives a condition for this. Recall $\bar{S}(\infty) := \lim_{y\to\infty} \bar{S}(y)$.

Lemma 9.5. *Assume A2.1 and A2.2. Let $y > 0$. Then the solution $(Y_t)_{t\geq 0}$ of equation* (3) *hits zero in finite time almost surely if and only if $\bar{S}(\infty) = \infty$. If $\bar{S}(\infty) < \infty$, then $(Y_t)_{t\geq 0}$ converges to infinity as $t \to \infty$ on the event $\{T_0 = \infty\}$ almost surely.*

Proof. On the event $\{Y_t \leq K\}$, we have that

$$\mathbf{P}^{Y_t}\big(\exists s\colon Y_s = 0\big) \geq \mathbf{P}^K\big(T_0 < \infty\big) > 0 \tag{188}$$

almost surely. The last inequality follows from Lemma 15.6.2 of [13] and Assumption A2.2. Therefore, Theorem 2 of Jagers [11] implies that, with probability one, either $(Y_t)_{t\geq 0}$ hits zero in finite time or converges to infinity as $t \to \infty$. With equation (21), we obtain

$$\mathbf{P}^y\big(\lim_{t\to\infty} Y_t = \infty\big) = \lim_{b\to\infty} \mathbf{P}^y\big(Y \text{ hits } b \text{ before } 0\big) = \lim_{b\to\infty} \frac{\bar{S}(y)}{\bar{S}(b)} = \frac{\bar{S}(y)}{\bar{S}(\infty)}. \tag{189}$$

This proves the assertion. ☐

The following lemma makes Assumption A2.4 more transparent. It proves that A2.4 holds if and only if the expected area under $\big(a(Y_t)\big)_{t\geq 0}$ is finite.

Lemma 9.6. *Assume A2.1 and A2.2. Assumption A2.4 holds if and only if*

$$\mathbf{E}^y\Big(\int_0^\infty a(Y_s)\,ds\Big) < \infty \qquad \forall\ y > 0. \tag{190}$$

If Assumption A2.4 holds, then $\bar{S}(\infty) = \infty$ and

$$\mathbf{E}^y\Big(\int_0^\infty f\left(Y_s\right)ds\Big) = \int_0^\infty \bar{S}\left(y \wedge z\right)\frac{f(z)}{g(z)\bar{s}(z)}\,dz < \infty \tag{191}$$

for all $y \geq 0$ and $f \in \mathbf{C}\big([0,\infty),[0,\infty)\big)$ with $c_f := \sup_{z>0} f(z)/z < \infty$.

Proof. Let c_1, c_2 be the constants from A2.1. In equation (180), let $b \to \infty$ and apply monotone convergence to obtain

$$\mathbf{E}^y\Big(\int_0^\infty f(Y_s)\,ds\Big) = \int_0^\infty \Big(f(z)\Big[1 - \frac{\bar{S}(y \vee z)}{\bar{S}(\infty)}\Big]\frac{\bar{S}(y \wedge z)}{g(z)\bar{s}(z)}\Big)\,dz. \tag{192}$$

Hence, if Assumption A2.4 holds, then Assumption A2.2 implies that the right-hand side of (192) is finite because $f(z) \leq c_f z \leq \frac{c_f}{c_1}a(z)$, $z > 0$. Therefore, the left-hand side of (192) with $f(\cdot)$ replaced by $a(\cdot)$ is finite. Together with $\lim_{x\to\infty} a(x) = \infty$, this implies that $(Y_t)_{t\geq 0}$ does not converge to infinity with positive probability as $t \to \infty$. Thus Lemma 9.5 implies $\bar{S}(\infty) = \infty$ and equation (192) implies (191).

