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Depinning of a polymer in a multi-interface medium*

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Abstract

In this paper we consider a model which describes a polymer chain interacting with an infinity of equi-spaced linear interfaces. The distance between two consecutive interfaces is denoted by $T = T_N$ and is allowed to grow with the size N of the polymer. When the polymer receives a positive reward for touching the interfaces, its asymptotic behavior has been derived in [3], showing that a transition occurs when $T_N \approx \log N$. In the present paper, we deal with the so–called *depinning case*, i.e., the polymer is repelled rather than attracted by the interfaces. Using techniques from renewal theory, we determine the scaling behavior of the model for large N as a function of $\{T_N\}_N$, showing that two transitions occur, when $T_N \approx N^{1/3}$ and when $T_N \approx \sqrt{N}$ respectively.

Key words: Polymer Model, Pinning Model, Random Walk, Renewal Theory, Localization/Delocalization Transition.

AMS 2000 Subject Classification: Primary 60K35, 60F05, 82B41.

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1 Introduction and main results

1.1 The model

We consider a $(1+1)$-dimensional model of a polymer depinned at an infinity of equi-spaced horizontal interfaces. The possible configurations of the polymer are modeled by the trajectories of the simple random walk $(i, S_i)_{i\geq 0}$, where $S_0 = 0$ and $(S_i - S_{i-1})_{i\geq 1}$ is an i.i.d. sequence of symmetric Bernouilli trials taking values 1 and -1, that is $P(S_i - S_{i-1} = +1) = P(S_i - S_{i-1} = -1) = \frac{1}{2}$. The polymer receives an energetic penalty $\delta < 0$ each times it touches one of the horizontal interfaces located at heights $\{kT : k \in \mathbb{Z}\}$, where $T \in 2\mathbb{N}$ (we assume that T is even for notational convenience). More precisely, the polymer interacts with the interfaces through the following Hamiltonian:

$$H_{N,\delta}^T(S) := \delta \sum_{i=1}^N \mathbf{1}_{\{S_i \in T\mathbb{Z}\}} = \delta \sum_{k\in\mathbb{Z}} \sum_{i=1}^N \mathbf{1}_{\{S_i = kT\}}, \tag{1.1}$$

where $N \in \mathbb{N}$ is the number of monomers constituting the polymer. We then introduce the corresponding polymer measure $\mathbf{P}_{N,\delta}^T$ (see Figure 1 for a graphical description) by

$$\frac{\mathrm{d}\mathbf{P}_{N,\delta}^T}{\mathrm{d}P}(S) := \frac{\exp\left(H_{N,\delta}^T(S)\right)}{Z_{N,\delta}^T}, \tag{1.2}$$

where the normalizing constant $Z_{N,\delta}^T = E[\exp(H_{N,\delta}^T(S))]$ is called the *partition function*.

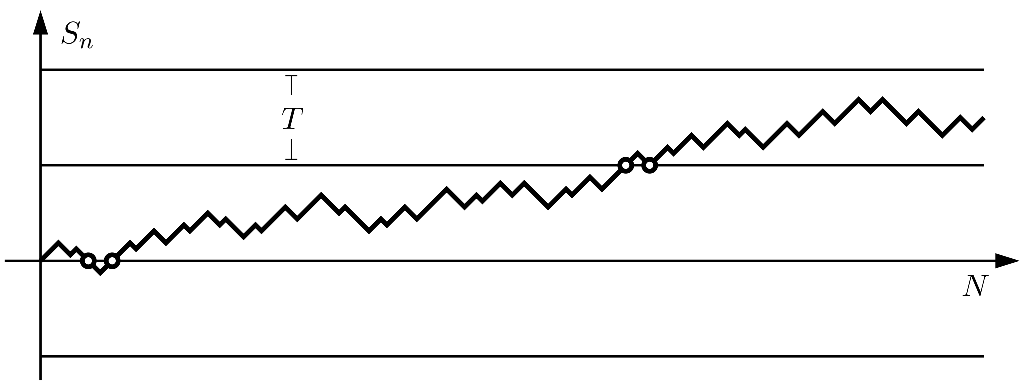

Figure 1: A typical path of $\{S_n\}_{0\leq n\leq N}$ under the polymer measure $\mathbf{P}_{N,\delta}^T$, for $N = 158$ and $T = 16$. The circles indicate the points where the polymer touches the interfaces, that are penalized by $\delta < 0$ each.

We are interested in the case where the interface spacing $T = \{T_N\}_{N\geq 1}$ is allowed to vary with the size N of the polymer. More precisely, we aim at understanding whether and how the asymptotic behavior of the polymer is modified by the interplay between the energetic penalty δ and the growth rate of T_N as $N \to \infty$. In the *attractive case* $\delta > 0$, when the polymer is rewarded rather than penalized to touch an interface, this question was answered in depth in a previous paper [3], to which we also refer for a detailed discussion on the motivation of the model and for an overview on the literature (see also §1.3 below). In the present paper we extend the analysis to the *repulsive case* $\delta < 0$, showing that the behavior of the model is sensibly different from the attractive case.

For the reader's convenience, and in order to get some intuition on our model, we recall briefly the result obtained in [3] for $\delta > 0$. We first set some notation: given a positive sequence $\{a_N\}_N$, we write $S_N \asymp a_N$ to indicate that, on the one hand, S_N/a_N is tight (for every $\varepsilon > 0$ there exists $M > 0$ such that $\mathbf{P}^{T_N}_{N,\delta}\big(|S_N/a_N| > M\big) \le \varepsilon$ for large N) and, on the other hand, that for some $\rho \in (0,1)$ and $\eta > 0$ we have $\mathbf{P}^{T_N}_{N,\delta}\big(|S_N/a_N| > \eta\big) \ge \rho$ for large N. This notation catches the rate of asymptotic growth of S_N somehow precisely: if $S_N \asymp a_N$ and $S_N \asymp b_N$, for some $\varepsilon > 0$ we must have $\varepsilon a_N \le b_N \le \varepsilon^{-1} a_N$, for large N.

Theorem 2 in [3] can be read as follows: for every $\delta > 0$ there exists $c_\delta > 0$ such that

$$S_N \text{ under } \mathbf{P}^{T_N}_{N,\delta} \asymp \begin{cases} \sqrt{N}\, e^{-\frac{c_\delta}{2} T_N}\, T_N & \text{if } T_N - \frac{1}{c_\delta}\log N \to -\infty \\ T_N & \text{if } T_N - \frac{1}{c_\delta}\log N = O(1) \\ 1 & \text{if } T_N - \frac{1}{c_\delta}\log N \to +\infty \end{cases} . \tag{1.3}$$

Let us give an heuristic explanation for these scalings. For fixed $T \in 2\mathbb{N}$, the process $\{S_n\}_{0\le n\le N}$ under $\mathbf{P}^{T}_{N,\delta}$ behaves approximately like a time-homogeneous Markov process (for a precise statement in this direction see §2.2). A quantity of basic interest is the first time $\hat\tau := \inf\{n > 0 : |S_n| = T\}$ at which the polymer visits a neighboring interface. It turns out that for $\delta > 0$ the typical size of $\hat\tau$ is of order $\approx e^{c_\delta T}$, so that until epoch N the polymer will make approximately $N/e^{c_\delta T}$ changes of interface.

Assuming that these arguments can be applied also when $T = T_N$ varies with N, it follows that the process $\{S_n\}_{0\le n\le N}$ jumps from an interface to a neighboring one a number of times which is approximately $u_N := N/e^{c_\delta T_N}$. By symmetry, the probability of jumping to the neighboring upper interface is the same as the probability of jumping to the lower one, hence the last visited interface will be approximately the square root of the number of jumps. Therefore, when $u_N \to \infty$, one expects that S_N will be typically of order $T_N \cdot \sqrt{u_N}$, which matches perfectly with the first line of (1.3). On the other hand, when $u_N \to 0$ the polymer will never visit any interface different from the one located at zero and, because of the attractive reward $\delta > 0$, S_N will be typically at finite distance from this interface, in agreement with the third line of (1.3). Finally, when u_N is bounded, the polymer visits a finite number of different interfaces and therefore S_N will be of the same order as T_N, as the second line of (1.3) shows.

1.2 The main results

Also in the repulsive case $\delta < 0$ one can perform an analogous heuristic analysis. The big difference with respect to the attractive case is the following: under $\mathbf{P}^{T}_{N,\delta}$, the time $\hat\tau$ the polymer needs to jump from an interface to a neighboring one turns out to be typically of order T^3 (see Section 2). Assuming that these considerations can be applied also to the case when $T = T_N$ varies with N, we conclude that, under $\mathbf{P}^{T_N}_{N,\delta}$, the total number of jumps from an interface to the neighboring one should be of order $v_N := N/T_N^3$. One can therefore conjecture that if $v_N \to +\infty$ the typical size of S_N should be of order $T_N \cdot \sqrt{v_N} = \sqrt{N/T_N}$, while if v_N remains bounded one should have $S_N \asymp T_N$. In the case $v_N \to 0$, the polymer will never exit the interval $(-T_N, +T_N)$. However, guessing the right scaling in this case requires some care: in fact, due to the repulsive penalty $\delta < 0$, the polymer will *not* remain close to the interface located at zero, as it were for $\delta > 0$, but it will rather spread in the interval $(-T_N, +T_N)$. We are therefore led to distinguish two cases: if $T_N = O(\sqrt{N})$ then S_N

should be of order T_N, while if $T_N \gg \sqrt{N}$ we should have $S_N \asymp \sqrt{N}$ (of course we write $a_N \ll b_N$ iff $a_N/b_N \to 0$ and $a_N \gg b_N$ iff $a_N/b_N \to +\infty$). We can sum up these considerations in the following formula:

$$S_N \asymp \begin{cases} \sqrt{N/T_N} & \text{if } T_N \ll N^{1/3} \\ T_N & \text{if } (const.)N^{1/3} \le T_N \le (const.)\sqrt{N} \\ \sqrt{N} & \text{if } T_N \gg \sqrt{N} \end{cases} . \tag{1.4}$$

It turns out that these conjectures are indeed correct: the following theorem makes this precise, together with some details on the scaling laws.

Theorem 1.1. *Let $\delta < 0$ and $\{T_N\}_{N\in\mathbb{N}} \in (2\mathbb{N})^{\mathbb{N}}$ be such that $T_N \to \infty$ as $N \to \infty$.*

(1) *If $T_N \ll N^{1/3}$, then $S_N \asymp \sqrt{N/T_N}$. More precisely, there exist two constants $0 < c_1 < c_2 < \infty$ such that for all $a, b \in \mathbb{R}$ with $a < b$ we have for N large enough*

$$c_1 P\big[a < Z \le b\big] \;\le\; \mathbf{P}_{N,\delta}^{T_N}\left(a < \frac{S_N}{C_\delta \sqrt{\frac{N}{T_N}}} \le b \right) \;\le\; c_2 P\big[a < Z \le b\big], \tag{1.5}$$

where $C_\delta := \pi/\sqrt{e^{-\delta} - 1}$ is an explicit positive constant and $Z \sim \mathcal{N}(0,1)$.

(2) *If $T_N \sim (const.)N^{1/3}$, then $S_N \asymp T_N$. More precisely, for every $\varepsilon > 0$ small enough there exist constants $M, \eta > 0$ such that $\forall N \in \mathbb{N}$*

$$\mathbf{P}_{N,\delta}^{T_N}\big(|S_N| \le M\, T_N\big) \;\ge\; 1-\varepsilon\,, \qquad \mathbf{P}_{N,\delta}^{T_N}\big(|S_N| \ge \eta\, T_N\big) \;\ge\; 1-\varepsilon\,. \tag{1.6}$$

(3) *If $N^{1/3} \ll T_N \le (const.)\sqrt{N}$, then $S_N \asymp T_N$. More precisely, for every $\varepsilon > 0$ small enough there exist constants $L, \eta > 0$ such that $\forall N \in \mathbb{N}$*

$$\mathbf{P}_{N,\delta}^{T_N}\big(0 < |S_n| < T_N\,,\ \forall n \in \{L,\dots,N\}\big) \;\ge\; 1-\varepsilon\,, \qquad \mathbf{P}_{N,\delta}^{T_N}\big(|S_N| \ge \eta\, T_N\big) \;\ge\; 1-\varepsilon\,. \tag{1.7}$$

(4) *If $T_N \gg \sqrt{N}$, then $S_N \asymp \sqrt{N}$. More precisely, for every $\varepsilon > 0$ small enough there exist constants $L, M, \eta > 0$ such that $\forall N \in \mathbb{N}$*

$$\mathbf{P}_{N,\delta}^{T_N}\big(0 < |S_n| < M\sqrt{N}\,,\ \forall n \in \{L,\dots,N\}\big) \;\ge\; 1-\varepsilon\,, \qquad \mathbf{P}_{N,\delta}^{T_N}\big(|S_N| \ge \eta\sqrt{N}\big) \;\ge\; 1-\varepsilon\,. \tag{1.8}$$

To have a more intuitive view on the scaling behaviors in (1.4), let us consider the concrete example $T_N \sim (const.)N^a$: in this case we have

$$S_N \asymp \begin{cases} N^{(1-a)/2} & \text{if } 0 \le a \le \frac{1}{3} \\ N^a & \text{if } \frac{1}{3} \le a \le \frac{1}{2} \\ N^{1/2} & \text{if } a \ge \frac{1}{2} \end{cases} . \tag{1.9}$$

As the speed of growth of T_N increases, in a first time (until $a = \frac{1}{3}$) the scaling of S_N decreases, reaching a minimum $N^{1/3}$, after which it increases to reattain the initial value $N^{1/2}$, for $a \ge \frac{1}{2}$.

We have thus shown that the asymptotic behavior of our model displays two transitions, at $T_N \approx \sqrt{N}$ and at $T_N \approx N^{1/3}$. While the first one is somewhat natural, in view of the diffusive behavior of the simple random walk, the transition happening at $T_N \approx N^{1/3}$ is certainly more surprising and somehow unexpected.

Let us make some further comments on Theorem 1.1.

- About regime (1), that is when $T_N \ll N^{1/3}$, we conjecture that equation (1.5) can be strengthened to a full convergence in distribution: $S_N/(C_\delta\sqrt{N/T_N}) \Longrightarrow \mathcal{N}(0,1)$. The reason for the slightly weaker result that we present is that we miss sharp renewal theory estimates for a basic renewal process, that we define in §2.2. As a matter of fact, using the techniques in [7] one can refine our proof and show that the full convergence in distribution holds true in the restricted regime $T_N \ll N^{1/6}$, but we omit the details for conciseness (see however the discussion following Proposition 2.3).

