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Coexistence in a two-dimensional Lotka-Volterra model

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Abstract

We study the stochastic spatial model for competing species introduced by Neuhauser and Pacala in two spatial dimensions. In particular we confirm a conjecture of theirs by showing that there is coexistence of types when the competition parameters between types are equal and less than, and close to, the within types parameter. In fact coexistence is established on a thorn-shaped region in parameter space including the above piece of the diagonal. The result is delicate since coexistence fails for the two-dimensional voter model which corresponds to the tip of the thorn. The proof uses a convergence theorem showing that a rescaled process converges to super-Brownian motion even when the parameters converge to those of the voter model at a very slow rate.

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1 Introduction and statement of results

Neuhauser and Pacala [12] introduced a stochastic spatial interacting particle model for two competing species. Each site in $\mathbb{Z}^d$ is occupied by one of two types of plants, designated 0 and 1. The state of the system at time t is represented by $\xi_t \in \{0,1\}^{\mathbb{Z}^d}$. A probability kernel p on $\mathbb{Z}^d$ will model both the dispersal of each type and competition between and within types. We assume throughout that p is symmetric and irreducible, has covariance matrix $\sigma^2 I$ (for some $\sigma > 0$) and satisfies $p(0) = 0$. Let

$$f_i(x,\xi) = \sum_y p(y-x)1\{\xi(y) = i\},\ i = 0,1$$

be the local density of type i near $x \in \mathbb{Z}^d$, and let $\alpha_0, \alpha_1 \geq 0$ be competition parameters. The Lotka-Volterra model with parameters (α_0, α_1), denoted $LV(\alpha_0, \alpha_1)$, is the spin flip system with rate functions

$$\begin{aligned} r_{0\to1}(x,\xi) &= f_1(f_0 + \alpha_0 f_1)(x,\xi) = f_1(x,\xi) + (\alpha_0 - 1)(f_1(x,\xi))^2 \\ r_{1\to0}(x,\xi) &= f_0(f_1 + \alpha_1 f_0)(x,\xi) = f_0(x,\xi) + (\alpha_1 - 1)(f_0(x,\xi))^2. \end{aligned}$$

It is easy to verify that these rates specify the generator of a unique $\{0,1\}^{\mathbb{Z}^d}$-valued Feller process whose law with initial state ξ_0 we denote $P^{\alpha}_{\xi_0}$ (or P_{ξ_0} if there is no ambiguity); see the discussion prior to Theorem 1.3 in [5]. We will be working with $d = 2$ in this work but for now will allow $d \geq 2$ to discuss our results in a wider context. For $d = 1$, see the original paper of Neuhauser and Pacala [12], and also Theorem 1.1 of [5].

The interpretation of the above rates is that $f_0 + \alpha_0 f_1(x,\xi)$ is the death rate of a 0 at site x in configuration ξ due to competition from nearby 1's ($\alpha_0 f_1$) and nearby 0's (f_0). Upon its death, the site is immediately recolonized by a randomly chosen neighbour. A symmetric explanation applies to the second rate function. Therefore $\alpha_0 p(y-x)$ represents the competitive intensity of a "neighbouring" 1 at y on a 0 at x and $\alpha_1 p(y-x)$ represents the competitive intensity of a "neighbouring" 0 at y on a 1 at x, while $p(y-x)$ is the competitive intensity of a particle at y on one of its own type at x.

Our interest lies in the related questions of survival of a rare type and coexistence of both types in the biologically important two-dimensional setting.

The case $\alpha_0 = \alpha_1 = 1$, where all intensities are the same, reduces to the well studied voter model (see Chapter V of Liggett [11]). In the second expression of the rates given above, one can therefore see that for α_0 and α_1 both close to one, we are dealing with a slight perturbation of the voter model rates. When both α_0 and α_1 are above 1, there is an alternative interpretation of the flip rates: as in the case of the voter model, a particle of type i is at rate 1 colonized by a randomly choosen neighbour, but in addition, at rate $(\alpha_i - 1)$, it is colonized by a randomly choosen neighbour *if* its type agrees with that of another randomly and independently choosen neighbour.

The voter model is dual to a system of coalescing random walks. This property makes it straightforward to prove that for $d \leq 2$, where a finite number of walks almost surely coalesce, the only invariant measures are convex combinations of the degenerate ones, which give positive probability only to configurations where all particles are of the same type (see Theorem 1.8 in Section V.1 of [11]). On the other hand when $d \geq 3$, transience of random walks makes it possible for non-degenerate invariant measures to exist.

When α_0 and α_1 are both greater than 1, the flip rate of a particle surrounded mostly by particles of the other type is increased. Therefore, intuitively, this setting should favour large clusters of particles of the same type, and at least for $d \leq 2$, invariant measures should remain degenerate.

On the other hand, as we will show when α_0 and α_1 are close to 1, both below 1 and when α_0 and α_1 are sufficiently close to one another, our model possesses in $d = 2$ quite a different asymptotic behaviour than that of the voter model (see Theorem 1.2 below). Indeed in this case, the death rate of a particle surrounded by particles of the other type is lowered, making it possible for sparse configurations to survive and non-degenerate invariant measures to appear.

Let $|\xi| = \sum_x \xi(x)$ and $|1-\xi| = \sum_x (1-\xi(x))$ denote the total number of 1's and 0's, respectively, in the population.

Definition 1.1. *We say that survival of ones (or just survival) holds iff*

$$P^{\alpha}_{\delta_0}(|\xi_t| > 0 \text{ for all } t \geq 0) > 0$$

and otherwise we say that extinction (of ones) holds. We say that coexistence holds iff there is a stationary law ν for $LV(\alpha_0, \alpha_1)$ such that

$$|\xi| = |1-\xi| = \infty \quad \nu - a.s. \tag{1}$$

Note that $\alpha_i < 1$ encourages coexistence as the interspecies competition is weaker than the intraspecies competition, while it is the other way around for $\alpha_i > 1$. The most favourable parameter values for coexistence should be $\alpha_0 = \alpha_1 < 1$ so that neither type is given any advantage over the other. In fact Conjecture 1 of [12] stated:

Conjecture. For $d \geq 2$ coexistence holds for $LV(\alpha_0, \alpha_1)$ whenever $\alpha_0 = \alpha_1 < 1$.

This was verified in [12] for $\alpha_0 = \alpha_1$ sufficiently small. For $d \geq 3$ coexistence was verified in Theorem 4 of [6] for $\alpha_i < 1$ and close to 1 as follows, where $m_0 \in (0,1)$ is a certain ratio of non-coalescing probabilities described in (4) below.

For any $0 < \eta < 1 - m_0$ there exists $r(\eta) > 0$ such that coexistence holds in the local cone-shaped region near $(1,1)$ given by

$$C^{d\geq 3}_{\eta} = \{(\alpha_0, \alpha_1) : 1 - r(\eta) < \alpha_0 < 1, (m_0+\eta)^{-1}(\alpha_0 - 1) \leq \alpha_1 - 1 \leq (m_0 + \eta)(\alpha_0 - 1)\}. \tag{2}$$

Coexistence in two dimensions is more delicate since it fails for the two-dimensional voter model, $\alpha_0 = \alpha_1 = 1$, and so in Theorem 1.2 below we are establishing coexistence for a class of perturbations of a model for which it fails. We also require an additional moment condition which was not required in higher dimensions or in [7], and which will be in force throughout this work:

$$\sum_{x\in\mathbb{Z}^2} |x|^3 p(x) < \infty. \tag{3}$$

Throughout $|x|$ will denote the Euclidean norm of a point $x \in \mathbb{R}^2$.

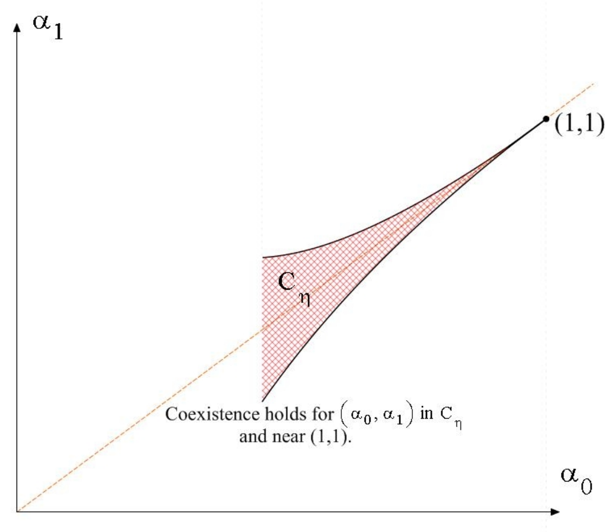
Our first result confirms the above Conjecture at least for $\alpha_0 = \alpha_1 < 1$ and sufficiently close to 1. In fact it gives coexistence on a thorn-shaped region including the piece of the diagonal just below $(1,1)$ (see the Figure below). The thorn actually widens out rather quickly. In the following Theorem γ and K are specific positive constants, which are determined by certain coalescing asymptotics–see (5) and (6) below.

Theorem 1.2. *Let $d = 2$. For $0 < \eta < K$, if*

$$C_\eta = \left\{ (\alpha_0, \alpha_1) \in (0,1)^2 : |\alpha_0 - \alpha_1| \le \frac{K-\eta}{\gamma} \frac{1-\alpha_0}{\left(\log \frac{1}{1-\alpha_0}\right)^2} \right\},$$

then there is an $r(\eta) > 0$ so that coexistence holds for $LV(\alpha_0, \alpha_0)$ whenever $(\alpha_0, \alpha_1) \in C_\eta$ and $1 - r(\eta) < \alpha_0$. Indeed in this case there is a translation invariant stationary law ν satisfying (1).

Note that it is plausible that coexistence for $\alpha_0 = \alpha_1 = \alpha$ implies coexistence for all $0 \le \alpha_0 = \alpha_1 = \alpha' \le \alpha$ since decreasing α should increase the affinity for the opposite type. If true, such a result would show that Theorem 1.2 implies the general Conjecture above.



We first recall the approach in [6] to coexistence for $d \ge 3$. Let $\{\hat\beta^x : x \in \mathbb{Z}^d\}$ be a system of continuous time rate 1 coalescing random walks with step kernel p and $\hat\beta^x_0 = x$, and let $\{e_i\}_{i \ge 1}$ be iid random variables with law p, independent of $\{\hat\beta^x\}$, all under a probability $\hat P$. In addition we set $e_0 = 0$. If $\tau(x,y) = \inf\{t \ge 0 : \hat\beta^x_t = \hat\beta^y_t\}$,

$$q_2 = \hat P(\tau(e_1, e_2) < \infty, \tau(0, e_1) = \tau(0, e_2) = \infty),$$

and

$$q_3 = \hat P(\tau(e_i, e_j) = \infty \text{ for all } i \ne j \in \{0,1,2\}),$$

then the constant m_0 appearing in the definition of $C_\eta^{d\ge 3}$ is

$$m_0 = \frac{q_2}{q_2 + q_3}. \tag{4}$$

Coexistence in $C_\eta^{d\ge3}$ for $d \ge 3$ was proved in [6] as a simple consequence of

Theorem A. (Theorem 1 of [6]) *Let $d \ge 3$. For $0 < \eta < m_0$, there is an $r(\eta) > 0$ so that survival holds if*

$$\alpha_1 - 1 < (m_0 + \eta)(\alpha_0 - 1) \text{ and } 1 - r(\eta) < \alpha_0 < 1.$$

Indeed by interchanging 0's and 1's it follows that survival of 1's and survival of 0's holds for $(\alpha_0, \alpha_1) \in C_\eta^{d\geq 3}$. The required stationary law ν is then easily constructed as a weak limit point of Cesaro averages of ξ_s over $[0, T]$ as $T \to \infty$ where ξ_0 has a Bernoulli $(1/2)$ distribution.

In two dimensions we have $q_2 = q_3 = 0$. If we write $f(t) \underset{t\to\infty}{\sim} g(t)$ when $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1$, the time dependent analogues of q_2, q_3 satisfy

$$q_2(t) \equiv \hat{P}(\tau(e_1, e_2) \leq t, \tau(0, e_1) \wedge \tau(0, e_2) > t) \underset{t\to\infty}{\sim} \frac{\gamma}{\log t}, \text{ where } \gamma \in (0, \infty), \tag{5}$$

and

$$q_3(t) \equiv \hat{P}(\min_{0\leq i\neq j\leq 2} \tau(e_i, e_j) > t) \underset{t\to\infty}{\sim} \frac{K}{(\log t)^3}, \text{ where } K \in (0, \infty). \tag{6}$$

Both positive constants γ and K only depend on the kernel p. (5) is proved in Proposition 2.1 of [7] along with an explicit formula for γ (denoted by γ^* in [7]), while (6) is a consequence of

Proposition 1.3. *Fix $n \in \mathbb{N}, n \geq 2$, and $x_i \in \mathbb{Z}^2, i \in \{1, ..., n\}$ be such that $x_i \neq x_j$ for all $1 \leq i < j \leq n$. There exists a constant $K_n(x_1, ..., x_n) > 0$ depending only on $p, x_1, ..., x_n$ such that*

$$q_{x_1,...,x_n}(t) \equiv \hat{P}(\min_{0\leq i<j\leq n} \tau(x_i, x_j) > t) \underset{t\to\infty}{\sim} \frac{K_n(x_1, ..., x_n)}{(\log t)^{\frac{n(n-1)}{2}}}. \tag{7}$$

Furthermore, there exists a constant $C_{1.3}$ depending only on p and n such that

$$K_n(x_1, ..., x_n) \leq C_{1.3} \left(\max_{\{1\leq i<j\leq n\}} \log\left(\left|x_i - x_j\right|\right)^{n(n-1)/2} + 1 \right). \tag{8}$$

Equation (7) above gives the asymptotics of the non-collision probability of n walks started at n distinct fixed points and is proved in Section 9. Equation (8) allows us to use dominated convergence and deduce that the first assertion continues to hold when the starting points are randomly chosen with law p, provided p is supported by at least $n - 1$ points. In particular by integrating (7) with respect to the law of $\{e_1, \ldots, e_n\}$ on the event that they are all distinct, we see there is a constant $K_n > 0$ depending only on p, n such that

$$(\log t)^{\frac{n(n-1)}{2}} \hat{P}(\min_{0\leq i<j\leq n-1} \tau(e_i, e_j) > t) \underset{t\to\infty}{\to} K_n, \tag{9}$$

the case $n = 3$ corresponding evidently to (6).

In this paper, we only make use of (6) which is also used in [8] in their PDE limit theorem for general voter model perturbations in two dimensions. However, the general result has the potential of handling other rescalings leading to different PDE limits.

These asymptotics show that $q_3(t) \ll q_2(t)$ as $t \to \infty$ and so suggest that m_0 should be 1 in two dimensions. These heuristics are further substantiated by Theorem 1.3 of [7], where the following is proved:

Theorem B. *If $d = 2$, the conclusion of Theorem A holds with $m_0 = 1$.*

Note that the above result will not give a coexistence result as it did for $d \geq 3$ because the intersection of the region in Theorem B with its mirror image across the line $\alpha_0 = \alpha_1$ will be the empty set. Of course one could hope that the result in Theorem B could be improved but in fact the result in Theorem A was shown to be sharp in [4] and we conjecture that the same ideas will verify the same is true of Theorem B, although the arguments here will be more involved. More specifically, introduce the critical curve

$$h(\alpha_0) = \sup\{\alpha_1 : \text{ survival holds for } LV(\alpha_0, \alpha_1)\}.$$

As is discussed in Section 1 of [6], $LV(\alpha_0, \alpha_1)$ is monotone for $\alpha_i \geq 1/2$, h is non-decreasing on $\alpha_0 \geq 1/2, h(\alpha_0) \geq 1/2$, and the region below or to the right of the graph of h are parameter values for which survival holds and the region above or to the left of the graph of h are parameter values for which extinction holds. Neuhauser and Pacala proved that $h(1) = 1$ and the sharpness of Theorem A mentioned above was proved in Corollary 1.6 of [4] as

$$\frac{d^-}{d\alpha_0 -} h(1) = m_0.$$

Given the conjectured sharpness of Theorem B, our only hope for extending the above approach to coexistence would be to find higher order asymptotics for h near 1 which would show survival holds on the diagonal $\alpha_0 = \alpha_1$ near $(1,1)$. This is what is shown in the next result which clearly refines Theorem B.

Theorem 1.4. *Let $d = 2$, and γ and K be as in* (5) *and* (6)*, respectively. For $0 < \eta < K$, if*

$$\mathscr{S}_\eta = \left\{ (\alpha_0, \alpha_1) \in (0,1)^2 : \alpha_1 \leq \alpha_0 + \frac{K - \eta}{\gamma} \frac{1 - \alpha_0}{\left(\log \frac{1}{1-\alpha_0}\right)^2} \right\},$$

then there exists $r(\eta) > 0$ so that survival of ones holds whenever $(\alpha_0, \alpha_1) \in \mathscr{S}_\eta$ and $1 - r(\eta) < \alpha_0$.

The reasoning described above for $d \geq 3$ will now allow us to use Theorem 1.4 to derive Theorem 1.2 (see Section 8.3 below).

Theorem B was proved in [7] using a limit theorem for a sequence of rescaled Lotka-Volterra models. The limit was a super-Brownian motion with drift and we use a similar strategy here to prove Theorem 1.4. $M_F(\mathbb{R}^d)$ is the space of finite measures on $\mathbb{R}^d$ with the topology of weak convergence. A d-dimensional super-Brownian motion with branching rate $b > 0$, diffusion coefficient $\sigma^2 > 0$, and drift $\theta \in \mathbb{R}$ (denoted $SBM(b, \sigma^2, \theta)$) is an $M_F(\mathbb{R}^d)$-valued diffusion X whose law is the unique solution of the martingale problem:

$$\text{(MP)} \quad \begin{cases} \forall\ \phi \in C_b^3(\mathbb{R}^d), \quad M_t(\phi) = X_t(\phi) - X_0(\phi) - \int_0^t X_s \left(\frac{\sigma^2}{2} \Delta\phi + \theta\phi \right) ds \\ \quad \text{is a continuous } \mathscr{F}_t^X\text{-martingale such that} \\ \quad \langle M(\phi) \rangle_t = \int_0^t X_s \left(b\phi^2 \right) ds. \end{cases}$$

(See Theorem II.5.1 and Remark II.5.13 of [13].) Here C_b^k is the space of bounded C^k functions with bounded derivatives of order k or less, $\mathscr{F}_t^X$ is the right-continuous completed filtration generated by X and $X_t(\phi)$ denotes integration of ϕ with respect to the random finite measure X_t.

Fix $\beta_0, \beta_1 \in \mathbb{R}$ and assume $\beta_i^N, N \in [3, \infty)$ satisfies

$$\lim_{N\to\infty} \beta_i^N = \beta_i,\ i = 0, 1, \quad \bar{\beta} = \sup_{N\geq 3} |\beta_0^N| \vee |\beta_1^N| < \infty. \tag{10}$$

(It will be convenient at times to have $\log N \geq 1$, whence the lower bound on N.) Define

$$\alpha_i^N = 1 - \frac{(\log N)^3}{N} + \beta_i^N \frac{\log N}{N},\ i = 0, 1,\ N \geq 3. \tag{11}$$

As we will only be concerned with N large we may assume

$$\alpha_i^N \in [1/2, 1]. \tag{12}$$

Let $\xi^{(N)}$ denote a $LV(\alpha_0^N, \alpha_1^N)$ process and consider the rescaled process

$$\xi_t^N(x) = \xi_{Nt}^{(N)}(x\sqrt{N}),\ x \in S_N := \mathbb{Z}^2/\sqrt{N}.$$

Finally define the associated measure-valued process

$$X_t^N = \frac{\log N}{N} \sum_{x\in S_N} \xi_t^N(x)\delta_x.$$

Theorem 1.5. *Assume $X_0^N \to X_0$ in $M_F(\mathbb{R}^2)$. If* (10) *holds, and γ and K are as in* (5) *and* (6)*, respectively, then $\{X^N\}$ converges weakly in the Skorokhod space $D(\mathbb{R}_+, M_F(\mathbb{R}^2))$ to two-dimensional $SBM(4\pi\sigma^2, \sigma^2, \theta)$, where $\theta = K + \gamma(\beta_0 - \beta_1)$.*

Note that the normalization by $\frac{\log N}{(\sqrt{N})^2}$ shows that the 1's are sparsely distributed over space and in fact occur with density $1/\log N$. Like the other constants in the above theorem, the branching rate of $4\pi\sigma^2$ also arises from the asymptotics of a coalescing probability, namely

$$\hat{P}(\tau(0, e_1) > t) \underset{t\to\infty}{\sim} \frac{2\pi\sigma^2}{\log t}. \tag{13}$$

See, for example, Lemma A.3(ii) of [3] for the above, and the discussion after Theorem 1.2 of the same reference for some intuition connecting the above with the limiting branching rate.

A bit of arithmetic shows that $(\alpha_0^N, \alpha_1^N) \in S_\eta$ for some $\eta > 0$ and N large iff $K + \gamma(\beta_0 - \beta_1) > 0$, that is iff the drift of the limiting super-Brownian motion is positive. The latter condition is necessary and sufficient for longterm survival of the super-Brownian motion and so we see that the derivation of Theorem 1.4 from Theorem 1.5 requires an interchange of the limits in N and t. This will be done using the standard comparison with super-critical $2K_0$-dependent oriented percolation. The key technical argument here is a bound on the mass killed to ensure the $2K_0$-dependence, and is proved in Lemma 8.1 in Section 8.

In [7] a limit theorem similar to Theorem 1.5 is proved but with $\alpha_i^N = 1 + \frac{\beta_i^N \log N}{N}$ and limiting drift $\theta = \gamma(\beta_0 - \beta_1)$ (unlike this work, in [7] $\alpha_i^N > 1$ was also considered). In order to analyze the higher order asymptotics of the survival region we need to consider (α_0^N, α_1^N) which are asymptotically farther from the voter model parameters $(1, 1)$. This results in some additional powers of $\log N$ multiplying some drift terms in the martingale decompositions of $X_t^N(\phi)$ and we must take advantage of some cancelative effects especially when dealing with the drift term $D^{N,3}$ in (29) below–see, for example, (39) below in Section 2.3. The delicate nature of the arguments will lead us to a number of refinements of the arguments in [7], and also will require a dual process which we describe in Section 2.

We conjecture that the results of Theorems 1.4 and 1.2 are both sharp.

Conjecture 1.6. *If* $0 < \eta$ *and*

$$\mathscr{E}_\eta = \left\{(\alpha_0, \alpha_1) \in (0,1)^2 : \alpha_1 \geq \alpha_0 + \frac{K+\eta}{\gamma} \frac{1-\alpha_0}{\left(\log \frac{1}{1-\alpha_0}\right)^2}\right\},$$

then there exists $r(\eta) > 0$ *so that extinction of ones holds whenever* $(\alpha_0, \alpha_1) \in \mathscr{E}_\eta$ *and* $1 - r(\eta) < \alpha_0$. *In fact infinitely many initial zero's will drive one's out of any compact set for large enough time a.s.*

The above strengthens Conjecture 2 of Neuhauser and Pacala [12]. Clearly this would also show that coexistence must fail and so, together with the symmetric result obtained by interchanging 0's and 1's, implies the sharpness of the thorn in Theorem 1.2. The analogous results for $d \geq 3$ (showing in particular that the coexistence region in (2) and survival region in Theorem A are "best possible") were proved in [4] using a PDE limit theorem. A corresponding limit theorem is derived in two dimensions in [8]. There the limit theorem is used to carry out an interesting analysis of the evolution of seed dispersal range.

Section 2 gives a construction of our particle system as a solution of a system of jump SDE's which leads to a description of the dual process mentioned above and the martingale problem solved by X^N. We conclude this Section with a brief outline of the proof of the convergence theorem including some hints as to why constants like K arise in the limit. Section 3 gives a preliminary analysis of the new drift term arising from the $(\log N)^3$ term and also sets out the first and second moment bounds required for tightness of $\{X^N\}$. The first moment bounds are established in Section 4, where the reader can see how many of the key ideas including the dual are used in a simpler setting. The more complex second moment bounds are given in Section 5 and tightness of $\{X^N\}$ is proved in Section 6. The limits of the drift terms are found in Section 7 and the identification of the limiting super-Brownian motion follows easily, thus completing the proof of Theorem 1.5. Theorems 1.2 and 1.4 are proved in Section 8. Finally Section 9 contains the proof of Proposition 1.3.

We will suppress dependency of constants on the kernel p (as for K and γ) and $\bar{\beta}$, but will mention or explicitly denote any other dependencies. Constants introduced in formula (k) will be denoted C_k while constants first appearing in Lemma i.j will be denoted $C_{i.j}$. Constants c, C are generic and may change from line to line.

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2 Preliminaries

2.1 Non-coalescing bounds and kernel estimates

Let $p_N(x)$ denote the rescaled kernel $p(\sqrt{N}x)$ for $x \in S_N$, and define

$$\begin{aligned} v_N &:= N\alpha_0^N = N - (\log N)^3 + \beta_0^N(\log N) \geq N/2, \\ \theta_N &:= N(1-\alpha_0^N) = (\log N)^3 - \beta_0^N(\log N) \geq 0, \end{aligned}$$

the inequalities by (12), and

$$\lambda_N := (\beta_1^N - \beta_0^N)(\log N).$$

Let $\{\hat{B}^{x,N}, x \in S_N\}$ denote a system of rate v_N coalescing random walks on S_N, with jump kernel p_N. For convenience we will drop the dependence in N from the notation $\hat{B}^{x,N}$. We slightly abuse our earlier notation and also let e_1, e_2 denote iid random variables with law p_N, independent of the collection $\{\hat{B}^x\}$, all under the probability $\hat{P}$. Throughout the paper we will work with

$$t_N := (\log N)^{-19}. \tag{14}$$

By (13)

$$C_{15} := \sup_{N\geq 3}(\log N)\hat{P}(\hat{B}_1^0 \neq \hat{B}_1^{e_1}) = \sup_{N\geq 3}(\log N)\hat{P}(\tau(0,e_1) > N) < \infty, \tag{15}$$

while (6) implies that

$$\bar{K} := \sup_{N\geq 3}(\log N)^3\hat{P}\big(\, \hat{B}_{t_N}^{e_i,N} \neq \hat{B}_{t_N}^{e_j,N} \;\; \forall i \neq j \in \{0,1,2\}\,\big) < \infty. \tag{16}$$

We will also need some kernel estimates. Set $p_t^N(x) = N\hat{P}(\hat{B}_t^0 = x)$. First, it is well-known (see for example (A7) of [3]) that there exists a constant $C_{17} > 0$ such that,

$$||p_t^N||_\infty \leq \frac{C_{17}}{t} \text{ for all } t > 0, N \geq 1. \tag{17}$$

The following bound on the spatial increments of p_t^N will be proved in Section 9.

Lemma 2.1. *There is a constant $C_{2.1}$ such that for any $x, y \in S_N$, $t > 0$, and $N \geq 1$,*

$$\left|p_t^N(x) - p_t^N(x+y)\right| \leq C_{2.1}|y|t^{-3/2}.$$

2.2 Construction of the process, the killed process, and their duals

To help understand our dual process below we construct the rescaled Lotka-Volterra process ξ^N as the solution to a system of stochastic differential equations driven by a collection of Poisson processes. If

$$f_i^N(x,\xi^N) = \sum_y p_N(y-x)1\{\xi^N(y) = i\},\ i = 0,1,$$

the rescaled flip rates of ξ^N can be written

$$\begin{aligned}r^N_{0\to 1}(x,\xi^N) &= Nf^N_1(x,\xi^N)+N(\alpha^N_0-1)f^N_1(x,\xi^N)^2\\ &= \nu_N f^N_1(x,\xi^N)+\theta_N f^N_1(x,\xi^N)f^N_0(x,\xi^N),\\ r^N_{1\to 0}(x,\xi^N) &= Nf^N_0(x,\xi^N)+N(\alpha^N_1-1)f^N_0(x,\xi^N)^2\\ &= \nu_N f^N_0(x,\xi^N)+\theta_N f^N_1(x,\xi^N)f^N_0(x,\xi^N)+\lambda_N f^N_0(x,\xi_N)^2. \end{aligned} \tag{18}$$

To ensure $\lambda_N \geq 0$ we assume

$$\beta^N_1 \geq \beta^N_0 \tag{19}$$

It is easy to modify the construction in the case $\beta^N_1 \leq \beta^N_0$.

We start at some initial configuration ξ^N_0 such that $|\xi^N_0| < \infty$. Let $\Lambda = \{\Lambda(x,y) : x,y \in S_N\}$, $\Lambda_r = \{\Lambda_r(x) : x \in S_N\}$, and $\Lambda_g = \{\Lambda_g(x) : x \in S_N\}$ be independent collections of independent Poisson point processes on $\mathbb{R}$, $\mathbb{R}\times S^2_N$ and $\mathbb{R}\times S^2_N$, respectively, with rates $\nu_N p_N(y-x)ds$, $\frac{\theta_N}{2}p_N(\cdot)p_N(\cdot)ds$ and $\lambda_N p_N(\cdot)p_N(\cdot)ds$, respectively. (It will be convenient at times to allow $s<0$.) Let

$$\mathscr{F}_t = \cap_{\varepsilon>0}\sigma(\Lambda(x,y)(A),\Lambda_r(x)(B_1),\Lambda_g(x)(B_2) : A\subset[0,t+\varepsilon], B_i\subset[0,t+\varepsilon]\times S^2_N, x,y\in S_N).$$

We consider the system of stochastic differential equations

$$\begin{aligned}\xi^N_t(x) = \xi^N_0(x) &+ \sum_y\int_0^t(\xi^N_{s-}(y)-\xi^N_{s-}(x))\Lambda(x,y)(ds) \\ &+\int_0^t\int\int 1(\xi^N_{s-}(x+e_1)\neq\xi^N_{s-}(x+e_2))(1-2\xi^N_{s-}(x))\Lambda_r(x)(ds,de_1,de_2)\\ &-\int_0^t\int\int 1(\xi^N_{s-}(x+e_1)=\xi^N_{s-}(x+e_2)=0)\xi^N_{s-}(x)\Lambda_g(x)(ds,de_1,de_2).\end{aligned} \tag{20}$$

Here the first integral represents the rate ν_N voter dynamics in (18), the second integral incorporates the rate $\theta_N f^N_0 f^N_1$ flips from the current state in (18), and the final integral represents the additional 1 to 0 flips with rate $\lambda_N(f^N_0)^2$. Note that the rate of Λ_r is proportional to $\theta_N/2$ to account for the fact that the randomly chosen neighbours can be distinct in two different ways. Although the above equation is not identical to (SDE)(I') in Proposition 2.1 of [6], the reasoning in parts (a) and (c) of that result applies. For example, the rates in (18) satisfy the hypotheses (2.1) and (2.3) of section 2 of [6]. As a result (20) has a pathwise unique $\mathscr{F}_t$-adapted solution whose law is that of the $\{0,1\}^{S_N}$-valued Feller process defined by the rates in (18), that is, our rescaled Lotka-Volterra process.

We first briefly recall the dual to the rate-ν_N rescaled voter model ζ^N on S_N defined by solving (20) without the Λ_r and Λ_g terms. We refer the reader to Section III.6 of [11] for more details. For every $s\in\Lambda(x,y)$ we draw a horizontal arrow from x to y at time s. A particle at (x,t) goes down at speed one on the vertical lines and jumps along every arrow it encounters. We denote by $\hat{B}^{(x,t)}$ the path of the particle started at (x,t). It is clear that for every $t>0$, $\{\hat{B}^{(x,t)}, x\in S_N\}$ is a system of rate-ν_N coalescing random walks on S_N with jump kernel p_N on a time interval of length t. $\zeta^N(x)$ adopts the type of y at each time $s\in\Lambda(x,y)$, and so if $\hat{B}^x_t := \hat{B}^{(x,t)}(t)$, then

$$\zeta^N_t(x) = \zeta^N_0(\hat{B}^x_t) \text{ for any given } t>0. \tag{21}$$

To construct a branching coalescing dual process to $(\xi_t^N, t \geq 0)$ we add arrows to the above dual corresponding to the points in Λ_r and Λ_g. For every $(s, e_1, e_2) \in \Lambda_r(x)$, respectively in $\Lambda_g(x)$, draw two horizontal red, respectively green, arrows from (x,s) towards $(x+e_1, s)$ and $(x+e_2, s)$. The dual process $\tilde{B}^{(x,t)}(s), s \leq t$ starting from $x = (x_1, \dots, x_m) \in S_N^m$ at time t is the branching coalescing system obtained by starting with locations x at time t and following the arrows backwards in time from t down to 0. Formally the dual takes values in

$$\mathscr{D} = \{(\tilde{B}^1, \tilde{B}^2, \dots) \in D([0,t], S_N \cup \{\infty\})^{\mathbb{N}} : \exists K : [0,t] \to \mathbb{N} \text{ non-decreasing } \\ s.t.\ \tilde{B}^k(t) = \infty\ \forall k > K(t)\}.$$

Here ∞ is added as a discrete point to S_N and $D([0,t], S_N \cup \{\infty\})$ is the Skorokhod space of cadlag paths. The dynamics of the dual are as follows.

1. $\tilde{B}_s^{(x,t)} = (\hat{B}_s^{(x_1,t)}, \dots, \hat{B}_s^{(x_m,t)}, \infty, \infty, \dots)$ and $K(s) = m$ for $s \leq t \wedge R_1$, where R_1 is the first time one of these coalescing random walks "meets" a coloured arrow. By this we mean that the walk is at y and there are coloured arrows from y to $y + e_i$, $i = 1, 2$. If $R_1 > t$ we are done.

2. Assume $R_1 \leq t$ and the above coloured arrows have colour $c(R_1)$ and go from $\mu(R_1)$ to $\mu(R_1) + e_i$, $i = 1, 2$. Then $K(R_1) = K(R_1-) + 2$ and $B_{R_1}^{(x,t)}$ is defined by adding the two locations $\mu(R_1) + e_i$, $i = 1, 2$ to the two new slots. If a particle already exists at such a location, the two particles at the location coalesce, that is, will henceforth move in unison.

3. For $s > R_1$ $\tilde{B}_s^{(x,t)}$ follows the coalescing system of random walks starting at $\tilde{B}_{R_1}^{(x,t)}$ until $t \wedge R_2$ where R_2 is the next time that one of these coalescing walks meets a coloured arrow. If $R_2 \leq t$ we repeat the previous step.

