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Convergence results and sharp estimates for the voter model interfaces

S. Belhaouari
École Polytéchnique
Féderale de Lausanne (EPFL)
1015, Lausanne, Switzerland
e-mail: samir.brahim@epfl.ch

T. Mountford
École Polytéchnique
Féderale de Lausanne (EPFL)
1015, Lausanne, Switzerland
e-mail: thomas.mountford@epfl.ch

Rongfeng Sun
EURANDOM, P.O. Box 513
5600 MB Eindhoven, The Netherlands
e-mail: rsun@euridice.tue.nl

G. Valle
École Polytéchnique
Féderale de Lausanne (EPFL)
1015, Lausanne, Switzerland
e-mail: glauco.valle@dme.ufrj.br

Abstract

We study the evolution of the interface for the one-dimensional voter model. We show that if the random walk kernel associated with the voter model has finite γth moment for some $\gamma > 3$, then the evolution of the interface boundaries converge weakly to a Brownian motion under diffusive scaling. This extends recent work of Newman, Ravishankar and Sun. Our result is optimal in the sense that finite γth moment is necessary for this convergence for all $\gamma \in (0, 3)$. We also obtain relatively sharp estimates for the tail distribution of the size of the equilibrium interface, extending earlier results of Cox and Durrett, and Belhaouari, Mountford and Valle

Key words: voter model interface, coalescing random walks, Brownian web, invariance principle

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1 Introduction

In this article we consider the one-dimensional voter model specified by a random walk transition kernel $q(\cdot,\cdot)$, which is an Interacting Particle System with configuration space $\Omega = \{0,1\}^{\mathbb{Z}}$ and is formally described by the generator $\mathcal{G}$ acting on local functions $F : \Omega \to \mathbb{R}$ (i.e., F depends on only a finite number of coordinates of $\mathbb{Z}$),

$$(\mathcal{G}F)(\eta) = \sum_{x\in\mathbb{Z}}\sum_{y\in\mathbb{Z}} q(x,y)\mathbf{1}\{\eta(x) \neq \eta(y)\}[F(\eta^x) - F(\eta)]\,, \quad \eta \in \Omega$$

where

$$\eta^x(z) = \begin{cases} \eta(z), & \text{if } z \neq x \\ 1-\eta(z), & \text{if } z = x\,. \end{cases}$$

By a result of Liggett (see [7]), $\mathcal{G}$ is the generator of a Feller process $(\eta_t)_{t\geq 0}$ on Ω. In this paper we will also impose the following conditions on the transition kernel $q(\cdot,\cdot)$:

(i) $q(\cdot,\cdot)$ is translation invariant, i.e., there exists a probability kernel $p(\cdot)$ on $\mathbb{Z}$ such that $q(x,y) = p(y-x)$ for all x, $y \in \mathbb{Z}$.

(ii) The probability kernel $p(\cdot)$ is irreducible, i.e., $\{x : p(x) > 0\}$ generates $\mathbb{Z}$.

(iii) There exists $\gamma \geq 1$ such that $\sum_{x\in\mathbb{Z}} |x|^\gamma p(x) < +\infty$.

Later on we will fix the values of γ according to the results we aim to prove. We also denote by μ the first moment of p

$$\mu := \sum_{x\in\mathbb{Z}} xp(x)\,,$$

which exists by (iii).

Let $\eta_{1,0}$ be the Heavyside configuration on Ω, i.e., the configuration:

$$\eta_{1,0}(z) = \begin{cases} 1, & \text{if } z \leq 0 \\ 0, & \text{if } z \geq 1\,, \end{cases}$$

and consider the voter model $(\eta_t)_{t\geq 0}$ starting at $\eta_{1,0}$. For each time $t > 0$, let

$$r_t = \sup\{x : \eta_t(x) = 1\} \quad \text{and} \quad l_t = \inf\{x : \eta_t(x) = 0\},$$

which are respectively the positions of the rightmost 1 and the leftmost 0. We call the voter model configuration between the coordinates l_t and r_t the *voter model interface*, and $r_t - l_t + 1$ is the *interface size*. Note that condition (iii) on the probability kernel $p(\cdot)$ implies that the interfaces are almost surely finite for all $t \geq 0$ and thus well defined. To see this, we first observe that the rate at which the interface size increases is bounded above by

$$\sum_{x<0<y} \{p(y-x) + p(x-y)\} = \sum_{z\in\mathbb{Z}} |z|p(z) < \infty\,. \tag{1.1}$$

Moreover this is the rate at which the system initially changes if it starts at $\eta_{1,0}$.

When $\gamma \geq 2$, Belhaouari, Mountford and Valle [1] proved that the interface is tight, i.e., the random variables $(r_t - l_t)_{t\geq 0}$ are tight. This extends earlier work of Cox and Durrett [4], which showed the tightness result when $\gamma \geq 3$. Belhaouari, Mountford and Valle also showed that, if $\sum_{x\in\mathbb{Z}} |x|^\gamma p(x) = \infty$ for some $\gamma \in (0, 2)$, then the tightness result fails. Thus second moment is, in some sense, optimal. Note that the tightness of the interface is a feature of the one-dimensional model. For voter models in dimension two or more, the so-called hybrid zone grows as $\sqrt{t}$ as was shown in [4].

In this paper we examine two questions for the voter model interface: the evolution of the interface boundaries, and the tail behavior of the equilibrium distribution of the interface which is known to exist whenever the interface is tight. Third moment will turn out to be critical in these cases.

From now on we will assume $p(\cdot)$ is symmetric, and in particular $\mu = 0$, which is by no means a restriction on our results since the general case is obtained by subtracting the drift and working with the symmetric part of $p(\cdot)$:

$$p^s(x) = \frac{p(x) + p(-x)}{2}.$$

The first question arises from the observation of Cox and Durrett [4] that, if $(r_t - \ell_t)_{t\geq 0}$ is tight, then the finite-dimensional distributions of

$$\left(\frac{r_{tN^2}}{N}\right)_{t\geq 0} \quad \text{and} \quad \left(\frac{l_{tN^2}}{N}\right)_{t\geq 0}$$

converge to those of a Brownian motion with speed

$$\sigma := \left(\sum_{z\in\mathbb{Z}} z^2 p(z)\right)^{1/2}. \tag{1.2}$$

As usual, let $D([0, +\infty), \mathbb{R})$ be the space of right continuous functions with left limits from $[0, +\infty)$ to $\mathbb{R}$, endowed with the Skorohod topology. The question we address is, as $N \to \infty$, whether or not the distributions on $D([0, +\infty), \mathbb{R})$ of

$$\left(\frac{r_{tN^2}}{N}\right)_{t\geq 0} \quad \text{and} \quad \left(\frac{l_{tN^2}}{N}\right)_{t\geq 0}$$

converge weakly to a one-dimensional σ-speed Brownian Motion, i.e, $(\sigma B_t)_{t\geq 0}$, where $(B_t)_{t\geq 0}$ is a standard one-dimensional Brownian Motion. We show:

Theorem 1.1. *For the one-dimensional voter model defined as above*

(i) If $\gamma > 3$, then the path distributions on $D([0, +\infty), \mathbb{R})$ of

$$\left(\frac{r_{tN^2}}{N}\right)_{t\geq 0} \quad \text{and} \quad \left(\frac{l_{tN^2}}{N}\right)_{t\geq 0}$$

converge weakly to a one-dimensional σ-speed Brownian Motion with σ defined in (1.2).

(ii) For $(\frac{r_{tN^2}}{N})_{t\geq 0}$ $\Big($resp. $(\frac{l_{tN^2}}{N})_{t\geq 0}\Big)$ to converge to a Brownian motion, it is necessary that

$$\sum_{x\in\mathbb{Z}} \frac{|x|^3}{\log^{\beta}(|x|\vee 2)} p\,(x) < \infty \qquad \text{for all } \beta > 1.$$

In particular, if for some $1 \leq \gamma < \tilde{\gamma} < 3$ we have $\sum_x |x|^{\tilde{\gamma}} p\,(x) = \infty$, then $\{(\frac{r_{tN^2}}{N})_{t\geq 0}\}$ $\Big($resp. $(\frac{l_{tN^2}}{N})_{t\geq 0}\Big)$ is not a tight family in $D([0,+\infty),\mathbb{R})$, and hence cannot converge in distribution to a Brownian motion.

Remark 1. *Theorem 1.1(i) extends a recent result of Newman, Ravishankar and Sun [9], in which they obtained the same result for $\gamma \geq 5$ as a corollary of the convergence of systems of coalescing random walks to the so-called Brownian web under a finite fifth moment assumption. The difficulty in establishing Theorem 1.1(i) and the convergence of coalescing random walks to the Brownian web lie both in tightness. In fact the tightness conditions for the two convergences are essentially equivalent. Consequently, we can improve the convergence of coalescing random walks to the Brownian web from a finite fifth moment assumption to a finite γth assumption for any $\gamma > 3$. We formulate this as a theorem.*

Theorem 1.2. *Let $\mathcal{X}_1$ denote the random set of continuous time rate 1 coalescing random walk paths with one walker starting from every point on the space-time lattice $\mathbb{Z}\times\mathbb{R}$, where the random walk increments all have distribution $p\,(\cdot)$. Let $\mathcal{X}_\delta$ denote $\mathcal{X}_1$ diffusively rescaled, i.e., scale space by δ/σ and time by δ^2. If $\gamma > 3$, then in the topology of the Brownian web [9], $\mathcal{X}_\delta$ converges weakly to the standard Brownian web $\bar{\mathcal{W}}$ as $\delta \to 0$. A necessary condition for this convergence is again $\sum_{x\in\mathbb{Z}} \frac{|x|^3}{\log^{\beta}(|x|\vee 2)} p\,(x) < \infty$ for all $\beta > 1$.*

It should be noted that the failure of convergence to a Brownian motion does not preclude the existence of $N_i \uparrow \infty$ such that $\left(\frac{r_{N_i^2 t}}{N_i}\right)_{t\geq 0}$ converges to a Brownian motion. Loss of tightness is due to "unreasonable" large jumps. Theorem 1.3 below shows that, when $2 < \gamma < 3$, tightness can be restored by suppressing rare large jumps near the voter model interface, and again we have convergence of the boundary of the voter model interface to a Brownian motion.

Before stating Theorem 1.3, we fix some notation and recall a usual construction of the voter model. We start with the construction of the voter model through the Harris system. Let $\{\mathcal{N}^{x,y}\}_{x,y\in\mathbb{Z}}$ be independent Poisson point processes with intensity $p(y-x)$ for each $x, y \in \mathbb{Z}$. From an initial configuration η_0 in Ω, we set at time $t \in \mathcal{N}^{x,y}$:

$$\eta_t(z) = \begin{cases} \eta_{t-}(z), & \text{if } z \neq x \\ \eta_{t-}(y), & \text{if } z = x\,. \end{cases}$$

From the same Poisson point processes, we construct the system of *coalescing random walks* as follows. We can think of the Poisson points in $\mathcal{N}^{x,y}$ as marks at site x occurring at the Poisson times. For each space-time point (x,t) we start a random walk $X^{x,t}$ evolving backward in time such that whenever the walk hits a mark in $\mathcal{N}^{u,v}$ (i.e., for $s \in (0,t)$, $(t-s) \in \mathcal{N}^{u,v}$ and $u = X^{x,t}_s$), it jumps from site u to site v. When two such random walks meet, which occurs because one walk jumps on top of the other walk, they coalesce into a single random walk starting from the space-time point where they first met. We define by ζ_s the Markov process which describes the positions of the coalescing particles at time s. If ζ_s starts at time t with one particle from every site of A for some $A \subset \mathbb{Z}$, then we use the notation

$$\zeta_s^t(A) := \{X_s^{x,t} : x \in A\},$$

where the superscript is the time in the voter model when the walks first started, and the subscript is the time for the coalescing random walks. It is well known that ζ_t is the dual process of η_t (see Liggett's book [7]), and we obtain directly from the Harris construction that

$$\{\eta_t(\cdot) \equiv 1 \text{ on } A\} = \{\eta_0(\cdot) \equiv 1 \text{ on } \zeta_t^t(A)\}$$

for all $A \subset \mathbb{Z}$.

Theorem 1.3. *Take $2 < \gamma < 3$ and fix $0 < \theta < \frac{\gamma-2}{\gamma}$. For $N \geq 1$, let $(\eta_t^N)_{t\geq 0}$ be described as the voter model according to the same Harris system and also starting from $\eta_{1,0}$ except that a flip from 0 to 1 at a site x at time t is suppressed if it results from the "influence" of a site y with $|x-y| \geq N^{1-\theta}$ and $[x \wedge y, x \vee y] \cap [r_{t-}^N - N, r_{t-}^N] \neq \phi$, where r_t^N is the rightmost 1 for the process $\eta_\cdot^N$. Then*

(i) $\left(\frac{r_{tN^2}^N}{N}\right)_{t\geq 0}$ *converge in distribution to a σ-speed Brownian Motion with σ defined in (1.2).*

(ii) As $N \to \infty$, the integral

$$\frac{1}{N^2}\int_0^{TN^2} I_{r_s^N \neq r_s} ds$$

tends to 0 in probability for all $T > 0$.