  In any case, equation (1.5) implies that the sequence $\{S_N/(C_\delta\sqrt{N/T_N})\}_N$ is *tight*, and the limit law of any converging subsequence must be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, with density bounded from above and from below by a multiple of the standard normal density.

- The case when $T_N \to T \in \mathbb{R}$ as $N \to \infty$ has not been included in Theorem 1.1 for the sake of simplicity. However a straightforward adaptation of our proof shows that in this case equation (1.5) still holds true, with C_δ replaced by a different (T-dependent) constant $\widehat{C}_\delta(T)$.

- We stress that in regimes (3) and (4) the polymer really touches the interface at zero a finite number of times, after which it does not touch any other interface.

1.3 A link with a polymer in a slit

It turns out that our model $\mathbf{P}_{N,\delta}^T$ is closely related to a model which has received quite some attention in the recent physical literature, the so-called *polymer confined between two walls and interacting with them* [1; 6; 8] (also known as polymer in a slit). The model can be simply described as follows: given $N, T \in 2\mathbb{N}$, take the first N steps of the simple random walk constrained not to exit the interval $\{0, T\}$, and give each trajectory a reward/penalization $\gamma \in \mathbb{R}$ each time it touches 0 or T (one can also consider two different rewards/penalties γ_0 and γ_T, but we will stick to the case $\gamma_0 = \gamma_T = \gamma$). We are thus considering the probability measure $Q_{N,\gamma}^T$ defined by

$$\frac{\mathrm{d}Q_{N,\gamma}^T}{\mathrm{d}P_N^{c,T}}(S) \propto \exp\left(\gamma \sum_{i=1}^N \mathbf{1}_{\{S_i=0 \text{ or } S_i=T\}}\right), \tag{1.10}$$

where $P_N^{c,T}(\cdot) := P(\cdot\,|\,0 \le S_i \le T \text{ for all } 0 \le i \le N)$ is the law of the simple random walk *constrained* to stay between the two walls located at 0 and T.

Consider now the simple random walk *reflected* on both walls 0 and T, which may be defined as $\{\Phi_T(S_n)\}_{n\ge0}$, where $(\{S_n\}_{n\ge0}, P)$ is the ordinary simple random walk and

$$\Phi_T(x) := \min\{[x]_{2T}, 2T - [x]_{2T}\}, \qquad \text{with} \qquad [x]_{2T} := 2T\left(\frac{x}{2T} - \left\lfloor\frac{x}{2T}\right\rfloor\right),$$

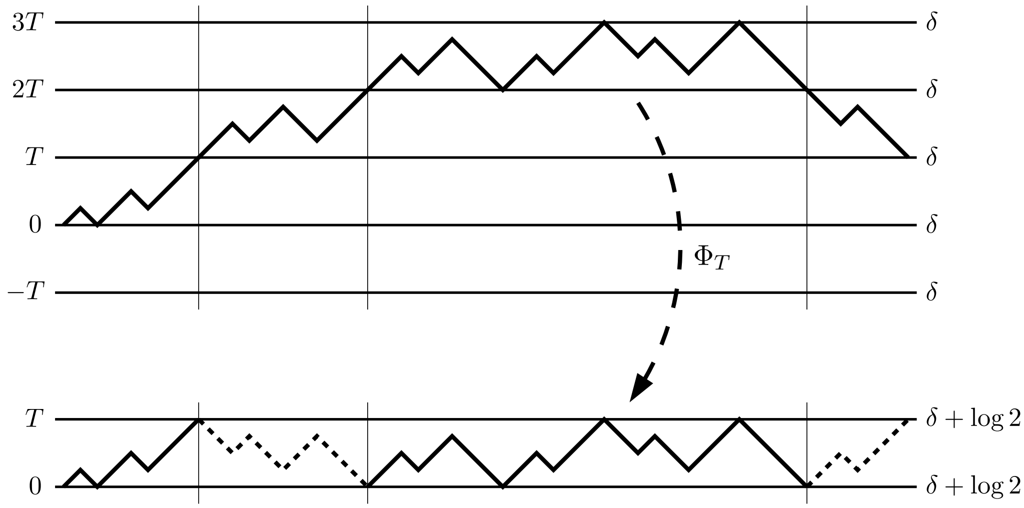

Figure 2: A polymer trajectory in a multi-interface medium transformed, after reflection on the interfaces 0 and T, in a trajectory of polymer in a slit. The dotted lines correspond to the parts of trajectory that appear upside-down after the reflection.

that is, $[x]_{2T}$ denotes the equivalence class of x modulo $2T$ (see Figure 2 for a graphical description). We denote by $P_N^{r,T}$ the law of the first N steps of $\{\Phi_T(S_n)\}_{n\geq 0}$. Of course, $P_N^{r,T}$ is different from $P_N^{c,T}$: the latter is the uniform measure on the simple random walk paths $\{S_n\}_{0\leq n\leq N}$ that stay in $\{0,T\}$, while under the former each such path has a probability which is proportional to $2^{\mathcal{N}_N}$, where $\mathcal{N}_N = \sum_{i=1}^N \mathbf{1}_{\{S_i=0 \text{ or } S_i=T\}}$ is the number of times the path has touched the walls. In other terms, we have

$$\frac{\mathrm{d}P_N^{c,T}}{\mathrm{d}P_N^{r,T}}(S) \;\propto\; \exp\left(-(\log 2)\sum_{i=1}^N \mathbf{1}_{\{S_i=0 \text{ or } S_i=T\}}\right). \tag{1.11}$$

If we consider the reflection under Φ_T of our model, that is the process $\{\Phi_T(S_n)\}_{0\leq n\leq N}$ under $\mathbf{P}_{N,\delta}^T$, whose law will be simply denoted by $\Phi_T(\mathbf{P}_{N,\delta}^T)$, then it comes

$$\frac{\mathrm{d}\Phi_T(\mathbf{P}_{N,\delta}^T)}{\mathrm{d}P_N^{r,T}}(S) \;\propto\; \exp\left(\delta\sum_{i=1}^N \mathbf{1}_{\{S_i=0 \text{ or } S_i=T\}}\right). \tag{1.12}$$

At this stage, a look at equations (1.10), (1.11) and (1.12) points out the link with our model: we have the basic identity $Q_{N,\delta+\log 2}^T = \Phi_T(\mathbf{P}_{N,\delta}^T)$, for all $\delta \in \mathbb{R}$ and $T, N \in 2\mathbb{N}$. In words, the polymer confined between two attractive walls is just the reflection of our model through Φ_T, up to a shift of the pinning intensity by $\log 2$. This allows a direct translation of all our results in this new framework.

Let us describe in detail a particular issue, namely, the study of the model $Q_{N,\gamma}^T$ when $T = T_N$ is allowed to vary with N (this is interesting, e.g., in order to interpolate between the two extreme cases when one of the two quantities T and N tends to ∞ before the other). This problem is considered in [8], where the authors obtain some asymptotic expressions for the partition function $Z_{n,w}(a,b)$ of a polymer in a slit, in the case of two different rewards/penalties (we are following their notation, in which $n = N$, $w = T$, $a = \exp(\gamma_0)$ and $b = \exp(\gamma_T)$) and with the boundary condition $S_N = 0$.

Focusing on the case $a = b = \exp(\gamma)$, we mention in particular equations (6.4)–(6.6) in [8], which for $a < 2$ read as

$$Z_{n,w}(a,a) \approx \frac{(const.)}{n^{3/2}} f_{\text{phase}}\left(\frac{\sqrt{n}}{w}\right), \tag{1.13}$$

where we have neglected a combinatorial factor 2^n (which just comes from a different choice of notation), and where the function $f_{\text{phase}}(x)$ is such that

$$f_{\text{phase}}(x) \to 1 \text{ as } x \to 0, \qquad f_{\text{phase}}(x) \approx x^3 e^{-\pi^2 x^2/2} \text{ as } x \to \infty. \tag{1.14}$$

The regime $a < 2$ corresponds to $\gamma < \log 2$, hence, in view of the correspondence $\delta = \gamma - \log 2$ described above, we are exactly in the regime $\delta < 0$ for our model $\mathbf{P}^T_{N,\delta}$. We recall (1.2) and, with the help of equation (2.11), we can express the partition function with boundary condition $S_N \in (2T)\mathbb{Z}$ as

$$Z^{T,\{S_N \in (2T)\mathbb{Z}\}}_{N,\delta} \sim O(1)\, Z^{T,\{S_N \in T\mathbb{Z}\}}_{N,\delta} \sim O(1)\, e^{\phi(\delta,T)N}\, \mathscr{P}_{\delta,T}(N \in \tau),$$

where, with some abuse of notation, we denote by $O(1)$ a quantity which stays bounded away from 0 and ∞ as $N \to \infty$. In this formula, $\phi(\delta, T)$ is the *free energy* of our model and $(\{\tau_n\}_{n\in\mathbb{Z}^+}, \mathscr{P}_{\delta,T})$ is a basic renewal process, introduced respectively in §2.1 and §2.2 below. In the case when $T = T_N \to \infty$, we can use the asymptotic expansion (2.3) for $\phi(\delta, T)$, which, combined with the bounds in (2.21), gives as $N, T \to \infty$

$$Z^{T,\{S_N \in (2T)\mathbb{Z}\}}_{N,\delta} = \frac{O(1)}{N^{3/2}} \max\left\{1, \left(\frac{\sqrt{N}}{T}\right)^3\right\} \exp\left(-\frac{\pi^2}{2}\frac{N}{T^2} + \frac{2\pi^2}{e^{-\delta}-1}\frac{N}{T^3} + o\left(\frac{N}{T^3}\right)\right).$$

Since $Z^{T,\{S_N \in (2T)\mathbb{Z}\}}_{N,\delta} = Z_{n,w}(a,a)$, we can rewrite this relation using the notation of [8]:

$$Z_{n,w}(a,a) \approx \frac{(const.)}{n^{3/2}} f_{\text{phase}}\left(\frac{\sqrt{n}}{w}\right) g\left(\frac{n^{1/3}}{w}\right), \quad \text{where } g(x) \approx e^{\frac{2\pi^2}{e^{-\delta}-1}x} \text{ as } x \to \infty.$$

We have therefore obtained a refinement of equations (1.13), (1.14). This is linked to the fact that we have gone beyond the first order in the asymptotic expansion of the free energy $\phi(\delta, T)$, making an additional term of the order N/T_N^3 appear. We stress that this new term gives a non-negligible (in fact, exponentially diverging!) contribution as soon as $T_N \ll N^{1/3}$ ($w \ll n^{1/3}$ in the notation of [8]). This corresponds to the fact that, by Theorem 1.1, the trajectories that touch the walls a number of times of the order N/T_N^3 are actually dominating the partition function when $T_N \ll N^{1/3}$. Of course, a higher order expansion of the free energy (cf. Appendix A.1) may lead to further correction terms.

1.4 Outline of the paper

Proving Theorem 1.1 requires to settle some technical tools, partially taken from [3], that we present in Section 2. More precisely, in §2.1 we introduce the free energy $\phi(\delta, T)$ of the polymer and we describe its asymptotic behavior as $T \to \infty$ (for fixed $\delta < 0$). In §2.2 we enlighten a basic correspondence between the polymer constrained to hit one of the interfaces at its right extremity and an explicit renewal process. In §2.3 we investigate further this renewal process, providing estimates on the renewal function, which are of crucial importance for the proof of Theorem 1.1. Sections 3, 4, 5 and 6 are dedicated respectively to the proof of parts (1), (2), (3) and (4) of Theorem 1.1. Finally, some technical results are proven in the appendices.

We stress that the value of $\delta < 0$ is kept fixed throughout the paper, so that the generic constants appearing in the proofs may be δ-dependent.

2 A renewal theory viewpoint

In this section we recall some features of our model, including a basic renewal theory representation, originally proven in [3], and we derive some new estimates.

2.1 The free energy

Considering for a moment our model when $T_N \equiv T \in 2\mathbb{N}$ is fixed, i.e., it does not vary with N, we define the *free energy* $\phi(\delta, T)$ as the rate of exponential growth of the partition function $Z_{N,\delta}^T$ as $N \to \infty$:

$$\phi(\delta, T) := \lim_{N\to\infty} \frac{1}{N} \log Z_{N,\delta}^T = \lim_{N\to\infty} \frac{1}{N} \log E\left(e^{H_{N,\delta}^T}\right). \tag{2.1}$$

Generally speaking, the reason for looking at this function is that the values of δ (if any) at which $\delta \mapsto \phi(\delta, T)$ is not analytic correspond physically to the occurrence of a *phase transition* in the system. As a matter of fact, in our case $\delta \mapsto \phi(\delta, T)$ is analytic on the whole real line, for every $T \in 2\mathbb{N}$. Nevertheless, the free energy $\phi(\delta, T)$ turns out to be a very useful tool to obtain a path description of our model, even when $T = T_N$ varies with N, as we explain in detail in §2.2. For this reason, we now recall some basic facts on $\phi(\delta, T)$, that were proven in [3], and we derive its asymptotic behavior as $T \to \infty$.

We introduce $\tau_1^T := \inf\{n > 0 : S_n \in \{-T, 0, +T\}\}$, that is the first epoch at which the polymer visits an interface, and we denote by $Q_T(\lambda) := E\big(e^{-\lambda \tau_1^T}\big)$ its Laplace transform under the law of the simple random walk. We point out that $Q_T(\lambda)$ is finite and analytic on the interval (λ_0^T, ∞), where $\lambda_0^T < 0$, and $Q_T(\lambda) \to +\infty$ as $\lambda \downarrow \lambda_0^T$ (as a matter of fact, one can give a closed explicit expression for $Q_T(\lambda)$, cf. equations (A.4) and (A.5) in [3]). A basic fact is that $Q_T(\cdot)$ is sharply linked to the free energy: more precisely, we have

$$\phi(\delta, T) = (Q_T)^{-1}(e^{-\delta}), \tag{2.2}$$

for every $\delta \in \mathbb{R}$ (see Theorem 1 in [3]). From this, it is easy to obtain an asymptotic expansion of $\phi(\delta, T)$ as $T \to \infty$, for $\delta < 0$, which reads as

$$\phi(\delta, T) = -\frac{\pi^2}{2T^2}\left(1 - \frac{4}{e^{-\delta} - 1}\,\frac{1}{T} + o\left(\frac{1}{T}\right)\right), \tag{2.3}$$

as we prove in Appendix A.1. We stress that this expansion is for *fixed* $\delta < 0$, in particular the term $o(1/T)$ in (2.3) does depend on δ.

