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Moderate deviations and laws of the iterated logarithm for the volume of the intersections of Wiener sausages *

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Abstract

Using the high moment method and the Feynman-Kac semigroup technique, we obtain moderate deviations and laws of the iterated logarithm for the volume of the intersections of two and three dimensional Wiener sausages.

Key words: Wiener sausage, moderate deviations, large deviations, laws of the iterated logarithm.

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1 Introduction

Let $\{\beta(t), t \geq 0\}$ be a standard Brownian motion. The Wiener sausage with radius $r > 0$ is the random process defined by

$$W_r(t) = \bigcup_{0 \leq s \leq t} B_r(\beta(s))$$

where $B_r(x)$ is the open ball with radius r around $x \in \mathbb{R}^d$. It is known that ([21],[27]) as $t \to \infty$,

$$\mathbf{E}|W_r(t)| \sim \begin{cases} \sqrt{\frac{8t}{\pi}} & if \ d = 1 \\ 2\pi t/\log t & if \ d = 2 \\ \kappa_r t & if \ d \geq 3 \end{cases}$$

where $\kappa_r = 2\pi^{d/2} r^{d-2} \Gamma((d-2)/2)$.

Let $p \geq 2$ be an integer and let $\{\beta_j(s), 1 \leq j \leq p\}$ be p independent standard Brownian motions. Write $W_r^j(t) = \bigcup_{0 \leq s \leq t} B_r(\beta_j(s))$ and define the volume of the intersections of p Wiener sausages as follows:

$$V_r(t) = |W_r^1(t) \cap W_r^2(t) \cap \cdots \cap W_r^p(t)|.$$

In the classical paper [17], Donsker and Varadhan studied asymptotic behavior of the Laplace-transform $\mathbf{E}(\exp\{-\lambda|W_r(t)|\})$ of the Wiener sausage $|W_r(t)|$ and solved a conjecture of Mark Kac concerning Wiener sausage. The fluctuation theorem of the Wiener sausage was obtained by Le Gall ([22]). The large deviation below the scale of the mean in the downward direction for the volume of the Wiener sausage: $P(|W_r(t)| \leq f(t))$ where $f(t) = o(\mathbf{E}|W_r(t)|)$, was studied in [10],[17] and [28]. The large deviation on the scale of the mean in the downward direction: $P(|W_r(t)| \leq c\mathbf{E}|W_r(t)|)$, was studied in [7]. The large deviation on the scale of the mean in the upward direction : $P(|W_r(t)| \geq c\mathbf{E}|W_r(t)|)$, was considered in [6],[9] and [25]. Strong approximations of three-dimensional Wiener sausages were studies in [15].

The asymptotic behavior of the volume of the intersections of Wiener sausages was obtained by Le Gall ([23], $d = 2$) and van den Berg ([5], $d \geq 3$). Van den Berg, Bolthausen and den Hollander [8] first considered the large deviations for the volume of the intersections of Wiener sausages. They obtained the following deep results: for any $c > 0$,

$$\lim_{t \to +\infty} \frac{1}{\log t} \log \mathbf{P}\left(V_r(ct) \geq \frac{t}{\log t}\right) = -I_2^{2\pi}(c) < 0 \tag{1.1}$$

as $d = 2$ and $p = 2$; and

$$\lim_{t \to +\infty} \frac{1}{t^{(d-2)/d}} \log \mathbf{P}\left(V_r(ct) \geq t\right) = -I_d^{\kappa_r}(c) < 0 \tag{1.2}$$

as $d \geq 3$ and $p = 2$. The intersection grows logarithmically with t in $d = 2$ and stays bounded in $d = 3$.

