Electronic Journal of Probability

Vol. 12 (2007), Paper no. 19, pages 573–590.

Journal URL
http://www.math.washington.edu/~ejpecp/

Dyson's Brownian motions, intertwining and interlacing

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Abstract

A reflected Brownian motion in the Gelfand-Tsetlin cone is used to construct Dyson's process of non-colliding Brownian motions. The key step of the construction is to consider two interlaced families of Brownian paths with paths belonging to the second family reflected off paths belonging to the first. Such families of paths are known to arise in the Arratia flow of coalescing Brownian motions. A determinantal formula for the distribution of coalescing Brownian motions is presented.

Key words: non-colliding Brownian motions; coalescing Brownian motions; intertwining; Gelfand-Tsetlin cone.

AMS 2000 Subject Classification: Primary 60J65; 60J35; 60J60.

Submitted to EJP on July 27 2006, final version accepted April 10 2007.

1 Introduction

The ordered eigenvalues $Y_1(t) \le Y_2(t) \le \ldots \le Y_n(t)$ of a Brownian motion in the space of $n \times n$ Hermitian matrices form a diffusion process which satisfies the stochastic differential equations,

$$Y_i(t) = y_i + \beta_i(t) + \sum_{j \neq i} \int_0^t \frac{ds}{Y_i(s) - Y_j(s)}, \tag{1}$$

where $\beta_1, \beta_2, \ldots, \beta_n$ are independent real Brownian motions. This is a result that goes back to Dyson [12] and we will refer to Y as a Dyson non-colliding Brownian motion. A number of important papers in recent years have developed a link between random matrices and certain combinatorial models, involving random permutations, last passage percolation, random tilings, random growth models and queueing systems, see Baik, Deift and Johansson, [1] and Johansson, [19], amongst many others. A recent survey is given by König [18]. At the heart of this connection lies the Robinson-Schensted-Knuth algorithm, a combinatorial procedure which has its origins in group representation theory, and using this the following remarkable formula, was observed by Gravner, Tracy and Widom, [15] and Baryshnikov [2], representing the largest eigenvalue $Y_n(t)$ (assuming $Y(0) = 0$) in terms of independent, real-valued, Brownian motions $B_1, B_2, \ldots, B_n$,

$$Y_n(t) \stackrel{dist}{=} \sup_{0=t_0 \le t_1 \le \ldots \le t_n = t} \sum_{i=1}^{n} \bigl\{ B_i(t_i) - B_i(t_{i-1}) \bigr\}. \tag{2}$$

O'Connell and Yor, [21], give a proof of this identity by considering reversibility properties of a queueing system, which in a subsequent paper, O'Connell [22], is shown to be linked to the RSK algorithm also. Another proof, again involving RSK, is given by Doumerc, [10]. A representation theoretic approach to the identity is taken by Bougerol and Jeulin, [4], see also Biane, Bougerol and O'Connell, [5].

In this paper a different proof of the identity (2) is given, based around the following construction. Let $\bigl(Y(t); t \ge 0\bigr)$ be a Dyson process, with components $Y_1, Y_2, \ldots Y_n$ solving (1). Let $\bigl(X(t); t \ge 0\bigr)$ be a process with $(n+1)$ components which are interlaced with those of Y, meaning,

$$X_1(t) \le Y_1(t) \le X_2(t) \le \ldots \le Y_n(t) \le X_{n+1}(t), \qquad \text{for all } t \ge 0, \tag{3}$$

and which satisfies the equations

$$X_i(t) = x_i + \gamma_i(t) + \bigl\{ L_i^-(t) - L_i^+(t) \bigr\}. \tag{4}$$

Here $\bigl(\gamma(t); t \ge 0\bigr)$ is a standard Brownian motion in $\mathbf{R}^{n+1}$, independent of the Brownian motion β which drives Y. The processes $(L_i^+(t); t \ge 0)$ and $(L_i^-(t); t \ge 0)$ are continuous non-decreasing processes that increase only at times when $X_i(t) = Y_i(t)$ and $X_i(t) = Y_{i-1}(t)$ respectively: they are twice the semimartingale local times at zero of $X_i - Y_i$ and $X_i - Y_{i-1}$. The two exceptional cases $L_1^-(t)$ and $L_{n+1}^+(t)$ are defined to be identically zero. Conditionally on Y the particles corresponding to X evolve as independent Brownian motions except when collisions occur with particles corresponding to Y. Think of the particles corresponding to the components of Y as being "heavy" so that in collisions with the "light" particles corresponding to components of X their motion is unaffected. On the other hand the light particles receive a singular drift from the collisions which maintains the interlacing. We will verify that is possible to start X and Y from the origin so that $x_i = y_j = 0$ for all i and j. Then, see Proposition 5, *the process X is distributed as a Dyson non-colliding process with $(n+1)$ particles.* Thus if we observe only the particles corresponding to the components of X, the singular drifts that these particles experience from collisions with the unseen particles corresponding to Y are somewhat magically transmuted into an electrostatic repulsion. This is a consequence of a relationship between the semigroup of the extended process (X, Y) and the semigroup of X that is called an intertwining relation.

The case $n = 1$ is directly related to a result obtained previously by Dubédat, [11]. If we define $U(t) = (X_2(t) - X_1(t))/\sqrt{2}$ and $V(t) = (X_1(t) + X_2(t) - 2Y_1(t))/\sqrt{6}$ then the process $\big((U(t), V(t)); t \geq 0\big)$ becomes a Brownian motion in a wedge of angle $\pi/3$ with certain oblique directions of reflection on the boundary. If the process starts from the origin, which is the apex of the cone, Dubédat has proved that $(U(t); t \geq 0)$ is distributed as a Bessel process of dimension three.

Intertwinings are intimately related to time reversal and concepts of duality between Markov processes. In the next section we present a duality between two systems of interlaced (driftless) Brownian motions $\big(X, Y\big)$ and $\big(\hat{X}, \hat{Y}\big)$. In the case of the first interlaced system, particles corresponding to components of X are reflected off particles corresponding to components of Y, for the second system $\big(\hat{X}, \hat{Y}\big)$ the interaction is the other way around. The duality can be checked explictly at the level of the transition semigroups, because, somewhat surprisingly, we are able to give explicit formulae for the transition densities. These formulae are proved directly in Section 2, but they were originally obtained through studying a system of coalescing Brownian motions, sometimes known as the Arratia flow. The interlaced systems of paths arise when we consider both forwards and backwards paths in the flow, and the transition densities for the interlaced systems can be determined from an explicit expression for the joint distribution of a system of coalescing Brownian motions. This is presented in Section 5.