2.2 A renewal theory interpretation

We now recall a basic renewal theory description of our model, that was proven in §2.2 of [3]. We have already introduced the first epoch τ_1^T at which the polymer visits an interface. Let us extend this definition: for $T \in 2\mathbb{N} \cup \{\infty\}$, we set $\tau_0^T = 0$ and for $j \in \mathbb{N}$

$$\tau_j^T := \inf\{n > \tau_{j-1}^T : S_n \in T\mathbb{Z}\} \qquad \text{and} \qquad \varepsilon_j^T := \frac{S_{\tau_j^T} - S_{\tau_{j-1}^T}}{T}, \tag{2.4}$$

where for $T = \infty$ we agree that $T\mathbb{Z} = \{0\}$. Plainly, τ_j^T is the j^{th} epoch at which S visits an interface and ε_j^T tells whether the j^{th} visited interface is the same as the $(j-1)^{\text{th}}$ ($\varepsilon_j^T = 0$), or the one above ($\varepsilon_j^T = 1$) or below ($\varepsilon_j^T = -1$). We denote by $q_T^j(n)$ the joint law of $(\tau_1^T, \varepsilon_1^T)$ under the law of the simple random walk:

$$q_T^j(n) := P\big(\tau_1^T = n, \varepsilon_1^T = j\big). \tag{2.5}$$

Of course, by symmetry we have that $q_T^1(n) = q_T^{-1}(n)$ for every n and T. We also set

$$q_T(n) := P\big(\tau_1^T = n\big) = q_T^0(n) + 2\, q_T^1(n). \tag{2.6}$$

Next we introduce a Markov chain $(\{(\tau_j, \varepsilon_j)\}_{j\geq 0}, \mathscr{P}_{\delta,T})$ taking values in $(\mathbb{N}\cup\{0\})\times\{-1,0,1\}$, defined in the following way: $\tau_0 := \varepsilon_0 := 0$ and under $\mathscr{P}_{\delta,T}$ the sequence of vectors $\{(\tau_j - \tau_{j-1}, \varepsilon_j)\}_{j\geq 1}$ is i.i.d. with marginal distribution

$$\mathscr{P}_{\delta,T}(\tau_1 = n, \varepsilon_1 = j) := e^{\delta}\, q_T^j(n)\, e^{-\phi(\delta,T)n}. \tag{2.7}$$

The fact that the r.h.s. of this equation indeed defines a probability law follows from (2.2), which implies that $Q(\phi(\delta,T)) = E(e^{-\phi(\delta,T)\tau_1^T}) = e^{-\delta}$. Notice that the process $\{\tau_j\}_{j\geq 0}$ alone under $\mathscr{P}_{\delta,T}$ is a (undelayed) *renewal process*, i.e. $\tau_0 = 0$ and the variables $\{\tau_j - \tau_{j-1}\}_{j\geq 1}$ are i.i.d., with step law

$$\mathscr{P}_{\delta,T}(\tau_1 = n) = e^{\delta}\, q_T(n)\, e^{-\phi(\delta,T)n} = e^{\delta}\, P(\tau_1^T = n)\, e^{-\phi(\delta,T)n}. \tag{2.8}$$

Let us now make the link between the law $\mathscr{P}_{\delta,T}$ and our model $\mathbf{P}_{N,\delta}^T$. We introduce two variables that count how many epochs have taken place before N, in the processes τ^T and τ respectively:

$$L_{N,T} := \sup\{n \geq 0 : \tau_n^T \leq N\}, \qquad L_N := \sup\{n \geq 0 : \tau_n \leq N\}. \tag{2.9}$$

We then have the following crucial result (cf. equation (2.13) in [3]): for all $N, T \in 2\mathbb{N}$ and for all $k \in \mathbb{N}$, $\{t_i\}_{1\leq i\leq k} \in \mathbb{N}^k$, $\{\sigma_i\}_{1\leq i\leq k} \in \{-1,0,+1\}^k$ we have

$$\begin{aligned}&\mathbf{P}_{N,\delta}^T\Big(L_{N,T} = k,\ (\tau_i^T, \varepsilon_i^T) = (t_i, \sigma_i),\ 1 \leq i \leq k \,\Big|\, N \in \tau^T\Big)\\ &\qquad = \mathscr{P}_{\delta,T}\Big(L_N = k,\ (\tau_i, \varepsilon_i) = (t_i, \sigma_i),\ 1 \leq i \leq k \,\Big|\, N \in \tau\Big),\end{aligned} \tag{2.10}$$

where $\{N \in \tau\} := \bigcup_{k=0}^{\infty}\{\tau_k = N\}$ and analogously for $\{N \in \tau^T\}$. In words, the process $\{(\tau_j^T, \varepsilon_j^T)\}_j$ under $\mathbf{P}_{N,\delta}^T(\cdot\,|N \in \tau^T)$ is distributed like the Markov chain $\{(\tau_j, \varepsilon_j)\}_j$ under $\mathscr{P}_{\delta,T}(\cdot\,|N \in \tau)$. It is precisely this link with renewal theory that makes our model amenable to precise estimates. Note that the law $\mathscr{P}_{\delta,T}$ carries no explicit dependence on N. Another basic relation we are going to use repeatedly is the following one:

$$E\left[e^{H_{k,\delta}^T(S)}\, \mathbf{1}_{\{k\in\tau^T\}}\right] = e^{\phi(\delta,T)k}\, \mathscr{P}_{\delta,T}\big(k \in \tau\big), \tag{2.11}$$

which is valid for all $k, T \in 2\mathbb{N}$ (cf. equation (2.11) in [3]).

2.3 Some asymptotic estimates

We now derive some estimates that will be used throughout the paper. We start from the asymptotic behavior of $P(\tau_1^T = n)$ as $n \to \infty$. Let us set

$$g(T) := -\log\cos\left(\frac{\pi}{T}\right) = \frac{\pi^2}{2T^2} + O\left(\frac{1}{T^4}\right), \qquad (T \to \infty). \tag{2.12}$$

We then have the following

Lemma 2.1. *There exist positive constants T_0, c_1, c_2, c_3, c_4 such that when $T > T_0$ the following relations hold for every $n \in 2\mathbb{N}$:*

$$\frac{c_1}{\min\{T^3, n^{3/2}\}}\, e^{-g(T)n} \leq P(\tau_1^T = n) \leq \frac{c_2}{\min\{T^3, n^{3/2}\}}\, e^{-g(T)n}, \tag{2.13}$$

$$\frac{c_3}{\min\{T, \sqrt{n}\}}\, e^{-g(T)n} \leq P(\tau_1^T > n) \leq \frac{c_4}{\min\{T, \sqrt{n}\}}\, e^{-g(T)n}. \tag{2.14}$$

The proof of Lemma 2.1 is somewhat technical and is deferred to Appendix B.1. Next we turn to the study of the renewal process $\left(\{\tau_n\}_{n\geq 0}, \mathscr{P}_{\delta,T}\right)$. It turns out that the law of τ_1 under $\mathscr{P}_{\delta,T}$ is essentially split into two components: the first one at $O(1)$, with mass e^{δ}, and the second one at $O(T^3)$, with mass $1 - e^{\delta}$ (although we do not fully prove these results, it is useful to keep them in mind). We start with the following estimates on $\mathscr{P}_{\delta,T}(\tau_1 = n)$, which follow quite easily from Lemma 2.1.

Lemma 2.2. *There exist positive constants T_0, c_1, c_2, c_3, c_4 such that when $T > T_0$ the following relations hold for every $m, n \in 2\mathbb{N} \cup \{+\infty\}$ with $m < n$:*

$$\frac{c_1}{\min\{T^3, k^{3/2}\}}\, e^{-(g(T)+\phi(\delta,T))k} \leq \mathscr{P}_{\delta,T}(\tau_1 = k) \leq \frac{c_2}{\min\{T^3, k^{3/2}\}}\, e^{-(g(T)+\phi(\delta,T))k} \tag{2.15}$$

$$\mathscr{P}_{\delta,T}(m \leq \tau_1 < n) \geq c_3 \left(e^{-(g(T)+\phi(\delta,T))m} - e^{-(g(T)+\phi(\delta,T))n}\right) \tag{2.16}$$

$$\mathscr{P}_{\delta,T}(\tau_1 \geq m) \leq c_4\, e^{-(g(T)+\phi(\delta,T))m}. \tag{2.17}$$

Proof. Equation (2.15) is an immediate consequence of equations (2.8) and (2.13). To prove (2.16), we sum the lower bound in (2.15) over k, estimating $\min\{T^3, k^{3/2}\} \leq T^3$ and observing that by (2.3) and (2.12), for every fixed $\delta < 0$, we have as $T \to \infty$

$$g(T) + \phi(\delta, T) = \frac{2\pi^2}{e^{-\delta} - 1}\, \frac{1}{T^3}\left(1 + o(1)\right). \tag{2.18}$$

To get (2.17), for $m \leq T^2$ there is nothing to prove (provided c_4 is large enough, see (2.18)), while for $m > T^2$ it suffices to sum the upper bound in (2.15) over k. □

Notice that equation (2.15), together with (2.18), shows indeed that the law of τ_1 has a component at $O(T^3)$, which is approximately geometrically distributed. Other important asymptotic relations are the following ones:

$$\mathcal{E}_{\delta,T}(\tau_1) \;=\; \frac{e^{\delta}(e^{-\delta}-1)^2}{2\pi^2}\, T^3 \;+\; o(T^3)\,, \tag{2.19}$$

$$\mathcal{E}_{\delta,T}(\tau_1^2) \;=\; \frac{e^{\delta}(e^{-\delta}-1)^3}{2\pi^4}\, T^6 \;+\; o(T^6)\,, \tag{2.20}$$

which are proven in Appendix A.2. We stress that these relations, together with equation (A.6), imply that, under $\mathcal{P}_{\delta,T}$, the time $\hat{\tau}$ needed to hop from an interface to a neighboring one is of order T^3, and this is precisely the reason why the asymptotic behavior of our model has a transition at $T_N \approx N^{1/3}$, as discussed in the introduction. Finally, we state an estimate on the renewal function $\mathcal{P}_{\delta,T}(n \in \tau)$, which is proven in Appendix B.2.

Proposition 2.3. *There exist positive constants T_0, c_1, c_2 such that for $T > T_0$ and for all $n \in 2\mathbb{N}$ we have*

$$\frac{c_1}{\min\{n^{3/2}, T^3\}} \;\le\; \mathcal{P}_{\delta,T}(n \in \tau) \;\le\; \frac{c_2}{\min\{n^{3/2}, T^3\}}\,. \tag{2.21}$$

Note that the large n behavior of (2.21) is consistent with the classical renewal theorem, because $1/\mathcal{E}_{\delta,T}(\tau_1) \approx T^{-3}$, by (2.19). One could hope to refine this estimate, e.g., proving that for $n \gg T^3$ one has $\mathcal{P}_{\delta,T}(n \in \tau) = (1+o(1))/\mathcal{E}_{\delta,T}(\tau_1)$: this would allow strengthening part (1) of Theorem 1.1 to a full convergence in distribution $S_N/(C_\delta\sqrt{N/T_N}) \Longrightarrow \mathcal{N}(0,1)$. It is actually possible to do this for $n \gg T^6$, using the ideas and techniques of [7], thus strengthening Theorem 1.1 in the restricted regime $T_N \ll N^{1/6}$ (we omit the details).

3 Proof of Theorem 1.1: part (1)

We are in the regime when $N/T_N^3 \to \infty$ as $N \to \infty$. The scheme of this proof is actually very similar to the one of the proof of part (i) of Theorem 2 in [3]. However, more technical difficulties arise in this context, essentially because, in the depinning case ($\delta < 0$), the density of contact between the polymer and the interfaces vanishes as $N \to \infty$, whereas it is strictly positive in the pinning case ($\delta > 0$). For this reason, it is necessary to display this proof in detail.

Throughout the proof we set $v_\delta = (1-e^{\delta})/2$ and $k_N = \lfloor N/\mathcal{E}_{\delta,T_N}(\tau_1)\rfloor$. Recalling (2.4) and (2.9), we set $Y_0^{T_N} = 0$ and $Y_i^{T_N} = \varepsilon_1^{T_N} + \cdots + \varepsilon_i^{T_N}$ for $i \in \{1,\ldots,L_{N,T_N}\}$. Plainly, we can write

$$S_N \;=\; Y^{T_N}_{L_{N,T_N}} \cdot T_N + s_N\,, \qquad \text{with} \quad |s_N| < T_N\,. \tag{3.1}$$

In view of equation (2.19), this relation shows that to prove (1.5) we can equivalently replace $S_N/(C_\delta\sqrt{N/T_N})$ with $Y^{T_N}_{L_{N,T_N}}/\sqrt{v_\delta k_N}$.

3.1 Step 1

Recall (2.7) and set $Y_n = \varepsilon_1 + \cdots + \varepsilon_n$ for all $n \geq 1$. The first step consists in proving that for all $a < b$ in $\overline{\mathbb{R}}$

$$\lim_{N\to\infty} \mathscr{P}_{\delta,T_N}\left(a < \frac{Y_{k_N}}{\sqrt{v_\delta k_N}} \leq b\right) \;=\; P(a < Z \leq b)\,, \tag{3.2}$$

that is, under $\mathscr{P}_{\delta,T_N}$ and as $N \to \infty$ we have $Y_{k_N}/\sqrt{v_\delta k_N} \Longrightarrow Z$, where "$\Longrightarrow$" denotes convergence in distribution.