Noting that $\frac{t}{\log t} \sim E(|W_r(t)|)/(2\pi)$ and $t \sim E(|W_r(t)|)/\kappa_r$, a natural problem is to consider

$$\mathbf{P}\left(V_r(ct) \geq f(t)\right) \tag{1.3}$$

where $f(t) = o(E(|W_r(t)|)$. This is a motivation of our paper. In fact, the problem for the intersection of the ranges of independent random walks was studied by Chen ([12][13]). In this paper, we prove that the volume of the intersection of Wiener sausages has the same large deviations as the intersection of the ranges of independent random walks. As an application, the corresponding laws of the iterated logarithm are obtained. The main results are as follows:

Theorem 1.1. *(1). Let $d = 2$ and $p \geq 2$ and let $b(t)$ be a positive function satisfying*

$$b(t) \longrightarrow +\infty,\ b(t) = o(\log t) \qquad t \to +\infty. \tag{1.4}$$

Then for any $\lambda > 0$,

$$\lim_{t\to+\infty} \frac{1}{b(t)} \log \mathbf{P}\left(V_r(t) \geq \frac{\lambda t}{(\log t)^p} b(t)^{p-1} \right) = -\frac{p}{2}(2\pi)^{-\frac{p}{p-1}} \kappa(2,p)^{-\frac{2p}{p-1}} \lambda^{\frac{1}{p-1}} \tag{1.5}$$

where $\kappa(d,p)$ is the best constant of the Gagliardo-Nirenberg inequality

$$\|f\|_{2p} \leq C \|\nabla f\|_2^{\frac{d(p-1)}{2p}} \|f\|_2^{1-\frac{d(p-1)}{2p}}, \qquad f \in W^{1,2}(\mathbb{R}^d) \tag{1.6}$$

and

$$W^{1,2}(\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d);\ \nabla f \in L^2(\mathbb{R}^d) \right\}.$$

(2). Let $d = 3$ and $p = 2$ and let $b(t)$ be a positive function satisfying

$$b(t) \longrightarrow +\infty,\ b(t) = o(t^{1/3}/(\log t)^2) \qquad t \to +\infty. \tag{1.7}$$

Then for any $\lambda > 0$,

$$\lim_{t\to+\infty} \frac{1}{b(t)} \log \mathbf{P}\left(V_r(t) \geq \lambda \sqrt{t b(t)^3} \right) = -(2\pi r)^{-4/3} \kappa(3,2)^{-8/3} \lambda^{2/3}. \tag{1.8}$$

Theorem 1.2. *Let $d = 2$ and $p \geq 2$. Then*

$$\limsup_{t\to+\infty} \frac{(\log t)^p}{t(\log\log t)^{p-1}} V_r(t) = (2\pi)^p \left(\frac{2}{p}\right)^{p-1} \kappa(2,p)^{2p} \qquad a.s. \tag{1.9}$$

Let $d = 3$ and $p = 2$. Then

$$\limsup_{t\to+\infty} \frac{1}{\sqrt{t(\log\log t)^3}} V_r(t) = (2\pi r)^2 \kappa(3,2)^4 \qquad a.s. \tag{1.10}$$

Remark 1.1. *(1). Let us compare Theorem 1.1 with the results of van den Berg et al. in [8]. Theorem 4 in [8] gives us the following estimates:*

$$\lim_{c\to\infty} c I_2^{2\pi}(c) = (2\pi)^{-2} \kappa(2,2)^{-4}, \quad \lim_{c\to\infty} c^{\frac{1}{3}} I_3^{\kappa_r}(c) = (2\pi r)^{-4/3} \kappa(3,2)^{-8/3}.$$

On the other hand, (1.1) and (1.2) can be written as: for $d = 2$ and $p = 2$,

$$\lim_{t\to+\infty} \frac{1}{\log t} \log \mathbf{P}\left(V_r(t) \geq \frac{t}{c \log t} \right) = -I_2^{2\pi}(c) \tag{1.11}$$

and for $d=3$ and $p=2$,

$$\lim_{t\to+\infty}\frac{c^{1/3}}{t^{1/3}}\log\mathbf{P}\left(V_r(t)\geq\frac{t}{c}\right)=-I_3^{\kappa_r}(c). \tag{1.12}$$