Now we prove that Assumption A2.4 holds if the left-hand side of (192) with $f(\cdot)$ replaced by $a(\cdot)$ is finite. Again, $\lim_{x\to\infty} a(x) = \infty$ and Lemma 9.5 imply $\bar{S}(\infty) = \infty$. Using monotonicity of S, we obtain for $x > 0$

$$\int_x^\infty \frac{a(z)}{g(z)\bar{s}(z)}\,dz \leq \frac{1}{\bar{S}(x)}\int_0^\infty a(z)\frac{\bar{S}(x \wedge z)}{g(z)\bar{s}(z)}\,dz. \tag{193}$$

The right-hand side is finite because (192) with $f(\cdot)$ replaced by $a(\cdot)$ is finite. Therefore, Assumption A2.4 holds. ☐

Lemma 9.7. *Assume A2.1, A2.3 and let $n \in \mathbb{N}_{\geq 1}$. If $\int_1^\infty \frac{y^n}{g(y)\bar{s}(y)}\,dy < \infty$, then*

$$\sup_{y\in(0,\infty)} \frac{y^n}{\bar{S}(y)} < \infty. \tag{194}$$

Proof. It suffices to prove $\liminf_{y\to\infty} \frac{\bar{S}(y)}{y^n} > 0$ because $\frac{y^n}{\bar{S}(y)}$ is locally bounded in $(0,\infty)$ and $\bar{S}'(0) \in (0,\infty)$ by Assumption A2.3. By Assumption A2.1, $g(y) \leq c_g y^2$ for all $y \geq 1$ and a constant $c_g < \infty$. Let $0 \leq x \mapsto \psi(x) := 1 - (1-x)^+ \wedge 1$. Thus,

$$\infty > \int_1^\infty \frac{y^n}{g(y)\bar{s}(y)}\,dy \geq \frac{1}{c_g}\int_1^\infty \frac{y^{n-1}}{y\bar{s}(y)}\,dy \geq \frac{1}{c_g}\int_1^\infty \frac{1}{y}\cdot\Big(1 - \psi\big(\frac{\bar{s}(y)}{y^{n-1}}\big)\Big)\,dy. \tag{195}$$

The last inequality follows from $\frac{1}{z} \geq \mathbb{1}_{z \leq 1} \geq 1 - \psi(z)$, $z > 0$. Consequently,

$$1 = \lim_{z\to\infty} \frac{\int_1^z \frac{1}{y}\psi\big(\frac{\bar{s}(y)}{y^{n-1}}\big)\,dy}{\log(z)} = \lim_{z\to\infty} \frac{\frac{1}{z}\psi\big(\frac{\bar{s}(z)}{z^{n-1}}\big)}{\frac{1}{z}} = \lim_{z\to\infty}\psi\big(\frac{\bar{s}(z)}{z^{n-1}}\big). \tag{196}$$

The proof of the second equation in (196) is similar to the proof of the lemma of L'Hospital. From (196), we conclude $\liminf_{y\to\infty} \frac{\bar{s}(y)}{y^{n-1}} \geq 1$ which implies

$$\liminf_{z\to\infty} \frac{\int_0^z \bar{s}(y)\,dy}{z^n} \geq \liminf_{z\to\infty} \frac{\int_0^z y^{n-1}\,dy}{z^n} = \frac{1}{n}. \tag{197}$$

This finishes the proof. □

Now we prove equation (178). Recall $\bar{S}(\infty) := \lim_{y\to\infty} \bar{S}(y)$. Define $\bar{w}_0 \equiv 1$ and

$$\bar{w}_1(z) := \int_0^\infty f(u) \frac{\bar{S}(z\wedge u)}{g(u)\bar{s}(u)}\,du, \quad z \geq 0 \tag{198}$$

for $f \in \mathbf{C}\big([0,\infty),[0,\infty)\big)$. If $\bar{S}(\infty) = \infty$, then $\bar{w}_1(z)$ is the monotone limit of the right-hand side of (180) as $b \to \infty$.