Each particle encounters a coloured arrow at rate $\theta_N + \lambda_N$ so $\lim_m R_m = \infty$ and the above definition terminates after finitely many branching events. We slightly abuse our notation and also write $\tilde{B}_s^{(x,t)}$ for the set of locations $\{\tilde{B}_s^{(x,t),i} : i \leq K(s)\}$. Some may prefer to refer to $\tilde{B}$ as a graphical representation of ξ^N but we prefer to use this terminology for the entire Poisson system of arrows implicit in $(\Lambda, \Lambda_r, \Lambda_g)$.

We will again write $\hat{P}, \hat{E}$ for the probability measure and the expectation with respect to the dual quantities $\hat{B}, \tilde{B}$. In particular recall that with respect to $\hat{P}$, e_1, e_2 will denote random variables chosen independently according to p_N. Also when the context is clear, we will often drop the dependence on t from the notation $\hat{B}^{(x,t)}, \tilde{B}^{(x,t)}$ and use the shorter $\hat{B}^x, \tilde{B}^x$.

It should now be clear from (20) how to construct $(\xi_t^N(x_1), \dots, \xi_t^N(x_m))$ from $\{\xi_0^N(y) : y \in \tilde{B}_t^x\}$, the dual $(\tilde{B}_s^x, s \leq t)$ and the sequence of "parent" locations and colours $\{(\mu(R_m), c(R_m)) : R_m \leq t\}$. First, $\xi_r^N(\tilde{B}_{t-r}^{x,j})$ remains constant until the first time a branching time of the dual (corresponding to a jump in K) is encountered at $r = t - R_n$. In particular,

$$\xi_t^N(x_i) = \xi_0^N(\hat{B}_t^{x_i}) = \xi_0^N(\tilde{B}_t^{x,i}),\ i = 1, \dots, m \tag{22}$$
$$\text{if no coloured arrows are encountered by any } \hat{B}^{x_i} \text{ on } [0,t].$$

If the above branch time $t - R_n$ is encountered we know the colour of the arrows, $c(R_n)$, and the locations of its endpoints, corresponding to $\mu(R_n)$ and the two last (non-∞) coordinates of the dual at time $t-R_n$. We also know the type at these locations since they are all dual locations at time R_n. Hence we will know to flip the type at $\mu(R_n)$ if the colour is red and the two endpoints have different types, or set it to 0 if the colour is green and the two endpoints are both 0's. Otherwise we keep the type at $\mu(R_n)$ and all other sites in $\tilde{B}^x_{R_n-}$ unchanged. Continuing on for $r>t-R_n$, the types $\xi^N_r(\tilde{B}^{x,j}_{t-r})$ remain fixed until the next branch time is encountered at $r=t-R_{n-1}$ and continue as above until we arrive at $\xi^N_t(x_i)=\xi^N_t(\tilde{B}^{(x,t),i}_0)$, $i=1,\dots,m$.

If $x=(x_1,\dots,x_m)\in S^m_N$, let $\hat{B}^{(x,t)}_s=\{\hat{B}^{(x_i,t)}_s : i=1,\dots,m\}$. Then it is clear from the above construction that

$$\hat{B}^{(x,t)}_s\subset\tilde{B}^{(x,t)}_s \text{ for all } s\le t. \tag{23}$$

It also follows from the above reconstruction of ξ^N_t that

$$\xi^N_0(y)=0 \text{ for all } y\in\tilde{B}^{(x,t)}_t \text{ implies } \xi^N_t(x_i)=0,\ i=1,\dots,m. \tag{24}$$

The above two results imply that for any $x\in S_N$ and $t\ge0$,

$$\xi^N_t(x)\le\sum_{y\in\tilde{B}^{(x,t)}_t}\xi^N_0(y) \text{ and } \xi^N_0(\hat{B}^{(x,t)}_t)\le\sum_{y\in\tilde{B}^{(x,t)}_t}\xi^N_0(y).$$

It follows from the above that if $z_1=1$, and $z_2,z_3\in\{0,1\}$, one has

$$\prod_{i=1}^3\mathbf{1}_{\{\xi^N_t(x_i)=z_i\}}\le\sum_{y\in\tilde{B}^{x_1}_t}\xi^N_0(y),\qquad\prod_{i=1}^3\mathbf{1}_{\{\xi^N_0(\hat{B}^{x_i}_t)=z_i\}}\le\sum_{y\in\tilde{B}^{x_1}_t}\xi^N_0(y),$$

and therefore, using (22) as well, we have

$$\begin{aligned}&\left|E\left[\prod_{i=1}^3\mathbf{1}_{\{\xi^N_t(x_i)=z_i\}}\right]-E\left[\prod_{i=1}^3\mathbf{1}_{\{\xi^N_0(\hat{B}^{x_i}_t)=z_i\}}\right]\right|\\&\le E\left[\mathbf{1}_{\mathscr{E}^{x_1,x_2,x_3}_t}\sum_{y\in\tilde{B}^{x_1}_t}\xi^N_0(y)\right],\end{aligned}\tag{25}$$

where

$$\mathscr{E}^{(x_i)_{i\in I}}_t:=\left\{\exists i\in I \text{ s.t. there is at least one coloured arrow encountered by } \hat{B}^{x_i} \text{ on } [0,t]\right\}.$$

If we ignore the coalescings in the dual and use independent copies of the random walks for particles landing on occupied sites, we may construct a branching random walk system with rate

$$r_N:=\theta_N+(\beta^N_1-\beta^N_0)(\log N),$$

denoted $\{\overline{B}^x_t, x\in S_N\}$, which stochastically dominates our branching coalescing random walk system, in the sense that for all $t\ge0$,

$$\forall x=(x_1,\dots,x_m)\in S^m_N\quad\hat{B}^x_t\subseteq\tilde{B}^x_t\subseteq\overline{B}^x_t:=\cup_{i=1}^m\overline{B}^{x_i}_t. \tag{26}$$

This observation leads to the following. Recall that $\hat{E}$ takes the expectation over the dual variables only.

Lemma 2.2. *For all $s > 0$, for any $0 \le u \le s$,*

$$\sum_{x\in S_N} \hat{E}\left[\mathbf{1}_{\{\mathscr{E}_u^{x,x+e_1,x+e_2}\}} \sum_{y\in\tilde{B}_u^{x,x+e_1,x+e_2}} \xi_{s-u}^N(y)\right] \le 9r_N u \exp(3r_N u) \sum_{y\in S_N} \xi_{s-u}^N(y).$$

Proof : We use (26) to see that for all $t \ge 0, \tilde{B}_t^{x,x+e_1,x+e_2} \subseteq \overline{B}_t^{x,x+e_1,x+e_2}$.
Let $N_t^{x,x+e_1,x+e_2} := \#\{\overline{B}_t^{x,x+e_1,x+e_2}(t)\}$, and define ρ as the first branching time of the three branching random walks started at $0, e_1, e_2$. Then, for a fixed $y \in S_N$, using the Markov property and translation invariance of p,

$$\begin{aligned}
\sum_{x\in S_N} \hat{E}\left(\mathbf{1}_{\{\mathscr{E}_u^{x,x+e_1,x+e_2}\}}\mathbf{1}_{\{y\in\tilde{B}_u^{x,x+e_1,x+e_2}\}}\right) &\le \sum_{x\in S_N} \hat{E}(\mathbf{1}_{\{N_u^{x,x+e_1,x+e_2}>3\}}\mathbf{1}_{\{y\in\overline{B}_u^{x,x+e_1,x+e_2}\}}) \\
&= \sum_{x\in S_N} \hat{E}(\mathbf{1}_{\{N_u^{0,e_1,e_2}>3\}}\mathbf{1}_{\{y-x\in\overline{B}_u^{0,e_1,e_2}\}}) \\
&= \hat{E}(\mathbf{1}_{\{N_u^{0,e_1,e_2}>3\}}N_u^{0,e_1,e_2}) \\
&\le \hat{E}(\mathbf{1}_{\{\rho<u\}}\hat{E}(N_{u-\rho}^{0,e_1,e_2}|\rho)) \\
&\le \hat{P}(\rho<u)\hat{E}(N_u^{0,e_1,e_2}) \\
&\le 3(1-\exp(-3r_N u))\exp(3r_N u) \\
&\le 9r_N u\exp(3r_N u),
\end{aligned}$$

and, after multiplying by $\xi_{s-u}^N(y)$ and summing over y we obtain the desired result. □

In Section 8 we will need to extend the above constructions to the setting where particles are killed, i.e., set to a cemetary state Δ, outside of an open rectangle I' in S_N. We consider initial conditions $\underline{\xi}_0^N$ such that $\underline{\xi}_0^N(x) = 0$ for $x \notin I'$ and the rates in (18) are modified so that $\underline{r}_{0\to1}^N(x, \underline{\xi}^N) = 0$ if $x \notin I'$. $\underline{\xi}^N$ may be again constructed as a pathwise unique solution of (20) for $x \in I'$ and $\underline{\xi}_t^N(x) = 0$ for $x \notin I'$. The argument is again as in Proposition 2.1 of [6].

The killed rescaled voter model $\underline{\zeta}^N$, corresponding to the rescaled voter rates on I set to be 0 outside I', may also be constructed as the pathwise unique solution of the above killed equation but now ignoring the Λ_r and Λ_g terms. To introduce its dual process, for each $(x,t) \in S_N \times \mathbb{R}_+$ and $s \le t$ let

$$\tau^{(x,t)} = \inf\{s \ge 0 : \hat{B}^{(x,t)}(s) \notin I'\},$$

and define the killed system of coalescing walks by

$$\underline{\hat{B}}^{(x,t)}(s) = \begin{cases} \hat{B}^{(x,t)}(s) \text{ if } s < \tau^{(x,t)} \\ \Delta \text{ otherwise .} \end{cases}$$

If we set $\underline{\zeta}_0^N(\Delta) = 0$, then the duality relation for $\underline{\zeta}$ is

$$\underline{\zeta}_t^N(x) = \underline{\zeta}_0^N(\underline{B}_t^{(x,t)}),\ t \ge 0,\ x \in S_N.$$

We also may define the dual $\{\underline{\tilde{B}}_s^{(t,x)} : s \le t\}$ of $\underline{\xi}^N$ as before but now using the killed coalescing random walks $\{\underline{\hat{B}}^{(x,t)}\}$ and allowing the dual coordinates to take on the value Δ. In step 2 of the above construction of the dual we reset locations $\mu(R_1)+e_i$ to Δ if they are outside I'. If $\underline{\xi}_s^N(\Delta)=0$, the reconstruction of $\underline{\xi}_t^N(x_i)$ from the dual variables is again valid as is the reasoning which led to (25). More specifically if $z_1=1$, $z_2,z_3\in\{0,1\}$, $x_i\in I'$, for $i=1,2,3$ and we define

$$\underline{\mathscr{E}}_t^{(x_i)_{i\le 3}} := \left\{\exists i\le 3 \text{ there is at least one coloured arrow encountered by } \underline{\hat{B}}^{x_i} \text{ on } [0,t]\right\},$$

then

$$\begin{aligned} &\left| E\left[\prod_{i=1}^3 \mathbf{1}_{\{\underline{\xi}_t^N(x_i)=z_i\}}\right] - E\left[\prod_{i=1}^3 \mathbf{1}_{\{\underline{\xi}_0^N(\underline{\hat{B}}_t^{x_i})=z_i\}}\right]\right| \\ &\le E\left[\mathbf{1}_{\underline{\mathscr{E}}_t^{x_1,x_2,x_3}} \sum_{y\in\underline{\tilde{B}}_t^{x_1}} \underline{\xi}_0^N(y)\right]. \end{aligned} \tag{27}$$

2.3 Martingale problem

By Proposition 3.1 of [7], we have, for any $\Phi\in C_b([0,T]\times S_N)$ such that $\dot\Phi := \frac{\partial\Phi}{\partial t}\in C_b([0,T]\times S_N)$,

$$X_t^N(\Phi(t,.))=X_0(\Phi(0,.))+M_t^N(\Phi)+D_t^{N,1}(\Phi)+D_t^{N,2}(\Phi)+D_t^{N,3}(\Phi), \tag{28}$$

where (the reader should be warned that the terms $D^{N,2}$ and $D^{N,3}$ below do not agree with the corresponding terms in [7] although their sums do agree)

$$\begin{aligned} D_t^{N,1}(\Phi) &= \int_0^t X_s^N(A_N\Phi(s,.)+\dot\Phi(s,.))ds, \\ A_N\Phi(s,x) &:= \sum_{y\in S_N} Np_N(y-x)\left(\Phi(s,y)-\Phi(s,x)\right), \end{aligned}$$

$$D_t^{N,2}(\Phi) := \int_0^t \frac{(\log N)^2}{N}\sum_{x\in S_N}\Phi(s,x)\left[\beta_0^N(1-\xi_s^N(x))f_1^N(x,\xi_s^N)^2-\beta_1^N\xi_s^N(x)f_0^N(x,\xi_s^N)^2\right]ds,$$

$$D_t^{N,3}(\Phi) := \int_0^t \frac{(\log N)^4}{N}\sum_{x\in S_N}\Phi(s,x)\left[\xi_s^N(x)f_0^N(x,\xi_s^N)^2-(1-\xi_s^N(x))f_1^N(x,\xi_s^N)^2\right]ds, \tag{29}$$

and $M_t^N(\Phi)$ is an $\mathscr{F}_t^{X^N}$, L^2-martingale such that

$$\langle M^N(\Phi)\rangle_t = \langle M^N(\Phi)\rangle_{1,t}+\langle M^N(\Phi)\rangle_{2,t}, \tag{30}$$

with

$$\langle M^N(\Phi)\rangle_{1,t} := \frac{(\log N)^2}{N}\int_0^t \sum_{x\in S_N}\Phi(s,x)^2\sum_{y\in S_N}p_N(y-x)(\xi_s^N(y)-\xi_s^N(x))^2ds, \tag{31}$$

and

$$\langle M^N(\Phi)\rangle_{2,t} := \frac{(\log N)^2}{N}\int_0^t \sum_{x\in S_N}\Phi(s,x)^2\Big[(\alpha_0^N-1)(1-\xi_s^N(x))f_1^N(x,\xi_s^N)^2 + (\alpha_1^N-1)\xi_s^N(x)f_0^N(x,\xi_s^N)^2\Big]ds. \quad (32)$$

Proposition 3.1 of [7] erroneously had a prefactor of $\frac{(\log N)^2}{N^2}$ on the final term. The error is insignificant as the $\alpha_i^N - 1$ terms in the above are still small enough to make this term approach 0 in L^1 as $N\to\infty$ both here and in [7].

Comparing (28) with the martingale problem (MP) for the super-Brownian motion limit, we see that to prove Theorem 1.5 we will need to establish the tightness of the laws of $\{X^N\}$ on $D(\mathbb{R}_+, M_F(\mathbb{R}^2))$, with all limit points continuous, and if $X^{N_k}\to X$ weakly, then as $N=N_k\to\infty$,

$$E\Big[\Big|D_t^{N,1}(\Phi) - \int_0^t X_s^N(\frac{\sigma^2}{2}\Delta\Phi_s + \dot\Phi_s)ds\Big|\Big]\to 0, \quad (33)$$

$$E\Big[\Big|D_t^{N,2}(\Phi) - \gamma(\beta_0-\beta_1)\int_0^t X_s^N(\Phi)ds\Big|\Big]\to 0, \quad (34)$$

$$E\Big[\Big|D_t^{N,3}(\Phi) - K\int_0^t X_s^N(\Phi)ds\Big|\Big]\to 0, \quad (35)$$

$$E\Big[\Big|\langle M^N(\Phi)\rangle_{1,t} - 4\pi\sigma^2\int_0^t X_s^N(\Phi^2)ds\Big|\Big]\to 0, \quad (36)$$

$$E\Big[\langle M^N(\Phi)\rangle_{2,t}\Big]\to 0. \quad (37)$$

Controlling the five terms in (28) in this manner will also be the main ingredients in establishing the above tightness. As already noted it is the presence of the higher powers of $\log N$ in $D^{N,3}$ which will make (35) particularly tricky.

We now give a brief outline of the proof of this particular result which is established in Section 7 (although many of the key ingredients are given in earlier Sections) and relies on the moment estimates in Sections 4 and 5. The remaining convergences (also treated in Section 7) are closer to the arguments in [7] although their proofs are also complicated by the greater values of $|\alpha_i^N - 1|$. In fact some of the new ingredients introduced here would have significantly simplified some of those proofs. Although we will be able to drop the time dependence in Φ for the convergence proof itself, it will be important for our moment bounds to keep this dependence.

If $D_t^{N,3}(\Phi) = \int_0^t d_s^{N,3}(\Phi)ds$, then a simple L^2 argument (see (67) in Section 5) will allow us to approximate $D_t^{N,3}(\Phi)$ with $\int_{t_N}^t E(d_s^{N,3}(\Phi)|\mathscr{F}_{s-t_N})\,ds$ (recall that $t_N = (\log N)^{-19}$). So we need to estimate

$$
\begin{aligned}
&E(d_s^{N,3}(\Phi)|\mathscr{F}_{s-t_N})\,ds\\
&=\frac{(\log N)^4}{N}\sum_{x\in S_N}\Phi(s,x)E(\xi_s^N(x)f_0^N(x,\xi_s^N)^2-(1-\xi_s^N(x))f_1^N(x,\xi_s^N)^2|\mathscr{F}_{s-t_N})\\
&=\frac{(\log N)^4}{N}\sum_{x\in S_N}\Phi(s,x)E(\xi_s^N(x)\prod_1^2(1-\xi_s^N(x+e_i))-(1-\xi_s^N(x))\prod_1^2\xi_s^N(x+e_i)|\mathscr{F}_{s-t_N}),
\end{aligned}
$$

where e_1,e_2 are chosen independently according to $p_N(\cdot)$. To proceed carefully we will need to use the dual $\tilde{B}$ of ξ^N but note that on the short interval $[s-t_N,s]$ we do not expect to find any red or green arrows since $(\log N)^3 t_N \ll 1$, and so we can use the voter dual $\hat{B}$ and the Markov property to calculate the above conditional expectation (see Lemma 3.6). This leads to the estimate (here $\hat{E}$ only integrates out the dual and (e_1,e_2))

$$
\begin{aligned}
E(d_s^{N,3}(\Phi)|\mathscr{F}_{s-t_N})\approx\frac{(\log N)^4}{N}\sum_{x\in S_N}\Phi(s,x)\hat{E}\Big(\xi_{s-t_N}^N(\hat{B}_{t_N}^x)\prod_1^2(1-\xi_{s-t_N}^N(\hat{B}_{t_N}^{x+e_i}))\\
-(1-\xi_{s-t_N}^N(\hat{B}_{t_N}^x))\prod_1^2\xi_{s-t_N}^N(\hat{B}_{t_N}^{x+e_i})\Big).
\end{aligned}
\tag{38}
$$

There is no contribution to the above expectation if $\hat{B}^x$ coalesces with $\hat{B}^{x+e_1}$ or $\hat{B}^{x+e_2}$ on $[0,t_N]$. On the event where $\hat{B}_{t_N}^{x+e_1}=\hat{B}_{t_N}^{x+e_2}$, the integrand becomes

$$
\begin{aligned}
&\xi_{s-t_N}^N(\hat{B}_{t_N}^x)(1-\xi_{s-t_N}^N(\hat{B}_{t_N}^{x+e_1}))-(1-\xi_{s-t_N}^N(\hat{B}_{t_N}^x))\xi_{s-t_N}^N(\hat{B}_{t_N}^{x+e_1})\\
&\quad=\xi_{s-t_N}^N(\hat{B}_{t_N}^x)-\xi_{s-t_N}^N(\hat{B}_{t_N}^{x+e_1}),
\end{aligned}
\tag{39}
$$

thanks to an elementary and crucial cancellation. Summing by parts and using the regularity of $\Phi(s,\cdot)$, one easily sees that the contribution to $E(d_s^{N,3}(\Phi)|\mathscr{F}_{s-t_N})$ from this term is negligible in the limit. Consider next the contribution from the remaining event $\{x\mid x+e_1\mid x+e_2\}$ on which there is no coalescing of $\{B^x,\hat{B}^{x+e_1},\hat{B}^{x+e_2}\}$ on $[0,t_N]$. Recall that the density of 1's in $\xi_{s-t_N}^N$ is $O(1/\log N)$ and so we expect the contribution from the second term in (38), requiring 1's at both distinct sites $\hat{B}_{t_N}^{x+e_i}$, $i=1,2$, to be negligible in the limit. The same reasoning shows the product in the first term in (38) should be close to 1 and so we expect

$$
\begin{aligned}
E(d_s^{N,3}(\Phi)|\mathscr{F}_{s-t_N})\approx&\frac{(\log N)^4}{N}\sum_{x\in S_N}\Phi(s,x)\hat{E}\Big(\xi_{s-t_N}^N(\hat{B}_{t_N}^x)1(\{x\mid x+e_1\mid x+e_2\})\Big)\\
\approx&\frac{(\log N)}{N}\sum_{x\in S_N}\Phi(s,x)\xi_{s-t_N}^N(x)(\log N)^3\hat{P}(\{0\mid e_1\mid e_2\})\\
\approx&X_{s-t_N}^N(\Phi_{s-t_N})K,
\end{aligned}
$$

where the second line is easy to justify by summing over the values of $\hat{B}_{t_N}^x$, and the last line is immediate from (6). Integrating over s we arrive at (35). The key ingredient needed to justify the above heuristic simplification based on the sparseness of 1's is Proposition 3.10 whose lengthy proof is in Section 5.

3 Intermediate results for the proof of Theorem 1.5

We use the classical strategy of proving tightness of $\{X^N\}$ in the space $D(\mathbb{R}_+, M_F(\mathbb{R}^2))$, and then identify the limits. The main difficulty will come from the above drift term $D_t^{N,3}(\Phi)$. Although it will be convenient to have Theorem 1.5 for a continuous parameter $N \geq 3$, it suffices to prove it for an arbitrary sequence approaching infinity and nothing will be lost by considering $N \in \mathbb{N}^{\geq 3}$. This condition will be in force thoughout the proof of Theorem 1.5, as will the assumption that ξ_0^N is deterministic and all the conditions of Theorem 1.5.

3.1 Tightness, moment bounds.

Recall that a sequence of processes with sample paths in $D(\mathbb{R}_+, S)$ for some Polish space S is C-tight in $D(\mathbb{R}_+, S)$ iff their laws are tight in $D(\mathbb{R}_+, S)$ and every limit point is continuous.

Proposition 3.1. *The sequence $\{X^N, N \in \mathbb{N}^{\geq 3}\}$ is C-tight in $D(\mathbb{R}_+, M_F(\mathbb{R}^2))$.*

Proposition 3.1 will follow from Jakubowski's theorem (see e.g. Theorem II.4.1 in [13]) and the two following lemmas.

Lemma 3.2. *For any function $\Phi \in C_b^3(\mathbb{R}^2)$, the sequence $\{X^N(\Phi), N \in \mathbb{N}^{\geq 3}\}$ is C-tight.*

Lemma 3.3. *For any $\epsilon > 0$, any $T > 0$ there exists $A > 0$ such that*

$$\sup_{N\geq 3} P\left(\sup_{t\leq T} X_t^N(B(0,A)^c) > \epsilon\right) < \epsilon.$$

Lemma 3.2 will be established by looking separately at each term appearing in (28). The difficulty will mainly lie in establishing tightness for the last term in (28). The proof of the two lemmas is given in Section 6.

An important step in proving tightness will be the derivation of bounds on the first and second moments. It will be assumed that $N \in \mathbb{N}^{\geq 3}$ until otherwise indicated.

Proposition 3.4. *There exists a $c_{3.4} > 0$, and for any $T > 0$ constants C_a, C_b, depending on T, such that for any $t \leq T$,*

$$\begin{aligned} &(a) \qquad E\left[X_t^N(1)\right] \leq \left(1 + C_a(\log N)^{-16}\right) X_0^N(1) \exp\left(c_{3.4} t\right), \\ &(b) \qquad E\left[(X_t^N(1))^2\right] \leq C_b \left(X_0^N(1) + (X_0^N(1))^2\right), \end{aligned}$$

Part (a) of the above Proposition is proved in Subsection 4.4, part (b) is proved in Subsection 5.1.

Note that (a) immediately implies the existence of a constant C'_a depending on T such that for any $t \leq T$, $E\left[X_t^N(1)\right] \leq C'_a X_0^N(1)$. Moreover, by (a), (b) and the Markov property, if we set $C_{ab} := C'_a C_b$, we have for any $s, t \in [0, T]$,

$$E\left[X_s^N(1) X_t^N(1)\right] \leq C_{ab}(X_0^N(1) + X_0^N(1)^2). \tag{40}$$

For establishing tightness of some of the terms of (28), and also for proving the compact containment condition, Lemma 3.3, we will need a space-time first moment bound. Recall that $t_N = (\log N)^{-19}$.

Suppose $\Phi : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$. Define $|\Phi|_{\text{Lip}}$, respectively $|\Phi|_{1/2}$, to be the smallest element in $\overline{\mathbb{R}}_+$ such that

$$\begin{cases} |\Phi(s,x) - \Phi(s,y)| \le |\Phi|_{\text{Lip}}|x-y|, & \forall s \ge 0,\ \ x \in \mathbb{R}^2,\ y \in \mathbb{R}^2, \\ |\Phi(s-t_N,x) - \Phi(s,x)| \le |\Phi|_{1/2}\sqrt{t_N}, & \forall\ s \ge t_N,\ x \in \mathbb{R}^2, \end{cases}$$

We will write $||\Phi||_{\text{Lip}} := ||\Phi||_\infty + |\Phi|_{\text{Lip}}$, $||\Phi||_{1/2} := ||\Phi||_\infty + |\Phi|_{1/2}$ and $||\Phi|| := ||\Phi||_\infty + |\Phi|_{\text{Lip}} + |\Phi|_{1/2}$. Obviously the definition of $||.||_{\text{Lip}}$ also applies to functions from $\mathbb{R}^2$ into $\mathbb{R}$.

Define $(P_t^N, t \ge 0)$ as the semigroup of the rate$-N$ random walk on S_N with jump kernel p_N.

Lemma 3.5. *There exist $\delta_{3.5} > 0$, $c_{3.5} > 0$ and for any $T > 0$, there is a $C_{3.5}(T)$, so that for all $t \le T$ and any $\Psi : \mathbb{R}^2 \to \mathbb{R}_+$ such that $||\Psi||_{\text{Lip}} \le T$,*

$$E\left[X_t^N(\Psi)\right] \le e^{c_{3.5}t} X_0^N\left(P_t^N(\Psi)\right) + C_{3.5}(\log N)^{-\delta_{3.5}}(X_0^N(1) + X_0^N(1)^2).$$

This lemma requires a key second moment estimate (see Proposition 3.10 below), it is proved in Subsection 5.3.

3.2 On the new drift term

If $\Phi : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$, let

$$d_s^{N,3}(\Phi, \xi^N) := \frac{(\log N)^4}{N} \sum_{x \in S_N} \Phi(s,x)\left[\xi_s^N(x) f_0(x, \xi_s^N)^2 - (1 - \xi_s^N(x)) f_1(x, \xi_s^N)^2\right],$$

so that (29) implies $D_t^{N,3}(\Phi) = \int_0^t d_s^{N,3}(\Phi, \xi^N)ds$. When the context is obvious we will drop ξ^N from the notation. For $s < t_N$ it will be enough to use the obvious bound

$$|d_s^{N,3}(\Phi)| \le 2(\log N)^3 ||\Phi||_\infty X_s^N(1). \tag{41}$$

On the other hand we are able, for $s \ge t_N$, to get good bounds on the projection of $d_s^{N,3}(\Phi)$ onto $\mathscr{F}_{s-t_N}$. Indeed on $[s - t_N, s]$, it is very unlikely to see a branching (i.e. red or green arrows) for the rate v_N random walks making up the dual process coming down from a given x at time s, and therefore, the dynamics of the rescaled Lotka-Volterra model on that scale should be very close to those of the voter model. Let

$$\begin{aligned}\hat{H}^N(\xi_{s-t_N}^N, x, t_N) := \hat{E}\Bigg(&\xi_{s-t_N}^N(\hat{B}_{t_N}^x)\prod_{i=1}^2\left(1 - \xi_{s-t_N}^N(\hat{B}_{t_N}^{x+e_i})\right) \\ &-(1 - \xi_{s-t_N}^N(\hat{B}_{t_N}^x))\prod_{i=1}^2 \xi_{s-t_N}^N(\hat{B}_{t_N}^{x+e_i})\Bigg).\end{aligned}$$

Lemma 3.6. *There exists a constant $C_{3.6}$ such that for any $s \ge t_N$, and any $\Phi : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$,*

$$\begin{aligned}\left| E\left[d_s^{N,3}(\Phi, \xi^N) \mid \mathscr{F}_{s-t_N}\right] - \frac{(\log N)^4}{N} \sum_{x \in S_N} \Phi(s - t_N, x)\hat{H}^N(\xi_{s-t_N}^N, x, t_N)\right| \\ \le C_{3.6}||\Phi||_{1/2} X_{s-t_N}^N(1)(\log N)^{-6}.\end{aligned}$$

We prove Lemma 3.6 in Section 4.

Let us now look a bit more closely at the term arising in Lemma 3.6. Note that in terms of the voter dual, $\hat{H}(\xi^N_{s-t_N}, x, t_N)$ disappears whenever the rate v_N walk started at x, coalesces before time t_N with either one or both of the rate v_N walks started respectively at $x+e_i, i=1,2$. The non zero contributions will come from two terms. The first corresponds to the event, which we will denote $\{x \mid x+e_1 \mid x+e_2\}_{t_N}$, that there is no collision between the three rate v_N walks started at $x, x+e_1, x+e_2$ up to time t_N. The second corresponds to the event, which we will denote $\{x \mid x+e_1 \sim x+e_2\}_{t_N}$, that the rate v_N walks started at $x+e_1, x+e_2$ coalesce before t_N, but that both do not collide with the walk started at x up to time t_N. For convenience, and when the context is clear, we will drop the subscript from these two notations. We can now write (recall $e_0=0$),

$$\begin{aligned}
&\hat{H}^N(\xi^N_{s-t_N}, x, t_N)\\
&= \hat{E}\left[\left(\xi^N_{s-t_N}(\hat{B}^x_{t_N})\left(1-\xi^N_{s-t_N}(\hat{B}^{x+e_1}_{t_N})\right) - (1-\xi^N_{s-t_N}(\hat{B}^x_{t_N}))\xi^N_{s-t_N}(\hat{B}^{x+e_1}_{t_N})\right)\mathbf{1}_{\{x|x+e_1\sim x+e_2\}_{t_N}}\right]\\
&+\hat{E}\left[\left(\xi^N_{s-t_N}(\hat{B}^x_{t_N}) + 2\prod_{i=0}^{2}\xi^N_{s-t_N}(\hat{B}^{x+e_i}_{t_N}) - \sum_{0\le i<j\le 2}\xi^N_{s-t_N}(\hat{B}^{x+e_i}_{t_N})\xi^N_{s-t_N}(\hat{B}^{x+e_j}_{t_N})\right)\mathbf{1}_{\{x|x+e_1|x+e_2\}_{t_N}}\right]\\
&= \hat{E}\left[\left(\xi^N_{s-t_N}(\hat{B}^x_{t_N}) - \xi^N_{s-t_N}(\hat{B}^{x+e_1}_{t_N})\right)\mathbf{1}_{\{x|x+e_1\sim x+e_2\}_{t_N}}\right] + \hat{E}\left[\xi^N_{s-t_N}(\hat{B}^x_{t_N})\mathbf{1}_{\{x|x+e_1|x+e_2\}_{t_N}}\right]\\
&+\hat{E}\left[\left(2\prod_{i=0}^{2}\xi^N_{s-t_N}(\hat{B}^{x+e_i}_{t_N}) - \sum_{0\le i<j\le 2}\xi^N_{s-t_N}(\hat{B}^{x+e_i}_{t_N})\xi^N_{s-t_N}(\hat{B}^{x+e_j}_{t_N})\right)\mathbf{1}_{\{x|x+e_1|x+e_2\}_{t_N}}\right]\\
&=: F^N_1(s-t_N, x, t_N) + F^N_2(s-t_N, x, t_N) + F^N_3(s-t_N, x, t_N)
\end{aligned}\tag{42}$$

Lemma 3.7. *There is a constant $C_{3.7}$ such that the following hold for any $u, v \ge 0$.*

$$\left|\frac{(\log N)^4}{N}\sum_{x\in S_N}\Phi(v,x)F^N_1(u,x,t_N)\right| \le C_{3.7}|\Phi|_{\text{Lip}}(\log N)^{-6}X^N_u(1), \tag{43}$$

$$\left|\frac{(\log N)^4}{N}\sum_{x\in S_N}\Phi(v,x)F^N_2(u,x,t_N) - (\log N)^3\hat{P}(\{0\mid e_1\mid e_2\}_{t_N})X^N_u(\Phi(v,.))\right| \le C_{3.7}|\Phi|_{\text{Lip}}(\log N)^{-6}X^N_u(1), \tag{44}$$

$$\left|\frac{(\log N)^4}{N}\sum_{x\in S_N}\Phi(v,x)F^N_3(u,x,t_N)\right| \le C_{3.7}||\Phi||_\infty X^N_u(1). \tag{45}$$

There exist $\delta_{3.7}, \eta_{3.7} \in (0,1)$ such that

$$\left|\frac{(\log N)^4}{N}\sum_{x\in S_N}\Phi(v,x)F^N_3(u,x,t_N)\right| \le C_{3.7}||\Phi||_\infty\left[\frac{1}{t_N(\log N)}\mathscr{I}^N_{\eta_{3.7}}(u) + X^N_u(1)(\log N)^{-\delta_{3.7}}\right], \tag{46}$$

where $\mathscr{I}^N_\eta(u) := \iint \mathbf{1}_{\{0<|x-y|<\sqrt{t_N(\log N)^{1-\eta}}\}}dX^N_u(x)dX^N_u(y)$.

We prove Lemma 3.7 in Section 4.