Remark 2. *There is no novelty in claiming that for $(\frac{r_{tN^2}}{N})_{t\geq 0}$, there is a sequence of processes $(\gamma_t^N)_{t\geq 0}$ which converges in distribution to a Brownian motion, such that with probability tending to 1 as N tends to infinity, γ_t^N is close to $\frac{r_{tN^2}}{N}$ most of the time. The value of the previous result is in the fact that there is a very natural candidate for such a process. Thus the main interest of Theorem 1.3 lies in the lower bound $\theta > 0$. By truncating jumps of size at least $N^{1-\theta}$ for some fixed $\theta > 0$, the tightness of the interface boundary evolution $\{(\frac{r_{tN^2}^N}{N})_{t\geq 0}\}_{N\in\mathbb{N}}$ is restored. The upper bound $\theta < \frac{\gamma}{\gamma-2}$ simply says that with higher moments, we can truncate more jumps without affecting the limiting distribution.*

Let $\{\Theta_x : \Omega \to \Omega, x \in \mathbb{Z}\}$ be the group of translations on Ω, i.e., $(\eta \circ \Theta_x)(y) = \eta(y+x)$ for every $x \in \mathbb{Z}$ and $\eta \in \Omega$. The second question we address concerns the equilibrium distribution of the *voter model interface* $(\eta_t \circ \Theta_{\ell_t})_{t\geq 0}$, when such an equilibrium exists. Cox and Durrett [4] observed that $(\eta_t \circ \Theta_{\ell_t}|\mathbb{N})_{t\geq 0}$, the configuration of $\eta_t \circ \Theta_{\ell_t}$ restricted to the positive coordinates, evolves as an irreducible Markov chain with countable state space

$$\tilde{\Omega} = \left\{\xi \in \{0,1\}^{\mathbb{N}} : \sum_{x\geq 1} \xi(x) < \infty\right\}.$$

Therefore a unique equilibrium distribution π exists for $(\eta_t \circ \Theta_{\ell_t}|\mathbb{N})_{t\geq 0}$ if and only if it is a positive recurrent Markov chain. Cox and Durret proved that, when the probability kernel $p(\cdot)$ has finite third moment, $(\eta_t \circ \Theta_{\ell_t}|\mathbb{N})_{t\geq 0}$ is indeed positive recurrent and a unique equilibrium π exists. Belhaouari, Mountford and Valle [1] recently extended this result to kernels $p(\cdot)$ with finite second moment, which was shown to be optimal.

Cox and Durrett also noted that if the equilibrium distribution π exists, then excluding the trivial nearest neighbor case, the equilibrium has $E_\pi[\Gamma] = \infty$ where $\Gamma = \Gamma(\xi) = \sup\{x : \xi(x) = 1\}$ for $\xi \in \tilde{\Omega}$ is the interface size. In fact, as we will see, under finite second moment assumpt ion on the probability kernel $p(\cdot)$, there exists a constant $C = C_p \in (0, \infty)$ such that

$$\pi\{\xi : \Gamma(\xi) \geq M\} \geq \frac{C_p}{M} \quad \text{for all } M \in \mathbb{N},$$

extending Theorem 6 of Cox and Durrett [4]. Furthermore, we show that M^{-1} is the correct order for $\pi\{\eta : \Gamma(\eta) \geq M\}$ as M tends to infinity if $p(\cdot)$ possesses a moment strictly higher than 3, but not so if $p(\cdot)$ fails to have a moment strictly less than 3.

Theorem 1.4. *For the non-nearest neighbor one-dimensional voter model defined as above*

(i) If $\gamma \geq 2$, then there exists $C_1 > 0$ such that for all $M \in \mathbb{N}$

$$\pi\{\xi : \Gamma(\xi) \geq M\} \geq \frac{C_1}{M}. \tag{1.3}$$

(ii) If $\gamma > 3$, then there exists $C_2 > 0$ such that for all $M \in \mathbb{N}$

$$\pi\{\xi : \Gamma(\xi) \geq M\} \leq \frac{C_2}{M}. \tag{1.4}$$

(iii) Let $\alpha = \sup\{\gamma : \sum_{x\in\mathbb{Z}} |x|^\gamma p(x) < \infty\}$. If $\alpha \in (2,3)$, then

$$\limsup_{n\to\infty} \frac{\log \pi\{\xi : \Gamma(\xi) \geq n\}}{\log n} \geq 2 - \alpha. \tag{1.5}$$

Furthermore, there exist choices of $p(\cdot) = p_\alpha(\cdot)$ with $\alpha \in (2,3)$ and

$$\pi\{\xi : \Gamma(\xi) \geq n\} \geq \frac{C}{n^{\alpha-2}} \tag{1.6}$$

for some constant $C > 0$.

This paper is divided in the following way: Sections 2, 3 and 4 are respectively devoted to the proofs of Theorems 1.1 and 1.2, 1.3, and 1.4. We end with section 5 with the statement and proof of some results needed in the previous sections.

2 Proof of Theorem 1.1 and 1.2

By standard results for convergence of distributions on the path space $D([0,+\infty),\mathbb{R})$ (see for instance Billingsley's book [3], Chapter 3), we have that the convergence to the σ-speed Brownian Motion in Theorem 1.1 is a consequence of the following results:

Lemma 2.1. *If $\gamma \geq 2$, then for every $n \in \mathbb{N}$ and $0 < t_1 < t_2 < ... < t_n$ in $[0,\infty)$ the finite-dimensional distribution*

$$\left(\frac{r_{t_1N^2}}{\sigma N\sqrt{t_1}}, \frac{r_{t_2N^2}-r_{t_1N^2}}{\sigma N\sqrt{t_2-t_1}}, \dots, \frac{r_{t_nN^2}-r_{t_{n-1}N^2}}{\sigma N\sqrt{t_n-t_{n-1}}}\right)$$

converges weakly to a centered n-dimensional Gaussian vector of covariance matrix equal to the identity. Moreover the same holds if we replace r_t by l_t.

Proposition 2.2. *If $\gamma > 3$, then for every $\epsilon > 0$ and $T > 0$*

$$\lim_{\delta\to 0}\limsup_{N\to\infty} \mathrm{P}\left[\sup_{\substack{|t-s|<\delta\\ s,t\in[0,T]}}\left|\frac{r_{tN^2}-r_{sN^2}}{N}\right| > \epsilon\right] = 0\,. \tag{2.1}$$

In particular if the finite-dimensional distributions of $\left(\frac{r_{tN^2}}{N}\right)_{t\geq 0}$ are tight, we have that the path distribution is also tight and every limit point is concentrated on continuous paths. The same holds if we replace r_t by l_t.

By Lemma 2.1 and Proposition 2.2 we have Theorem 1.1.

Lemma 2.1 is a simple consequence of the Markov property, the observations of Cox and Durrett [4] and Theorem 2 of Belhaouari-Mountford-Valle [1] where it was shown that for $\gamma \geq 2$ the distribution of $\frac{r_{tN^2}}{\sigma N}$ converges to a standard normal random variable (see also Theorem 5 in Cox and Durrett [4] where the case $\gamma \geq 3$ was initially considered).

We are only going to carry out the proof of (2.1) for r_t since the result of the proposition follows for l_t by interchanging the roles of 0's and 1's in the voter model.

Note that by the right continuity of r_t, the event in (2.1) is included in

$$\bigcup_{0\leq i\leq \lfloor \frac{T}{\delta}\rfloor}\left\{\sup_{s\in[i\delta,(i+1)\delta)}\left|\frac{r_{sN^2}-r_{i\delta N^2}}{N}\right| > \frac{\epsilon}{4}\right\}.$$

By the Markov property, the attractivity of the voter model and the tightness of the voter model interface, (2.1) is therefore a consequence of the following result: for all $\epsilon > 0$

$$\limsup_{\delta\to 0}\delta^{-1}\limsup_{N\to+\infty}\mathrm{P}\left[\sup_{0\leq t\leq N^2\delta}|r_t| \geq \epsilon N\right] = 0\,. \tag{2.2}$$

Let us first remark that in order to show (2.2) it is sufficient to show that

$$\limsup_{\delta\to 0} \delta^{-1} \limsup_{N\to+\infty} \mathrm{P}\left[\sup_{0\le t\le N^2\delta} r_t \ge \epsilon N\right] = 0\,. \tag{2.3}$$

Indeed, from the last equation we obtain

$$\limsup_{\delta\to 0} \delta^{-1} \limsup_{N\to+\infty} \mathrm{P}\left[\inf_{0\le t\le N^2\delta} r_t \le -\epsilon N\right] = 0\,. \tag{2.4}$$

To see this note that $r_t \ge l_t - 1$, thus (2.4) is a consequence of

$$\limsup_{\delta\to 0} \delta^{-1} \limsup_{N\to+\infty} \mathrm{P}\left[\inf_{0\le t\le N^2\delta} l_t \le -\epsilon N\right] = 0\,, \tag{2.5}$$

which is equivalent to (2.3) by interchanging the 0's and 1's in the voter model.

The proof of (2.3) to be presented is based on a chain argument for the dual coalescing random walks process. We first observe that by duality, (2.3) is equivalent to showing that for all $\epsilon > 0$,

$$\lim_{\delta\to 0} \delta^{-1} \limsup_{N\to+\infty} \mathrm{P}\left[\zeta_t^t([\epsilon N, +\infty)) \cap (-\infty, 0] \neq \phi \;\text{ for some } t \in [0, \delta N^2]\right] = 0\,.$$

Now, if we take $R := R(\delta, N) = \sqrt{\delta} N$ and $M = \epsilon/\sqrt{\delta}$, we may rewrite the last expression as

$$\lim_{M\to+\infty} M^2 \limsup_{R\to+\infty} \mathrm{P}\left[\zeta_t^t([MR, +\infty)) \cap (-\infty, 0] \neq \phi \;\text{ for some } t \in [0, R^2]\right] = 0\,,$$

which means that we have to estimate the probability that no dual coalescing random walk starting at a site in $[MR, +\infty)$ at a time in the interval $[0, R^2]$ arrives at time $t = 0$ at a site to the left of the origin. It is easy to check that the condition above, and hence Proposition 2.2 is a consequence of the following:

Proposition 2.3. *If $\gamma > 3$, then for $R > 0$ sufficiently large and $2^b \le M < 2^{b+1}$, for some $b \in \mathbb{N}$ the probability*

$$\mathrm{P}\left[\,\zeta_t^t([MR, +\infty)) \cap (-\infty, 0] \neq \phi \;\text{ for some } t \in [0, R^2]\,\right]$$

is bounded above by a constant times

$$\sum_{k\ge b} \left\{ \frac{1}{2^{2k} R^{\frac{\gamma-3}{2}}} + e^{-c2^k} + 2^k R^4 e^{-c 2^{k(1-\beta)} R^{\frac{(1-\beta)}{2}}} + 2^k e^{-c 2^{2k}} \right\} \tag{2.6}$$

for some $c > 0$ and $0 < \beta < 1$.

Proof:

The proof is based on a chain argument which we first describe informally. Without loss of generality we fix $M = 2^b$. The event stated in the proposition is a union of the events that

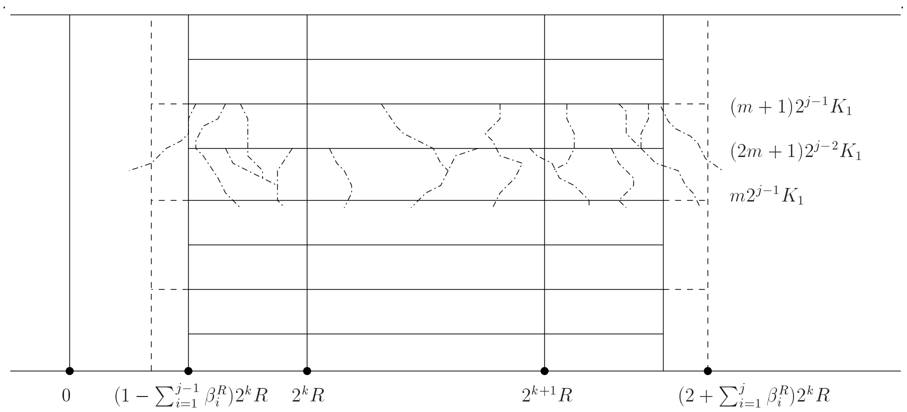

Figure 1: Illustration of the j-th step of the chain argument.

some backward random walk starting from $[2^kR, 2^{k+1}R] \times [0, R^2]$ $(k \geq b)$ hits the negative axis at time 0. Therefore it suffices to consider such events.

The first step is to discard the event that at least one of the backward coalescing random walks $X^{x,s}$ starting in $I_{k,R} = [2^kR, 2^{k+1}R] \times [0, R^2]$ has escaped from a small neighborhood around $I_{k,R}$ before reaching time level $K_1\lfloor \frac{s}{K_1} \rfloor$, where $\lfloor x \rfloor = \max\{m \in \mathbb{Z} : m \leq x\}$. The constant K_1 will be chosen later. We call this small neighborhood around $I_{k,R}$ the *first-step interval*, and the times $\{nK_1\}_{0 \leq n \leq \lfloor \frac{R^2}{K_1} \rfloor}$ the *first-step times*. So after this first step we just have to consider the system of coalescing random walks starting on each site of the first-step interval at each of the first-step times.

In the second step of our argument, we let these particles evolve backward in time until they reach the *second-step times*: $\{n(2K_1)\}_{0 \leq n \leq \lfloor \frac{R^2}{2K_1} \rfloor}$. I.e., if a walk starts at time lK_1, we let it evolve until time $(l-1)K_1$ if l is odd, and until time $(l-2)K_1$ if l is even. We then discard the event that either some of these particles have escaped from a small neighborhood around the first-step interval, which we call the *second-step interval*, or the density of the particles alive at each of the second-step times in the second-step interval has not been reduced by a fixed factor $0 < p < 1$.

We now continue by induction. In the jth-step, (see Figure 1) we have particles starting from the $(j-1)$th-step interval with density at most p^{j-2} at each of the $(j-1)$th-step times. We let these particles evolve backward in time until the next *jth-step times*: $\{n(2^{j-1}K_1)\}_{0 \leq n \leq \lfloor \frac{R^2}{2^{j-1}K_1} \rfloor}$. We then discard the event that either some of these particles have escaped from a small neighborhood around the $(j-1)$th-step interval, which we call the *jth-step interval*, or the density of the particles alive at each of the jth-step times in the jth-step interval has not been reduced below p^{j-1}.

We repeat this procedure until the Jth-step with J of order $\log R$, when the only *Jth-step time* left in $[0, R^2]$ is 0. The rate p will be chosen such that at the Jth-step, the number of particles alive at time 0 is of the order of a constant which is uniformly bounded in R but which still depends on k. The *Jth-step interval* will be chosen to be contained in $[0, 3 \cdot 2^kR]$.

