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Chains with Complete Connections and One-Dimensional Gibbs Measures

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Abstract: We discuss the relationship between one-dimensional Gibbs measures and discrete-time processes (chains). We consider finite-alphabet (finite-spin) systems, possibly with a grammar (exclusion rule). We establish conditions for a stochastic process to define a Gibbs measure and vice versa. Our conditions generalize well known equivalence results between ergodic Markov chains and fields, as well as the known Gibbsian character of processes with exponential continuity rate. Our arguments are purely probabilistic; they are based on the study of regular systems of conditional probabilities (specifications). Furthermore, we discuss the equivalence of uniqueness criteria for chains and fields and we establish bounds for the continuity rates of the respective systems of finite-volume conditional probabilities. As an auxiliary result we prove a (re)construction theorem for specifications starting from single-site conditioning, which applies in a more general setting (general spin space, specifications not necessarily Gibbsian).

Keywords and phrases: Discrete-time stochastic processes, chains with complete connections, Gibbs measures, Markov chains.

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1 Introduction

One dimensional systems are simultaneously the object of the theory of stochastic processes and the theory of Gibbs measures. The complementarity of both approaches has yet to be fully exploited. Stochastic processes are defined on the basis of transition probabilities. A consistent chain is one for which these probabilities are a realization of the single-site conditional probabilities given the past. A Gibbs measure is defined in terms of specifications, which determine its finite-volume conditional probabilities given the exterior of the volume. In one dimension this implies conditioning both the past and the future. In this paper we study conditions under which a stochastic process defines, in fact, a Gibbs measure and, in the opposite direction, when a Gibbs measures can be seen as a stochastic process.

This type of questions has been completely elucidated for Markov processes and fields. See, for instance, Chapter 11 of the treatise by Georgii (1988). The equivalence, however, is obtained by eigenvalue-eigenvector considerations which are not readily applicable to non-Markovian processes. Another approach, based on entropy considerations, has been used by Goldstein et al (1989) to prove that cellular automata —a class of Markov processes with alphabets of the form $S^{\mathbb{Z}^d}$ with S finite— are indeed Gibbs states. This approach, however, is restricted to translation-invariant, or periodic, processes. The Gibbsian character of processes with exponentially decreasing continuity rate is also known. It follows from Bowen's characterization of Gibbs measures (Theorem 5.2.4 in Keller, 1998, for instance). No result seems to be available on the opposite direction, namely on the characterization of a one-dimensional Gibbs measure for an exponentially summable interaction as a stochastic process.

In our paper we present both a generalization and an alternative to this previous work. We directly establish consistency-preserving maps between specifications and transition probabilities. More precisely, these applications are between specifications and their analogous for stochastic processes, which we call *left-interval specifications* (LIS). The description in terms of LIS is equivalent to that in terms of transition probabilities, but it offers a setting that mirrors the statistical mechanical setting of Gibbs measures. In fact, the use of LIS allows us to "import", in a painless manner, concepts and results from statistical mechanics into the theory of stochastic processes. This will be further exploited in a companion paper (Fernández and Maillard, 2003).

We consider systems with a finite alphabet, possibly with a grammar, that is, with exclusion rules such that the non-excluded configurations form a compact set. We do not assume translation invariance either of the kernels or of the consistent measures. Such a level of generality is consistent with our goal. It is well known that specifications can have non-invariant extremal consistent mesures even if the specifications are translation-invariant themselves. An analogous phenomenon is expected for processes. Furthermore, in that case, the maps to or from specifications could be measure-dependent and could lead to non-invariant kernels. As a matter of fact, regimes of this sort lie outside the scope of the results of this paper, but nevertheless we allow non-invariance in our formalismin in the hope of future use.

The main limitation of our work is that, in order to insure that the necessary limits are uniquely defined, specifications and processes are required to satisfy a strong uniqueness condition called *hereditary uniqueness condition* (HUC). A second property, called *good future* (GF) is demanded for stochastic processes to guarantee some control of the conditioning with respect to the future. HUC is verified, for instance, by specifications satisfying Dobrushin and boundary-uniformity criteria (reviewed below). Both GF and HUC are satisfied by a large family of processes, for instance by the chains with summable variations studied by Harris (1955), Ledrappier (1974), Walters (1975),Lalley (1986), Berbee (1987), Bressaud et al (1999), ...

Our results show that under these hypotheses there exist: (i) a map that to each LIS associates a specification such that the process consistent with the former is a Gibbs measure consistent with the latter (Theorem 4.16), and (ii) a map that to each specification associates a LIS such that the Gibbs measure consistent with the former is a process consistent with the latter (Theorem 4.19). If domain and image match, these maps are inverses of each other. This happens, in particular, in the case of exponentially decreasing continuity rates (Theorem 4.22). As part of the proofs, we obtain estimates linking the continuity rates of LIS and specifications related by these maps (Theorem 4.21). We also show that the validity of the Dobrushin and boundary-uniformity criteria for the specification implies the validity of analogous criteria for the associated stochastic process (Theorem 4.20). Finally, in Appendix A we show that a system of single-site normalized kernels, satisfying order-consistency and boundedness properties with respect to an a-priori measure, can be extended, in a unique manner, to a full specification. This generalizes the reconstruction Theorem 1.33 in Georgii (1988). As this theorem may be of independent interest, we have stated it in rather general terms, for arbitrary spin space and non-necessarily Gibbsian kernels (Theorem A.4).

2 Notation and preliminary definitions

We consider a finite alphabet $\mathcal{A}$ endowed with the discrete topology and σ-algebra, and Ω a compact subset of $\mathcal{A}^{\mathbb{Z}}$. The space Ω is endowed with the projection $\mathcal{F}$ of the product σ-algebra associated to $\mathcal{A}^{\mathbb{Z}}$. The space Ω represents admissible "letter configurations", where the admissibility is defined, for instance by some exclusion rule as in Ruelle (1978) or by a "grammar" (subshift of finite type) as in Walters (1975). For each $\Lambda \subset \mathbb{Z}$, and each configuration $\sigma \in \mathcal{A}^{\mathbb{Z}}$ we denote σ_Λ its projection on Λ, namely the family $(\sigma_i)_{i\in\Lambda} \in \mathcal{A}^\Lambda$. We denote

$$\Omega_\Lambda \triangleq \Big\{\sigma_\Lambda \in \mathcal{A}^\Lambda : \exists\, \omega \in \Omega \text{ with } \omega_\Lambda = \sigma_\Lambda\Big\} . \tag{2.1}$$

We denote $\mathcal{F}_\Lambda$ the corresponding sub-σ-algebra of $\mathcal{F}$. When Λ is an interval, $\Lambda = [k, n]$ with $-\infty \le k \le n \le +\infty$, we shall use the "sequence" notation: $\omega_k^n \triangleq \omega_{[k,n]} = \omega_k, \ldots, \omega_n$, $\Omega_k^n \triangleq \Omega_{[k,n]}$, etc. The notation $\omega_\Lambda\, \sigma_\Delta$, where $\Lambda \cap \Delta = \emptyset$, indicates the configuration on $\Lambda \cup \Delta$ coinciding with ω_i for $i \in \Lambda$ and with σ_i for $i \in \Delta$. In particular, $\omega_k^n\, \sigma_{n+1}^m = \omega_k, \ldots, \omega_n, \sigma_{n+1}, \ldots, \sigma_m$. For $\omega, \sigma \in \mathcal{A}^{\mathbb{Z}}$, we note

$$\sigma \overset{\neq j}{=} \omega \quad \Longleftrightarrow \quad \sigma_i = \omega_i \;,\; \forall\, i \neq j \tag{2.2}$$

("σ equal to ω off j").

We denote $\mathcal{S}$ the set of finite subsets of $\mathbb{Z}$ and $\mathcal{S}_b$ the set of finite intervals of $\mathbb{Z}$. For every $\Lambda \in \mathcal{S}_b$ we denote $l_\Lambda \triangleq \min \Lambda$ and $m_\Lambda \triangleq \max \Lambda$, $\Lambda_- =]-\infty, l_\Lambda - 1]$, $\Lambda_+ = [m_\Lambda + 1, +\infty[$ and $\Lambda_+^{(k)} = [m_\Lambda + 1, m_\Lambda + k]$ for all $k \in \mathbb{N}^*$. The expression $\lim_{\Lambda\uparrow V}$ will be used in two senses. For kernels associated to a LIS (defined below), $\lim_{\Lambda\uparrow V} f_\Lambda$ is the limit of the net $\{f_\Lambda, \{\Lambda\}_{\Lambda\in\mathcal{S}_b,\, \Lambda\subset V}, \subset\}$, for V an infinite interval of $\mathbb{Z}$. For kernels associated to a specification , $\lim_{\Lambda\uparrow V} \gamma_\Lambda$ is the limit of the net $\{\gamma_\Lambda, \{\Lambda\}_{\Lambda\in\mathcal{S},\, \Lambda\subset V}, \subset\}$, for V an infinite subset of $\mathbb{Z}$. To lighten up formulas involving probability kernels, we will freely use $\rho(h)$ instead of $E_\rho(h)$ for ρ a measure on Ω and h a $\mathcal{F}$-measurable function. Also $\rho(\sigma_\Lambda)$ will mean $\rho(\{\omega \in \Omega : \omega_\Lambda = \sigma_\Lambda\})$ for $\Lambda \subset \mathbb{Z}$ and $\sigma_\Lambda \in \Omega_\Lambda$.

We start by briefly reviewing the well known notion of specification.

Definition 2.3
A **specification** γ *on* $(\Omega, \mathcal{F})$ *is a family of probability kernels* $\{\gamma_\Lambda\}_{\Lambda\in\mathcal{S}}$, $\gamma_\Lambda : \mathcal{F} \times \Omega \to [0, 1]$ *such that for all* Λ *in* $\mathcal{S}$,

(a) *For each* $A \in \mathcal{F}$, $\gamma_\Lambda(A \mid \cdot) \in \mathcal{F}_{\Lambda^c}$.

(b) *For each* $B \in \mathcal{F}_{\Lambda^c}$ *and* $\omega \in \Omega$, $\gamma_\Lambda(B \mid \omega) = \mathbb{1}_B(\omega)$.

(c) *For each* $\Delta \in \mathcal{S} : \Delta \supset \Lambda$,

$$\gamma_\Delta \, \gamma_\Lambda \;=\; \gamma_\Delta . \tag{2.4}$$

The specification is:

(i) **Continuous** *on* Ω *if for all* $\Lambda \in \mathcal{S}$ *and all* $\sigma_\Lambda \in \Omega_\Lambda$ *the functions*

$$\Omega \ni \omega \;\longrightarrow\; \gamma_\Lambda \, (\sigma_\Lambda \mid \omega) \tag{2.5}$$

are continuous.

(ii) **Non-null** *on* Ω *if* $\gamma_\Lambda(\omega_\Lambda \mid \omega) > 0$ *for each* $\omega \in \Omega$ *and* $\Lambda \in \mathcal{S}$.

Property c) is usually referred to as *consistency*. There and in the sequel we adopt the standard notation for composition of probability kernels (or of a probability kernel with a measure). For instance, (2.4) is equivalently to

$$\iint h(\xi) \, \gamma_\Lambda(d\xi \mid \sigma) \, \gamma_\Delta(d\sigma \mid \omega) \;=\; \int h(\sigma) \, \gamma_\Delta(d\sigma \mid \omega)$$

for each $\mathcal{F}$-measurable function h and configuration $\omega \in \Omega$.

Remarks

2.6 A *Markov specification of range* k corresponds to the particular case in which the applications (2.5) are in fact $\mathcal{F}_{\partial_k \Lambda}$-measurable, where $\partial_k \Lambda = \{i \in \Lambda^c : |i - j| \le k \text{ for some } j \in \Lambda\}$.

2.7 In the sequel, we find useful to consider also the natural extension of the kernels γ_Λ to functions $\mathcal{F} \times \mathcal{A}^{\mathbb{Z}} \to [0,1]$ such that $\gamma_\Lambda(\,\cdot \mid \omega) = 0$ if $\omega \notin \Omega$. We shall not distinguish notationally both types of kernels.

Definition 2.8

A probability measure μ *on* $(\Omega, \mathcal{F})$ *is said to be* **consistent** *with a specification* γ *if*

$$\mu \, \gamma_\Lambda \;=\; \mu \quad \forall \, \Lambda \in \mathcal{S}. \tag{2.9}$$

The family of these measures will be denoted $\mathcal{G}(\gamma)$.