Therefore, for any $\lambda>0$,

$$\lim_{c\to\infty}\lim_{t\to\infty}\frac{c}{\log t}\log\mathbf{P}\left(V_r(t)\geq\frac{\lambda t}{c\log t}\right)=-(2\pi)^{-2}\kappa(2,2)^{-4}\lambda$$

as $d=2$ and $p=2$; and

$$\lim_{c\to\infty}\lim_{t\to\infty}\frac{c^{2/3}}{t^{1/3}}\log\mathbf{P}\left(V_r(t)\geq\frac{\lambda t}{c}\right)=-(2\pi r)^{-4/3}\kappa(3,2)^{-8/3}\lambda^{2/3}$$

as $d=3$ and $p=2$. The large deviation results in [8] give us some hints for the moderate deviations results and also suggest that the optimal conditions on $b(t)$ are $b(t)=o(\log t)$ for $d=2$ and $b(t)=o(t^{1/3})$ for $d=3$. Similarly, we can also guess the moderate deviations for the intersection of Wiener sausages in $d=4$ case from results in [8] (also see [13]).

(2). Theorem 1.1 is a complement of van den Berg, Bolthausen and den Hollander's results ([8]). In the case of $d=2$, (1) in Theorem 1.1 answers the problem (1.3). In the case of $d=3$, there exists a gap for answer of the problem (1.3) since we require a slightly stronger condition $b(t)=o(t^{1/3}/(\log t)^2)$ than $b(t)=o(t^{1/3})$. In [13], the author has obtained a complete result for the intersection of the range of the random walk in which the fact that the range $R_n\leq n$ is used. In the case of Wiener sausage, we can not find a proper upper bound for $|W_r(t)|$.

(3). The intersection local time of Brownian motions plays an important role in our proof. By classical results of Dvoretzky, Erdös, Kakutani and Taylor ([29]), we know

$$S=\bigcap_{i=1}^{p}\{x\in\mathbf{R}^d : x=\beta_i(t)\text{ for some } t\in(0,\infty)\},$$

contains points different from the starting point if and only if $p(d-2)<d$. So, the intersection local time of Brownian motions does not exist in the case of $p(d-2)\geq d$. The corresponding moderate deviations need a further study.

The proof of Theorem 1.1 is based on the weak and L^p-convergence results for the Wiener sausage in [20] and the high moment method that were developed in [2], [12],[14] and [26]. The key component is the moment estimates and Feynman-Kac semigroup approach. The proofs of the main results are given respectively in Section 2 and Section 3. The proofs of some technical lemmas are delayed to Sections 4, 5 and 6. Our proofs draw on some ideas and techniques in [12].

2 Moderate deviations

In this section, we give the proof of Theorem 1.1. Since $V_r(t)$ is a non-negative random variable, by a version of Gärtner-Ellis theorem advised by Chen ([12]), in order to prove Theorem 1.1, we need only to prove the following result.

Theorem 2.1. *(1). Let $d=2$ and $p \geq 2$ and let $b(t)$ be a positive function satisfying (1.4). Then for any $\theta > 0$,*

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{b(t)} \log \sum_{m=0}^{\infty} \frac{\theta^m}{m!} \left(\frac{b(t)(\log t)^p}{t} \right)^{\frac{m}{p}} (\mathbf{E}V_r^m(t))^{\frac{1}{p}} \\
=&\frac{1}{p} \left(\frac{2(p-1)}{p} \right)^{p-1} (2\pi\theta)^p \kappa(2,p)^{2p}.
\end{aligned} \tag{2.1}$$

(2). Let $d=3$ and $p=2$ and let $b(t)$ be a positive function satisfying (1.7). Then for any $\theta > 0$,

$$\lim_{t\to\infty} \frac{1}{b(t)} \log \sum_{m=0}^{\infty} \frac{\theta^m}{m!} \left(\frac{b(t)}{t} \right)^{\frac{m}{4}} \sqrt{\mathbf{E}V_r^m(t)} = 2\left(\frac{3}{4}\right)^3 (2\pi\theta r)^4 \kappa(3,2)^8. \tag{2.2}$$

We only prove the case of $d=2$. The proof of the case of $d=3$ is analogous. For simplicity, we denote by $l_t = \frac{t}{b(t)}$.