Remark 3.8. *If we set $s = t_N$ in* (42) *and $u = 0$ in the above Lemma and combine* (42), (43), (44), *and* (46) *we see there is a $\delta_{3.8} > 0$ and $C_{3.8}$ so that for any $\xi_0^N \in S_N$*

$$\left|\frac{(\log N)^4}{N}\sum_x \Phi(v,x)\Big(\hat{H}(\xi_0^N,x,t_N) - \xi_0^N(x)P(\{0|e_1|e_2\}_{t_N})\Big)\right|$$
$$\leq C_{3.8}\|\Phi\|_{\text{Lip}}\left[\frac{1}{t_N\log N}\mathscr{I}^N_{\eta_{3.7}}(0) + (\log N)^{-\delta_{3.8}}X_0^N(1)\right]. \quad (47)$$

We now deduce two immediate consequences of the above. Firstly, for any $s \geq t_N$, combining Lemma 3.6, (42), the first three estimates of the above Lemma with $u = s - t_N$, and (6), we obtain

$$\left|E\left[d_s^{N,3}(\Phi,\xi^N) \mid \mathscr{F}_{s-t_N}\right]\right| \leq C_{48}\|\Phi\|X^N_{s-t_N}(1). \quad (48)$$

This, along with (41), allows us to bound the total mass of this new drift term and conclude that

$$E\left[D_t^{N,3}(1)\right] \leq C_{48}E\left[\int_0^{(t-t_N)^+} X_s^N(1)ds\right] + 2(\log N)^3 E\left[\int_0^{t\wedge t_N} X_s^N(1)ds\right]. \quad (49)$$

Secondly, estimates (43), (44), (46) used with $u = s - t_N$ allow us to refine the estimate of Lemma 3.6 on the conditional expectation of $d_s^{N,3}(\Phi)$. That is we have :

Lemma 3.9. *There is a positive constant $C_{3.9}$ such that for any $s \geq t_N$,*

$$\left|E\left[d_s^{N,3}(\Phi) \mid \mathscr{F}_{s-t_N}\right] - (\log N)^3\hat{P}(\{0 \mid e_1 \mid e_2\}_{t_N})X^N_{s-t_N}(\Phi(s-t_N,.))\right|$$
$$\leq C_{3.9}\|\Phi\|(\log N)^{-\delta_{3.7}}X^N_{s-t_N}(1) + \frac{C_{3.7}\|\Phi\|_\infty}{t_N(\log N)}\mathscr{I}^N_{\eta_{3.7}}(s-t_N).$$

3.3 A key second moment estimate

In order to exploit this last result we need to bound the second term arising in the upper bound of Lemma 3.9.

Proposition 3.10. *Let $(\delta_N)_{N\geq 3}$ be a positive sequence such that $\lim_{N\to\infty}\delta_N = 0$ and $\liminf_{N\to\infty}\sqrt{N}\delta_N > 0$. For any T, there exists $C_{3.10}$ depending on T and $\{\delta_N\}$ such that for any $t \leq T$, for any $N \geq 3$,*

$$E\left[\iint \mathbf{1}_{\{|x-y|\leq\sqrt{\delta_N}\}}dX_t^N(x)dX_t^N(y)\right]$$
$$\leq C_{3.10}\left(X_0^N(1) + X_0^N(1)^2\right)\left[\delta_N\left(1 + \log\left(1 + \frac{t}{\delta_N}\right)\right) + \frac{\delta_N}{t+\delta_N}\right].$$

The proof of Proposition 3.10 is somewhat long and technical. We will address it in Section 5. Those familiar with the Hausdorff measure results for planar super-Brownian motion (see e.g., [10]) should be able to predict the key $\delta_N\log(1/\delta_N)$ term in the above bound from the limit theorem we are trying to prove. Although that reference gives $\phi(r) = r^2 \log(1/r) \log\log\log(1/r)$ as the exact Hausdorff measure function for the limiting super-Brownian motion, the proofs show that the typical mass in a ball of radius r centered at a point chosen at random according to super-Brownian motion is $r^2 \log(1/r)$. The triple log term arises from the limsup behaviour as $r \downarrow 0$. Now set $r = \sqrt{\delta_N}$ to arrive at the above term.

A similar result to the above is proved in [7] (Proposition 7.2) but in that setting one has $N|\alpha_i^N - 1| = O(\log N)$ and this allows us to bound X^N by a rescaled biased voter model. It then suffices to obtain the above for the biased voter model and this calculation is much easier due to its elementary dual process. We have $N|\alpha_i^N - 1| = O((\log N)^3)$, making some of the drift terms much harder to bound, and also forcing us to use the more complicated dual $\tilde{B}$.

Corollary 3.11. *If $T > 0$ and $\eta \in (0,1)$, there exists a constant $C_{3.11}$ depending on T and η such that for any $t \leq T$,*

$$\frac{1}{t_N(\log N)} E\left[\int_{t_N}^{t \vee t_N} \mathscr{I}_\eta^N(s - t_N)ds\right] \leq C_{3.11}(\log N)^{-\eta/2}(X_0^N(1) + X_0^N(1)^2).$$

Proof : Without loss of generality consider $T \geq 1$. Use Proposition 3.10 with $\delta_N = t_N(\log N)^{1-\eta}$ to obtain

$$\begin{aligned}
&\frac{1}{t_N(\log N)} E\left[\int_{t_N}^{t \vee t_N} \mathscr{I}_\eta^N(s - t_N)ds\right] \\
&\leq C_{3.10}(\log N)^{-\eta}(X_0^N(1)+X_0^N(1)^2)\int_0^T \left(1 + \log\left(1 + \frac{s}{t_N(\log N)^{1-\eta}}\right) + \frac{1}{s + t_N(\log N)^{1-\eta}}\right)ds \\
&= C_{3.10}(\log N)^{-\eta}(X_0^N(1)+X_0^N(1)^2)(T + 1 + (\log N)^{-18-\eta})\log[1 + T(\log N)^{18+\eta}],
\end{aligned}$$

and the result follows. □

4 Estimates on the drift terms, first moment bounds

We start this section by establishing the first moment estimates on the new drift term in Lemmas 3.6, 3.7, and 3.9. The corresponding estimates for the second drift term $D^{N,2}$ are easier, and follow from a subset of the arguments used for $D^{N,3}$. We will state the estimates and outline their proofs in paragraph 4.3. Finally, we will apply these results to proving the bound on the mean total mass Proposition 3.4 (a).

4.1 Proof of Lemma 3.6

We first observe that

$$\frac{(\log N)}{N} \sum_{x \in S_N} \left|\hat{H}^N(\xi_{s-t_N}^N, x, t_N)\right| \leq 2X_{s-t_N}^N(1). \tag{50}$$

Letting

$$H^N(s,x,t_N) := E\left[\hat{E}\left[\xi_s^N(x)\prod_{i=1}^{2}(1-\xi_s^N(x+e_i)) - (1-\xi_s^N(x))\prod_{i=1}^{2}\xi_s^N(x+e_i)\right] \,\middle|\, \mathscr{F}_{s-t_N}\right],$$

where the expectation $\hat{E}$ is now only over (e_1, e_2), we have

$$\begin{aligned}
&\left|E\left[d_s^{N,3}(\Phi) \mid \mathscr{F}_{s-t_N}\right] - \frac{(\log N)^4}{N}\sum_{x\in S_N}\Phi(s-t_N,x)\hat{H}^N(\xi^N_{s-t_N},x,t_N)\right| \\
&\le \frac{(\log N)^4}{N}\sum_{x\in S_N}|\Phi(s,x)|\left|H^N(s,x,t_N)-\hat{H}^N(\xi^N_{s-t_N},x,t_N)\right| \\
&\qquad + \frac{(\log N)^4}{N}\sum_{x\in S_N}\left|\Phi(s,x)-\Phi(s-t_N,x)\right| \times \left|\hat{H}^N(\xi^N_{s-t_N},x,t_N)\right|. \qquad (51)
\end{aligned}$$

By (50) the second sum in (51) is bounded by $2|\Phi|_{1/2}(\log N)^3\sqrt{t_N}X^N_{s-t_N}(1)$. Moreover, using (25) twice and then Lemma 2.2 in the second line below, we obtain

$$\begin{aligned}
\sum_{x\in S_N}\left|H^N(s,x,t_N)-\hat{H}^N(\xi^N_{s-t_N},x,t_N)\right| &\le \sum_{x\in S_N}\hat{E}\left[\mathbf{1}_{\{\mathscr{E}_{t_N}^{x,x+e_1,x+e_2}\}}\sum_{y\in\tilde{B}_{t_N}^{x,x+e_1,x+e_2}}\xi^N_{s-t_N}(y)\right] \\
&\le 9r_N t_N \exp(3r_N t_N)\sum_{y\in S_N}\xi^N_{s-t_N}(y).
\end{aligned}$$

Since $r_N \sim_{N\to\infty} (\log N)^3$, while $t_N = (\log N)^{-19}$, (51) and the above two bounds clearly imply Lemma 3.6. □

4.2 Proof of Lemma 3.7

We first establish (43). We have

$$\begin{aligned}
&\left|\frac{(\log N)^4}{N}\sum_{x\in S_N}\Phi(v,x)F_1^N(u,x,t_N)\right| \\
&=\left|\frac{(\log N)^4}{N}\sum_{x,w\in S_N}\Phi(v,x)\xi_u^N(w)\Big(\hat{P}\left(\hat{B}^x_{t_N}=w,\{x\mid x+e_1\sim x+e_2\}_{t_N}\right)\right. \\
&\qquad\qquad\left. -\hat{P}\left(\hat{B}^{x+e_1}_{t_N}=w,\{x\mid x+e_1\sim x+e_2\}_{t_N}\right)\Big)\right| \\
&=\left|\frac{(\log N)^4}{N}\sum_{w\in S_N}\xi_u^N(w)\sum_{x\in S_N}\Phi(v,x)\Big(\hat{P}\left(\hat{B}^0_{t_N}=w-x,\{0\mid e_1\sim e_2\}_{t_N}\right)\right. \\
&\qquad\qquad\left. -\hat{P}\left(\hat{B}^{e_1}_{t_N}=w-x,\{0\mid e_1\sim e_2\}_{t_N}\right)\Big)\right| \\
&\leq \frac{(\log N)^4}{N}\sum_{w\in S_N}\xi_u^N(w)\left|\hat{E}\left[\left(\Phi(v,w-\hat{B}^0_{t_N})-\Phi(v,w-\hat{B}^{e_1}_{t_N}\right)\mathbf{1}_{\{0|e_1\sim e_2\}_{t_N}}\right]\right| \\
&\leq (\log N)^3|\Phi|_{\mathrm{Lip}}X_u^N(1)\hat{E}\left[\left|\hat{B}^0_{t_N}-\hat{B}^{e_1}_{t_N}\right|\right] \\
&\leq C_{43}|\Phi|_{\mathrm{Lip}}(\log N)^3\left(\sqrt{t_N}+\frac{1}{\sqrt{N}}\right)X_u^N(1),
\end{aligned}$$

where we used translation invariance at the third line above. The first bound of Lemma 3.7 then follows from our choice of $t_N=(\log N)^{-19}$.

We now turn to proving (44). We have

$$\begin{aligned}
&\left|\frac{(\log N)^4}{N}\sum_{x\in S_N}\Phi(v,x)F_2^N(u,x,t_N)-(\log N)^3\hat{P}(\{0\mid e_1\mid e_2\}_{t_N})X_u^N(\Phi(v,.))\right| \\
&\leq \frac{(\log N)^4}{N}\left|\sum_{x,w\in S_N}\xi_u^N(w)\Phi(v,x)\hat{P}\left(\hat{B}^x_{t_N}=w,\{x\mid x+e_1\mid x+e_2\}_{t_N}\right)\right. \\
&\qquad\qquad\left. -\hat{P}\left(\{0\mid e_1\mid e_2\}_{t_N}\right)\sum_{w\in S_N}\xi_u(w)\Phi(v,w)\right| \\
&\leq \frac{(\log N)^4}{N}\sum_{w\in S_N}\xi_u^N(w)\hat{E}\left[\left|\Phi(v,w-\hat{B}^0_{t_N})-\Phi(v,w)\right|\mathbf{1}_{\{0|e_1|e_2\}_{t_N}}\right] \\
&\leq (\log N)^3|\Phi|_{\mathrm{Lip}}X_u^N(1)\hat{E}\left[\left|\hat{B}^0_{t_N}\right|\right]\leq C_{44}|\Phi|_{\mathrm{Lip}}(\log N)^{-6}X_u^N(1),
\end{aligned}$$

which is (44).

We next establish (45). We first observe that $F_3^N(u,x,t_N)$ is a sum of three terms. Ignoring possible cancellations, we are simply going to bound each one of them separately. In fact, we will prove such a bound for one of them, the other estimates can be derived in a very similar way. For instance,

$$\begin{aligned}\mathscr{T}_N(\Phi) &:= \frac{(\log N)^4}{N}\left|\sum_{x\in S_N}\Phi(v,x)\hat{E}\left[\xi_u^N(\hat{B}^x_{t_N})\xi_u^N(\hat{B}^{x+e_1}_{t_N})\mathbf{1}_{\{x|x+e_1|x+e_2\}_{t_N}}\right]\right| \\ &= \frac{(\log N)^4}{N}\left|\sum_{x,w,z\in S_N,w\neq z}\Phi(v,x)\xi_u^N(w)\xi_u^N(z)\hat{P}\left[\hat{B}^x_{t_N}=w,\hat{B}^{x+e_1}_{t_N}=z,\{x\mid x+e_1\mid x+e_2\}_{t_N}\right]\right| \\ &\leq ||\Phi||_\infty\frac{(\log N)^4}{N}\sum_{w,z\in S_N,w\neq z}\xi_u^N(w)\xi_u^N(z)\hat{P}\left[\hat{B}^0_{t_N}-\hat{B}^{e_1}_{t_N}=w-z,\{0\mid e_1\mid e_2\}_{t_N}\right],\end{aligned} \tag{52}$$

where we used translation invariance of p to obtain the last equality. From the above and the definition of $\bar{K}$ in (16) we get

$$\begin{aligned}\mathscr{T}_N(\Phi) &\leq ||\Phi||_\infty\frac{(\log N)^4}{N}\sum_{w\in S_N}\xi_u^N(w)\sum_{z\in S_N}\hat{P}\left[\hat{B}^0_{t_N}-\hat{B}^{e_1}_{t_N}=w-z,\{0\mid e_1\mid e_2\}_{t_N}\right] \\ &\leq \bar{K}||\Phi||_\infty X_u^N(1).\end{aligned}$$

This and the similar estimates for the three other terms yield (45).

We finally establish (46). As before, we bound each of the three terms summed in $F_3^N(u,x)$ separately, and will only give the proof for the term $\mathscr{T}_N(\Phi)$. From (17), we get that for any $y\in S_N\setminus\{0\}$,

$$\hat{P}(\hat{B}^{e_1}_{t_N/2}-\hat{B}^0_{t_N/2}=y)\leq\hat{P}(\hat{B}^{e_1}_{t_N}=y)\leq\frac{C_{17}}{Nt_N}.$$

Here note that the random walk on the left is absorbed at the origin, whence the inequality and restriction to $y\neq 0$. Thus, using the Markov property at time $t_N/2$ for the walks $\hat{B}^0,\hat{B}^{e_1}$ along with (6) and the above inequality, we obtain

$$\begin{aligned}&\hat{P}(\hat{B}^0_{t_N}-\hat{B}^{e_1}_{t_N}=w-z,\{0\mid e_1\mid e_2\}_{t_N}) \\ &\leq\sum_{x\neq y,x,y\in S_N}\hat{P}\left[\{0\mid e_1\mid e_2\}_{t_N/2},\hat{B}^0_{t_N/2}=x,\hat{B}^{e_1}_{t_N/2}=y,\hat{B}^0_{t_N}-\hat{B}^{e_1}_{t_N}=w-z\right] \\ &\leq\frac{C_{17}\bar{K}}{Nt_N}(\log N)^{-3}.\end{aligned}$$

Although this bound is valid for any w,z, we will only use it when w and z are close enough that the above term effectively contributes to $\mathscr{T}_N(\Phi)$. For $\eta\in(0,1)$,

$$\begin{aligned}&\frac{(\log N)^4}{N}\sum_{w,z\in S_N:\ 0<|w-z|<\sqrt{t_N(\log N)^{1-\eta}}}\xi_u^N(w)\xi_u^N(z)\hat{P}(\hat{B}^0_{t_N}-\hat{B}^{e_1}_{t_N}=w-z,\{0\mid e_1\mid e_2\}_{t_N}) \\ &\leq\frac{(\log N)}{N}\frac{C_{17}\bar{K}}{Nt_N}\sum_{w,z\in S_N:\ 0<|w-z|<\sqrt{t_N(\log N)^{1-\eta}}}\xi_u^N(w)\xi_u^N(z)=\frac{C_{17}\bar{K}}{t_N(\log N)}\mathscr{I}_\eta^N(u).\end{aligned} \tag{53}$$

The contribution to $\mathcal{T}_N(\Phi)$ of those w and z that are sufficiently far apart is bounded as follows. Using (16) we have

$$\begin{aligned}
&\frac{(\log N)^4}{N} \sum_{w,z\in S_N,\ |w-z|>\sqrt{(\log N)^{1-\eta} t_N}} \xi_u^N(w)\xi_u^N(z)\hat{P}(\hat{B}^0_{t_N} - \hat{B}^{e_1}_{t_N} = w - z, \{0 \mid e_1 \mid e_2\}_{t_N}) \\
&\le \bar{K}\frac{\log N}{N} \sum_{w\in S_N} \xi_u^N(w) \sum_{y\in S_N} 1(|y| > \sqrt{(\log N)^{1-\eta} t_N})\hat{P}(\hat{B}^0_{t_N} - \hat{B}^{e_1}_{t_N} = y|\{0 \mid e_1 \mid e_2\}_{t_N}) \\
&\le \bar{K}X_u^N(1)\hat{P}\left(\left|\hat{B}^0_{t_N} - \hat{B}^{e_1}_{t_N}\right| > \sqrt{(\log N)^{1-\eta} t_N} \,\middle|\, \{0 \mid e_1 \mid e_2\}_{t_N}\right) \\
&\le \bar{K}C_{9.1}(\log N)^{-(1-\eta)}X_u^N(1),
\end{aligned}$$

where the last line comes from Lemma 9.1 and the facts that $v_N \le N$ (by(12)) and $(\log N) \ge \log(v_N t_N)$. Combining (52), (53) and the above gives the desired bound on $\mathcal{T}_N(\Phi)$. The two other terms are handled in a similar way, and we finally obtain (46).

4.3 Second drift estimates

We may write $D_t^{N,2}(\Phi) = \int_0^t d_s^{N,2}(\Phi, \xi^N)ds$, where

$$d_s^{N,2}(\Phi,\xi^N) := \frac{(\log N)^2}{N} \sum_{x\in S_N} \Phi(s,x)\left[\beta_0^N(1-\xi_s^N(x))(f_1^N(x,\xi_s^N))^2 - \beta_1^N\xi_s^N(x)(f_0^N(x,\xi_s^N))^2\right].$$

Again, when context is clear we drop ξ^N from this notation. When $s \le t_N$ we will use

$$|d_s^{N,2}(\Phi)| \le 2(\log N)\overline{\beta}||\Phi||_\infty X_s^N(1). \tag{54}$$

For $s \ge t_N$, the same reasoning we used to establish Lemma 3.6 yields

$$\begin{aligned}
&\Bigg|E\left[d_s^{N,2}(\Phi) \mid \mathcal{F}_{s-t_N}\right] - \frac{(\log N)^2}{N}\sum_{x\in S_N}\Phi(s-t_N,x)\hat{E}\Bigg[\beta_0^N(1-\xi^N_{s-t_N}(\hat{B}^x_{t_N}))\prod_{i=1}^2 \xi^N_{s-t_N}(\hat{B}^{x+e_i}_{t_N}) \\
&\qquad\qquad -\beta_1^N\xi^N_{s-t_N}(\hat{B}^x_{t_N}))\prod_{i=1}^2(1-\xi^N_{s-t_N}(\hat{B}^{x+e_i}_{t_N}))\Bigg]\Bigg| \\
&\le C_{55}||\Phi||_{1/2}X^N_{s-t_N}(1)(\log N)^{-8}.
\end{aligned} \tag{55}$$

The summand in the above expression vanishes whenever the walk started at x coalesces before time t_N with either of the two walks started at $x+e_1$, $x+e_2$. We may therefore write

$$\begin{aligned}
&\hat{E}\left[\beta_0^N(1-\xi^N_{s-t_N}(\hat{B}^x_{t_N}))\prod_{i=1}^2\xi^N_{s-t_N}(\hat{B}^{x+e_i}_{t_N})-\beta_1^N\xi^N_{s-t_N}(\hat{B}^x_{t_N}))\prod_{i=1}^2(1-\xi^N_{s-t_N}(\hat{B}^{x+e_i}_{t_N})))\right]\\
&=\hat{E}\left[\left(\beta_0^N\xi^N_{s-t_N}(\hat{B}^{x+e_1}_{t_N})-\beta_1^N\xi^N_{s-t_N}(\hat{B}^x_{t_N})\right)\mathbf{1}_{\{x|x+e_1\sim x+e_2\}_{t_N}}\right]\\
&+\hat{E}\left[(\beta_1^N-\beta_0^N)\xi^N_{s-t_N}(\hat{B}^x_{t_N})\xi^N_{s-t_N}(\hat{B}^{x+e_1}_{t_N})\mathbf{1}_{\{x|x+e_1\sim x+e_2\}_{t_N}}\right]\\
&+\hat{E}\left[\left(-\beta_1^N\xi^N_{s-t_N}(\hat{B}^x_{t_N})+\beta_1^N\left(\sum_{i=1}^2\xi^N_{s-t_N}(\hat{B}^x_{t_N})\xi^N_{s-t_N}(\hat{B}^{x+e_i}_{t_N})\right)\right.\right.\\
&\qquad\left.\left.+\beta_0^N\xi^N_{s-t_N}(\hat{B}^{x+e_1}_{t_N})\xi^N_{s-t_N}(\hat{B}^{x+e_2}_{t_N})-(\beta_1^N+\beta_0^N)\prod_{i=0}^2\xi^N_{s-t_N}(\hat{B}^{x+e_i}_{t_N})\right)\mathbf{1}_{\{x|x+e_1|x+e_2\}_{t_N}}\right]\\
&=:G_1^N(s-t_N,x)+G_2^N(s-t_N,x)+G_3^N(s-t_N,x).
\end{aligned}$$

For $u\geq 0$, in the expression $\frac{(\log N)^2}{N}\sum_{x\in S_N}\Phi(u,x)\sum_{i=1}^3 G_i(u,x)$, only the first term gives a non-negligible contribution. Indeed, an argument similar to the one we used to establish Lemma 3.7 provides the following estimates for some constants $\delta,\eta\in(0,1)$:

$$\begin{aligned}
&\left|\frac{(\log N)^2}{N}\sum_{x\in S_N}\Phi(u,x)G_1^N(u,x)-(\beta_0^N-\beta_1^N)(\log N)\hat{P}(\{0\mid e_1\sim e_2\}_{t_N})X_u^N(\Phi(u,.))\right|\\
&\quad\leq C_{56}(\log N)^{-8}\|\Phi\|_{\mathrm{Lip}}X_u^N(1),
\end{aligned}\tag{56}$$

$$\left|\frac{(\log N)^2}{N}\sum_{x\in S_N}\Phi(u,x)(G_2^N(u,x)+G_3^N(u,x))\right|\leq C_{57}\|\Phi\|_\infty X_u^N(1),\tag{57}$$

$$\begin{aligned}
&\left|\frac{(\log N)^2}{N}\sum_{x\in S_N}\Phi(u,x)(G_2^N(u,x)+G_3^N(u,x))\right|\\
&\qquad\leq C_{58}\|\Phi\|_\infty\left[\frac{1}{t_N(\log N)}\mathscr{I}_\eta^N(u)+X_u^N(1)(\log N)^{-\delta}\right].
\end{aligned}\tag{58}$$

We recall from (5) that the quantity $(\log N)\hat{P}(\{0\mid e_1\sim e_2\}_{t_N})$, which appears in (56), converges to a positive limit as $N\to\infty$. For $s\geq t_N$, by (55), and (56)-(57) used with $u=s-t_N$, we obtain

$$\left|E\left[d_s^{N,2}(\Phi)\mid\mathscr{F}_{s-t_N}\right]\right|\leq C_{59}\|\Phi\|X^N_{s-t_N}(1)\tag{59}$$

Along with (54) this provides

$$E[D_t^{N,2}(1)]\leq 2\overline{\beta}(\log N)E\left[\int_0^{t\wedge t_N}X_s^N(1)ds\right]+C_{59}E\left[\int_0^{(t-t_N)^+}X_s^N(1)ds\right]\tag{60}$$

Finally, from (55), and (56), (58) used with $u = s - t_N$ we deduce

$$\left| E\left[d_s^{N,2}(\Phi) \mid \mathscr{F}_{s-t_N} \right] - (\beta_0^N - \beta_1^N)(\log N)\hat{P}(\{0 \mid e_1 \sim e_2\}_{t_N}) X_{s-t_N}^N(\Phi(s - t_N, .)) \right| \tag{61}$$
$$\leq C_{61}(\log N)^{-\delta} ||\Phi|| X_{s-t_N}^N(1) + C_{61}||\Phi||_\infty \frac{1}{t_N(\log N)} \mathscr{I}_\eta^N(s - t_N).$$

4.4 Bounding the total mass : proof of Proposition 3.4 (a)

Since $D_t^{N,1}(1) = 0$, we deduce from (28), (49) and (60) that

$$E[X_t^N(1)] \leq X_0^N(1) + 2(\overline{\beta} + 1)E\left[\int_0^{t \wedge t_N} (\log N)^3 X_s^N(1) ds \right]$$
$$+ (C_{59} + C_{48})E\left[\int_0^{(t-t_N)^+} X_s^N(1) ds \right].$$

Therefore, whenever $t \leq t_N$,

$$\begin{aligned} E[X_t^N(1)] &\leq X_0^N(1) + 2(1 + \overline{\beta})(\log N)^3 \int_0^t E[X_s^N(1)] ds \\ &\leq X_0^N(1) \exp(2(1 + \overline{\beta})(\log N)^3 t) \leq C_{62} X_0^N(1) \end{aligned} \tag{62}$$

where at the last line, we first used Gronwall's lemma, then the fact that $t \leq t_N = (\log N)^{-19}$.
On the other hand when $t > t_N$, we find, setting $c_{3.4} := C_{59} + C_{48}$,

$$\begin{aligned} E[X_t^N(1)] &\leq c_{3.4} \int_0^t E[X_s^N(1)] ds + 2(\overline{\beta} + 1)(\log N)^3 \int_0^{t_N} E[X_s^N(1)] ds + X_0^N(1) \\ &\leq c_{3.4} \int_0^t E[X_s^N(1)] ds + 2C_{62}(\overline{\beta} + 1) X_0^N(1)(\log N)^{-16} + X_0^N(1) \end{aligned}$$

Therefore, using again Gronwall's lemma, we obtain

$$E[X_t^N(1)] \leq X_0^N(1)(1 + 2C_{62}(\overline{\beta} + 1)(\log N)^{-16}) \exp(c_{3.4} t),$$

which is Proposition 3.4 (a).

5 Second moment estimates, proof of Proposition 3.10

5.1 Proof of Proposition 3.4 (b)

Use that for real a_i, $i \geq 1$,

$$\left(\sum_{i=1}^n a_i \right)^2 \leq n \sum_{i=1}^n a_i^2 \tag{63}$$

to deduce from (28) that

$$E[X_t^N(1)^2] \le 4\left[E[X_0^N(1)^2] + E[\langle M^N(1)\rangle_t] + E[D_t^{N,2}(1)^2] + E[D_t^{N,3}(1)^2]\right]. \tag{64}$$

We are going to bound each term on the right-hand side of (64) separately.

We first deal with the expected square predictable function of the martingale. It satisfies a similar decomposition as that given in Lemma 4.8 of [7]. We give this decomposition for the more general $\langle M^N(\Phi)\rangle_t$, as we will also need it later in the proof of tightness. Recall formulas (30), (31) and (32), which gave the square predictible function of the martingale $M^N(\Phi)$ as a sum of two terms.

Lemma 5.1. *There is a constant $C_{5.1}$ such that for $t \le T$, and for any $\Phi : [0,T]\times S_N \to \mathbb{R}$ bounded measurable,*

$$(a) \quad \langle M^N(\Phi)\rangle_{2,t} = \int_0^t m_{2,s}^N(\Phi)ds, \quad \textit{where} \quad |m_{2,s}^N(\Phi)| \le C_{5.1}\frac{||\Phi(s,\cdot)||_\infty^2(\log N)^4}{N}X_s^N(1),$$

$$(b) \quad \langle M^N(\Phi)\rangle_{1,t} = 2\int_0^t X_s^N((\log N)\Phi(s,.)^2 f_0^N(\xi_s^N))ds + \int_0^t m_{1,s}^N(\Phi)ds,$$

$$\textit{where} \quad |m_{1,s}^N(\Phi)| \le C_{5.1}\frac{||\Phi||_{\mathrm{Lip}}^2(\log N)}{\sqrt{N}}X_s^N(1).$$

The proof is almost identical to that of Lemma 4.8 of [7] (with the missing factor of N mentioned at the end of Section 2.3 now included). The only difference comes from the fact that, in the formula for $\langle M^N(\Phi)\rangle_{2,t}$ given in Section 2.3, the terms $(\alpha_i^N - 1)$ bring in a multiplicative factor of $(\log N)^3$ compared to the $(\log N)$ factor in [7]. □

We will need to estimate the expectation of the first term of the sum in the right-hand side of Lemma 5.1 (b). The following bound, corresponding to Proposition 4.5 of [7], is not optimal, but will be all that we need.

Lemma 5.2. *There is a positive constant $C_{5.2}$ such that for any $\Phi : \mathbb{R}^2 \to \mathbb{R}_+$ measurable,*

$$E\left[X_{t_N}^N((\log N)\Phi f_0^N(.,\xi_{t_N}^N))\right] \le C_{5.2}\left(||\Phi||_{\mathrm{Lip}}(\log N)^{-8}X_0^N(1) + X_0^N(\Phi)\right).$$

<u>Proof</u> : Using a reasoning similar to (25), we find

$$E\left[X_{t_N}^N((\log N)\Phi f_0^N(.,\xi_{t_N}^N))\right] = \frac{(\log N)^2}{N}\sum_{x,e_1\in S_N}\Phi(x)p_N(e_1)E\left[\xi_{t_N}^N(x)(1-\xi_{t_N}^N(x+e_1))\right]$$

$$\le \quad \frac{(\log N)^2}{N}\sum_{x\in S_N}\Phi(x)\hat{E}\left[\xi_0^N(\hat{B}_{t_N}^x)(1-\xi_0^N(\hat{B}_{t_N}^{x+e_1}))\right]$$

$$+\frac{(\log N)^2||\Phi||_\infty}{N}\sum_{x\in S_N}\hat{E}\left[\mathbf{1}_{\mathscr{E}_{t_N}^{x,x+e_1}}\sum_{y\in\tilde{B}_{t_N}^x}\xi_0^N(y)\right]$$

For the first term in the sum above, we can use the voter model estimate (5.8) of [7] (when applying this result note that our scale change leads to a factor of $1/\sqrt{N}$ in front of the Lipschitz constant of Φ). The second term in the sum above is bounded using Lemma 2.2. This leads to the desired conclusion. □

We return to the proof of Proposition 3.4 (b) and to bounding the square predictable function of the martingale term. We have

$$E\left[\int_0^t X_s^N\left((\log N)f_0^N(\xi_s^N)\right)ds\right]$$
$$\leq \quad E\left[\int_0^{t\wedge t_N} X_s^N\left((\log N)f_0^N(\xi_s^N)\right)ds\right]+E\left[\int_{t\wedge t_N}^t E\left[X_s^N\left((\log N)f_0^N(\xi_s^N)\mid \mathscr{F}_{s-t_N}^N\right)\right]ds\right].$$

By Proposition 3.4 (a), the first term in the sum above is bounded by $C_{\mathrm{a}}' t_N(\log N)X_0^N(1)$. Moreover, by the Markov property and Lemma 5.2 we see that, for any $s\geq t_N$,

$$E\left[X_s^N\left((\log N)f_0^N(\xi_s^N)\right)\mid \mathscr{F}_{s-t_N}^N\right]\leq 2C_{5.2}E[X_{s-t_N}^N(1)].$$

We finally deduce from Lemma 5.1 and the above bounds, then Proposition 3.4 (a) that there exists C_{65} depending on T such that for $t\leq T$,

$$E[\langle M^N(1)\rangle_t]\leq C_{\mathrm{a}}' t_N(\log N)X_0^N(1)+2(C_{5.1}+C_{5.2})\int_0^t E[X_s^N(1)]ds\leq C_{65}X_0^N(1). \tag{65}$$

It remains to deal with the drift terms. Our goal is to bound $E[(D_t^{N,i}(1))^2]$, for $i=2,3$. Here again, anticipating the proof of tightness, we will rather consider the more general $D_t^{N,i}(\Phi)$, $i=2,3$, for a $\Phi:[0,T]\times S_N\to\mathbb{R}$ such that $||\Phi||_{\mathrm{Lip}}<\infty$.
We first observe, using (54), (41) and Jensen's inequality, that

$$E\left[D_{t\wedge t_N}^{N,i}(\Phi)^2\right]\leq(2\overline{\beta}\vee 1)^2||\Phi||_\infty^2(\log N)^6 t_N E\left[\int_0^{t\wedge t_N}(X_s^N(1))^2ds\right]. \tag{66}$$

For $t_N\leq t_1<t_2\leq T$, we are then going to bound $E\left[\left(D_{t_2}^{N,i}(\Phi)-D_{t_1}^{N,i}(\Phi)\right)^2\right]$.