2 A duality between interlaced Brownian motions

Consider a continuous, adapted, $\mathbf{R}^{n+1} \times \mathbf{R}^n$-valued process $\big(X(t), Y(t); t \geq 0\big)$ having components $X_1(t), X_2(t), \dots X_{n+1}(t)$ and $Y_1(t), Y_2(t), \dots, Y_n(t)$ which is defined on a filtered probability space $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{Q}^n_{x,y}\big)$ satisfying, for all $t \geq 0$, the interlacing condition

$$X_1(t) \leq Y_1(t) \leq X_2(t) \leq \dots \leq Y_n(t) \leq X_{n+1}(t),$$

and the equations

$$Y_i(t) = y_i + \beta_i(t \wedge \tau), \tag{5}$$

$$X_i(t) = x_i + \gamma_i(t \wedge \tau) + L_i^-(t \wedge \tau) - L_i^+(t \wedge \tau), \tag{6}$$

where,

τ is the stopping time given by $\tau = \inf\big\{t \geq 0 : Y_i(t) = Y_{i+1}(t) \text{ for some } i \in \{1, 2, \dots, n-1\}\big\}$,

$\beta_1, \beta_2, \dots \beta_n, \gamma_1, \gamma_2, \dots \gamma_{n+1}$ are independent $\mathcal{F}_t$-Brownian motions,

$L_1^-(t) = L_{n+1}^+(t) = 0$ for all $t \geq 0$, otherwise the processes L_i^+ and L_i^- are continuous, non-decreasing and increase only when $X_i = Y_i$ and $X_i = Y_{i-1}$ respectively,

$$L_i^+(t) = \int_0^t \mathbf{1}\big(X_i(s) = Y_i(s)\big) dL_i^+(s) \qquad L_i^-(t) = \int_0^t \mathbf{1}\big(X_i(s) = Y_{i-1}(s)\big) dL_i^-(s).$$

The process just defined is called a stopped, semimartingale reflecting Brownian motion. For general results on such processes see, for example, Dai and Williams, [7]. In this case it is not difficult to give a pathwise construction starting from the Brownian motions β_i, for $i \in \{1, 2, \ldots, n\}$, and γ_i for $i \in \{1, 2, \ldots, n+1\}$, together with the choice of initial co-ordinates $x_1 \le y_1 \le x_2 \ldots \le y_n \le x_{n+1}$. We obtain Y_i immediately. X_i is constructed by alternately using the usual Skorokhod construction to push X_i up from Y_{i-1} and down from Y_i. For more details see Section 3 of [26], where a similar construction is used. In fact by the same argument as Lemma 6 in Soucaliuc, Toth and Werner, [26] pathwise uniqueness holds, and hence the law of $\big(X, Y\big)$ is uniquely determined. This uniqueness implies, by standard methods, that the process is Markovian, and in fact we are able to give an explicit formula for its transition probabilities.

We denote by ϕ_t the centered Gaussian density with variance t. Φ_t is the corresponding distribution function

$$\Phi_t(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi t}} \exp\{-z^2/(2t)\} dz,$$

and

$$\phi_t'(y) = \frac{-y}{\sqrt{2\pi t^3}} \exp\{-y^2/(2t)\}.$$

Let $W^{n+1,n} = \{(x, y) \in \mathbf{R}^{n+1} \times \mathbf{R}^n : x_1 \le y_1 \le x_2 \le \ldots \le y_n \le x_{n+1}\}$. Define $q_t^n\big((x, y), (x', y')\big)$ for $(x, y), (x', y') \in W^{n+1,n}$ and $t > 0$ to be equal to determinant of the $(2n + 1) \times (2n + 1)$ matrix

$$\begin{pmatrix} A_t(x, x') & B_t(x, y') \\ C_t(y, x') & D_t(y, y') \end{pmatrix}$$

where

$A_t(x, x')$ is an $(n + 1) \times (n + 1)$ matrix with (i, j)th element $\phi_t(x_j' - x_i)$;

$B_t(x, y')$ is an $(n + 1) \times n$ matrix with (i, j)th element $\Phi_t(y_j' - x_i) - \mathbf{1}(j \ge i)$.

$C_t(y, x')$ is an $n \times (n + 1)$ matrix with (i, j)th element $\phi_t'(x_j' - y_i)$;

$D_t(y, y')$ is an $n \times n$ matrix with (i, j)th element $\phi_t(y_j' - y_i)$.

Lemma 1. *For any $f : W^{n+1,n} \to \mathbf{R}$ which is bounded and continuous, and zero in a neighbourhood of the boundary of $W^{n+1,n}$,*

$$\lim_{t \downarrow 0} \int_{W^{n+1,n}} q_t^n\big(w, w'\big) f(w') dw' = f(w),$$

uniformly for all $w = (x, y) \in W^{n+1,n}$.

The proof of this lemma is postponed to Section 6.

Proposition 2. $(q_t^n; t > 0)$ *are a family of transition densities for the process $\big(X, Y\big)$ killed at the instant τ, that is to say, for $t > 0$ and $(x, y), (x', y') \in W^{n+1,n}$,*

$$q_t^n\big((x, y), (x', y')\big) dx' dy' = \mathbf{Q}_{x,y}^n\big(X_t \in dx', Y_t \in dy'; t < \tau\big).$$

Proof. For any choice of $z' \in \mathbf{R}$, each of the functions $(t,z) \mapsto \Phi_t(z'-z)$, $(t,z) \mapsto \phi_t(z'-z)$ and $(t,z) \mapsto \phi'_t(z'-z)$ satisfies the heat equation on $(0,\infty) \times \mathbf{R}$. Thus, by differentiating the determinant, we find that,

$$\tfrac{1}{2}\sum_{i=1}^{2n+1} \frac{\partial^2 q_t^n}{\partial w_i^2}(w,w') = \frac{\partial q_t^n}{\partial t}(w,w') \quad (t,w,w') \in (0,\infty) \times \mathbf{R}^{2n+1} \times \mathbf{R}^{2n+1}. \tag{7}$$

We need to identify certain boundary conditions. We treat $w' = (x', y') \in W^{n+1,n}$ as fixed. First consider $(x,y) \in \partial W^{n+1,n}$ satisfying $y_i = y_{i+1}$ for some $i \in \{1, 2, \ldots n-1\}$. We see that the ith and $(i+1)$th rows of both $C_t(y,x')$ and $D_t(y,y')$ are equal, and hence $q_t^n\big((x,y),(x',y')\big) = 0$. Next consider $(x,y) \in \partial W^{n+1,n}$ satisfying $x_i = y_i$ for some $i \in \{1,2,\ldots n\}$. Calculate $\frac{\partial}{\partial x_i} q_t^n((x,y),(x',y'))$ by differentiating the ith rows of $A_t(x,x')$ and $B_t(x,y')$. Notice that, under our assumption that $x_i = y_i$, the ith row of $\frac{\partial}{\partial x_i} A_t(x,x')$ is equal to the ith row of $-C_t(y,x')$. Likewise the ith row of $\frac{\partial}{\partial x_i} B_t(x,y')$ is equal to the ith row of $-D_t(y,x')$. Thus we deduce that $\frac{\partial}{\partial x_i} q_t^n((x,y),(x',y')) = 0$. Finally consider $(x,y) \in \partial W^{n+1,n}$ satisfying $x_{i+1} = y_i$ for some $i \in \{1,2,\ldots n\}$. Similarly to the previous case we obtain $\frac{\partial}{\partial x_{i+1}} q_t^n((x,y),(x',y')) = 0$.