We now give the details. In our approach the factor p is taken to be $2^{-1/2}$. The constant $K_1 = 7K_0$ where K_0 is the constant satisfying Proposition 5.4, which is necessary to guarantee the reduction in the number of particles. Note that K_1 is independent of k and R. The j th-step interval is obtained from the $(j-1)$th-step intervals by adding intervals of length $\beta_j^R 2^k R$, where

$$\beta^R_{J_R-j} = \frac{1}{2(j+1)^2},$$

and

$$J_R = 1 + \left\lceil \frac{1}{\log 2} \log\left(\frac{R^2}{K_1}\right)\right\rceil$$

is taken to be the last step in the chain argument. Here $\lceil x\rceil = \min\{m \in \mathbb{Z} : m \geq x\}$. We have chosen J_R because it is the step when $2^{J_R-1}K_1$ first exceeds R^2 and the only J_Rth-step time in $[0, R^2]$ is 0. With our choice of β_j^R, we have that the J_Rth-step interval lies within $[0, 3(2^k R)]$, and except for the events we discard, no random walk reaches level 0 before time 0.

Let us fix $\gamma = 3 + \epsilon$ in Theorem 1.1. The first step in the chain argument described above is carried out by noting that the event we reject is a subset of the event

$$\Big\{ \text{For some } k \geq b \text{ and } (x,s) \in [2^k R, 2^{k+1}R] \times [0, R^2],$$
$$|X_u^{x,s} - x| \geq \beta_1^R 2^k R \text{ for some } 0 \leq u \leq s - K_1 \left\lfloor \frac{s}{K_1} \right\rfloor \Big\}.$$

Since $\beta_1^R = 1/(2J_R^2) \geq C/(\log R)^2$, Lemma 5.5 implies that the probability of the above event is bounded by

$$\sum_{k\geq b} \frac{CK_1(\log R)^{2(3+\epsilon)}}{2^{2k+3\epsilon}R^{\epsilon}} \tag{2.7}$$

for R sufficiently large. Therefore, for each $k \geq b$, instead of considering all the coalescing random walks starting from $[2^k R, 2^{k+1}R] \times [0, R^2]$, we just have to consider coalescing random walks starting from $[(1-\beta_1^R)2^k R, (2+\beta_1^R)2^k R] \times \{nK_1\}$ where $\{nK_1\}_{0\leq n\leq \lfloor \frac{R^2}{K_1}\rfloor}$ are the first-step times. By this observation, we only need to bound the probability of the event

$$A^{k,R} = \Big\{ X_u^{x,nK_1} \leq 0 \text{ for some } n = 1, ..., \left\lfloor \frac{R^2}{K_1} \right\rfloor,\ u \in [0, nK_1]$$
$$\text{and } x \in \left[\left(1-\beta_1^R\right) 2^k R, \left(2+\beta_1^R\right) 2^k R\right] \Big\}.$$

We start by defining events which will allow us to write $A^{k,R}$ in a convenient way. For $n_1 := n \in \mathbb{N}$ and for each $1 \leq j \leq J_R - 1$, define recursively

$$n_{j+1} = \begin{cases} \left\lfloor \frac{n_j - 1}{2^j} \right\rfloor 2^j, & \text{if } \left\lfloor \frac{n_j-1}{2^j}\right\rfloor 2^j \geq 0 \\ 0\,, & \textit{otherwise}\,. \end{cases}$$

For a random walk starting at time nK_1 in the dual voter model, $n_j K_1$ is its time coordinate after the jth step of our chain argument. Then define

$$W_1^{k,R} = \Big\{ |X_u^{x,nK_1} - x| \geq \beta_2^R 2^k R \ \text{ for some } n = 1, ..., \left\lfloor \frac{R^2}{K_1}\right\rfloor,$$
$$u \in [0, (n-n_2)K_1] \ \text{ and } \ x \in \left[\left(1-\beta_1^R\right) 2^k R, \left(2+\beta_1^R\right) 2^k R\right] \Big\},$$

and for each $2 \le j \le J_R - 1$

$$W_j^{k,R} = \Big\{ \big|X^{x,nK_1}_{(n-n_j)K_1+u} - X^{x,nK_1}_{(n-n_j)K_1}\big| \ge \beta^R_{j+1} 2^k R \text{ for some } n = 1, ..., \left\lfloor \frac{R^2}{K_1} \right\rfloor, \\ u \in [0, (n_j - n_{j+1})K_1] \text{ and } x \in \Big[\big(1-\beta_1^R\big) 2^k R, \big(2+\beta_1^R\big) 2^k R \Big] \Big\}.$$

Note that $W_j^{k,R}$ is the event that in the $(j+1)$th step of the chain argument, some random walk starting from a jth-step time makes an excursion of size $\beta^R_{j+1} 2^k R$ before it reaches the next $(j+1)$th-step time. Then we have

$$A^{k,R} \subset \bigcup_{j=1}^{J_R-1} W_j^{k,R},$$

since on the complement of $\bigcup_{j=1}^{J_R-1} W_j^{k,R}$ the random walks remain confined in the interval

$$\left[\left(1 - \sum_{i=1}^{J_R} \beta_i^R \right) 2^k R, \left(2 + \sum_{i=1}^{J_R} \beta_i^R \right) 2^k R \right] \subset [0, 3 \cdot 2^k R].$$

Now let $U_j^{k,R}$, $1 \le j \le J_R - 1$, be the event that for some $0 \le n \le \lfloor \frac{R^2}{2^j K_1} \rfloor$ the density of coalescing random walks starting at $(x,s) \in \big[\big(1-\beta_1^R\big) 2^k R, \big(2+\beta_1^R\big) 2^k R\big] \times \{lK_1 : l_{j+1} = n2^j\}$ that are alive in the $(j+1)$th-step interval at time $n2^j K_1$ is greater than $2^{-\frac{j}{2}}$. In other words, $U_j^{k,R}$ is the event that after the $(j+1)$th-step of the chain argument, the density of particles in the $(j+1)$th-step interval at some of the $(j+1)$th-step times $\{n2^j K_1\}_{0 \le n \le \lfloor \frac{R^2}{2^j K_1} \rfloor}$ is greater than $2^{-\frac{j}{2}}$. The chain argument simply comes from the following decomposition:

$$\begin{aligned} \bigcup_{j=1}^{J_R-1} W_j^{k,R} &\subset \bigcup_{j=1}^{J_R-1} \Big(W_j^{k,R} \cup U_j^{k,R} \Big) \\ &= \bigcup_{j=1}^{J_R-1} \Big((W_j^{k,R} \cup U_j^{k,R}) \cap \bigcap_{i=1}^{j-1} \big(W_i^{k,R} \cup U_i^{k,R}\big)^c \Big) \\ &= \bigcup_{j=1}^{J_R-1} \Big(W_j^{k,R} \cap \bigcap_{i=1}^{j-1} \big(W_i^{k,R} \cup U_i^{k,R}\big)^c \Big) \qquad (2.8) \\ &\quad \cup \bigcup_{j=1}^{J_R-1} \Big(U_j^{k,R} \cap \bigcap_{i=1}^{j-1} \big(W_i^{k,R} \cup U_i^{k,R}\big)^c \Big). \qquad (2.9) \end{aligned}$$

We are going to estimate the probability of the events in (2.8) and (2.9).

We start with (2.9). It is clear from the definitions that the events $U_i^{k,R}$ were introduced to obtain the appropriate reduction on the density of random walks at each step of the chain argument. The event $U_j^{k,R} \cap \bigcap_{i=1}^{j-1} \big(W_i^{k,R} \cup U_i^{k,R}\big)^c$ implies the existence of jth-step times $t_1 = (2m+1)2^{j-1}K_1$ and $t_2 = (2m+2)2^{j-1}K_1$ such that, after the jth-step of the chain argument, the walks at ti me

t_1 and t_2 are inside the jth-step interval with density at most $2^{-\frac{j-1}{2}}$, and in the $(j+1)$th-step these walks stay within the $(j+1)$th-step interval until the $(j+1)$th-step time $t_0 = m2^jK_1$, when the density of remaining walks in the $(j+1)$th-step interval exceeds $2^{-\frac{j}{2}}$. We estimate the probability of this last event by applying three times Proposition 5.4 with $p = 2^{-\frac{1}{2}}$ and L equal to the size of the $(j+1)$th-step interval, which we denote by $L^{k,R}_{j+1}$.

We may suppose that at most $2^{-\frac{j-1}{2}} L^{k,R}_{j+1}$ random walks are leaving from times t_1 and t_2. We let both sets of walks evolve for a dual time interval of length $7^{-1} \cdot 2^{j-1}K_1 = 2^{j-1}K_0$. By applying Proposition 5.4 with $\gamma = 2^{-\frac{j-1}{2}}$, the density of particles starting at times t_1 or t_2 is reduced by a factor of $2^{-\frac{1}{2}}$ with large probability. Now we let the particles evolve further for a t ime interval of length 2^jK_0. Apply Proposition 5.4 with $\gamma = 2^{-\frac{j}{2}}$, the density of remaining particles is reduced by another factor of $2^{-\frac{1}{2}}$ with large probability. By a last application of Proposition 5.4 for another time interval of length $2^{j+1}K_0$ with $\gamma = 2^{-\frac{j+1}{2}}$ we obtain that the total density of random walks originating from the two jth-step times t_1 (resp. t_2) remaining at time t_0 (resp. t_1) has been reduced by a factor $2^{-\frac{3}{2}}$. Finally we let the random walks remaining at time t_1 evolve un till the $(j+1)$th-step time t_0, at which time the density of random walks has been reduced by a factor $2 \cdot 2^{-\frac{3}{2}} = 2^{-\frac{1}{2}}$ with large probability. By a decomposition similar to (2.8) and (2.9) and using the Markov property, we can assume that before each application of Proposition 5.4, the random walks are all confined within the $(j+1)$th-step interval. All the events described above have probability at least $1 - Ce^{-c\frac{2^kR}{2^{j/2}}}$. Since there are $(\lfloor \frac{R^2}{2^jK_1} \rfloor + 1)$ $(j+1)$th-step times, the probability of the event in (2.9) is bounded by

$$C \sum_{j=0}^{J_R} \frac{R^2}{2^jK_1} \exp\left\{-c\frac{2^kR}{2^{j/2}}\right\} .$$

It is simple to verify that this last expression is bounded above by

$$C \int_1^{+\infty} u^2 e^{-c2^ku} du \le Ce^{-c2^k} .$$

Now we estimate the probability of the event in (2.8). For every $j = 1, ..., J_R - 1$,

$$W^{k,R}_j \cap \bigcap_{i=1}^{j-1} (W^{k,R}_i)^c \cap \bigcap_{i=1}^{j-1} (U^{k,R}_i)^c$$

is contained in the event that at the jth-step times $\{n2^{j-1}K_1\}_{1 \le n \le \lfloor \frac{R^2}{2^{j-1}K_1} \rfloor}$, the random walks are contained in the jth-step interval with density at most $2^{-\frac{j-1}{2}}$, and some of these walks move by more than $\beta^R_{j+1}\, 2^kR$ in a time interval of length 2^jK_1. If X_t denotes a random walk with transition kernel $q(x, y) = p(y - x)$ starting at 0, then the probability of the above event is bounded by

$$\frac{R^2}{2^{j-1}K_1} \frac{2^kR}{2^{\frac{j-1}{2}}} \mathrm{P}\left(\sup_{0 \le t \le 2^jK_1} |X_t| \ge \beta^R_{j+1}\, 2^kR \right) , \tag{2.10}$$

since

$$\frac{R^2}{2^{j-1}K_1} \frac{2^kR}{2^{\frac{j-1}{2}}} \tag{2.11}$$

bounds the number of walks we are considering. By Lemma 5.1 the probability in (2.10) is dominated by a constant times

$$\exp\left\{-c\left(\beta_{j+1}^R\, 2^k R\right)^{1-\beta}\right\}+\exp\left\{-c\frac{\left(\beta_{j+1}^R\, 2^k R\right)^2}{2^j K_1}\right\}+\left(\frac{1}{\beta_{j+1}^R\, 2^k R}\right)^{3+\epsilon} 2^j K_1\,.$$

Then multiplying by (2.11) and summing over $1 \leq j \leq J_R$, we obtain by straightforward computations that if R is sufficiently large, then there exist constants $c > 0$ and $c' > 1$ such that the probability of the event in (2.8) is bounded above by a constant times

$$2^k R^4 e^{-c2^{(1-\beta)k} R^{\frac{(1-\beta)}{2}}} + 2^k \int_1^\infty u^3 e^{-\frac{c2^{2k}u^2}{\log(c'u)}} du + \frac{1}{2^{(2+\epsilon)k} R^{\frac{\epsilon}{2}}}\,. \tag{2.12}$$

Adjusting the terms in the last expression we complete the proof of the proposition. □

Proof of (ii) in Theorem 1.1:

For the rescaled voter model interface boundaries $\frac{l_{tN^2}}{N}$ and $\frac{r_{tN^2}}{N}$ to converge to a σ-speed Brownian motion, it is necessary that the boundaries cannot wander too far within a small period of time, i.e., we must have

$$\lim_{t\to 0}\limsup_{N\to\infty} \mathrm{P}\left[\sup_{0\leq s\leq t} \frac{r_{sN^2}}{N} > \epsilon\right] = \lim_{t\to 0}\limsup_{N\to\infty} \mathrm{P}\left[\inf_{0\leq s\leq t} \frac{l_{sN^2}}{N} < -\epsilon\right] = 0. \tag{2.13}$$