Remarks

2.10 A *Markov field of range* k is a measure consistent with a Markov specification of range k.

2.11 A **Gibbs measure** on $(\Omega, \mathcal{F})$ is a measure μ consistent with a specification that is continuous and non-null on Ω. The SRB measures (Bowen, 1975) are particular one-dimensional examples.

We now introduce the analogous notion for processes. Due to the nature of the defining transition probabilities, the corresponding finite-region kernels must apply to functions measurable only with respect to the region and its past. Furthermore, finite intervals already suffice.

Definition 2.12
A **left interval-specification (LIS)** f *on* $(\Omega, \mathcal{F})$ *is a family of probability kernels* $\{f_\Lambda\}_{\Lambda \in \mathcal{S}_b}$, $f_\Lambda : \mathcal{F}_{\leq m_\Lambda} \times \Omega \longrightarrow [0,1]$ *such that for all* Λ *in* $\mathcal{S}_b$,

(a) *For each* $A \in \mathcal{F}_{\leq m_\Lambda}$, $f_\Lambda(A \mid \cdot\,)$ *is* $\mathcal{F}_{\Lambda_-}$*-measurable.*

(b) *For each* $B \in \mathcal{F}_{\Lambda_-}$ *and* $\omega \in \Omega$, $f_\Lambda(B \mid \omega) = 1\!\!1_B(\omega)$.

(c) *For each* $\Delta \in \mathcal{S}_b : \Delta \supset \Lambda$,

$$f_\Delta\, f_\Lambda \;=\; f_\Delta \quad \text{over } \mathcal{F}_{\leq m_\Lambda}, \tag{2.13}$$

that is, $(f_\Delta f_\Lambda)(h \mid \omega) = f_\Delta(h \mid \omega)$ *for each* $\mathcal{F}_{\leq \max \Lambda}$*-measurable function* h *and configuration* $\omega \in \Omega$.

The LIS is:

(i) **Continuous** *on* Ω *if for all* $\Lambda \in \mathcal{S}_b$ *and all* $\sigma_\Lambda \in \Omega_\Lambda$ *the functions*

$$\Omega \ni \omega \;\longrightarrow\; f_\Lambda(\sigma_\Lambda \mid \omega) \tag{2.14}$$

are continuous.

(ii) **Non-null** *on* Ω *if* $f_\Lambda(\omega_\Lambda \mid \omega_{\Lambda_-}) > 0$ *for all* $\Lambda \in \mathcal{S}_b$ *and* $\omega \in \Omega_{-\infty}^{m_\Lambda}$.

(iii) **Weakly non-null** *on* Ω *if for all* $\Lambda \in \mathcal{S}_b$, *there exists a* $\sigma_\Lambda \in \Omega_\Lambda$ *such that* $f_\Lambda(\sigma_\Lambda \mid \omega_{-\infty}^{l_\Lambda - 1}) > 0$ *for all* $\omega_{-\infty}^{l_\Lambda-1} \in \Omega_{-\infty}^{l_\Lambda-1}$ *such that* $\sigma_\Lambda\, \omega_{-\infty}^{l_\Lambda-1} \in \Omega_{-\infty}^{m_\Lambda}$.

Remarks

2.15 A *Markov LIS of range* k is a LIS such that each of the functions (2.14) is measurable with respect to $\mathcal{F}_{[l_\Lambda - k,\, l_\Lambda - 1]}$.

2.16 As for specifications, in the sequel we shall not distinguish notationally the kernels f_Λ from their extensions on $\mathcal{F}_{m_\Lambda} \times \mathcal{A}^{\mathbb{Z}} \to [0,1]$ such that $f_\Lambda(\,\cdot \mid \omega) = 0$ if $\omega \notin \Omega$.

Definition 2.17
A probability measure μ *on* $(\Omega,\, \mathcal{F})$ *is said to be* **consistent** *with a LIS* f *if for each* $\Lambda \in \mathcal{S}_b$

$$\mu\, f_\Lambda \;=\; \mu \quad \text{over } \mathcal{F}_{\leq m_\Lambda}. \tag{2.18}$$

The family of these measures will be denoted $\mathcal{G}(f)$.

Remarks

2.19 A *Markov chain* of range k is a measure consistent with a Markov LIS of range k.

2.20 Measures consistent with general, non-necessarily Markovian LIS were initially called *Chains with complete connections* by Onicescu and Mihoc (1935). These objects have been reintroduced several times in the literature under a variety of names: *chains of infinite order* (Harris, 1955), *g-measures* (Keane, 1972), *uniform martingales* (=random Markov processes) (Kalikow, 1990), ...

Finally, we introduce a strong notion of uniqueness needed in the sequel.

Definition 2.21

1) *A specification γ satisfies a* **hereditary uniqueness condition** *(HUC)* **for a family** *$\mathcal{H}$ of subsets of $\mathbb{Z}$ if for all (possibly infinite) sets $V \in \mathcal{H}$ and all configurations $\omega \in \Omega$, the specification $\gamma^{(V,\omega)}$ defined by*

$$\gamma_\Lambda^{(V,\omega)}(\,\cdot \mid \xi) = \gamma_\Lambda(\,\cdot \mid \omega_{V^c}\,\xi_V)\;, \quad \forall\,\Lambda \in \mathcal{S},\ \Lambda \subset V,\ \ \forall\,\omega_{V^c}\,\xi_V \in \Omega\;, \tag{2.22}$$

admits a unique Gibbs measure. The specification satisfies a HUC if it satisfies a HUC for $\mathcal{H} = \mathcal{P}(\mathbb{Z})$.

2) *A LIS f satisfies a* **hereditary uniqueness condition** *(HUC) if for all intervals of the form $V = [i, +\infty[$, $i \in \mathbb{Z}$, or $V = \mathbb{Z}$, and all configurations $\omega \in \Omega$, the LIS $f^{(V,\omega)}$ defined by*

$$f_\Lambda^{(V,\omega)}(\,\cdot \mid \xi) = f_\Lambda(\,\cdot \mid \omega_{V_-}\,\xi_V)\;, \quad \forall\,\Lambda \in \mathcal{S}_b,\ \Lambda \subset V,\ \forall\,\omega_{V^c}\,\xi_V \in \Omega\;, \tag{2.23}$$

admits a unique consistent chain.

3 Preliminary results

Let us summarize a number of useful properties of LIS and specifications. First we introduce functions associated to LIS singletons. For a LIS f and a configuration $\omega \in \Omega$, let

$$f_i(\omega) \;\triangleq\; f_{\{i\}}(\omega_i \mid \omega_{-\infty}^{i-1}). \tag{3.1}$$

In the shift-invariant case, the function f_0 is a g-function in the sense of Keane (1972).

The following theorem expresses the equivalence between the description in terms of LIS and the usual description in terms of transition probabilities (=LIS singletons).

Theorem 3.2 (singleton consistency for chains)
Let $(g_i)_{i\in\mathbb{Z}}$ be a family of measurable functions over $(\Omega, \mathcal{F})$ which enjoy the following properties

(a) *Measurability: for every i in $\mathbb{Z}$, g_i is $\mathcal{F}_{\leq i}$-measurable.*

(b) *Normalization: for every i in $\mathbb{Z}$ and $\omega \in \Omega_{-\infty}^{i-1}$,*

$$\sum_{\sigma_i \in \mathcal{A}\,:\,\omega\,\sigma_i \in \Omega_{-\infty}^{i}} g_i\left(\omega\,\sigma_i\right) \;=\; 1. \tag{3.3}$$

Then there exists a unique left interval-specification $f \triangleq (f_\Lambda)_{\Lambda\in\mathcal{S}_b}$ such that $f_i = g_i$, for all i in $\mathbb{Z}$. Furthermore:

(i) *f satisfies (in fact, it is defined by) the property*

$$f_{[l,m]} = f_{[l,n]} f_{[n+1,m]} \qquad \text{over } \mathcal{F}_{\leq n} \tag{3.4}$$

for each $l, m, n \in \mathbb{Z} : l \leq n < m$.

(ii) *f is non-null on Ω if, and only if, so are the functions g_i, that is, if and only if $g_i(\omega) > 0$ for each $i \in \mathbb{Z}$ and each $\omega \in \Omega_{-\infty}^{i}$.*

(iii) *f is weakly non-null on Ω if, and only if, so are the functions g_i, that is, if and only if for each $i \in \mathbb{Z}$ there exists $\sigma_i \in \Omega_{\{i\}}$ such that $g_i(\sigma_i\,\omega_{-\infty}^{i-1}) > 0$ for all $\omega_{-\infty}^{i-1} \in \Omega_{-\infty}^{i-1}$ such that $\sigma_i\,\omega_{-\infty}^{i-1} \in \Omega_{-\infty}^{i}$.*

(iv) *$\mathcal{G}(f) = \{\mu : \mu f_i = \mu,\ \text{for all } i \text{ in } \mathbb{Z}\}$.*

The proof of this result is rather simple (it is spelled up in Fernández and Maillard, 2003). The following theorem is the analogous result for specifications. Let us consider the following functions associated to an specification γ:

$$\gamma_\Lambda(\omega) \;\triangleq\; \gamma_{\{\Lambda\}}(\omega_\Lambda \mid \omega_{\Lambda^c}) \quad , \quad \gamma_i(\omega) \;\triangleq\; \gamma_{\{i\}}(\omega)\ . \tag{3.5}$$

Theorem 3.6 (singleton consistency for Gibbs measure)
Let $(\rho_i)_{i\in\mathbb{Z}}$ be a family of measurable functions over $(\mathcal{A}^{\mathbb{Z}}, \mathcal{F})$ which enjoys the following properties

(a) *Non-nullness on Ω: for every i in $\mathbb{Z}$, $\rho_i(\omega) = 0 \Longleftrightarrow \omega \notin \Omega$.*

(b) *Order-consistency on Ω: for every $i, j \in \mathbb{Z}$ and $\omega \in \Omega$,*

$$\frac{\rho_i(\omega)}{\displaystyle\sum_{\sigma_i\in\mathcal{A}} \rho_i\left(\sigma_i\,\omega_{\{i\}^c}\right)\,\rho_j^{-1}\left(\sigma_i\,\omega_{\{i\}^c}\right)} \;=\; \frac{\rho_j(\omega)}{\displaystyle\sum_{\sigma_j\in\mathcal{A}} \rho_j\left(\sigma_j\,\omega_{\{j\}^c}\right)\,\rho_i^{-1}\left(\sigma_j\,\omega_{\{j\}^c}\right)}. \tag{3.7}$$

(c) *Normalization on Ω: for every $i \in \mathbb{Z}$ and $\omega \in \Omega$,*

$$\sum_{\sigma_i \in \mathcal{A}} \rho_i \left(\sigma_i \, \omega_{\{i\}^c}\right) = 1. \tag{3.8}$$

Then there exists a unique specification γ on $(\Omega, \mathcal{F})$ such that $\gamma_i(\omega) = \rho_i(\omega)$, for all i in $\mathbb{Z}$. Furthermore,

(i) *γ is non-null on Ω: For each $\Lambda \in \mathcal{S}$, $\gamma_\Lambda(\omega) = 0 \Longleftrightarrow \omega \notin \Omega$.*

(ii) *γ satisfies an order-independent prescription: For each $\Lambda, \Gamma \in \mathcal{S}$ with $\Gamma \subset \Lambda^c$*

$$\gamma_{\Lambda \cup \Gamma}(\omega) \;=\; \frac{\gamma_\Lambda(\omega)}{\displaystyle\sum_{\sigma_\Lambda} \gamma_\Lambda \left(\sigma_\Lambda \, \omega_{\Lambda^c}\right) \, \gamma_\Gamma^{-1} \left(\sigma_\Lambda \, \omega_{\Lambda^c}\right)} \tag{3.9}$$

for all $\omega \in \Omega$.

(iii) *$\mathcal{G}(\gamma) = \{\mu : \mu \, \gamma_i = \mu$ for all $i \in \mathbb{Z}\}$.*

This result will be proved in Appendix A in a more general setting.

For completeness, we list now several, mostly well known, sufficient conditions for hereditary uniqueness. They refer to different ways to bound continuity rates of transition kernels. We start with the relevant definitions.