Let $f : W^{n+1,n} \to \mathbf{R}$ be a bounded and continuous, and zero in a neighbourhood of the boundary. Then define a smooth function F on $(0,\infty) \times W^{n+1,n}$ via

$$F(t,w) = \int_{W^{n+1,n}} q_t^n(w,w') f(w') dw'.$$

By virtue of the above observations regarding q_t^n, and differentiating through the integral, we find that

$$\tfrac{1}{2}\sum_{i=1}^{2n+1} \frac{\partial^2 F}{\partial w_i^2}(t,w) = \frac{\partial F}{\partial t}(t,w) \qquad \text{on } (0,\infty) \times W^{n+1,n},$$

with the boundary conditions

$$\begin{aligned} &F(t,w) = 0 \text{ whenever } w = (x,y) \text{ satisfies } y_i = y_{i+1} \\ &\frac{\partial F}{\partial x_i}(t,w) = 0 \text{ whenever } w = (x,y) \text{ satisfies } x_i = y_i \\ &\frac{\partial F}{\partial x_{i+1}}(t,w) = 0 \text{ whenever } w = (x,y) \text{ satisfies } x_{i+1} = y_i \end{aligned}$$

Fix $T, \epsilon > 0$. Applying Itô's formula, we find that the process $\big(F\big(T+\epsilon-t,(X_t,Y_t)\big); t \in [0,T]\big)$ is a local martingale, which is easily seen to be bounded and hence is a true martingale. Thus

$$\begin{aligned} F\big(T+\epsilon,(x,y)\big) &= \mathbf{Q}^n_{x,y}\big[F\big(\epsilon,(X_T,Y_T)\big)\big] \\ &= \mathbf{Q}^n_{x,y}\big[F\big(\epsilon,(X_T,Y_T)\big)\mathbf{1}(T<\tau)\big]. \end{aligned}$$

Appealing to the previous lemma, we may let $\epsilon \downarrow 0$ and so obtain,

$$F\big(T,(x,y)\big) = \mathbf{Q}^n_{x,y}\big[f(X_T,Y_T))\mathbf{1}(T<\tau)\big].$$

Since the part of the distribution of (X_T, Y_T) that charges the boundary of $W^{n+1,n}$ exactly corresponds to the event $\{T \geq \tau\}$ this suffices to prove the proposition. □

We now consider a second reflected semimartingale Brownian motion $(\hat{X}, \hat{Y})$ having components $\hat{X}_1(t), \hat{X}_2(t), \ldots \hat{X}_{n+1}(t)$ and $\hat{Y}_1(t), \hat{Y}_2(t), \ldots, \hat{Y}_n(t)$ which is defined on filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \hat{\mathbf{Q}}^n_{x,y})$ satisfying, for all $t \geq 0$, the interlacing condition

$$\hat{X}_1(t) \leq \hat{Y}_1(t) \leq \hat{X}_2(t) \leq \ldots \leq \hat{Y}_n(t) \leq \hat{X}_{n+1}(t),$$

and the equations

$$\hat{Y}_i(t) = y_i + \beta_i(t \wedge \hat{\tau}) + L^-_i(t \wedge \hat{\tau}) - L^+_i(t \wedge \hat{\tau}), \tag{8}$$

$$\hat{X}_i(t) = x_i + \gamma_i(t \wedge \hat{\tau}), \tag{9}$$

where,

$\hat{\tau}$ is the stopping time given by $\hat{\tau} = \inf\{t \geq 0 : \hat{X}_i(t) = \hat{X}_{i+1}(t) \text{ for some } i \in \{1, 2, \ldots, n\}\}$,

$\beta_1, \beta_2, \ldots \beta_n, \gamma_1, \gamma_2, \ldots \gamma_{n+1}$ are independent $\mathcal{F}_t$-Brownian motions,

the processes L^+_i and L^-_i are continuous, non-decreasing and increase only when $\hat{Y}_i = \hat{X}_{i+1}$ and $\hat{Y}_i = \hat{X}_i$ respectively,

$$L^+_i(t) = \int_0^t \mathbf{1}\big(\hat{Y}_i(s) = \hat{X}_{i+1}(s)\big) dL^+_i(s) \qquad L^-_i(t) = \int_0^t \mathbf{1}\big(\hat{Y}_i(s) = \hat{X}_i(s)\big) dL^-_i(s).$$

Notice the difference between this process and (X, Y) is the reflection rule: here $\hat{Y}$ is pushed off $\hat{X}$ whereas it was X that was pushed off Y.

Define $(\hat{q}^n_t; t > 0)$ via

$$\hat{q}^n_t\big((x, y), (x', y')\big) = q^n_t\big((x', y'), (x, y)\big) \qquad \text{for } (x, y), (x', y') \in W^{n+1,n}. \tag{10}$$

The following proposition is proved by arguments exactly parallel to those just given in proof of Proposition 2.

Proposition 3. $(\hat{q}^n_t; t > 0)$ *are a family of transition densities for the process* $(\hat{X}, \hat{Y})$ *killed at the instant* $\hat{\tau}$*, that is to say, for* $t > 0$ *and* $(x, y), (x', y') \in W^{n+1,n}$*,*

$$\hat{q}^n_t\big((x, y), (x', y')\big) dx' dy' = \hat{\mathbf{Q}}^n_{x,y}\big(\hat{X}_t \in dx', \hat{Y}_t \in dy'; t < \hat{\tau}\big).$$

The duality, represented by (10), between the transition semigroups of (X, Y) and $(\hat{X}, \hat{Y})$ is not unexpected. It is consistent with general results, see for example DeBlassie [8], and Harrison and Williams [16], which show that, in a variety of contexts, the dual of a reflected Brownian motion is another reflected Brownian motion where the direction of reflection at the boundary is obtained by reflecting the original direction of reflection across the normal vector. This is precisely the relationship holding between (X, Y) and $(\hat{X}, \hat{Y})$ here.

3 An intertwining involving Dyson's Brownian motions

It is known that Dyson's non-colliding Brownian motions can be obtained by means of a Doob h-transform. Let $W^n = \{y \in \mathbf{R}^n : y_1 \le y_2 \le \ldots \le y_n\}$. Suppose that $\big(Y(t); t \ge 0\big)$ is, when governed by the probability measure $\mathbf{P}_y^n$, a standard Brownian motion in $\mathbf{R}^n$, relative to a filtration $\{\mathcal{F}_t; t \ge 0\}$, starting from a point $y \in W^n$ and stopped at the instant $\tau = \inf\{t \ge 0 : Y_i(t) = Y_j(t) \text{ for some } i \ne j\}$. The transition probabilities of Y killed at the time τ are given explicitly by the Karlin-McGregor formula, [17],

$$\mathbf{P}_y^n\big(Y_t \in dy'; t < \tau\big) = p_t^n(y, y')dy', \tag{11}$$

for $y, y' \in W^n$, where, with ϕ_t again denoting the Gaussian kernel with variance t,

$$p_t^n(y, y') = \det\big\{\phi_t(y'_j - y_i); 1 \le i, j \le n\big\}. \tag{12}$$

If the initial co-ordinates y satisfy $y_1 < y_2 < \ldots < y_n$, then we may define a new probability measure by the absolute continuity relation

$$\mathbf{P}_y^{n,+} = \frac{h_n(Y(t \wedge \tau))}{h_n(y)} \cdot \mathbf{P}_y^n \quad \text{ on } \mathcal{F}_t, \tag{13}$$

for $t > 0$, where h_n is the function given by

$$h_n(y) = \prod_{i<j}(y_j - y_i). \tag{14}$$

Under $\mathbf{P}_y^{n,+}$ the process Y evolves as a Dyson non-colliding Brownian motion, that is to say τ is almost surely infinite and the stochastic differential equations (1) hold. The transition probabilities

$$\mathbf{P}_y^{n,+}\big(Y(t) \in dy'\big) = p_t^{n,+}(y, y')dy', \tag{15}$$

are related to those for the killed process by an h-transform

$$p_t^{n,+}(y, y') = \frac{h_n(y')}{h_n(y)} p_t^n(y, y'), \tag{16}$$

for $y, y' \in W^n \setminus \partial W^n$. Finally we recall, see O'Connell and Yor, [21], that we may describe $\mathbf{P}_0^{n,+}$, the measure under which the non-colliding Brownian motion issues from the origin by specifying that it is Markovian with transition densities $\big(p_t^{n,+}; t > 0\big)$ and with the entrance law

$$\mathbf{P}_0^{n,+}\big(Y(t) \in dy\big) = \mu_t^n(y)dy, \tag{17}$$

for $t > 0$, given by

$$\mu_t^n(y) = \frac{1}{Z_n} t^{-n^2/2} \exp\left\{-\sum_i y_i^2/(2t)\right\} \left\{\prod_{i<j}(y_j - y_i)\right\}^2, \tag{18}$$

with the normalising constant being $Z_n = (2\pi)^{n/2} \prod_{j<n} j!$.