2.1 Upper bound

We apply the high moment method to prove the upper bound (cf. [12]). The proof is based on L^p-convergence results for the Wiener sausage in [20] and the high moment estimates of the volume of intersections of Wiener sausages.

By the scaling property of Brownian motion, we can easily get the following L^p-convergence results from Corollary 3.2 in [20].

Lemma 2.1. *As $d=2$, $p \geq 2$, $m = 1, 2, \cdots$,*

$$\lim_{t\to\infty} \frac{(\log t)^{pm}}{t^m} \mathbf{E}V_r^m(t) = (2\pi)^{pm} \mathbf{E}(\alpha([0,1]^p))^m \tag{2.3}$$

and as $d=3$, $p=2$, $m=1,2,\cdots$,

$$\lim_{t\to\infty} \frac{1}{t^{m/2}} \mathbf{E}V_r^m(t) = (2\pi r)^{2m} \mathbf{E}(\alpha([0,1]^p))^m \tag{2.4}$$

where $\alpha([0,1])^p$ is the p-multiple intersection local time for Brownian motions, the quantity formally written as

$$\alpha([0,1]^p) = \int_{\mathbb{R}^d} \left(\prod_{j=1}^{p} \int_0^1 \delta_x(\beta_j(s)) ds \right) dx.$$

The key estimates in the upper bound are the following moment estimates. Their proofs will be given in Section 4.

Lemma 2.2. *For any integer $a \geq 1$, let $t_1, t_2, \cdots, t_a$ be positive real numbers satisfying $t_1 + \cdots + t_a = t$. Then for any integer $m \geq 1$,*

$$(\mathbf{E}V_r^m(t))^{\frac{1}{p}} \leq \sum_{\substack{k_1+k_2+\cdots+k_a=m,\\ k_1,k_2,\cdots,k_a \geq 0}} \frac{m!}{k_1!k_2!\cdots k_a!} \prod_{i=1}^{a} (\mathbf{E}(V_r^{k_i}(t_i)))^{\frac{1}{p}}. \tag{2.5}$$

Consequently, for any $\theta > 0$,

$$\sum_{m=0}^{\infty} \frac{\theta^m}{m!} (\mathbf{E} V_r^m(t))^{\frac{1}{p}} \le \prod_{i=1}^{a} \sum_{m=0}^{\infty} \frac{\theta^m}{m!} (\mathbf{E}(V_r^m(t_i)))^{\frac{1}{p}}. \tag{2.6}$$

Lemma 2.3. *There is a constant C depending only on d and p such that:*

(1). When $d = 2$ and $p \ge 2$,

$$\sup_{t\ge 3} \sup_{y_1,y_2,\cdots,y_p} \mathbf{E}^{(y_1,y_2,\cdots,y_p)} \exp\left\{ C \left(\frac{(\log t)^p}{t} \right)^{1/(p-1)} (V_r(t))^{1/(p-1)} \right\} < \infty. \tag{2.7}$$

(2). When $d = 3$ and $p = 2$,

$$\sup_{t\ge 3} \sup_{y_1,y_2} \mathbf{E}^{y_1,y_2} \exp\left\{ \frac{C}{t^{1/3}} V_r(t)^{2/3} \right\} < \infty. \tag{2.8}$$

The proof of the upper bound

Let $s > 0$ be fixed. By Lemma 2.2, we get

$$\begin{aligned} &\sum_{m=0}^{\infty} \frac{\theta^m}{m!} \left(\frac{b(t)(\log t)^p}{t} \right)^{\frac{m}{p}} (\mathbf{E} V_r^m(t))^{\frac{1}{p}} \\ \le &\left(\sum_{m=0}^{\infty} \frac{\theta^m}{m!} \left(\frac{b(t)(\log t)^p}{t} \right)^{\frac{m}{p}} (\mathbf{E}(V_r^m(sl_t)) \right)^{\frac{1}{p}} \Bigg)^{[\frac{t}{sl_t}]+1}. \end{aligned}$$