Definition 3.10

(i) *The k-**variation** of a $\mathcal{F}_{\{i\}}$-measurable function f_i is defined by*

$$\mathrm{var}_k(f_i) \;\triangleq\; \sup\Big\{ \big| f_i(\omega_{-\infty}^i) - f_i(\sigma_{-\infty}^i) \big| : \omega_{-\infty}^i, \sigma_{-\infty}^i \in \Omega_{-\infty}^i \, , \, \omega_{i-k}^i = \sigma_{i-k}^i \Big\} \, .$$

(ii) *The **interdependence coefficients** for a family of probability kernels $\pi = \left(\pi_{\{i\}}\right)_{i \in \mathbb{Z}}$, $\pi_{\{i\}} : \mathcal{F}_i \times \Omega \to [0,1]$ are defined by*

$$C_{ij}(\pi) \;\triangleq\; \sup_{\substack{\xi, \eta \in \Omega \\ \xi \stackrel{\neq j}{=} \eta}} \left\| \overset{\circ}{\pi}_{\{i\}} \left(\cdot \mid \xi \right) - \overset{\circ}{\pi}_{\{i\}} \left(\cdot \mid \eta \right) \right\| \tag{3.11}$$

for all $i, j \in \mathbb{Z}$. Here we use the variation norm and $\overset{\circ}{\pi}_{\{i\}}$ is the projection of $\pi_{\{i\}}$ over $\{i\}$ that is $\overset{\circ}{\pi}_{\{i\}} \left(A \mid \omega\right) \triangleq \pi_{\{i\}} \left(\{\sigma_i \in A\} \mid \omega\right)$ for all $A \in \mathcal{F}_i$ and $\omega \in \Omega$.

A LIS f on $(\Omega, \mathcal{F})$ satisfies a HUC if it satisfies one of the following conditions:

- **Harris** (Harris 1955; Coelho and Quas, 1998): The LIS f is stationary, weakly non-null on Ω and

$$\sum_{n\geq 1}\prod_{k=1}^{n}\left(1-\frac{|\mathcal{A}|}{2}\mathrm{var}_k(f_0)\right)=+\infty.$$

- **Berbee** (1987): The LIS f is stationary, non-null and

$$\sum_{n\geq 1}\exp\left(-\sum_{k=1}^{n}\mathrm{var}_k(\log f_0)\right)=+\infty.$$

- **Stenflo** (2002): The LIS f is stationary, non-null and

$$\sum_{n\geq 1}\prod_{k=1}^{n}\Delta_k(f_0)=+\infty$$

where $\Delta_k(f_0)\triangleq\inf\{\sum_{\omega_0\in\mathcal{A}}\min\left(f_0\left(\omega_{-\infty}^{0}\right),f_0\left(\sigma_{-\infty}^{-1}\,\omega_0\right)\right):\omega_{-k}^{-1}=\sigma_{-k}^{-1}\}$.

- **Johansson and Öberg** (2002): The LIS f is stationary, non-null and

$$\sum_{k\geq 0}\mathrm{var}_k^2(\log f_0)<+\infty.$$

- **One-sided Dobrushin** (Fernàndez and Maillard, 2003): For each $i\in\mathbb{Z}$, $\sum_{j<i}C_{ij}(f)<1$ and f is continuous on Ω.

- **One-sided boundary-uniformity** (Fernàndez and Maillard, 2003): There exists a constant $K>0$ so that for every cylinder set $A=\{x_l^m\}\in\Omega_l^m$ there exists an integer n such that

$$f_{[n,m]}(A\mid\xi)\geq K\,f_{[n,m]}(A\mid\eta)\quad\text{for all }\xi,\eta\in\Omega\,. \tag{3.12}$$

The last two conditions, proven in a companion paper (Fernández and Maillard, 2003), are in fact adaptations of the following well known criteria for specifications.

A specification γ on $(\Omega,\mathcal{F})$ satisfies a HUC if it satisfies one of the following conditions:

- **Dobrushin** (1968), Lanford (1973): $\sup_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}C_{ij}(\gamma)<1$ and γ continuous.

- **Georgii** (1974) **boundary-uniformity**: There exists a constant $K > 0$ so that for every cylinder set $A \in \mathcal{F}$ there exists $\Lambda \in \mathcal{S}_b$ such that

$$\gamma_\Lambda(A \mid \xi) \geq K\, \gamma_\Lambda(A \mid \eta) \quad \text{for all } \xi, \eta \in \Omega\ .$$

We remark that the conditions involve no non-nullness assumption.

4 Main results

For a LIS f on Ω let us denote, for each $\Lambda \in \mathcal{S},\ k \geq m_\Lambda$ and $\omega \in \Omega_{-\infty}^k$,

$$c_\Lambda^\omega(f_k) \;\triangleq\; \inf_{\sigma_\Lambda \in \Omega_\Lambda} \left\{ f_k\left(\sigma_\Lambda\, \omega_{]-\infty,k]\setminus\Lambda}\right) : \sigma_\Lambda\, \omega_{]-\infty,k]\setminus\Lambda} \in \Omega_{-\infty}^k \right\} \tag{4.1}$$

and

$$\delta_\Lambda^\omega(f_k) \;\triangleq\; \sum_{j\in\Lambda} \sup \left\{ \left| f_k\left(\omega_{-\infty}^k\right) - f_k\left(\sigma_{-\infty}^k\right) \right| : \sigma \in \Omega_{-\infty}^k,\ \sigma \overset{\neq j}{=} \omega \right\}. \tag{4.2}$$

Similarly, for a specification γ on Ω, let us denote, for each $\omega \in \Omega$ and each $k, j \in \mathbb{Z}$

$$c_j^\omega(\gamma_k) \;\triangleq\; \min_{\sigma_j \in \Omega_{\{j\}}} \left\{ \gamma_k\left(\sigma_j\, \omega_{\{j\}^c}\right) : \sigma_j\, \omega_{\{j\}^c} \in \Omega \right\} \tag{4.3}$$

and

$$\delta_j^\omega\left(\gamma_k\right) \;\triangleq\; \sup \left\{ \left|\gamma_k(\omega) - \gamma_k(\sigma)\right| : \sigma \in \Omega,\ \sigma \overset{\neq j}{=} \omega \right\}\ . \tag{4.4}$$

Definition 4.5

(i) *A LIS f on Ω is said to have a* **good future (GF)** *if it is non-null on Ω and for each $\Lambda \in \mathcal{S}$, there exists a sequence $\left\{\varepsilon_k^\Lambda\right\}_{k\in\mathbb{N}}$ of positive numbers such that $\sum_k \varepsilon_k^\Lambda < +\infty$ for which*

$$\sup_{\omega \in \Omega_{-\infty}^k} \; c_\Lambda^\omega(f_k)^{-1}\, \delta_\Lambda^\omega(f_k) \;\leq\; \varepsilon_k^\Lambda \tag{4.6}$$

for each $k \geq m_\Lambda$.

(ii) *A LIS f on Ω is said to have an* **exponentially-good future (EGF)** *if it is non-null on Ω and there exists a real $a > 1$ such that*

$$\limsup_{k\to\infty}\, a^{|k-j|} \sup_{\omega \in \Omega_{-\infty}^k} \; c_j^\omega(f_k)^{-1}\, \delta_j^\omega(f_k) \;<\; \infty \tag{4.7}$$

for all $j \in \mathbb{Z}$.

(iii) *A specification γ on Ω is said to have an* **exponentially-good future (EGF)** *if it is non-null on Ω and there exists a real $a > 1$ such that*

$$\limsup_{k\to\infty} a^{|k-j|} \sup_{\omega\in\Omega} c_j^\omega(\gamma_k)^{-1}\, \delta_j^\omega(\gamma_k) \;<\; \infty \tag{4.8}$$

for all $j \in \mathbb{Z}$.

The results in the sequel refer to the following families of LIS

$$\begin{aligned} \Theta &\triangleq \{\text{LIS } f \text{continuous and non-null on } \Omega\} \\ \Theta_1 &\triangleq \{f \in \Theta : f \text{ has a GF}\} \\ \Theta_2 &\triangleq \{f \in \Theta : f \text{ satisfies a HUC}\} \\ \Theta_3 &\triangleq \{f \in \Theta_2 : f \text{ has an EGF}\} \end{aligned} \tag{4.9}$$

and to the following families of specifications

$$\begin{aligned} \Pi &\triangleq \{\text{specifications } \gamma \text{ continuous and non-null on } \Omega\} \\ \Pi_1 &\triangleq \{\gamma \in \Pi : |\mathcal{G}(\gamma)| = 1\} \\ \Pi_2 &\triangleq \{\gamma \in \Pi_1 : \gamma \text{ satisfies a HUC over all } [i, +\infty[,\ i \in \mathbb{Z}\} \\ \Pi_3 &\triangleq \{\gamma \in \Pi_2 : \gamma \text{ has an EGF}\} \end{aligned} \tag{4.10}$$

Remarks

4.11 A stationary non-null LIS is in Θ_1 if it is non-null and its oscillations $\delta_j(f_0) \triangleq \sup_\omega \delta_j^\omega(f_0)$ are summable. Since for all $k \geq 1$, $\mathrm{var}_k(f_0) \geq \delta_k(f_0)$, Θ_1 includes the set of stationary non-null LIS with summable variations.

4.12 The translation-invariant non-null LIS with $\sum_j \delta_j(f_0) < 1$ are in $\Theta_1 \cap \Theta_2$. This is a consequence of the precedent remark and the one-sided Dobrushin criterium.

4.13 Each of the LIS or specifications in (4.9) or (4.10) has at least one consistent measure. This is because the (interesting part of) the configuration space is compact and the LIS or specifications are assumed to be continuous. Indeed, as the space of probability measures on a compact space is weakly compact, every sequence of measures $\gamma_{\Lambda_n}(\,\cdot \mid \omega^{\{n\}})$ or $f_{\Lambda_n}(\,\cdot \mid \omega^{\{n\}})$, for (Λ_n) an exhausting sequence of regions and $(\omega^{\{n\}})$ a sequence of configurations, has a weakly convergent subsequence. By continuity of the transitions the limit is respectively a Gibbs measure or a consistent chain.

Consider the function

$$F_{\Lambda,n}\left(\omega_\Lambda \mid \omega\right) \;\triangleq\; \frac{f_{[l_\Lambda,n]}\left(\omega^n_{l_\Lambda} \mid \omega_{\Lambda_-}\right)}{f_{[l_\Lambda,n]}\left(\omega_{\Lambda^c\cap[l_\Lambda,n]} \mid \omega_{\Lambda_-}\right)} \tag{4.14}$$

for all $\Lambda \in \mathcal{S}$, $n \geq m_\Lambda$ and $\omega \in \Omega$. The continuity of f implies that the functions $F_{\Lambda,n}(\omega_\Lambda \mid \cdot)$ are continuous on Ω_{Λ^c} for each $\omega_\Lambda \in \Omega_\Lambda$. We use these functions to introduce the map

$$b : \Theta_1 \to \Pi\;,\;\; f \mapsto \gamma^f$$

defined by

$$\gamma^f_\Lambda\left(\omega_\Lambda \mid \omega\right) \;\triangleq\; \lim_{n\to+\infty} F_{\Lambda,n}(\omega_\Lambda \mid \omega) \tag{4.15}$$

for all $\Lambda \in \mathcal{S}$ and $\omega \in \Omega$.

Theorem 4.16 (LIS $\rightsquigarrow$ specification)

1) *The map b is well defined. That is, for $f \in \Theta_1$*

- **(a)** *the limit (4.15) exists for all $\Lambda \in \mathcal{S}$ and $\omega \in \Omega$.*
- **(b)** *γ^f is a specification on $(\Omega, \mathcal{F})$.*
- **(c)** *$\gamma^f \in \Pi$.*

2)

- **(a)** *For each (finite or infinite) interval V and each $\omega \in \Omega$, $\mathcal{G}\left(f^{(V,\omega)}\right) \subset \mathcal{G}\left(\left(\gamma^f\right)^{(V,\omega)}\right)$.*
- **(b)** *For $f \in b^{-1}(\Pi_1)$, $\mathcal{G}(f) = \mathcal{G}(\gamma^f) = \left\{\mu^f\right\}$, where μ^f is the only chain consistent with f.*
- **(c)** *The map b restricted to $b^{-1}\left(\Pi_1\right)$ is one-to-one.*

Consider now the map

$$c : \Pi_2 \to \Theta_2\;,\;\; \gamma \mapsto f^\gamma \tag{4.17}$$

defined by

$$f^\gamma_\Lambda\left(A \mid \omega_{\Lambda_-}\right) \;\triangleq\; \lim_{k\to+\infty} \gamma_{\Lambda\cup\Lambda^{(k)}_+}\left(A \mid \omega\right) \tag{4.18}$$

for all $\Lambda \in \mathcal{S}_b$, $A \in \mathcal{F}_\Lambda$, and $\omega \in \Omega$ for which the limit exists.