Whenever $s_2>s_1+t_N$, $\left(d_{s_1}^{N,i}(\Phi)-E[d_{s_1}^{N,i}(\Phi)\mid\mathscr{F}_{s_1-t_N}]\right)$ is $\mathscr{F}_{s_2-t_N}$-measurable, and therefore in this case,

$$E\left[\left(d_{s_1}^{N,i}(\Phi)-E[d_{s_1}^{N,i}(\Phi)\mid\mathscr{F}_{s_1-t_N}]\right)\left(d_{s_2}^{N,i}(\Phi)-E[d_{s_2}^{N,i}(\Phi)\mid\mathscr{F}_{s_2-t_N}]\right)\right]=0.$$

For $t_2>t_1\geq t_N$, it follows that

$$E\left[\left(\int_{t_1}^{t_2} d_s^{N,i}(\Phi)-E[d_s^{N,i}(\Phi)\mid\mathscr{F}_{s-t_N}]ds\right)^2\right]$$
$$=2\int_{t_1}^{t_2}\int_{s_1}^{(s_1+t_N)\wedge t_2}E\left[\left(d_{s_1}^{N,i}(\Phi)-E[d_{s_1}^{N,i}(\Phi)\mid\mathscr{F}_{s_1-t_N}]\right)\left(d_{s_2}^{N,i}(\Phi)-E[d_{s_2}^{N,i}(\Phi)\mid\mathscr{F}_{s_2-t_N}]\right)\right]ds_1ds_2.$$

Using again (54), (41) in the above, we deduce

$$
\begin{aligned}
&E\left[\left(\int_{t_1}^{t_2} d_s^{N,i}(\Phi) - E[d_s^{N,i}(\Phi) \mid \mathscr{F}_{s-t_N}]ds\right)^2\right] \\
&\le 32(\overline{\beta} \vee 1)^2 ||\Phi||_\infty^2 (\log N)^6 E\left[\iint_{t_1 \le s_1 < s_2 \le (s_1+t_N)\wedge t_2} X_{s_1}^N(1) X_{s_2}^N(1) ds_1 ds_2\right]
\end{aligned}
$$

Moreover

$$
E[X_{s_1}^N(1)X_{s_2}^N(1)] = E[E[X_{s_2}^N(1) \mid \mathscr{F}_{s_1}]X_{s_1}^N(1)] \le \left(1 + C_{\text{a}}(\log N)^{-16}\right) e^{c(s_2-s_1)} E[(X_{s_1}^N(1))^2],
$$

by using the Markov property at time s_1 along with Proposition 3.4 (a). Using this last inequality in the preceding one leads to

$$
\begin{aligned}
&E\left[\left(\int_{t_1}^{t_2} d_s^{N,i}(\Phi) - E[d_s^{N,i}(\Phi) \mid \mathscr{F}_{s-t_N}]ds\right)^2\right] \\
&\le C_{67} ||\Phi||_\infty^2 (\log N)^6 E\left[\int_{t_1}^{t_2} X_s^N(1)^2 (t_N \wedge (t_2 - s)) ds\right]. \qquad (67)
\end{aligned}
$$

Moreover, using Cauchy-Schwarz inequality,

$$
E\left[\left(\int_{t_1}^{t_2} E[d_s^{N,i}(\Phi) \mid \mathscr{F}_{s-t_N}]ds\right)^2\right] \quad \le \quad E\left[\int_{t_1}^{t_2} \left(E[d_s^{N,i}(\Phi) \mid \mathscr{F}_{s-t_N}]\right)^2 ds\right](t_2 - t_1).
$$

Using (59) for $i = 2$, (48) for $i = 3$, we deduce

$$
E\left[\left(\int_{t_1}^{t_2} E[d_s^{N,i}(\Phi) \mid \mathscr{F}_{s-t_N}]ds\right)^2\right] \le (C_{59} \vee C_{48})^2 ||\Phi||^2 (t_2 - t_1) \int_{t_1}^{t_2} E\left[X_{s-t_N}(1)^2\right] ds. \qquad (68)
$$

Combining (67) and (68), we obtain, for $i = 2, 3$ and $t_2 > t_1 \ge t_N$,

$$
\begin{aligned}
&E\left[\left(D_{t_2}^{N,i}(\Phi) - D_{t_1}^{N,i}(\Phi)\right)^2\right] \\
\le\ & C_{69}\left(||\Phi||^2(t_2 - t_1) + ||\Phi||_\infty^2 (\log N)^6 (t_N \wedge (t_2 - t_1))\right) \int_{t_1}^{t_2} E[X_s^N(1)^2] ds. \qquad (69)
\end{aligned}
$$

By plugging (65), (66) and the above into (64) it follows in particular that there exists C_{70} depending on T such for any $t \le T$,

$$
E[X_t^N(1)^2] \le 4X_0^N(1)^2 + C_{65} X_0^N(1) + C_{70} \int_0^t E[X_s^N(1)^2] ds, \qquad (70)
$$

and a simple use of Gronwall's lemma finishes the proof of Proposition 3.4 (b).

5.2 Proof of Proposition 3.10

Recall that $p_t^N(x) = NP(\hat{B}_t^{N,0} = x)$ and set $p_t^{N,z}(x) := p_t^N(z-x)$. By assumption $N\delta_N \to \infty$ as $N \to \infty$ so we can use the local limit theorem for the walk $\hat{B}^N$ (see e.g., Lemma 7.3 (c) of [7]) to obtain the existence of a constant C_{71} such that

$$\mathbf{1}_{\{|x-y|\le\sqrt{\delta_N}\}} \le C_{71}\delta_N p_{2\delta_N}^N(y-x). \tag{71}$$

Therefore the desired result will follow if we can bound

$$E\left[\int_{\mathbb{R}^2} \delta_N p_{2\delta_N}^N(y-x)dX_t^N(x)dX_t^N(y)\right]$$

as in the Proposition. By Chapman-Kolmogorov we have

$$\begin{aligned}
&\frac{\delta_N}{N}\sum_{z\in S_N} E\left[\left(X_t^N(p_{\delta_N}^{N,z})\right)^2\right] \\
&= E\left[\frac{(\log N)^2}{N^2}\sum_{x,y\in S_N}\xi_t^N(x)\xi_t^N(y)\delta_N\left(\frac{1}{N}\sum_{z\in S_N}p_{\delta_N}^N(z-x)p_{\delta_N}^N(z-y)\right)\right] \\
&= E\left[\frac{(\log N)^2}{N^2}\sum_{x,y\in S_N}\xi_t^N(x)\xi_t^N(y)\delta_N p_{2\delta_N}^N(y-x)\right] \\
&= E\left[\int_{\mathbb{R}^2}\delta_N p_{2\delta_N}^N(y-x)dX_t^N(x)dX_t^N(y)\right].
\end{aligned} \tag{72}$$

Set $\phi_s^z = p_{t-s+\delta_N}^{N,z}$, which satisfies $A_N\phi_s^z + \dot{\phi}_s^z = 0$, so that, from (28), we deduce that

$$E\left[\left(X_t^N(p_{\delta_N}^{N,z})\right)^2\right] \le 4\left[E[(X_0^N(\phi_0^z))^2] + \langle M(\phi^z)\rangle_t + (D_t^{N,2}(\phi^z))^2 + (D_t^{N,3}(\phi^z))^2\right].$$

Using (72) and the above, then again Chapman-Kolmogorov, we obtain the bound:

$$E\left[\int_{\mathbb{R}^2}\delta_N p_{2\delta_N}^N(y-x)dX_t^N(x)dX_t^N(y)\right] \le 4[\mathscr{T}_0 + \mathscr{T}_1 + \mathscr{T}_2 + \mathscr{T}_3 + \mathscr{T}_4],$$

where

$$\mathcal{T}_0 := \frac{\delta_N}{N}\sum_{z\in S_N} E\left[\left(X_0^N(p_{t+\delta_N}^{N,z})\right)^2\right] = E\left[\iint_{\mathbb{R}^2} \delta_N p^N_{2(t+\delta_N)}(y-x)dX_0^N(x)dX_0^N(y)\right],$$

$$\mathcal{T}_1 := \frac{\delta_N}{N}\sum_{z\in S_N} E[\langle M(\phi^z)\rangle_{1,t}]$$

$$= E\left[\frac{\delta_N(\log N)^2}{N}\int_0^t p^N_{2(t-s+\delta_N)}(0)\sum_{x,y\in S_N} p_N(y-x)(\xi_s^N(x)-\xi_s^N(y))^2 ds\right],$$

$$\mathcal{T}_2 := \frac{\delta_N}{N}\sum_{z\in S_N} E[\langle M(\phi^z)\rangle_{2,t}],$$

$$\mathcal{T}_3 := E\Bigg[\frac{\delta_N(\log N)^4}{N^2}\int_0^t\int_0^t ds_1 ds_2 \sum_{x_1,x_2\in S_N} p^N_{2t-s_1-s_2+2\delta_N}(x_2-x_1)$$
$$\times\prod_{i=1}^2\left(\beta_0^N(1-\xi_{s_i}^N(x_i))f_1^N(x_i,\xi_{s_i}^N)^2 - \beta_1^N\xi_{s_i}^N(x_i)f_0^N(x_i,\xi_{s_i}^N)^2\right)\Bigg],$$

$$\mathcal{T}_4 := E\Bigg[\frac{\delta_N(\log N)^8}{N^2}\int_0^t\int_0^t ds_1 ds_2 \sum_{x_1,x_2\in S_N} p^N_{2t-s_1-s_2+2\delta_N}(x_2-x_1)$$
$$\times\prod_{i=1}^2\left(\xi_{s_i}^N(x_i)f_0^N(x_i,\xi_{s_i}^N)^2 - (1-\xi_{s_i}^N(x_i))f_1^N(x_i,\xi_{s_i}^N)^2\right)\Bigg].$$

We will handle each of these terms separately. Using (17), we immediately obtain

$$\mathcal{T}_0 \le \frac{C_{17}\delta_N}{t+\delta_N}X_0^N(1)^2. \tag{73}$$

We then consider $\mathcal{T}_1$ which will turn out to be the main contribution, although not the most difficult to handle. As we already did before, we will condition back a bit in order to be able to use the voter estimates. This time however, we will need to condition back by $u_N := \frac{\delta_N}{2}\wedge(\log N)^{-11}$. Let

$$\Delta_N(s,x) := \Bigg| E\left[\sum_{e\in S_N} p_N(e)\xi_s^N(x)(1-\xi_s^N(x+e)) \,\Big|\, \mathcal{F}_{s-u_N}\right]$$
$$-\hat{E}\left[\sum_{e\in S_N} p_N(e)\xi_{s-u_N}^N(\hat{B}^x_{u_N})(1-\xi^N_{s-u_N}(\hat{B}^{x+e}_{u_N}))\right]\Bigg|.$$

By the obvious analogues of (25) and Lemma 2.2,

$$\begin{aligned}\frac{(\log N)}{N}\sum_{x\in S_N}\Delta_N(s,x) &\le \frac{(\log N)}{N}\sum_{x\in S_N}\hat{E}\left[\mathbf{1}_{(\mathcal{E}^{x,x+e}_{u_N})}\sum_{y\in\tilde{B}^x_{u_N}}\xi^N_{s-u_N}(y)\right]\\ &\le 4u_N r_N\exp(2u_N r_N)X^N_{s-u_N}(1)\\ &\le C_{74}(\log N)^{-8}X^N_{s-u_N}(1).\end{aligned} \tag{74}$$

Moreover, from the fact that $\xi_s^N(x) \in \{0,1\}$,

$$\sum_{x,e\in S_N} p_N(e)(\xi_s^N(x)-\xi_s^N(x+e))^2 = 2\sum_{x,e\in S_N} p_N(e)\xi_s^N(x)(1-\xi_s^N(x+e)) \le 2\frac{N}{(\log N)}X_s^N(1).$$

Therefore, using (17),

$$\begin{aligned}
\mathcal{T}_1 &\le C_{17}\delta_N(\log N)E\left[\int_0^t (t-s+\delta_N)^{-1}\frac{(\log N)}{N}2\sum_{x\in S_N, e\in S_N} p_N(e)\xi_s^N(x)(1-\xi_s^N(x+e))ds\right] \\
&\le 2C_{17}\delta_N(\log N)E\left[\int_0^{u_N}(t-s+\delta_N)^{-1}X_s^N(1)ds\right] \\
&\quad +2C_{17}\delta_N(\log N)E\left[\int_{u_N}^{t\vee u_N}(t-s+\delta_N)^{-1}\frac{(\log N)}{N}\sum_{x\in S_N}\hat{E}\left[\xi_{s-u_N}^N(\hat{B}_{u_N}^x)(1-\xi_{s-u_N}^N(\hat{B}_{u_N}^{x+e}))\right]ds\right] \\
&\quad +2C_{17}C_{74}\delta_N(\log N)^{-7}E\left[\int_{u_N}^{t\vee u_N}(t-s+\delta_N)^{-1}X_{s-u_N}^N(1)ds\right] \\
&=: \quad \mathcal{T}_{1,1}+\mathcal{T}_{1,2}+\mathcal{T}_{1,3},
\end{aligned} \tag{75}$$

where we used (74) for $s \ge u_N$ in the last inequality. Then using Proposition 3.4 (a), we obtain

$$\mathcal{T}_{1,1} \le C_{17}C'_{\text{a}}(\log N)u_N X_0^N(1),$$

and as we will see this term is negligible compared to the bound we will obtain for $\mathcal{T}_{1,2}$.
Notice that

$$\xi_{s-u_N}^N(\hat{B}_{u_N}^x)(1-\xi_{s-u_N}^N(\hat{B}_{u_N}^{x+e})) \le \sum_{w\in S_N}\xi_{s-u_N}^N(w)\mathbf{1}_{\{\hat{B}_{u_N}^x=w\}}\mathbf{1}_{\{x|x+e\}_{u_N}}.$$

Thus, by translation invariance of our kernel p_N,

$$\begin{aligned}
\mathcal{T}_{1,2} &\le 2C_{17}\delta_N(\log N)E\left[\int_{u_N}^{t\vee u_N}(t-s+\delta_N)^{-1}X_{s-u_N}^N(1)\sum_{y\in S_N}\hat{P}\left(\hat{B}_{u_N}^0=y,\{0\mid e\}_{u_N}\right)\right] \\
&\le C_{76}\delta_N\log\left(1+\frac{T}{\delta_N}\right)X_0^N(1),
\end{aligned} \tag{76}$$

where the above line was obtained using Proposition 3.4 (a) and (13). Finally, using once again Proposition 3.4 (a),

$$\mathcal{T}_{1,3} \le 2C_{17}C_{74}C'_{\text{a}}\delta_N(\log N)^{-7}\log\left(1+\frac{T}{\delta_N}\right)X_0^N(1),$$

so that this term is also negligible compared to $\mathcal{T}_{1,2}$. From (75) and the above estimates we deduce

$$\mathcal{T}_1 \le C_{77}\delta_N\log\left(1+\frac{T}{\delta_N}\right)X_0^N(1). \tag{77}$$

We now turn to bound $\mathcal{T}_2$. By summing over z in (32), and then using (17) and Lemma 5.1 (a), we easily see that

$$\mathcal{T}_2 \le \frac{C_{5.1}C_{17}(\log N)^4}{N^2}\int_0^t X_s^N(1)\frac{1}{t-s+\delta_N}ds$$

Therefore, using Proposition 3.4 (a) we obtain

$$\mathscr{T}_2 \le \frac{C_{78}(\log N)^4}{N^2}\log\left(1+\frac{T}{\delta_N}\right)X_0^N(1), \tag{78}$$

which is negligible compared to the right-hand side of (77) as $N\to\infty$, by our assumption that $\liminf_{N\to\infty}\sqrt{N}\delta_N>0$.

We turn to bound $\mathscr{T}_4$, which comes from the new drift term. We have

$$\begin{aligned}
\mathscr{T}_4 \le\ & 2\frac{\delta_N}{N}\sum_{z\in S_N}E\left[\left(\int_{u_N}^{t\vee u_N}\left(d_s^{N,3}(\phi_s^z)-E[d_s^{N,3}(\phi_s^z)\mid\mathscr{F}_{s-u_N}]\right)ds\right)^2\right]\\
&+2\frac{\delta_N}{N}\sum_{z\in S_N}E\left[\left(\int_{u_N}^{t\vee u_N}E[d_s^{N,3}(\phi_s^z)\mid\mathscr{F}_{s-u_N}]\,ds\right)^2\right]\\
&+2E\Bigg[\int_0^{u_N}\int_0^{s_1}\frac{\delta_N(\log N)^8}{N^2}\sum_{x_1\in S_N,x_2\in S_N}p^N_{2(t+\delta_N)-s_1-s_2}(x_2-x_1)\\
&\qquad\times\prod_{i=1}^2 ds_i\left[\xi^N_{s_i}(x_i)f_0^N(x_i,\xi^N_{s_i})^2-(1-\xi^N_{s_i}(x_i))f_1^N(x_i,\xi^N_{s_i})^2\right]\Bigg]\\
=:\ & \mathscr{T}_{4,1}+\mathscr{T}_{4,2}+\mathscr{T}_{4,3}.
\end{aligned}$$

We first handle $\mathscr{T}_{4,1}$. Note that if $s_2-u_N>s_1$,

$$E\left[\prod_{i=1}^2\left(d_{s_i}^{N,3}(\phi_{s_i}^z)-E\left[d_{s_i}^{N,3}(\phi_{s_i}^z)\mid\mathscr{F}_{s_i-u_N}\right]\right)\right]=0,$$

therefore

$$\mathscr{T}_{4,1}=4\frac{\delta_N}{N}\sum_{z\in S_N}E\Big[\int_{u_N}^{t\vee u_N}\int_{s_1}^{s_1+u_N}ds_1ds_2\prod_{i=1}^2\left(d_{s_i}^{N,3}(\phi_{s_i}^z)-E\left[d_{s_i}^{N,3}(\phi_{s_i}^z)\mid\mathscr{F}_{s_i-u_N}\right]\right)\Big].$$

We have the evident bound on $d^{N,3}$:

$$|d_s^{N,3}(\phi_s^z)|\le\frac{(\log N)^4}{N}\sum_{x\in S_N}\phi_s^z(x)\Xi_s^N(x). \tag{79}$$

where we wrote $\Xi_s^N(x):=\xi_s^N(x)+\sum_{e\in S_N}p_N(e)\xi_s^N(x+e)$. Therefore, using Chapman-Kolmogorov once again, and then (17), we find

$$\begin{aligned}
\mathscr{T}_{4,1}\ \le\ & 4C_{17}\delta_N E\Bigg[\int_{u_N}^{t\vee u_N}\int_{s_1}^{s_1+u_N}\frac{(\log N)^8}{N^2}\sum_{x_1,x_2}(2(t+\delta_N)-s_1-s_2)^{-1}\\
&\times\prod_{i=1}^2\left(\Xi_{s_i}^N(x_i)+E\left[\Xi_{s_i}^N(x_i)\mid\mathscr{F}_{s_i-u_N}\right]\right)\Bigg]ds_1ds_2.
\end{aligned} \tag{80}$$

Using the Markov property and Proposition 3.4 (a) it comes easily that

$$\frac{(\log N)}{N}\sum_{x_i\in S_N} E\left[\Xi^N_{s_i}(x_i)\mid \mathscr{F}_{s_i-u_N}\right]\le 2C'_{\text{a}}X^N_{s_i-u_N}(1).$$

We may therefore expand the product in (80) and bound each of the terms using the Markov property and Proposition 3.4. It follows that there exist constants C_{81}, C'_{81} depending on T such that, if $t\ge u_N$,

$$\begin{aligned}
\mathscr{T}_{4,1} &\le C_{81}\delta_N(\log N)^6(X^N_0(1)+X^N_0(1)^2)\int_{u_N}^{t}\int_{s_1}^{s_1+u_N} ds_1ds_2(2(t+\delta_N)-s_1-s_2)^{-1}\\
&\le C_{81}\delta_N u_N(\log N)^6(X^N_0(1)+X^N_0(1)^2)\int_{u_N}^{t} ds_1(2(t+\delta_N)-2s_1-u_N)^{-1}\\
&\le C_{81}u_N^{2/3}(\log N)^6\delta_N(X^N_0(1)+X^N_0(1)^2)\delta_N^{1/3}\log\left(\frac{2t+2\delta_N-3u_N}{2\delta_N-u_N}\right)\\
&\le C_{81}(\log N)^{-1}\delta_N(X^N_0(1)+X^N_0(1)^2)\delta_N^{1/3}\log\left(1+\frac{2(t-u_N)}{\delta_N}\right)\\
&\le C'_{81}(\log N)^{-1}\delta_N(X^N_0(1)+X^N_0(1)^2), \qquad (81)
\end{aligned}$$

where we used that $u_N\le(\delta_N/2)\wedge(\log N)^{-11}$ in the third line above, and the assumption $\delta_N\to 0$ in the last.

We now turn to the more difficult bound on $\mathscr{T}_{4,2}$. Recall the notation $\hat H^N(\xi^N_{s-u_N},x,u_N)$ from Subsection 3.2, and $H^N(s,x,u_N)$ from Subsection 4.1. Using Chapman-Kolmogorov again, we see that

$$\mathscr{T}_{4,2}=E\left[2\int_{u_N}^{t}\int_{u_N}^{s_1} ds_1ds_2\frac{(\log N)^8\delta_N}{N^2}\sum_{x_1,x_2\in S_N}p^N_{2(t+\delta_N)-s_1-s_2}(x_2-x_1)\prod_{i=1}^{2}H^N(s_i,x_i,u_N)\right]. \qquad (82)$$

However, by (25),

$$\left|H^N(s,x,u_N)-\hat H^N(\xi^N_{s-u_N},x,u_N)\right|\le \hat E\left[\mathbf{1}_{\{\mathscr{E}_{x,x+e_1,x+e_2}\}}\sum_{y\in\tilde B^{x,x+e_1}_{u_N}}\xi^N_{s-u_N}(y)\right],$$

and Lemma 2.2 thus implies

$$\frac{(\log N)}{N}\sum_{x\in S_N}\left|H^N(s,x,u_N)-\hat H^N(\xi^N_{s-u_N},x,u_N)\right|\le C_{83}X^N_{s-u_N}(1)u_N(\log N)^3. \qquad (83)$$

From Proposition 3.4 (a) and (50), we see that

$$(\log N)N^{-1}\sum_{x\in S_N}|H^N(s,x,u_N)|\le C'_{\text{a}}X^N_{s-u_N}(1).$$

Using this, (17) and (83), we deduce

$$E\left[2\int_{u_N}^{t}\int_{u_N}^{s_1} ds_2 ds_1 \frac{(\log N)^8\delta_N}{N^2}\sum_{x_1,x_2\in S_N} p^N_{2(t+\delta_N)-s_1-s_2}(x_2-x_1)|H^N(s_1,x_1,u_N)| \times\left|H^N(s_2,x_2,u_N)-\hat{H}^N(\xi^N_{s_2-u_N},x_2,u_N)\right|\right]$$
$$\le 2C_{17}C_{83}C'_{\rm a}(\log N)^9 u_N\delta_N\int_{u_N}^{t}\int_{u_N}^{s_1} ds_2 ds_1(2(t+\delta_N)-s_1-s_2)^{-1}E[X^N_{s_1-u_N}(1)X^N_{s_2-u_N}(1)].$$

Therefore by Proposition 3.4 and the fact that $(\log N)^9 u_N \le (\log N)^{-2}$, the above is bounded by

$$2C_{17}C_{83}C'_{\rm a}C_{\rm ab}(\log N)^{-2}\delta_N(X^N_0(1)+X^N_0(1)^2).$$

Similarly,

$$\frac{(\log N)}{N}\sum_{x\in S_N}|\hat{H}(\xi^N_{s-u_N},x,u_N)| \le \frac{(\log N)}{N}\sum_{x\in S_N}\hat{E}\left[\xi^N_{s-u_N}(\hat{B}^x_{u_N})+\xi^N_{s-u_N}(\hat{B}^{x+e_1}_{u_N})\right] \le 2X^N_{s-u_N}(1),$$

so that, using (17) and (83),

$$E\left[2\int_{u_N}^{t}\int_{u_N}^{s_1} ds_1 2ds_1 \frac{(\log N)^8\delta_N}{N^2}\sum_{x_1,x_2\in S_N} p^N_{2(t+\delta_N)-s_1-s_2}(x_2-x_1)|\hat{H}^N(\xi^N_{s_2-u_N},x_2,u_N)| \times\left|H^N(s_1,x_1,u_N)-\hat{H}^N(s_1,x_1,u_N)\right|\right]$$
$$\le 2C'_{\rm a}C_{17}C_{83}(\log N)^9 u_N\delta_N\int_{u_N}^{t}\int_{u_N}^{s_1} ds_2 ds_1(2(t+\delta_N)-s_1-s_2)^{-1}E[X^N_{s_1-u_N}(1)X^N_{s_2-u_N}(1)]$$
$$\le C(T)(\log N)^{-2}\delta_N(X^N_0(1)+X^N_0(1)^2).$$

Therefore, we see from (82) and the above bounds, for $C_{84}=C_{84}(T)$, that

$$\mathcal{T}_{4,2} \le C_{84}(X^N_0(1)+X^N_0(1)^2)(\log N)^{-2}\delta_N + E\left[\int_{u_N}^{t}\int_{u_N}^{s_1}\frac{(\log N)^8\delta_N}{N^2}\sum_{x_1,x_2\in S_N} p^N_{2(\delta_N+t)-s_1-s_2}(x_2-x_1)\prod_{i=1}^{2} ds_i\hat{H}^N(\xi^N_{s_i-u_N},x_i,u_N)\right]. \tag{84}$$

Recall from Subsection 3.2 that $\hat{H}^N(\xi^N_{s-u_N},x,u_N)=\sum_{i=1}^3 F^N_i(s-u_N,x,u_N)$. First, by translation invariance of p, for $i=1,2$,

$$\begin{aligned}
&\frac{(\log N)}{N}\left|\sum_{x_i\in S_N} p^N_{2(t+\delta_N)-s_1-s_2}(x_2-x_1)F_1^N(s_i-u_N,x_i,u_N)\right| \\
&=\frac{(\log N)}{N}\Bigg|\sum_{w\in S_N}\xi^N_{s_i-u_N}(w)\sum_{x_i\in S_N} p^N_{2(t+\delta_N)-s_1-s_2}(x_2-x_1) \\
&\qquad\times\left[\hat{P}(\hat{B}^0_{u_N}=w-x_i,\{0\mid e_1\sim e_2\}_{u_N})-\hat{P}(\hat{B}^{e_1}_{u_N}=w-x_i,\{0\mid e_1\sim e_2\}_{u_N})\right]\Bigg| \\
&\le \frac{(\log N)}{N}\sum_{w\in S_N}\xi^N_{s_i-u_N}(w)\hat{E}\Bigg[\mathbf{1}_{\{0|e_1\sim e_2\}_{u_N}}\Bigg|p^N_{2(t+\delta_N)-s_1-s_2}(w-\hat{B}^0_{u_N}-x_{3-i}) \\
&\qquad\qquad -p^N_{2(t+\delta_N)-s_1-s_2}(w-\hat{B}^{e_1}_{u_N}-x_{3-i})\Bigg|\Bigg]
\end{aligned}$$

Hence, by Lemma 2.1, for $i=1,2$,

$$\begin{aligned}
&\frac{(\log N)}{N}\left|\sum_{x_i\in S_N} p^N_{2(t+\delta_N)-s_1-s_2}(x_2-x_1)F_1^N(s_i-u_N,x_i,u_N)\right| \\
&\le C_{2.1}\frac{(\log N)}{N}\sum_{w\in S_N}\xi^N_{s_i-u_N}(w)\hat{E}\left[\left|\hat{B}^0_{u_N}-\hat{B}^{e_1}_{u_N}\right|\right](2(t+\delta_N)-s_1-s_2)^{-3/2} \\
&\le C_{85}\sqrt{u_N}X^N_{s_i-u_N}(1)(2(t+\delta_N)-s_1-s_2)^{-3/2}. \qquad (85)
\end{aligned}$$

Furthermore, using again translation invariance of p and (13),

$$\begin{aligned}
&\frac{(\log N)}{N}\sum_{x_1\in S_N}\left|F_1^N(s_1-u_N,x_1,u_N)\right| \\
\le\;& \frac{(\log N)}{N}\sum_{x_1\in S_N}\hat{E}\left[(\xi^N_{s_1-u_N}(\hat{B}^{x_1}_{u_N})+\xi^N_{s_1-u_N}(\hat{B}^{x_1+e_1}_{u_N}))\mathbf{1}_{\{x_1|x_1+e_1\sim x_1+e_2\}_{u_N}}\right] \\
\le\;& \frac{(\log N)}{N}\sum_{w\in S_N}\xi^N_{s_1-u_N}(w)\Bigg(\sum_{x_1\in S_N}\hat{P}(\hat{B}^0_{u_N}=w-x_1,\{0\mid e_1\sim e_2\}_{u_N}) \\
&\qquad +\hat{P}(\hat{B}^{e_1}_{u_N}=w-x_1,\{0\mid e_1\sim e_2\}_{u_N})\Bigg) \\
\le\;& 2C_{15}X^N_{s_1-u_N}(1)(\log(Nu_N))^{-1}\le C_{86}X^N_{s_1-u_N}(1)(\log N)^{-1}. \qquad (86)
\end{aligned}$$

We next handle $F_2^N(s-u_N,x,u_N)$ and $F_3^N(s-u_N,x,u_N)$ together. Using translation invariance of p

and (17),

$$\begin{aligned}
&\frac{(\log N)}{N}\sum_{x_2\in S_N} p^N_{2(t+\delta_N)-s_1-s_2}(x_2-x_1)(|F^N_2(s_2-u_N,x_2,u_N)|+|F^N_3(s_2-u_N,x_2,u_N)|)\\
&\le \frac{(\log N)}{N}\sum_{x_2\in S_N} p^N_{2(t+\delta_N)-s_1-s_2}(x_2-x_1)\\
&\qquad\qquad \times 9\hat{E}\left[\left(\xi^N_{s_2-u_N}(\hat{B}^{x_2}_{u_N})+\xi^N_{s_2-u_N}(\hat{B}^{x_2+e_1}_{u_N})\right)\mathbf{1}_{\{x_2|x_2+e_1|x_2+e_2\}_{u_N}}\right]\\
&\le 9C_{17}(2(t+\delta_N)-s_1-s_2)^{-1}\frac{(\log N)}{N}\sum_{w\in S_N}\xi^N_{s_2-u_N}(w)\\
&\qquad\qquad \times\sum_{x_2\in S_N}\sum_{i=0}^{1}\hat{P}(\hat{B}^{e_i}_{u_N}=w-x_2,\{0\mid e_1\mid e_2\}_{u_N})\\
&\le C_{87}(\log N)^{-3}(2(t+\delta_N)-s_1-s_2)^{-1}X^N_{s_2-u_N}(1),
\end{aligned} \tag{87}$$

where we used (6) in the last inequality above, and have set $C_{87}:=9C_{17}\bar{K}$, $\bar{K}$ as in (16). For $i=1,2$, a similar argument leads to

$$\frac{(\log N)}{N}\sum_{x_i\in S_N}(|F^N_2(s_i-u_N,x_i,u_N)|+|F^N_3(s_i-u_N,x_i,u_N)|)\le C_{88}(\log N)^{-3}X^N_{s_i-u_N}(1). \tag{88}$$

We now use equation (85), first for $i=2$, then for $i=1$, and then equation (87) to obtain

$$\begin{aligned}
&\frac{(\log N)^8\delta_N}{N^2}\int_{u_N}^{t}\int_{u_N}^{s_1} ds_2\,ds_1\left|E\left[\sum_{x_1,x_2\in S_N} p^N_{2(t+\delta_N)-s_1-s_2}(x_2-x_1)\prod_{i=1}^{2}\left(\sum_{j=1}^{3}F^N_j(s_i-u_N,x_i,u_N)\right)\right]\right|\\
&\le (\log N)^6\delta_N\int_{u_N}^{t}\int_{u_N}^{s_1}E\left[\frac{(\log N)}{N}\sum_{x_1\in S_N}\left(\left|F^N_1(s_1-u_N,x_1,u_N)\right|+\left|\sum_{j=2}^{3}F^N_j(s_1-u_N,x_1,u_N)\right|\right)\right.\\
&\qquad\qquad\left.\times C_{85}\sqrt{u_N}X^N_{s_2-u_N}(1)\right]\left(2(t+\delta_N)-s_1-s_2\right)^{-3/2}ds_2\,ds_1\\
&\quad+(\log N)^6\delta_N\int_{u_N}^{t}\int_{u_N}^{s_1}E\left[\frac{(\log N)}{N}\sum_{x_2\in S_N}\left|\sum_{j=2}^{3}F^N_j(s_2-u_N,x_2,u_N)\right|C_{85}\sqrt{u_N}X^N_{s_1-u_N}(1)\right]\\
&\qquad\qquad\times\left(2(t+\delta_N)-s_1-s_2\right)^{-3/2}ds_2\,ds_1\\
&\quad+C_{87}(\log N)^3\delta_N\int_{u_N}^{t}\int_{u_N}^{s_1}E\left[X^N_{s_2-u_N}(1)\frac{(\log N)}{N}\sum_{x_1\in S_N}\left|\sum_{j=2}^{3}F^N_j(s_1-u_N,x_1,u_N)\right|\right]\\
&\qquad\qquad\times\left(2(t+\delta_N)-s_1-s_2\right)^{-1}ds_2\,ds_1.
\end{aligned}$$

Using (86), (88) it follows from the above inequality that

$$\begin{aligned}
&\frac{(\log N)^8\delta_N}{N^2}\int_{u_N}^t\int_{u_N}^{s_1} ds_1 ds_2\left|E\left[\sum_{x_1,x_2\in S_N} p^N_{2(t+\delta_N)-s_1-s_2}(x_2-x_1)\prod_{i=1}^2\Big(\sum_{j=1}^3 F_j^N(s_i-u_N,x_i,u_N)\Big)\right]\right| \\
&\le C_{85}\left(\frac{C_{86}}{(\log N)}+\frac{2C_{88}}{(\log N)^3}\right)\sqrt{u_N}(\log N)^6\delta_N\int_{u_N}^t\int_{u_N}^{s_1}\frac{E\left[X^N_{s_1-u_N}(1)X^N_{s_2-u_N}(1)\right]}{\left(2(t+\delta_N)-s_1-s_2\right)^{3/2}}ds_2 ds_1 \\
&\quad +C_{87}C_{88}\delta_N\int_{u_N}^t\int_{u_N}^{s_1} E\left[X^N_{s_1-u_N}(1)X^N_{s_2-u_N}(1)\right]\left(2(t+\delta_N)-s_1-s_2\right)^{-1}ds_2 ds_1.
\end{aligned}$$

Using Proposition 3.4 and the subsequent (40) in the above, it then follows from (84) that

$$\begin{aligned}
\mathscr{T}_{4,2} &\le C_{89}(X_0^N(1)+X_0^N(1)^2)\delta_N\left[(\log N)^{-2}+(\log N)^5\sqrt{u_N}+1\right] \\
&\le 2C_{89}(X_0^N(1)+X_0^N(1)^2)\delta_N. \qquad (89)
\end{aligned}$$

The term $\mathscr{T}_{4,3}$ is much easier to handle. Indeed, using (17) and the trivial bound

$$\prod_{i=1}^2\left|\left[\xi^N_{s_i}(x_i)f_0^N(x_i,\xi^N_{s_i})^2-(1-\xi^N_{s_i}(x_i))f_1^N(x_i,\xi^N_{s_i})^2\right]\right|\le\prod_{i=1}^2(\xi^N_{s_i}(x_i)+f_1^N(x,\xi^N_{s_i})),$$

we find

$$\mathscr{T}_{4,3}\le 8C_{17}(\log N)^6\delta_N\int_0^{u_N}\int_0^{s_1}\left(2(t+\delta_N)-s_1-s_2\right)^{-1}E\left[X^N_{s_1}(1)X^N_{s_2}(1)\right]ds_2 ds_1.$$

Using $u_N\le\delta_N/2$, we see that

$$\int_0^{u_N}\int_0^{s_1} ds_2 ds_1(2(t+\delta_N)-s_1-s_2)^{-1}\le u_N\log\left(\frac{2(t+\delta_N)}{2(t+\delta_N-u_N)}\right)\le u_N\log\left(\frac{2(t+\delta_N)}{2t+\delta_N}\right).$$

Thus, using (40) and our choice of u_N,

$$\begin{aligned}
\mathscr{T}_{4,3} &\le 8C_{17}C_{\text{ab}}(\log N)^6\delta_N(X_0^N(1)+X_0^N(1)^2)u_N\log\left(\frac{2t+2\delta_N}{2t+\delta_N}\right) \\
&\le C_{90}(\log N)^{-5}\delta_N(X_0^N(1)+X_0^N(1)^2). \qquad (90)
\end{aligned}$$

Recall $\mathscr{T}_4=\sum_{i=1}^3\mathscr{T}_{4,i}$. By grouping (81), (89), (90), we finally obtain

$$\mathscr{T}_4\le C(T)\delta_N(X_0^N(1)+X_0^N(1)^2). \qquad (91)$$

We finish by providing an upper bound for $\mathscr{T}_3$. We give less details, because the method is very similar to the one we used for $\mathscr{T}_4$, and the smaller power of $(\log N)$ makes this term easier to handle. As we did for $\mathscr{T}_4$, we may bound $\mathscr{T}_3$ by the sum of three terms :

$$\mathcal{T}_3 \le C\frac{\delta_N}{N}\sum_{z\in S_N}\Bigg(E\Bigg[\Bigg(\int_{u_N}^{t\vee u_N}\Big(d_s^{N,2}(\phi_s^z)-E[d_s^{N,2}(\phi_s^z)\mid\mathcal{F}_{s-u_N}]\Big)\,ds\Bigg)^2\Bigg]$$
$$+E\Bigg[\Bigg(\int_{u_N}^{t\vee u_N}E[d_s^{N,2}(\phi_s^z)\mid\mathcal{F}_{s-u_N}]ds\Bigg)^2\Bigg]\Bigg)$$
$$+CE\Bigg[\int_0^{u_N}\int_0^{s_1}\frac{(\log N)^4}{N^2}\delta_N\sum_{x_1\in S_N,x_2\in S_N}p^N_{2(t+\delta_N)-s_1-s_2}(x_2-x_1)\prod_{i=1}^2 ds_i\Xi^N_{s_i}(x_i)\Bigg]$$
$$=:\mathcal{T}_{3,1}+\mathcal{T}_{3,2}+\mathcal{T}_{3,3},$$

where Ξ^N is defined after (79), and we used $f_0^N(x,\xi_s^N)^2\le 1, f_1^N(x,\xi_s^N)^2\le f_1^N(x,\xi_s^N)$ to bound the integral on $[0,u_N]^2$.