By (2.3), Lemma 2.3, and dominated convergence theorem, we have

$$\begin{aligned} &\lim_{t\to\infty} \sum_{m=0}^{\infty} \frac{\theta^m}{m!} \left(\frac{b(t)(\log t)^p}{t} \right)^{\frac{m}{p}} (\mathbf{E}(V_r^m(sl_t)))^{\frac{1}{p}} \\ = &\sum_{m=0}^{\infty} \frac{(2\pi\theta)^m}{m!} s^{\frac{m}{p}} (\mathbf{E}(\alpha([0,1]^p)^m))^{\frac{1}{p}}. \end{aligned} \tag{2.9}$$

So,

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{b(t)} \log \sum_{m=0}^{\infty} \frac{\theta^m}{m!} \left(\frac{b(t)(\log t)^p}{t} \right)^{\frac{m}{p}} (\mathbf{E} V_r^m(t))^{\frac{1}{p}} \\ \le &\frac{1}{s} \log \sum_{m=0}^{\infty} \frac{(2\pi\theta)^m}{m!} s^{\frac{m}{p}} (\mathbf{E}\alpha([0,1]^p)^m)^{\frac{1}{p}}. \end{aligned}$$

Now applying the large deviation result for Brownian intersection local time given by Chen (Theorem 2.1, [11]), we obtain the upper bound:

$$\begin{aligned} &\lim_{s\to\infty} \frac{1}{s} \log \sum_{m=0}^{\infty} \frac{(2\pi\theta)^m}{m!} s^{\frac{m}{p}} (\mathbf{E}\alpha([0,1]^p)^m)^{\frac{1}{p}} \\ = &\sup_{\lambda>0} \left\{ 2\pi\theta\lambda^{\frac{1}{p}} - \frac{1}{2}\kappa(2,p)^{-\frac{2p}{p-1}} \lambda^{\frac{1}{p-1}} \right\} \\ = &\frac{1}{p} \left(\frac{2(p-1)}{p} \right)^{p-1} (2\pi\theta)^p \kappa(2,p)^{2p}. \end{aligned}$$

2.2 Lower bound

We will use the Feynman-Kac semigroup method (cf. [12]) to prove the lower bound. The key is to find a proper Feynman-Kac type operator and obtain a good lower bound.

Notice that for any non-negative, bounded and uniformly continuous function f on $\mathbb{R}^2$ with $||f||_q = 1$, for any integer $m \geq 1$,

$$\begin{aligned}
&\mathbf{E}\left(\int_{\mathbb{R}^2} f\left(\sqrt{\frac{b(t)}{t}}x\right) I_{\{x\in W_r(t)\}}dx\right)^m \\
=&\left(\frac{t}{b(t)}\right)^m \int_{\mathbb{R}^{2m}} \prod_{k=1}^m f(x_k)\mathbf{E}\left(\prod_{k=1}^m I_{\{\sqrt{\frac{t}{b(t)}}x_k\in W_r(t)\}}\right) dx_1\cdots dx_m \\
\leq&||f||_q^m \left(\frac{t}{b(t)}\right)^m \left(\int_{\mathbb{R}^{2m}} \left(\mathbf{E}\prod_{k=1}^m I_{\{\sqrt{\frac{t}{b(t)}}x_k\in W_r(t)\}}\right)^p dx_1\cdots dx_m\right)^{1/p} \qquad (2.10)\\
=&\left(\frac{t}{b(t)}\right)^{\frac{p-1}{p}m} \left(\int_{\mathbb{R}^{2m}} \mathbf{E}\prod_{j=1}^p\prod_{k=1}^m I_{\{x_k\in W_r^j(t)\}}dx_1\cdots dx_m\right)^{1/p} \\
=&\left(\frac{t}{b(t)}\right)^{\frac{p-1}{p}m} (\mathbf{E}V_r^m(t))^{1/p}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{m=0}^\infty \frac{\theta^m}{m!}\left(\frac{b(t)(\log t)^p}{t}\right)^{m/p} (\mathbf{E}V_r^m(t))^{1/p} \\
\geq&\sum_{m=0}^\infty \frac{\theta^m}{m!}\left(\frac{b(t)\log t}{t}\right)^m \mathbf{E}\left(\int_{\mathbb{R}^2} f\left(\sqrt{\frac{b(t)}{t}}x\right) I_{\{x\in W_r(t)\}}dx\right)^m \qquad (2.11)\\
=&\mathbf{E}\left(\exp\left\{\theta\frac{b(t)\log t}{t}\int_{\mathbb{R}^2} f\left(\sqrt{\frac{b(t)}{t}}x\right) I_{\{x\in W_r(t)\}}dx\right\}\right).
\end{aligned}$$