Theorem 4.19 (specification $\rightsquigarrow$ LIS)

1) *The map c is well defined. That is, for $\gamma \in \Pi_2$*

- **(a)** *the limit (4.18) exists for all $\Lambda \in \mathcal{S}_b$, $A \in \mathcal{F}_\Lambda$, $\omega_{\Lambda_-} \in \Omega_{\Lambda_-}$ and is independent of ω_{Λ_+}.*

(b) f^γ *is a LIS on* $(\Omega, \mathcal{F})$.

(c) $f^\gamma \in \Theta_2$.

2) **(a)** $\mathcal{G}(f^\gamma) = \mathcal{G}(\gamma) = \{\mu^\gamma\}$, *where* μ^γ *is the only Gibbs measure consistent with* γ.

(b) *The map* c *is one-to-one.*

In addition a LIS of the form f^γ satisfies the following properties.

Theorem 4.20
Let $\gamma \in \Pi_2$.

(a) *If* γ *satisfies Dobrushin uniqueness condition, then so does* f^γ.

(b) *If* γ *satisfies the boundary-uniformity uniqueness condition, then so does* f^γ.

Theorem 4.21 (Continuity rates)
Let $\omega \in \Omega$ *and* $j \in \mathbb{Z}$.

1) *For* $f \in \Theta_1$ *and* $\Lambda \in \mathcal{S}$

(a) *if* $j > m_\Lambda$ *then* $\delta_j^\omega\left(\gamma_\Lambda^f\right) \leq 2\sum_{i \geq j} c_k^\omega(f_i)^{-1}\,\delta_k^\omega(f_i)$.

(b) *if* $j < l_\Lambda$ *then* $\delta_j^\omega(\gamma_\Lambda^f) \leq 1 - \prod_{i=l_\Lambda}^{+\infty} \frac{1 - c_j^\omega(f_i)^{-1}\,\delta_j^\omega(f_i)}{1 + c_j^\omega(f_i)^{-1}\,\delta_j^\omega(f_i)}$.

2) *For* $\gamma \in \Pi_2$, $\Lambda \in \mathcal{S}_b$ *and* $j < l_\Lambda$, $\delta_j^\omega(f_\Lambda^\gamma) \leq 1 - \prod_{i=l_\Lambda}^{+\infty} \frac{1 - c_j^\omega(\gamma_k)^{-1}\,\delta_j^\omega(\gamma_i)}{1 + c_j^\omega(\gamma_k)^{-1}\,\delta_j^\omega(\gamma_i)}$.

Under suitable conditions the maps b and c are reciprocal.

Theorem 4.22 (LIS $\leftrightsquigarrow$ specification)

(a) $b \circ c = \mathrm{Id}$ *over* $c^{-1}(\Theta_1)$ *and* $\mathcal{G}(f^\gamma) = \mathcal{G}(\gamma) = \{\mu^\gamma\}$.

(b) $c \circ b = \mathrm{Id}$ *over* $b^{-1}(\Pi_2)$ *and* $\mathcal{G}(\gamma^f) = \mathcal{G}(f) = \left\{\mu^f\right\}$.

(c) b *and* c *establish a one-to-one correspondence between* Θ_3 *and* Π_3 *that preserves the consistent measure.*

We remark that Θ_3 includes the well studied processes with Holdërian transition rates (see, for instance, Lalley, 1986, or Keller, 1998). Part (c) of the theorem shows, in particular, the equivalence between such processes and Bowen's Gibbs measures.

5 Proofs

We start with a collection of results used for several proofs.

Lemma 5.1
Consider $\Lambda \in \mathcal{S}$, $\Delta \subset \Lambda^c$ *and* π *a probability kernel over* $(\mathcal{F}_\Lambda \otimes \mathcal{F}_\Delta\,,\,\Omega_\Lambda \times \Omega_\Delta)$ *such that* $\pi\,(A_\Delta \mid \cdot\,) = \mathbb{1}_{A_\Delta}(\,\cdot\,)$, $\forall\, A_\Delta \in \mathcal{F}_\Delta$. *Then, for all* $\omega \in \Omega_\Lambda \times \Omega_\Delta$,

$$\pi(\,\cdot \mid \omega) \;=\; [\overset{\circ}{\pi}_\Lambda\,(\,\cdot \mid \omega) \otimes \delta_{\omega_\Delta}](\,\cdot\,)$$

where $\overset{\circ}{\pi}_\Lambda\,(\,\cdot \mid \omega)$ *is the restriction of* $\pi(\,\cdot \mid \omega)$ *to* $\mathcal{F}_\Lambda$ *and* δ_{ω_Δ} *is the Dirac mass at* ω_Δ.

Proof If $A = A_\Lambda \times A_\Delta \in \mathcal{F}_\Lambda \otimes \mathcal{F}_\Delta$ and $\omega \in \Omega_\Lambda \times \Omega_\Delta$

$$\pi\left(A_\Delta \,\middle|\, \omega\right) \;=\; \pi\left(A_\Lambda \times A_\Delta \,\middle|\, \omega\right) + \pi\left(A_\Lambda^c \times A_\Delta \,\middle|\, \omega\right)$$

and

$$\pi\left(A_\Lambda \,\middle|\, \omega\right) \;=\; \pi\left(A_\Lambda \times A_\Delta \,\middle|\, \omega\right) + \pi\left(A_\Lambda \times A_\Delta^c \,\middle|\, \omega\right)\;.$$

Hence,

$$\pi\,(A_\Lambda \times A_\Delta) \;\le\; \pi\,(A_\Lambda) \,\wedge\, \pi\,(A_\Delta) \;\le\; \pi\,(A_\Lambda)\; \mathbb{1}_{A_\Delta}.$$

Analogously,

$$\pi\,(A_\Lambda \times A_\Delta^c) \;\le\; \pi\,(A_\Lambda)\; \mathbb{1}_{A_\Delta^c}\;.$$

On the other hand,

$$\pi\,(A_\Lambda \times A_\Delta) + \pi\,(A_\Lambda \times A_\Delta^c) \;=\; \pi\,(A_\Lambda)\; \mathbb{1}_{A_\Delta} + \pi\,(A_\Lambda)\; \mathbb{1}_{A_\Delta^c}\;.$$

The last three displays imply

$$\pi(A) = \pi\,(A_\Lambda)\;\; \mathbb{1}_{A_\Delta} \quad\Longleftrightarrow\quad \pi\,(A_\Delta) = \;\mathbb{1}_{A_\Delta}\;. \;\square$$

In particular, LIS and specifications are completely defined by the families of their restrictions.

Proposition 5.2
Let $\omega \in \Omega$, $\Lambda \in \mathcal{S}$ *and* $n \in \mathbb{Z}$, $n \ge m_\Lambda$.

(a) *For any* $\beta \in \Omega$,

$$f_{n+1}\left(\omega_{n+1} \mid \omega_{]-\infty,n]}\right) \;\substack{\le \\ \ge}\; f_{n+1}\left(\omega_{n+1} \mid \beta_\Lambda\, \omega_{]-\infty,n]\setminus\Lambda}\right)\,\left[1 \,\pm\, \frac{\delta_\Lambda^\omega(f_{n+1})}{c_\Lambda^\omega(f_{n+1})}\right]\,. \tag{5.3}$$

(b) *For $j < l_\Lambda$ and $\sigma \in \Omega$ with $\sigma \overset{\neq j}{=} \omega$*

$$F_{\Lambda,n}(\omega_\Lambda \mid \sigma) - F_{\Lambda,n}(\omega_\Lambda \mid \omega) \quad \begin{array}{c}\le\\ \ge\end{array}$$
$$\pm \left[1 - \prod_{i=l_\Lambda}^{n} \frac{1 - c_j^\omega(f_i)^{-1}\,\delta_j^\omega(f_i)}{1 + c_j^\omega(f_i)^{-1}\,\delta_j^\omega(f_i)}\right] \times \left\{\begin{array}{l} F_{\Lambda,n}(\omega_\Lambda \mid \sigma) \\ F_{\Lambda,n}(\omega_\Lambda \mid \omega)\end{array}\right. . \qquad (5.4)$$

(c)

$$F_{\Lambda,n+1}(\omega_\Lambda \mid \omega) \begin{array}{c}\le\\ \ge\end{array} F_{\Lambda,n}(\omega_\Lambda \mid \omega)\left[1 \pm \frac{\delta_\Lambda^\omega(f_{n+1})}{c_\Lambda^\omega(f_{n+1})}\right] \qquad (5.5)$$

Proof If we telescope $f_{n+1}\left(\omega_{n+1} \mid \xi_\Lambda\,\omega_{]-\infty,n]\setminus\Lambda}\right) - f_{n+1}\left(\omega_{n+1} \mid \eta_\Lambda\,\omega_{]-\infty,n]\setminus\Lambda}\right)$, transforming ξ_Λ into η_Λ letter by letter, we have

$$f_{n+1}\left(\omega_{n+1} \mid \xi_\Lambda\,\omega_{]-\infty,n]\setminus\Lambda}\right) - f_{n+1}\left(\omega_{n+1} \mid \eta_\Lambda\,\omega_{]-\infty,n]\setminus\Lambda}\right) \;\le\; \delta_\Lambda^\omega(f_{n+1})$$

for any $\xi_\Lambda, \eta_\Lambda \in \Omega_\Lambda$. To obtain **(5.3)** we simply use this inequality twice, assigning ω_Λ to ξ_Λ, β_Λ to η_Λ, and vice versa.

To prove **(5.4)** we use definition (4.14)

$$F_{\Lambda,n}(\omega_\Lambda \mid \omega) = \frac{f_{[l_\Lambda,n]}\left(\omega_{l_\Lambda}^n \mid \omega_{\Lambda_-}\right)}{\sum_{\beta_\Lambda} f_{[l_\Lambda,n]}\left(\beta_\Lambda\,\omega_{\Lambda^c\cap[l_\Lambda,n]} \mid \omega_{\Lambda_-}\right)}$$

and the factorization

$$f_{[k,n]}\left(\omega_k^n \mid \omega_{-\infty}^{k-1}\right) = \prod_{i=k}^{n} f_i\left(\omega_i \mid \omega_{-\infty}^{i-1}\right) .$$

We then apply inequalities (5.3) to bound each of the factors by similar factors with conditioning configuration σ.

To obtain **(5.5)** we apply the LIS-reconstruction formula (3.4) with $m = n+1$ which yields

$$F_{\Lambda,n+1}(\omega_\Lambda \mid \omega) = \frac{f_{[l_\Lambda,n]}\left(\omega_{l_\Lambda}^n \mid \omega_{\Lambda_-}\right)\; f_{n+1}\left(\omega_{n+1} \mid \omega_{]-\infty,n]}\right)}{\sum_{\beta_\Lambda}\left[f_{[l_\Lambda,n]}\left(\beta_\Lambda\,\omega_{\Lambda^c\cap[l_\Lambda,n]} \mid \omega_{\Lambda_-}\right)\; f_{n+1}\left(\omega_{n+1} \mid \beta_\Lambda\,\omega_{]-\infty,n]\setminus\Lambda}\right)\right]} .$$

In the denominator, only β_Λ with $f_{n+1}\left(\omega_{n+1} \mid \omega_{]-\infty,n]\setminus\Lambda}\,\beta_\Lambda\right) \neq 0$ contribute. We use inequalities (5.3) for these. $\square$

Lemma 5.6
Let $f \in \Theta_1,\ \omega \in \Omega$ and $\Lambda \in \mathcal{S}$. Consider the sequence $F_{\Lambda,n}\left(\omega_\Lambda \mid \omega\right),\, n \geq m_\Lambda$ defined by (4.14). Then the limit (4.15) exits and satisfies

$$\left|\gamma_\Lambda^f\left(\omega_\Lambda \mid \omega_{\Lambda^c}\right) - F_{\Lambda,n}\left(\omega_\Lambda \mid \omega\right)\right| \leq \sum_{k \geq n+1} c_\Lambda^\omega(f_k)^{-1}\, \delta_\Lambda^\omega(f_k). \tag{5.7}$$

Proof ¿From (5.5), plus the fact that $F_{\Lambda,n} \in [0,1]\ \forall\, n \geq m$, we obtain

$$\left|F_{\Lambda,n+1}\left(\omega_\Lambda \mid \omega\right) - F_{\Lambda,n}\left(\omega_\Lambda \mid \omega\right)\right| \leq c_\Lambda^\omega(f_{n+1})^{-1}\, \delta_\Lambda^\omega(f_{n+1}).$$

Therefore the summability of $c_\Lambda^\omega(f_k)^{-1}\, \delta_\Lambda^\omega(f_k)$ for $k \geq m$ implies the summability of the sequence $\left|F_{\Lambda,k+1}\left(\omega_\Lambda \mid \omega\right) - F_{\Lambda,k}\left(\omega_\Lambda \mid \omega\right)\right|$. In particular $\big(F_{\Lambda,n}(\omega_\Lambda \mid \omega)\big)_{n \geq m}$ is a Cauchy sequence so the limit $\lim_{n\to+\infty} F_{\Lambda,n}\left(\omega_\Lambda \mid \omega\right) \triangleq \gamma_\Lambda^f\left(\omega_\Lambda \mid \omega\right)$ exits and satisfies (5.7) for each $\omega \in \Omega$. □

5.1 LIS ⇝ specification

Proof of Theorem 4.16

Lemma (5.6) proves Item **1) (a)**.