The random variables $(\varepsilon_1, \ldots, \varepsilon_N)$, defined under $\mathscr{P}_{\delta,T_N}$, are symmetric and i.i.d. . Moreover, they take their values in $\{-1, 0, 1\}$, which together with (A.6) entails

$$\mathscr{E}_{\delta,T_N}(|\varepsilon_1|^3) \;=\; \mathscr{E}_{\delta,T_N}((\varepsilon_1)^2) \;\longrightarrow\; v_\delta \qquad \text{as } N \to \infty. \tag{3.3}$$

Observe that $k_N \to \infty$ as $N \to \infty$ and $\mathscr{E}_{\delta,T_N}(\tau_1) = O(T_N^3)$, by (2.19). Thus, we can apply the Berry Esseèn Theorem that directly proves (3.2) and completes this step. □

3.2 Step 2

Henceforth, we fix a sequence of integers $(V_N)_{N\geq 1}$ such that $T_N^3 \ll V_N \ll N$. In this step we prove that, for all $a < b \in \overline{\mathbb{R}}$, the following convergence occurs, uniformly in $u \in \{0, \ldots, 2V_N\}$:

$$\lim_{N\to\infty} \mathscr{P}_{\delta,T_N}\left(a < \frac{Y_{L_{N-u}}}{\sqrt{v_\delta k_N}} \leq b\right) \;=\; P(a < Z \leq b)\,. \tag{3.4}$$

To obtain (3.4), it is sufficient to prove that, as $N \to \infty$ and under the law $\mathscr{P}_{\delta,T_N}$,

$$U_N := \frac{Y_{k_N}}{\sqrt{v_\delta k_N}} \Longrightarrow Z \qquad \text{and} \qquad G_N := \sup_{u\in\{0,\ldots,2V_N\}} \left|\frac{Y_{L_{N-u}} - Y_{k_N}}{\sqrt{v_\delta k_N}}\right| \Longrightarrow 0\,. \tag{3.5}$$

Step 1 gives directly the first relation in (3.5). To deal with the second relation, we must show that $\mathscr{P}_{\delta,T_N}(G_N \geq \varepsilon) \to 0$ as $N \to \infty$, for all $\varepsilon > 0$. To this purpose, notice that $\{G_N \geq \varepsilon\} \subseteq A_\eta^N \cup B_{\eta,\varepsilon}^N$, where for $\eta > 0$ we have set

$$A_\eta^N := \{L_N - k_N \geq \eta k_N\} \cup \{L_{N-2V_N} - k_N \leq -\eta k_N\} \tag{3.6}$$

$$B_{\eta,\varepsilon}^N := \left\{ \sup\left\{\left|\frac{Y_{k_N+i} - Y_{k_N}}{\sqrt{v_\delta k_N}}\right|,\ i \in \{-\eta k_N, \ldots, \eta k_N\}\right\} \geq \varepsilon \right\}. \tag{3.7}$$

Let us focus on $\mathscr{P}_{\delta,T_N}(A_\eta^N)$. Introducing the centered variables $\widetilde{\tau_k} := \tau_k - k \cdot \mathscr{E}_{\delta,T_N}(\tau_1)$, for $k \in \mathbb{N}$, by the Chebychev inequality we can write (assuming that $(1-\eta)k_N \in \mathbb{N}$ for notational convenience)

$$\begin{aligned}
\mathscr{P}_{\delta,T_N}\big(L_{N-2V_N} - k_N < -\eta k_N\big) \;&=\; \mathscr{P}_{\delta,T_N}\big(\tau_{(1-\eta)k_N} > N - 2V_N\big) \\
&=\; \mathscr{P}_{\delta,T_N}\big(\widetilde{\tau}_{(1-\eta)k_N} > N - 2V_N - (1-\eta)k_N \mathscr{E}_{\delta,T_N}(\tau_1) = \eta N - 2V_N\big) \\
&\leq\; \frac{(1-\eta)k_N \operatorname{Var}_{\delta,T_N}(\tau_1)}{(\eta N - 2V_N)^2} \;\leq\; \frac{N\ \operatorname{Var}_{\delta,T_N}(\tau_1)}{(\eta N - 2V_N)^2\, \mathscr{E}_{\delta,T_N}(\tau_1)}\,.
\end{aligned} \tag{3.8}$$

With the help of the estimates in (2.19), (2.20), we can assert that $\mathrm{Var}_{\delta,T_N}(\tau_1)/\mathscr{E}_{\delta,T_N}(\tau_1) = O(T_N^3)$. Since $N \gg V_N$ and $N \gg T_N^3$, the r.h.s. of (3.8) vanishes as $N \to \infty$. With a similar technique, we prove that $\mathscr{P}_{\delta,T_N}\big(L_N - k_N > \eta k_N\big) \to 0$ as well, and consequently $\mathscr{P}_{\delta,T_N}(A_\eta^N) \to 0$ as $N \to \infty$.

At this stage it remains to show that, for every fixed $\varepsilon > 0$, the quantity $\mathscr{P}_{\delta,T_N}\big(B_{\eta,\varepsilon}^N\big)$ vanishes as $\eta \to 0$, *uniformly in* N. This holds true because $\{Y_n\}_n$ under $\mathscr{P}_{\delta,T_N}$ is a symmetric random walk, and therefore $\{(Y_{k_N+j} - Y_{k_N})^2\}_{j\geq 0}$ is a submartingale (and the same with $j \mapsto -j$). Thus, the maximal inequality yields

$$\mathscr{P}_{\delta,T_N}\big(B_{\eta,\varepsilon}^N\big) \;\leq\; \frac{2}{\varepsilon}\,\frac{\mathscr{E}_{\delta,T_N}\big((Y_{k_N+\eta k_N} - Y_{k_N})^2\big)}{v_\delta k_N} \;\leq\; \frac{2\,\eta\,\mathscr{E}_{\delta,T_N}(\varepsilon_1^2)}{\varepsilon v_\delta} \;\leq\; \frac{2\,\eta}{\varepsilon\, v_\delta}\,. \tag{3.9}$$

We can therefore assert that the r.h.s in (3.9) tends to 0 as $\eta \to 0$, uniformly in N. This completes the step. □

3.3 Step 3

Recall that $k_N = \lfloor N/\mathscr{E}_{\delta,T_N}(\tau_1)\rfloor$. In this step we assume for simplicity that $N \in 2\mathbb{N}$, and we aim at switching from the free measure $\mathscr{P}_{\delta,T_N}$ to $\mathscr{P}_{\delta,T_N}\big(\cdot\,\big|N \in \tau\big)$. More precisely, we want to prove that there exist two constants $0 < c_1 < c_2 < \infty$ such that for all $a < b \in \overline{\mathbb{R}}$ there exists $N_0 > 0$ such that for $N \geq N_0$ and for all $u \in \{0,\ldots,V_N\} \cap 2\mathbb{N}$

$$c_1\, P(a < Z \leq b) \;\leq\; \mathscr{P}_{\delta,T_N}\left(a < \frac{Y_{L_{N-u}}}{\sqrt{v_\delta k_N}} \leq b \,\middle|\, N-u \in \tau\right) \;\leq\; c_2\, P(a < Z \leq b)\,. \tag{3.10}$$

A first observation is that we can safely replace L_{N-u} with $L_{N-u-T_N^3}$ in (3.10). To prove this, since $k_N \to \infty$, the following bound is sufficient: for every $N, M \in 2\mathbb{N}$

$$\sup_{u\in\{0,\ldots,V_N\}\cap 2\mathbb{N}} \mathscr{P}_{\delta,T_N}\left(\Big|Y_{L_{N-u}} - Y_{L_{N-u-T_N^3}}\Big| \geq M \,\middle|\, N-u \in \tau\right) \;\leq\; \frac{(const.)}{M}\,. \tag{3.11}$$

Note that the l.h.s. is bounded above by $\mathscr{P}_{\delta,T_N}\big(\#\{\tau \cap [N-u-T_N^3, N-u)\} \geq M \,\big|\, N-u \in \tau\big)$. By time-inversion and the renewal property we then rewrite this as

$$\begin{aligned} \mathscr{P}_{\delta,T_N}\big(\#\{\tau \cap (0,T_N^3]\} \geq M \,\big|\, N-u\in\tau\big) \;&=\; \mathscr{P}_{\delta,T_N}\big(\tau_M \leq T_N^3 \,\big|\, N-u \in \tau\big) \\ &\leq\; \sum_{n=1}^{T_N^3} \mathscr{P}_{\delta,T_N}\big(\tau_M = n\big) \cdot \frac{\mathscr{P}_{\delta,T_N}\big(N-u-n \in \tau\big)}{\mathscr{P}_{\delta,T_N}\big(N-u\in\tau\big)}\,. \end{aligned} \tag{3.12}$$

Recalling that $N \gg V_N \gg T_N^3$ and using the estimate (2.21), we see that the ratio in the r.h.s. of (3.12) is bounded above by some constant, uniformly for $0 \leq n \leq T_N^3$ and $u \in \{0,\ldots,V_N\} \cap 2\mathbb{N}$. We are therefore left with estimating $\mathscr{P}_{\delta,T_N}\big(\tau_M \leq T_N^3\big)$. Recalling the definition $\widetilde{\tau}_k := \tau_k - k \cdot \mathscr{E}_{\delta,T}(\tau_1) \sim \tau_k - ckT^3$ as $T \to \infty$, where $c > 0$ by (2.19), it follows that for large $N \in \mathbb{N}$ we have

$$\begin{aligned} \mathscr{P}_{\delta,T_N}(\tau \cap [0,T_N^3] \geq M) \;&=\; \mathscr{P}_{\delta,T_N}(\tau_M \leq T_N^3) \\ &\leq\; \mathscr{P}_{\delta,T_N}\left(\widetilde{\tau}_M \leq -\frac{c}{2}\,M\,T_N^3\right) \;\leq\; \frac{4M\;\mathrm{Var}_{\delta,T_N}(\tau_1)}{c^2\,M^2\,T_N^6} \;\leq\; \frac{(const.)}{M}\,, \end{aligned}$$

having applied the Chebychev inequality and (2.20). This proves (3.11).

Let us come back to (3.10). By summing over the last point in τ before $N-u-T_N^3$ (call it $N-u-T_N^3-t$) and the first point in τ after $N-u-T_N^3$ (call it $N-u-T_N^3+r$), using the Markov property we obtain

$$\begin{aligned} &\mathscr{P}_{\delta,T_N}\left(a < \frac{Y_{L_{N-u-T_N^3}}}{\sqrt{v_\delta k_N}} \le b \,\middle|\, N-u \in \tau \right) \\ &= \sum_{t=0}^{N-T_N^3-u} \mathscr{P}_{\delta,T_N}\left(a < \frac{Y_{L_{N-u-T_N^3-t}}}{\sqrt{v_\delta k_N}} \le b \,,\, N-u-T_N^3-t \in \tau \right) \cdot \mathscr{P}_{\delta,T_N}\big(\tau_1 > t\big) \cdot \Theta^u_{\delta,N}(t), \end{aligned} \tag{3.13}$$

where $\Theta^u_{\delta,N}$ is defined by

$$\Theta^u_{\delta,N}(t) \;:=\; \frac{\sum_{r=1}^{T_N^3} \mathscr{P}_{\delta,T_N}\big(\tau_1 = t+r\big)\cdot \mathscr{P}_{\delta,T_N}\big(T_N^3 - r \in \tau\big)}{\mathscr{P}_{\delta,T_N}\big(N-u\in\tau\big)\cdot \mathscr{P}_{\delta,T_N}\big(\tau_1 > t\big)}\,. \tag{3.14}$$

Let us set $\mathscr{I}_N^u := \{0,\dots,N-u-T_N^3\}$. Notice that replacing $\Theta^u_{\delta,N}(t)$ by the constant 1 in the r.h.s. of (3.13), the latter becomes equal to

$$\mathscr{P}_{\delta,T_N}\left(a < \frac{Y_{L_{N-u-T_N^3}}}{\sqrt{v_\delta k_N}} \le b \right). \tag{3.15}$$

Since $u+T_N^3 \le 2V_N$ for large N (because $V_N \gg T_N^3$), equation (3.4) implies that (3.15) converges as $N\to\infty$ to $P(a<Z\le b)$, uniformly for $u\in\{0,\dots,V_N\}\cap 2\mathbb{N}$. Therefore, equation (3.10) will be proven (completing this step) once we show that there exists N_0 such that $\Theta^u_{\delta,N}(t)$ is bounded from above and below by two constants $0<l_1<l_2<\infty$, for $N\ge N_0$ and for all $u\in\{0,\dots,V_N\}$ and $t\in\mathscr{I}_N^u$.

Let us set $K_N(n) := \mathscr{P}_{\delta,T_N}(\tau_1 = n)$ and $u_N(n) := \mathscr{P}_{\delta,T_N}(n\in\tau)$. The lower bound is obtained by restricting the sum in the numerator of (3.14) to $r\in\{1,\dots,T_N^3/2\}$. Recalling that $N\gg V_N\gg T_N^3$, and applying the upper (resp. lower) bound in (2.21) to $u_N(N-u)$ (resp. $u_N(T_N^3-r)$), we have that for large N, uniformly in $u\in\{0,\dots,V_N\}$ and $t\in\mathscr{I}_N^u$,

$$\Theta^u_{\delta,N}(t) \;\ge\; \frac{\sum_{r=1}^{T_N^3/2} K_N(t+r)\cdot u_N\big(T_N^3-r\big)}{u_N\big(N-u\big)\cdot \sum_{j=1}^{\infty} K_N(t+j)} \;\ge\; \frac{c_1}{c_2}\cdot \frac{\sum_{r=1}^{T_N^3/2} K_N(t+r)}{\sum_{j=1}^{\infty} K_N(t+j)}\,. \tag{3.16}$$

Then, we use (2.16) to bound from below the numerator in the r.h.s. of (3.16) and we use (2.17) to bound from above its denominator. This allows to write

$$\Theta^u_{\delta,N}(t) \;\ge\; \frac{c_1\,c_3\,(1-e^{-(g(T_N)+\phi(\delta,T_N))\frac{T_N^3}{2}})}{c_2\,c_4}\,. \tag{3.17}$$

Moreover, (2.18) shows that there exists $m_\delta>0$ such that $g(T_N)+\phi(\delta,T_N)\sim m_\delta/T_N^3$ as $N\to\infty$, which proves that the r.h.s. of (3.17) converges to a constant $c>0$ as N tends to ∞. This completes the proof of the lower bound.