Now suppose that $\big(X, Y\big)$ is governed by the probability measure $\mathbf{Q}^n_{x,y}$ defined in the previous section. Recall that $\big(q^n_t; t > 0\big)$ are the transition densities of the process killed at the time $\tau = \inf\{t \geq 0 : Y_i(t) = Y_j(t) \text{ for some } i \neq j\}$. Suppose the initial co-ordinates y of Y satisfy $y_1 < y_2 < \ldots < y_n$, then we may define a new probability measure $\mathbf{Q}^{n,+}_{x,y}$ by the absolute continuity relation

$$\mathbf{Q}^{n,+}_{x,y} = \frac{h_n(Y(t \wedge \tau))}{h_n(y)} \cdot \mathbf{Q}^n_{x,y} \quad \text{on } \mathcal{F}_t, \tag{19}$$

for $t > 0$. It follows from the fact that under $\mathbf{Q}^n_{x,y}$ the process Y evolves as a Brownian motion stopped at the instant τ, that $h_n(Y(t \wedge \tau))$ is a martingale, and that this definition is hence consistent as t varies. Under the measure $\mathbf{Q}^{n,+}_{x,y}$, the process Y now evolves as a non-colliding Brownian motion satisfying the stochastic differential equation (1), whilst the process X satisfies (4). The corresponding transition densities $\big(q^{n,+}_t; t > 0\big)$ are obtained from those for the killed process by the h-transform

$$q^{n,+}_t\big((x, y), (x', y')\big) = \frac{h_n(y')}{h_n(y)} q^n_t\big((x, y), (x', y')\big) \tag{20}$$

for $(x, y), (x', y') \in W^{n+1,n}$ with the components of y all distinct.

Lemma 4. *The family of probability measures with densities given by* $\big(\nu^n_t; t > 0\big)$ *on* $W^{n+1,n}$, *given by*

$$\nu^n_t(x, y) = \frac{n!}{Z_{n+1}} t^{-(n+1)^2/2} \exp\left\{-\sum_i x_i^2/(2t)\right\} \left\{\prod_{i<j}(x_j - x_i)\right\} \left\{\prod_{i<j}(y_j - y_i)\right\},$$

form an entrance law for $\big(q^{n,+}_t; t > 0\big)$, *that is to say, for* $s, t > 0$

$$\nu^n_{t+s}(w') = \int_{W^{n+1,n}} \nu^n_s(w) q^{n,+}_t(w, w') dw.$$

Accordingly we may define a probability measure $\mathbf{Q}^{n,+}_{0,0}$, under which the process $\big(X, Y\big)$ is Markovian with transition densities $\big(q^{n,+}_t; t > 0\big)$ and with the entrance law

$$\mathbf{Q}^{n,+}_{0,0}\big(X_t \in dx, Y_t \in dy\big) = \nu^n_t(x, y) dx dy. \tag{21}$$

It is easy to see that under this measure $\big(X, Y\big)$ satisfies the equations (1) and (4), starting from the origin $x = 0, y = 0$. Presumably any solution to (1) and (4) starting from the origin has the same law, but we do not prove this.

We may now state the main result of this section.

Proposition 5. *Suppose the process* $\big(X_t, Y_t; t \geq 0\big)$ *is governed by* $\mathbf{Q}^{n,+}_{0,0}$ *then the process* $\big(X_t; t \geq 0\big)$ *is distributed as under* $\mathbf{P}^{n+1,+}_0$, *that is as a Dyson non-colliding Brownian motion in* W^{n+1} *starting from the origin.*

This result is proved by means of a criterion described by Rogers and Pitman [24] for a function of a Markov process to be Markovian. Carmona, Petit and Yor, [6], describe some further examples of intertwinings, and discuss the relationship between intertwinings and various notions of duality. For $x \in W^{n+1}$ let $W^n(x) = \{y \in \mathbf{R}^n : x_1 \le y_1 \le \ldots \le y_n \le x_{n+1}\}$, and define

$$\lambda^n(x,y) = n! \frac{h_n(y)}{h_{n+1}(x)}, \tag{22}$$

for $x \in W^{n+1} \setminus \partial W^{n+1}$ and $y \in W^n(x)$. The normalising constant being chosen so that $\lambda^n(x,\cdot)$ is the density of a probability measure on $W^n(x)$. This follows from the equality

$$\int_{W^n(x)} h_n(y) dy = \frac{1}{n!} h_{n+1}(x), \tag{23}$$

which is easily verified by writing $h_n(y) = \det\{y_i^{j-1}; 1 \le i,j \le n\}$. The proof of Proposition 5 depends on the following intertwining relation between $\big(q_t^{n,+}; t>0\big)$ and $\big(p_t^{n+1,+}; t>0\big)$, for all $t>0$, $x \in W^{n+1} \setminus \partial W^{n+1}$, and $(x',y') \in W^{n+1,n}$,

$$\int_{W^n(x)} \lambda^n(x,y) q_t^{n,+}\big((x,y),(x',y')\big) dy = p_t^{n+1,+}(x,x') \lambda^n(x',y'). \tag{24}$$

This may be verified directly using the explicit formula for q_t^n given in the previous section. Alternatively the following derivation is enlightening. Recall that if $\big(\hat{X}_t, \hat{Y}_t; t \ge 0\big)$ is governed by $\hat{\mathbf{Q}}^n_{x,y}$ then the process $\big(\hat{X}_t; t \ge 0\big)$ is a Brownian motion stopped at the instant $\hat{\tau} = \inf\{t \ge 0; \hat{X}_i = \hat{X}_j \text{ for some } i \ne j\}$. Consequently the transition probabilities of the killed process satisfy

$$\int_{W^n(x')} \hat{q}_t^n\big((x,y),(x',y')\big) dy' = p_t^{n+1}(x,x'). \tag{25}$$

Now using the duality between q_t^n and $\hat{q}_t^n$ and the symmetry of p_t^{n+1} we may re-write this as

$$\int_{W^n(x')} q_t^n\big((x',y'),(x,y)\big) dy' = p_t^{n+1}(x',x). \tag{26}$$

Finally to obtain (24) we swop the roles of (x,y) and (x',y') and use the expressions for $q_t^{n,+}$ and $p_t^{n+1,+}$ as h-transforms. As a first application of the intertwining we have the following.

Proof of Lemma 4. Notice that $\nu_t^n(x,y) = \mu_t^{n+1}(x)\lambda^n(x,y)$. Hence, by virtue of the intertwining and the fact that $\big(\mu_t^{n+1}; t>0\big)$ is an entrance law for $\big(p_t^{n+1,+}; t>0\big)$ we have,

$$\begin{aligned}
&\int_{W^{n+1,n}} dxdy\, \nu_s^n(x,y) q_t^{n,+}\big((x,y),(x',y')\big) = \\
&\qquad \int_{W^{n+1}} dx\, \mu_s^{n+1}(x) \int_{W^n(x)} dy\, \lambda^n(x,y) q_t^{n,+}\big((x,y),(x',y')\big) = \\
&\qquad \int_{W^{n+1}} dx\, \mu_s^{n+1}(x) p_t^{n+1,+}(x,x') \lambda^n(x',y') = \mu_{t+s}^{n+1}(x')\lambda^n(x',y') = \nu_{t+s}^n(x',y').
\end{aligned}$$

□

A similar argument, following [24], proves the proposition.