In terms of the dual system of coalescing random walks, this is equivalent to

$$\lim_{t\to 0}\limsup_{N\to\infty} \mathrm{P}\left\{\zeta_s^s([\epsilon N, +\infty)) \cap (-\infty, 0] \neq \phi \ \text{ for some } s \in [0, tN^2]\right\} = 0 \tag{2.14}$$

and the same statement for its mirror event. If some random walk jump originating from the region $[\epsilon\sigma N, \infty) \times [0, tN^2]$ jumps across level 0 in one step (which we denote as the event $D_N(\epsilon, t)$), then with probability at least α for some $\alpha > 0$ depending only on the random walk kernel $p(\cdot)$, that random walk will land on the negative axis at time 0 (in the dual voter model). Thus (2.14) implies that

$$\lim_{t\to 0}\limsup_{N\to\infty} \mathrm{P}[D_N(\epsilon, t)] = 0 \tag{2.15}$$

and the same statement for its mirror event. Since random walk jumps originating from $(-\infty, -\epsilon N] \cup [\epsilon N, +\infty)$ which crosses level 0 in one step occur as a Poisson process with rate $\sum_{k=\epsilon N}^{\infty} F(k)$ where $F(k) = \sum_{|x|\geq k} p(x)$, condition (2.15) implies that

$$\limsup_{N\to\infty} N^2 \sum_{k=\epsilon N}^{\infty} F(k) \leq C < +\infty.$$

In particular,

$$\sup_{N\in\mathbb{Z}^+} N^2 \sum_{k=N}^{\infty} F(k) \leq C_\epsilon < +\infty. \tag{2.16}$$

Let $H(y) = y^3 \log^{-\beta}(y \vee 2)$ for some $\beta > 0$. Let $H^{(1)}(k) = H(k) - H(k-1)$ and $H^{(2)}(k) = H^{(1)}(k) - H^{(1)}(k-1) = H(k) + H(k-2) - 2H(k-1)$, which are the discrete gradient and laplacian of H. Then for $k \geq k_0$ for some $k_0 \in \mathbb{Z}^+$, $0 < H^{(2)}(k) < 8k \log^{-\beta} k$. Denote $G(k) = \sum_{i=k}^{\infty} F(i)$. Then (2.16) is the same as $G(k) \leq \frac{C_\epsilon}{k^2}$ for all $k \in \mathbb{Z}^+$. Recall that $p^s(k) = \frac{p(k)+p(-k)}{2}$, we have by summation by parts

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} H(|k|)p(k) &= \sum_{k=1}^{\infty} 2H(k)p^s(k) \\
&= \sum_{k=1}^{k_0-1} 2H(k)p^s(k) + H(k_0)F(k_0) + \sum_{k=k_0+1}^{\infty} H^{(1)}(k)F(k) \\
&= \sum_{k=1}^{k_0-1} 2H(k)p^s(k) + H(k_0)F(k_0) \\
&\qquad + H^{(1)}(k_0+1)G(k_0+1) + \sum_{k=k_0+2}^{\infty} H^{(2)}(k)G(k) \\
&\leq \sum_{k=1}^{k_0-1} 2H(k)p^s(k) + H(k_0)F(k_0) \\
&\qquad + H^{(1)}(k_0+1)G(k_0+1) + \sum_{k=k_0+2}^{\infty} \frac{8k}{\log^{\beta} k} \cdot \frac{C_\epsilon}{k^2} \\
&< \infty
\end{aligned}$$

for $\beta > 1$. This concludes the proof. □

We end this section with

Proof of Theorem 1.2: In [5, 6], the standard Brownian web $\bar{\mathcal{W}}$ is defined as a random variable taking values in the space of compact sets of paths (see [5, 6] for more details), which is essentially a system of one-dimensional coalescing Brownian motions with one Brownian path starting from every space-time point. In [9], it was shown that under diffusive scaling, the random set of coalescing random walk paths with one walker starting from every point on the space-time lattice $\mathbb{Z} \times \mathbb{Z}$ converges to $\bar{\mathcal{W}}$ in the topology of the Brownian web (the details for the continuous time walks case is given in [11]), provided that the random walk jump kernel $p(\cdot)$ has finite fifth moment. To improve their result from finite fifth moment to finite γ-th moment for any $\gamma > 3$, we only need to verify the tightness criterion (T_1) formulated in [9], the other convergence criteria require either only finite second moment or tightness.

Recall the tightness criteria (T_1) in [9],

$$(T_1) \qquad \lim_{t\downarrow 0} \frac{1}{t} \limsup_{\delta\downarrow 0} \sup_{(x_0,t_0)\in\Lambda_{L,T}} \mu_\delta(A_{t,u}(x_0,t_0)) = 0, \quad \forall\, u > 0,$$

where $\Lambda_{L,T} = [-L, L] \times [-T, T]$, μ_δ is the distribution of $\mathcal{X}_\delta$, $R(x_0, t_0; u, t)$ is the rectangle $[x_0 - u, x_0 + u] \times [t_0, t_0 + t]$, and $A_{t,u}(x_0, t_0)$ is the event that (see Figure 2) the random set

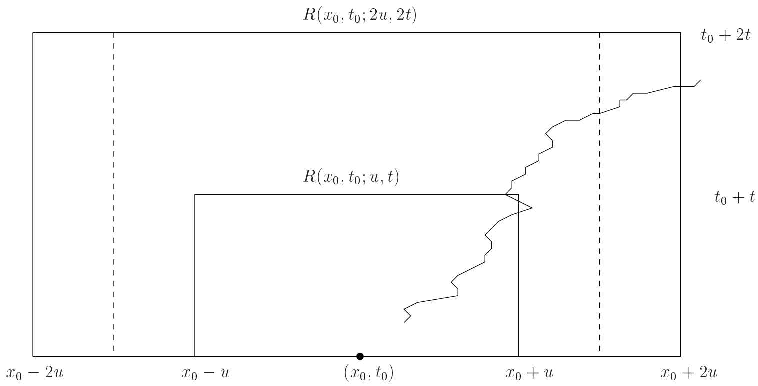

Figure 2: Illustration of the event $A_{t,u}(x_0, t_0)$.

of coalescing walk paths contains a path touching both $R(x_0, t_0; u, t)$ and (at a later time) the left or right boundary of the bigger rectangle $R(x_0, t_0; 2u, 2t)$. In [9], in order to guarantee the continuity of paths, the random walk paths are taken to be the interpolation between consecutive space-time points where jumps take place. Thus the contribution to the event $A_{t,u}(x_0, t_0)$ is either due to interpolated line segments intersecting the inner rectangle $R(x_0, t_0; u, t)$ and then not landing inside the intermediate rectangle $R(x_0, t_0; 3u/2, 2t)$, which can be shown to have 0 probability in the limit $\delta \to 0$ if $p(\cdot)$ has finite third moment; or it is due to some random walk originating from inside $R(x_0, t_0; 3u/2, 2t)$ and then reaches either level $-2u$ or $2u$ before time $2t$. In terms of the unscaled random walk paths, and note the symmetry between left and right boundaries, condition (T_1) reduces to

$$\lim_{t\downarrow 0} \frac{1}{t} \limsup_{\delta\to 0} \mathbb{P}\left\{ \zeta_{s_2}^{s_1}([\frac{u\sigma}{2\delta}, \frac{7u\sigma}{2\delta}]) \cap (-\infty, 0] \neq \phi \text{ for some } 0 \leq s_2 < s_1 \leq \frac{t}{\delta^2} \right\} = 0,$$

which by the reflection principle for random walks is further implied by

$$\lim_{t\downarrow 0} \frac{1}{t} \limsup_{\delta\to 0} \mathbb{P}\left\{ \zeta_s^s([\frac{u\sigma}{2\delta}, \frac{7u\sigma}{2\delta}]) \cap (-\infty, 0] \neq \phi \text{ for some } 0 \leq s \leq \frac{t}{\delta^2} \right\} = 0,$$

which is a direct consequence of Proposition 2.3. This establishes the first part of Theorem 1.2. It is easily seen that the tightness of $\{\mathcal{X}_\delta\}$ imposes certain equicontinuity conditions on the random walk paths, and the condition in (2.15) and its mirror statement are also necessary for the tightness of $\{\mathcal{X}_\delta\}$, and hence the convergence of $\mathcal{X}_\delta$ (with $\delta = \frac{1}{N}$) to the standard Brownian web $\bar{\mathcal{W}}$. Therefore, we must also have $\sum_{x\in\mathbb{Z}} \frac{|x|^3}{\log^\beta(|x|\vee 2)} p(x) < \infty$ for all $\beta > 1$. $\square$

3 Proof of Theorem 1.3

In this section we assume that $2 < \gamma < 3$ and we fix $0 < \theta < \frac{\gamma-2}{\gamma}$.

We recall the definition of $(\eta_t^N)_{t\geq 0}$ on Ω. The evolution of this process is described by the same Harris system on which we constructed $(\eta_t)_{t\geq 0}$, i.e., the family of Poisson point processes $\{\mathcal{N}^{x,y}\}_{x,y\in\mathbb{Z}}$, except that if $t \in \mathcal{N}^{x,y} \cup \mathcal{N}^{y,x}$, for some $y > x$ with $y - x \geq N^{1-\theta}$ and $[x, y] \cap$

$[r^N_{t-} - N, r^N_{t-}] \neq \phi$, then a flip from 0 to 1 at x or y, if it should occur, is suppressed. We also let $(\eta^N_t)_{t\geq 0}$ start from the Heavyside configuration $\eta_{1,0}$. We also recall that we denote by r^N_t the position of its rightmost "1".

Since $(\eta_t)_{t\geq 0}$ and $(\eta^N_t)_{t\geq 0}$ are generated by the same Harris system and they start with the same configuration, it is natural to believe that $r^N_t = r_t$ for "most" $0 \leq t \leq N^2$ with high probability. To see this we use the additive structure of the voter model to show (ii) in Theorem 1.3.

For a fixed realization of the process $(\eta^N_t)_{t\geq 0}$, we denote by $t_1 < ... < t_k$ the times of the suppressed jumps in the time interval $[0, TN^2]$ and by $x_1, ..., x_k$ the target sites, i.e., the sites where the suppressed flips should have occurred. Now let $(\eta^{t_i,x_i}_t)_{t\geq 0}$ be voter models constructed on the same Harris system starting at time t_i with a single 1 at site x_i. As usual we denote by r^{t_i,x_i}_t, $t \geq t_i$, the position of the rightmost "1". It is straightforward to verify that

$$0 \leq r_t - r^N_t = \max_{\substack{1\leq i\leq k \\ t_i \leq t}} (r^{t_i,x_i}_t - r^N_t) \vee 0 .$$

The random set of times $\{t_i\}$ is a Poisson point process on $[0, N^2]$ with rate at most

$$\sum_{\substack{[x,y]\cap[-N,0]\neq\phi \\ y-x\geq N^{1-\theta}}} \{p(y-x) + p(x-y)\} \leq \sum_{|x|\geq N^{1-\theta}} |x|p(x) + (N+1) \sum_{|x|\geq N^{1-\theta}} p(x) ,$$

which is further bounded by

$$\frac{2\sum_{x\in\mathbb{Z}} |x|^\alpha p(x)}{N^{(1-\theta)\alpha-1}}$$

for every $\alpha > 1$. Therefore if we take $\alpha = \gamma$, then by the choice of θ and the assumption that the γ-moment of the transition probability is finite, we have that the rate decreases as $N^{-(1+\epsilon)}$ for $\epsilon = (1-\theta)\gamma - 2 > 0$.

Lemma 3.1. *Let $\{(t_i, x_i)\}_{i\in\mathbb{N}}$ with $t_1 < t_2 < \cdots$ denote the random set of space-time points in the Harris system where a flip is suppressed in $(\eta^N_t)_{t\geq 0}$. Let $K = \max\{i \in \mathbb{N} : t_i \leq TN^2\}$, and let*

$$\tau_i = \inf\{t \geq t_i : \eta^{t_i,x_i}_t \equiv 0 \text{ on } \mathbb{Z}\} - t_i .$$

Then

$$\mathrm{P}[\tau_i \geq N^2 \text{ for some } 1 \leq i \leq K] \to 0 \quad \text{as } N \to \infty ,$$

and for all $i \in \mathbb{N}$,

$$E[\tau_i; \tau_i \leq N^2] \leq CN .$$

Moreover, from these estimates we have that

$$N^{-2} E\left[\sum_{i=1}^{K} \tau_i \middle| \tau_i \leq N^2 \text{ for all } 1 \leq i \leq K\right] \to 0 \quad \text{as } N \to \infty .$$

Proof:

The proof is basically a corollary of Lemma 5.6, which gives that the lifetime τ of a single particle voter model satisfies

$$\mathrm{P}[\tau \geq t] \leq \frac{C}{\sqrt{t}}$$

for some $C > 0$. Thus, by the strong Markov Property

$$\begin{aligned} \mathrm{P}[\tau_i \geq N^2 \text{ for some } 1 \leq i \leq K] &\leq \sum_{k=0}^{+\infty} \mathrm{P}[\tau_k \geq N^2 |\ t_k \leq TN^2]\ \mathrm{P}[t_k \leq TN^2] \\ &= \mathrm{P}[\tau_1 \geq N^2]\ \mathbb{E}[K] \\ &\leq \frac{C}{N} \cdot TN^2 \cdot \frac{2\sum_{x\in\mathbb{Z}} |x|^\gamma p(x)}{N^{(1-\theta)\gamma-1}} = \frac{C'}{N^\epsilon}\,, \end{aligned}$$

which gives the first assertion in the lemma. The verification of $E[\tau_i; \tau_i \leq N^2] \leq CN$ is trivial. Now from the first two assertions in the lemma we obtain easily the third one. $\square$

Now to complete the proof of (ii) in Theorem 1.3, observe that if $s \in [0, TN^2]$ then $r_s^N \neq r_s$ only if $s \in \cup_{i=1}^K [t_i, (\tau_i + t_i) \wedge TN^2)$, and then

$$\int_0^{TN^2} I_{r_s^N \neq r_s} ds \leq \sum_{i=1}^K ((\tau_i + t_i) \wedge TN^2) - t_i) \leq \sum_{i=1}^K (\tau_i \wedge TN^2)\,.$$

The result follows from the previous lemma by usual estimates.