The upper bound is obtained by splitting the r.h.s. of (3.14) into

$$R_N + D_N \ := \ \frac{\sum_{r=1}^{T_N^3/2} K_N(t+r)\cdot u_N(T_N^3-r)}{u_N\big(N-u\big)\cdot\sum_{j=1}^{\infty}K_N(t+j)} + \frac{\sum_{r=1}^{T_N^3/2} K_N(t+T_N^3-r)\cdot u_N(r)}{u_N\big(N-u\big)\cdot\sum_{j=1}^{\infty}K_N(t+j)}. \tag{3.18}$$

The term R_N can be bounded from above by a constant by simply applying the upper bound in (2.21) to $u_N(T_N^3-r)$ for all $r\in\{1,\dots,T_N^3/2\}$ and the lower bound to $u_N(N-u)$. To bound D_N from above, we use the upper bound in (2.15), which, together with the fact that $g(T_N)+\phi(\delta,T_N)\sim m_\delta/T_N^3$, shows that there exists $c>0$ such that for N large enough and $r\in\{1,\dots,T_N^3/2\}$ we have

$$K_N(t+T_N^3-r)\le \frac{c}{T_N^3}\, e^{-(g(T_N)+\phi(\delta,T_N))\,t}. \tag{3.19}$$

Notice also that by (2.16) we can assert that

$$\sum_{j=1}^{\infty} K_N(t+j)\ge c_3 e^{-(g(T_N)+\phi(\delta,T_N))\,t}. \tag{3.20}$$

Finally, (3.19), (3.20) and the fact that $u_N(N-u)\ge c_1/T_N^3$ for all $u\in\{0,\dots,V_N\}$ (by (2.21)) allow to write

$$D_N \ \le\ \frac{c\sum_{r=1}^{T_N^3/2} u_N(r)}{c_1 c_3}. \tag{3.21}$$

By applying the upper bound in (2.21), we can check easily that $\sum_{r=1}^{T_N^3/2} u_N(r)$ is bounded from above uniformly in $N\ge 1$ by a constant. This completes the proof of the step. □

3.4 Step 4

In this step we complete the proof of Theorem 1.1 (1), by proving equation (1.5), that we rewrite for convenience: there exist $0<c_1<c_2<\infty$ such that for all $a<b\in\overline{\mathbb{R}}$ and for large $N\in 2\mathbb{N}$ (for simplicity)

$$c_1\, P(a<Z\le b)\ \le\ \mathbf{P}_{N,\delta}^{T_N}\left(a<\frac{Y_{L_N}^{T_N}}{\sqrt{v_\delta k_N}}\le b\right)\ \le\ c_2\, P(a<Z\le b). \tag{3.22}$$

We recall (2.4) and we start summing over the location $\mu_N:=\tau_{L_{N,T_N}}^{T_N}$ of the last point in τ^{T_N} before N:

$$\mathbf{P}_{N,\delta}^{T_N}\left(a<\frac{Y_{L_{N,T_N}}^{T_N}}{\sqrt{v_\delta k_N}}\le b\right)\ =\ \sum_{\ell=0}^{N}\mathbf{P}_{N,\delta}^{T_N}\left(a<\frac{Y_{L_{N,T_N}}^{T_N}}{\sqrt{v_\delta k_N}}\le b\,\middle|\,\mu_N=N-\ell\right)\cdot\mathbf{P}_{N,\delta}^{T_N}\big(\mu_N=N-\ell\big). \tag{3.23}$$

Of course, only the terms with ℓ even are non-zero. We want to show that the sum in the r.h.s. of (3.23) can be restricted to $\ell\in\{0,\dots,V_N\}$. To that aim, we need to prove that $\sum_{\ell=V_N}^{N}\mathbf{P}_{N,\delta}^{T_N}\big(\mu_N=N-\ell\big)$ tends to 0 as $N\to\infty$. We start by displaying a lower bound on the partition function $Z_{N,\delta}^{T_N}$.

Lemma 3.1. *There exists a constant $c > 0$ such that for N large enough*

$$Z_{N,\delta}^{T_N} \geq \frac{c}{T_N} e^{\phi(\delta,T_N)N}. \tag{3.24}$$

Proof. Summing over the location of μ_N and using the Markov property, together with (2.11), we have

$$\begin{aligned} Z_{N,\delta}^{T_N} &= E\left[e^{H_{N,\delta}^{T_N}(S)}\right] = \sum_{r=0}^{N} E\left[e^{H_{N,\delta}^{T_N}(S)} \mathbf{1}_{\{\mu_N=r\}}\right] \\ &= \sum_{r=0}^{N} E\left[e^{H_{r,\delta}^{T_N}(S)} \mathbf{1}_{\{r\in\tau^{T_N}\}}\right] P(\tau_1^{T_N} > N-r) \\ &= \sum_{r=0}^{N} e^{\phi(\delta,T_N)r} \mathscr{P}_{\delta,T_N}(r\in\tau) P(\tau_1^{T_N} > N-r). \end{aligned} \tag{3.25}$$

From (3.25) and the lower bounds in (2.14) and (2.21), we obtain for N large enough

$$Z_{N,\delta}^{T_N} \geq (const.)\, e^{\phi(\delta,T_N)N} \sum_{r=0}^{N} \frac{e^{-[\phi(\delta,T_N)+g(T_N)](N-r)}}{\min\{\sqrt{N-r+1}, T_N\} \min\{(r+1)^{3/2}, T_N^3\}}. \tag{3.26}$$

At this stage, we recall that $\phi(\delta, T) + g(T) = m_\delta/T^3 + o(1/T^3)$ as $T \to \infty$, with $m_\delta > 0$, by (2.18). Since $T_N^3 \ll N$, we can restrict the sum in (3.26) to $r \in \{N - T_N^3, \ldots, N - T_N^2\}$, for large N, obtaining

$$Z_{N,\delta}^{T_N} \geq (const.) \frac{e^{\phi(\delta,T_N)N}}{T_N^4} \sum_{r=N-T_N^3}^{N-T_N^2} e^{-\left(\frac{m_\delta}{T_N^3} + o\left(\frac{1}{T_N^3}\right)\right)(N-r)} \geq (const.') \frac{e^{\phi(\delta,T_N)N}}{T_N}, \tag{3.27}$$

because the geometric sum gives a contribution of order T_N^3. □

We can now bound from above (using the Markov property and (2.11))

$$\begin{aligned} \sum_{l=0}^{N-V_N} \mathbf{P}_{N,\delta}^{T_N}(\mu_N = \ell) &= \sum_{\ell=0}^{N-V_N} \frac{E\big(\exp\big(H_{\ell,\delta}^{T_N}(S)\big)\mathbf{1}_{\{\ell\in\tau\}}\big) \cdot P\big(\tau_1 > N-\ell\big)}{Z_{N,\delta}^{T_N}} \\ &= \sum_{\ell=0}^{N-V_N} \frac{\mathscr{P}_{\delta,T_N}(\ell\in\tau)\, e^{\phi(\delta,T_N)\ell}\, P\big(\tau_1 > N-\ell\big)}{Z_{N,\delta}^{T_N}} \\ &\leq (const.) \sum_{\ell=0}^{N-V_N} \frac{T_N}{\min\{(\ell+1)^{3/2}, T_N^3\}} \cdot \frac{e^{-[\phi(\delta,T_N)+g(T_N)](N-\ell)}}{\min\{\sqrt{N-\ell}, T_N\}}, \end{aligned} \tag{3.28}$$

where we have used Lemma 3.1 and the upper bounds in (2.14) and (2.21). For notational convenience we set $d(T_N) = \phi(\delta, T_N) + g(T_N)$. Then, the estimate (2.18) and the fact that $V_N \gg T_N^3$ imply that

$$\begin{aligned} \sum_{\ell=0}^{N-V_N} \mathbf{P}_{n,\delta}^{T_N}(\mu_N = \ell) &\leq (const.) e^{-d(T_N)V_N} \sum_{\ell=0}^{N-V_N} \frac{e^{-d(T_N)(N-V_N-\ell)}}{\min\{(\ell+1)^{3/2}, T_N^3\}} \\ &\leq (const.') e^{-d(T_N)V_N} \left(\sum_{\ell=0}^{\infty} \frac{1}{(l+1)^{3/2}} + \sum_{\ell=0}^{\infty} \frac{e^{-d(T_N)(\ell)}}{T_N^3} \right). \end{aligned} \tag{3.29}$$

Since $d(T_N) \sim m_\delta / T_N^3$, with $m_\delta > 0$, and $V_N \gg T_N^3$ we obtain that the l.h.s. of (3.29) tends to 0 as $N \to \infty$.

Thus, we can write

$$\begin{aligned}
&\mathbf{P}_{N,\delta}^{T_N}\left(a < \frac{Y_{L_{N,T_N}}^{T_N}}{\sqrt{v_\delta k_N}} \le b \right) \\
&\quad = \sum_{\ell=0}^{V_N} \mathbf{P}_{N,\delta}^{T_N}\left(a < \frac{Y_{L_{N,T_N}}^{T_N}}{\sqrt{v_\delta k_N}} \le b \,\middle|\, \mu_N = N - \ell \right) \mathbf{P}_{N,\delta}^{T_N}\big(\mu_N = N - \ell\big) \;+\; \varepsilon_N(a,b),
\end{aligned} \tag{3.30}$$

where $\varepsilon_N(a,b)$ tends to 0 as $N \to \infty$, uniformly over $a, b \in \mathbb{R}$. At this stage, by using the Markov property and (2.10) we may write

$$\begin{aligned}
\mathbf{P}_{N,\delta}^{T_N}\left(a < \frac{Y_{L_{N,T_N}}^{T_N}}{\sqrt{v_\delta k_N}} \le b \,\middle|\, \mu_N = N - \ell \right) \;&=\; \mathbf{P}_{N,\delta}^{T_N}\left(a < \frac{Y_{L_{N-\ell,T_N}}^{T_N}}{\sqrt{v_\delta k_N}} \le b \,\middle|\, N - \ell \in \tau^T \right) \\
&= \mathscr{P}_{\delta,T_N}\left(a < \frac{Y_{L_{N-\ell}}}{\sqrt{v_\delta k_N}} \le b \,\middle|\, N - \ell \in \tau \right).
\end{aligned}$$

Plugging this into (3.30), recalling (3.10) and the fact that $\sum_{\ell=0}^{V_N} \mathbf{P}_{N,\delta}^{T_N}(\mu_N = N - \ell) \to 1$ (by (3.29)), it follows that equation (3.22) is proven, and the proof is complete. □

4 Proof of Theorem 1.1: part (2)

We assume that $T_N \sim (const.) N^{1/3}$ and we start proving the first relation in (1.6), that we rewrite as follows: for every $\varepsilon > 0$ we can find $M > 0$ such that for large N

$$\mathbf{P}_{N,\delta}^{T_N}\big(|S_N| > M \cdot T_N\big) \;\le \varepsilon\,.$$

Recalling that L_N^T is the number of times the polymer has touched an interface up to epoch N, see (2.9), we have $|S_N| \le T_N \cdot (L_{N,T_N} + 1)$, hence it suffices to show that

$$\mathbf{P}_{N,\delta}^{T_N}\big(L_{N,T_N} > M\big) \;\le \varepsilon\,. \tag{4.1}$$

By using (2.10) we have

$$\begin{aligned}
\mathbf{P}_{N,\delta}^{T_N}\big(L_{N,T} > M\big) \;&=\; \frac{1}{Z_{N,\delta}^{T_N}}\, E\Big[e^{H_{N,\delta}^{T_N}(S)}\, \mathbf{1}_{\{L_{N,T_N} > M\}} \Big] \\
&=\; \frac{1}{Z_{N,\delta}^{T_N}} \sum_{r=0}^{N} E\Big[e^{H_{r,\delta}^{T_N}(S)}\, \mathbf{1}_{\{L_{r,T_N} > M\}}\, \mathbf{1}_{\{r \in \tau^{T_N}\}} \Big]\, P(\tau_1^{T_N} > N - r) \\
&=\; \frac{1}{Z_{N,\delta}^{T_N}} \sum_{r=0}^{N} e^{\phi(\delta,T_N) r}\, \mathscr{P}_{\delta,T_N}\big(L_{r,T_N} > M,\, r \in \tau^{T_N}\big)\, P(\tau_1^{T_N} > N - r)\,.
\end{aligned}$$

By (2.14) and (2.12) it follows easily that

$$Z^{T_N}_{N,\delta} \;\geq\; P(\tau_1^{T_N} > N) \;\geq\; \frac{(const.)}{T_N}\, e^{-\frac{\pi^2}{2T_N^2}N} \tag{4.2}$$

(note that this bound holds true whenever we have $(const.)N^{1/4} \leq T_N \leq (const.')\sqrt{N}$ for large N). Using this lower bound on $Z^{T_N}_{N,\delta}$, together with the upper bound in (2.14), the asymptotic expansions in (2.18) and (2.12), we obtain

$$\mathbf{P}^{T_N}_{N,\delta}\big(L_{N,T} > M\big) \;\leq\; (const.)\, T_N \sum_{r=0}^{N} \mathscr{P}_{\delta,T_N}\big(L_{r,T_N} > M,\, r \in \tau^{T_N}\big)\, \frac{1}{\min\{\sqrt{N-r+1}, T_N\}}\,.$$

The contribution of the terms with $r > N - T_N^2$ is bounded with the upper bound (2.21):

$$T_N \sum_{r=N-T_N^2}^{N} \frac{1}{T_N^3}\, \frac{1}{\sqrt{N-r+1}} \;\leq\; \frac{(const.)}{T_N} \;\longrightarrow\; 0 \qquad (N \to \infty)\,,$$

while for the terms with $r \leq N - T_N^2$ we get

$$T_N \sum_{r=0}^{N} \mathscr{P}_{\delta,T_N}\big(L_{r,T_N} > M,\, r \in \tau^{T_N}\big)\, \frac{1}{T_N} \;=\; \mathscr{E}_{\delta,T_N}\big((L_{N,T_N} - M)\mathbf{1}_{\{L_{N,T_N} > M\}}\big)\,.$$

Finally, we simply observe that $\{L_{N,T_N} = k\} \subseteq \bigcap_{i=1}^{k} \{\tau_i - \tau_{i-1} \leq N\}$, hence

$$\mathscr{P}_{\delta,T_N}(L_{N,T_N} = k) \;\leq\; \big(\mathscr{P}_{\delta,T_N}(\tau_1 \leq N)\big)^k \;\leq\; c^k\,,$$

with $0 < c < 1$, as it follows from (2.16) and (2.18) recalling that $N = O(T_N^3)$. Putting together the preceding estimates, we have

$$\begin{aligned}
\mathbf{P}^{T_N}_{N,\delta}\big(L_{N,T_N} > M\big) \;&\leq\; (const.)\, \mathscr{E}_{\delta,T_N}\big((L_{N,T_N} - M)\mathbf{1}_{\{L_{N,T_N} > M\}}\big) \\
&=\; (const.) \sum_{k=M+1}^{\infty} (k - M)\, \mathscr{P}_{\delta,T_N}(L_{N,T_N} = k) \\
&\leq\; (const.) \sum_{k=M+1}^{\infty} (k - M)\, c^k \;\leq\; (const.')\, c^M\,,
\end{aligned}$$

and (4.1) is proven by choosing M sufficiently large.