Lemma 9.8. *Assume A2.1, A2.2 and $\bar{S}(\infty) = \infty$. Let $f \in \mathbf{C}\big([0,\infty),[0,\infty)\big)$. Then*

$$\int \Big(\int_0^\infty f(\chi_s)\,ds\Big)^m \bar{Q}_Y(d\chi) = \int_0^\infty f(z)\frac{m\bar{w}_{m-1}(z)}{g(z)\bar{s}(z)}\,dz \tag{199}$$

$$\int \Big(\int_0^\infty s f(\chi_s)\,ds\Big)\bar{Q}_Y(d\chi) = \int_0^\infty \bar{w}_1(z)\frac{1}{g(z)\bar{s}(z)}\,dz \tag{200}$$

for $m = 1, 2$. If A2.4 holds and if $f(z)/z$ is bounded, then (199) *is finite for $m = 1$. If A2.5 holds and if $f(z)/z$ is bounded, then* (199) *is finite for $m = 2$.*

Proof. Choose $f_\varepsilon \in \mathbf{C}\big([0,\infty),[0,\infty)\big)$ with compact support in $(0,\infty)$ for every $\varepsilon > 0$ such that $f_\varepsilon \uparrow f$ as $\varepsilon \to 0$. Fix $\varepsilon > 0$ and $b \in (0,\infty)$. Lemma 9.2 proves that

$$\lim_{y\to 0} \frac{1}{\bar{S}(y)} \mathbf{E}^y\Big[\Big(\int_0^{T_b} f_\varepsilon(Y_s)\,ds\Big)^m\Big] = \int \Big(\int_0^{T_b} f_\varepsilon(\chi_s)\,ds\Big)^m \bar{Q}_Y(d\chi). \tag{201}$$

Let $w_m^b(y)$ be as in (181) with τ replaced by T_b and f replaced by f_ε. Fix $m \in \{1,2\}$. Lemma 9.4 and Lemma 9.3 provide us with an expression for the left-hand side of equation (201). Hence,

$$\begin{aligned}
&\int \Big(\int_0^{T_b} f_\varepsilon(\chi_s)\,ds\Big)^m \bar{Q}_Y(d\chi)\\
&= \lim_{y\to 0}\frac{1}{\bar{S}(y)}\int_0^b f_\varepsilon(z) m\, w_{m-1}^b(z) \frac{\bar{S}(b) - \bar{S}(y\vee z)}{\bar{S}(b)} \frac{\bar{S}(y\wedge z)}{g(z)\bar{s}(z)}\,dz \\
&= \int_0^b f_\varepsilon(z)\, m\, w_{m-1}^b(z)\Big(1 - \frac{\bar{S}(z)}{\bar{S}(b)}\Big)\frac{1}{g(z)\bar{s}(z)}\,dz.
\end{aligned} \tag{202}$$

The last equation follows from dominated convergence and Assumption A2.2. Note that the hitting time $T_b\big((\chi_t)_{t\geq 0}\big) \to \infty$ as $b \to \infty$ for every continuous path $(\chi_t)_{t\geq 0}$. By Lemma 9.3 and the monotone convergence theorem, $w^b_{m-1}(y) \nearrow \bar{w}_{m-1}(y)$ as $b \nearrow \infty$. Let $b \to \infty$, $\varepsilon \to 0$ and apply monotone convergence to arrive at equation (199).

Similar arguments prove (200). Instead of (201), consider

$$\lim_{y\to 0}\frac{1}{\bar{S}(y)}\mathbf{E}^y\Big(\int_0^{T_b} s f_\varepsilon(Y_s)\,ds\Big) = \int \Big(\int_0^{T_b} s f_\varepsilon(\chi_s)\,ds\Big)\bar{Q}_Y(d\chi) \tag{203}$$

which is implied by Lemma 9.2. Furthermore, instead of applying Lemma 9.3 to equation (201), apply equation (182) together with equation (180).