Now we show (i) in Theorem 1.3. The convergence of the finite-dimensional distributions follows from a similar argument as the proof of (ii) in Theorem 1.3, which treats η_t^N as a perturbation of η_t. We omit the details. Similar to (2.1) — (2.3) in the proof of Theorem 1.1, tightness can be reduced to showing that for all $\epsilon > 0$,

$$\limsup_{\delta\to 0} \delta^{-1} \limsup_{N\to+\infty} \mathrm{P}\left[\sup_{0\leq t\leq \delta N^2} r_t^N \geq \epsilon N\right] = 0, \tag{3.1}$$

for which we can adapt the proof of Theorem 1.1. As the next lemma shows, it suffices to consider the system of coalescing random walks with jumps of size greater than or equal to $N^{1-\theta}$ suppressed.

Lemma 3.2. *For almost every realization of the Harris system in the time interval $[0, \delta N^2]$ with $\sup_{0\leq t\leq \delta N^2} r_t^N \geq \epsilon N$ for some $0 < \epsilon < 1$, there exists a dual backward random walk starting from some site in $\{\mathbb{Z} \cap [\epsilon N, +\infty)\} \times [0, \delta N^2]$ which attains the left of the origin before time 0, where all jumps of size greater than or equal to $N^{1-\theta}$ in the Harris system have been suppressed.*

Proof:

Since $(\eta_t^N)_{t\geq 0}$ starts from the Heavyside configuration, for a realization of the Harris system with $\sup_{0\leq s\leq \delta N^2} r_s^N \geq \epsilon N$, by duality, in the same Harris system with jumps that are discarded in the definition of $(\eta_t^N)_{t\geq 0}$ suppressed, we can find a backward random walk which starts from some site $(x, s) \in \{\mathbb{Z} \cap [\epsilon N, +\infty)\} \times [0, \delta N^2]$ with $\eta_s^N(x) = 1$ and attains the left of the origin before reaching time 0. If by the time the walk first reaches the left of the origin, it has made no jumps of size greater than or equal to $N^{1-\theta}$, we are done; otherwise when the first large jump occurs the ra ndom walk must be to the right of the origin, and by the definition of η_t^N, either the jump does not induce a flip from 0 to 1, in which case we can ignore this large jump and continue tracing backward in time; or the rightmost 1 must be at least at a distance N to the right of the position of the random walk before the jump, in which case since $\epsilon < 1$, at this time there is a dual random walk i n $\mathbb{Z} \cap [\epsilon N, +\infty)$ which also attains the left of the origin before reaching time 0. Now either this second random walk makes no jump of size greater than or equal to $N^{1-\theta}$ before it reaches time 0, or we repeat the previous argument to find another random walk starting in $\{\mathbb{Z} \cap [\epsilon N, +\infty)\} \times [0, \delta N^2]$ which also att ains the left of the origin before reaching time 0. For almost surely all realizations of the Harris system, the above procedure can only be iterated a finite number of times. The lemma then follows. □

Lemma 3.2 reduces (3.1) to an analogous statement for a system of coalescing random walks with jumps larger than or equal to $N^{1-\theta}$ suppressed.

Take $0 < \sigma < \theta$ and let $\epsilon' := \frac{(1-\theta)(3-\gamma)}{\sigma}$. Then

$$\sum_{|x| \le N^{1-\theta}} |x|^{3+\epsilon'} p(x) \le N^{(1-\theta)(3+\epsilon'-\gamma)} \sum_{x \in \mathbb{Z}} |x|^\gamma p(x) \le C N^{(1-\theta+\sigma)\epsilon'} . \tag{3.2}$$

The estimate required here is the same as in the proof of Theorem 1.1, except that as we increase the index N, the random walk kernel also changes and its $(3+\epsilon')$th-moment increases as $CN^{(1-\theta+\sigma)\epsilon'}$. Therefore it remains to correct the exponents in Proposition 2.3. Denote by ζ^N the system of coalescing random walks with jumps larger than or equal to $N^{1-\theta}$ suppressed, and recall that $R = \sqrt{\delta} N$ and $M = \epsilon/\sqrt{\delta}$ in our argument, (3.1) then follows from

Proposition 3.3. *For $R > 0$ sufficiently large and $2^b \le M < 2^{b+1}$ for some $b \in \mathbb{N}$, the probability*

$$\mathrm{P}\left\{\zeta_t^{N,t}([MR, +\infty)) \cap (-\infty, 0] \neq \phi \ \ \text{for some } t \in [0, R^2]\right\}$$

is bounded above by a constant times

$$\sum_{k \ge b} \left\{ \frac{1}{2^{2k} \delta^{\epsilon'} R^{\frac{(\theta-\sigma)\epsilon'}{2}}} + e^{-c2^k} + 2^k R^4 e^{-c2^{k(1-\beta)} R^{\frac{(1-\beta)}{2}}} + 2^k e^{-c2^{2k}} \right\} \tag{3.3}$$

for some $c > 0$ and $0 < \beta < 1$.

The only term that has changed from Proposition 2.3 is the first term, which arises from the application of Lemma 5.5. We have incorporated the fact that the $3+\epsilon'$ moment of the random walk with large jumps suppressed grows as $CN^{(1-\theta+\sigma)\epsilon'}$, and we have employed a tighter bound for the power of R than stated in Proposition 2.3. The other three terms remain unchanged because the second term comes from the particle reduction argument derived from applications of Proposition 5.4, while the third and forth terms come from the Gaussian correction on Lemma 5.1. The constants in these three terms only depend on the second moment of the truncated random walks which is uniformly bounded. The verification of this last assertion only need some more concern in the case of the second term due to applications of Lemma 5.2. But if we go through the proof of Theorem T1 in section 7 and Proposition P4 in Section 32 of [10], we see that in order to obtain uniformity in Lemma 5.2 for a family of random walks, we only need uniform bounds on the characteristic functions associated to the walks in the family, which are clearly satisfied by the family of random walks with suppressed jumps. This concludes the proof of Theorem 1.3. □

4 Proof of Theorem 1.4

4.1 Proof of (i) in Theorem 1.4

We start by proving (i) in Theorem 1.4. Since $(\eta_t \circ \Theta_{\ell_t}|\mathbb{N})_{t\geq 0}$ is a positive recurrent Markov chain on $\tilde{\Omega}$, by usual convergence results, we only have to show that starting from the Heavyside configuration for every t and M sufficiently large

$$P(r_t - l_t \geq M) \geq \frac{C}{M}\,,$$

for some $C > 0$ independent of M ant t. Now fix $\lambda > 0$, this last probability is bounded below by

$$\begin{aligned}&P(r_t - l_t \geq M,\ r_{t-\lambda M^2} - l_{t-\lambda M^2} \leq M)\\ &\quad = P(r_t - l_t \geq M | r_{t-\lambda M^2} - l_{t-\lambda M^2} \leq M) P(r_{t-\lambda M^2} - l_{t-\lambda M^2} \leq M)\,,\end{aligned}$$

which by tightness is bounded below by

$$\frac{1}{2} P(r_t - l_t \geq M | r_{t-\lambda M^2} - l_{t-\lambda M^2} \leq M)$$

for M sufficiently large. To estimate the last probability we introduce some notation first, let $(X_t^{-M})_{t\geq 0}$ and $(X_t^M)_{t\geq 0}$ be two independent random walks starting respectively at $-M$ and M at time 0 with transition probability $p(\cdot)$. Denote $Z_t^M = X_t^M - X_t^{-M}$. For every set $A \subset \mathbb{Z}$, let τ_A be the stopping time

$$\inf\{t \geq 0 : Z_t^M \in A\}\,.$$

If $A = \{x\}$, we denote τ_A simply by τ_x. Then by duality and the Markov property after translating the system to have the leftmost 0 at the origin by time $t - \lambda M^2$ we obtain that

$$P(r_t - l_t \geq 2M | r_{t-\lambda M^2} - l_{t-\lambda M^2} \leq M) \geq P(\tau_0 > \lambda M^2; X_{\lambda M^2}^{-M} \geq M; X_{\lambda M^2}^M \leq -M)\,.$$

Part (i) of Theorem 1.4 then follows from the next result:

Lemma 4.1. *If $p(\cdot)$ is a non-nearest neighbor transition probability and has zero mean and finite second moment, then we can take λ sufficiently large such that for some $C > 0$ independent of M and for all M sufficiently large,*

$$P(\tau_0 > \lambda M^2; X_{\lambda M^2}^{-M} \geq M; X_{\lambda M^2}^M \leq -M) \geq \frac{C}{M}. \tag{4.1}$$

Let $A_s(M,k,x)$ be the event

$$\{\tau_0^{x,x+k} > \lambda M^2 - s;\ X^{x+k}_{\lambda M^2-s} \geq M; X^x_{\lambda M^2-s} \leq -M\},$$

where as before, for every x and y, $(X^x_t)_{t\geq 0}$ and $(X^y_t)_{t\geq 0}$ denote two independent random walks starting respectively at x and y with transition probability $p(\cdot)$, and

$$\tau_0^{x,x+k} = \inf\{t \geq 0 : X^{x+k}_t - X^x_t = 0\}.$$

To prove Lemma 4.1 we apply the following result:

Lemma 4.2. *Let $K \in \mathbb{N}$ be fixed. For all $l \in \mathbb{N}$ sufficiently large, there exists some $C > 0$ such that for all $s \leq \lambda M^2/2$, $|x| < lM$ and $0 < k \leq K$, and M sufficiently large*

$$P(A_s(M,k,x)) > \frac{C}{M}.$$

Proof of Lemma 4.1:

Let $\mathbb{Z}^-$ denote $\mathbb{Z} \cap (-\infty, 0]$. The probability in (4.1) is then bounded below by

$$P\left(\tau_{\mathbb{Z}^-} < \frac{\lambda M^2}{2}; \tau_0 > \lambda M^2;\ X^{-M}_{\lambda M^2} \geq M; X^M_{\lambda M^2} \leq -M\right)$$

which by the Strong Markov property is greater than or equal to

$$\sum_{|x|\leq lM}\sum_{1\leq k\leq K}\int_0^{\lambda M^2/2} P\left(\tau_{\mathbb{Z}^-} \in ds; X^{-M}_s = x+k, X^M_s = x\right) P(A_s(M,k,x)),$$

where $l \in \mathbb{N}$ is some fixed large constant. Now applying Lemma 4.2 we have that the probability in (4.1) is bounded below by

$$\frac{C}{M}\sum_{|x|\leq lM}\sum_{1\leq k\leq K} P\left(\tau_{\mathbb{Z}^-} < \frac{\lambda M^2}{2}; X^{-M}_{\tau_{\mathbb{Z}^-}} = x+k, X^M_{\tau_{\mathbb{Z}^-}} = x\right)$$

Thus to finish the proof we have to show that

$$\sum_{|x|\leq lM}\sum_{1\leq k\leq K} P\left(\tau_{\mathbb{Z}^-} < \frac{\lambda M^2}{2}; X^{-M}_{\tau_{\mathbb{Z}^-}} = x+k, X^M_{\tau_{\mathbb{Z}^-}} = x\right) \tag{4.2}$$

is bounded below uniformly over M by some positive constant.

Let $D = \{(x, x+k) :\ 1 \leq k \leq K \text{ and } |x| < lM\}$, then this last expression can be rewritten as

$$\begin{aligned}
&P\left(\tau_{\mathbb{Z}^-} \leq \frac{\lambda M^2}{2};\ \left(X^M_{\tau_{\mathbb{Z}^-}}, X^{-M}_{\tau_{\mathbb{Z}^-}}\right) \in D\right)\\
&\geq\ P\left(\tau_{\mathbb{Z}^-} \leq \frac{\lambda M^2}{2}\right) - P\left(\tau_{\mathbb{Z}^-} \leq \frac{\lambda M^2}{2};\ X^{-M}_{\tau_{\mathbb{Z}^-}} - X^M_{\tau_{\mathbb{Z}^-}} = 0 \text{ or } > K\right)\\
&\quad -P\left(\tau_{\mathbb{Z}^-} \leq \frac{\lambda M^2}{2};\ \left|X^M_{\tau_{\mathbb{Z}^-}}\right| \geq lM\right)\\
&\geq\ P\left(\tau_{\mathbb{Z}^-} \leq \frac{\lambda M^2}{2}\right) - P\left(Z^M_{\tau_{\mathbb{Z}^-}} = 0 \text{ or } < -K\right) - P\left(\sup_{0\leq t\leq \lambda M^2/2}\left|X^M_t\right| \geq lM\right).
\end{aligned}$$

We claim that the second term can be bounded uniformly away from 1 for large M by taking K large. This follows from a standard result for random walks (see, e.g., Proposition 24.7 in [10]), which states that: if a mean zero random walk Z_t^M starting from $2M > 0$ has finite second moment, then the overshoot $Z^M_{\tau_{\mathbb{Z}^-}}$ converges to a limiting probability distribution on $\mathbb{Z}^-$ as $2M \to +\infty$. The distribution is concentrated at 0 only if the random walk is nearest-neighbor. Then by Donsker's invariance principle, the first term can be made arbitrarily close to 1 uniformly over large M by taking λ large, and finally the last term can be made arbitrarily close to 0 uniformly over large M by taking l sufficiently large. With appropriate choices for K, λ and l, we can guarantee that (4.2) is bounded below by a positive constant uniformly for large M, which completes the proof of the Lemma. □

It remains to prove Lemma 4.2.

Proof of Lemma 4.2:

By the Markov property the probability of $A_s(M,k,x)$ is greater than or equal to

$$\sum_{(l_1,l_2)\in D_1} P\left(\tau_0^{x,x+k} > \lambda M^2/4,\ X^x_{\lambda M^2/4} = l_1;\ X^{x+k}_{\lambda M^2/4} = l_2\right) P(B_s(l_1,l_2,M)),$$

where

$$D_1 = \{(l_1,l_2):\ l_2 - l_1 > 2M;\ l_2 < 2lM;\ l_1 > -2lM\}.$$

and for $r = r(M,s) := 3\lambda M^2/4 - s$

$$B_s(l_1,l_2,M) = \{\tau_0^{l_1,l_2} > r(M,s),\ X^{l_2}_{r(M,s)} \geq M;\ X^{l_1}_{r(M,s)} \leq -M\}$$

The proof is then complete with the following two claims.