Proof of Proposition 5. For a sequence of times $0 < t_1 < t_2 < \ldots < t_n$, repeated use of the intertwining relation gives,

$$\mathbf{Q}^{n,+}_{0,0}\big(X_{t_1} \in A_1, X_{t_2} \in A_2, \ldots, X_{t_n} \in A_n\big) =$$
$$\int_{A_1} dx_1 \ldots \int_{A_n} dx_n \int_{W^n(x_1)} dy_1 \ldots \int_{W^n(x_n)} dy_n\ \nu^n_{t_1}(x_1, y_1) q^{n,+}_{t_2-t_1}\big((x_1, y_1), (x_2, y_2)\big) \ldots$$
$$\ldots q^{n,+}_{t_n-t_{n-1}}\big((x_{n-1}, y_{n-1}), (x_n, y_n)\big) =$$
$$\int_{A_1} dx_1 \ldots \int_{A_n} dx_n \int_{W^n(x_1)} dy_1 \ldots \int_{W^n(x_n)} dy_n\ \mu^{n+1}_{t_1}(x_1)\lambda^n(x_1, y_1) q^{n,+}_{t_2-t_1}\big((x_1, y_1), (x_2, y_2)\big) \ldots$$
$$\ldots q^{n,+}_{t_n-t_{n-1}}\big((x_{n-1}, y_{n-1}), (x_n, y_n)\big) =$$
$$\int_{A_1} dx_1 \ldots \int_{A_n} dx_n \int_{W^n(x_2)} dy_2 \ldots \int_{W^n(x_n)} dy_n\ \mu^{n+1}_{t_1}(x_1) p^{n+1,+}_{t_2-t_1}\big(x_1, x_2\big)\lambda^n(x_2, y_2) \ldots$$
$$\ldots q^{n,+}_{t_n-t_{n-1}}\big((x_{n-1}, y_{n-1}), (x_n, y_n)\big) =$$
$$\int_{A_1} dx_1 \ldots \int_{A_n} dx_n \int_{W^n(x_n)} dy_n\ \mu^{n+1}_{t_1}(x_1) p^{n+1,+}_{t_2-t_1}\big(x_1, x_2\big) \ldots p^{n+1,+}_{t_n-t_{n-1}}\big(x_{n-1}, x_n\big)\lambda^n(x_n, y_n) =$$
$$\int_{A_1} dx_1 \ldots \int_{A_n} dx_n\ \mu^{n+1}_{t_1}(x_1) p^{n+1,+}_{t_2-t_1}\big(x_1, x_2\big) \ldots p^{n+1,+}_{t_n-t_{n-1}}\big(x_{n-1}, x_n\big).$$

□

Notice that in the above proof, if we integrate y_n over some smaller set than $W^n(x_n)$ we find that

$$\mathbf{Q}^{n,+}_{0,0}\big(Y_{t_n} \in A | X_{t_1}, X_{t_2}, \ldots, X_{t_n}\big) = \int_{A \cap W^n(X_{t_n})} \lambda^n(X_{t_n}, y) dy. \tag{27}$$

This may be interpreted as the following filtering property: the conditional distribution of Y_t given $\big(X_s; s \leq t\big)$ is given by the density $\lambda^n(X_t, \cdot)$ on $W^n(X_t)$.

4 Brownian motion in the Gelfand-Tsetlin cone

Proposition 5 lends itself to an iterative procedure. Let $\mathbf{K}$ be the cone of points $\mathbf{x} = \big(x^1, x^2, \ldots x^n\big)$ with $x^k = \big(x^k_1, x^k_2, \ldots, x^k_k\big) \in \mathbf{R}^k$ satisfying the inequalities

$$x^{k+1}_i \leq x^k_i \leq x^{k+1}_{i+1}. \tag{28}$$

$\mathbf{K}$ is sometimes called the Gelfand-Tsetlin cone, and arises in representation theory. We will consider a process $\mathbf{X}(t) = \big(X^1(t), X^2(t), \ldots X^n(t)\big)$ taking values in $\mathbf{K}$ so that

$$X^k_i(t) = x^k_i + \gamma^k_i(t) + L^{k,-}_i(t) - L^{k,+}_i(t), \tag{29}$$

where $\big(\gamma^k_i(t); t \geq 0\big)$ for $1 \leq k \leq n, 1 \leq i \leq k$ are independent Brownian motions, and $\big(L^{k,+}_i(t); t \geq 0\big)$ and $\big(L^{k,-}_i(t); t \geq 0\big)$ are continuous, increasing processes growing only when $X^k_i(t) = X^{k-1}_i(t)$ and $X^k_i(t) = X^{k-1}_{i-1}(t)$ respectively, the exceptional cases $L^{k,+}_k(t)$ and $L^{k,-}_1(t)$ being identically zero for all k. For initial co-ordinates satisfying $x^k_i < x^k_{i+1}$ for all k and i, we may give a pathwise construction, as in Section 2, based on alternately using the Skorokhod construction to reflect X_i^k downwards from X_i^{k-1} and upwards from X_{i-1}^{k-1}. The potential difficulty that X_i^{k-1} meets X_{i-1}^{k-1} does not arise.

In order to construct $\mathbf{X}$ starting from the origin we use a different method. First we note that if the pair of processes $\big(X, Y\big)$, governed by the measure $\mathbf{Q}_{0,0}^{n,+}$ satisfies equations (1) and (4), then Y is measurable with respect to the Brownian motion β, and consequently,

$$\text{the Brownian motion } \gamma = (\gamma_1, \gamma_2, \ldots \gamma_{n+1}) \text{ is independent of } Y. \tag{30}$$

By repeated application of Proposition 5, there exists a process $\big(\mathbf{X}(t); t \geq 0\big)$, starting from the origin, such that

the process $\big(X^k(t); t \geq 0\big)$ is distributed as $\mathbf{P}_0^{k,+}$, for $k = 1, 2, \ldots, n$,

the pair of processes $\big(X^{k+1}(t), X^k(t); t \geq 0\big)$ are distributed as $\mathbf{Q}_{0,0}^{k,+}$, for $k = 1, \ldots, n-1$,

for $k = 2, \ldots, n-1$ the process $\big(X^{k+1}(t); t \geq 0\big)$ is conditionally independent of $\big(X^1(t), \ldots, X^{k-1}(t); t \geq 0\big)$ given $\big(X^k(t); t \geq 0\big)$.

By its very construction the process $\mathbf{X}$ satisfies the equations (29), for some Brownian motions γ_i^k, which by the observation (30) are independent. Even starting from the origin, pathwise uniqueness, and hence uniqueness in law hold for $\mathbf{X}$. Consequently we may state the following proposition.