For the rest of the proof, assume that $f(z)/z$ is bounded by c_f. Let c_1, c_2 be the constants from A2.1. Note that $f(z) \leq c_f z \leq \frac{c_f}{c_1} a(z)$. Consider the right-hand side of (199). If $m = 1$, then the integral over $[1,\infty)$ is finite by Assumption A2.4. If $m = 2$, then the integral over $[1,\infty)$ is finite by Assumption A2.5. The integral over $[0,1)$ is finite because of A2.2 and

$$a(z) \leq c_2 z \leq \bar{c}\bar{S}(z) \quad z \in [0,1], \tag{204}$$

where $\bar{c}$ is a finite constant. The last inequality in (204) follows from Lemma 9.7. □

The convergence (24) of Theorem 1 also holds for $(\chi_s)_{s\geq 0} \mapsto f(\chi_t)$, t fixed, if $f(y)/y$ is a bounded function. For this, we first estimate the moments of $(Y_t)_{t\geq 0}$.

Lemma 9.9. *Assume A2.1. Let $(Y_t)_{t\geq 0}$ be a solution of equation* (3) *and let T be finite. Then, for every $n \in \mathbb{N}_{\geq 2}$, there exists a constant c_T such that*

$$\sup_{t\leq T}\mathbf{E}^y\left[Y_{\tau\wedge t}\right] \leq c_T y, \quad \mathbf{E}^y\left[\sup_{t\leq T} {Y_t}^n\right] \leq c_T(y+y^n) \tag{205}$$

for all $y \geq 0$ and every stopping time τ.

Proof. The proof is fairly standard and uses Itô's formula and Doob's L_p-inequality. □

Lemma 9.10. *Assume A2.1, A2.2 and A2.3. Let $f : [0,\infty) \to [0,\infty)$ be a continuous function such that $f(y) \leq c_f y \vee y^n$ for some $n \in \mathbb{N}_{\geq 1}$, some constant $c_f < \infty$ and for all $y \geq 0$. If $\int_1^\infty \frac{y^n}{g(y)\bar{S}(y)}\,dy < \infty$, then*

$$\int f(\chi_t)\bar{Q}_Y(d\chi) = \lim_{y\to 0}\frac{1}{\bar{S}(y)}\mathbf{E}^y f(Y_t) = \mathbf{E}^0\Big[\frac{1}{\bar{S}(Y^\uparrow_t)}f(Y^\uparrow_t)\mathbb{1}_{t<T_\infty}\Big] \tag{206}$$

is bounded in $t > 0$.

Proof. Choose $f_\varepsilon \in \mathbf{C}\big([0,\infty),[0,\infty)\big)$ with compact support in $(0,\infty)$ for every $\varepsilon > 0$ such that $f_\varepsilon \uparrow f$ pointwise as $\varepsilon \to 0$. By Theorem 1,

$$\int f_\varepsilon(\chi_t)\bar{Q}_Y(d\chi) = \lim_{y\to 0}\frac{1}{\bar{S}(y)}\mathbf{E}^y f_\varepsilon(Y_t). \tag{207}$$

The left-hand side of (207) converges to the left-hand side of (206) as $\varepsilon \to 0$ by the monotone convergence theorem. Hence, the first equality in (206) follows from (207) if the limits $\lim_{\varepsilon\to 0}$ and $\lim_{y\to 0}$ can be interchanged. For this, we prove the second equality in (206).

Let $b \in (0,\infty)$. The $\uparrow$-diffusion is a strong Markov process. Thus, by (161),

$$\begin{aligned}
&\lim_{y\to 0}\frac{1}{\bar{S}(y)}\mathbf{E}^y\Big[f(Y_t)\mathbb{1}_{t<T_b}\Big] = \lim_{y\to 0}\mathbf{E}^y\Big[\frac{f(Y_t^\uparrow)}{\bar{S}(Y_t^\uparrow)}\mathbb{1}_{t<T_b}\Big] \\
&= \mathbf{E}^0\Big[\lim_{y\to 0}\frac{f(Y_{t+T_y}^\uparrow)}{\bar{S}(Y_{t+T_y}^\uparrow)}\mathbb{1}_{t+T_y<T_b}\Big] = \mathbf{E}^0\Big[\frac{f(Y_t^\uparrow)}{\bar{S}(Y_t^\uparrow)}\mathbb{1}_{t<T_b}\Big].
\end{aligned} \tag{208}$$