As for $\mathcal{T}_{4,1}$, we have

$$\mathcal{T}_{3,1}=C\frac{\delta_N}{N}\sum_{z\in S_N}E\Big[\int_{u_N}^{t\vee u_N}\int_{s_1}^{s_1+u_N}ds_1ds_2\prod_{i=1}^2\Big(d_s^{N,2}(\phi_{s_i}^z)-E\left[d_s^{N,2}(\phi_{s_i}^z)\mid\mathcal{F}_{s_i-u_N}\right]\Big)\Big].$$

We then use the following bound on $d^{N,2}$ (compare (79)):

$$|d_s^{N,2}(\phi_s^z)|\le\overline{\beta}\frac{(\log N)^2}{N}\sum_{x\in S_N}\phi_s^z(x)\Xi_s^N(x).$$

Now we may reason exactly as for $\mathcal{T}_{4,1}$ to obtain (compare (81))

$$\mathcal{T}_{3,1}\le C_{92}(\log N)^{-5}\delta_N(X_0^N(1)+X_0^N(1)^2). \tag{92}$$

Let us now deal with $\mathcal{T}_{3,2}$. For $s\ge u_N$, we let

$$\mathcal{H}^N(s,x,u_N):=E\left[\xi_s^N(x)f_0^N(x,\xi_s^N)+(1-\xi_s^N(x))f_1^N(x,\xi_s^N)\mid\mathcal{F}_{s-u_N}\right],$$
$$\hat{\mathcal{H}}^N(s,x,u_N):=\hat{E}\left[\xi^N_{s-u_N}(\hat{B}^x_{u_N})(1-\xi^N_{s-u_N}(\hat{B}^{x+e_1}_{u_N}))+(1-\xi^N_{s-u_N}(\hat{B}^x_{u_N}))\xi^N_{s-u_N}(\hat{B}^{x+e_1}_{u_N})\right].$$

As for $H,\hat{H}$, we get by the analogues of (25) and Lemma 2.2 that

$$\frac{(\log N)}{N}\sum_{x\in S_N}\left|\mathcal{H}^N(s,x,u_N)-\hat{\mathcal{H}}^N(s,x,u_N)\right|\le C_{93}X^N_{s-u_N}(1)u_N(\log N)^3. \tag{93}$$

Argue as when dealing with $\mathcal{T}_{4,2}$ to get that for some $C(T)>0$,

$$E\Big[2\int_{u_N}^t\int_{u_N}^{s_1}ds_1ds_2\frac{(\log N)^4\delta_N}{N^2}\sum_{x_1,x_2\in S_N}p^N_{2(t+\delta_N)-s_1-s_2}(x_2-x_1)\mathcal{H}^N(s_1,x_1,u_N)$$
$$\times\left|\mathcal{H}^N(s_2,x_2,u_N)-\hat{\mathcal{H}}^N(s_2,x_2,u_N)\right|\Big]$$
$$\le C(T)(\log N)^{-6}\delta_N(X_0^N(1)+X_0^N(1)^2),$$

and

$$E\Big[2\int_{u_N}^{t}\int_{u_N}^{s_1}ds_1ds_2\frac{(\log N)^4\delta_N}{N^2}\sum_{x_1,x_2\in S_N}p^N_{2(t+\delta_N)-s_1-s_2}(x_2-x_1)\hat{\mathscr{H}}^N(s_1,x_1,u_N)$$
$$\times\Big|\mathscr{H}^N(s_2,x_2,u_N)-\hat{\mathscr{H}}^N(s_2,x_2,u_N)\Big|\Big]$$
$$\le C(T)(\log N)^{-6}\delta_N(X_0^N(1)+X_0^N(1)^2),$$

and therefore

$$\mathscr{T}_{3,2}\le C_{94}(X_0^N(1)+X_0^N(1)^2)(\log N)^{-6}\delta_N \tag{94}$$
$$+C_{94}E\left[\int_{u_N}^{t}\int_{u_N}^{s_1}\frac{(\log N)^4\delta_N}{N^2}\sum_{x_1,x_2\in S_N}p^N_{2(\delta_N+t)-s_1-s_2}(x_2-x_1)\prod_{i=1}^{2}ds_i\hat{\mathscr{H}}^N(s_i,x_i,u_N)\right].$$

Then,

$$\begin{aligned}
&\frac{(\log N)^2}{N}\sum_{x\in S_N}\hat{\mathscr{H}}^N(s,x,u_N)\\
=\ &2\frac{(\log N)^2}{N}\sum_{x\in S_N}\hat{E}\left[\xi^N_{s-u_N}(\hat{B}^x_{u_N})(1-\xi^N_{s-u_N}(\hat{B}^{x+e_1}_{u_N}))\mathbf{1}_{\{x|x+e_1\}_{u_N}}\right]\\
\le\ &2\frac{(\log N)^2}{N}\sum_{x,w\in S_N}\xi^N_{s-u_N}(w)\hat{P}(\hat{B}^0_{u_N}=w-x,\{0\mid e_1\}_{u_N})\\
\le\ &2C_{15}\log N(\log(Nu_N))^{-1}X^N_{s-u_N}(1),
\end{aligned}$$

by the definition of C_{15} in (15). We deduce from the above, (94) and (17) that

$$\begin{aligned}
\mathscr{T}_{3,2}\ \le\ &C_{94}(X_0^N(1)+X_0^N(1)^2)(\log N)^{-6}\delta_N\\
&+C_{95}\delta_N E\left[\int_{u_N}^{t}\int_{u_N}^{s_1}X^N_{s_1-u_N}(1)X^N_{s_2-u_N}(1)ds_1ds_2\right]\\
\le\ &C'_{95}(X_0^N(1)+X_0^N(1)^2)\delta_N,
\end{aligned}\tag{95}$$

where we used Proposition 3.4 in the last line.

Finally, using the same method as for bounding $\mathscr{T}_{4,3}$,

$$\mathscr{T}_{3,3}\le C_{96}(\log N)^{-9}\delta_N(X_0^N(1)+X_0^N(1)^2). \tag{96}$$

Grouping (92), (95) and (96), we find

$$\mathscr{T}_3\le C_{97}\delta_N(X_0^N(1)+X_0^N(1)^2). \tag{97}$$

We may now conclude the proof of Proposition 3.10. Indeed, using (73), (77), (78), (91), (97),

$$\begin{aligned}
&E\left[\iint\int\delta_N p^N_{2\delta_N}(y-x)dX^N_t(x)dX^N_t(y)\right]\\
\le\ &C(T)(X_0^N(1)+X_0^N(1)^2)\delta_N\left(1+\log\left(1+\frac{T}{\delta_N}\right)+\frac{1}{t+\delta_N}\right),
\end{aligned}$$

which, as explained in the beginning of the proof, is the desired bound.

5.3 Space-time first moment bound : proof of Lemma 3.5

Let $\Psi : \mathbb{R}^2 \to \mathbb{R}$ be bounded Lipschitz. Recall that P^N is the semigroup of the rate-N random walk on S_N with jump kernel p_N, and introduce the function

$$\Phi(s,x) := P^N_{t-s}\Psi(x)\exp(-c_{3.5}s),$$

where $c_{3.5} := 1 \vee \sigma \vee (\bar{K} + \left(\sup_{N\geq 3}(\beta_0^N - \beta_1^N)^+\right) C_{15})$, and $\bar{K}, C_{15}$ were introduced in Section 2.1. Note that $\Psi(x) = \exp(c_{3.5}t)\Phi(t,x)$.

It is straightforward that $||\Phi|| \leq 2c_{3.5}||\Psi||_{\text{Lip}}$. Indeed, we first obviously have that $||\Phi||_\infty \leq ||\Psi||_\infty$. Furthermore, on the one hand,

$$\begin{aligned}|\Phi(s,x+h) - \Phi(s,x)| &\leq \exp(-c_{3.5}s)E\left[\left|\Psi(x+h+B^0_{t-s}) - \Psi(x+B^0_{t-s})\right|\right]\\ &\leq \exp(-c_{3.5}s)|\Psi|_{\text{Lip}}|h|,\end{aligned}$$

hence $|\Phi|_{\text{Lip}} \leq |\Psi|_{\text{Lip}}$. On the other hand,

$$\begin{aligned}|\Phi(s+h,x) - \Phi(s,x)| &\leq \exp(-c_{3.5}(s+h))E\left[\left|\Psi(x+B^0_{t-s-h}) - \Psi(x+B^0_{t-s})\right|\right]\\ &\qquad + \left|\exp(-c_{3.5}(s+h)) - \exp(-c_{3.5}s)\right| E\left[\left|\Psi(x+B^0_{t-s}\right|\right]\\ &\leq |\Psi|_{\text{Lip}}\sigma\sqrt{h} + c_{3.5}|h|||\Psi||_\infty,\end{aligned}$$

and it follows that

$$|\Phi|_{1/2} \leq \sigma|\Psi|_{\text{Lip}} + c_{3.5}||\Psi||_\infty \leq c_{3.5}||\Psi||_{\text{Lip}}.$$

Use Lemma 3.9 to obtain, for some $\eta, \delta > 0$ and $T \geq s \geq t_N$,

$$\begin{aligned}E[d_s^{N,3}(\Phi)] &\leq (\log N)^3\hat{P}(\{0 \mid e_1 \mid e_2\}_{t_N})E[X^N_{s-t_N}(\Phi(s-t_N,.))] + C_{3.9}||\Phi||(\log N)^{-\delta}E[X^N_{s-t_N}(1)]\\ &\quad + \frac{C_{3.7}||\Phi||_\infty}{t_N(\log N)}E[\mathscr{I}^N_\eta(s-t_N)].\end{aligned}$$

From (61) we can get a similar bound for the expected first drift integrand, $E[d_s^{N,2}(\Phi)]$. Only the first term in the right-hand side above should be replaced with

$$\left(\sup_{N\in\mathbb{N}}(\beta_0^N - \beta_1^N)^+\right) \times (\log N)\hat{P}(\{0 \mid e_1 \sim e_2\}_{t_N})E[X^N_{s-t_N}(\Phi(s-t_N,.))].$$

From our definition of $c_{3.5}$, these two bounds and Proposition 3.4 (a), we deduce that for large enough N,

$$\begin{aligned}E[d_s^{N,3}(\Phi) + d_s^{N,2}(\Phi)] &\leq c_{3.5}E[X^N_{s-t_N}(\Phi(s-t_N,.))] + 2C_{3.9}C'_{\text{a}}c_{3.5}||\Psi||_{\text{Lip}}(\log N)^{-\delta}E[X_0^N(1)]\\ &\quad + \frac{(C_{3.7}+C_{61})||\Phi||_\infty}{t_N(\log N)}E[\mathscr{I}^N_\eta(s-t_N)].\end{aligned}$$

We now integrate and use Corollary 3.11 to conclude that there is a constant C_{98}, depending on T and $||\Psi||_{\text{Lip}}$, such that for $t_N \leq t \leq T$,

$$\begin{aligned}&E\left[\int_{t_N}^t (d_s^{N,3}(\Phi) + d_s^{N,2}(\Phi))ds\right]\\ &\leq c_{3.5}\int_0^t E[X^N_s(\Phi(s,.))]ds + C_{98}(\log N)^{-\delta_2}(X_0^N(1) + X_0^N(1)^2),\end{aligned} \tag{98}$$

where $\delta_2 := \eta/2 \wedge \delta$. On the other hand, when $t \le t_N$, we use (41), (54), and Proposition 3.4 (a) to get

$$E\left[\int_0^t (d_s^{N,3}(\Phi) + d_s^{N,2}(\Phi))ds\right] \le C||\Phi||_\infty (\log N)^3 \int_0^t E[X_s^N(1)]ds \\ \le C'||\Psi||_\infty (\log N)^3 t_N X_0^N(1).$$

Combining the above and (98), we obtain

$$E\left[\int_0^t \left(d_s^{N,3}(\Phi) + d_s^{N,2}(\Phi)\right) ds\right] \\ \le c_{3.5}\int_0^t E[X_s^N(\Phi(s,.))]ds + C_{98}(\log N)^{-\delta_2}(X_0^N(1)+X_0^N(1)^2) \\ +C'||\Psi||_\infty(\log N)^{-16}X_0^N(1). \tag{99}$$

We have chosen Φ so it satisfies $A_N\Phi(s,.)+\dot{\Phi}(s,.) = -c_{3.5}\Phi(s,.)$. Therefore using the above inequality and (28) we obtain that for any $t \le T$,

$$E[X_t^N(\Phi(t,.))] \le X_0^N(\Phi(0,.)) + (C_{98} + C'||\Psi||_\infty)(\log N)^{-\delta_2}(X_0^N(1) + X_0^N(1)^2),$$

which yields ($C_{3.5}$ may depend on T)

$$E[X_t^N(\Psi)] \le \exp(c_{3.5}t)X_0^N(P_t^N\Psi) + C_{3.5}(\log N)^{-\delta_2}(X_0^N(1) + X_0^N(1)^2),$$

and completes the proof of Lemma 3.5. □

6 Proof of the tightness of the sequence

In this paragraph we establish Lemmas 3.2 and 3.3. We already explained how Proposition 3.1 follows from these two results.

By (28), Lemma 3.2 is a simple consequence of the following.

Lemma 6.1. *For any bounded Lipschitz $\Phi : \mathbb{R}^2 \to \mathbb{R}$,*

(a) *the sequences $(D^{N,2}(\Phi))_{N\in\mathbb{N}}, (D^{N,3}(\Phi))_{N\in\mathbb{N}}$ are C-tight,*

(b) *if in addition Φ is in $C_b^3(\mathbb{R}^2)$, the sequence $(D^{N,1}(\Phi))_N$ is C-tight,*

(c) *the sequence $(M^N(\Phi))_{N\in\mathbb{N}}$ is C-tight.*

Proof **(a)** Plugging in Proposition 3.4 (b) into equation (69), we deduce that for $T \ge t_2 > t_1 \ge t_N$, $i = 2, 3$,

$$E\left[\left(D_{t_2}^{N,i}(\Phi) - D_{t_1}^{N,i}(\Phi)\right)^2\right] \\ \le C_{69}C_b||\Phi||_{\text{Lip}}^2(X_0^N(1)^2 + X_0^N(1))\Big(((t_2-t_1)^2 + (\log N)^6(t_N \wedge (t_2-t_1))(t_2-t_1)\Big).$$

Recall $t_N = (\log N)^{-19}$ so that,

$$(\log N)^6(t_N \wedge (t_2 - t_1)) \le (\log N)^6 \sqrt{t_N}\sqrt{t_2 - t_1} \le \sqrt{t_2 - t_1}.$$

It follows that for $t_N \le t_1 \le t_2 \le T$, $i = 2, 3$,

$$\begin{aligned} & E\left[\left(D^{N,i}_{t_2}(\Phi) - D^{N,i}_{t_1}(\Phi)\right)^2\right] \\ \le\ & C_{100}||\Phi||^2_{\text{Lip}}(X^N_0(1)^2 + X^N_0(1))(t_2 - t_1)^{3/2}. \end{aligned} \tag{100}$$

Moreover,

$$\begin{aligned} E\left[\int_0^{t_N} |d^{N,i}_s(\Phi)|ds\right] &\le (\overline{\beta} \vee 1)(\log N)^3||\Phi||_\infty E\left[\int_0^{t_N} X^N_s(1)ds\right] \\ &\le C_{101}t_N(\log N)^3||\Phi||_\infty X^N_0(1) \underset{N\to\infty}{\longrightarrow} 0, \end{aligned} \tag{101}$$

where we used Proposition 3.4 (a) in the last line. Lemma 6.1 (a) thus follows from (100), (101) and Kolmogorov's criterion (see Theorem 12.4 in [1]). Here note that for a sequence of $\mathbb{R}$-valued processes $\{Y^N\}$, if $\{Y^N_{\cdot\vee t_N}\}$ is C-tight and $\sup_{t\le t_N}|Y^N_t| \to 0$ in probability, then $\{Y^N\}$ is C-tight.

(b) By Lemma 2.6 of [3] we have

$$\sup_N ||A_N\Phi||_\infty \le C_{102}(\Phi).$$

Therefore, using Cauchy-Schwarz, we obtain that for $0 \le t_1 < t_2 \le T$,

$$E\left[\left(\int_{t_1}^{t_2} X^N_s(A_N\Phi)ds\right)^2\right] \le C_{102}(\Phi)(t_2 - t_1)E\left[\int_{t_1}^{t_2} X^N_s(1)^2 ds\right], \tag{102}$$

and the conclusion follows from Proposition 3.4 and Kolmogorov's criterion (see Theorem 12.4 in [1]).

(c) We now turn to the martingale term. As in the proof of Proposition 3.7 of [5], it is enough to show that for any bounded Lipschitz Φ, $\{\langle M^N(\Phi)\rangle\}_N$ is C-tight.
Let $t_N \le t_1 \le t_2 \le T$. Using Lemma 5.1 along with (30) and (63), we obtain

$$\begin{aligned} E[\left(\langle M^N(\Phi)\rangle_{t_2} - \langle M^N(\Phi)\rangle_{t_1}\right)^2] &\le 3E\left[\left(\int_{t_1}^{t_2} m^N_{1,s}(\Phi)ds\right)^2\right] + 3E\left[\left(\int_{t_1}^{t_2} m^N_{2,s}(\Phi)ds\right)^2\right] \\ &\quad +12E\left[\left(\int_{t_1}^{t_2} X^N_s((\log N)\Phi^2 f^N_0(\xi^N_s))ds\right)^2\right] \\ &\le C_{103}\frac{||\Phi||^4_{\text{Lip}}(\log N)^2}{N}(X^N_0(1) + X^N_0(1)^2)(t_2 - t_1)^2 \\ &\quad +12||\Phi||^4_\infty E\left[\left(\int_{t_1}^{t_2} X^N_s((\log N)f^N_0(\xi^N_s))ds\right)^2\right]. \end{aligned} \tag{103}$$

To bound the last term in the right-hand side above, we proceed in a very similar fashion as when dealing with the drift terms. More precisely, writing $Y(s) := X_s^N((\log N)f_0^N(\xi_s^N))$, we obtain

$$
\begin{aligned}
E\left[\left(\int_{t_1}^{t_2} X_s^N((\log N)f_0^N(\xi_s^N))ds\right)^2\right] &:= E\left[\left(\int_{t_1}^{t_2} Y(s)ds\right)^2\right] \\
\leq\quad E\left[4\int_{t_1}^{t_2}\int_{s_1}^{(s_1+t_N)\wedge t_2} ds_2 ds_1 \prod_{i=1}^{2}\Big(Y(s_i)-E[Y(s_i)\mid \mathscr{F}_{s_i-t_N}]\Big)\right] \\
+2E\left[\left(\int_{t_1}^{t_2} E[Y(s)\mid \mathscr{F}_{s-t_N}]ds\right)^2\right]. \qquad (104)
\end{aligned}
$$

We may proceed as in the proof of (100). First, use the fact that $Y(s)\leq(\log N)X_s^N(1)$ and Proposition 3.4 to get

$$
E\left[4\int_{t_1}^{t_2}\int_{s_1}^{(s_1+t_N)\wedge t_2} ds_2\, ds_1 \prod_{i=1}^{2}\Big(Y(s_i)-E[Y(s_i)\mid \mathscr{F}_{s_i-t_N}]\Big)\right]\leq C_{105}(X_0^N(1)+X_0^N(1)^2)(t_2-t_1)^{3/2}. \quad (105)
$$

Then, use Cauchy-Schwarz, Lemma 5.2 and Proposition 3.4 to see that

$$
2E\left[\left(\int_{t_1}^{t_2} E[Y(s)\mid \mathscr{F}_{s-t_N}]ds\right)^2\right]\leq C_{106}(X_0^N(1)+X_0^N(1)^2)(t_1-t_2)^2. \qquad (106)
$$

Finally, it is straightforward that $\langle M^N(\Phi)\rangle_{t_N}\to 0$ in probability as $N\to\infty$. Therefore by the Kolmogorov criterion, $\{\langle M^N(\Phi)\rangle\}_N$ is C-tight, and the proof of Lemma 6.1 is complete. □

In order to finish the proof of Proposition 3.1, it remains to establish the compact containment condition Lemma 3.3.

Let $T>0$ and $\left\{h_n:\mathbb{R}^2\to\mathbb{R}, n\geq 0\right\}$ be bounded Lipschitz functions such that

$$
\mathbf{1}_{\{|x|>n+1\}}\leq h_n(x)\leq \mathbf{1}_{\{|x|>n\}}, \quad \sup_{n\in\mathbb{N}}||h_n||_{\mathrm{Lip}}\leq C_{107}. \qquad (107)
$$

Our goal is to show that

$$
\lim_{n\to\infty}\sup_{N\in\mathbb{N}} P(\sup_{t\leq T} X_T^N(h_n)\geq\varepsilon)=0 \text{ for all } \varepsilon>0. \qquad (108)
$$

By (28),

$$
\sup_{t\leq T} X_t^N(h_n)\leq \sup_{t\leq T} M_t^N(h_n)+\sup_{t\leq T} Y_t^N(h_n), \qquad (109)
$$

where

$$
Y_t^N(h_n):=X_0^N(h_n)+\int_0^t X_s^N(A_N h_n)ds+D_t^{N,2}(h_n)+D_t^{N,3}(h_n)=:X_0^N(h_n)+\int_0^t y_s^N(h_n)ds.
$$

We let $\{\epsilon_N\}$ and $\{\eta_n\}$ denote sequences converging to 0, which may depend on T and may change from line to line. Let us first handle the martingale part. By Lemma 5.1 and Proposition 3.4 (a), we deduce that

$$E(\langle M^N(h_n)\rangle_T) \le E\left[\int_0^T X_s^N\left(2(\log N)h_n^2 f_0^N(\xi_s^N)\right)ds\right] + \epsilon_N.$$

We also may bound the first term in the above by

$$2\left\{E\left[\int_0^{t_N} X_s^N\left(h_n^2(\log N)f_0^N(\xi_s^N)\right)ds\right] + E\left[\int_{t_N}^T E\left[X_s^N\left(h_n^2(\log N)f_0^N(\xi_s^N)\right) \mid \mathscr{F}_{s-t_N}^N\right]ds\right]\right\}$$
$$\le 2\Big\{C_{\rm a}(\log N)^{-18}X_0^N(1) + \frac{C_{5.2}\|h_n^2\|_{\rm Lip}}{(\log N)^8}\int_{t_N}^T E[X_{s-t_N}^N(1)]ds + C_{5.2}\int_{t_N}^T E[X_{s-t_N}^N(h_n^2)]ds\Big\},$$

where we used Lemma 5.2 and the Markov property to bound the second term of the first line above. Now use Proposition 3.4 (a), Lemma 3.5, and the convergence of $\{X_0^N\}$ to deduce there exists C_{110}, depending on T, such that

$$E(\langle M^N(h_n)\rangle_T) \le \epsilon_N + C_{110}\int_0^T X_0^N(P_s^N(h_n^2))ds. \tag{110}$$

The tightness of $\{X_0^N\}$ shows that

$$\lim_{n\to\infty}\sup_N \sup_{t\le T} X_0^N(P_t^N(h_n^2)) = 0. \tag{111}$$

It follows from (110), (111), and Doob's strong L^2 inequality that

$$\lim_{n\to\infty}\sup_N E(\sup_{t\le T} M_t^N(h_n)^2) = 0. \tag{112}$$

Consider now the other terms of the sum in (109).

Claim 6.2. *Let $T > 0$, and Φ be bounded Lipschitz on $\mathbb{R}^2$ such that $\|\Phi\|_{\rm Lip} \le C_{107}$. There exists $N_0(\omega, N) \in \mathbb{N}$ and a positive constant $C_{6.2} \ge 1$ depending only on T, C_{107} and $\sup_N X_0^N(1)$ such that*

- $\forall t_1, t_2 \in [t_N, T], \quad |t_1 - t_2| \le 2^{-N_0(\omega,N)} \Rightarrow \left|Y_{t_1}^N(\Phi) - Y_{t_2}^N(\Phi)\right| \le |t_1 - t_2|^{1/8}$
- $\sup_N E[2^{N_0(\omega,N)/8}] \le C_{6.2}$.

Proof of Claim. By equations (100) and (102), there is a constant C_{113} depending only on T, C_{107} and $\sup_N X_0^N(1)$ such that for any N, for any $t_N \le t_1 \le t_2 \le T$,

$$E[\left|Y_{t_1}^N(\Phi) - Y_{t_2}^N(\Phi)\right|^2] \le C_{113}|t_2 - t_1|^{3/2}. \tag{113}$$

The result follows from the usual dyadic expansion proof of Kolmogorov's criterion. □

We now return to the compact containment proof. We have

$$\begin{aligned}
&E(\mathbf{1}_{\{N_0<M\}}\sup_{t\le T}|Y_t^N(h_n)|)\\
&\le X_0^N(h_n)+E\left[\sup_{t\le t_N}|Y_t^N(h_n)-X_0^N(h_n)|\right]+E\left(\max_{t_N\le j2^{-M}\le T}|Y_{j2^{-M}}^N(h_n)|\right)\\
&\quad+E\left(\mathbf{1}_{\{N_0<M\}}\max_{t_N\le j2^{-M}\le T}\sup_{s\in[j2^{-M},(j+1)2^{-M}]}|Y_s^N(h_n)-Y_{j2^{-M}}^N(h_n)|\right)\\
&\le X_0^N(h_n)+E\left[\int_0^{t_N}|y_s^N(h_n)|ds\right]+2^M T\max_{0<j2^{-M}\le T}E[|Y_{j2^{-M}}^N(h_n)|]+2^{-M/8},
\end{aligned}$$

where, to obtain the last inequality, we used Claim 6.2 to bound the last term. Using (101), and the corresponding easier bound on $D_t^{N,1}(h_n)$ we see that the second term in the above sum will be bounded, uniformly in n, by ϵ_N. The tightness of $\{X_0^N\}$ bounds the first term, uniformly in N, by η_n. Next use

$$|Y_t^N(h_n)|\le|M_t^N(h_n)|+X_t^N(h_n),$$

(112), and Lemma 3.5 to see that

$$\max_{0<j2^{-M}\le T}E[|Y_{j2^{-M}}^N(h_n)|]\le\eta_n+\epsilon_N+e^{c_{3.5}T}\sup_{t\le T}X_0^N(P_t^N(h_n)).$$

An application of (111) shows that the last term above may also be bounded uniformly in N by η_n and so combining the above bounds gives us

$$E(\mathbf{1}_{\{N_0<M\}}\sup_{t\le T}|Y_t^N(h_n)|)\le(\eta_n+\epsilon_N)(1+2^M)+2^{-M/8}, \tag{114}$$

and from Claim 6.2 and Markov's inequality we have

$$P(N_0\ge M)\le C_{6.2}2^{-M/8}.$$

The above two bounds easily imply

$$\lim_{n\to\infty}\sup_N P(\sup_{t\le T}|Y_t^N(h_n)|>\varepsilon)=0\quad\text{for all }\varepsilon>0. \tag{115}$$

To see this fix $\varepsilon,\delta>0$ and then choose M so that $P(N_0\ge M)<\delta/2$ (by the above bound). For n large enough and $N\ge N_1$ the upper bound in (114) is at most $\varepsilon\delta/2$. It then follows that for n large enough,

$$\sup_{N\ge N_1}P(\sup_{t\le T}|Y_t^N(h_n)|>\varepsilon)<\delta$$

and (115) follows since the limit in (115) is trivially 0 for each fixed N. Finally, use (115) and (112) in (109) to complete the derivation of (108) and hence Lemma 3.3. □

7 Identifying the limit

In the previous section we established that the sequence $(X^N)_{N\in\mathbb{N}}$ is C-tight in the space $D(\mathbb{R}_+,M_F(\mathbb{R}^2))$. It remains to identify possible limit points.

Using the Skorokhod representation we may suppose without loss of generality that on an appropriate probability space and for an increasing sequence of integers (N_k) we have

$$X^{N_k} \xrightarrow[k\to\infty]{a.s.} X \in C(\mathbb{R}_+, M_F(\mathbb{R}^2)).$$

Furthermore, if we write $\hat{q}_3(N) := (\log N)^3 \hat{P}(\{0|e_1|e_2\}_{t_N})$ we have (cf (6)) that $\lim_{N\to\infty} \hat{q}_3(N) = K$.

Dropping the dependence in k in the notation, we have for $0 \le t \le T$ and $\Phi \in C_b^3(\mathbb{R}^2)$,

$$\begin{aligned}
& E\left[\left|D_t^{N,3}(\Phi) - K\int_0^t X_s^N(\Phi)ds\right|\right] \\
&\le E\left[\left|\int_0^{t_N\wedge t} d_s^{N,3}(\Phi)ds\right|\right] + E\left[\left(\int_{t_N}^{t_N\vee t} d_s^{N,3}(\Phi) - E[d_s^{N,3}(\Phi) \mid \mathscr{F}_{s-t_N}]ds\right)^2\right]^{1/2} \\
&\quad + E\left[\int_{t_N}^{t_N\vee t} \left|E[d_s^{N,3}(\Phi) \mid \mathscr{F}_{s-t_N}] - \hat{q}_3(N) X_{s-t_N}^N(\Phi)\right| ds\right] \\
&\quad + \left|\hat{q}_3(N) - K\right| ||\Phi||_\infty E\left[\int_0^{(t-t_N)^+} X_s^N(1)ds\right] + K||\Phi||_\infty E\left[\int_{(t-t_N)^+}^t X_s^N(1)ds\right],
\end{aligned}$$

where we used Cauchy-Schwarz to get the second term in the sum above. Use (101) to bound the first term in the sum above, (67) and Proposition 3.4 (b) to bound the second, Lemma 3.9, Proposition 3.4 (a) and Corollary 3.11 to handle the third, and finally Proposition 3.4 (a) to deal with the two last. We obtain for some constants C_{116}, C'_{116} depending on T, that for any $t \le T$,

$$\begin{aligned}
& E\left[\left|D_t^{N,3}(\Phi) - K\int_0^t X_s^N(\Phi)ds\right|\right] \\
&\le C_{116}||\Phi||\Big(t_N(\log N)^3 X_0^N(1) + \sqrt{t_N}(\log N)^3 (X_0^N(1) + X_0^N(1)^2)^{1/2} + (\log N)^{-\delta_2} X_0^N(1) \\
&\qquad + C_{3.11}(\log N)^{-\eta/2}(X_0^N(1) + X_0^N(1)^2) + \left|\hat{q}_3(N) - K\right| X_0^N(1) + K t_N X_0^N(1)\Big) \\
&\le C'_{116}||\Phi||(X_0^N(1) + X_0^N(1)^2 + 1)\left((\log N)^{-\eta/2} + \left|\hat{q}_3(N) - K\right|\right). \qquad (116)
\end{aligned}$$

The above clearly goes to 0 as $N = N_k$ goes to infinity.