Finally, we prove at the same time the second relations in (1.6) and (1.7), by showing that for every $\varepsilon > 0$ there exists $\eta > 0$ such that for large N

$$\mathbf{P}^{T_N}_{N,\delta}\big(|S_N| \leq \eta\, T_N\big) \;\leq\; \varepsilon\,, \tag{4.3}$$

whenever T_N satisfies $(const.)N^{1/3} \leq T_N \leq (const.')\sqrt{N}$ for large N. Letting P_k denote the law of the simple random walk starting at $k \in \mathbb{N}$ and τ_1^∞ its first return to zero, it follows by Donsker's invariance principle that there exists $c > 0$ such that $\inf_{0 \le k \le \eta T_N} P_k(\tau_1^\infty \le \eta^2 T_N^2\,,\ S_i < T_N\ \forall i \le \tau_1^\infty) \ge c$ for large N. Therefore we may write

$$\begin{aligned}
c\,\mathbf{P}_{N,\delta}^{T_N}\big(|S_N| \le \eta\, T_N\big) \;&=\; \frac{c}{Z_{N,\delta}^{T_N}} \sum_{k=0}^{\eta T_N} E\Big[e^{H_{N,\delta}^{T_N}(S)}\, \mathbf{1}_{\{|S_N|=k\}}\Big] \\
&\le\; \frac{1}{Z_{N,\delta}^{T_N}} \sum_{k=0}^{\eta T_N} \sum_{u=0}^{\eta^2 T_N^2} E\Big[e^{H_{N,\delta}^{T_N}(S)}\, \mathbf{1}_{\{|S_N|=k\}}\Big]\, P_k(\tau_1^\infty = u\,,\ S_i < T_N\ \forall i \le u) \\
&=\; \frac{1}{Z_{N,\delta}^{T_N}} \sum_{k=0}^{\eta T_N} \sum_{u=0}^{\eta^2 T_N^2} E\Big[e^{H_{N+u,\delta}^{T_N}(S)}\, \mathbf{1}_{\{|S_N|=k\}}\, \mathbf{1}_{\{|S_{N+i}|<T_N\ \forall i\le u\}}\, \mathbf{1}_{\{S_{N+u}=0\}}\Big]\,.
\end{aligned}$$

Performing the sum over k, dropping the second indicator function and using equations (2.11), (2.21) and (2.3), we obtain the estimate

$$\begin{aligned}
\mathbf{P}_{N,\delta}^{T_N}\big(|S_N| \le \eta\, T_N\big) \;&\le\; \frac{1}{c\, Z_{N,\delta}^{T_N}} \sum_{u=0}^{\eta^2 T_N^2} E\Big[e^{H_{N+u,\delta}^{T_N}(S)}\, \mathbf{1}_{\{N+u \in \tau^{T_N}\}}\Big] \\
&\le\; \frac{1}{c\, Z_{N,\delta}^{T_N}} \sum_{u=0}^{\eta^2 T_N^2} e^{\phi(\delta,T_N)(N+u)}\, \mathscr{P}_{\delta,T_N}(N+u \in \tau) \;\le\; (const.)\, \frac{\eta^2\, T_N^2}{Z_{N,\delta}^{T_N}\, T_N^3}\, e^{-\frac{\pi^2}{2T_N^2} N}\,.
\end{aligned}$$

Then (4.2) shows that equation (4.3) holds true for η small, and we are done. □

5 Proof of Theorem 1.1: part (3)

We now give the proof of part (3) of Theorem 1.1. More precisely, we prove the first relation in (1.7), because the second one has been proven at the end of Section 4 (see (4.3) and the following lines). We recall that we are in the regime when $N^{1/3} \ll T_N \le (const.)\sqrt{N}$, so that in particular

$$C \;:=\; \inf_{N\in\mathbb{N}} \frac{N}{T_N^2} \;>\; 0\,. \tag{5.1}$$

We start stating an immediate corollary of Proposition 2.3.

Corollary 5.1. *For every $\varepsilon > 0$ there exist $T_0 > 0$, $M_\varepsilon \in 2\mathbb{N}$, $d_\varepsilon > 0$ such that for $T > T_0$*

$$\sum_{k=M_\varepsilon}^{d_\varepsilon T^3} \mathscr{P}_{\delta,T}\big(k \in \tau\big) \;\le\; \varepsilon\,.$$

Note that we can restate the first relation in (1.7) as $\mathbf{P}_{N,\delta}^{T_N}\big(\tau_{L_{N,T_N}}^{T_N} \le L\big) \ge 1-\varepsilon$. Let us define three intermediate quantities, by setting for $l \in \mathbb{N}$

$$B_1(l,N) \;=\; \mathbf{P}_{N,\delta}^{T_N}\big(\tau_{L_{N,T_N}}^{T_N} \le l\big)\, Z_{N,\delta}^{T_N}\,, \tag{5.2}$$

$$B_2(l,N) \;=\; \mathbf{P}_{N,\delta}^{T_N}\big(l < \tau_{L_{N,T_N}}^{T_N} \le N - \eta T_N^2\big)\, Z_{N,\delta}^{T_N}\,, \tag{5.3}$$

$$B_3(N) \;=\; \mathbf{P}_{N,\delta}^{T_N}\big(\tau_{L_{N,T_N}}^{T_N} > N - \eta T_N^2\big)\, Z_{N,\delta}^{T_N}\,, \tag{5.4}$$

where we fix $\eta := C/2$, so that $\eta T_N^2 \le N/2$. The first relation in (1.7) will be proven once we show that for all $\varepsilon > 0$, there exists $l_\varepsilon \in \mathbb{N}$ such that for large N we have

$$\frac{B_2(l_\varepsilon, N)}{B_1(l_\varepsilon, N)} \le \varepsilon \qquad \text{and} \qquad \frac{B_3(N)}{B_1(l_\varepsilon, N)} \le \varepsilon\,. \tag{5.5}$$

We start giving a simple lower bound of B_1: since $\{\tau^{T_N}_{L_{N,T_N}} \le l\} \supseteq \{\tau_1^{T_N} > N\}$, we have

$$B_1(l,N) \ge E\Big[e^{H^{T_N}_{N,\delta}(S)} \mathbf{1}_{\{\tau_1^{T_N} > N\}}\Big] = P\big(\tau_1^{T_N} > N\big) \ge \frac{(const.)}{T_N} e^{-\frac{\pi^2}{2T_N^2}N}, \tag{5.6}$$

having applied the lower bound in (2.14). Next we consider B_2. Summing over the possible values of $\tau^{T_N}_{L_{N,T_N}}$ and using (2.11), we have

$$\begin{aligned} B_2(l,N) &= \sum_{n=l+1}^{N-\eta T_N^2} E\Big[e^{H^{T_N}_{n,\delta}(S)} \mathbf{1}_{\{n\in\tau^{T_N}\}}\Big] \cdot P\big(\tau_1^{T_N} > N-n\big) \\ &= \sum_{n=l+1}^{N-\eta T_N^2} \mathscr{P}_{\delta,T_N}(n\in\tau)\, e^{\phi(\delta,T_N)n}\, P\big(\tau_1^{T_N} > N-n\big) \\ &\le \frac{(const.)}{T_N} e^{-\frac{\pi^2}{2T_N^2}N} \left(\sum_{n=l+1}^{N} \mathscr{P}_{\delta,T_N}(n\in\tau) \right), \end{aligned} \tag{5.7}$$

where we have applied the upper bound in (2.14) and the equalities (2.3) and (2.12) (we also assume that $\eta T_N^2 \in \mathbb{N}$ for simplicity). Since $N \ll T_N^3$, by Corollary 5.1 we can fix $l = l_\varepsilon$ depending only on ε such that $B_2/B_1 \le \varepsilon$ (recall (5.6)). Finally we analyze $B_3(N)$: in analogy with (5.7) we write

$$\begin{aligned} B_3(N) &\le \sum_{n=N-\eta T_N^2+1}^{N} \mathscr{P}_{\delta,T_N}(n\in\tau)\, e^{\phi(\delta,T_N)n}\, P\big(\tau_1^{T_N} > N-n\big) \\ &\le e^{-\frac{\pi^2}{2T_N^2}N} \frac{(const.)}{T_N^3} \sum_{n=N-\eta T_N^2+1}^{N} \frac{(const.')}{\sqrt{N-n+1}} \le (const.'')\, e^{-\frac{\pi^2}{2T_N^2}N} \frac{1}{T_N^2}, \end{aligned}$$

where we have applied the upper bounds in (2.14) and (2.21) (note that $n \ge (C/2)\,T_N^2$). Therefore $B_3/B_1 \le \varepsilon$ for N large, and the first relation in (1.7) is proven.

6 Proof of Theorem 1.1: part (4)

We now assume that $T_N \gg \sqrt{N}$, that is

$$\lim_{N\to\infty} \frac{N}{T_N^2} = 0\,. \tag{6.1}$$

The proof is analogous to the proof of part (3), given in Section 5. We set for $l \in \mathbb{N}$

$$B_1(l,N) \;=\; \mathbf{P}_{N,\delta}^{T_N}\big(\tau_{L_{N,T_N}}^{T_N} < l\big)\, Z_{N,\delta}^{T_N}\,, \tag{6.2}$$

$$B_2(l,N) \;=\; \mathbf{P}_{N,\delta}^{T_N}\big(l \le \tau_{L_{N,T_N}}^{T_N} \le N/2\big)\, Z_{N,\delta}^{T_N}\,, \tag{6.3}$$

$$B_3(N) \;=\; \mathbf{P}_{N,\delta}^{T_N}\big(\tau_{L_{N,T_N}}^{T_N} > N/2\big)\, Z_{N,\delta}^{T_N}\,, \tag{6.4}$$

and we first show that for every $\varepsilon > 0$ we can choose $l_\varepsilon \in \mathbb{N}$ such that for large N

$$\frac{B_2(l_\varepsilon,N)}{B_1(l_\varepsilon,N)} \;\le\; \varepsilon \qquad \text{and} \qquad \frac{B_3(N)}{B_1(l_\varepsilon,N)} \;\le\; \varepsilon\,. \tag{6.5}$$

We start with a lower bound: since $\{\tau_{L_{N,T_N}}^{T_N} \le l\} \supseteq \{\tau_1^{T_N} > N\}$, by (2.14) we have

$$B_1(l,N) \;\ge\; E\Big[e^{H_{N,\delta}^{T_N}(S)}\,\mathbf{1}_{\{\tau_1^{T_N} > N\}}\Big] \;=\; P\big(\tau_1^{T_N} > N\big) \;\ge\; \frac{(const.)}{\sqrt{N}}\,. \tag{6.6}$$

Next consider B_2. Summing over the possible values of $\tau_{L_{N,T_N}}^{T_N}$ and using (2.11), we have

$$\begin{aligned} B_2(l,N) \;&=\; \sum_{k=l}^{N/2} E\Big[e^{H_{k,\delta}^{T_N}(S)}\,\mathbf{1}_{\{k\in\tau^{T_N}\}}\Big]\cdot P\big(\tau_1^{T_N} > N-k\big) \\ &=\; \sum_{k=l}^{N/2} \mathscr{P}_{\delta,T_N}(k\in\tau)\, e^{\phi(\delta,T_N)k}\, P\big(\tau_1^{T_N} > N-k\big) \end{aligned} \tag{6.7}$$

(we assume that $N/2 \in \mathbb{N}$ for notational convenience). By the upper bound in (2.14) we have $P\big(\tau_1^{T_N} > N-k\big) \le (const.')/\sqrt{N-k}$. Since $\phi(\delta,T_N) \le 0$, we obtain

$$B_2(l,N) \;\le\; \frac{(const.)}{\sqrt{N}} \left(\sum_{k=l}^{N} \mathscr{P}_{\delta,T_N}(k\in\tau)\right)\,,$$

which can be made arbitrarily small by fixing $l = l_\varepsilon$, thanks to Corollary 5.1, hence we have proven that $B_2/B_1 \le \varepsilon$ for large N. In a similar fashion, for B_3 we can write

$$\begin{aligned} B_3 \;&=\; \sum_{n=N/2+1}^{N} \mathscr{P}_{\delta,T_N}(n\in\tau)\, e^{\phi(\delta,T_N)n}\, P\big(\tau_1^{T_N} > N-n\big) \\ &\le\; (const.) \sum_{n=N/2+1}^{N} \frac{1}{n^{3/2}}\,\frac{1}{\sqrt{N-n+1}} \;\le\; \frac{(const.)}{(N/2)^{3/2}} \sum_{n=N/2+1}^{N} \frac{1}{\sqrt{N-n+1}} \;\le\; \frac{(const.')}{N}\,, \end{aligned}$$

where we have used the upper bounds in (2.21) and (2.14) as well as the fact that $\phi(\delta,T_N)n = o(1)$ uniformly in $n \le N$, by (2.3). Therefore for large N we have $B_3/B_1 \le \varepsilon$ and equation (6.5) is proven. This implies that, for every $\varepsilon > 0$, there exists $l_\varepsilon \in \mathbb{N}$ such that for large N

$$\mathbf{P}_{N,\delta}^{T_N}\big(\tau_{L_{N,T_N}}^{T_N} < l_\varepsilon\big) \ge 1-\varepsilon\,. \tag{6.8}$$

Next we turn to the proof of the both relations in (1.8) at the same time. In view of (6.8), it suffices to show that, for every $\varepsilon > 0$, we can choose $M \in \mathbb{N}$ and $\eta > 0$ such that for large N

$$\mathbf{P}_{N,\delta}^{T_N}\left(\left\{ \tau_{L_{N,T_N}}^{T_N} < l_\varepsilon \right\} \cap \left(\left\{ \sup_{n \le N} |S_n| > M\sqrt{N} \right\} \cup \left\{ |S_N| \le \eta\sqrt{N} \right\} \right) \right) \le \varepsilon . \tag{6.9}$$

Summing over the values of $\tau_{L_{N,T_N}}^{T_N}$ and using (2.11), the l.h.s. of (6.9) is bounded from above by

$$\sum_{u=0}^{l_\varepsilon - 1} \mathscr{P}_{\delta,T_N}(u \in \tau)\, e^{\phi(\delta,T_N)u} A_{N,u}(M,\eta),$$

where

$$A_{N,u}(M,\eta) := \frac{P\left(\{\tau_1^{T_N} > N-u\} \cap \left(\{\sup_{n\le N-u}|S_n| > M\sqrt{N}\} \cup \{|S_{N-u}| \le \eta\sqrt{N}\}\right)\right)}{Z_{N,\delta}^{T_N}} .$$

Therefore equation (6.9) will be proven once we show that we can chose M, η such that $A_{N,u}(M,\eta) \le \varepsilon/l_\varepsilon$, for N large. For the partition function appearing in the denominator, applying (6.2) and (6.6) we easily obtain $Z_{N,\delta}^{T_N} \ge (const.)/\sqrt{N}$. Setting $N_u := N - u$ for short, the numerator in the definition of $A_{N,u}(M,\eta)$ can be bounded from above by

$$P\left(|S_i| > 0,\, \forall i \le N_u\right) \cdot P\left(\left\{ \sup_{n \le N_u} |S_n| > M\sqrt{N} \right\} \cup \left\{ |S_{N_u}| \le \eta\sqrt{N} \right\} \,\middle|\, |S_i| > 0,\, \forall i \le N_u \right).$$