The second equality follows from the dominated convergence theorem because of

$$\sup_{0<y\le b}\frac{f(y)}{\bar{S}(y)} \le c_f \sup_{0<y\le b}\frac{y\vee y^n}{\bar{S}(y)} < \infty. \tag{209}$$

Right-continuity of the function $t \mapsto \frac{f(Y_t^\uparrow)}{\bar{S}(Y_t^\uparrow)}\mathbb{1}_{t<T_b}$ implies the last equality in (208). Now we let $b\to\infty$ in (208) and apply monotone convergence to obtain

$$\lim_{b\to\infty}\lim_{y\to 0}\frac{1}{\bar{S}(y)}\mathbf{E}^y\Big[f(Y_t)\mathbb{1}_{t<T_b}\Big] = \mathbf{E}^0\Big[\frac{f(Y_t^\uparrow)}{\bar{S}(Y_t^\uparrow)}\mathbb{1}_{t<T_\infty}\Big]. \tag{210}$$

The following estimate justifies the interchange of the limits $\lim\limits_{b\to\infty}$ and $\lim\limits_{y\to 0}$

$$\begin{aligned}
&\Big|\lim_{y\to 0}\frac{1}{\bar{S}(y)}\mathbf{E}^y f(Y_t) - \lim_{b\to\infty}\lim_{y\to 0}\frac{1}{\bar{S}(y)}\mathbf{E}^y\big[f(Y_t)\mathbb{1}_{t<T_b}\big]\Big| \\
&\le c_f \lim_{b\to\infty}\sup_{y\le 1}\frac{1}{\bar{S}(y)}\mathbf{E}^y\Big[Y_t\vee Y_t^n\mathbb{1}_{\sup_{s\le t}Y_s\ge b}\Big] \\
&\le c_f\lim_{b\to\infty}\frac{1}{b}\sup_{y\le 1}\frac{y}{\bar{S}(y)}\sup_{y\le 1}\frac{1}{y}\mathbf{E}^y\Big[\sup_{s\le t}\big(Y_s^2+Y_s^{n+1}\big)\Big] = 0.
\end{aligned} \tag{211}$$

The last equality follows from $\bar{S}^{'}(0)\in(0,\infty)$ and from Lemma 9.9. Putting (211) and (210) together, we get

$$\lim_{y\to 0}\frac{1}{\bar{S}(y)}\mathbf{E}^y\big[f(Y_t)\big] = \lim_{b\to\infty}\lim_{y\to 0}\frac{1}{\bar{S}(y)}\mathbf{E}^y\big[f(Y_t)\mathbb{1}_{t<T_b}\big] = \mathbf{E}^0\Big[\frac{f(Y_t^\uparrow)}{\bar{S}(Y_t^\uparrow)}\mathbb{1}_{t<T_\infty}\Big]. \tag{212}$$

Note that (212) is bounded in $t>0$ because of $f(y)\le c_f\, y\vee y^n$ and Lemma 9.7.

We finish the proof of the first equality in (206) by proving that the limits $\lim\limits_{\varepsilon\to 0}$ and $\lim\limits_{y\to 0}$ on the right-hand side of (207) interchange.

$$\begin{aligned}
&\Big|\lim_{\varepsilon\to 0}\lim_{y\to 0}\frac{1}{\bar{S}(y)}\mathbf{E}^y f_\varepsilon(Y_t) - \lim_{y\to 0}\frac{1}{\bar{S}(y)}\mathbf{E}^y f(Y_t)\Big| \\
&\le \lim_{\varepsilon\to 0}\lim_{y\to 0}\frac{1}{\bar{S}(y)}\mathbf{E}^y\big[f(Y_t)-f_\varepsilon(Y_t)\big] = \lim_{\varepsilon\to 0}\mathbf{E}^0\Big[\frac{f(Y_t^\uparrow)-f_\varepsilon(Y_t^\uparrow)}{\bar{S}(Y_t^\uparrow)}\mathbb{1}_{t<T_\infty}\Big] = 0.
\end{aligned} \tag{213}$$