The drift term $D^{N,2}(\Phi)$ is handled in a similar manner; use (61) in place of Lemma 3.9. We find that

$$E\left[\left|D_t^{N,2}(\Phi) - \gamma(\beta_0 - \beta_1)\int_0^t X_s^N(\Phi)ds\right|\right] \underset{k\to\infty}{\longrightarrow} 0.$$

For the term $\int_0^t X_s^N(A_N\Phi)ds$ we have by Lemma 2.6 of [3] :

$$\sup_{s\le T}\left|\left|A_N\Phi - \sigma^2\frac{\Delta\Phi}{2}\right|\right|_\infty \underset{N\to\infty}{\longrightarrow} 0,$$

and therefore

$$E\left[\left|\int_0^t X_s^N(A_N\Phi)ds - \int_0^t X_s\left(\sigma^2\frac{\Delta\Phi}{2}\right)ds\right|\right] \underset{k\to\infty}{\longrightarrow} 0.$$

It finally remains to deal with the predictable square function of the martingale term. By Lemma 5.1 there is an $\epsilon_N = \epsilon_N(t) \to 0$ as $N \to \infty$ so that

$$\begin{aligned}&E(|\langle M^N(\Phi)\rangle_t - \int_0^t 4\pi\sigma^2 X_s^N(\Phi^2)ds|)\\ &\le E\left(\left|2\int_0^t X_s^N(\log N\Phi^2 f_0^N(\xi_s^N)) - 4\pi\sigma^2 X_s^N(\Phi^2)ds\right|\right) + \epsilon_N.\end{aligned}$$

The proof that the first term goes to zero as $N \to \infty$ proceeds as in the argument above for $D^{N,2}(\Phi)$. In the analogue of (61) one gets $(\log N)P(\{0|e_1\}_{t_N}) \to 2\pi\sigma^2$ (see (13)) in place of $(\log N)P(\{0|e_1 \sim e_2\}) \to \gamma$. It is now routine to take limits in (28) for Φ as above to obtain the martingale problem (MP) characterizing super-Brownian motion. The details are just as in the proof of Proposition 3.2 of [5]. We have completed the proof of Theorem 1.5. □

8 Proof of Theorems 1.4, 1.2.

8.1 Outline of the proof of survival, Theorem 1.4

We proceed as in Section 9 of [7] and we establish analogues of Lemma 9.1 and Proposition 9.2 of that reference. Given these results, the proof of Theorem 4.1 of [6] then goes through to give Theorem 1.4 just as before.

The stochastic equation for ξ^N and the associated process $\underline{\xi}^N$, killed outside of an open square I', used in Subsection 2.2 was convenient to define the duals but does not give the natural ordering of the two processes. For this we use the equation in Proposition 2.1 of [6]. Let $\{N^{x,i}, x \in S_N, i = 0,1\}$ be independent Poisson point processes on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $Nds \times du$ (Lebesgue measure). $N^{x,i}$ will be used to switch the type at x from i to $1-i$. Let $\{\mathscr{F}_t, t \ge 0\}$ be the natural right-continuous filtration generated by these processes and recall the flip rates in (18). Let ξ_0^N be a deterministic initial condition with $|\xi_0^N| < \infty$. By Proposition 2.1 of [6] there is a unique $(\mathscr{F}_t)$-adapted solution to

$$\xi_t^N(x) = \xi_0^N(x) + \sum_{i=0}^{1}(-1)^{1-i}\int_0^t\int_0^\infty \mathbf{1}\{\xi_{s-}^N(x) = i\}\mathbf{1}\{u \le r_{i\to(1-i)}^N(x,\xi_{s-})\}N^{x,i}(ds,du) \quad (117)$$

for all $t \ge 0$, $x \in S_N$ which is the unique Feller process associated with the rates in (18). $\underline{\xi}^N$ is defined in the same filtration and still satisfies equation (117) for all $t \ge 0$, $x \in I'$, but satisfies $\underline{\xi}_t^N(x) = 0$ for all $t \ge 0, x \in S_N \setminus I'$. It follows from the monotonicity of $LV(\alpha_0,\alpha_1)$ for $\alpha_i \ge 1/2$ (see Section 1 of [6]) and (12) that we may apply Proposition 2.1(b)(i) of [6] to conclude that

$$\text{if } \underline{\xi}_0^N \le \xi_0^N \text{ then } \underline{\xi}_t^N \le \xi_t^N \ \forall t \ge 0 \ \text{ almost surely.} \quad (118)$$

Let $K_0 > 2, L > 3, I = [-L, L]^2$ and $I' = (-K_0 L, K_0 L)^2$. Assume $\{\xi_t^N(x), x \in S_N, t \geq 0\}$ and $\{\underline{\xi}_t^N(x), x \in S_N, t \geq 0\}$ are as above with $\underline{\xi}_0^N(x) = \xi_0^N(x)$ if $x \in S_N \cap I'$, and $\underline{\xi}_0^N(x) = 0$ otherwise. Therefore by the above $\underline{\xi}_t^N \leq \xi_t^N$ for all $t \geq 0$ a.s. We still write P^α for the joint distribution of $(\xi^N, \underline{\xi}^N)$. We let $B^{x,N}$ be a rate-v_N continuous time random walk with step distribution p_N, started at x. If $T'_x = \inf\{t \geq 0 : B_t^{x,N} \notin I'\}$ then define

$$\underline{B}_t^{x,N} = \begin{cases} B_t^{x,N} & \text{if } t < T'_x \\ \Delta & \text{if } t \geq T'_x. \end{cases}$$

The associated semigroup is denoted $\left\{\underline{P}_t^N, t \geq 0\right\}$. We will often drop the dependence in N from these notations. We define the measure-valued process $X_t^N \in M_F(\mathbb{R}^2)$ as before, and $\underline{X}_t^N \leq X_t^N$ by

$$\underline{X}_t^N = \frac{(\log N)}{N} \sum_{x \in S_N} \underline{\xi}_t^N(x) \delta_x.$$

Although we are now using a different stochastic equation to couple ξ^N and $\underline{\xi}^N$ than those in Section 2.2, we may still use the distributional results derived in Section 2.2 for each separately, such as (25) and (27), since the individual laws of ξ^N and $\underline{\xi}^N$ remain unchanged.

Here is the version of Lemma 9.1 of [7] we will need.

Lemma 8.1. *Let $T > 0$. There is a $C_{8.1}$, depending on $T, \overline{\beta}$, and a universal constant $\delta_{8.1}$, such that if $X_0^N = \underline{X}_0^N$ is supported on I, then for $N > 1, K_0 > 2$, $L > 3$, and $0 \leq t \leq T$,*

$$E\left(X_t^N(1) - \underline{X}_t^N(1)\right) \leq C_{8.1} X_0^N(1) \left[P(\sup_{s \leq t} |B_s^0| > (K_0 - 1)L - 3) + (1 \vee X_0^N(1))(\log N)^{-\delta_{8.1}}\right].$$

For $\alpha_0 \in [\frac{1}{2}, 1)$, a bit of calculus shows there exists a unique $N = N(\alpha_0) > e^3$ so that

$$1 - \alpha_0 = \frac{(\log N)^3}{N}. \tag{119}$$

Let $I_{-1} = (-2L, 0)e_1 + I, I_1 = (0, 2L)e_1 + I$. Here is the analogue of Proposition 9.2 of [7] we will need. It will imply that the Lotka-Volterra process dominates a $2K_0$-dependent supercritical oriented percolation and therefore, survives. The role of the additional killing for $\underline{X}^N$ is to ensure the finite range of dependence need for the associated oriented percolation. Recall that K is as in (6).

Proposition 8.2. *Assume $0 < \eta < K$. There are $L > 3, K_0 > 2, J \geq 1$ all in $\mathbb{N}$, $T \geq 1$, and $r \in (0, e^{-4})$, all depending on η such that if*

$$\alpha_0 \leq \alpha_1 \leq \alpha_0 + \frac{K - \eta}{\gamma} \frac{1 - \alpha_0}{\left(\log \frac{1}{1-\alpha_0}\right)^2}, \quad 0 < 1 - \alpha_0 < r \text{ and } N = N(\alpha_0),$$

then $\underline{X}_0^N(1) = \underline{X}_0^N(I) \geq J$ implies

$$P^\alpha\left(\underline{X}_T^N(I_1) \wedge \underline{X}_T^N(I_{-1}) \geq J\right) \geq 1 - 6^{-4(2K_0+1)^2}.$$

Remark: The parameters β_0^N, β_1^N are now defined by N, α_0, α_1 and (11). Indeed our choice of $N = N(\alpha_0)$ in the above implies $\beta_0^N = 0$ and

$$\beta_1^N = \frac{N}{(\log N)}(\alpha_1 - \alpha_0) \in \left[0, \frac{K-\eta}{\gamma}\frac{(\log N)^2}{((\log N) - 3(\log(\log N)))^2}\right].$$

Therefore, for $r(\eta)$ small enough we will have

$$\beta_1^N \in \left[0, \frac{K-\eta/2}{\gamma}\right].$$

<u>Proof</u> : Given Lemma 8.1, Proposition 8.2 is proved by making trivial changes to the proof of Proposition 9.2 in [7]. We omit the details but the intuition should be clear. By Theorem 1.5, X^N converges to a super-Brownian motion with drift $\theta = K - \gamma\beta_1$. By the above remark this quantity is bounded below by $\eta/2 > 0$ for a good choice of r. Super-Brownian motion with positive drift and large enough initial mass will continue to grow exponentially up to time T with high probability, and so the same should be true for X^N for N large. Finally, Lemma 8.1 bounds $X_T^N - \underline{X}_T^N$ and allows us to make the same conclusion for $\underline{X}_T^N$. □

To complete the proof of survival it remains to establish Lemma 8.1.

8.2 Proof of Lemma 8.1.

Choose $h : \mathbb{R}^2 \cup \Delta \to [0,1]$ such that

$$[-K_0L+3, K_0L-3]^2 \subseteq \{h=1\} \subseteq \mathrm{Supp}(h) \subseteq [-K_0L+2, K_0L-2]^2, \quad |h|_{\mathrm{Lip}} \leq 1. \tag{120}$$

We then define for $s \leq t, x \in S_N$ the function

$$\Psi(s,x) := \underline{P}_{t-s}^N h(x) = E[h(\underline{B}_{t-s}^x)] = E[h(B^x_{(t-s)\wedge T'_x})].$$

As in the proof of Lemma 3.2 of [6], if $f : S_N \cup \Delta \to \mathbb{R}$ and $\Phi(s,x) = \underline{P}_{t-s}^N f(x), s \leq t$ we have

$$\underline{X}_t^N(f) = \underline{X}_0^N(\underline{P}_t^N f) + \underline{M}_t^N(\Phi) + \int_0^t \left(d_s^{N,2}(\Phi, \underline{\xi}_s^N) + d_s^{N,3}(\Phi, \underline{\xi}_s^N)\right) ds, \tag{121}$$

where $\underline{M}_t^N(\Phi)$ is a square-integrable, mean zero martingale. Apply (28) and (121) to obtain

$$\begin{aligned} E\left[X_t^N(1) - \underline{X}_t^N(1)\right] &\leq E\left[X_t^N(1) - \underline{X}_t^N(h)\right] \\ &= X_0^N(1 - \underline{P}_t^N(h)) + E\left[\int_0^t \sum_{i=2}^3 (d_s^{N,i}(1, \xi_s^N) - d_s^{N,i}(\Psi, \underline{\xi}_s^N))ds\right]. \end{aligned} \tag{122}$$

We now state and prove four intermediate results.

Lemma 8.3. *Let $\{S_n\}$ be a mean zero random walk on $\mathbb{Z}$ starting at $x \in \mathbb{Z}$ under P_x and such that $E_0(|S_1|^3) < \infty$. There is a $C_{8.3} > 0$ such that if*

$$U_M = \inf\{n \geq 0 : S_n \notin (0, M)\}, \quad M \in \mathbb{N},$$

then

$$P_x(S_{U_M} \geq M) \leq \frac{x + C_{8.3}}{M} \text{ for all } x \in \mathbb{Z} \cap [0, M].$$

Proof. Let $p(j) = P_0(|S_1| = j)$. The inequality in the middle of p. 255 of [14] and the inequality P7 on p. 253 of the same reference imply that for x as in the statement, and some $c > 0$,

$$\begin{aligned}
\left|P_x(S_{U_M} \geq M) - \frac{x}{M}\right| &\leq \frac{c}{M}\Big[\sum_{s=0}^{M}(1+s)E_0(|S_1|\mathbf{1}_{\{|S_1|>s\}})\Big] \\
&= \frac{c}{M}\Big[\sum_{j=1}^{M+1}\sum_{k\geq j} jkp(k)\Big] \\
&\leq \frac{c}{M}\Big[\sum_{k=1}^{\infty}\frac{k^2(k+1)}{2}p(k)\Big] \\
&\leq \frac{c}{M}E_0(|S_1|^3).
\end{aligned}$$

The result follows. □

To use Lemma 3.5 and (61), we need the following.

Lemma 8.4. *There exists a constant $C_{8.4}$ such that*

$$|\Psi|_{1/2} \leq \sqrt{2}\sigma, \qquad |\Psi|_{\mathrm{Lip}} \leq C_{8.4}.$$

Proof : The first statement is easy to verify. Indeed, if $s < s' \leq t, x \in S_N$, then using $|h|_{\mathrm{Lip}} \leq 1$, applying the Markov property to B^x at time $t - s'$ and then Cauchy-Schwarz,

$$\begin{aligned}
\left|\Psi(s', x) - \Psi(s, x)\right| &\leq E\left[\left|B^x_{(t-s')\wedge T'_x} - B^x_{(t-s)\wedge T'_x}\right|\mathbf{1}_{\{T'_x > t-s'\}}\right] \\
&\leq \sup_{z \in I' \cap S_N} E\left[\left|B^z_{(s'-s)\wedge T'_z} - z\right|^2\right]^{1/2} \leq \sqrt{2}\sigma|s'-s|^{1/2}.
\end{aligned}$$

Let us now turn to the proof of the second statement of Lemma 8.4. Let $x, x' \in I' \cap S_N$ be such that $0 < |x - x'| \leq 1$. In this argument we couple the walk $B^{x'}$ with B^x by setting $B^{x'}_s := x' - x + B^x_s$. We have

$$\begin{aligned}
&\left|\Psi(t-s, x) - \Psi(t-s, x')\right| = \left|E\left[h(B^x_{s\wedge T'_x}) - h(B^{x'}_{s\wedge T'_{x'}})\right]\right| \\
\leq\ & E\left[\mathbf{1}_{\{T'_x \wedge T'_{x'} \geq s\}}|h(B^x_s) - h(B^{x'}_s)|\right] + E\left[\mathbf{1}_{\{T'_x < s \leq T'_{x'}\}}h(B^{x'}_s) + \mathbf{1}_{\{T'_{x'} < s \leq T'_x\}}h(B^x_s)\right] \\
\leq\ & |x - x'| + E\left[\mathbf{1}_{\{T'_x < s \leq T'_{x'}\}}h(B^{x'}_s)\right] + E\left[\mathbf{1}_{\{T'_{x'} < s \leq T'_x\}}h(B^x_s)\right], \qquad (123)
\end{aligned}$$

where we used $|h|_{\mathrm{Lip}} \leq 1$ to bound the first term in the sum. The other two terms are symmetric, so we may as well only handle the first of the two.

On the event $\{T'_x < s \leq T'_{x'}\}$, since $\left|B^{x'}_{T'_x} - B^x_{T'_x}\right| = |x' - x| \leq 1$, it is easy to see that

$$B^{x'}_{T'_x} \in I', \qquad \inf_{z \in (I')^c}\left|B^{x'}_{T'_x} - z\right| \leq |x - x'|.$$

So one of the components, say i, of $B^{x'}_{T'_x}$ is within $|x - x'|$ of $(-K_0L, K_0L)^c$. Let us assume without loss of generality (by symmetry) that $B^{x',i}_{T'_x} \in [K_0L - |x - x'|, K_0L]$, and let $M_u = B^{x',i}_{T'_x+u}$. From (120), for $h(B_s^{x'})$ to be positive we need that M hits $(-\infty, K_0L-2]$ before $[K_0L, \infty)$. Therefore if we set $\nu := \inf\{u \geq 0 : M_u \notin (K_0L-2, K_0L)\}$ and S_n is the embedded discrete time rescaled random walk on $\mathbb{Z}$, then

$$\begin{aligned} E\left[\mathbf{1}_{\{T'_x < s \leq T'_{x'}\}} h(B_s^{x'})\right] &\leq P\left(M_\nu \leq K_0L-2 | M_0 \geq K_0L - |x-x'|\right) \\ &\leq P_{\lceil\sqrt{N}|x-x'|\rceil}(S_{U_{\lceil 2\sqrt{N}\rceil}} \geq \lceil 2\sqrt{N}\rceil) \\ &\leq \frac{C_{8.3}}{\sqrt{N}} + |x-x'| \leq (C_{8.3}+1)|x'-x|, \end{aligned} \tag{124}$$

where we used Lemma 8.3 and then $|x'-x| \geq N^{-1/2}$ in the last line. Therefore by (123) and (124) we deduce that for any $x, x' \in S_N \cap I'$ such that $|x-x'| \leq 1$ we have

$$\left|\Psi(s,x) - \Psi(s,x')\right| \leq (2C_{8.3}+3)|x-x'|.$$

Since $\Psi(s,.)$ is supported on I' the second assertion of Lemma 8.4 follows. □

The next intermediate result we need for proving Lemma 8.1 is a version of Lemma 3.6 for the killed process. Let

$$\begin{aligned} \underline{\hat{H}}(\underline{\xi}^N_{s-t_N}, x, t_N) := \hat{E}&\left[\xi^N_{s-t_N}(\underline{\hat{B}}^x_{t_N}) \prod_{i=1}^{2}(1 - \underline{\xi}^N_{s-t_N}(\underline{\hat{B}}^{x+e_i}_{t_N}))\right] \\ &- \hat{E}\left[(1-\underline{\xi}^N_{s-t_N}(\underline{\hat{B}}^x_{t_N})) \prod_{i=1}^{2} \underline{\xi}^N_{s-t_N}(\underline{\hat{B}}^{x+e_i}_{t_N})\right]. \end{aligned}$$

Lemma 8.5. *There is a $C_{8.5}$ such that for any $s \geq t_N$ and $\Phi : \mathbb{R}^2 \to \mathbb{R}$ bounded measurable,*

$$\begin{aligned} &\left| E\left[d_s^{N,3}(\Phi, \underline{\xi}^N_s) \mid \mathscr{F}_{s-t_N}\right] - \frac{(\log N)^4}{N} \sum_{x \in S_N} \Phi(s-t_N, x) \underline{\hat{H}}(\underline{\xi}^N_{s-t_N}, x, t_N) \right| \\ &\qquad \leq C_{8.5} ||\Phi||_{1/2} \underline{X}^N_{s-t_N}(1)(\log N)^{-6}. \end{aligned}$$

<u>Proof</u> : This is the same as the proof of Lemma 3.6, only we use (27) in place of (25). □

One can similarly obtain the obvious analogue of (55) for the killed process. We leave details to the reader.

We will finally need an alternative form of Lemma 3.5. It is a cruder estimate but gives better bounds for test functions which are small in an L^1 sense. Let $\log^+(x) := \log(1 \vee x)$.

Lemma 8.6. *Assume $f : S_N \to [0,1]$ and $|f|_1 = \frac{1}{N}\sum_{x \in S_N} f(x) < \infty$. There exist constants $c_{8.6}$ and $C_{8.6}$, depending on $\overline{\beta}$, such that*

$$E[X_t^N(f)] \leq X_0^N(P_t^N f) + C_{8.6} \exp(c_{8.6} t) X_0^N(1)(\log N)^3 |f|_1 \left[1 + \log^+\left(\frac{1}{|f|_1}\right)\right].$$

<u>Proof</u> : Let $\Phi(s,x) = P^N_{t-s} f(x)$ and use (28) to get

$$E[X_t^N(f)] \leq X_0^N(P_t^N(f)) + E\left[\sum_{i=2}^{3} |D_t^{N,i}(\Phi)|\right]. \tag{125}$$

Moreover,

$$
\begin{aligned}
&E\left[\sum_{i=2}^{3}|D_t^{N,i}(\Phi)|\right] \\
&\leq E\left[\int_0^t \frac{(\log N)^2}{N}\left[\sum_{x\in S_N}\Phi(s,x)((\log N)^2+\overline{\beta})(\xi_s^N(x)+\sum_{e\in S_N}p_N(e)\xi_s^N(x+e))\right]\right] \\
&\leq (1+\overline{\beta})E\left[\int_0^t (\log N)^3\left(X_s^N(P_{t-s}^N f)+X_s^N(p_N*(P_{t-s}^N f))\right)\right].
\end{aligned}
$$

By (A7) of [3], there exists C_{126} such that

$$
P_{t-s}^N f(x) \leq \left[\sum_{y\in S_N} C_{126}(1+N(t-s))^{-1}f(y)\right]\wedge 1 \leq \left(C_{126}(t-s)^{-1}|f|_1\right)\wedge 1. \tag{126}
$$

Therefore,

$$
\begin{aligned}
&E\left[\sum_{i=2}^{3}|D_t^{N,i}(\Phi)|\right] \\
&\leq C_{126}(2+2\overline{\beta})(\log N)^3\int_0^t \left([(t-s)^{-1}|f|_1]\wedge 1\right)E[X_s^N(1)]ds \\
&\leq C_{126}(2+2\overline{\beta})(1+C_{\mathrm{a}}(\log N)^{-16})\exp(c_{3.4}t)(\log N)^3 X_0^N(1)\left[\int_{|f|_1\wedge t}^t \frac{|f|_1}{u}du+|f|_1\right],
\end{aligned}
$$

where we used Proposition 3.4 (a) at the last line. Lemma 8.6 now easily follows from (125) and the above. □

We now turn to the actual proof of Lemma 8.1. By (122),

$$
\begin{aligned}
E\left[X_t^N(1)-\underline{X}_t^N(1)\right] \leq\ & X_0^N(1-\underline{P}_t^N(h))+E\left[\int_0^{t_N\wedge t}\sum_{i=2}^{3}(d_s^{N,i}(1,\xi_s^N)-d_s^{N,i}(\Psi,\underline{\xi}_s^N))\right]ds \\
&+E\left[\sum_{i=2}^{3}\int_{t_N\wedge t}^t d_s^{N,i}(1-\Psi,\xi_s^N)ds\right]+E\left[\int_{t_N\wedge t}^t (d_s^{N,2}(\Psi,\xi_s^N)-d_s^{N,2}(\Psi,\underline{\xi}_s^N))ds\right] \\
&+E\left[\int_{t_N\wedge t}^t d_s^{N,3}(\Psi,\xi_s^N)-d_s^{N,3}(\Psi,\underline{\xi}_s^N)ds\right] \\
&=: \mathscr{U}_0+\mathscr{U}_1+\mathscr{U}_2+\mathscr{U}_3+\mathscr{U}_4.
\end{aligned} \tag{127}
$$

Below we establish bounds on each of the above terms, and $|.|_\infty$ denotes the L^∞-norm on $\mathbb{R}^2$. We claim there is a constant C_{129}, depending on T, and a $\delta_{130}>0$ such that for any $t\leq T$,

$$
\mathscr{U}_0 \leq X_0^N(1)P(\sup_{u\leq t}|B_u^0|_\infty > (K_0-1)L-3), \tag{128}
$$

$$
\left|\mathscr{U}_1\right| \leq C_{129}X_0^N(1)(\log N)^3 t_N, \tag{129}
$$

$$\left|\mathscr{U}_2\right| \le C_{129}(X_0^N(1)+X_0^N(1)^2)\left((\log N)^{-\delta_{130}}+P(\sup_{u\le t}|B_u^{0,N}|_\infty > (K_0-1)L-3)\right), \tag{130}$$

$$\left|\mathscr{U}_3\right| \le C_{129}\left((\log N)^{-\delta_{130}}(X_0^N(1)+X_0^N(1)^2)+\int_0^t E[X_s^N(1)-\underline{X}_s^N(1)]ds\right), \tag{131}$$

$$\left|\mathscr{U}_4\right| \le C_{129}\left((\log N)^{-\delta_{130}}(X_0^N(1)+X_0^N(1)^2)+\int_0^t E[X_s^N(1)-\underline{X}_s^N(1)]ds\right). \tag{132}$$

We start with a bound which will be useful for proving both (128) and (130). We have

$$\begin{aligned} 0\le 1-\Psi(s,y)=1-\underline{P}_{t-s}^N h(y) &\le P(T_y'\le t-s)+P(T_y'>t-s,|B_{t-s}^y|_\infty > K_0L-3)\\ &\le P(\sup_{u\le t-s}|B_u^y|_\infty > K_0L-3). \end{aligned} \tag{133}$$

Since we assumed $\mathrm{supp}(X_0^N)\subset I$, it follows that for any $s\ge 0$

$$X_0^N\left(P_s^N(1-\Psi(s,.))\right)\le X_0^N(1)P(\sup_{u\le t}|B_u^0|_\infty > (K_0-1)L-3) \tag{134}$$

Using (134) for $s=0$ we obtain

$$\mathscr{U}_0 = X_0^N(1-\Psi(0,.))\le X_0^N(1)P(\sup_{u\le t}|B_u^0|_\infty > (K_0-1)L-3),$$

which is (128).

We next show (129). Using (54) for $i=2$, (41) for $i=3$, and then Proposition 3.4 (a), we obtain

$$\begin{aligned} |\mathscr{U}_1| &\le 2(1\vee\overline{\beta})E\left[\int_0^{t_N\wedge t}((\log N)^3+(\log N))X_s^N(1)ds\right]\\ &\le C_{129}X_0^N(1)(\log N)^3 t_N, \end{aligned}$$

and we are done. Here the fact that $\underline{\xi}^N\le\xi^N$ means the above bounds trivially apply to $\underline{\xi}^N$ as well.

Let us turn to the proof of (130). Use (61) for $i=2$, Lemma 3.9 for $i=3$ to get for suitable $\delta,\eta>0$ and $\bar{K}$ as in (16),

$$\begin{aligned} |\mathscr{U}_2| \le (2\overline{\beta}C_{15}+\bar{K})&\int_0^{(t-t_N)^+} E[X_s^N(1-\Psi(s,.))]ds\\ &+(C_{3.9}+C_{61})||1-\Psi||_{\mathrm{Lip}}(\log N)^{-\delta}\int_0^{(t-t_N)^+}E[X_s^N(1)]ds\\ &+\frac{(C_{3.7}+C_{61})||1-\Psi||_\infty}{t_N(\log N)}E\left[\int_{t_N}^{t\vee t_N}\mathscr{I}_\eta^N(s-t_N)ds\right]. \end{aligned} \tag{135}$$

Use Lemma 3.5 to bound the first term in (135) :

$$\begin{aligned}
&\int_0^{(t-t_N)^+} E[X_s^N(1-\Psi(s,.))]ds \\
\le\ &\int_0^{(t-t_N)^+} \exp(c_{3.5}s)X_0^N(P_s^N(1-\Psi(s,.))) + C_{3.5}(\log N)^{-\delta_{3.5}}(X_0^N(1)+X_0^N(1)^2)ds \\
\le\ &\int_0^{(t-t_N)^+} \big(\exp(c_{3.5}s)X_0^N(1)(P(\sup_{u\le t}|B_u^{0,N}|_\infty > (K_0-1)L-3)) \\
&\qquad\qquad +C_{3.5}(\log N)^{-\delta_{3.5}}(X_0^N(1)+X_0^N(1)^2)\big)ds,
\end{aligned}$$

where, to obtain the last inequality, we applied (134). Inequality (130) then follows by using the above, Proposition 3.4 (a), Corollary 3.11 and Lemma 8.4 in (135).

We now turn to the proof of the critical (132), and leave the proof of (131) to the reader, as it uses a similar method.

Use the fact that $\underline{X}_s^N \le X_s^N$ with Lemmas 3.9 and 8.5 to get

$$\begin{aligned}
|\mathscr{U}_4| \le &\left(C_{3.9}(\log N)^{-\delta_{3.7}} + C_{8.5}(\log N)^{-6}\right)||\Psi|| \int_0^{(t-t_N)^+} E(X_s^N(1))ds \\
&+\frac{C_{3.7}||\Psi||_\infty}{t_N(\log N)}\int_0^{(t-t_N)^+} E[\mathscr{I}^N_{\eta_{3.7}}(s)]ds \\
&+(\log N)^3 P(\{0\mid e_1\mid e_2\}_{t_N})\int_0^{(t-t_N)^+} E(X_s^N(\Psi(s,.)) - \underline{X}_s^N(\Psi(s,.)))ds \\
&+\left|\frac{(\log N)^4}{N}E\Big[\sum_{x\in S_N}\int_0^{(t-t_N)^+}\Psi(s,x)\left(\underline{\xi}_s^N(x)P(\{0\mid e_1\mid e_2\}_{t_N}) - \underline{\hat{H}}^N(\underline{\xi}_s^N,x,t_N)\right)ds\Big]\right|.
\end{aligned}$$

Using the above, Lemma 8.4, Proposition 3.4 (a) and Corollary 3.11 together with $\underline{X}^N \le X^N$, we deduce there exists a C_{136} and $\delta_{136}>0$ such that

$$\begin{aligned}
|\mathscr{U}_4| \le C_{136}\Bigg[&(\log N)^{-\delta_{136}}(X_0^N(1)+X_0^N(1)^2) + \bar{K}\int_0^t E[X_s^N(1)-\underline{X}_s^N(1)]ds \\
&+\left|E\left[\frac{(\log N)^4}{N}\sum_{x\in S_N}\int_0^{(t-t_N)^+}\Psi(s,x)\left(\hat{H}^N(\underline{\xi}_s^N,x,t_N) - \underline{\hat{H}}^N(\underline{\xi}_s^N,x,t_N)\right)ds\right]\right| \\
&+\left|E\left[\frac{(\log N)^4}{N}\sum_{x\in S_N}\int_0^{(t-t_N)^+}\Psi(s,x)\left(\underline{\xi}_s^N(x)P(\{0\mid e_1\mid e_2\}_{t_N}) - \hat{H}^N(\underline{\xi}_s^N,x,t_N)\right)ds\right]\ \Bigg]\right|. \quad (136)
\end{aligned}$$

By (47) in Remark 3.8 with $\xi_0^N = \underline{\xi}_s^N$, along with the fact that $\underline{X}^N \le X^N$, the last term in the sum in

(136) is

$$\left|E\left[\frac{(\log N)^4}{N}\sum_{x\in S_N}\int_0^{(t-t_N)^+}\Psi(s,x)\left(\underline{\xi}_s^N(x)P(\{0\mid e_1\mid e_2\}_{t_N})-\hat{H}^N(\underline{\xi}_s^N,x,t_N)\right)ds\right]\right|$$

$$\leq C_{3.8}(\log N)^{-\delta_{3.8}}\,||\Psi||E\left[\int_0^t X_s^N(1)ds\right]+\frac{C_{3.8}||\Psi||}{t_N(\log N)}E\left[\int_0^t \mathscr{I}^N_{\eta_{3.7}}(s)ds\right] \tag{137}$$

$$\leq C_{137}(X_0^N(1)+X_0^N(1)^2)(\log N)^{-(\delta_{3.8}\wedge\eta_{3.7}/2)}, \tag{138}$$

where we used Proposition 3.4 (a), Corollary 3.11 and Lemma 8.4 to get the second inequality above.