It is well-known [4] that $P\left(|S_i| > 0,\, \forall i \le n\right) \le (const.)/\sqrt{n}$. Recalling the weak convergence of the random walk conditioned to stay positive toward the Brownian meander [2], we conclude that for every fixed $u \le l_\varepsilon$ and for large N we have the bound

$$A_{N,u}(M,\eta) \le (const.)\, P\left(\left\{ \sup_{0 \le t \le 1} m_t > M \right\} \cup \left\{ m_1 \le \eta \right\} \right). \tag{6.10}$$

We can then choose M large and η small so as to satisfy the desired bound $A_{N,u}(M,\eta) \le \varepsilon/l_\varepsilon$, and the proof is completed. □

A On the free energy

A.1 Free energy estimates

We determine the asymptotic behavior of $\phi(\delta, T)$ as $T \to \infty$, for fixed $\delta < 0$. By Theorem 1 in [3], we have $Q_T\left(\phi(\delta,T)\right) = e^{-\delta}$, and furthermore

$$Q_T(\lambda) = 1 + \sqrt{e^{-2\lambda} - 1} \cdot \frac{1 - \cos\left(T \arctan\sqrt{e^{-2\lambda}-1}\right)}{\sin\left(T \arctan\sqrt{e^{-2\lambda}-1}\right)},$$

see, e.g., equation (A.5) in [3]. If we set

$$\gamma = \gamma(\delta, T) := \arctan \sqrt{e^{-2\phi(\delta,T)} - 1}\,, \tag{A.1}$$

we can then write

$$\widetilde{Q}_T\big(\gamma(\delta,T)\big) \;=\; e^{-\delta} \qquad \text{where} \qquad \widetilde{Q}_T(\gamma) \;=\; 1 + \tan\gamma \cdot \frac{1-\cos(T\gamma)}{\sin(T\gamma)}\,. \tag{A.2}$$

Note that $\gamma \mapsto \widetilde{Q}_T(\gamma)$ is an increasing function with $\widetilde{Q}_T(0) = 1$ and $\widetilde{Q}_T(\gamma) \to +\infty$ as $\gamma \uparrow \frac{\pi}{T}$, hence $0 < \gamma(\delta,T) < \frac{\pi}{T}$. So we have to study the equation $\widetilde{Q}_T(\gamma) = e^{-\delta}$ for $0 < \gamma < \frac{\pi}{T}$. An asymptotic expansion yields

$$(1+o(1))\,\gamma \cdot \frac{1-\cos(T\gamma)}{\sin(T\gamma)} \;=\; e^{-\delta} - 1\,,$$

where here and in the sequel $o(1)$ is to be understood as $T \to \infty$ with $\delta < 0$ fixed. Setting $x = T\gamma$ gives

$$(1+o(1))\,x \cdot \frac{1-\cos x}{\sin x} \;=\; T(e^{-\delta} - 1)\,,$$

where $0 < x < \pi$. Since the r.h.s. diverges as $T \to \infty$, x must tend to π and a further expansion yields

$$(1+o(1))\,\frac{2\pi}{\pi - x} = T(e^{-\delta} - 1)\,,$$

from which we get $x = \pi - \frac{2\pi}{e^{-\delta}-1}\frac{1}{T}(1+o(1))$ and hence, since $\gamma(\delta,T) = \frac{x}{T}$,

$$\gamma(\delta,T) \;=\; \frac{\pi}{T} - \frac{2\pi}{e^{-\delta}-1}\,\frac{1}{T^2}(1+o(1))\,. \tag{A.3}$$

Recalling (A.1), we have

$$\sqrt{e^{-2\phi(\delta,T)} - 1} \;=\; \tan\left(\frac{\pi}{T} - \frac{2\pi}{e^{-\delta}-1}\,\frac{1}{T^2}(1+o(1))\right).$$

Since the function $\lambda \mapsto \arctan\sqrt{e^{-2\lambda} - 1}$ is decreasing and continuously differentiable, with non-vanishing first derivative, it follows that

$$\phi(\delta,T) \;=\; -\frac{\pi^2}{2T^2}\left(1 - \frac{4}{e^{-\delta}-1}\,\frac{1}{T} + o\left(\frac{1}{T}\right)\right), \tag{A.4}$$

so that equation (2.3) is proven. □

A.2 Further estimates

We now derive some asymptotic properties of the variables (τ_1, ε_1) under $\mathscr{P}_{\delta,T}$, as $T \to \infty$ and for fixed $\delta < 0$.

We first focus on $Q_T^1(\phi(\delta,T))$, where $Q_T^1(\lambda) := E(e^{-\lambda\tau_1^T}\,\mathbf{1}_{\{\varepsilon_1^T = 1\}}) = \sum_{n\in\mathbb{N}} e^{-\lambda n} q_T^1(n)$. In analogy with the computations above, by equation (A.5) in [3] we can write

$$Q_T^1(\phi(\delta,T)) \;=\; \widetilde{Q}_T^1(\gamma(\delta,T))\,, \qquad \text{where} \qquad \widetilde{Q}_T^1(\gamma) \;:=\; \frac{\tan\gamma}{2\,\sin(T\gamma)}\,,$$

so that from (A.3) we obtain as $T \to \infty$

$$Q_T^1(\phi(\delta,T)) \;=\; \frac{\pi}{T}\,\frac{1}{2\cdot\frac{2\pi}{e^{-\delta}-1}\,\frac{1}{T}}\,(1+o(1)) \;\longrightarrow\; \frac{e^{-\delta}-1}{4}\,. \tag{A.5}$$

In particular, by (2.7) we can write as $T \to \infty$

$$\mathscr{E}_{\delta,T}(\varepsilon_1^2) \;=\; 2\,\mathscr{P}_{\delta,T}(\varepsilon_1=+1) \;=\; 2\,e^{\delta}\,Q_T^1(\phi(\delta,T)) \;\longrightarrow\; \frac{1-e^{\delta}}{2}\,. \tag{A.6}$$

Next we determine the asymptotic behavior of $\mathscr{E}_{\delta,T}(\tau_1)$ as $T \to \infty$ for fixed $\delta < 0$. Recalling (2.8) we can write

$$\mathscr{E}_{\delta,T}(\tau_1) \;=\; e^{\delta}\sum_{n\in\mathbb{N}} n\,q_T(n)\,e^{-\phi(\delta,T)n} \;=\; -e^{\delta}\cdot Q_T'(\phi(\delta,T))\,, \tag{A.7}$$

$$\mathscr{E}_{\delta,T}(\tau_1^2) \;=\; e^{\delta}\sum_{n\in\mathbb{N}} n^2\,q_T(n)\,e^{-\phi(\delta,T)n} \;=\; e^{\delta}\cdot Q_T''(\phi(\delta,T))\,, \tag{A.8}$$

hence the problem is to determine $Q_T'(\lambda)$ for $\lambda = \phi(\delta,T)$. Introducing the function $\gamma(\lambda) := \arctan\sqrt{e^{-2\lambda}-1}$ and recalling (A.2), since $Q_T = \widetilde{Q}_T \circ \gamma$ it follows that

$$Q_T'(\lambda) \;=\; \widetilde{Q}_T'(\gamma(\lambda))\cdot\gamma'(\lambda)\,, \tag{A.9}$$

$$Q_T''(\lambda) \;=\; \gamma''(\lambda)\cdot\widetilde{Q}_T'(\gamma(\lambda)) + (\gamma'(\lambda))^2\cdot\widetilde{Q}_T''(\gamma(\lambda))\,. \tag{A.10}$$

By direct computation

$$\widetilde{Q}_T'(\gamma) \;=\; \frac{1\cos(T\gamma)}{\sin(T\gamma)}\cdot\left(\frac{1}{\cos^2\gamma} \;+\; \frac{T\tan\gamma}{\sin(T\gamma)}\right), \tag{A.11}$$

$$\widetilde{Q}_T''(\gamma) \;=\; \frac{1-\cos(T\gamma)}{\sin(T\gamma)}\cdot\left(\frac{2T}{\sin(T\gamma)\cos^2 x} \;+\; \frac{2\sin\gamma}{\cos^3 x} + \frac{T^2\tan\gamma}{\sin^2(T\gamma)}(1-\cos(T\gamma))\right), \tag{A.12}$$

and

$$\gamma'(\lambda) \;=\; -\frac{1}{\sqrt{e^{-2\lambda}-1}}\,, \qquad \gamma''(\lambda) \;=\; -\frac{e^{-2\lambda}}{(e^{-2\lambda}-1)^{3/2}}\,.$$

Recalling (A.7) and (A.1), we have

$$\mathscr{E}_{\delta,T}(\tau_1) \;=\; -e^{\delta}\cdot\widetilde{Q}_T'(\gamma(\delta,T))\cdot\gamma'(\phi(\delta,T))\,.$$

Now the asymptotic behaviors (A.3) and (A.4) give

$$\widetilde{Q}_T'(\gamma(\delta,T)) \;=\; \frac{e^{-\delta}-1}{\pi}\,T \;+\; \frac{(e^{-\delta}-1)^2}{2\pi}\,T^2 \;+\; o(T)\,, \qquad \gamma'(\phi(\delta,T)) \;=\; -\frac{T}{\pi} \;+\; o(T)\,,$$

and

$$\widetilde{Q}_T''(\gamma(\delta,T)) \;=\; \frac{(e^{-\delta}-1)^3}{2\pi^2}\,T^4 \;+\; o(T^4)\,, \qquad \gamma''(\phi(\delta,T)) \;=\; -\frac{T^3}{\pi^3} \;+\; o(T^3)\,.$$

Combining the preceding relations, we obtain

$$\mathscr{E}_{\delta,T}(\tau_1) \;=\; \frac{e^{\delta}(e^{-\delta}-1)^2}{2\pi^2}\,T^3 \;+\; \frac{1-e^{\delta}}{\pi^2}\,T^2 \;+\; o(T^2)\,,$$

$$\mathscr{E}_{\delta,T}(\tau_1^2) \;=\; \frac{e^{\delta}(e^{-\delta}-1)^3}{2\pi^4}\,T^6 \;+\; o(T^6)\,,$$

which show that equations (2.19) and (2.20) hold true. □

B Renewal theory estimates

This section collects the proofs of Lemma 2.1 and Proposition 2.3.

B.1 Proof of Lemma 2.1

We recall that, by equation (5.8) in Chapter XIV of [4], we have the following explicit formula for $q_T^j(n)$ (defined in (2.5)):

$$
\begin{aligned}
q_T^0(n) &= \left(\frac{2}{T} \sum_{\nu=1}^{\lfloor (T-1)/2 \rfloor} \cos^{n-2}\left(\frac{\pi\nu}{T}\right) \sin^2\left(\frac{\pi\nu}{T}\right)\right) \cdot \mathbf{1}_{\{n \text{ is even}\}}\,, \\
q_T^1(n) &= \left(\frac{1}{T} \sum_{\nu=1}^{\lfloor (T-1)/2 \rfloor} (-1)^{\nu+1} \cos^{n-2}\left(\frac{\pi\nu}{T}\right) \sin^2\left(\frac{\pi\nu}{T}\right)\right) \cdot \mathbf{1}_{\{n-T \text{ is even}\}}\,,
\end{aligned}
\tag{B.1}
$$

hence $q_T(n) = P(\tau_1^T = n) = q_T^0(n) + 2q_T^1(n)$ is given for n and T even by

$$
q_T(n) = \frac{4}{T} \sum_{\nu=1}^{\lfloor (T+2)/4 \rfloor} \cos^{n-2}\left(\frac{(2\nu-1)\pi}{T}\right) \sin^2\left(\frac{(2\nu-1)\pi}{T}\right), \tag{B.2}
$$

(notice that $\lfloor (T-1)/2 \rfloor = T/2 - 1$ for T even).
We split (B.2) in the following way: we fix $\varepsilon > 0$ and we write

$$
P(\tau_1^T = n) = V_0(n) + V_1(n) + V_2(n), \tag{B.3}
$$

where we set

$$
\begin{aligned}
V_0(n) &:= \frac{4}{T} \cos^{n-2}\left(\frac{\pi}{T}\right) \sin^2\left(\frac{\pi}{T}\right), \\
V_1(n) &:= \frac{4}{T} \sum_{\nu=2}^{\lfloor \varepsilon T \rfloor} \cos^{n-2}\left(\frac{(2\nu-1)\pi}{T}\right) \sin^2\left(\frac{(2\nu-1)\pi}{T}\right), \\
V_2(n) &:= \frac{4}{T} \sum_{\nu=\lfloor \varepsilon T \rfloor+1}^{\lfloor (T+2)/4 \rfloor} \cos^{n-2}\left(\frac{(2\nu-1)\pi}{T}\right) \sin^2\left(\frac{(2\nu-1)\pi}{T}\right).
\end{aligned}
$$

Plainly, as $T \to \infty$ we have

$$
V_0(n) = \frac{4\pi^2}{T^3} (1 + o(1))\, e^{-g(T)n}, \tag{B.4}
$$

where $o(\cdot)$ refer as $T \to \infty$, *uniformly in* n. Next we focus on V_1: for ε small enough and $x \in [0, \pi\varepsilon]$ we have $\log(\cos(x)) \le -\frac{x^2}{3}$, and since $\sin(x) \le x$ we have

$$
\begin{aligned}
V_1(n) &\le \frac{4\pi^2}{T} \sum_{\nu=2}^{\lfloor \varepsilon T \rfloor} \left(\frac{2\nu-1}{T}\right)^2 e^{-\frac{(n-2)\pi^2}{3}(\frac{2\nu-1}{T})^2} \le (const.) \int_{2/T}^{\infty} x^2\, e^{-\frac{\pi^2}{3} n x^2}\, \mathrm{d}x \\
&= \frac{(const.)}{n^{3/2}} \int_{2\sqrt{n}/T}^{\infty} y^2\, e^{-\frac{\pi^2}{3} y^2}\, \mathrm{d}y \le \frac{(const.')}{n^{3/2}} e^{-\frac{\pi^2 n}{T^2}} \le \frac{(const.')}{n^{3/2}} e^{-g(T)n},
\end{aligned}
\tag{B.5}
$$

where the last inequality holds for T large by (2.12). The upper bound on V_2 is very rough: since $\sin(x) \le x$ and $\cos(x) \le \cos(\pi\varepsilon)$ for $x \in [\pi\varepsilon, \pi/2]$, we can write

$$V_2(n) \;\le\; \frac{16\pi^2}{T^3}\cos^{n-2}(\pi\varepsilon) \sum_{\nu=\lfloor\varepsilon T\rfloor+1}^{\lfloor (T+2)/4\rfloor} \nu^2 \;\le\; (const.)\cos^n(\pi\varepsilon)\,. \tag{B.6}$$

Finally, we get a lower bound on $V_1 + V_2$, but only when $400 \le n \le T^2$. Since $\log(\cos(x)) \ge -\frac{2}{3}x^2$ and $\sin(x) \ge \frac{x}{2}$ for $x \in [0, \pi/4]$, we can write

$$\begin{aligned} V_1(n)+V_2(n) \;&\ge\; \frac{\pi^2}{T}\sum_{\nu=2}^{\lfloor T/8\rfloor}\left(\frac{2\nu-1}{T}\right)^2 e^{-\frac{2n\pi^2}{3}(\frac{2\nu-1}{T})^2} \;\ge\; \frac{\pi^2}{2}\int_{4/T}^{1/4} x^2\, e^{-\frac{2\pi^2}{3}nx^2}\,\mathrm{d}x \\ &=\; \frac{\pi^2}{2n^{3/2}}\int_{4\sqrt{n}/T}^{\sqrt{n}/4} y^2\, e^{-\frac{\pi^2}{3}y^2}\,\mathrm{d}y \;\ge\; \frac{\pi^2}{2n^{3/2}}\int_4^5 y^2\, e^{-\frac{\pi^2}{3}y^2}\,\mathrm{d}y \;=\; \frac{(const.)}{n^{3/2}}\,. \end{aligned} \tag{B.7}$$

Putting together (B.4), (B.5) and (B.6), it is easy to see that the upper bound in (2.13) holds true (consider separately the cases $n \le T^2$ and $n > T^2$), while the lower bound follows analogously from (B.4) and (B.7). To see that also equation (2.14) holds it is sufficient to sum the bounds in (2.13) over n, and the proof is completed. □

B.2 Proof of Proposition 2.3

For convenience, we split the proof in two parts, distinguishing between the two regimes $n \le T^3$ and $n \ge T^3$.