The first equality is (212) with f replaced by $f-f_\varepsilon$. The last equality follows from the dominated convergence theorem. The function $f_\varepsilon/\bar{S}$ converges to $f/\bar{S}$ for every $y>0$ as $\varepsilon\to 0$. Note that $Y_t^\uparrow>0$ almost surely for $t>0$. Integrability of $\frac{f(Y_t^\uparrow)}{\bar{S}(Y_t^\uparrow)}\mathbb{1}_{t<T_\infty}$ follows from finiteness of (212). □

We have settled equation (178) in Lemma 9.8 (together with Lemma 9.6). A consequence of the finiteness of this equation is that $\liminf_{t\to\infty} \int \chi_t \, d\bar{Q}_Y = 0$. In the proof of the extinction result for the Virgin Island Model, we will need that $\int \chi_t \, d\bar{Q}_Y$ converges to zero as $t \to \infty$. This convergence will follow from equation (178) if $[0,\infty) \ni t \mapsto \int \chi_t \, d\bar{Q}_Y$ is globally upward Lipschitz continuous. We already know that this function is bounded in t by Lemma 9.10.

Lemma 9.11. *Assume A2.1, A2.2 and A2.3. Let $n \in \mathbb{N}_{\geq 1}$. If $\int_1^\infty \frac{y^n}{g(y)\bar{s}(y)} \, dy < \infty$, then*

$$\lim_{t\to\infty} \int \chi_t^n \, \bar{Q}_Y(d\chi) = 0. \tag{214}$$

Proof. We will prove that the function $[0,\infty) \ni t \mapsto \int \chi_t^n d\bar{Q}_Y$ is globally upward Lipschitz continuous. The assertion then follows from the finiteness of (199) with $f(z)$ replaced by z^n and with $m = 1$. Recall τ_K, c_h and c_g from the proof of Lemma 9.9. From (3) and Itô's lemma, we obtain for $y > 0$ and $0 \leq s \leq t$

$$\frac{1}{\bar{S}(y)} \mathbf{E}^y \big(Y^n_{t\wedge\tau_K}\big) - \frac{1}{\bar{S}(y)} \mathbf{E}^y \left(Y^n_{s\wedge\tau_K}\right) \leq \tilde{c} \int_s^t \frac{1}{\bar{S}(y)} \mathbf{E}^y \big(Y^n_{r\wedge\tau_K} + Y^{n-1}_{r\wedge\tau_K}\big) \, dr \tag{215}$$

where $\tilde{c} := n\big(c_h + (n-1)c_g\big)$. Letting $K \to \infty$ and then $y \to 0$, we conclude from the dominated convergence theorem, Lemma 9.9 and Lemma 9.10 that

$$\int \chi_t^n - \chi_s^n \, \bar{Q}_Y(d\chi) \leq \tilde{c} \int_s^t \mathbf{E}^0 \Big[\frac{\big(Y_r^\uparrow\big)^n + \big(Y_r^\uparrow\big)^{n-1}}{\bar{S}(Y_r^\uparrow)} \mathbb{1}_{r<T_\infty} \Big] \, dr \leq \tilde{c} c_S |t-s| \tag{216}$$

for some constant c_S. The last inequality follows from Lemma 9.7. Inequality (216) implies upward Lipschitz continuity which finishes the proof. ◻

10 Proof of Theorem 2, Theorem 3 and of Theorem 4

We will derive Theorem 2 from Theorem 5 and Theorem 3 from Theorem 6. Thus, we need to check that Assumptions A4.1, A4.2 and A4.3 with $\phi(t,\chi) := \chi_t$, $\nu := \mathcal{L}^x(Y)$ and $Q := Q_Y$ hold under A2.1, A2.2, A2.3 and A2.4. Recall that $Q_Y = \bar{S}'(0)\bar{Q}_Y$ and $\bar{s}(0) = \bar{S}'(0)s(0)$. Assumption A4.1 follows from Lemma 9.9 and Lemma 9.10. Lemma 9.6 and Lemma 9.8 imply A4.2. Lemma 9.5 together with Lemma 9.6 implies that $(Y_t)_{t\geq 0}$ hits zero in finite time almost surely. The second assumption in A4.3 is implied by Lemma 9.11 with $n = 1$ and Assumption A2.4. By Theorem 5, we now know that the total mass process $(V_t)_{t\geq 0}$ dies out if and only if condition (43) is satisfied. However, by Lemma 9.8 with $m = 1$ and $f(\cdot) = a(\cdot)$, condition (43) is equivalent to condition (28). This proves Theorem 2