Claim 1: There exists $C > 0$ such that

$$\inf_{(l_1,l_2)\in D_1} \inf_{s\leq \lambda M^2/2} P(B_s(l_1,l_2,M)) \geq C$$

uniformly over M sufficiently large.

Claim 2: There exists $C > 0$ such that

$$\inf_{1\leq k\leq K} \inf_{|x|\leq lM} \sum_{(l_1,l_2)\in D_1} P\left(\tau_0^{x,x+k} > \lambda M^2/4,\ X^x_{\lambda M^2/4} = l_1,\ X^{x+k}_{\lambda M^2/4} = l_2\right) \geq \frac{C}{M} \tag{4.3}$$

uniformly over M sufficiently large.

Proof of claim 1:

Since $B_s(l_1,l_2,M)$ contains

$$\left\{\max_{m<r(M,s)} X^{l_1}_m < l_1 + M;\ X^{l_1}_{r(M,s)} < l_1 - (2l-1)M\right\}$$
$$\cap \left\{\min_{m<r(M,s)} X^{l_2}_m > l_2 - M;\ X^{l_2}_{r(M,s)} > l_2 + (2l-1)M\right\},$$

by independence and reflection symmetry,

$$P(B_s(l_1,l_2,M)) \geq P\left\{\min_{t<r(M,s)} X^0_t > -M;\ X^0_{r(M,s)} > (2l-1)M\right\}^2.$$

Since $\lambda M^2/4 \leq r(M,s) \leq 3\lambda M^2/4$, by Donsker's invariance principle the above quantity is uniformly bounded below by some $C>0$ for M sufficiently large. This establishes Claim 1.

Proof of Claim 2:
We write the sum in (4.3) as

$$CP\left(\tau_0^{x,x+k} > \lambda M^2/4;\ (X^x_{\lambda M^2/4}, X^{x+k}_{\lambda M^2/4}) \in D_1\right)$$

which by the definition of D_1 is greater than or equal to

$$\begin{aligned} &P\left(\tau_0^{x,x+k} > \lambda M^2/4;\ X^{x+k}_{\lambda M^2/4} - X^x_{\lambda M^2/4} > 2M\right) \\ &\quad -P\left(\tau_0^{x,x+k} > \lambda M^2/4;\ X^{x+k}_{\lambda M^2/4} > 2lM \text{ or } X^x_{\lambda M^2/4} < -2lM\right). \end{aligned}$$

The first term in this expression is bounded below by C/M for some constant $C>0$, dependent only on K. This follows from Theorem B in the Appendix of [4], which states that the conditional distribution of $Z^k_{\lambda M^2}/M := (X^{x+k}_{\lambda M^2} - X^x_{\lambda M^2})/M$ conditioned on $\tau_0 > \lambda M^2$ converges to a two-sided Rayleigh distribution. For the second term, we apply Lemma 2 in [1] and then Lemma 5.2, to dominate it by

$$CP(\tau_0^{x,x+k} > \lambda M^2) P\left(\sup_{0\leq t\leq \frac{\lambda M^2}{2}} X^0_t > lM\right) \leq \frac{C}{M} P\left(\sup_{0\leq t\leq \frac{\lambda M^2}{2}} X^0_t > lM\right),$$

where C depends only on K. Since $P\left(\sup_{0\leq t\leq \frac{\lambda M^2}{2}} X^0_t > lM\right)$ can be made arbitrarily small uniformly for large M if l is sufficiently large, and $1\leq k\leq K$, we obtain the desired uniform bound in Claim 2. $\square$

4.2 Proof of (ii) in Theorem 1.4

We still consider the voter model $(\eta_t : t\geq 0)$ starting from the initial Heavyside configuration. Under the assumption $\gamma > 3$, $\mathrm{P}(r_t - \ell_t \geq M)$ converges to $\pi(\xi : \Gamma(\xi) \geq M)$ as $t \to +\infty$. Therefore, to prove Theorem 1.4 (ii), it suffices to show that, for every $M>0$, if t is sufficiently large, then

$$\mathrm{P}(r_t - l_t \geq M) \leq \frac{C}{M}$$

for some $C>0$ independent on M and t.

We now fix $N\in\mathbb{N}$ and assume $M = 2^N$, which is not a restriction to the result since $2^N \leq M < 2^{N+1}$ for some $N\in\mathbb{N}$ and the inequality (1.4) remains valid by replacing C_2 with $2C_2$. In

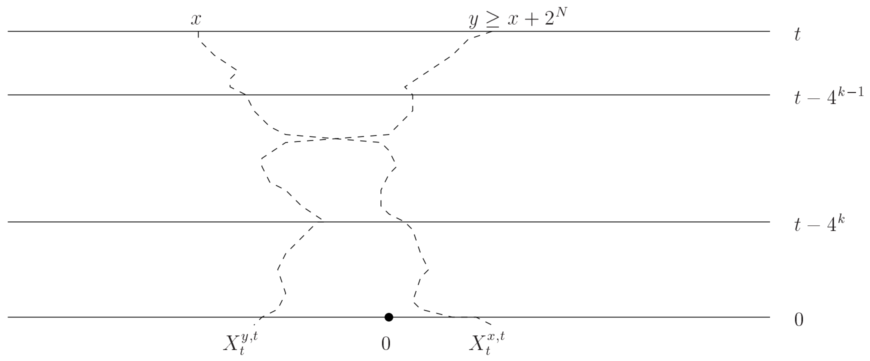

Figure 3: Illustration of the event V_k^N.

the following t will be $>> 2^{2N}$. Let $\Delta_t(s)$, for $s < t$, be the event that a crossing of two dual coalescing random walks starting at time t (in the voter model) occurs in the dual time interval $(s, t]$ and by the dual time t they are on opposite sides of the origin, i.e, there exists $u, v \in \mathbb{Z}$ with $X_s^{u,t} < X_s^{v,t}$ and $X_t^{v,t} \leq 0 < X_t^{u,t}$.

From the estimates in the proof of lemma 5 in Cox and Durrett [4], one can show that $\mathrm{P}(\Delta_t(s)) \leq C/\sqrt{s}$, if we have that $\mathrm{P}(0 \in \zeta_s^s(\mathbb{Z})) \leq C/\sqrt{s}$, which holds if $p(\cdot)$ has finite second moment (see Lemma 5.6). Therefore, all we have to show is that

$$\mathrm{P}\left(\{r_t - l_t \geq 2^N\} \cap (\Delta_t(4^N))^c\right) \leq \frac{C}{2^N} \tag{4.4}$$

for some C independent of t and N. We denote the event $\{r_t - l_t \geq 2^N\} \cap (\Delta_t(4^N))^c$ by V^N which is a subset of $\cup_{k=0}^N V_k^N$ where V_k^N is the event that (see Figure 3) there exists $x, y \in \mathbb{Z}$ with $y - x \geq 2^N$ such that, for the coalescing walks $X_s^{x,t}$ and $X_s^{y,t}$,

(i) $X_s^{x,t} < X_s^{y,t}$ for every $0 \leq s \leq 4^{k-1}$;

(ii) There exists $s \in (4^{k-1}, 4^k]$ with $X_s^{x,t} > X_s^{y,t}$;

(iii) $X_t^{x,t} > 0$ and $X_t^{y,t} \leq 0$.

For $k = 0$ we replace 4^{k-1} by 0. We will obtain suitable bounds on V_k^N which will enable us to conclude that $\sum_{k=0}^N P(V_k^N) \leq \frac{C}{2^N}$.

Fix $0 \leq k \leq N$. For $0 \leq s \leq t$ and $y \in \mathbb{Z}$, we call

$$R_y(s) := \begin{cases} \sup_{x \in \mathbb{Z}}\{|x - y| : X_s^{x,t} = y\} & \text{, if there exists } x \text{ such that } X_s^{x,t} = y \\ 0 & \text{, otherwise} \end{cases}$$

the *range* of the coalescing random walk at $(s, y) \in (0, t] \times \mathbb{Z}$. Obviously V_k^N is contained in the event that there exists x, y in $\zeta_{4^{k-1}}^t(\mathbb{Z})$ with $x < y$ such that

(i) $R_x(4^{k-1}) + R_y(4^{k-1}) + |y - x| \geq 2^N$;

(ii) There exists $s \in (4^{k-1}, 4^k]$ with $X_{s-4^{k-1}}^{x,t-4^{k-1}} > X_{s-4^{k-1}}^{y,t-4^{k-1}}$;

(iii) $X^{x,t-4^{k-1}}_{t-4^{k-1}} > 0$, $X^{y,t-4^{k-1}}_{t-4^{k-1}} \leq 0$,

which we denote by $\tilde{V}_k^N$.

We call the crossing between two coalescing random walks a *relevant crossing* if it satisfies conditions (i) and (ii) in the definition of $\tilde{V}_k^N$ up to the time of the crossing. We are interested in the density of relevant crossings between random walks in the time interval $(4^{k-1}, 4^k]$ and (as is also relevant) the size of the overshoot, i.e., the distance between the random walks just after crossing. To begin we consider separately three cases:

(i) The random walks at time 4^{k-1} are at $x < y$ with $|x - y| \leq 2^{k-1}$ (so it is "reasonable" to expect the random walks to cross in the time interval $(4^{k-1}, 4^k]$, and either $R_x(4^{k-1})$ or $R_y(4^{k-1})$ must exceed 2^{N-2}).

(ii) The random walks are separated at time 4^{k-1} by at least 2^{k-1} but no more than 2^{N-1} (so either $R_x(4^{k-1})$ or $R_y(4^{k-1})$ must exceed 2^{N-2}).

(iii) The random walks are separated at time 4^{k-1} by at least 2^{N-1}. In this case we disregard the size of the range.

Before dealing specifically with each case, we shall consider estimates on the density of particles in $\zeta^t_{4^k}(\mathbb{Z})$ with range greater than $m2^k$. We first consider the density of random walks at time 4^k which move by more than $m2^k$ in the time interval $(4^k, 4^{k+1}]$. By Lemma 5.6, the density of particles in $\zeta^t_{4^k}(\mathbb{Z})$ is bounded by $\frac{C}{2^k}$. By the Markov property and Lemma 5.1, we obtain the following result:

Lemma 4.3. *For every $0 < \beta < 1$, there exists $c, C \in (0, \infty)$ such that for every $k \in \mathbb{N}$ and $m \geq 1$, the probability that a fixed site $y \in \mathbb{Z}$ satisfies $y \in \zeta^t_{4^k}(\mathbb{Z})$, and the backward random walk starting at $(y, t - 4^k)$ makes an excursion of size at least $m2^k$ before reaching time level $t - 4^{k+1}$ is bounded by*

$$\frac{C}{2^k}\left(e^{-c(m2^k)^{1-\beta}} + e^{-cm^2} + \frac{1}{m^{3+\epsilon}2^{k(1+\epsilon)}}\right).$$

As a corollary, we have

Lemma 4.4. *For every $0 < \beta < 1$, there exists $c, C \in (0, \infty)$ so that for every $k \in \mathbb{N}$ and $m \geq 1$, the density of $y \in \zeta^t_{2^{2k}}(\mathbb{Z})$ whose range is greater than $m2^k$ is bounded by*

$$\frac{C}{2^k}\left(2^k e^{-c(m2^k)^{1-\beta}} + e^{-cm^2} + \frac{1}{m^{3+\epsilon}2^{k(1+\epsilon)}}\right).$$

Proof:

Let $d_{l,k}$ be the density of coalescing random walks remaining at time 4^l, which on interval $(4^l, 4^{l+1}]$ move by more than

$$\left(\sum_{r=1}^{\infty}\frac{1}{r^2}\right)^{-1}\frac{m2^k}{(k-l)^2}.$$

By Lemma 4.3 we have that $d_{l,k}$ is bounded above by

$$\frac{C}{2^l}\left[e^{-c\left(\frac{m2^k}{(k-l)^2}\right)^{1-\beta}}+e^{\frac{-c(m2^{k-l})^2}{(k-l)^4}}+\frac{(k-l)^{2(3+\epsilon)}}{(m2^{k-l})^{3+\epsilon}2^{l(1+\epsilon)}}\right].$$

It is not difficult to see that $\sum_{l<k} d_{l,k}$ provides an upper bound for the density we seek. Summing the above bounds for $d_{l,k}$ establishes the lemma. □

We can now estimate the relevant crossing densities and overshoot size in cases (i), (ii) and (iii) above. More precisely, we will estimate the expectation of the overshoot between two random walks starting at $x < y$ at time 4^{k-1} restricted to the event that: $x, y \in \zeta^t_{4^{k-1}}(\mathbb{Z})$, R_x and R_y are compatible with $y - x$ as stated in cases (i) –(iii), and the two walks cross before time 4^k. From now on, we fix $\beta \in (0, 1)$.

Case (i): Since if the two events $\{x \in \zeta^t_{4^{k-1}}(\mathbb{Z})\} \cap \{R_x(4^{k-1}) > 2^{N-2}\}$ and $\{y \in \zeta^t_{4^{k-1}}(\mathbb{Z})\}$ both occur, they always occur on disjoint trajectories of random walks in the dual time interval $[0, 4^{k-1}]$, we may apply the van den Berg-Kesten-Reimer inequality (see Lemma 4 in [2] and the discussion therein) which together with the previous lemma implies that the probability that $x, y \in \zeta^t_{4^{k-1}}(\mathbb{Z})$ and at least one has range 2^{N-2} is less than

$$\frac{C}{4^k}\left(2^k e^{-c2^{N(1-\beta)}}+e^{-c4^{N-k}}+\frac{4^k}{2^{N(3+\epsilon)}}\right).$$

Moreover the expectation of the overshoot (see [4])

$$X^{x,t-4^{k-1}}_\tau - X^{y,t-4^{k-1}}_\tau$$

on the event $\tau \le 4^k - 4^{k-1} = 3 \cdot 4^{k-1}$ where

$$\tau = \inf\{s > 0 : X^{x,t-4^{k-1}}_s - X^{y,t-4^{k-1}}_s \ge 0\}$$

is the time of crossing, is uniformly bounded over k and $y - x$.