B.2.1 The regime $n \le T^3$

The lower bound in (2.21) for $n \le T^3$ follows easily from $\mathscr{P}_{\delta,T}(n \in \tau) \ge \mathscr{P}_{\delta,T}(\tau_1 = n)$ together with the lower bound in (2.15). The upper bound requires more work. We set for $k, n \in \mathbb{N}$

$$K_k(n) \;=\; K_k^T(n) \;:=\; \mathscr{P}_{\delta,T}(\tau_k = n)\,,$$

and we note that, by (2.16) and (2.18), there exists $T_0 > 0$ and $\alpha < 1$ such that $\sum_{n=1}^{T^3} K_1(n) \;\le\; \alpha$, for every $T > T_0$. Since $K_{k+1}(n) = \sum_{m=1}^{n-1} K_k(m)K_1(n-m)$, an easy induction argument yields

$$\sum_{n=1}^{T^3} K_k(n) \;\le\; \alpha^k\,, \qquad \forall k \in \mathbb{N}\,. \tag{B.8}$$

Next we turn to a pointwise upper bound on $K_k(n)$. From the upper bound in (2.15), we know that there exists $C > 0$ such that $K_1(n) \le C/\min\{n^{3/2}, T^3\}$ for every $n \le T^3$. We now claim that

$$K_k(n) \;\le\; k^3\,\alpha^{k-1}\,\frac{C}{\min\{n^{3/2}, T^3\}}\,, \qquad \forall k \in \mathbb{N}\,,\; \forall n \le T^3\,. \tag{B.9}$$

We argue by induction: we have just observed that this formula holds true for $k=1$. Assuming now that the formula holds for $k=1,\dots,2m-1$, we can write for $n \le T^3$

$$K_{2m}(n) \;\le\; 2\sum_{i=1}^{\lceil n/2\rceil} K_m(i) K_m(n-i) \;\le\; 2\sum_{i=1}^{\lceil n/2\rceil} K_m(i) \left(m^3\, \alpha^{m-1} \frac{C}{\min\{(n-i)^{3/2}, T^3\}} \right),$$

and since $\min\{(n-i)^{3/2}, T^3\} \ge \min\{(n/2)^{3/2}, T^3\} \ge 2^{-3/2} \min\{n^{3/2}, T^3\}$ for i in the range of summation, from (B.8) we get

$$K_{2m}(n) \;\le\; 2^{5/2}\, m^3\, \alpha^{m-1} \frac{C}{\min\{n^{3/2}, T^3\}} \sum_{i=1}^{\lceil n/2\rceil} K_m(i) \;\le\; (2m)^3\, \alpha^{2m-1} \frac{C}{\min\{n^{3/2}, T^3\}},$$

so that (B.9) is proven (we have only checked it when $k=2m$, but the case $k=2m+1$ is completely analogous). For $n \le T^3$ we can then write

$$\mathscr{P}_{\delta,T}(n\in\tau) \;=\; \sum_{k=0}^{\infty} K_k(n) \;\le\; \frac{C}{\min\{n^{3/2}, T^3\}} \sum_{k=0}^{\infty} k^3\, \alpha^{k-1} \;=\; \frac{(const.)}{\min\{n^{3/2}, T^3\}},$$

hence the upper bound in (2.21) is proven. □

B.2.2 The regime $n \ge T^3$

We start proving the lower bound in (2.21) for $n \ge T^3$. Setting $\gamma_m := \inf\{k \ge m : k \in \tau\}$ we can write

$$\begin{aligned} &\mathscr{P}_{\delta,T}(n\in\tau) \;\ge\; \mathscr{P}_{\delta,T}\big(\tau\cap[n-T^3, n-1] \neq \emptyset,\, n\in\tau\big) \\ &= \sum_{k=n-T^3}^{n-1} \mathscr{P}_{\delta,T}\big(\mu_{n-T^3}=k\big)\, \mathscr{P}_{\delta,T}\big(n-k\in\tau\big) \;\ge\; \frac{(const.)}{T^3} \mathscr{P}_{\delta,T}\big(\tau\cap[n-T^3, n-1]\neq\emptyset\big), \end{aligned}$$

where we have applied the lower bound in (2.21) to $\mathscr{P}_{\delta,T}\big(n-k\in\tau\big)$, because $n-k \le T^3$. It then suffices to show that there exist $c, T_0 > 0$ such that for $T > T_0$ and $n \ge T^3$

$$\mathscr{P}_{\delta,T}\big(\tau\cap[n-T^3, n-1]\neq\emptyset\big) \;>\; c\,.$$

We are going to prove the equivalent statement

$$\mathscr{P}_{\delta,T}\big(\tau\cap[n-T^3, n-1]\neq\emptyset\big) \;\ge\; C\, \mathscr{P}_{\delta,T}\big(\tau\cap[n-T^3, n-1]=\emptyset\big), \tag{B.10}$$

for a suitable $C > 0$. We have

$$\begin{aligned} \mathscr{P}_{\delta,T}\big(\tau\cap[n-T^3, n-1]\neq\emptyset\big) \;&=\; \sum_{\ell=0}^{n-T^3-1} \mathscr{P}_{\delta,T}\big(\ell\in\tau\big) \sum_{k=n-T^3}^{n-1} \mathscr{P}_{\delta,T}(\tau_1 = k-\ell) \\ \ge\; (const.) &\sum_{\ell=0}^{n-T^3-1} \mathscr{P}_{\delta,T}\big(\ell\in\tau\big) \left(e^{-(\phi(\delta,T)+g(T))(n-T^3-\ell)} - e^{-(\phi(\delta,T)+g(T))(n-\ell)} \right), \end{aligned} \tag{B.11}$$

having applied (2.16). Analogously, applying (2.17) we get

$$\begin{aligned}\mathscr{P}_{\delta,T}\big(\tau\cap[n-T^3,n-1]=\emptyset\big) &= \sum_{\ell=0}^{n-T^3-1}\mathscr{P}_{\delta,T}\big(\ell\in\tau\big)\sum_{k=n}^{\infty}\mathscr{P}_{\delta,T}(\tau_1=k-\ell)\\ &\le (const.)\sum_{\ell=0}^{n-T^3-1}\mathscr{P}_{\delta,T}\big(\ell\in\tau\big)\,e^{-(\phi(\delta,T)+g(T))(n-\ell)},\end{aligned}\tag{B.12}$$

having used the upper bound in (2.13). However we have

$$\frac{e^{-(\phi(\delta,T)+g(T))(n-T^3-\ell)}-e^{-(\phi(\delta,T)+g(T))(n-\ell)}}{e^{-(\phi(\delta,T)+g(T))(n-\ell)}} = e^{T^3(\phi(\delta,T)+g(T))}-1 \xrightarrow{T\to\infty} e^{\frac{2\pi^2}{(e^{-\delta}-1)}}-1\,,$$

thanks to (2.18), so that (B.10) is proven.

It remains to prove the upper bound in (2.21) for $n\ge T^3$. Notice first that

$$\begin{aligned}\mathscr{P}_{\delta,T}\big(\tau\cap[n-T^3,n-T^2]\neq\emptyset,\,n\in\tau\big) &= \sum_{k=n-T^3}^{n-T^2}\mathscr{P}_{\delta,T}(\gamma_{n-T^3}=k)\mathscr{P}_{\delta,T}(n-k\in\tau)\\ &\le \frac{(const.)}{T^3}\mathscr{P}_{\delta,T}\big(\tau\cap[n-T^3,n-T^2]\neq\emptyset\big) \le \frac{(const.)}{T^3}\,,\end{aligned}$$

having applied the upper bound in (2.21) to $\mathscr{P}_{\delta,T}(n-k\in\tau)$, because $T^2\le n-k\le T^3$. If we now show that there exist $c,T_0>0$ such that for $T>T_0$ and for $n>T^3$

$$\mathscr{P}_{\delta,T}\big(\tau\cap[n-T^3,n-T^2]\neq\emptyset\,\big|\,n\in\tau\big) \ge c\,,\tag{B.13}$$

it will follow that

$$\mathscr{P}_{\delta,T}(n\in\tau) \le \frac{1}{c}\mathscr{P}_{\delta,T}\big(\tau\cap[n-T^3,n-T^2]\neq\emptyset,\,n\in\tau\big) \le \frac{(const.')}{T^3}\,,$$

and we are done. Instead of (B.13), we prove the equivalent relation

$$\mathscr{P}_{\delta,T}\big(\tau\cap[n-T^3,n-T^2]\neq\emptyset,\,n\in\tau\big) \ge C\mathscr{P}_{\delta,T}\big(\tau\cap[n-T^3,n-T^2]=\emptyset,\,n\in\tau\big)\,,\tag{B.14}$$

for some $C>0$. We start considering the l.h.s.:

$$\begin{aligned}&\mathscr{P}_{\delta,T}\big(\tau\cap[n-T^3,n-T^2]\neq\emptyset,\,n\in\tau\big)\\ &\quad= \sum_{m=0}^{n-T^3-1}\mathscr{P}_{\delta,T}(m\in\tau)\sum_{\ell=n-T^3}^{n-T^2}\mathscr{P}_{\delta,T}(\tau_1=\ell-m)\mathscr{P}_{\delta,T}(n-\ell\in\tau).\end{aligned}\tag{B.15}$$

Notice that $\mathscr{P}_{\delta,T}(n-\ell\in\tau)\ge(const.)/T^3$ for $n-\ell\in 2\mathbb{N}$ by the lower bound in (2.21). Equation (2.16) then yields

$$\begin{aligned}&\sum_{\ell=n-T^3}^{n-T^2}\mathscr{P}_{\delta,T}(\tau_1=\ell-m) \ge (const.)\left(e^{-(\phi(\delta,T)+g(T))(n-T^3-m)}-e^{-(\phi(\delta,T)+g(T))(n-T^2-m)}\right)\\ &= (const.)\,e^{-(\phi(\delta,T)+g(T))(n-T^3-m)}\big(1-e^{-(\phi(\delta,T)+g(T))(T^3-T^2)}\big)\\ &\ge (const.')\,e^{-(\phi(\delta,T)+g(T))(n-T^3-m)} \ge (const.'')\,e^{-(\phi(\delta,T)+g(T))(n-m)}\,,\end{aligned}$$

having used repeatedly (2.18). Coming back to (B.15), we obtain

$$\mathscr{P}_{\delta,T}\big(\tau \cap [n-T^3, n-T^2] \neq \emptyset,\, n \in \tau\big) \\ \geq \frac{(const.)}{T^3} \sum_{m=0}^{n-T^3-1} \mathscr{P}_{\delta,T}(m \in \tau)\, e^{-(\phi(\delta,T)+g(T))(n-m)} . \tag{B.16}$$

Next we focus on the r.h.s. of (B.14):

$$\mathscr{P}_{\delta,T}\big(\tau \cap [n-T^3, n-T^2] = \emptyset,\, n \in \tau\big) \\ = \sum_{m=0}^{n-T^3-1} \mathscr{P}_{\delta,T}(m \in \tau) \sum_{\ell=n-T^2}^{n} \mathscr{P}_{\delta,T}(\tau_1 = \ell - m)\, \mathscr{P}_{\delta,T}(n-\ell \in \tau) . \tag{B.17}$$

Since $\ell - m \geq T^3 - T^2$, from the upper bound in (2.15) we get

$$\mathscr{P}_{\delta,T}(\tau_1 = \ell - m) \;\leq\; \frac{(const.)}{T^3}\, e^{-(\phi(\delta,T)+g(T))(\ell-m)} \;\leq\; \frac{(const.')}{T^3}\, e^{-(\phi(\delta,T)+g(T))(n-m)} ,$$

because $n-\ell \leq T^2$ (recall (2.18)). Furthermore, by the upper bound in (2.21) applied to $\mathscr{P}_{\delta,T}(n-\ell \in \tau)$, for $n-\ell \leq T^2$, we obtain

$$\sum_{\ell=n-T^2}^{n} \mathscr{P}_{\delta,T}(n-\ell \in \tau) \;\leq\; (const.) \sum_{\ell=n-T^2}^{n} \frac{1}{(n-\ell)^{3/2}} \;\leq\; (const.') ,$$

and coming back to (B.17) we get

$$\mathscr{P}_{\delta,T}\big(\tau \cap [n-T^3, n-T^2] = \emptyset,\, n \in \tau\big) \\ \leq \frac{(const.')}{T^3} \sum_{m=0}^{n-T^3-1} \mathscr{P}_{\delta,T}(m \in \tau)\, e^{-(\phi(\delta,T)+g(T))(n-m)} . \tag{B.18}$$

Comparing (B.16) and (B.18) we see that (B.14) is proven and this completes the proof. □

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