For an application of Theorem 6, note that f^ν and f^Q are integrable by Lemma 9.6 and Lemma 9.8, respectively. In addition, Lemma 9.6 and Lemma 9.8 show that

$$w_{id}(x) = \mathbf{E}^x \int_0^\infty Y_t \, dt = \int_0^\infty f^\nu(t) \, dt \text{ and } w_a'(0) = \int \left(\int_0^\infty a(\chi_s) \, ds \right) Q_Y(d\chi).$$

Similar equations hold for $w_{id}^{'}(0)$ and $w_a(x)$. Moreover, the denominators in (33) and (48) are equal by Lemma 9.8, equation (200), together with Lemma 9.6. Therefore, equations (32) and (33) follow from equations (47) and (48), respectively. In the supercritical case, (50) holds because of

$$\sum_{k=0}^{\infty} \sup_{k\le t\le k+1} e^{-\alpha t} \int \chi_t Q(d\chi) \le \sup_{t\ge 0} \int \chi_t Q(d\chi) \sum_{k=0}^{\infty} e^{-\alpha(k+1)} \tag{217}$$

and Lemma 9.11 with $n = 1$ together with Assumption A2.4. Furthermore, Lemma 9.10 together with Lemma 9.7 and the dominated convergence theorem implies continuity of f^Q. Therefore, Theorem 6 implies (51) which together with (52) reads as (35).

Theorem 4 is a corollary of Theorem 7. For this, we need to check A4.4. The expression in (55) is finite because of Lemma 9.10 with $f(\cdot) = (a(\cdot))^2$ and Assumption A2.5. Assumption A2.1 provides us with $c_1 y \le a(y)$ for all $y \ge 0$ and some $c_1 > 0$. Thus,

$$\int_1^{\infty} \frac{y^2}{g(y)\bar{s}(y)}\, dy \le \frac{1}{c_1} \int_1^{\infty} a(y) \frac{y + w_a(y)}{g(y)\bar{s}(y)}\, dy \tag{218}$$

which is finite by A2.5. Lemma 9.11 with $n = 2$ and Lemma 9.9 show that $\int \chi_t^2 \bar{Q}_Y(d\chi)$ is bounded in $t \ge 0$. Furthermore, Hölder's inequality implies

$$\Big(\int \Big[\chi_t \int_0^t a(\chi_s)\, ds\Big] \bar{Q}_Y(d\chi)\Big)^2 \le \int \chi_t^2 \bar{Q}_Y(d\chi) \int \Big(\int_0^{\infty} a(\chi_s)\, ds\Big)^2 \bar{Q}_Y(d\chi) \tag{219}$$

which is bounded in $t \ge 0$ because of Lemma 9.8 with $m = 2$, $f(\cdot) = a(\cdot)$ and because of Assumption A2.5. Therefore, we may apply Theorem 7. Note that the limiting variable is not identically zero because of

$$\int \Big(A_\alpha \log^+(A_\alpha)\Big) dQ \le \int (A_\alpha)^2 dQ \le \int \Big(\int_0^{\infty} a(\chi_s)\, ds\Big)^2 \bar{Q}_Y(d\chi) < \infty. \tag{220}$$

The right-hand side is finite because of Lemma 9.8 with $m = 2$, $f(\cdot) = a(\cdot)$ and because of Assumption A2.5.

Acknowledgments: We thank Anton Wakolbinger and Don Dawson for inspiring discussions and valuable comments. Also we thank two anonymous referees for their helpful comments and suggestions.

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