Case (ii): In this case we must also take into account that the probability of the two random walks crossing before time 4^k is small. We analyze this by dividing up the crossing into two cases. In the first case the two random walks halve the distance between them before crossing. In the second case the crossing occurs due to a jump of order $y - x$.

Let

$$\tau' = \inf\left\{s > 0 : X^{y,t-4^{k-1}}_s - X^{x,t-4^{k-1}}_s < \frac{y-x}{2}\right\}.$$

Then as in Case (i),

$$E[X^{x,t-4^{k-1}}_\tau - X^{y,t-4^{k-1}}_\tau \mid \tau' < \tau]$$

is uniformly bounded by some constant $C > 0$. Therefore

$$
\begin{aligned}
&E\left[X_\tau^{x,t-4^{k-1}} - X_\tau^{y,t-4^{k-1}};\ \tau' < \tau \le 3\cdot 4^{k-1};\ x,y \in \zeta^t_{4^{k-1}}(\mathbb{Z});\ R_x \text{ or } R_y \ge 2^{N-2}\right] \\
&\le C\ P\Big(\tau' < 3\cdot 4^{k-1}\Big)\ \ P\Big(x,y \in \zeta^t_{4^{k-1}}(\mathbb{Z});\ R_x \text{ or } R_y \ge 2^{N-2}\Big) \\
&\le C\,P\left(x,y \in \zeta^t_{4^{k-1}}(\mathbb{Z});\ R_x \text{ or } R_y \ge 2^{N-2}\right) \\
&\qquad\qquad\qquad\qquad \times \left(e^{-c|x-y|^{1-\beta}} + e^{-c\frac{(x-y)^2}{4^k}} + \frac{4^k}{|x-y|^{3+\epsilon}}\right) \\
&\le \frac{C}{4^k}\left(2^k e^{-c2^{N(1-\beta)}} + e^{-c4^{N-k}} + \frac{4^k}{2^{N(3+\epsilon)}}\right) \\
&\qquad\qquad\qquad\qquad \times \left(e^{-c|x-y|^{1-\beta}} + e^{-c\frac{(x-y)^2}{4^k}} + \frac{4^k}{|x-y|^{3+\epsilon}}\right).
\end{aligned}
$$

On the other hand it is easily seen (by estimating the rates at which a large jump occurs, see Section 3 for details) that

$$E[X_\tau^{x,t-4^{k-1}} - X_\tau^{yt-4^{k-1}}, \tau = \tau' < 3\cdot 4^{k-1}] \le C\frac{4^k}{|x-y|^{2+\epsilon}}$$

and so we have a contribution

$$\frac{C}{4^k}\left(2^k e^{-c2^{(1-\beta)N}} + e^{-c4^{N-k}} + \frac{4^k}{2^{N(3+\epsilon)}}\right)\frac{4^k}{|x-y|^{2+\epsilon}}.$$

Case (iii): In this case we argue as in (ii) except the factor

$$\left(2^k e^{-c2^{(1-\beta)N}} + e^{-c4^{N-k}} + \frac{4^k}{2^{N(3+\epsilon)}}\right)$$

is dropped as we make no assumption on the size of R_x or R_y. So our bound is

$$\frac{C}{4^k}\left(\frac{4^k}{|x-y|^{2+\epsilon}} + e^{-c|x-y|^{1-\beta}} + e^{-c\frac{(x-y)^2}{4^k}} + \frac{4^k}{|x-y|^{(3+\epsilon)}}\right).$$

From the three cases above, we can sum over $y \in \mathbb{Z}$ and verify that, for a given site $x \in \mathbb{Z}$, the total expected overshoot associated with relevant crossings in the time interval $(4^{k-1}, 4^k]$ involving $(x, 4^{k-1})$ and $(y, 4^{k-1})$ for all possible $y \in \mathbb{Z}$ is bounded by

$$C\left(\frac{1}{2^{N(1+\epsilon)}} + e^{-c2^{N(1-\beta)}} + \frac{e^{-c4^{N-k}}}{2^k}\right). \tag{4.5}$$

We say a d-crossover ($d \in \mathbb{N}$) occurs at site $x \in \mathbb{Z}$ at time $s \in (4^{k-1}, 4^k]$ if at this time (dual time, for coalescing random walks) a relevant crossing occurs leaving particles at sites x and $x+d$ immediately after the crossing. We de note the indicator function for such a crossover by

$I_k(s,x,d)$. By translation invariance, the distribution of $\{I_k(s,x,d)\}_{s\in(4^{k-1},4^k]}$ is independent of $x \in \mathbb{Z}$.

Let X_s^x and X_s^{x+d} be two independent random walks with transition probability $p(\cdot)$ starting at x and $x+d$ at time 0, and let $\tau_{x,x+d} = \inf\{s : X_s^x = X_s^{x+d}\}$. Then

$$\begin{aligned} P(\tilde{V}_k^N) &\le \sum_{d\in\mathbb{N}}\sum_{x\in\mathbb{Z}} E\left[\int_{4^{k-1}}^{4^k} I_k(s,x,d) P\left(X_{t-s}^x \le 0 < X_{t-s}^{x+d}, \tau_{x,x+d} > t-s\right) ds\right] \\ &= \sum_{d\in\mathbb{N}} E\left[\int_{4^{k-1}}^{4^k} I_k(s,0,d) \sum_{x\in\mathbb{Z}} P\left(X_{t-s}^x \le 0 < X_{t-s}^{x+d}, \tau_{x,x+d} > t-s\right) ds\right]. \end{aligned}$$

If we know that

$$\sum_{x\in\mathbb{Z}} P\left(X_{t-s}^x \le 0 < X_{t-s}^{x+d}, \tau_{x,x+d} > t-s\right) \le Cd \tag{4.6}$$

for some $C > 0$ independent of k, d, s, t and N, and

$$E\left[\sum_{d\in\mathbb{N}} d \int_{4^{k-1}}^{4^k} I_k(s,0,d)ds\right] \le C\left(\frac{1}{2^{N(1+\epsilon)}} + e^{-c2^{N(1-\beta)}} + \frac{e^{-c4^{N-k}}}{2^k}\right), \tag{4.7}$$

then substituting (4.6) and (4.7) into the bound for $P(\tilde{V}_k^N)$ gives

$$\sum_{k=0}^N P(\tilde{V}_k^N) \le C\sum_{k=0}^N \left(\frac{1}{2^{N(1+\epsilon)}} + e^{-c2^{N(1-\beta)}} + \frac{e^{-c4^{N-k}}}{2^k}\right) \le \frac{C'}{2^N}$$

for some $C' > 0$ uniformly over all large t and N and we are done.

If we denote $Z_{s'}^d = X_{s'}^{x+d} - X_{s'}^x$, $(Z_{s'}^d)^+ = Z_{s'}^d \vee 0$ and $\tau_0 = \inf\{s' : Z_{s'}^d = 0\}$, then by translation invariance, it is not difficult to see that

$$\sum_{x\in\mathbb{Z}} P\left(X_{t-s}^x \le 0 < X_{t-s}^{x+d}, \tau_{x,x+d} > t-s\right) = E[(Z_{t-s}^d)^+, \tau_0 > t-s] \le Cd,$$

where the inequality with $C > 0$ uniform over d and t is a standard result for random walks (see Lemma 2 in [4]).

Finally, to show (4.7), we note that the left hand side is the expected overshoot of relevant crossings where one of the two random walks after the crossing is at 0. By translation invariance this is bounded above by the expected overshoot associated with relevant crossings in the time interval $(4^{k-1}, 4^k]$ involving $(0, 4^{k-1})$ and $(y, 4^{k-1})$ for every $y > 0$, which is estim ated in (4.5). Indeed, let $F_k(x, y; m, m+d)$ be the indicator function of the event that a relevant crossover occurs before time 4^k due to random walks starting at sites x and y at time 4^{k-1}, and immediately after the crossover the walks are at positions m and $m+d$. Then by translation invariance and a change of variable

$$\begin{aligned}
E\left[\sum_{d\in\mathbb{N}} d\int_{4^{k-1}}^{4^k} I_k(s,0,d)ds\right] &\le E\left[\sum_{d\in\mathbb{N}} d\sum_{x<y} F_k(x,y;0,d)\right]\\
&= E\left[\sum_{d\in\mathbb{N}} d\sum_{x\in\mathbb{Z},y>0} F_k(x,x+y;0,d)\right]\\
&= \sum_{y>0} E\left[\sum_{d\in\mathbb{N}} d\sum_{x\in\mathbb{Z}} F_k(0,y;-x,-x+d)\right].
\end{aligned}$$

4.3 Proof of (iii) in Theorem 1.4

We know from [1] that if $\gamma \ge 2$, then the voter model interface evolves as a positive recurrent chain, and hence the equilibrium distribution π exists. In particular, $\pi\{\xi_0\} > 0$ where ξ_0 is the trivial interface of the Heavyside configuration $\eta_{1,0}$. Let ξ_t denote the interface configuration at time t starting with ξ_0, and let ν denote its distribution. Then

$$\pi\{\xi : \Gamma(\xi) \ge n\} > \pi\{\xi_0\}\nu\{\Gamma(\xi_t) \ge n\} \tag{4.8}$$

for all $t > 0$. To prove (1.5), it then suffices to show

$$\limsup_{n\to\infty} \frac{\log \nu\{\Gamma(\xi_{n^2}) \ge n\}}{\log n} \ge 2 - \alpha. \tag{4.9}$$

Let X_t^{2n} and X_t^{5n} denote the positions at time t of two independent random walks with transition probability $p(\cdot)$ starting at $2n$ and $5n$ at time 0. Let A denote the event that $X_t^{2n} \in [n, 3n]$ for all $t \in [0, n^2]$, and let B_s, $s \in [0, n^2]$, denote the event that $X_t^{5n} \in [4n, 6n]$ for all $t \in [0, s)$ and $X_t^{5n} \in (-\infty, -n]$ for all $t \in [s, n^2]$. Event B_s can only occur if X_t^{5n} makes a large negative jump at time s. By duality between voter models and coalescing random walks,

$$\begin{aligned}
\nu\{\Gamma(\xi_{n^2}) \ge 3n\} &\ge P\left\{\eta_{n^2}(2n) = 0, \eta_{n^2}(5n) = 1\right\}\\
&\ge P\left\{\bigcup_{s\in[0,n^2]} (A \cap B_s)\right\} = P(A)\; P\left\{\bigcup_{s\in[0,n^2]} B_s\right\}.
\end{aligned}$$

Condition on X_t^{5n} staying inside $[4n, 6n]$ before time s and making a negative jump of size at least $-8n$ at time s, we have by the strong Markov property that

$$\begin{aligned}
P\left\{\bigcup_{s\in[0,n^2]} B_s\right\} \ge \int_0^{n^2} & P\left\{\bigcap_{t\in[0,s)} \left\{X_t^{5n} \in [4n, 6n]\right\}\right\}\left(\sum_{y\le -8n} p(y)\right)\\
&\cdot P\left\{\bigcap_{t\in[s,n^2]} \left\{X_t^{5n} \le -n\right\} \,\middle|\, X_s^{5n} \le -2n\right\} ds.
\end{aligned}$$

By Donsker's invariance principle, the probability of each of the three events: A, $\bigcap_{t\in[0,s)} \Big\{X_t^{5n} \in [4n,6n]\Big\}$ and $\Big\{\bigcap_{t\in[s,n^2]} \{X_t^{5n} \le -n\} \,|\, X_s^{5n} \le -2n\Big\}$, is at lea st β for some $\beta > 0$ independent of n and $s \in [0, n^2]$. Therefore,

$$\nu\{\Gamma(\xi_{n^2}) \ge n\} \ge \nu\{\Gamma(\xi_{n^2}) \ge 3n\} \ge \beta^3 n^2 \left(\sum_{y\le -8n} p(y)\right),$$

which we may symmetrize to obtain

$$\nu\{\Gamma(\xi_{n^2}) \ge n\} \ge \frac{\beta^3}{2} n^2 \left(\sum_{|y|\ge 8n} p(y)\right). \tag{4.10}$$

If (4.9) fails, then there exists some $n_0 \in \mathbb{N}$ and $\epsilon > 0$ such that, for all $n \ge n_0$,

$$\sum_{|y|\ge 8n} p(y) \le \frac{2}{\beta^3 n^2} \nu\{\Gamma(\xi_{n^2}) \ge n\} \le \frac{C}{n^{\alpha+\epsilon}},$$

which implies that

$$\sum_{y\in\mathbb{Z}} |y|^{\alpha+\frac{\epsilon}{2}} p(y) < \infty,$$

contradicting our assumption. This proves the first part of (iii) in Theorem 1.4. To find random walk jump kernel $p(\cdot)$ satisfying (1.6), we can choose $p(\cdot)$ with $\sum_{|y|\ge n} p(y) \sim Cn^{-\alpha}$ for some $C > 0$. (1.6) then follows directly from (4.8) and (4.10). $\square$

5 Technical Estimates

The following lemmas for random walks will be needed.

Lemma 5.1. *Let X_t be a centered continuous time one-dimensional random walk starting at the origin and with finite $3+\epsilon$ moment for some $\epsilon > 0$. Then for every $0 < \beta < 1$, there exists $c, C > 0$ such that*

$$\mathbb{P}\left(\sup_{t\le T} |X_t| \ge M\right) \le \begin{cases} C\left(e^{-cT} + e^{-\frac{cM^2}{T}} + \frac{T}{M^{3+\epsilon}}\right), & T > M \\ C\left(e^{-cM^{1-\beta}} + \frac{T}{M^{3+\epsilon}}\right), & T \le M \end{cases}$$

for all $T, M > 0$. In particular

$$\mathbb{P}\left(\sup_{t\le T} |X_t| \ge M\right) \le C\left(e^{-cM^{1-\beta}} + e^{-\frac{cM^2}{T}} + \frac{T}{M^{3+\epsilon}}\right)$$

for all $T, M > 0$.