To prove item **1) (b)**, we observe that $\gamma_\Lambda^f(A \mid \cdot\,)$ is clearly $\mathcal{F}_{\Lambda^c}$-measurable for every $\Lambda \in \mathcal{S}$ and every $A \in \mathcal{F}$. Moreover condition (b) of Definition 2.12 together with the presence of the indicator function $1\!\!1_{\omega_{\Lambda^c \cap [l_\Lambda, m_\Lambda]}}$ in the denominator of (4.14) imply that $\gamma_\Lambda^f(B \mid \cdot\,) = 1\!\!1_B(\,\cdot\,)$ for every $\Lambda \in \mathcal{S}$ and every $B \in \mathcal{F}_{\Lambda^c}$. Therefore it suffices to show that

$$\sum_{\omega_{\Delta\setminus\Lambda}} \gamma_\Lambda^f\left(\omega_\Lambda \mid \omega_{\Lambda^c}\right) \gamma_\Delta^f\left(\omega_{\Delta\setminus\Lambda} \mid \omega_{\Delta^c}\right) \;=\; \gamma_\Delta^f\left(\omega_\Lambda \mid \omega_{\Delta^c}\right) \tag{5.8}$$

for each $\Lambda, \Delta \in \mathcal{S}$ such that $\Lambda \subset \Delta$ and each $\omega \in \Omega$. Let us denote, for each $\Gamma \subseteq \Delta$, each integer $n \geq l_\Gamma$ and $\omega_{-\infty}^n \in \Omega_{-\infty}^n$,

$$G_{\Gamma,n}\left(\,\cdot \mid \omega_{\Gamma^c}\right) \;\triangleq\; f_{[l_\Gamma,n]}\Big(\,\cdot\; 1\!\!1_{\omega_{\Gamma^c \cap [l_\Gamma,n]}} \,\Big|\, \omega_{\Gamma_-}\Big).$$

Definition (4.14)–(4.15) becomes

$$\gamma_\Delta\left(h \mid \omega\right) \;=\; \lim_{n\to+\infty} \frac{G_{\Delta,n}\left(\, 1\!\!1_{\omega_\Delta} \mid \omega_{\Delta^c}\right)}{G_{\Delta,n}\left(1 \mid \omega_{\Delta^c}\right)}. \tag{5.9}$$

Using the reconstruction property (3.4) of LIS with $l = l_\Delta,\ n = l_\Lambda - 1$ and $m = n$, we obtain

$$G_{\Delta,n}(\, 1\!\!1_{\omega_{\Delta\setminus\Lambda}} \mid \omega_{\Delta^c}) \;=\; G_{\Delta,l_\Lambda-1}(\, 1\!\!1_{\omega_{\Delta\setminus\Lambda}} \mid \omega_{\Delta^c}) \times G_{\Lambda,n}(1 \mid \omega_{\Lambda^c})$$

and

$$G_{\Lambda,n}(\,\mathbb{1}_{\omega_\Lambda} \mid \omega_{\Lambda^c}) \times G_{\Delta,l_\Lambda-1}(\,\mathbb{1}_{\omega_{\Delta\setminus\Lambda}} \mid \omega_{\Delta^c}) \;=\; G_{\Delta,n}(\,\mathbb{1}_{\omega_\Delta} \mid \omega_{\Delta^c})\;.$$

Therefore

$$\frac{G_{\Lambda,n}(\,\mathbb{1}_{\omega_\Lambda} \mid \omega_{\Lambda^c})}{G_{\Lambda,n}(1 \mid \omega_{\Lambda^c})} \times \frac{G_{\Delta,n}(\,\mathbb{1}_{\omega_{\Delta\setminus\Lambda}} \mid \omega_{\Delta^c})}{G_{\Delta,n}(1 \mid \omega_{\Delta^c})} \;=\; \frac{G_{\Delta,n}(\,\mathbb{1}_{\omega_\Delta} \mid \omega_{\Delta^c})}{G_{\Delta,n}(1 \mid \omega_{\Delta^c})}\;. \tag{5.10}$$

Identity (5.8) follows from (5.9) and (5.10).

We proceed with item **1) (c)**. By (5.7) and the summability of the bound ϵ_k [defined in (4.6)], $F_{\Lambda,n}(\omega_\Lambda \mid \cdot)$ converges uniformly to $\gamma_\Lambda(\omega_\Lambda \mid \cdot)$. As each $F_{\Lambda,n}$ is continuous on Ω, so is γ^f_Λ. Let us fix k_0 such that $\epsilon_k < 1$ for $k \geq k_0$. By (4.6) and the lower bound in (5.5)

$$\gamma^f_\Lambda \;\geq\; F_{\Lambda,k_0} \prod_{k=k_0}^{\infty} (1-\epsilon_k)\;.$$

The right-hand side is strictly positive on Ω due to the non-nullness of F_{Λ,k_0} and the summability of the ϵ_k. Hence γ^f_Λ is non-null on Ω.

To prove assertion **2)(a)** we consider $\mu \in \mathcal{G}\left(f^{(V,\omega)}\right)$ and denote

$$G^{(V,\omega)}_{\Lambda,n}(\,\cdot \mid \sigma_{\Lambda^c}) \;\triangleq\; f_{[l_\Lambda,n]}\Big(\,\cdot\;\mathbb{1}_{\sigma_{\Lambda^c\cap[l_\Lambda,n]}}\,\Big|\,\omega_{V_-}\,\sigma_{\Lambda_-\setminus V_-}\Big)$$

for all $\Lambda \in \mathcal{S}_b : \Lambda \subset V$ and $\omega,\sigma : \omega_{V_-}\,\sigma_{\Lambda_-\setminus V_-} \in \Omega_{\Lambda_-}$. By a straightforward extension of (5.9), the dominated convergence theorem and the consistency of μ with respect to $G^{(V,\omega)}_{\Lambda,n}$

$$\begin{aligned}\mu\,\gamma^f_\Lambda\,(\,\mathbb{1}_{\omega_\Lambda}) &= \lim_{n\to+\infty} \mu\,G^{(V,\omega)}_{\Lambda,n}\left(\frac{G^{(V,\omega)}_{\Lambda,n}\,(\,\mathbb{1}_{\omega_\Lambda} \mid \cdot\,)}{G^{(V,\omega)}_{\Lambda,n}\,(1 \mid \cdot\,)}\right)\\ &= \lim_{n\to+\infty} \mu\,G^{(V,\omega)}_{\Lambda,n}\,(\,\mathbb{1}_{\omega_\Lambda})\,.\end{aligned}$$

Applying the consistency hypothesis a second time we obtain $\mu\,\gamma^f_\Lambda = \mu$.

Assertion **2)(b)** is an immediate consequence of 2) (a) and of the fact that $|\mathcal{G}(\gamma)| = 1$ for all $\gamma \in \Pi_1$.

Finally we prove **2) (c)**. Let f^1 and f^2 be two LIS on $(\Omega,\mathcal{F})$, both in $b^{-1}(\Pi_1)$, and such that $\gamma^{f^1} = \gamma^{f^2}$. By 2) (c), $\mu^{f^1} = \mu^{f^2} \triangleq \mu$. The non-nullness of f^1 and f^2 on Ω implies that μ charges all open sets in Ω. Therefore, f^1_Λ and f^2_Λ coincide, on Ω, with the unique continuous realization of $E_\mu\left(\,\cdot \mid \mathcal{F}_{\Lambda_-}\right)$.
□

5.2 Specification $\rightsquigarrow$ LIS

Let us introduce the *spread* of a (bounded) function h on Ω:

$$\mathrm{Spr}(h) \;=\; \sup(h) - \inf(h)\;.$$

Lemma 5.11

1) *Let γ be a specification on Ω.*

(a) *If there exists an exhausting sequence of regions $\Lambda_n \subset \mathbb{Z}$ such that*

$$\lim_{n\to+\infty} \mathrm{Spr}\,(\gamma_{\Lambda_n} h) = 0 \tag{5.12}$$

for each continuous $\mathcal{F}$-measurable function h, then $|\mathcal{G}(\gamma)| \le 1$.

(b) *If γ is continuous and $|\mathcal{G}(\gamma)| \le 1$, then (5.12) holds for all exhausting sequences of regions $\Lambda_n \subset \mathbb{Z}$ and all continuous $\mathcal{F}$-measurable function h.*

2) *Let f be a LIS on Ω*

(a) *If for each $i \in \mathbb{Z}$ and each continuous $\mathcal{F}_{\le i}$-measurable continuous function h*

$$\lim_{n\to+\infty} \mathrm{Spr}\left(f_{[i-n,i]} h\right) = 0\;, \tag{5.13}$$

then $|\mathcal{G}(f)| \le 1$.

(b) *If f is continuous and $|\mathcal{G}(f)| \le 1$, then (5.13) is verified for all $i \in \mathbb{Z}$ and all continuous $\mathcal{F}_{\le i}$-measurable continuous function h.*

Proof We proof part 1), the proof of 2) is similar. The obvious spread-reducing relation

$$\inf_{\widetilde\omega\in\Omega} h(\widetilde\omega) \;\le\; (\gamma_\Lambda h)(\omega) \;\le\; \sup_{\widetilde\omega\in\Omega} h(\widetilde\omega)\;,$$

valid for every bounded measurable function h on Ω and every configuration $\omega \in \Omega$, plus the consistency condition (2.4) imply that the sequence $\{\sup(\gamma_{\Lambda_n} h)\}$ is decreasing (and bounded below by $\inf h$), while the sequence $\{\inf(\gamma_{\Lambda_n} h)\}$ is increasing (and bounded above by $\sup h$). Therefore, if $\mu, \nu \in \mathcal{G}(\gamma)$,

$$\mu(h) - \nu(h) \;\le\; \sup(\gamma_{\Lambda_n} h) - \inf(\gamma_{\Lambda_n} h) \tag{5.14}$$

for every n, which yields

$$|\mu(h) - \nu(h)| \;\le\; \mathrm{Spr}(\gamma_{\Lambda_n} h) \tag{5.15}$$

for every n. This proves item **1)(a)**.

Regarding **1)(b)**, we observe that, as Ω is compact, there exist optimizing boundary conditions $\{\sigma^{(n)}\}$ and $\{\eta^{(n)}\}$ such that $(\gamma_{\Lambda_n} h)\left(\sigma^{(n)}\right) = \sup(\gamma_{\Lambda_n} h)$. and $(\gamma_{\Lambda_n} h)\left(\eta^{(n)}\right) = \inf(\gamma_{\Lambda_n} h)$. (Of course, both sequences of boundary conditions depend on h). Let $\overline{\rho}$ and $\underline{\rho}$ be respective accumulation point of the sequences of measures $\{\gamma_{\Lambda_n}(\,\cdot \mid \sigma^{(n)})\}$ and $\{\gamma_{\eta_n}(\,\cdot \mid \sigma^{(n)})\}$ (they exist by compactness). Then, $\overline{\rho}, \underline{\rho} \in \mathcal{G}(\gamma)$ (due to the continuity of γ) and

$$\lim_{n} \operatorname{Spr}(\gamma_{\Lambda_n} h) \;\leq\; \overline{\rho}(h) - \underline{\rho}(h)\,. \tag{5.16}$$

Hence the uniqueness of the consistent measure implies (5.12). We learnt this argument from Michael Aizenman (private communication). □

Our last auxiliary result refers to the following notion.