Let us now turn to bound the next-to-last term in the sum in (136). For $\delta\geq 0$ we let

$$\mathscr{A}(\delta):=\{x\in S_N\cap I':\inf_{y\in(I')^c}|x-y|\leq\delta\},\quad \mathscr{A}(\delta)':=\{x\in S_N\cap I':\inf_{y\in(I')^c}|x-y|>\delta\}$$

For $x\in\mathscr{A}(t_N^{1/3})$, it is enough to use the straightforward bound

$$\left|\hat{H}^N(\underline{\xi}_s^N,x,t_N)-\underline{\hat{H}}^N(\underline{\xi}_s^N,x,t_N)\right|\leq 2\hat{E}(\underline{\xi}_s^N(\hat{B}_{t_N}^x)+\underline{\xi}_s^N(\hat{B}_{t_N}^{x+e_1}))$$

to obtain

$$\left|E\left[\frac{(\log N)^4}{N}\sum_{x\in\mathscr{A}(t_N^{1/3})}\int_0^{(t-t_N)^+}\Psi(s,x)\left(\hat{H}^N(\underline{\xi}_s^N,x,t_N)-\underline{\hat{H}}^N(\underline{\xi}_s^N,x,t_N)\right)ds\right]\right|$$

$$\leq 2E\left[\int_0^t\frac{(\log N)^4}{N}\sum_{x\in\mathscr{A}(t_N^{1/3})}\sum_{w\in S_N}\underline{\xi}_s^N(w)\left(\hat{P}(\hat{B}_{t_N}^0=w-x)+\hat{P}(\hat{B}_{t_N}^{e_1}=w-x)\right)ds\right]$$

$$\leq 2E\left[\int_0^t\frac{(\log N)^4}{N}\left[\sum_{w\in\mathscr{A}(2t_N^{1/3})^c}\underline{\xi}_s^N(w)\left(\hat{P}(|\hat{B}_{t_N}^0|_\infty>t_N^{1/3})+P(|\hat{B}_{t_N}^{e_1}|_\infty>t_N^{1/3})\right)+\sum_{w\in\mathscr{A}(2t_N^{1/3})}2\underline{\xi}_s^N(w)\right]\right]$$

$$\leq C_{139}E\left[(\log N)^3\int_0^t\left(X_s^N(1)(N^{-1}+t_N)t_N^{-2/3}+X_s^N(\mathscr{A}(2t_N^{1/3}))\right)ds\right], \tag{139}$$

where we used $\underline{X}^N\leq X^N$ and Chebychev's inequality to obtain the last line above. Use Lemma 8.6, then the fact that $\mathrm{supp}(X_0^N)\subset I$ and the bound (17) to get for any $t\leq T$,

$$\begin{aligned}
&E\left[(\log N)^3\int_0^t X_s^N(\mathscr{A}(2t_N^{1/3}))ds\right]\\
\leq\;& C(T)(\log N)^3\int_0^t\left(X_0^N(P(B_s^{\cdot}\in\mathscr{A}(2t_N^{1/3})))+(\log N)^3X_0^N(1)t_N^{1/3}[1+\log^+(t_N^{-1/3})]\right)ds\\
\leq\;& C(T)X_0^N(1)(\log N)^3\int_0^t\frac{t_N^{1/3}}{s}\wedge P(B_s^0>(K_0-1)L-2t_N^{1/3})ds\\
&+t(\log N)^6X_0^N(1)t_N^{1/3}(1+7\log(\log N))\\
\leq\;& C_{140}X_0^N(1)\left((\log N)^3\int_0^t\left(\frac{t_N^{1/3}}{s}\wedge s\right)ds+(\log N)^{-1/6}\right),
\end{aligned} \tag{140}$$

for some C_{140} depending on T. We then use (140) in (139) to deduce

$$\left| E\left[\frac{(\log N)^4}{N} \sum_{x\in\mathscr{A}(t_N^{1/3})} \int_0^{(t-t_N)^+} \Psi(s,x)\left(\hat{H}^N(\underline{\xi}_s^N,x,t_N) - \underline{\hat{H}}^N(\underline{\xi}_s^N,x,t_N)\right) ds \right] \right| \leq C_{141} X_0^N(1)(\log N)^{-1/6} \tag{141}$$

For $x \in \mathscr{A}(t_N^{1/3})'$, we claim that

$$\left|\hat{H}^N(\underline{\xi}_s^N,x,t_N) - \underline{\hat{H}}^N(\underline{\xi}_s^N,x,t_N)\right| \leq \sum_{i=0}^{2} \hat{E}\left[\underline{\xi}_s^N(\hat{B}_{t_N}^{x+e_i})\mathbf{1}\left\{ \sup_{u\leq t_N} |\hat{B}_u^{x+e_i} - x| > t_N^{1/3} \right\} \right] \tag{142}$$

First the obvious coupling shows that for $i = 0,1,2$, $\underline{\xi}_s^N(\underline{\hat{B}}_{t_N}^{x+e_i}) \leq \underline{\xi}_s^N(\hat{B}_{t_N}^{x+e_i})$. Furthermore, if the left side of (142) is non zero, we must have $0 = \underline{\xi}_s^N(\underline{\hat{B}}_{t_N}^{x+e_i}) < \underline{\xi}_s^N(\hat{B}_{t_N}^{x+e_i}) = 1$ for some $i \in \{0,1,2\}$. For this choice of i we must have $\hat{B}_u^{x+e_i} \notin I'$ for some $u \leq t_N$, and since we supposed $x \in \mathscr{A}(t_N^{1/3})'$ this implies $\sup_{u\leq t_N} |\hat{B}_u^{x+e_i} - x| > t_N^{1/3}$. As the left side of (142) is at most 1, (142) is proved. Now use (142), then Proposition 3.4(a), and finally the weak L^1-inequality for submartingales to obtain for $t \leq T$,

$$\begin{aligned}
&\left| E\left[\frac{(\log N)^4}{N} \sum_{x\in\mathscr{A}(t_N^{1/3})'} \int_0^{(t-t_N)^+} \Psi(s,x)\left(\hat{H}^N(\underline{\xi}_s^N,x,t_N) - \underline{\hat{H}}^N(\underline{\xi}_s^N,x,t_N)\right) ds \right] \right| \\
&\leq E\left[\frac{(\log N)^4}{N} \int_0^t \sum_{w\in S_N} \sum_{x\in\mathscr{A}(t_N^{1/3})'} \sum_{i=0}^{2} \underline{\xi}_s^N(w)\hat{P}(\hat{B}_{t_N}^{e_i} = w - x, \sup_{u\leq t_N} |\hat{B}_u^{e_i}| > t_N^{1/3}) ds \right] \\
&\leq C'_{\text{a}} t X_0^N(1)(\log N)^3 \sum_{i=0}^{2} \hat{P}(\sup_{u\leq t_N} |\hat{B}_u^{e_i}|^2 > t_N^{2/3}) \leq C_{143} X_0^N(1)(\log N)^{-3}.
\end{aligned} \tag{143}$$

We finally combine (136), (137), (141), and (143) to deduce (132).

Finally, use (127)—(132) to conclude

$$\begin{aligned}
E\left[X_t^N(1) - \underline{X}_t^N(1)\right] \leq C_{144}(X_0^N(1) + X_0^N(1)^2)\Bigg[&P(\sup_{u\leq t} |B_t^{0,N}|_\infty > (K_0-1)L - 3) \\
&+ (\log N)^{-(\delta_{130}\wedge(1/6))} + \int_0^t E\left[X_s^N(1) - \underline{X}_s^N(1)\right] ds \Bigg],
\end{aligned} \tag{144}$$

and therefore Gronwall's lemma completes the proof of Lemma 8.1. □

8.3 Proof of coexistence, Theorem 1.2.

This follows from Theorem 1.4 just as in the proof of Theorem 6.1 of [6]. Let $(\alpha_0, \alpha_1) \in \mathbb{R}_+^2$ and $(1-\alpha_0) \wedge (1-\alpha_1) < r(\eta)$ where $r(\eta)$ is as in Theorem 1.4. Let $0 < q < 1$ and ξ^q be the Lotka-Volterra process such that $\xi_0^q(x), x \in \mathbb{Z}^2$ are independent Bernoulli(q). By Theorem I.1.8 in [11] there is a sequence $v_n \to \infty$ such that $\frac{1}{v_n}\int_0^{v_n} \xi_s^q ds \overset{(w)}{\Rightarrow} \xi_\infty$ as $n \to \infty$, and ξ_∞ is a translation invariant stationary law. Arguing just as in Lemma 8.4 of [6] we see that Theorem 1.4 and its proof imply $\sum_{x\in\mathbb{Z}^2} \xi_\infty(x) = \infty$ almost surely.

Now interchange the roles of 0's and 1's. This means we interchange α_0 with α_1 and replace q with $1-q$. All the hypotheses of the previous case still hold and so we also get $\sum_{x\in\mathbb{Z}^2}(1-\xi_\infty(x)) = \infty$ almost surely. ☐

9 Asymptotics of the non-collision probability for n planar walks

Our main goal in this section is to establish Proposition 1.3 which gives the asymptotic behaviour of the non-collision probability for $n \geq 2$ independent walks started at fixed points.

For $x_1, ..., x_n$ elements of $\mathbb{Z}^2$, we write $P_{x_1,...,x_n}$ for the probability measure under which $B^1, ..., B^n$ are independent rate-1 continuous-time random walks on $\mathbb{Z}^2$, with jump kernel p, started at $x_1, ..., x_n$. When the context is clear we drop $x_1, ..., x_n$ from the notation.

We let $B = (B^1, ..., B^n)$, and $Y = (Y^1, ..., Y^{n(n-1)/2}) = (B^1 - B^2, ..., B^1 - B^n, B^2 - B^3, ..., B^2 - B^n, ..., B^{n-1} - B^n)$, whose starting points are denoted $y_1, ..., y_{n(n-1)/2}$. With a slight abuse we will sometimes write $P_{y_1,...,y_{n(n-1)/2}}$ for $P_{x_1,...,x_n}$ when working with the differences Y. Note that for any $i \in \{1, ..., n(n-1)/2\}, Y^i$ is a rate-2 continuous-time random walk with jump kernel p, but the coordinates of Y are no longer independent.

Define the non-collision events :

$$D^{(n)}_{[t',t]} := \left\{\forall\ 1 \leq i < j \leq n,\ \forall s \in [t',t],\ B^i_s \neq B^j_s\right\}, \quad D^{(n)}_t = D^{(n)}_{[0,t]} = \left\{\min_{0\leq i<j\leq n} \tau(x_i, x_j) > t\right\}$$

Clearly, $P_{x_1,...,x_n}(D^{(n)}_t)$ decreases as t increases, and, since we are in $\mathbb{Z}^2$, goes to 0 as $t \to \infty$.

As noted prior to Proposition 1.3, the case $n = 2$ of Proposition 1.3 is well-known. Since $D^{(n)}_t$ requires the non-collision of $n(n-1)/2$ pairs of random walks, the $\log(t)^{-n(n-1)/2}$ decay in this Proposition is perhaps not surprising heuristically. However, even getting a rather crude power law lower bound requires some work; see Claim 9.7 below. The logarithmic decay for $n = 2$ suggests that walks will not collide between large times t and $2t$ with probability close to one (see Claim 9.4 for a proof). Conditioning not to collide in the first time interval should only help this event–see Lemma 9.10 for a preliminary and quantitative version of this. These facts suggest, and help show, that walks conditioned to avoid one another behave at large times, roughly speaking, like independent ones; Claim 9.11 gives a coupling of these walks which provides a precise and quantitative version of this result. The fact that on this large interval, $[t, 2t]$, collision events involving only one pair are far more likely than collision events involving at least three walks will allow us to give a precise asymptotic expansion of $P(D^{(n)}_{[t,2t]})$ as $t \to \infty$ (see Lemma 9.12 below), through a careful study of the case $n = 2$, which will be our first task in subsection 9.1.3 below. Proposition 1.3 will follow easily from this expansion.

Finally, we used the following result in the proof of Lemma 3.7 (and would have used it in the omitted proof of (58)). Its own proof will use the coupling mentioned above to provide information on the distribution of the non-colliding walks at a large time.

Lemma 9.1. *Let $n \geq 2$. For any $\delta \in (0,1/2]$ there is a constant $C_{9.1}$ depending only on p, n, δ such that for all $t \geq e$*

$$\hat{P}(|Y_t^1| \geq \sqrt{t}(\log t)^{\delta} \mid D_t^{(n)}) \leq C_{9.1}(\log t)^{-2\delta}.$$

We prove Lemma 9.1 in subsection 9.3.

9.1 Preliminaries

Before starting the proof of Proposition 1.3, we need some preliminary results. A key tool is a strong version of a local limit theorem for the walk with jump kernel p (see subsection 9.1.1 below). It will allow us to establish bounds on the jump kernel and on its spatial increments, and in particular we will prove Lemma 2.1 in subsection 9.1.2. In subsection 9.1.3, we first use the local limit theorem to show the usual asymptotic bounds on the collision probability of *two* walks up to time t, when started at points whose distance is a function of t. Second, although the case $n = 2$ of assertion (9) is well-known, we also need a bound on the error.

Although these preliminaries are either direct easy adaptations of classical results, or follow from well known ones in a straightforward way, we provide short proofs, for the sake of completeness.

9.1.1 Local limit theorem

Let

$$p_t(x,y) = p_t(x-y) = P(B_t^1 = x - y), \quad \overline{p}_t(x,y) = \overline{p}_t(x-y) = \frac{1}{2\pi\sigma^2 t}\exp\left(-\frac{|x-y|^2}{2\sigma^2 t}\right),$$

and $E(t,x) = |p_t(x) - \overline{p}_t(x)|$. It is well known that uniformly in x, $E(t,x)$ goes to 0 as $t \to \infty$, but we need an error bound.

Proposition 9.2. *There exists a constant $C_{9.2}$ depending only on p such that for any $x \in \mathbb{Z}^2$, for any $t > 0$,*

$$E(t,x) \leq \frac{C_{9.2}}{t^{3/2}}.$$

Proposition 9.2 is a simple adaptation of the first assertion of Theorem 1.2.1 in [9], and its proof goes along the same lines. The proof also extends to d dimensions leading to a bound of $Ct^{-(d+1)/2}$.

Proof : We may assume $t \geq 1$. For $\theta \in \mathbb{Z}^2$, let $\phi(\theta) := \sum_{x \in \mathbb{Z}^2} p(x)\exp(i\theta x)$ be the characteristic function of the step distribution. Since p is symmetric, the function ϕ takes real values, and the fact that p is irreducible guarantees that $\{\theta \in [-\pi,\pi]^2 : \phi(\theta) = 1\} = \{0\}$. Furthermore, since p has three moments, ϕ has a Taylor expansion about the origin:

$$\phi(\theta) = 1 - \frac{\sigma^2|\theta|^2}{2} + O(|\theta|^3). \tag{145}$$

Thus, there exists $r \in (0,\pi)$ and $\rho < 1$ depending on r such that

$$\phi(\theta) \leq 1 - \frac{\sigma^2|\theta|^2}{4} \;\; \forall |\theta| \in (-r,r)^2 := B(0,r), \qquad |\phi(\theta)| \leq \rho \;\; \forall \theta \in [-\pi,\pi]^2 \setminus B(0,r). \tag{146}$$

By the inversion formula,

$$p_t(x) = \frac{1}{4\pi^2}\int_{[-\pi,\pi]^2} \exp(-ix\cdot\theta)\exp(-t(1-\phi(\theta)))d\theta. \tag{147}$$

Therefore

$$\begin{aligned} p_t(x) &= \frac{1}{4\pi^2 t}\int_{[-\pi\sqrt{t},\pi\sqrt{t}]^2} \exp\left(-it^{-1/2}x\cdot\alpha\right)\exp(-t(1-\phi(\alpha t^{-1/2})))d\alpha \\ &=: \frac{1}{4\pi^2 t}(I(t,x)+J(t,x)) \end{aligned}$$

where $I(t,x)=\int_{B(0,rt^{1/8})}\exp\left(-it^{-1/2}x\cdot\alpha\right)\exp(-t(1-\phi(\alpha t^{-1/2})))d\alpha$ and, using (146),

$$\begin{aligned} |J(t,x)| &\leq \int_{B(0,r\sqrt{t})\setminus B(0,rt^{1/8})} \exp(-\sigma^2|\alpha|^2/4)d\alpha + \int_{[-\pi\sqrt{t},\pi\sqrt{t}]^2\setminus B(0,r\sqrt{t})} \exp(-(1-\rho)t)d\alpha \\ &= O(\exp(-\sigma^2 r^2 t^{1/4}/4)) + O(t\exp(-(1-\rho)t)). \end{aligned}$$

Recall $t\geq 1$. For $|\alpha|\leq rt^{1/8}$, using (145) leads to

$$\exp(-t(1-\phi(\alpha t^{-1/2}))) = \exp(-\sigma^2|\alpha|^2/2)(1+|\alpha|^3 O(t^{-1/2})),$$

and thus

$$\begin{aligned} I(t,x) &= \int_{B(0,rt^{1/8})} \exp\left(-it^{-1/2}x\cdot\alpha\right)\exp(-\sigma^2|\alpha|^2/2)d\alpha \\ &\quad + \int_{B(0,rt^{1/8})} \exp\left(-it^{-1/2}x\cdot\alpha\right)\left[\exp(-t(1-\phi(\alpha t^{-1/2})))) - \exp(-\sigma^2|\alpha|^2/2)\right]d\alpha \\ &= \frac{2\pi}{\sigma^2}\exp\left(-\frac{|x|^2}{2\sigma^2 t}\right) + O(\exp(-\sigma^2 r^2 t^{1/4}/2)) \\ &\quad + O(t^{-1/2})\int_{B(0,rt^{1/8})} |\alpha|^3\exp(-\sigma^2|\alpha|^2/2)\exp\left(-it^{-1/2}x\cdot\alpha\right)d\alpha \\ &= \frac{2\pi}{\sigma^2}\exp\left(-\frac{|x|^2}{2\sigma^2 t}\right) + O(t^{-1/2}), \end{aligned}$$

where the last display holds because $\int_{\mathbb{R}^2}|\alpha|^3\exp(-\sigma^2|\alpha|^2/2)<\infty$ and $O(e^{ct^{1/4}})\leq O(t^{-1/2})$ as $t\geq 1$. Combining the bound on $|J(t,x)|$ and the estimate on $|I(t,x)|$ we deduce Proposition 9.2. □

9.1.2 Further kernel estimates

It follows directly from Proposition 9.2 that for any $t>0$,

$$p_t(x) \leq \frac{C_{9.2}}{t^{3/2}} + \frac{1}{2\pi\sigma^2 t}\exp(-\frac{|x|^2}{2\sigma^2 t}), \tag{148}$$

which refines (17). Lemma 2.1 is also a direct consequence of Proposition 9.2.

Proof of Lemma 2.1 : Recall, for $x \in \mathbb{Z}^2/\sqrt{N}$, that $p_t^N(x) = Np_{Nt}(\sqrt{N}x)$, and observe that $\overline{p}_t^N(x) := N\overline{p}_{Nt}(\sqrt{N}x) = \overline{p}_t(x)$. For $x, y \in \mathbb{Z}^2/\sqrt{N}$, we deduce from Proposition 9.2 that

$$\left|p_t^N(x) - p_t^N(x+y)\right| \leq \left|\overline{p}_t(x) - \overline{p}_t(x+y)\right| + \frac{2C_{9.2}}{\sqrt{N}t^{3/2}}. \tag{149}$$

Moreover, a bit of calculus leads to the existence of a constant C_{150} such that

$$\left|\exp\left(-\frac{|x|^2}{2\sigma^2 t}\right) - \exp\left(-\frac{|x+y|^2}{2\sigma^2 t}\right)\right| \leq C_{150}\frac{|y|}{\sqrt{t}}. \tag{150}$$

Combining (149) and (150) yields Lemma 2.1. □

9.1.3 The case n=2

We use the notation

$$A_t := D_t^{(2)} = \{Y_s^1 \neq 0 \quad \forall s \leq t\}, \qquad A_{[t',t]} := D_{[t',t]}^{(2)}.$$

Assertion (9) in this case is well known, and $K_2 = 2\pi\sigma^2$ (see for instance Lemma A3 (ii) in [3]). The proof of (9) in the case $n = 2$ relies on the last-exit formula at the origin, for the rate−2 walk Y^1 started at a point $y \in \mathbb{Z}^2$ (see, e.g., Lemma A.2(ii) of [3]),

$$1 = P_y(\forall s \leq t \ Y_s^1 \neq 0) + p_{2t}(y) + 2\int_0^t p_{2s}(y)\hat{P}(A_{t-s})ds. \tag{151}$$

Note that when $y = 0$ the first term on the right side of the above sum disappears. Also, note that for any x_1, x_2 such that $x_1 - x_2 = y$,

$$P_y(\forall s \leq t \ Y_s^1 \neq 0) = P_{x_1,x_2}(\forall s \leq t \ X_s^1 \neq X_s^2) = P_{x_1,x_2}(A_t).$$

We first provide a bound on the *collision* probability for two walks started at arbitrary points.

Claim 9.3. *There exist a constant $C_{9.3}$ depending only on p such that for any $x_1, x_2 \in \mathbb{Z}^2$, for any $t > 1$,*

$$P_{x_1,x_2}((D_t^{(2)})^c) \leq \frac{C_{9.3}}{\log t}\left(1 + (\log t - 2\log(|x_2 - x_1|))^+\right).$$

The above claim only will be useful when the starting points vary with t and are such that $|x_1(t) - x_2(t)| >> t^\beta$ for some $\beta > 0$.

We introduce

$$\tau(t) = t - t(\log t)^{-2}. \tag{152}$$

Proof of Claim 9.3 : Let $y = x_1 - x_2 \neq 0$ (or the result is trivial). For a rate-2 random walk with jump kernel p, note that

$$P(Y_s^1 = x - y) =: p_s^{(2)}(x - y) = p_{2s}(x - y).$$

The last-exit formula (151) yields

$$\begin{aligned} P_{x_1,x_2}((D_t^{(2)})^c) &= P_y(\exists s \le t\ Y_s^1 = 0) \\ &= p_{2t}(y) + 2\int_0^t p_{2s}(y)\hat{P}(A_{t-s})ds. \end{aligned}$$

We may use (17) to bound the first term as required, and proving the claim therefore reduces to bounding the above integral. By (9) for $n = 2$, for any $s \le \tau(t)$, we have $\hat{P}(A_{t-s}) \le C(\log t)^{-1}$. By Chebychev's inequality, $p_{2s}(y) \le 4\sigma^2 s|y|^{-2}$, and so

$$\int_0^{|y|^{1/2}\wedge\tau(t)} p_{2s}(y)\hat{P}(A_{t-s})ds \le C\sigma^2|y|^{-1}(\log t)^{-1}.$$

Moreover, by Proposition 9.2,

$$\int_{|y|^{1/2}\wedge\tau(t)}^{\tau(t)} |p_{2s}(y) - \overline{p}_{2s}(y)|\hat{P}(A_{t-s})ds \le \frac{C}{\log t}|y|^{-1/4} \le \frac{C}{\log t},$$

and by (17),

$$\int_{\tau(t)}^{t} p_{2s}(y)ds \le C_{17}(\log t)^{-2}.$$

A change of variables shows that $\int_{|y|^{1/2}\wedge\tau(t)}^{|y|^2} \overline{p}_{2s}(y)ds$ is bounded by a constant, while $\int_{|y|^2\wedge\tau(t)}^{\tau(t)} \overline{p}_{2s}(y)ds$ is bounded by $C(\log t - 2\log(|y|))$ (use $\overline{p}_{2s} \le C/s$). We deduce the claim from the above bounds. □

One can adapt the proof in [3] of (9) in the case $n = 2$ to get sharper asymptotics.

Claim 9.4. *There exists a constant $C_{9.4}$ depending only on p such that for every $t \ge e$,*

$$\left|\hat{P}(A_t) - \frac{2\pi\sigma^2}{\log t}\right| \le \frac{C_{9.4}}{(\log t)^{3/2}}.$$

Proof of Claim 9.4 : Let $G(2t) := 2\int_0^t p_s^{(2)}(0)ds = \int_0^{2t} p_s(0)ds$. Proposition 9.2 yields

$$G(2t) = \frac{\log t}{2\pi\sigma^2}\left(1 + O\left((\log t)^{-1}\right)\right). \tag{153}$$

On the one hand, we have by (151) that $1 \ge G(2t)\hat{P}(A_t)$, thus (153) yields the desired upper bound on $\hat{P}(A_t)$.

On the other hand, applying the last-exit formula (151) at the origin and at time $2t$, we find

$$1 \le G(2t)\hat{P}(A_t) + p_{4t}(0) + 2\int_t^{2t} p_{2s}(0)\hat{P}(A_{2t-s})ds. \tag{154}$$

By (17), $p_{4t}(0)$ is negligible in the above sum, and we need to bound the above integral. Using the fact that $\hat{P}(A_{2t-s})$ increases with s and (17), we get

$$\int_t^{2t-\exp(\sqrt{\log t})} p_{2s}(0)\hat{P}(A_{2t-s})ds \le \int_t^{2t-\exp(\sqrt{\log t})} p_{2s}(0)\hat{P}(A_{\exp(\sqrt{\log t})})ds \le \frac{C_{155}}{(\log t)^{1/2}}, \tag{155}$$

where we also used the well-known case $n = 2$ of (9) in the last inequality above. Moreover, using (17), we have

$$\int_{2t-\exp(\sqrt{\log t})}^{2t} p_{2s}(0)ds \le \frac{C_{17}\exp(\sqrt{\log t})}{t}.$$

We therefore obtain the desired lower bound on $\hat{P}(A_t)$, and hence Claim 9.4, from the above integral bounds, (153), and (154). □

A first consequence of Claim 9.4 is an asymptotic expansion of $P_{x_1,x_2}(A_{[t,2t]})$.

Claim 9.5. *There exists a constant $C_{9.5}$ depending only on p such that for any $x_1 \neq x_2 \in \mathbb{Z}^2$, $t \ge |x_1 - x_2|^4 \vee e^4$,*

$$\left| P_{x_1,x_2}(A_{[t,2t]}) - 1 + \frac{\log(2)}{\log t} \right| \le \frac{C_{9.5}}{(\log t)^{3/2}}.$$

Proof of Claim 9.5 : Recall $y = x_1 - x_2$. In terms of the difference walk, $P_{x_1,x_2}(A_{[t,2t]}) = P_y(\forall s \in [t,2t]\ Y_s^1 \neq 0)$. The last-exit formula (151) for the difference walk started at $x_1 - x_2 = y$, over the time interval $[t,2t]$ (use the Markov property at time t), yields

$$1 - P_y(\forall s \in [t,2t]\ Y_s^1 \neq 0) = p_{4t}(y) + 2\int_0^t p_{2t+2s}(y)\hat{P}(A_{t-s})ds. \tag{156}$$

By (17), uniformly in y, $p_{4t}(y)$ is $O(t^{-1})$, and the contribution to the above integral for s between $\tau(t)$ and t is $O((\log t)^{-2})$, uniformly in y. Therefore we only need to find an asymptotic expansion of $2\int_0^{\tau(t)} p_{2t+2s}(y)\hat{P}(A_{t-s})ds$.

By Claim 9.4, uniformly in $s \in [0,\tau(t)]$, $\hat{P}(A_{t-s}) = 2\pi\sigma^2(\log t)^{-1} + O((\log t)^{-3/2})$ ($t \ge e^4$ ensures that $t - \tau(t) \ge e$), thus, thanks to (17) again we may as well look at $\frac{4\pi\sigma^2}{\log t}\int_0^{\tau(t)} p_{2t+2s}(y)ds$. Then, by Proposition 9.2, and the definition of $\tau(t)$,

$$\left| \frac{4\pi\sigma^2}{\log t}\int_0^{\tau(t)} p_{2t+2s}(y)ds - \frac{4\pi\sigma^2}{\log t}\int_0^t \overline{p}_{2t+2s}(y)ds \right| \le \frac{4\pi\sigma^2 C_{9.2}}{\sqrt{t}(\log t)} + \frac{1}{(\log t)^3}.$$

Using our assumption that $t \ge |y|^4$ and a Taylor expansion of the exponential yields

$$\left| \frac{4\pi\sigma^2}{\log t}\int_0^t \overline{p}_{2t+2s}(y)ds - \frac{\log(2)}{\log t} \right| \le \frac{1}{4\sigma^2\sqrt{t}}.$$

Claim 9.5 follows. □

Finally, we we will need a bound on certain spatial increments of $P_{0,x}(A_t)$.

Claim 9.6. *There is a constant $C_{9.6}$ depending only on p so that if x,y are such that $|x| \geq \sqrt{t}(\log t)^{-1}$, $|x-y| \leq 2\sqrt{t}(\log t)^{-2}$ and $t \geq e$, then*

$$|P_{0,x}(A_t) - P_{0,y}(A_t)| \leq C_{9.6}\frac{1}{(\log t)^{3/2}}.$$

<u>Proof of Claim 9.6</u> : By (151) and the remark which follows it, we see that

$$|P_{0,x}(A_t) - P_{0,y}(A_t)| \leq \left|p_{2t}(x) - p_{2t}(y)\right| + 2\int_0^t |p_{2s}(x) - p_{2s}(y)|\hat{P}(A_{t-s})ds.$$

As usual we may use (17) to see that the first term in the above sum is less than the required bound, and we can focus on the integral. Fix $\alpha \in (0,1/2)$, and let us look at the above integral for values of s smaller than t^α. By Chebychev's inequality and the case $n=2$ of (9),

$$\int_0^{t^\alpha} |p_{2s}(x) - p_{2s}(y)|\hat{P}(A_{t-s})ds \leq \frac{C_{157}t^{2\alpha}}{(|x|^2 \wedge |y|^2)\log t} \leq C'_{157}(\log t)^{-2}, \tag{157}$$

where we used our assumptions on $|x|, |x-y|$ to deduce the last inequality above.

For larger values of s, we use Proposition 9.2 to obtain

$$\begin{aligned}
&\int_{t^\alpha}^t |p_{2s}(x) - p_{2s}(y)|\hat{P}(A_{t-s})ds \\
\leq\ &\int_{t^\alpha}^t |\overline{p}_{2s}(x) - \overline{p}_{2s}(y)|\hat{P}(A_{t-s})ds + \int_{t^\alpha}^t \left|E(2s,x) - E(2s,y)\right|\hat{P}(A_{t-s})ds \\
\leq\ &\int_{t^\alpha}^t |\overline{p}_{2s}(x) - \overline{p}_{2s}(y)|\hat{P}(A_{t-s})ds + Ct^{-\alpha/2}.
\end{aligned}$$

It remains to bound the above integral. Values of $s \in [t^\alpha, (|x|^2(\log t)^{-1}) \wedge t]$ are easy to deal with, as the exponential factors produce a negative power of t. The integral over $s \in [\tau(t), t]$ will be $O((\log t)^{-2})$. We are left with values of $s \in [|x|^2(\log t)^{-1}, \tau(t)]$, for which we know that $\hat{P}(A_{t-s}) \leq C(\log t)^{-1}$. To finish the proof, it remains to check that (set the integrals below to 0 if the lower bound of integration exceeds the upper bound)

$$\int_{|x|^2(\log t)^{-1}}^{\sigma^{-2}|x|^2(\log\log t)^{-1}/8} |\overline{p}_{2s}(x) - \overline{p}_{2s}(y)|ds + \int_{\sigma^{-2}|x|^2(\log\log t)^{-1}/8}^{\tau(t)} |\overline{p}_{2s}(x) - \overline{p}_{2s}(y)|ds = O((\log t)^{-1/2}).$$

The first term is easy, since each of the exponential terms produce a term of $O((\log t)^{-2+\delta})$ for $t \geq t(\delta)$ (use the conditions on x,y here). For the second term, our hypotheses on x,y and lower bound on s give $s^{-1}||x|^2 - |y|^2| \leq K(\log\log t)(\log t)^{-1}$, and then use a Taylor expansion and the lower bound on $|x|$ to obtain the desired bound. □

9.2 Proof of Proposition 1.3.

In this section when the context is clear we write P for $P_{x_1,\ldots,x_n}$, and drop the exponent n from the notation $D_t^{(n)}$.

We first obtain a crude lower bound on $P(D_t)$.

Claim 9.7. *There exist a positive constant α depending only on p,n such that for any $t \geq 2$, for any distinct $x_1,...,x_n$,*

$$P_{x_1,...,x_n}(D_t) \geq t^{-\alpha}.$$

Proof of Claim 9.7 : Let $w_1,...,w_n \in \mathbb{Z}$ denote the first coordinates of $x_1,...,x_n$ respectively. Without loss of generality we may assume that $w_1 \leq w_2 \leq ... \leq w_n$. We also let $W_t^1,...,W_t^n$ denote the first coordinates of $B_t^1,...,B_t^n$ respectively. Our assumptions on p guarantee that for all $j \in \{1,...,n\}$, $(W_t^j, t \geq 0)$ is a rate-1 one-dimensional symmetric random walk whose jump kernel has finite variance.

We are going to argue that if $W^1,...,W^n$ do not collide, neither can $X^1,...,X^n$. There is a small technical obstruction in the fact that $w_1,...,w_n$ are not necessarily distinct even if $x_1,...,x_n$ are. However, irreducibility of p ensures that there is a positive constant C_{158} depending only on p,n such that

$$P\left(D_1,\ \forall j \in \{1,...,n\}\ W_1^j \in (w_j + \frac{4^j}{2}, w_j + 2\cdot 4^j)\right) \geq C_{158}. \tag{158}$$

We now bound the probability that the one-dimensional walks $W^1,...,W^n$ do not collide on the time interval $[1,2^k]$ by placing them in sequences of disjoint intervals. For $j \in \{1,...,n\}$, $n \in \mathbb{N}$, and $i \in \{1,...,n\}$, we let

$$\begin{aligned}\mathscr{E}(k,i,j) \quad := \quad & \Bigg\{\forall s \in [2^{k-i}, 2^{k-i+1}], W_s^j \in \left(w_j + \frac{4^j\sqrt{2}^{k-i}}{2}, w_j + 2\cdot 4^j\sqrt{2}^{k-i}\right), \\ & W_{2^{k-i+1}}^j \in \left(w_j + \frac{(10/9)4^j\sqrt{2}^{k-i+1}}{2}, w_j + 2\cdot 4^j\sqrt{2}^{k-i}\right)\Bigg\}.\end{aligned}$$

Clearly, on the event $\bigcap_{i=1}^k \bigcap_{j=1}^n \mathscr{E}(k,i,j)$, the n walks do not collide on the time interval $[1,2^k]$. By the Functional Central Limit Theorem, there is a positive constant C_{159} depending only on p,n such that for any $k \geq 2$, for any $i \in \{1,...,k-1\}$, for any $j \in \{1,...,n\}$,

$$P\left(\mathscr{E}(k,i,j) \mid \mathscr{E}(k,i+1,j)\right) \geq C_{159}. \tag{159}$$

By redefining C_{158} we may also assume that

$$P(\cap_{j=1}^k \mathscr{E}(k,k,j)) \geq C_{158}.$$

It follows from the above that for any $k \in \mathbb{N}$,

$$P(D_{2^k}) \geq C_{158}^2 \prod_{i=1}^{k-1} (C_{159})^n = C_{158}^2 (C_{159})^{n(k-1)}.$$

For $t \geq 2$ let $k(t) = \lfloor \log t / \log(2) \rfloor$. Since $t \to P(D_t)$ is decreasing we have

$$P(D_t) \geq P(D_{2^{k(t)+1}}) \geq C_{158}^2 (C_{159})^{n(k(t))},$$

which yields Claim 9.7. □

Claim 9.7 only provides a rather bad lower bound on the non-collision probability, but it is sufficient to prove that some events which are very unlikely for independent walks, are also very unlikely for non-colliding walks.

We introduce $Z_t := \inf_{k \in \{1,...,n(n-1)/2\}} |Y_t^k|$.