Proposition 6. *The process $\big(\mathbf{X}(t); t \geq 0\big)$, satisfying (29), if started from the origin, satisfies for each $k = 1, 2, \ldots, n$,*

$$\big(X^{(k)}(t); t \geq 0\big) \textit{ is distributed as under } \mathbf{P}_0^{k,+}.$$

It is not difficult to see that for any $t > 0$, and $k = 2, \ldots, n-1$ the process $\big(X^{k+1}(s); 0 \leq s \leq t\big)$ is conditionally independent of $\big(X^1(s), \ldots X^{k-1}(s); 0 \leq s \leq t\big)$ given $\big(X^k(s); 0 \leq s \leq t\big)$. For any $x^k \in W^k$ we will denote by $\mathbf{K}(x^k)$ the set of all $(x^1, x^2, \ldots x^{k-1})$ such that for all i and j, $x_i^{j+1} \leq x_i^j \leq x_{i+1}^{j+1}$. The $k(k-1)/2$-dimensional volume of $\mathbf{K}(x^k)$ is given by

$$\frac{1}{\prod_{j<k} j!} h_k(x^k).$$

Recall from (27) that the conditional distribution of $X^k(t)$ given $\big(X^{k+1}(s); 0 \leq s \leq t\big)$ has the density $\lambda^k(X^{k+1}(t), \cdot)$ on $W^k(X^{k+1}(t))$. Combining this with the conditional independence property noted above we deduce that the conditional distribution of $\big(X^1(t), X^2(t) \ldots X^k(t)\big)$ given $\big(X^{k+1}(s); 0 \leq s \leq t\big)$ is uniform on $\mathbf{K}(X^{k+1}(t))$. Finally using the fact that the distribution of $X^n(t)$ is given by the density μ_t^n on W^n we deduce that the distribution of $\mathbf{X}(t)$ has the density

$$\boldsymbol{\mu}_t^n(\mathbf{x}) = (2\pi)^{-n/2} t^{-n^2/2} \exp\left\{ -\sum_i (x_i^n)^2/(2t) \right\} \left\{ \prod_{i<j} (x_j^n - x_i^n) \right\}, \tag{31}$$

with respect to Lebesgue measure on $\mathbf{K}$. Baryshnikov, [2], studies this distribution in some detail. Let $\big(H(t); t \geq 0\big)$ be a Brownian motion in the space of $n \times n$ Hermitian matrices, and consider the process $\big(H^1(t), H^2(t), \ldots H^n(t); t \geq 0\big)$ where $H^k(t)$ is the $k \times k$ minor of $H(t) = H^n(t)$ obtained by deleting the last $n-k$ rows and columns. It is a classical result that the eigenvalues of $H^{k-1}(t)$ are interlaced with those of $H^k(t)$. Baryshnikov shows that, at any fixed instant $t > 0$, the distribution of the eigenvalues of $H^1(t), H^2(t), \ldots H^n(t)$ is given by the density (31). However it is not the case that the eigenvalue process is distributed as the process $\big(\mathbf{X}(t); t \geq 0\big)$.

O'Connell, [22], describes another process $\big(\mathbf{\Gamma}(t); t \geq 0\big)$ taking values in $\mathbf{K}$ which is constructed via certain explicit path transformations. This process arises as the scaling limit of the RSK correspondence. The process $\mathbf{X}$ described above has several features in common with $\mathbf{\Gamma}$. For each k, the subprocess $\big(\Gamma^k(t); t \geq 0\big)$ evolves as Dyson k-tuple starting from zero. Additionally

$$\big(X_1^1(t), X_2^2(t), \ldots, X_n^n(t); t \geq 0\big) \overset{dist}{=} \big(\Gamma_1^1(t), \Gamma_2^2(t), \ldots, \Gamma_n^n(t); t \geq 0\big), \tag{32}$$

but remarkably all other components Γ_l^k with $l < k$ are given by explicit deterministic transformations applied to the processes $\Gamma_1^1, \Gamma_2^2, \ldots \Gamma_n^n$. A feature that $\mathbf{X}$ certainly does not share.

Notice that, for $k \geq 2$,

$$X_k^k(t) = \gamma_k^k(t) + L_k^{k,-}(t), \tag{33}$$

where $L_k^{k,-}(t)$ grows only when $X_k^k(t) = X_{k-1}^{k-1}(t)$. On applying the Skorokhod lemma, see Chapter VI of [23], we find that

$$L_k^{k,-}(t) = \sup_{s \leq t}\big(X_{k-1}^{k-1}(s) - \gamma_k^k(s)\big). \tag{34}$$

Iterating this relation we obtain

$$X_k^k(t) = \sup_{0=t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_k = t} \sum_{i=1}^{k} \big\{\gamma_i^i(t_i) - \gamma_i^i(t_{i-1})\big\}, \tag{35}$$

which in the light of Proposition 6 proves the identity (2). This is essentially the same argument for (2) as given by O'Connell and Yor, [21], with Proposition 6 replacing the corresponding statement about $\mathbf{\Gamma}$.

We close this section by giving an explicit formula for the transition densities of the Markov process $\big(X_1^1(t), X_2^2(t), \ldots, X_n^n(t); t \geq 0\big)$. This is continuous analogue of a formula obtained Schutz, [25], for the totally asymmetric exclusion process. For $n \geq 1$ let $\Phi_t^{(n)}$ denote the nth iterated integral of the Gaussian density ϕ_t,

$$\Phi_t^{(n)}(y) = \int_{-\infty}^{y} \frac{(y-x)^{n-1}}{(n-1)!}\phi_t(x)dx, \tag{36}$$

and for $n \geq 0$ let $\Phi_t^{(-n)}$ denote the nth derivative of ϕ_t. Define for $x, x' \in W^n$,

$$r_t(x, x') = \det\big\{\Phi_t^{(i-j)}(x_j' - x_i); 1 \leq i, j \leq n\big\}. \tag{37}$$

Lemma 7. *For any $f : W^n \to \mathbf{R}$ which is bounded and continuous and zero in a neighbourhood of the boundary of W^n,*

$$\lim_{t \downarrow 0} \int_{W^n} r_t\big(x, x'\big) f(x') dx' = f(x),$$

uniformly for all $x \in W^n$.

The proof of the lemma is given in Section 6.

Proposition 8. *The process* $\big(X_1^1(t), X_2^2(t), \dots, X_n^n(t); t \geq 0\big)$ *satisfying* (33) *is Markovian with transition densities given by* $r_t(x, x')$.

Proof. For a fixed $x' \in \mathbf{R}$, and any n, the function $(t, x) \mapsto \Phi_t^{(n)}(x' - x)$ solves the heat equation on $(0, \infty) \times \mathbf{R}$. From this we easily see that for a fixed $x' \in \mathbf{R}^n$, that the function $(t, x) \mapsto r_t(x' - x)$ solves the heat equation on $(0, \infty) \times \mathbf{R}^n$. Moreover if $x_i = x_{i-1}$ for any $i = 2, 3, \dots, n$ then the ith and $(i-1)$th rows of the determinant defining $\frac{\partial}{\partial x_i} r_t(x, x')$ are equal and hence this quantity is zero.