Definition 5.17
A **global specification** *γ over $(\Omega, \mathcal{F})$ is a family of probability kernels $\{\gamma_V\}_{V \subset \mathbb{Z}}$, $\gamma_V : \mathcal{F} \times \Omega \to [0,1]$ such that for all $V \subset \mathbb{Z}$*

(a) *For each $A \in \mathcal{F}$, $\gamma_V(A \mid \cdot\,) \in \mathcal{F}_{V^c}$.*

(b) *For each $B \in \mathcal{F}_{V^c}$ and $\omega \in \Omega$, $\gamma_V(B \mid \omega) = \mathbb{1}_B(\omega)$.*

(c) *For each $W \subset \mathbb{Z} : W \supset V$, $\gamma_W \gamma_V = \gamma_W$.*

Proposition 5.18
Let γ be a continuous specification over $(\Omega, \mathcal{F})$ which satisfies a HUC. Then γ can be extended into a continuous global specification such that for every subset $V \subset \mathbb{Z}$,

$$\gamma_V\left(h \mid \omega_{V^c}\right) \;\triangleq\; \lim_{\Lambda \uparrow V} \gamma_\Lambda\left(h \mid \omega\right) \tag{5.19}$$

for all continuous functions $h \in \mathcal{F}$ and all $\omega \in \Omega$. Moreover for all $V \subset \mathbb{Z}$ and all $\omega \in \Omega$,

$$\mathcal{G}\left(\gamma^{(V,\omega)}\right) \;=\; \{\gamma_V\left(\,\cdot \mid \omega\right)\}\,. \tag{5.20}$$

Georgii (1988) gives a proof of this proposition in the Dobrushin regime (Theorem 8.23). The same proof extends, with minor adaptations, under a HUC (see Fernández and Pfister, 1997).

Proof of Theorem 4.19

Items **1) (a)–(b)** are proven in Proposition (5.18).

There are three things to prove regarding **1) (c)**:

(i) Continuity of f^γ. This is, in fact, an application of Proposition (5.18).

(ii) Non-nullness of f^γ. Consider $\Lambda \in \mathcal{S}$, $\omega \in \Omega$, $n \geq m_\Lambda$ and $k \geq 0$. By the non-nullness and the continuity of γ and the compactness of Ω_{Λ^c}, there exists $\widetilde{\omega} \in \Omega_{\Lambda^c}$ such that

$$0 \ < \ \gamma_\Lambda \left(\omega_\Lambda \mid \widetilde{\omega}\right) \ = \ \inf_{\omega \in \Omega_\Lambda^c} \gamma_\Lambda \left(\omega_\Lambda \mid \omega\right) \ \triangleq \ c\left(\Lambda, \omega_\Lambda\right).$$

Therefore by the consistency of γ

$$\begin{aligned} f_\Lambda^\gamma \left(\omega_\Lambda \mid \omega_{\Lambda_-}\right) \ &= \ \lim_{k\to\infty} \gamma_{[l_\Lambda, n+k]} \left(\omega_\Lambda \mid \omega\right) \ = \ \lim_{k\to\infty} \Big(\gamma_{[l_\Lambda, n+k]} \gamma_\Lambda\Big) \left(\omega_\Lambda \mid \omega\right) \\ &\geq \ c(\Lambda, \omega_\Lambda) \ > \ 0 \,. \end{aligned}$$

(iii) Hereditary uniqueness. Let us fix $\omega \in \Omega$ and $V \in \{[j, +\infty[, \, j \in \mathbb{Z}\} \cup \mathbb{Z}$. For each $i \in \mathbb{Z}$ and $h \in \mathcal{F}_{\leq i}$. We have

$$\mathrm{Spr}\left(f_{[i-k,i]}^{\gamma\,(V,\omega)} h\right) \ \leq \ \lim_{n\to+\infty} \mathrm{Spr}\left(\gamma_{[i-k,i+n]}^{(V,\omega)} h\right) \ .$$

As, by hypothesis, each specification $\gamma^{(V,\omega)}$ admits an unique Gibbs measure, it follows from lemma 5.11 1) (b) that

$$\lim_{k\to+\infty} \mathrm{Spr}\left(f_{[i-k,i]}^{\gamma\,(V,\omega)} h\right) \ = \ 0 \,.$$

This proves that $\left|\mathcal{G}\left(f^{\gamma\,(V,\omega)}\right)\right| = 1$ by lemma 5.11 2) (a).

The uniqueness part of assertion **2) (a)** is contained in the just proven hereditary uniqueness. To show that $\mu^\gamma \in \mathcal{G}(f^\gamma)$, consider $\Lambda \in \mathcal{S}_b$ and h a continuous $\mathcal{F}_{\leq m_\Lambda}$-measurable function. By the dominated convergence theorem

$$\mu^\gamma f_\Lambda^\gamma(h) = \lim_{n\to+\infty} \int \gamma_{\Lambda\cup\Lambda_+^{(n)}} \left(h \mid \xi\right) \mu^\gamma(d\xi).$$

The consistency of μ^γ with respect to γ implies, hence, that $\mu^\gamma \in \mathcal{G}(f^\gamma)$.

To prove assertion **2) (b)**, let γ^1 and γ^2 such that $f^{\gamma^1} = f^{\gamma^2}$. By 2) (a) $\mu^{\gamma^1} = \mu^{\gamma^2} \triangleq \mu$. The non-nullness of γ^1 and γ^2 implies that μ charges all open sets on Ω. Therefore for each $\Lambda \in \mathcal{S}$, γ_Λ^1 and γ_Λ^2 coincide with the unique continuous realization of $E_\mu\left(\,\cdot \mid \mathcal{F}_{\Lambda^c}\right)$. $\square$

Proof of Theorem 4.20

To prove item **(a)**, let us recall one of the equivalent definitions of the variational distance between probability measures over $(\Omega_i, \mathcal{F}_i)$

$$\|\mu - \nu\| \ = \ \sup_{h\in\mathcal{F}_i} \frac{|\mu(h) - \nu(h)|}{\mathrm{Spr}(h)} \ .$$

For a proof of this result see for example Georgii (1988) (section 8.1). By the consistency of $\overset{\circ}{\gamma}_i$ with respect to $\gamma_{[i,i+k]}$, $k \geq 0$

$$\overset{\circ}{f}{}_i^{\gamma}\left(\,\cdot \mid \omega_{\infty}^{i-1}\right) \;\triangleq\; \lim_{k\to+\infty} \overset{\circ}{\gamma}_{[i,i+k]}\,(\,\cdot \mid \omega) \;=\; \lim_{k\to+\infty} \gamma_{[i,i+k]}\; \overset{\circ}{\gamma}_i\,(\,\cdot \mid \omega)\;.$$

Therefore, by dominated convergence,

$$C_{ij}(f^{\gamma}) \;\leq\; \sup_{\substack{\xi,\eta\in\Omega \\ \xi_{-\infty}^{i-1} \overset{\neq j}{=} \eta_{-\infty}^{i-1}}} \left\| \overset{\circ}{\gamma}_i\,(\,\cdot \mid \xi) - \overset{\circ}{\gamma}_i\,(\,\cdot \mid \eta) \right\| . \tag{5.21}$$

Since γ is continuous, we can do an infinite telescoping of (5.21) to obtain

$$C_{ij}(f^{\gamma}) \;\leq\; \sum_{\substack{k=j \\ \text{or } k>i}} C_{ik}(\gamma)\;.$$

Thus

$$\sum_{j:j<i} C_{ij}(f^{\gamma}) \;\leq\; \sum_{j:j\neq i} C_{ij}(\gamma) < 1\;.$$

To show assertion **(b)**, consider $\gamma \in \Pi_2$ for which there exists a constant $K > 0$ such that for every cylinder set $A = \{x_l^m\} \in \Omega_l^m$ there exist integers n, p satisfying

$$\gamma_{[n,p]}(A \mid \xi) \;\geq\; K\,\gamma_{[n,p]}(A \mid \eta) \quad \text{for all } \xi, \eta \in \Omega\;.$$

Hence, by consistency of γ, we have that for some fixed $\sigma \in \Omega$ and for each $k \geq 0$

$$\gamma_{[n,p+k]}(A \mid \xi) \;=\; \int \gamma_{[n,p]}(A \mid \omega)\,\gamma_{[n,p+k]}(d\omega \mid \xi) \;\geq\; K\,\gamma_{[n,p]}(A \mid \sigma)\;.$$

In a similar way we obtain

$$\gamma_{[n,p+k]}(A \mid \eta) \;\leq\; \frac{1}{K}\,\gamma_{[n,p]}(A \mid \sigma)\;.$$

We conclude that for each $k \geq 0$

$$\gamma_{[n,p+k]}(A \mid \xi) \;\geq\; K^2\,\gamma_{[n,p+k]}(A \mid \eta)\;.$$

Letting $k \to \infty$ we obtain, due to definition (4.18), that $f_{[n,m]}^{\gamma}(A \mid \xi) \geq K^2 f_{[n,m]}^{\gamma}(A \mid \eta)$. $\square$

5.3 LIS $\leftrightsquigarrow$ specification

Proof of Theorem 4.21
Assertion **1) (a)** is a direct consequence of inequality (5.7) of Lemma 5.6 with $\Lambda = \{k\}$ and $n = j - 1$.

Assertion **1) (b)** follows from the $n \to \infty$ limit of inequalities (5.4) and the fact that $0 \le F_{\Lambda,n} \le 1$.

To prove assertion **2)**, let $k, j \in \mathbb{Z}$ such that $j < k$ and consider $\omega, \sigma \in \Omega$ such that $\omega \overset{\neq j}{=} \sigma$. As a direct consequence of definitions 4.3–4.4 we have that, for all $i \ge k$,

$$\left(1 - c_j^{\omega}(\gamma_i)^{-1}\, \delta_j^{\omega}(\gamma_i)\right) \times \gamma_i(\omega_i \mid \omega_{-\infty}^{i-1}\, \omega_{i+1}^{+\infty}) \;\le\; \gamma_i(\sigma_i \mid \sigma_{-\infty}^{i-1}\, \sigma_{i+1}^{+\infty}) \tag{5.22}$$

and

$$\gamma_i(\sigma_i \mid \sigma_{-\infty}^{i-1}\, \sigma_{i+1}^{+\infty}) \;\le\; \left(1 + c_j^{\omega}(\gamma_i)^{-1}\, \delta_j^{\omega}(\gamma_i)\right) \times \gamma_i(\omega_i \mid \omega_{-\infty}^{i-1}\, \omega_{i+1}^{+\infty})\,. \tag{5.23}$$

By the specification reconstruction formula (3.9) with $\Lambda = \{n+1\}$ and $\Gamma = [l_\Lambda, n]$ we have

$$\gamma_{[l_\Lambda, n+1]}\left(\sigma_\Lambda \mid \sigma_{\Lambda_-}\, \sigma_{m_\Lambda+1}^{+\infty}\right) \;=\; \sum_{\sigma_{m_\Lambda+1}^{n}} \frac{\gamma_{n+1}\left(\sigma_{n+1} \mid \sigma_{-\infty}^{n}\, \sigma_{n+2}^{+\infty}\right)}{\displaystyle\sum_{\xi_{n+1}} \frac{\gamma_{n+1}\left(\xi_{n+1} \mid \sigma_{-\infty}^{n}\, \sigma_{n+2}^{+\infty}\right)}{\gamma_{[l_\Lambda,n]}\left(\sigma_{l_\Lambda}^{n} \mid \sigma_{\Lambda_-}\, \xi_{n+1}\, \sigma_{n+2}^{+\infty}\right)}}.$$

Using (5.22) and (5.23) it is easy to show, by induction over $n \ge m_\Lambda + 1$, that

$$\gamma_{[l_\Lambda,n]}\left(\omega_\Lambda \mid \xi_{\Lambda_-}\, \omega_{n+1}^{+\infty}\right) \;\le\; \gamma_{[l_\Lambda,n]}\left(\omega_\Lambda \mid \eta_{\Lambda_-}\, \omega_{n+1}^{+\infty}\right) \times \prod_{i=k}^{n} \frac{1 - c_j^{\omega}(\gamma_k)^{-1}\, \delta_j^{\omega}(\gamma_i)}{1 + c_j^{\omega}(\gamma_k)^{-1}\, \delta_j^{\omega}(\gamma_i)}$$

for all $\xi, \eta \in \Omega : \xi \overset{\neq j}{=} \eta \overset{\neq j}{=} \omega$. Taking the limit when n tends to infinity, we obtain 2). $\square$

Proof of Theorem 4.22
For the proof of item **(a)** we consider $\gamma \in \Pi_2$ such that $f^\gamma \in \Theta_1$ and fix $\Lambda \in \mathcal{S}$ and $\omega \in \Omega$. By definition of the maps b and c [see (4.14)–(4.15) and (4.18)], we have that

$$\gamma_\Lambda^{f^\gamma}(\omega_\Lambda \mid \omega) \;=\; \lim_{n\to+\infty} \lim_{k\to+\infty} \frac{\gamma_{[l_\Lambda,n+k]}\left(\omega_\Lambda\, \omega_{\Lambda^c \cap [l_\Lambda,n]} \mid \omega_{\Lambda_-}\, \omega_{n+k+1}^{+\infty}\right)}{\gamma_{[l_\Lambda,n+k]}\left(\omega_{\Lambda^c \cap [l_\Lambda,n]} \mid \omega_{\Lambda_-}\, \omega_{n+k+1}^{+\infty}\right)}\,. \tag{5.24}$$