Proof: By the reflection principle for random walks, we only have to show that for every $0<\beta<1$, there exists $c, C>0$ such that

$$\mathbb{P}\left(|X_T|\geq M\right)\leq\begin{cases} C\left(e^{-cT}+e^{-\frac{cM^2}{T}}+\frac{T}{M^{3+\epsilon}}\right), & T>M \\ C\left(e^{-cM^{1-\beta}}+\frac{T}{M^{3+\epsilon}}\right), & T\leq M \end{cases} \tag{5.1}$$

for all $M,T>0$. To prove this inequality, we consider the following usual representation of X_t: there exist centered i.i.d. random variables $(Y_n)_{n\geq 1}$ on $\mathbb{Z}$ with finite $3+\epsilon$ moment and a Poisson process $(N_t)_{t\geq 0}$ of parameter 1 independent of the Y_n's, such that

$$X_t=\sum_{j=0}^{N_t}Y_j:=S_{N_t},$$

where $Y_0=0$. The analogue of (5.1) for discrete time random walks appears as corollary 1.8 in [8], from which we obtain

$$\mathbb{P}\left(|S_n|\geq M\right)\leq C\left(e^{-\frac{cM^2}{n}}+\frac{nE[|Y_1|^{3+\epsilon}]}{M^{3+\epsilon}}\right). \tag{5.2}$$

It then follows that

$$\begin{aligned}\mathbb{P}\left(|X_T|\geq M\right)&=\sum_k P(|S_k|\geq M)P(N_T=k)\\ &\leq C\sum_k\left(e^{-\frac{cM^2}{k}}+\frac{kE[|Y_1|^{3+\epsilon}]}{M^{3+\epsilon}}\right)P(N_T=k)\\ &\leq C\left(P(N_T\geq 3T)+e^{-\frac{cM^2}{3T}}+\frac{E[N_T]E[|Y_1|^{3+\epsilon}]}{M^{3+\epsilon}}\right).\end{aligned} \tag{5.3}$$

By basic large deviations results for Poisson distribution, we have $P(N_T\geq 3T)\leq C'e^{-c'T}$ for some $c',C'>0$. Then after adjusting the constants, we obtain

$$\mathbb{P}\left(|X_T|\geq M\right)\leq C\left(e^{-cT}+e^{\frac{-cM^2}{T}}+\frac{TE[|Y_1|^{3+\epsilon}]}{M^{3+\epsilon}}\right)$$

for every $M>0$ and $T>0$.

We now suppose $T\leq M$. Back to the term after the first inequality in equation (5.3),

$$\begin{aligned}&C\sum_k\left(e^{-\frac{cM^2}{k}}+\frac{kE[|Y_1|^{3+\epsilon}]}{M^{3+\epsilon}}\right)P(N_T=k)\\ \leq\ &C\left(P(N_T\geq M^{1+\beta})+e^{-cM^{1-\beta}}+\frac{TE[|Y_1|^{3+\epsilon}]}{M^{3+\epsilon}}\right).\end{aligned}$$

Since

$$P(N_T\geq M^{1+\beta})=e^{-T}\sum_{k\geq M^{1+\beta}}\frac{T^k}{k!}\ \leq\ \frac{T^{M^{1+\beta}}}{(M^{1+\beta})!}\ \leq\ \frac{M^{M^{1+\beta}}}{(M^{1+\beta})!}.$$

By Stirling's formula, we can choose $C>0$ large enough such that for all $M>0$, $P(N_T\geq M^{1+\beta})\leq Ce^{-cM^{1-\beta}}$, thus concluding the proof. □

Lemma 5.2. *Let X_t^x and X_t^y be two independent identically distributed continuous time homogeneous random walks with finite second moments starting from positions x and y at time 0. Let $\tau_{x,y} = \inf\{t > 0 : X_t^x = X_t^y\}$ be the first meeting time of the two walks. Then there exists $C_0 > 0$ such that*

$$P(\tau_{x,y} > T) \leq \frac{C_0}{\sqrt{T}}|x-y|$$

for all x, y and $T > 0$.

Proof. This is a standard result. See, e.g., Proposition P4 in Section 32 of [10], or Lemma 2.2 of [9]. Both results are stated for discrete time random walks, but the continuous time analogue follows readily from a standard large deviation estimate for Poisson processes.

Lemma 5.3. *Given a system of $2J$ coalescing random walks indexed by their starting positions $\{x_1^{(1)}, x_2^{(1)}, \ ..., \ x_1^{(J)}, x_2^{(J)}\}$ at time 0, if*

$$x_1^{(1)} < x_2^{(1)} < \cdots < x_1^{(i)} < x_2^{(i)} < \cdots < x_1^{(J)} < x_2^{(J)},$$

and $\sup_i |x_1^{(i)} - x_2^{(i)}| \leq M$ for some $M > 0$, then for any fixed time $T > C_0^2 M^2$ with C_0 satisfying Lemma 5.2, the number of coalesced walks by time T stochastically dominates the sum of J independent Bernoulli random variables $\{Y_1, ..., Y_J\}$, each with parameter $1 - C_0 M/\sqrt{T}$. In particular

$$P(\text{the number of coalesced particles by time } T \text{ is smaller than } N)$$
$$\leq P\left(\sum_{i=1}^{J} Y_i \leq N\right).$$

Proof: To prove the lemma, we construct the system of coalescing random walks from the system of independent walks. Given the trajectories of a system of independent walks starting from positions $\{x_1^{(1)}, x_2^{(1)}, \ ..., \ x_1^{(J)}, x_2^{(J)}\}$ at time 0, the first time some walk, say $x_1^{(i)}$, jumps to the position of another walk, say $x_2^{(j)}$, the walk $x_1^{(i)}$ is considered *coalesced*, i.e., from that time on, it follows the same trajectory as walk $x_2^{(j)}$, while the trajectory of walk $x_2^{(j)}$ remains unchanged. Among the remaining distinct trajectories, we iterate this procedure until no more coalescing takes place. Note that this construction is well defined, since almost surely no two random walk jumps take place at the same time. The resulting collection of random walk trajectories is distributed as a system of coalescing random walks.

In the above construction, almost surely, the number of *coalesced* walks by time T in the coalescing system is bounded from below by the number of pairs $\{x_1^{(i)}, x_2^{(i)}\}$ $(1 \leq i \leq J)$ for which $x_1^{(i)}$ and $x_2^{(i)}$ meet before time T in the independent system. If $x_1^{(i)}$ meets $x_2^{(i)}$ in the independent system at time $t \leq T$, then in the coalescing system, either $x_1^{(i)}$ and $x_2^{(i)}$ haven't *coalesced* with other walks before time t, in which case the two will coalesce at time t; or one of the two walks has *coalesced* with another walk before time t. In either case, whenever $x_1^{(i)}$ and $x_2^{(i)}$ meet in the independent system, at least one of them will be *coalesced* in the coalescing system. The asserted stochastic domination then follows by noting that Lemma 5.2 implies that each pair $\{x_1^{(i)}, x_2^{(i)}\}$ has probability at least $1 - C_0 M/\sqrt{T}$ of meeting before time T in the independent system. □

Proposition 5.4. *Let $\frac{1}{2} < p < 1$ be fixed. Consider a system of coalescing random walks starting with at most γL particles inside an interval of length L at time 0. Let $K_0 = \frac{64C_0^2}{(2p-1)^4}$, where C_0 is as in Lemma 5.2. If $\gamma L \geq \frac{8}{2p-1}$, then there exist constants C, c depending only on p such that, the probability that the number of particles alive at time $T = \frac{K_0}{\gamma^2}$ is greater than $p\gamma L$ is bounded above by $Ce^{-c\gamma L}$.*

Proof: The basic idea is to apply Lemma 5.3 and large deviation bounds for Bernoulli random variables. The choice of the constants K_0 and T will become apparent in the proof.

Without loss of generality, we assume $\gamma L \in \mathbb{N}$. We only need to consider a system starting with γL particles. If the initial number of particles is less than γL, we can always add extra particles to the system which only increases the probability of having $p\gamma L$ particles survive by time T.

Let M be a positive integer to be determined later. Since the γL particles partition the interval of length L into $\gamma L + 1$ pieces, the number of adjacent pairs of particles of distance at most $M-1$ apart is at least $\gamma L - 1 - \frac{L}{M}$. Therefore the number of *disjoint* pairs of adjacent particles of distance at most $M-1$ apart is at least $\frac{1}{2}(\gamma L - 2 - \frac{L}{M})$. Each such pair coalesces before time T with probability at least $1 - C_0 M/\sqrt{T}$. By Lemma 5.3, the number of coalesced particles stochastically dominates the sum of $m := \frac{1}{2}(\gamma L - 2 - \frac{L}{M})$ i.i.d. Bernoulli random variables with parameter $1 - C_0M/\sqrt{T}$, which we denote by $Y_1, \cdots, Y_m$. If by time T, more than $p\gamma L$ particles survive, then we must have

$$\sum_{i=1}^{m} Y_i \;\leq\; (1-p)\gamma L. \tag{5.4}$$

Let $p = \frac{1+\epsilon}{2}$ with $\epsilon \in (0,1)$, then we can rewrite (5.4) as

$$\frac{1}{m}\sum_{i=1}^{m} Y_i \;\leq\; \frac{(1-p)\gamma L}{\frac{1}{2}(\gamma L - 2 - \frac{L}{M})} \;=\; \frac{1-\epsilon}{1 - \frac{2}{\gamma L} - \frac{1}{\gamma M}}. \tag{5.5}$$

By our assumption $\frac{2}{\gamma L} \leq \frac{1}{4}(2p-1) = \frac{\epsilon}{4}$. If we choose $M = \frac{4}{\epsilon\gamma}$, and let $T = (\frac{2C_0M}{\epsilon})^2 = \frac{64C_0^2}{\epsilon^4\gamma^2} = \frac{K_0}{\gamma^2}$, then we have

$$\frac{1}{m}\sum_{i=1}^{m} Y_i \leq \frac{1-\epsilon}{1-\epsilon/2} < 1 - C_0M/\sqrt{T} = 1 - \epsilon/2.$$

By standard large deviation estimates for Bernoulli random variables with parameter $1 - \epsilon/2$, the probability of the event in (5.4) is bounded above by $Ce^{-c'm}$ for some C, c' depending only on p. Since $m = \frac{1}{2}(\gamma L - 2 - \frac{L}{M}) \geq (1/2 - \epsilon/4)\gamma L$ by our choice of M and the assumption $\gamma L \geq \frac{8}{2p-1}$, we have $Ce^{-c'm} \leq Ce^{-c'(1/2-\epsilon/4)\gamma L} = Ce^{-c\gamma L}$, which concludes the proof of the lemma. $\square$

The next result allows us to carry out the first step in the chain argument of section 2.

Lemma 5.5. *In the system of backward coalescing random walks $\{X^{x,s}\}_{(x,s)\in\mathbb{Z}\times\mathbb{R}}$ dual to the voter model, assume the random walk increment distribution $p(\cdot)$ has finite $3+\epsilon$ moment. Then* *there exists $C>0$ depending only on $p(\cdot)$, such that for all $K\geq 1$,*

$$P\Big\{ \text{ for some } (x,s)\in[2^kR,2^{k+1}R]\times[0,R^2], \\ |X_u^{x,s}-x|\geq \frac{2^kR}{(\log R)^2} \text{ for some } 0\leq u\leq s-K\lfloor\frac{s-1}{K}\rfloor\Big\}$$

is bounded above by

$$\frac{CK(\log R)^{2(3+\epsilon)}}{2^{2k+3\epsilon}R^{\epsilon}}$$

for all R sufficiently large.

Proof: Let $V_{x,s}$ be the event as above but concerning only the random walk $X_u^{x,s}$, then denote the event in the statement by V which is the union of $V_{x,s}$ over all $(x,s)\in[2^kR,2^{k+1}R]\times[0,R^2]$. Due to the coalescence, event V occurs only if $V_{x,s}$ occurs either for some (x,s) with $s\in\{K,2K,\cdots,\lfloor\frac{R^2}{K}\rfloor K\}\cup\{R^2\}$, or for some (x,s) which is a Poisson point in the Harris representation of the voter model detailed in Section 1. Therefore we can bound $P(V)$ by the expected number of such points, which by the Strong Markov property of Poisson processes can in turn be bounded by

$$\left(2^kR(\frac{R^2}{K}+1)+2^kR^3\right)P\left(|X_u|\geq\frac{2^kR}{(\log R)^2}\quad\text{for some } 0\leq u\leq K\right),$$

where X_u is a random walk starting at the origin with transition probability $p(\cdot)$. By our assumption that $p(\cdot)$ has finite $3+\epsilon$ moment, we can apply Lemma 5.1 and obtain

$$P(V)\leq\left(2^kR(\frac{R^2}{K}+1)+2^kR^3\right)\ C\cdot\left(e^{-c(\frac{2^kR}{\log^2 R})^{1-\beta}}+\frac{K}{(\frac{2^kR}{\log^2 R})^{3+\epsilon}}\right),$$

where C depends only on $p(\cdot)$. The Lemma then follows if we take R sufficiently large. $\square$

We finish by stating a result on the lifetime of a single particle voter model.

Lemma 5.6. *Let $\zeta_t^{\mathbb{Z}}$ be the process of coalescing random walks starting from $\mathbb{Z}$ at time 0 where all random walk increments are distributed according to a transition probability $p(\cdot)$ with finite second moment. Then for all $t>0$*

$$\mathbb{P}(0\in\xi_t^{\mathbb{Z}})\leq\frac{C}{\sqrt{t}}$$

for some $C>0$.

Proof: See Lemma 2.0.7 and the remark that follows it in [11].

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G. Valle's current address:

Instituto de Matemática
Universidade Federal do Rio de Janeiro (DME-IM-UFRJ)
Caixa Postal 68530, 21945-970
Rio de Janeiro, Brazil