Lemma 9.8. *Suppose that the positive h satisfies*

$$\lim_{t\to\infty} h(t)^{3/2}\log t = 0, \quad \lim_{t\to\infty} th(t)^2 = +\infty.$$

For $t > 0$, we introduce the stopping time $T_h(t) := \inf\{s \geq 0 : Z_s \geq \sqrt{t}h(t)\}$. There exists a constant $C_{9.8}$ depending only on p, n, h such that for any distinct starting points and any $t > 0$,

$$P(T_h(t) > t\sqrt{h(t)} \mid D_t) \leq C_{9.8} t^{-2}.$$

Proof of Lemma 9.8: Let $t > 0$, and $A = 5n(n-1)C_{17}$. We may assume $t \geq 2$ and $\sqrt{t}h(t) \geq 1$ without loss of generality. For this choice of A we have

$$\begin{aligned}
&\sup_{y_1,\dots,y_{n(n-1)/2}\in\mathbb{Z}^2} P_{y_1,\dots,y_{n(n-1)/2}}\left(\exists i \in \{1,\dots,n(n-1)/2\} : |Y^i_{Ath(t)^2}| \leq \sqrt{t}h(t)\right) \\
&\leq n(n-1)/2 \sup_{x\in\mathbb{Z}^2} \sum_{|y|\leq\sqrt{t}h(t)} p_{2Ath(t)^2}(0, y-x) \leq \frac{(2\sqrt{t}h(t)+1)^2}{20th(t)^2} \leq \frac{1}{2},
\end{aligned}$$

where we used (17) to obtain the second inequality above. Thus, using the Markov property at times $kAth(t)^2, k \in \{2,\dots,\lfloor h(t)^{-3/2}/A\rfloor\}$,

$$\begin{aligned}
P(T_h(t) > t\sqrt{h(t)}) &= P(\sup_{s\in[0,t\sqrt{h(t)}]} Z_s < \sqrt{t}h(t)) \\
&\leq P\left(\forall k \in \left\{1,\dots,\lfloor h(t)^{-3/2}/A\rfloor\right\},\ Z_{kAth(t)^2} \leq \sqrt{t}h(t)\right) \\
&\leq \prod_{k=1}^{\lfloor h(t)^{-3/2}/A\rfloor} P\left(Z_{kAth(t)^2} \leq \sqrt{t}h(t) \mid Z_{(k-1)Ath(t)^2} \leq \sqrt{t}h(t)\right) \\
&\leq \prod_{k=1}^{\lfloor h(t)^{-3/2}/A\rfloor} \sup_{y_1,\dots,y_{n(n-1)/2}\in\mathbb{Z}^2} P_{y_1,\dots,y_{n(n-1)/2}}\left(\exists i \in \{1,\dots,n(n-1)/2\} : |Y^i_{Ath(t)^2}| \leq \sqrt{t}h(t)\right) \\
&\leq \left(\frac{1}{2}\right)^{\lfloor h(t)^{-3/2}/A\rfloor}.
\end{aligned}$$

The above and our assumption on h imply $P(T_h(t) > t\sqrt{h(t)}) \leq t^{-\beta}$ for any $\beta > 0$, which, along with Claim 9.7, yields Lemma 9.8. □

As a consequence of the above, we will be able to bound the probability that two of the non-colliding walks find themselves unusually close at time t.

Lemma 9.9. *There exists a constant $C_{9.9}$ depending only on p, n such that for all initial points and any $t \geq e$,*

$$P\left(Z_t \leq \frac{\sqrt{t}}{\log t} \mid D_t\right) \leq \frac{C_{9.9}}{(\log t)^2}.$$

Proof of Lemma 9.9 : Using Lemma 9.8 with $h(t) = (\log t)^{-1}$, then the fact that

$$\left\{D_t, T_h(t) \leq t(\log t)^{-1/2}\right\} \subset \left\{D_{T_h(t)}, T_h(t) \leq t(\log t)^{-1/2}\right\},$$

we see that for $t \geq e$,

$$
\begin{aligned}
& P(Z_t \leq \sqrt{t}(\log t)^{-1} \mid D_t) \qquad (160)\\
\leq \quad & C_{9.8} t^{-2} + \frac{P(Z_t \leq \sqrt{t}(\log t)^{-1}, T_h(t) \leq t(\log t)^{-1/2}, D_t)}{P(D_t)} \\
\leq \; & C_{9.8} t^{-2} + \frac{P(Z_t \leq \sqrt{t}(\log t)^{-1}, T_h(t) \leq t(\log t)^{-1/2}, D_{T_h(t)})}{P(D_t)} \\
= \; & C_{9.8} t^{-2} + P(Z_t \leq \sqrt{t}(\log t)^{-1} \mid T_h(t) \leq t(\log t)^{-1/2}, D_{T_h(t)}) \frac{P(D_{T_h(t)}, T_h(t) \leq t(\log t)^{-1/2})}{P(D_t)}.
\end{aligned}
$$

Using the strong Markov property at time $T_h(t)$, we have

$$
\begin{aligned}
P(Z_t \leq \sqrt{t}(\log t)^{-1} \quad \mid \quad & T_h(t) \leq t(\log t)^{-1/2}, D_{T_h(t)}) \\
\leq \; & \sup_{\substack{y_1,\dots,y_n \in (B(0,\sqrt{t}(\log t)^{-1})^c) \\ s \in [t - t(\log t)^{-1/2}, t]}} P_{y_1,\dots,y_{n(n-1)/2}} \left(Z_s \leq \sqrt{t}(\log t)^{-1} \right) \\
\leq \; & \sup_{y_1 \in \mathbb{Z}, s \in [t - t(\log t)^{-1/2}, t]} \frac{n(n-1)}{2} P_{y_1} \left(|Y_s^1| \leq \sqrt{t}(\log t)^{-1} \right) \\
\leq \; & C(\log t)^{-2}, \qquad (161)
\end{aligned}
$$

where we have used (17) in the last line. Recall the notation $A_t = D_t^{(2)}$ from the beginning of Section 9.1.3 and again use the strong Markov property at time $T_h(t)$ to see that

$$
\begin{aligned}
P(D_t)/P(D_{T_h(t)}, & T_h(t) \leq t(\log t)^{-1/2}) \\
& \geq P(D_t | D_{T_h(t)}, T_h(t) \leq t(\log t)^{-1/2}) \\
& \geq \inf_{x_1,\dots,x_n \in B(0,\sqrt{t}(\log t)^{-1})^c} P_{x_1,\dots,x_n}(D_t) \\
& \geq 1 - \sup_{x_1 \in B(0,\sqrt{t}(\log t)^{-1})^c} \frac{n(n-1)}{2} P_{x_1}(A_t^c) \\
& \geq 1 - \frac{C}{\log t} \left[1 + \log t - 2 \log \left[\frac{\sqrt{t}}{\log t} \vee 1 \right] \right] \quad \text{(by Claim 9.3)} \\
& \geq 1 - \frac{C}{\log t} [1 + 2 \log \log t] \geq 1/2, \qquad (162)
\end{aligned}
$$

providing $t \geq t_0$. Use (161) and (162) in (160) to obtain the required inequality for $t \geq t_0$ and hence for $t \geq e$ by adjusting $C_{9.9}$. □

Lemma 9.10. *Fix $\delta \in (0,1)$ and let $g : (1,\infty) \to (0,\infty)$ be such that $g(t)t^{-\delta} \to \infty$ and $t - g(t) \to \infty$ when $t \to \infty$. There exists a constant $C_{9.10} \geq 0$ depending on p, q, g such that for any t such that $g(t) \geq 2e$ and for any starting points $x_1, \dots, x_n$,*

$$
P(D_t^c \mid D_{g(t)}) \leq \frac{C_{9.10}(\log t - \log(g(t)) + 2 \log \log(g(t)))}{\log t}.
$$

Proof of Lemma 9.10 : By Lemma 9.9, applied at time $g(t)$, there is a constant C_{163} such that for $g(t) \geq e$,

$$P(Z_{g(t)} \leq \sqrt{g(t)}(\log(g(t)))^{-1} \mid D_{g(t)}) \leq C_{163}(\log t)^{-2}. \tag{163}$$

Therefore,

$$\begin{aligned} P(D_t^c \mid D_{g(t)}) &\leq C_{163}(\log t)^{-2} + P(D_t^c, Z_{g(t)} > \sqrt{g(t)}(\log(g(t)))^{-1} \mid D_{g(t)}) \\ &\leq C_{163}(\log t)^{-2} + P(D_t^c \mid Z_{g(t)} > \sqrt{g(t)}(\log(g(t)))^{-1}, D_{g(t)}). \end{aligned} \tag{164}$$

Denote by $A_t^{(i)}, i \in \{1, ..., \frac{n(n-1)}{2}\}$ the event that there is no collision for the i-th pair up to time t, that is $A_t^{(i)} := \{\forall s \in [0,t] \;\; Y_s^i \neq 0\}$. We have $D_t^c = \bigcup_{i=1}^{n(n-1)/2}(A_t^{(i)})^c$, and it follows that

$$P(D_t^c \mid D_{g(t)}) \leq C_{163}(\log t)^{-2} + \frac{n(n-1)}{2} \sup_{y \notin B(0, \sqrt{g(t)}(\log(g(t)))^{-1})} P_{0,y}((A_{t-g(t)}^{(i)})^c).$$

where we used the Markov property at time $g(t)$. Using Claim 9.3 in the above, we get Lemma 9.10 . In the last we may assume without loss of generality that t is large enough so that $t - g(t) > 1$. □

Choose $\beta \in (0,1)$ such that $2C_{9.10}(1-\beta) = 1$ and apply Lemma 9.10 with $g(t) = t^\beta$ to see that for any $t \geq (2e)^{1/\beta}$, $P(D_t^c \mid D_{t^\beta}) \leq 1/2$. For $t \geq 2e$ define the sequence $u_0 = t, u_{p+1} = (u_p)^\beta$ and $N(t) = \inf\{p : u_p < 2e\}$. We have $P(D_t) \geq \frac{1}{2^{N(t)}}$, and it is straightforward to deduce that there exist constants $k \geq 2, C_{165}$ depending only on p, n such that for any $t > 1$,

$$P(D_t) \geq C_{165}(\log t)^{-k}. \tag{165}$$

Choose $\gamma \geq 2k+4$. Applying Lemma 9.10 with $g(t) = t(\log t)^{-\gamma}$, we obtain that for a constant C_{166} depending only on p, n,

$$P(D_t^c \mid D_{t(\log t)^{-\gamma}}) \leq \frac{C_{166} \log\log t}{\log t} \;\text{ whenever } g(t) \geq 2e. \tag{166}$$

Both independent and non-colliding walks should not travel very far up to time $t(\log t)^{-\gamma}$. First, using Markov's inequality and the fact that p has two moments, we obtain that for constants C_{167}, C_{168} depending only on p, n, for any $t > 1$,

$$\begin{aligned} & P_{y_1,\dots,y_{n(n-1)/2}}\left(\sup_{j \in \{1,\dots,n(n-1)/2\}} |Y^j_{t(\log t)^{-\gamma}} - y_j| > \sqrt{t}(\log t)^{-\gamma/4}\right) \\ \leq \;& \frac{n(n-1)}{2} \log(t)^{-\gamma/2} E\left[\frac{\left(Y^1_{t(\log t)^{-\gamma}} - y_1\right)^2}{t(\log t)^{-\gamma}}\right] \\ \leq \;& C_{167}(\log t)^{-\gamma/2}. \end{aligned} \tag{167}$$

Then, from (167), (165), and our choice of γ, for all $t \geq e$,

$$P_{y_1,\dots,y_{n(n-1)/2}}\left(\sup_{j \in \{1,\dots,n(n-1)/2\}} |Y^j_{t(\log t)^{-\gamma}} - y_j| > \sqrt{t}(\log t)^{-\gamma/4} \mid D_{t(\log t)^{-\gamma}}\right) \leq C_{168}(\log t)^{-2}. \tag{168}$$

Fix $t > e$. We may couple walks $B, \tilde{B}$ in the following way :

- both are started at points $x_1, ..., x_n$, which we suppose distinct.
- over the time interval $[0, t(\log t)^{-\gamma}]$, B, $\tilde{B}$ are run independently, and respectively under $P(\cdot \mid D_{t(\log t)^{-\gamma}})$, and P.
- the increments of B and $\tilde{B}$ over the time interval $[t(\log t)^{-\gamma}, t]$ coincide.

This coupling, along with (167) and (168), then yields the following observation. Note that these two assertions guarantee that the constant $C_{9.11}$ below only depends on p, n.

Claim 9.11. *For any $t > e$ there exists a probability measure Q_1^t on the set of $2n$-tuplets of walk paths such that the following holds. Let $(B, \tilde{B})$ be defined under Q_1^t, and both started at $x_1, ..., x_n$. Let Y, respectively $\tilde{Y}$ be the corresponding $n(n-1)/2$-tuplet of differences. Then,*

- *the distribution of B under Q_1^t is $P_{x_1,...,x_n}(\cdot \mid D_{t(\log t)^{-\gamma}})$, that is, the n first coordinates are walks which do not collide up to $t(\log t)^{-\gamma}$.*
- *the distribution of $\tilde{B}$ under Q_1^t is $P_{x_1,...,x_n}$, that is, the n last coordinates are independent walks.*
- $Q_1^t\left(|Y_t - \tilde{Y}_t|_\infty > \frac{2\sqrt{t}}{(\log t)^2}\right) \leq \frac{C_{9.11}}{(\log t)^2}$.

For the last, recall that $\gamma \geq 8$.

We are now in position to compute an asymptotic expansion of $P(D_{2t} \mid D_t)$.

Lemma 9.12. *There exists a constant $C_{9.12}$ depending only on p, n such that for any distinct $x_1, ..., x_n$, for any $t \geq \max_{j\in\{1,...,n(n-1)/2\}} |y_j|^4 \vee e^4$,*

$$\forall i \in \left\{1, ..., \frac{n(n-1)}{2}\right\}, \qquad \left|P(A_{2t}^{(i)} \mid D_t) - 1 + \frac{\log(2)}{\log t}\right| \leq \frac{C_{9.12}}{(\log t)^{3/2}}, \tag{169}$$

$$\forall (i_1, i_2) \in \left\{1, ..., \frac{n(n-1)}{2}\right\}^2, i_1 \neq i_2, \qquad P\left(\bigcap_{j=1}^{2} (A_{2t}^{(i_j)})^c \mid D_t\right) \leq \frac{C_{9.12}}{(\log t)^{3/2}}, \tag{170}$$

$$\left|P(D_{2t} \mid D_t) - 1 + \frac{n(n-1)\log(2)}{2\log t}\right| \leq \frac{C_{9.12}}{(\log t)^{3/2}}. \tag{171}$$

Proof of Lemma 9.12 : We start by proving (169). We choose γ as above and use the notation $g(t) := t(\log t)^{-\gamma}$ throughout the proof. Without loss of generality we may take t large enough that $g(t) \geq e$, and only consider the case when $i = 1$.

We are looking for an asymptotic expansion of $1 - P(A_{2t}^{(1)} \mid D_t) = P((A_{[t,2t]}^{(1)})^c \mid D_t)$. In order to exploit the coupling Claim 9.11, we are first going to establish that there exists a constant C_{172} depending only on p, n, γ such that

$$P\left(\left(A_{[t,2t]}^{(1)}\right)^c \mid D_{g(t)}\right) - \frac{C_{172}}{(\log t)^{3/2}} \leq P\left(\left(A_{[t,2t]}^{(1)}\right)^c \mid D_t\right) \leq P\left(\left(A_{[t,2t]}^{(1)}\right)^c \mid D_{g(t)}\right)\left(1 - \frac{C_{166}\log(\log t)}{(\log t)}\right)^{-1} \tag{172}$$

The upper bound in (172) follows directly from (166). Let us now establish the lower bound. We have

$$\begin{aligned} P\left(\left(A^{(1)}_{[t,2t]}\right)^c \mid D_t\right) &\geq P\left(\left(A^{(1)}_{[t,2t]}\right)^c \cap D_{[g(t),t]} \mid D_{g(t)}\right) \\ &= P\left(\left(A^{(1)}_{[t,2t]}\right)^c \mid D_{g(t)}\right) - P\left(\left(A^{(1)}_{[t,2t]}\right)^c \cap D^c_{[g(t),t]} \mid D_{g(t)}\right). \end{aligned}$$

Since $D^c_{[g(t),t]} = \bigcup_{j=1}^{n(n-1)/2}\left(A^{(j)}_{[g(t),t]}\right)^c$,

$$P\left(D^c_{[g(t),t]} \cap \left(A^{(1)}_{[t,2t]}\right)^c \mid D_{g(t)}\right) \leq \sum_{j=1}^{n(n-1)/2} P\left(\left(A^{(j)}_{[g(t),t]}\right)^c \cap \left(A^{(1)}_{[t,2t]}\right)^c \mid D_{g(t)}\right).$$

Denote $\Upsilon_t := \{\mathscr{Y} = (y_1, ..., y_{n(n-1)/2}) \in (\mathbb{Z}^2)^{n(n-1)/2} : \min|y_i| \geq \sqrt{t}(\log t)^{-\gamma}\}$ and fix $j \in \{1, ..., n(n-1)/2\}$. Using Lemma 9.9 at time $g(t)$ and the Markov property at time $g(t)$,

$$P\left(\left(A^{(j)}_{[g(t),t]}\right)^c \cap \left(A^{(1)}_{[t,2t]}\right)^c \mid D_{g(t)}\right) \leq \frac{C'_{9.9}}{\log(t)^2} + \sup_{\mathscr{Y}\in\Upsilon_t} P_{\mathscr{Y}}\left(\left(A^{(j)}_{[0,t-g(t)]}\right)^c \cap \left(A^{(1)}_{[t-g(t),2t-g(t)]}\right)^c\right)$$

for some $C'_{9.9}$ depending only on p, q, γ.

On $\left(A^{(j)}_{[0,t-g(t)]}\right)^c$, either the j-th pair of walks collides on the time interval $[0, t-2g(t)]$ or on $[t-2g(t), t-g(t)]$. Thus for any $\mathscr{Y} \in \Upsilon_t$,

$$\begin{aligned} & P_{\mathscr{Y}}\left(\left(A^{(j)}_{[0,t-g(t)]}\right)^c \cap \left(A^{(1)}_{[t-g(t),2t-g(t)]}\right)^c\right) \\ \leq\ & P_{\mathscr{Y}}\left(\left(A^{(j)}_{[t-2g(t),t-g(t)]}\right)^c\right) + P_{\mathscr{Y}}\left(\left(A^{(j)}_{[0,t-2g(t)]}\right)^c\right) \times \sup_{\mathscr{Y}'\in(\mathbb{Z}^2)^{n(n-1)/2}} P_{\mathscr{Y}'}\left(\left(A^{(1)}_{[g(t),t+g(t)]}\right)^c\right) \end{aligned}$$

where we used the Markov property under $P_{\mathscr{Y}}$ at time $t-2g(t)$. The first term above is easily bounded by $C_{17}(\log t)^{-\gamma}$, by using a last-exit formula and (17). More specifically, use the analogue of (156) with $[t-2g(t), t-g(t)]$ in place of $[t, 2t]$. Moreover, by Claim 9.3 and our definition of Υ_t, for any $\mathscr{Y} \in \Upsilon_t$,

$$P_{\mathscr{Y}}\left(\left(A^{(j)}_{[0,t-2g(t)]}\right)^c\right) \leq \frac{3\gamma C_{9.3}\log(\log t)}{\log t}.$$

Furthermore, by (17)

$$\sup_{\mathscr{Y}'\in(\mathbb{Z}^2)^{n(n-1)/2}} P_{\mathscr{Y}'}\left(|Y^1_{g(t)}| \leq \sqrt{t}(\log t)^{-\gamma}\right) \leq C(\log t)^{-\gamma}.$$

Therefore, using the Markov property under $P_{\mathscr{Y}'}$ at time $g(t)$ and Claim 9.3 we get

$$\sup_{\mathscr{Y}'\in(\mathbb{Z}^2)^{n(n-1)/2}} P_{\mathscr{Y}'}\left(\left(A^{(1)}_{[g(t),t+g(t)]}\right)^c\right) \leq C(\log t)^{-\gamma} + \frac{C\log(\log t)}{\log t}.$$

Since $\gamma > 2$, combining the above inequalities yields the existence of a constant C_{173} depending only on p, n, γ such that

$$\sup_{\mathscr{Y}\in\Upsilon_t} P_{\mathscr{Y}}\left(\left(A^{(j)}_{[0,t-g(t)]}\right)^c \cap \left(A^{(1)}_{[t-g(t),2t-g(t)]}\right)^c\right) \leq \frac{C_{173}\log(\log t)^2}{(\log t)^2}, \tag{173}$$

which completes the proof of (172).

Now using Claim 9.11, $P\left(\left(A^{(1)}_{[t,2t]}\right)^c \mid D_{t(\log t)^{-\gamma}}\right) = E_{Q_1^t}\left[P_{B_t}\left((A_t^{(1)})^c\right)\right]$, and

$$
\begin{aligned}
&E_{Q_1^t}\left[\left|P_{B_t}\left((A_t^{(1)})^c\right) - P_{\tilde{B}_t}\left((A_t^{(1)})^c\right)\right|\right]\\
&\le C_{9.11}(\log t)^{-2} + Q_1^t[\tilde{Z}_t \le \sqrt{t}(\log t)^{-1}]\\
&\quad + E_{Q_1^t}\left[\left|P_{\tilde{B}_t}\left((A_t^{(1)})^c\right) - P_{B_t}\left((A_t^{(1)})^c\right)\right| \mathbf{1}_{\{\tilde{Z}_t > \sqrt{t}(\log t)^{-1}\}} \mathbf{1}_{\{|Y_t - \tilde{Y}_t|_\infty \le 2\sqrt{t}(\log t)^{-2}\}}\right]. \qquad (174)
\end{aligned}
$$

The second term of the sum above is bounded by $C(\log t)^{-2}$, by (17). The third term is bounded by $C(\log t)^{-3/2}$ thanks to Claim 9.6. Finally, $E_{Q_1^t}[P_{\tilde{B}_t}\left((A_t^{(1)})^c\right)] = P_{x_1,x_2}(A^c_{[t,2t]})$, and our assumptions on t guarantee we can use Claim 9.5 to deduce that

$$
\left| E_{Q_1^t}\left[P_{\tilde{B}_t}\left((A_t^{(1)})^c\right)\right] - \frac{\log(2)}{\log(t)} \right| \le C_{9.5}(\log t)^{-3/2}.
$$

By (172), (174) and the above bound, we conclude to (169).

We now turn to the proof of (170). Without loss of generality we can treat the case $i_1 = 1, i_2 \in \{2, ..., n(n-1)/2\}$, so that $Y^{i_1} = B^1 - B^2$ and $Y^{i_2} = B^{k_1} - B^{k_2}$ with $k_2 \ge 3$, $k_1 \ne k_2$. Either $k_1 \ne 1$ or $k_1 \ne 2$. For definiteness assume the latter. Then

$$
B^{k_2} \text{ is independent of } (Y^{i_1}, B^{k_1}) \text{ and } B^2 \text{ is independent of } (Y^{i_2}, B^1). \qquad (175)
$$

By Lemma 9.9,

$$
P(Z_t \le \sqrt{t}(\log t)^{-1} \mid D_t) \le C_{9.9}(\log t)^{-2}.
$$

Therefore, to establish (170), by the Markov property at time t it suffices to consider n independent walks started at points further apart from each other than $\sqrt{t}(\log t)^{-1}$, and bound the probability that the first and i_2th pairs of walks collide by time t. More precisely, if we let

$$
\Xi_t := \left\{ (y_1, ..., y_{n(n-1)/2}) \in (\mathbb{Z}^2)^{n(n-1)/2} : \min |y_i| \ge \frac{\sqrt{t}}{\log t} \right\},
$$

we need to establish

$$
\sup_{\xi \in \Xi_t} P_\xi((A_t^{(1)})^c \cap (A_t^{(i_2)})^c) \le \frac{C_{176}}{(\log t)^{3/2}} \quad \forall t \ge e^4. \qquad (176)
$$

Let us define the stopping times

$$
\tau_j := \inf\{t \ge 0 : Y_t^{i_j} = 0\},\ j = 1, 2,\ \tau = \tau_1 \wedge \tau_2.
$$

Note that $\{\tau \le t\} = (A_t^{(1)})^c \cup (A_t^{(i_2)})^c$, and thus, by Claim 9.3, there exists a constant C_{177} such that for any $\xi \in \Xi_t$,

$$
P_\xi(\tau \le t) \le \frac{C_{177} \log\log t}{\log t} \quad \text{for all } t \ge e^2. \qquad (177)
$$

By Doob's weak maximal inequality for $\xi \in \Xi_t$,

$$\begin{aligned}
P_\xi(\tau \le t(\log t)^{-4}) &\le 2 \sup_{|y_1| \ge \sqrt{t}(\log t)^{-1}} P_{y_1}(|Y_s| = 0 \text{ for some } s \le t(\log t)^{-4}) \\
&\le 2P_0(\sup_{s \le t(\log t)^{-4}} (|Y_s| \ge \sqrt{t}(\log t)^{-1}) \\
&\le C(\log t)^{-2}.
\end{aligned}$$

Therefore for $\xi \in \Xi_t$ and $t \ge e^2$, using the independence in (175), we have

$$\begin{aligned}
& P_\xi\left(\max(|Y^1_\tau|, |Y^{i_2}_\tau|) \le \sqrt{t}(\log t)^{-3}\right) \\
\le\ & P_\xi(\tau \le t(\log t)^{-4}) + E(1(\tau_1 > t(\log t)^{-4}) P(B^{k_2}_{\tau_1} \in B(B^{k_1}_{\tau_1}, \sqrt{t}(\log t)^{-3}) | Y^{i_1}, B^{k_1})) \\
& \qquad + E(1(\tau_2 > t(\log t)^{-4}) P(B^{2}_{\tau_2} \in B(B^{1}_{\tau_2}, \sqrt{t}(\log t)^{-3}) | Y^{i_2}, B^{1})) \\
\le\ & C(\log t)^{-2} + \sup_{s > t(\log t)^{-4}, x \in \mathbb{Z}^2} P(B^{k_2}_s \in B(x, \sqrt{t}(\log t)^{-3}) \\
& \qquad + \sup_{s > t(\log t)^{-4}, x \in \mathbb{Z}^2} P(B^{2}_s \in B(x, \sqrt{t}(\log t)^{-3}) \\
\le\ & C(\log t)^{-2}, \qquad (178)
\end{aligned}$$

the last by (17). Thus, by (178), and the strong Markov property at time τ, it follows that for any $\xi \in \Xi_t$ and $t \ge e^2$, we have

$$\begin{aligned}
P_\xi((A^{(1)}_t)^c \cap (A^{(i_2)}_t)^c) &\le C_{178}(\log t)^{-2} + P_\xi(\tau \le t) \left(\max_{y \in B(0, \sqrt{t}(\log t)^{-3})^c} P_{0,y}(A^c_t) \right) \\
&\le \frac{C_{179}(\log\log t)^2}{(\log t)^2}, \qquad (179)
\end{aligned}$$

where we used (177) and Claim 9.3 to get the last inequality above. This completes the proof of (170).

Finally, inequality (171) is a simple consequence of (169), (170), and the inclusion-exclusion formula applied to $D^c_{2t} = \bigcup_{i=1}^{\frac{n(n-1)}{2}} (A^{(i)}_{2t})^c$. This finishes the proof of Lemma 9.12. □

It remains to show how Proposition 1.3 follows from Lemma 9.12.

Define $f(t) := (\log t)^{\frac{n(n-1)}{2}} P(D_t)$, and $k(t) := \max\{i \in \mathbb{N} : 2^i \le t\}$. Since $P(D_t)$ is a decreasing function of t, an easy consequence of (171) is that $f(t)/f(2^{k(t)}) \underset{t\to\infty}{\to} 1$.
Therefore, to establish the first assertion of Proposition 1.3, it suffices to show that the sequence $(f(2^m))_{m \in \mathbb{N}}$ converges to a positive limit.
Let $m_0 := \inf\{m \in \mathbb{N} : 2^m \ge \max_{j \in \{1,\dots,n(n-1)/2\}} |y_j|^4 \vee e^4\}$. If $m \in \mathbb{N}, m \ge m_0$, and $m' \in \mathbb{N}$,

$$f(2^{m+m'}) = f(2^m) \prod_{i=0}^{m'-1} \frac{((m+i+1))^{\frac{n(n-1)}{2}}}{((m+i))^{\frac{n(n-1)}{2}}} P(D_{2^{m+i+1}} \mid D_{2^{m+i}}).$$

We may use (171) since $m \geq m_0$, and we deduce that uniformly in $x_1, ..., x_n$,

$$f(2^{m+m'}) = f(2^m) \prod_{i=0}^{m'-1} \left(1 + \frac{n(n-1)}{2} \frac{1}{(m+i)} + O((m+i)^{-2})\right)$$
$$\times \left(1 - \frac{n(n-1)}{2} \frac{1}{(m+i)} + O((m+i)^{-3/2})\right).$$

The first term in the product comes from a second order expansion of $(1+x)^p$ for x near 0 and $p = n(n-1)/2$. It follows easily that the sequence $(f(2^m))$ is Cauchy. Since $f(2^{m_0}) > 0$, we see by applying the above to $m = m_0, m' \to \infty$ that the limit of $(f(2^m))$ has to be positive.

Moreover, $f(2^{m_0}) \leq c_n \max_{j \in \{1,...,n(n-1)/2\}} (4 \log |y_j|)^{n(n-1)/2} + 1)$, which establishes the second part of Proposition 1.3. □

We end this section with an interesting coupling between non-colliding and independent walks. Although we do not use it here, we feel it may be of future use and is natural in view of the construction in Claim 9.11. Choose γ as in the paragraph preceding Claim 9.11. The constant $C_{9.13}$ below depends only on p, n, γ.

Claim 9.13. *For any $t > e$ there exists a probability measure Q_2^t on the set of $2n$-tuplets of walk paths such that the following holds. Let $(B, \tilde{B})$ be defined under Q_2^t, and both started at $x_1, ..., x_n$. Let Y, respectively $\tilde{Y}$ be the corresponding $n(n-1)/2$-tuplet of differences. Then,*

- *the distribution of B under Q_2^t is $P_{x_1,...,x_n}(\cdot \mid D_t)$, that is, the n first coordinates are walks which do not collide up to t.*
- *the distribution of $\tilde{B}$ under Q_2^t is $P_{x_1,...,x_n}$, that is, the n last coordinates are independent walks.*
- $Q_2^t \left(|Y_t - \tilde{Y}_t|_\infty > \frac{2\sqrt{t}}{(\log t)^{\gamma/4}} \right) \leq \frac{C_{9.13} \log \log t}{\log t}$.

Proof of Claim 9.13 : Fix $t > e$, and distinct $x_1, ..., x_n$. We construct $B, \tilde{B}$ in the following way :

- Both are started at $x_1, ..., x_n$.
- Over the time interval $[0, t]$, we run $\tilde{B}$ under P.
- Over $[0, t(\log t)^{-\gamma}]$, we run B under $P(. \mid D_t)$.
- Define $\overline{B}$ on $[0, t]$ as follows.
  For all $s \in [0, t(\log t)^{-\gamma}]$, $\overline{B}_s := B_s$.
  For all $s \in [t(\log t)^{-\gamma}, t]$, $\overline{B}_s - \overline{B}_{t(\log t)^{-\gamma}} := \tilde{B}_s - \tilde{B}_{t(\log t)^{-\gamma}}$.
  Furthermore introduce the non-collision event
  $\overline{D}_t := \{\forall s \in [0, t] \; \forall i, j \in \{1, ..., n\}^2, i \neq j, \overline{B}^i \neq \overline{B}^j\}$.
- On $\overline{D}_t$ set $B = \overline{B}$ on $[0, t]$.
- On $\overline{D}_t^c$, we run B on $[t(\log t)^{-\gamma}, t]$ under $P(\cdot \mid D_t)$, independently of $\tilde{B}$.

The second assertion of the claim is obvious from the construction. The first follows from the fact that the distribution of $\overline{B}$ conditioned on $\overline{D}_t$ is, thanks to the Markov property at time $t(\log t)^{-\gamma}$, precisely $P(\cdot \mid D_t)$. It remains to establish the last assertion of the claim.
Let $\overline{Z}_s := \inf_{i,j\in\{1,...,n\}^2, i\neq j}\left\{\left|\overline{B}^i_s - \overline{B}^j_s\right|\right\}$. By (17), (165), and our choice of γ,

$$P\left(\overline{Z}_{t(\log t)^{-\gamma}} \leq \sqrt{t}(\log t)^{-\gamma}\right) \leq C_{180}(\log t)^{-k-4}. \tag{180}$$

Using the same method as for establishing Lemma 9.10 we deduce that

$$P(\overline{D}^c_t) = P\left(\exists s \in [t(\log t)^{-\gamma}, t]\ \exists i \neq j\ :\ \overline{B}^i_s = \overline{B}^j_s\right) \leq \frac{C_{181}\log(\log t)}{\log t}. \tag{181}$$

Moreover, by (168), (166), and our choice of γ, for all $t \geq e^2$,

$$P_{y_1,...,y_{n(n-1)/2}}\left(\sup_{j\in\{1,...,n(n-1)/2\}} |Y^j_{t(\log t)^{-\gamma}} - y_j| > \sqrt{t}(\log t)^{-\gamma/4} \mid D_t\right) \leq C_{182}(\log t)^{-2}. \tag{182}$$

From the above construction, $(B_t - \tilde{B}_t)\mathbf{1}_{\overline{D}_t} = \left(B_{t\log(t)^{-\gamma}} - \tilde{B}_{t\log(t)^{-\gamma}}\right)\mathbf{1}_{\overline{D}_t}$. Therefore, (167), (182) and (181) imply the last assertion of Claim 9.13. □

9.3 Proof of Lemma 9.1

Let Q^t_1 be as in Claim 9.11 and γ be as in (167) and (168). Then (166) shows that for any starting points $x_1, ..., x_n$

$$P(D_t|D_{t(\log t)^{-\gamma}}) \geq \frac{1}{2} \text{ for } t \geq t_0 \geq e^4.$$

So for $t \geq t_0$ and $\delta \in (0, \frac{1}{2}]$, and any starting points $x_1, ..., x_n$

$$\begin{aligned}
& P(|Y^1_t - y_1| \geq \sqrt{t}(\log t)^{\delta}/2 \mid D_t) \\
\leq\ & P(|Y^1_t - y^1| \geq \sqrt{t}(\log t)^{\delta}/2 \mid D_{t(\log t)^{-\gamma}})P(D_{t(\log t)^{-\gamma}})/P(D_t) \\
\leq\ & Q^t_1(|Y^1_t - y_1| \geq \sqrt{t}(\log t)^{\delta}/2) \times 2 \\
\leq\ & \frac{2C_{9.11}}{(\log t)^2} + 2Q^t_1\left(|\tilde{Y}^1_t - y_1| \geq \sqrt{t}(\log t)^{\delta}/2 - \frac{2\sqrt{t}}{(\log t)^2}\right) \\
\leq\ & \frac{2C_{9.11}}{(\log t)^2} + C(\log t)^{-2\delta},
\end{aligned}$$

where Chebychev's inequality is used in the last line and the constant C depends only on p. It follows from the above that for some constant C' depending only on p, n, for any $\delta \in (0, 1/2]$ and $t \geq e$,

$$P(|Y^1_t| \geq \sqrt{t}(\log t)^{\delta} \mid D_t) \leq C' \log(|y_1|)(\log t)^{-2\delta}.$$

Here note that the inequality is trivial if $|y_1| > \sqrt{t}/2$ as $\delta \leq 1/2$. Lemma 9.1 follows by dominated convergence. □

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