The consistency of γ_Λ and $\gamma_{[l_\Lambda,n+k]}$ implies

$$\gamma_{[l_\Lambda,n+k]}\left(\omega_\Lambda\,\omega_{\Lambda^c\cap[l_\Lambda,n]}\mid\omega_{\Lambda_-}\,\omega_{n+k+1}^{+\infty}\right) \;=\;$$
$$\sum_{\xi_{n+1}^{n+k}}\gamma_\Lambda\left(\omega_\Lambda\mid\omega_{\Lambda^c\cap]-\infty,n]}\,\xi_{n+1}^{n+k}\,\omega_{n+k+1}^{+\infty}\right)\,\gamma_{[l_\Lambda,n+k]}\left(\omega_{\Lambda^c\cap[l_\Lambda,n]}\,\xi_{n+1}^{n+k}\mid\omega_{\Lambda_-}\,\omega_{n+k+1}^{+\infty}\right)\;. \tag{5.25}$$

By continuity of $\gamma_\Lambda\left(\omega_\Lambda\mid\cdot\,\right)$ we have that, for each $\varepsilon>0$,

$$\left|\gamma_\Lambda\left(\omega_\Lambda\mid\omega_{\Lambda^c\cap]-\infty,n]}\,\xi_{n+1}^{n+k}\,\omega_{n+k+1}^{+\infty}\right)-\gamma_\Lambda\left(\omega_\Lambda\mid\omega\right)\right|\;<\;\varepsilon$$

for n large enough uniformly in k. Combining this with (5.24)–(5.25) we conclude that

$$\left|\gamma_\Lambda^{f^\gamma}\left(\omega_\Lambda\mid\omega\right)-\gamma_\Lambda\left(\omega_\Lambda\mid\omega\right)\right|\;<\;\varepsilon$$

for every $\varepsilon>0$. Therefore $\gamma^{f^\gamma}=\gamma$.

To prove item **(b)**, consider $f\in\Theta_1$ such that $\gamma^f\in\Pi_2$ and fix $\Lambda\in\mathcal{S}_b$ and $\omega\in\Omega$. Let us denote $V=[l_\Lambda,+\infty[$. Since γ^f satisfies a HUC equation, (5.19) and definition (4.18) yield

$$f_\Lambda^{\gamma^f}\left(\omega_\Lambda\mid\omega_{\Lambda_-}\right)\;\triangleq\;\lim_{n\to+\infty}\gamma^f_{[l_\Lambda,m_\Lambda+n]}\left(\omega_\Lambda\mid\omega\right)\;=\;\gamma_V^f\left(\omega_\Lambda\mid\omega_{\Lambda_-}\right)\;. \tag{5.26}$$

Combining (5.20) with assertion 2) (a) of Theorem 4.16 we obtain that

$$\mathcal{G}\left(f^{(V,\omega)}\right)=\left\{\gamma_V^f\left(\,\cdot\mid\omega_{\Lambda_-}\right)\right\}.$$

Therefore

$$\gamma_V^f\left(\omega_\Lambda\mid\omega_{\Lambda_-}\right)=\gamma_\Lambda^f\left(f_\Lambda^{(V,\omega)}\left(\omega_\Lambda\mid\cdot\,\right)\mid\omega_{\Lambda_-}\right)=f_\Lambda\left(\omega_\Lambda\mid\omega_{\Lambda_-}\right).$$

The last equality is a consequence of the definition (2.23). By (5.26) this implies that

$$f_V^{\gamma^f}\left(\omega_\Lambda\mid\omega_{\Lambda_-}\right)=f_\Lambda\left(\omega_\Lambda\mid\omega_{\Lambda_-}\right).$$

Item **(c)** is a direct consequence of Theorem 4.21 and the following result. □

Lemma 5.27
Let $h:\mathbb{R}^+\to\mathbb{R}^+$ be a decreasing function and $(u_i)_{i\in\mathbb{N}}$ be a sequence taking values in $]0,1[$ for which there exists $m\geq 0$ such that $u_i\leq m\,h(i)$. Then there exists $M\geq 0$ such that

$$1-\prod_{i=k}^{+\infty}\frac{1-u_i}{1+u_i}\;\leq\;M\,H(k-1)\;,$$

where $H(x)=\displaystyle\int_x^{+\infty}h(t)\,dt$.

The proof is left to the reader. □

A Singleton consistency for Gibbs measures

In this appendix we work in a more general setting than in the paper. We consider a general measurable space $(E, \mathcal{E})$ (not necessarily finite or even compact) and a subset Ω of $E^{\mathbb{Z}^d}$ for a given $d \geq 1$. The space Ω is endowed with the projection $\mathcal{F}$ of the product σ-algebra associated to $E^{\mathbb{Z}^d}$. We also consider a family of *a priori* measures $\lambda = (\lambda^i)_{i\in\mathbb{Z}^d}$ in $\mathcal{M}(E, \mathcal{E})$ and their products $\lambda^\Lambda \triangleq \bigotimes_{i\in\Lambda} \lambda^i$ for $\Lambda \subset \mathbb{Z}^d$. We denote by $(\lambda_\Lambda)_{\Lambda\in\mathcal{S}}$ the family of measure kernels defined over $(\Omega, \mathcal{F})$ by

$$\lambda_\Lambda(h \mid \omega) = \left(\lambda^\Lambda \otimes \delta_{\omega_{\Lambda^c}}\right)(h) \tag{A.1}$$

for every measurable function h and configuration ω. These kernels satisfy the following identities for every $\Lambda \in \mathcal{S}$:

$$\lambda_\Lambda(B \mid \cdot\,) \;=\; 1\!\!1_B(\,\cdot\,),\ \forall\, B \in \mathcal{F}_{\Lambda^c} \tag{A.2}$$

and

$$\lambda_{\Lambda\cup\Delta} \;=\; \lambda_\Lambda\lambda_\Delta\ , \quad \forall\, \Delta \in \mathcal{S} : \Lambda \cup \Delta = \emptyset. \tag{A.3}$$

Theorem A.4
Let λ be as above and $(\gamma_i)_{i\in\mathbb{Z}^d}$ be a family of probability kernels on $\mathcal{F} \times \Omega_i$ such that

1) *For each $i \in \mathbb{Z}^d$ and for some measurable function ρ_i,*

$$\gamma_i \;=\; \rho_i\lambda_i\ . \tag{A.5}$$

2) *The following properties hold:*

- **(a)** *Normalization on Ω: for every i in $\mathbb{Z}^d$,*

$$\left(\lambda_i\left(\rho_i\right)\right)(\omega) \;=\; 1\ , \quad \forall\, \omega \in \Omega\ . \tag{A.6}$$

- **(b)** *Bounded-positivity on Ω: for every $i, j \in \mathbb{Z}^d$,*

$$\inf_{\omega\in\Omega} \lambda_j\left(\rho_j\,\rho_i^{-1}\right)(\omega) \;>\; 0 \tag{A.7}$$

and

$$\sup_{\omega\in\Omega} \lambda_j\left(\rho_j\,\rho_i^{-1}\right)(\omega) \;<\; +\infty\ . \tag{A.8}$$

(c) *Order-consistency on Ω: for every i, j in $\mathbb{Z}^d$ and every $\omega \in \Omega$,*

$$\rho_{ij}(\omega) \;=\; \frac{\rho_i}{\lambda_i\left(\rho_i\,\rho_j^{-1}\right)}(\omega) \;=\; \frac{\rho_j}{\lambda_j\left(\rho_j\,\rho_i^{-1}\right)}(\omega)\,. \tag{A.9}$$

Then there exists a unique family $\rho = \{\rho_\Lambda\}_{\Lambda\in\mathcal{S}}$ of positive measurable functions on $(\Omega, \mathcal{F})$ such that

(i) $\gamma \triangleq \{\rho_\Lambda\lambda_\Lambda\}_{\Lambda\in\mathcal{S}}$ *is a specification on $(\Omega, \mathcal{F})$ with $\gamma_{\{i\}} = \gamma_i$ for each $i \in \mathbb{Z}^d$.*

(ii) $\rho_{\Lambda\cup\Gamma} = \dfrac{\rho_\Lambda}{\lambda_\Lambda\left(\rho_\Lambda\,\rho_\Gamma^{-1}\right)}$, *for all $\Lambda, \Gamma \in \mathcal{S}$ such that $\Gamma \subset \Lambda^c$.*

(iii) $\mathcal{G}(\gamma) = \left\{\mu \in \mathcal{P}(\Omega, \mathcal{F}) : \mu\gamma_i = \mu \text{ for all } i \in \mathbb{Z}^d\right\}$.

(iv) *For each $\Lambda \in \mathcal{S}$ there exist constants $C_\Lambda, D_\Lambda > 0$ such that $C_\Lambda\,\rho_k(\omega) \le \rho_\Lambda(\omega) \le D_\Lambda\,\rho_k(\omega)$ for all $k \in \Lambda$ and all $\omega \in \Omega$).*

Remarks

A.10 This theorem is a strengthening of the reconstruction result given by Theorem 1.33 in Georgii (1988). In the latter, the order-consistency condition (A.9) is replaced by the requirement that the singletons come from a pre-existing specification (which the prescription reconstructs). For finite E, Nahapetian and Dachian (2001) have presented an alternative approach where (A.9) is replaced by a more detailed pointwise condition. Their non-nullness hypotheses are also different from ours.

A.11 Identity (ii) can be used, in fact, to inductively define the family ρ by adding one site at a time. In fact, this is what is done in the proof below. The inequalities (iv) relate the non-nullness properties of ρ to those of the original family $\{\rho_i\}_{i\in\mathbb{Z}^d}$.

A.12 In the case E countable, λ_i=counting measure, the order-consistency requirement (A.9) is automatically verified if the singletons are defined through a measure μ on $\mathcal{F}$ in the form

$$\rho_i(\omega) \;=\; \lim_{n\to\infty} \frac{\mu(\omega_{V_n})}{\mu(\omega_{V_n\setminus\{i\}})}$$

for an exhausting sequence of volumes $\{V_n\}$. Indeed, a simple computation shows that the last two terms in (A.9) coincide with

$$\lim_{n\to\infty} \frac{\mu(\omega_{V_n})}{\mu(\omega_{V_n\setminus\{i,j\}})}\,.$$

Proof In the following all functions are defined on Ω or on a projection of Ω over a subset of $\mathbb{Z}^d$.

Initially we define ρ by choosing a total order for $\mathbb{Z}^d$ and prescribing, inductively, that for each $\Lambda \in \mathcal{S}$ with $|\Lambda| \geq 2$ and each $\omega \in \Omega$

$$\rho_\Lambda(\omega) \;=\; \frac{\rho_k}{\lambda_k\left(\rho_k\, \rho^{-1}_{\Lambda^*_k}\right)}(\omega)\;, \tag{A.13}$$

where $k = \max \Lambda$ and $\Lambda^*_k = \Lambda \setminus \{k\}$. For each $\Lambda, \Gamma \in \mathcal{S}$ such that $\Gamma \subset \Lambda^c$, we will prove, by induction over $|\Lambda \cup \Gamma|$, that the functions so defined satisfy the following properties:

(I1) $\inf_{\omega\in\Omega} \lambda_\Lambda\left(\rho_\Lambda\, \rho_\Gamma^{-1}\right)(\omega) > 0$ and $\sup_{\omega\in\Omega} \lambda_\Lambda\left(\rho_\Lambda\, \rho_\Gamma^{-1}\right)(\omega) < +\infty$.

(I2) $\rho_{\Lambda\cup\Gamma} = \dfrac{\rho_\Lambda}{\lambda_\Lambda\left(\rho_\Lambda\, \rho_\Gamma^{-1}\right)}$.

(I3) $\lambda_\Lambda\left(\rho_\Lambda\right) = 1$.

(I4) If μ is a probability measure on $(\Omega, \mathcal{F})$ such that $\mu\left(\rho_i\, \lambda_i\right) = \mu,\ \forall\, i \in \Lambda$, then $\mu\left(\rho_\Lambda\, \lambda_\Lambda\right) = \mu$.

(I5) $\left(\rho_{\Lambda\cup\Gamma}\, \lambda_{\Lambda\cup\Gamma}\right)\left(\rho_i\, \lambda_i\right) = \rho_{\Lambda\cup\Gamma}\, \lambda_{\Lambda\cup\Gamma},\ \forall\, i \in \Lambda \cup \Gamma$.