Let $f : W^n \to \mathbf{R}$ be a bounded, continuous and are in a neighbourhood of the boundary of W^n. Then define a smooth function F on $(0, \infty) \times W^n$ via

$$F(t, x) = \int_{W^n} r_t(x, x') f(x') dx'.$$

By virtue of the above observations regarding r_t, and differentiating through the integral, we find that

$$\tfrac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 F}{\partial x_i^2}(t, x) = \frac{\partial F}{\partial t}(t, x) \qquad \text{on } (0, \infty) \times W^n,$$

with the boundary conditions

$$\frac{\partial F}{\partial x_i}(t, x) = 0 \text{ whenever } x_i = x_{i-1} \text{ for some } i = 2, 3, \dots, n.$$

Let X denote a process governed by a probability $\mathbf{R}_x$, with components $X_1(t) \leq X_2(t) \leq \dots \leq X_n(t)$ satisfying the equations $X_k(t) = x_k + \gamma_k(t) + L^k(t)$, where γ_k are independent Brownian motions and L^k is an increasing process growing only when $X_k(t) = X_{k-1}(t)$, with L^1 being identically zero. Fix $T, \epsilon > 0$. Applying Itô's formula, we find that the process $\big(F\big(T + \epsilon - t, X_t\big); t \in [0, T]\big)$ is a local martingale, which being bounded is a true martingale. Thus

$$F\big(T + \epsilon, x\big) = \mathbf{R}_x\big[F\big(\epsilon, X(T)\big)\big].$$

Appealing to the previous lemma, we may let $\epsilon \downarrow 0$ and so obtain,

$$F(T, x) = \mathbf{R}_x\big[f(X(T))\big],$$

which, since it is clear the distribution of $X(T)$ does not charge the boundary of W^n, proves the proposition. □

In view of Proposition 6, we obtain from r_t by a simple integration the following expression for the distribution function of the largest eigenvalue of $H(t)$:

$$\mathbf{P}_0^{n,+}\big(X_n(t) \leq x\big) = \det\big\{\Phi_t^{(i-j+1)}(x)\big\}. \tag{38}$$

It is possible to verify this equality directly by using the Cauchy-Binet formula (personal communication, K. Johansson).

5 Coalescing Brownian motions

In this section we consider the joint distribution of a family of coalescing Brownian motions. Fix $z_1 \leq z_2 \leq \ldots \leq z_n$ and consider the process of n coalescing Brownian Motions,

$$t \mapsto Z_t = \big(Z_t(z_1), \ldots Z_t(z_n)\big),$$

where each process $\big(Z_t(z_i); t \geq 0\big)$ is a Brownian motion (relative to some common filtration) starting from $Z_0(z_i) = z_i$, with for each distinct pair $i \neq j$ the process

$$t \mapsto \frac{1}{\sqrt{2}} |Z_t(z_i) - Z_t(z_j)|$$

being a standard Brownian motion on the half-line $[0, \infty)$ with an absorbing barrier at 0. Thus informally $\big(Z_t(z_i); t \geq 0\big)$ and $\big(Z_t(z_j); t \geq 0\big)$ evolve independently until they first meet, after which they coalesce and move together. Such families of coalescing Brownian motions have been well-studied; for some recent works concerning them see [14] and [20]. The main result of this section is a simple formula for the joint distribution function of $\big(Z_t(z_1), \ldots Z_t(z_n)\big)$. A different "exact" solution for coalescing Brownian motion, is derived in [3] by the empty interval method.

For a fixed $t > 0$, the distribution of $Z_t(z)$ is supported on W^n. That part of the distribution supported on the boundary of W^n corresponds to the event that coalescence has occurred. Whereas the restriction of the distribution to the interior W^n (corresponding to no coalescence) is given by Karlin-McGregor formula :

$$\mathbf{P}\big(Z_t(z_i) \in dz_i' \text{ for all } i\big) = \det\big\{\phi_t(z_j' - z_i)\big\} dz'. \tag{39}$$

In fact we can bootstrap from this result to a complete determination of the law of $Z_t(z)$, which can be expressed in the following neat way.

Proposition 9. *For $z, z' \in W^n$, the probability*

$$\mathbf{P}\big(Z_t(z_i) \leq z_i' \textit{ for } 1 \leq i \leq n\big)$$

is given by the determinant of an $n \times n$ matrix with (i,j)th element given by

$$\begin{aligned} &\Phi_t(z_j' - z_i) && \textit{if } i \geq j, \\ &\Phi_t(z_j' - z_i) - 1 && \textit{if } i < j, \end{aligned}$$

where

$$\Phi_t(z) = \int_{-\infty}^{z} \frac{dy}{\sqrt{2\pi t}} \exp\{-y^2/(2t)\}.$$

Proof. First we note that by integrating the Karlin-McGregor formula we obtain

$$\mathbf{P}\big(Z_t(z_1) \leq z_1' < Z_t(z_2) \leq z_2' < \ldots \leq z_{n-1}' < Z_t(z_n) \leq z_n'\big) = \det\big\{\Phi_t(z_j' - z_i)\big\}. \tag{40}$$

We are going to obtain the desired result by showing how the indicator function of the event of interest

$$\big\{Z_t(z_1) \leq z_1', Z_t(z_2) \leq z_2', \ldots, Z_t(z_n) \leq z_n'\big\}$$

can be expanded in terms of the indicator functions of the events of the form

$$\{Z_t(z_{i(1)}) \leq z'_{j(1)} < Z_t(z_{i(2)}) \leq z'_{j(2)} < \ldots < z'_{j(s-1)} < Z_t(z_{i(s)}) \leq z'_{j(s)}\},$$

for increasing subsequences of indices $i(1), i(2), \ldots, i(s)$ and $j(1), j(2), \ldots, j(s)$. To this end I claim firstly that, whenever $z, z' \in W^n$,

$$\det\big\{\mathbf{1}(z_i \leq z'_j)\big\} = \mathbf{1}(z_1 \leq z'_1 < z_2 \leq z'_2 < \ldots < z_n \leq z'_n). \tag{41}$$

I claim secondly that

$$\det\left\{\begin{matrix} \mathbf{1}(z_i \leq z'_j) & i \geq j \\ -\mathbf{1}(z'_j < z_i) & i < j \end{matrix}\right\} = \mathbf{1}(z_1 \leq z'_1, z_2 \leq z'_2, \ldots, z_n \leq z'_n). \tag{42}$$

To prove the first claim take the matrix $M = \big\{\mathbf{1}(z_i \leq z'_j)\big\}$, and subtract from each column (other than the first) the values of the preceding column. The diagonal elements of this new matrix are

$$\mathbf{1}(z_i \leq z'_i) - \mathbf{1}(z_i \leq z'_{i-1}) = \mathbf{1}(z'_{i-1} < z_i \leq z'_i);$$

adopting the convention that $z'_0 = -\infty$. Thus the product of these diagonal elements gives the desired result. We have to check that in the expansion of the determinant this is the only contribution. Suppose that ρ is a permutation, not the identity. Then we can find $i < j$ with $\rho(i) > i$ and $\rho(j) \leq i$. Consider the product of the $(i, \rho(i))$th and $(j, \rho(j))$th elements of the matrix (after the column operations). We obtain

$$\mathbf{1}(z'_{\rho(i)-1} < z_i \leq z'_{\rho(i)})\mathbf{1}(z'_{\rho(j)-1} < z_j \leq z'_{\rho(j)}).$$

This can only be non-zero if both $z'_{\rho(i)-1} < z_i$ and $z_j \leq z'_{\rho(j)}$; but $z_i \leq z_j$ so this would imply $z'_{\rho(i)-1} < z'_{\rho(j)}$. In view of the fact $\rho(i) - 1 \geq \rho(j)$ this is impossible.

Consider the matrix N appearing in the second claim. The product of its diagonal elements gives the desired result. To show that this is the only contribution to the determinant, take ρ a permutation, not equal to the identity and $i < j$ with $\rho(i) > i$ and $\rho(j) \leq i$, as before. Then the product of the $(i, \rho(i))$th and $(j, \rho(j))$th elements of the matrix is

$$-\mathbf{1}(z'_{\rho(i)} < z_i)\mathbf{1}(z_j \leq z'_{\rho(j)})$$

Since $z_i \leq z_j$, for this to be non-zero we would have to have $z'_{\rho(i)} < z'_{\rho(j)}$, which is impossible for $\rho(i) > \rho(j)$.