Let us first comment why these properties imply the theorem. It is clear that properties (I3)–(I5), together with the deterministic character of (λ_Λ) on $\mathcal{F}_{\Lambda^c}$ [property (A.2)], imply that $\left(\rho_\Lambda \lambda_\Lambda\right)_{\Lambda\in\mathcal{S}}$ verifies assertions (i)–(iv). Furthermore, if $\widetilde{\gamma}$ is a specification such that $\widetilde{\gamma}_{\{i\}} = \gamma_i$ for all $i \in \mathbb{Z}^d$ then, by consistency, $\widetilde{\gamma}_\Lambda(\,\cdot \mid \omega)\, \gamma_i = \widetilde{\gamma}_\Lambda(\,\cdot \mid \omega)$ for every $\Lambda \in \mathcal{S},\ i \in \Lambda$ and $\omega \in \Omega$. Therefore property (I5) implies that $\widetilde{\gamma}_\Lambda(\,\cdot \mid \omega)\ (\rho_\Lambda \lambda_\Lambda) = \widetilde{\gamma}_\Lambda(\,\cdot \mid \omega)$, that is $\rho_\Lambda \lambda_\Lambda(\,\cdot \mid \omega) = \widetilde{\gamma}_\Lambda(\,\cdot \mid \omega)$. So the construction is unique.

Initial inductive step The first non-trivial case is when $|\Lambda \cup \Gamma| = 2$. This implies that $|\Lambda| = |\Gamma| = 1$ and hence (I1)–(I3) coincide with hypotheses (A.6)–(A.9) while (I4) is trivially true. To prove (I5), assume that $\Lambda = \{i\}$ and $\Gamma = \{j\}$. By (A.3) and (A.9), we have

$$\left(\rho_{ij}\, \lambda_{ij}\right)\left(\left(\rho_i\, \lambda_i\right)(h)\right) \;=\; \lambda_j\left[\left(\frac{\rho_i\, \lambda_i}{\lambda_i\left(\rho_i\, \rho_j^{-1}\right)}\right)\left(\left(\rho_i\, \lambda_i\right)(h)\right)\right]\;.$$

As the factor $\left(\rho_i\, \lambda_i\right)(h)/\lambda_i\left(\rho_i\, \rho_j^{-1}\right)$ is independent of the configuration at $\{i\}$, the remaining integration with respect to the measure $\rho_i \lambda_i$ disappears due to the normalization condition (A.6). We obtain

$$(\rho_{ij}\,\lambda_{ij})\,((\rho_i\,\lambda_i)\,(h)) \;=\; \lambda_j\left[(\rho_i\,\lambda_i)\left(\frac{h}{\lambda_i\left(\rho_i\,\rho_j^{-1}\right)}\right)\right] \;=\; (\rho_{ij}\,\lambda_{ij})\,(h)$$

Inductive step We suppose the assertions true for $|\Lambda\cup\Gamma| = n, (n \geq 2)$, and consider Λ, Γ such that $\Gamma \subset \Lambda^c$ and $|\Lambda\cup\Gamma| = n+1$.

(I1) Assume first that $|\Gamma| = 1$ and let $k = \max\Lambda$. Combining the definition (A.13) and the property (A.3), we obtain

$$\lambda_\Lambda\left(\rho_\Lambda\,\rho_\Gamma^{-1}\right) \;=\; \lambda_{\Lambda_k^*}\left(\frac{\lambda_k\left(\rho_k\,\rho_\Gamma^{-1}\right)}{\lambda_k\left(\rho_k\,\rho_{\Lambda_k^*}^{-1}\right)}\right)\,. \tag{A.14}$$

If $|\Gamma| \geq 2$ we consider $l \triangleq \max\Gamma$ and apply the definition (A.13) to obtain

$$\lambda_\Lambda\left(\rho_\Lambda\,\rho_\Gamma^{-1}\right) \;=\; \lambda_\Lambda\left(\rho_\Lambda\,\rho_l^{-1}\,\lambda_l\left(\rho_l\,\rho_{\Gamma_l^*}^{-1}\right)\right)\,. \tag{A.15}$$

We can now apply the inductive hypothesis (I1) to the right-hand side of (A.14) and (A.15) to prove (I1) at the next inductive level.

(I2) The argument is symmetric in Λ and Γ, so we can assume without loss that $k = \max(\Lambda\cup\Gamma)$ belongs to Λ. We will use the inductive hypothesis to write the RHS of (I2) as a sequence of similar expressions where Λ is successively deprived of one of its sites which becomes "attached" to Γ. At the end we shall obtain (A.13) with $\Lambda\cup\Gamma$ instead of Λ. This will prove (I2).

If $|\Lambda| = 1$ (I2) is just the definition (A.13) applied to $\Lambda\cup\Gamma$. We assume, hence, that $|\Lambda| \geq 2$ and consider $j \in \Lambda$ such that $j \neq k$. By the inductive assumption (I2) we have

$$\rho_\Lambda \;=\; \frac{\rho_{\Lambda_j^*}}{\lambda_{\Lambda_j^*}\left(\rho_{\Lambda_j^*}\,\rho_j^{-1}\right)} \;=\; \frac{\rho_j}{\lambda_j\left(\rho_j\,\rho_{\Lambda_j^*}^{-1}\right)}\,. \tag{A.16}$$

We first combine the rightmost preceding expression with the factorization property (A.3) to write

$$\lambda_\Lambda\left(\rho_\Lambda\,\rho_\Gamma^{-1}\right) \;=\; \lambda_{\Lambda_j^*}\left(\frac{\lambda_j\left(\rho_j\,\rho_\Gamma^{-1}\right)}{\lambda_j\left(\rho_j\,\rho_{\Lambda_j^*}^{-1}\right)}\right)\,. \tag{A.17}$$

We now apply once more the inductive assumption (I2) in the form

$$\lambda_j\left(\rho_j\,\rho_\Gamma^{-1}\right) \;=\; \rho_{\Gamma\cup\{j\}}^{-1}\,\rho_j \tag{A.18}$$

in combination with the rightmost identity in (A.16), to obtain

$$\lambda_j \left(\rho_j \, \rho^{-1}_{\Lambda^*_j}\right) \;=\; \rho^{-1}_{\Lambda^*_j} \, \rho_j \, \lambda_{\Lambda^*_j} \left(\rho_{\Lambda^*_j} \, \rho_j^{-1}\right) \, . \tag{A.19}$$

¿From (A.17)–(A.19) we get

$$\lambda_\Lambda \left(\rho_\Lambda \, \rho_\Gamma^{-1}\right) \;=\; \frac{\lambda_{\Lambda^*_j} \left(\rho_{\Lambda^*_j} \, \rho^{-1}_{\Gamma\cup\{j\}}\right)}{\lambda_{\Lambda^*_j} \left(\rho_{\Lambda^*_j} \, \rho_j^{-1}\right)} \, .$$

We now use this relation together with the first identity in (A.16) to conclude that

$$\frac{\rho_\Lambda}{\lambda_\Lambda \left(\rho_\Lambda \, \rho_\Gamma^{-1}\right)} \;=\; \frac{\rho_{\Lambda^*_j}}{\lambda_{\Lambda^*_j} \left(\rho_{\Lambda^*_j} \, \rho^{-1}_{\Gamma\cup\{j\}}\right)} \, .$$

We iterate this formula $\left|\Lambda^*_j\right| - 1$ times and we arrive to

$$\frac{\rho_\Lambda}{\lambda_\Lambda \left(\rho_\Lambda \, \rho_\Gamma^{-1}\right)} \;=\; \frac{\rho_k}{\lambda_k \left(\rho_k \, \rho^{-1}_{(\Lambda\cup\Gamma)^*_k}\right)}$$

which is precisely $\rho_{\Lambda\cup\Gamma}$ according to our definition (A.13).

(I3) We assume that $|\Lambda| \geq 2$, otherwise (I3) is just the normalization hypothesis (A.6). Let $k = \max \Lambda$. Definition A.13 and property A.3 yield

$$\lambda_\Lambda \left(\rho_\Lambda\right) \;=\; \lambda_k \left(\frac{\lambda_{\Lambda^*_k} \left(\rho_{\Lambda^*_k}\right)}{\lambda_{\Lambda^*_k} \left(\rho_{\Lambda^*_k} \, \rho_k^{-1}\right)} \right) \;=\; \lambda_k \left(\frac{1}{\lambda_{\Lambda^*_k} \left(\rho_{\Lambda^*_k} \, \rho_k^{-1}\right)} \right)$$

where the last identity follows from the inductive hypothesis (I3). But, as in (A.18),

$$\lambda_{\Lambda^*_k} \left(\rho_{\Lambda^*_k} \, \rho_k^{-1}\right) \;=\; \lambda_k \left(\rho_k \, \rho^{-1}_{\Lambda^*_k}\right) \, \rho_{\Lambda^*_k} \, \rho_k^{-1} \, ,$$

therefore

$$\lambda_\Lambda \left(\rho_\Lambda\right) \;=\; \frac{\lambda_k \left(\rho_k \, \rho^{-1}_{\Lambda^*_k}\right)}{\lambda_k \left(\rho_k \, \rho^{-1}_{\Lambda^*_k}\right)} \;=\; 1 \, .$$

(I4) To avoid a triviality we assume that $|\Lambda| \geq 2$. Let μ be a probability measure on $(\Omega, \mathcal{F})$ such that $\mu\left(\rho_i \, \lambda_i\right) = \mu$ for all $i \in \Lambda$. Consider $k = \max \Lambda$ and a measurable function h. By the factorization property (A.3) of λ_Λ and the definition (A.13) of ρ_Λ, we have

$$\mu\Big(\left(\rho_\Lambda \lambda_\Lambda\right)(h)\Big) \;=\; \mu \left[\lambda_{\Lambda^*_k} \left(\left(\frac{\rho_k}{\lambda_k \left(\rho_k \, \rho^{-1}_{\Lambda^*_k}\right)} \, \lambda_k \right)(h) \right) \right] \, .$$

By the inductive hypothesis (I4) μ is consistent with $\rho_{\Lambda_k^*}\lambda_{\Lambda_k^*}$ and with $\rho_k\lambda_k$, thus

$$\mu\Big(\,(\rho_\Lambda\lambda_\Lambda)\,(h)\Big) \;=\; \mu\left[(\rho_k\,\lambda_k)\left(\rho_{\Lambda_k^*}^{-1}\left(\frac{\rho_k}{\lambda_k\left(\rho_k\,\rho_{\Lambda_k^*}^{-1}\right)}\,\lambda_k\right)(h)\right)\right]\;.$$

But, in the right-hand side, the two innermost integrals with respect to λ_k commute with the external one, so we have

$$\begin{aligned}\mu\Big((\rho_\Lambda\lambda_\Lambda)\,(h)\Big) \;&=\; \mu\left[(\rho_k\,\lambda_k)\left(\frac{h}{\lambda_k\left(\rho_k\,\rho_{\Lambda_k^*}^{-1}\right)}\quad\lambda_k\left(\rho_k\,\rho_{\Lambda_k^*}^{-1}\right)\right)\right]\\ &=\; \mu\Big((\rho_k\,\lambda_k)\,(h)\Big)\;,\end{aligned}$$

which proves (I4).

(I5) Denote $\Delta = \Lambda\cup\Gamma$ and pick $i, j \in \Delta$, $i \neq j$ and a measurable function h. By (A.3) and (I2) we have

$$\Big(\rho_\Delta\,\lambda_\Delta\Big)\Big((\rho_i\,\lambda_i)\,(h)\Big) \;=\; \lambda_j\left[\frac{\Big(\rho_{\Delta_j^*}\,\lambda_{\Delta_j^*}\Big)\Big((\rho_i\,\lambda_i)\,(h)\Big)}{\lambda_{\Delta_j^*}\left(\rho_{\Delta_j^*}\,\rho_j^{-1}\right)}\right]\;.$$

Therefore, applying inductive assumption (I5) we obtain

$$\begin{aligned}\Big(\rho_\Delta\,\lambda_\Delta\Big)\Big((\rho_i\,\lambda_i)\,(h)\Big) \;&=\; \lambda_j\left[\frac{\Big(\rho_{\Delta_j^*}\,\lambda_{\Delta_j^*}\Big)\,(h)}{\lambda_{\Delta_j^*}\left(\rho_{\Delta_j^*}\,\rho_j^{-1}\right)}\right]\\ &=\; (\rho_\Delta\,\lambda_\Delta)\,(h)\;. \quad \square \end{aligned} \tag{A.21}$$

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