Let $T = \{-\mathbf{1}(j > i)\}$ be the upper triangular matrix so that $N = M + T$ and consider the Laplace expansion of $\det(M + T)$ in terms of minors. For increasing vectors of subscripts $\mathbf{i}$ and $\mathbf{j}$ let $M[\mathbf{i}, \mathbf{j}]$ denote the corresponding minor of M and let $\tilde{T}[\mathbf{i}, \mathbf{j}]$ be the complementary minor of T so that

$$\det(N) = \det(M + T) = \sum_{\mathbf{i},\mathbf{j}} (-1)^{s(\mathbf{i},\mathbf{j})} M[\mathbf{i}, \mathbf{j}]\tilde{T}[\mathbf{i}, \mathbf{j}],$$

for appropriate signs $s(\mathbf{i}, \mathbf{j})$. Evaluating $\det(N)$ via the second claim, and the minors $M[\mathbf{i}, \mathbf{j}]$ via (general versions of) the first claim we have obtained an expansion of $\mathbf{1}(z_1 \leq z'_1, z_2 \leq z'_2, \ldots, z_n \leq z'_n)$ as a linear combination of terms of the form $\mathbf{1}(z_{i(1)} \leq z'_{j(1)} < z_{i(2)} \leq z'_{j(2)} < \ldots \leq z'_{j(s)})$.

To complete the proof replace, in the above expansion, z_i by $Z_t(z_i)$ and take expectations. On the lefthandside we obtain $\mathbf{P}\big(Z_t(z_1) \le z'_1, Z_t(z_2) \le z'_2, \ldots, Z_t(z_n) \le z'_n\big)$. On the righthandside we have a linear combination of probabilities: $\mathbf{P}\big(Z_t(z_{i(1)}) \le z'_{j(1)} < Z_t(z_{i(2)}) \le z'_{j(2)} < \ldots < z'_{j(s-1)} < Z_t(z_{i(s)}) \le z'_{j(s)}\big)$ each of which can re-written by means of the integrated Karlin-McGregor formula as a minor of the determinant $\det\{\Phi(z'_j - z_i)\}$. And to finish we notice that the righthandside is now the Laplace expansion of the determinant of the sum of matrices $\{\Phi(z'_j - z_i)\}$ and $\{-\mathbf{1}(j > i)\}$. □

The expression just obtained for the distribution of coalescing Brownian motions is closely related to the formula for the transition density of the interlaced Brownian motions given by Proposition 2. In fact differentiating the formula in Proposition 9 and comparing it with the definition of q_t^n, we find that,

$$q_t^n\big((x,y),(x',y')\big) \;=\; (-1)^n \frac{\partial^n}{\partial y_1 \ldots \partial y_n} \frac{\partial^{n+1}}{\partial x'_1 \ldots \partial x'_{n+1}} \mathbf{P}\big(Z_t(x_i) \;\le\; x'_i, Z_t(y_j) \;\le\; y'_j \text{ for all } i,j\big). \tag{43}$$

This represents a duality between the the interlaced Brownian motions and coalescing Brownian motions which generalises the well-known duality between Brownian motion on the half-line $[0,\infty)$ with a reflecting barrier at zero, and Brownian motion on the half-line with an absorbing barrier at zero.

There is interesting alternative way of expressing the equality (43). The Arratia flow or Brownian web is an infinite family of coalescing Brownian motions, with a path starting from every point in space-time. Let $t \in [s,\infty) \mapsto Z_{s,t}(x)$ denote the path starting from (s,x). It is possible to define on the same probability space a dual flow with paths running backwards in time: $s \in (-\infty,t] \mapsto \hat{Z}_{s,t}(x)$ being the path beginning at (t,x). For the details of this construction see [28] and [14]. The flow Z and its dual $\hat{Z}$ are such that for any s,t,x and y, the two events $Z_{s,t}(x) \le y$ and $\hat{Z}_{s,t}(y) \ge x$ differ by a set of zero probability. Using this we may rewrite (43) as

$$q_t^n\big((x,y),(x',y')\big)dx'dy = \mathbf{P}\big(Z_{0,t}(x_i) \in dx'_i, \hat{Z}_{0,t}(y'_j) \in dy_j \text{ for all } i,j\big). \tag{44}$$

An alternative approach to proving this equality would be to combine Proposition 2 with results from [26] and [27] which state that the paths of the dual flow $\hat{Z}$ are Brownian motions reflected off paths of Z. Closely related duality results for the Arratia flow are given in [9] and [13].

6 Proofs of two lemmas

Proof of Lemma 7. The contribution to the determinant defining $r_t(x,x')$ coming from the principal diagonal is equal to the standard heat kernel in $\mathbf{R}^n$. The lemma will follow if we can show all other contributions to the determinant are uniformly negligible as t tends down to 0. Choose $\epsilon > 0$ so that the function f is zero in an 2ϵ-neighbourhood of the boundary of W^n. Then consider a contribution to the determinant corresponding to some permutation ρ which is not the identity. There exist $i < j$ with $\rho(i) > i$ and $\rho(j) \le i$, and the contribution corresponding to ρ consequently contains factors of $\Phi_t^{(i-\rho(i))}(x'_{\rho(i)} - x_i)$ and $\Phi_t^{(j-\rho(j))}(x'_{\rho(j)} - x_j)$. Noting that $j-\rho(j) > 0$ and $i - \rho(i) < 0$ we see that on the set $\{x'_{\rho(i)} - x_i > \epsilon\} \cup \{x'_{\rho(j)} - x_j < -\epsilon\}$ at least one of these factors, and indeed the entire contribution, tends to zero uniformly as t tends down to zero. But on the complement of this set we have $x'_{\rho(i)} \le x_i + \epsilon \le x_j + \epsilon \le x'_{\rho(j)} + 2\epsilon$, and $\rho(j) \le \rho(i)$ implies that $x'_{\rho(j)} \le x'_{\rho(i)}$, so we see that x' is within the 2ϵ-neighbourhood of the boundary of W^n, and does not belong to the support of f. This proves the lemma. □

Proof of Lemma 1. It is convenient to write $z_1 = x_1, z_3 = x_2, \ldots z_{2n+1} = x_n$, and $z_2 = y_1, z_4 = y_2, \ldots, z_{2n} = y_n$, with a corresponding change of notation for x'_i and y'_i also. Now reorder the columns and rows of the determinant defining q^n_t so that the (i,j)th entry is a function of the difference $z'_j - z_i$. We may now argue in the same way as in the preceding proof. Choose $\epsilon > 0$ so that the function f is zero in an 2ϵ-neighbourhood of the boundary of $W^{n+1,n}$. Consider a contribution to the determinant corresponding to some permutation ρ which is not the identity. There exist $i < j$ with $\rho(i) > i$ and $\rho(j) \le i$, and the contribution corresponding to ρ consequently contains factors which are functions of $z'_{\rho(i)} - z_i$ and $z'_{\rho(j)} - z_j$. Noting that $j - \rho(j) > 0$ and $i - \rho(i) < 0$, and checking the entries of the determinant above and below the diagonal we see that on the set $\{z'_{\rho(i)} - z_i > \epsilon\} \cup \{z'_{\rho(j)} - z_j < -\epsilon\}$ at least one of these factors, and indeed the entire contribution, tends to zero uniformly as t tends down to zero. As above, this proves the lemma. □

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