Therefore, the lower bound transfer to estimate the following linear operator T on $L^2(\mathbb{R}^d)$ defined by

$$(T\xi)(x) = \mathbf{E}_x\left(\exp\left\{\int_{\mathbb{R}^d} f\left(\sqrt{\frac{b(t)}{t}}y\right) I_{\{y\in W_r(t)\}}dy\right\}\xi(\beta(t))\right), \quad \xi\in L^2(\mathbb{R}^d).$$

Lemma 2.4. *T is a self-adjoint operator on $L^2(\mathbb{R}^d)$.*

Proof. For any $\xi, \eta \in L^2(\mathbb{R}^d)$,

$$\begin{aligned}
\langle \eta, T\xi \rangle =& \int_{\mathbb{R}^d} \eta(x) \mathbf{E}_x \left(\exp\left\{ \int_{\mathbb{R}^d} f\left(\sqrt{l_t^{-1}} y\right) I_{\{y \in W_r(t)\}} dy \right\} \xi(\beta(t)) \right) dx \\
=& \int_{\mathbb{R}^d} \eta(x) \mathbf{E} \left(\exp\left\{ \int_{\mathbb{R}^d} f\left(\sqrt{l_t^{-1}} (x+y)\right) I_{\{y \in W_r(t)\}} dy \right\} \xi(x+\beta(t)) \right) dx \\
=& \mathbf{E} \left(\int_{\mathbb{R}^d} \eta(x) \exp\left\{ \int_{\mathbb{R}^d} f\left(\sqrt{l_t^{-1}} (x+y)\right) I_{\{y \in W_r(t)\}} dy \right\} \xi(x+\beta(t)) dx \right) \\
=& \mathbf{E} \left(\int_{\mathbb{R}^d} \eta(x-\beta(t)) \exp\left\{ \int_{\mathbb{R}^d} f\left(\sqrt{l_t^{-1}} (x+y-\beta(t))\right) I_{\{y \in W_r(t)\}} dy \right\} \xi(x) dx \right) \\
=& \mathbf{E} \left(\int_{\mathbb{R}^d} \eta(x+\beta'(t)) \exp\left\{ \int_{\mathbb{R}^d} f\left(\sqrt{l_t^{-1}} (x+y)\right) I_{\{y \in W'_r(t)\}} dy \right\} \xi(x) dx \right) \\
=& \mathbf{E} \left(\int_{\mathbb{R}^d} \eta(x+\beta(t)) \exp\left\{ \int_{\mathbb{R}^d} f\left(\sqrt{l_t^{-1}} (x+y)\right) I_{\{y \in W_r(t)\}} dy \right\} \xi(x) dx \right) \\
=& \langle T\eta, \xi \rangle
\end{aligned}$$

where $\{\beta'(s) = -\beta(t) + \beta(t-s), 0 \le s \le t\} \overset{d}{=} \{\beta(s), 0 \le s \le t\}$, and $W'_r(t)$ is the Wiener sausage associated with $\beta'(s)$.

□

The following lemma plays an important role in the lower bound. Its proof is given in section 6.

Lemma 2.5. *Let f be bounded and continuous on $\mathbb{R}^d$.*
(1). If $d = 2$ and $b(t)$ satisfies (1.4), then

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{b(t)} \log \mathbf{E} \left(\exp\left\{ \frac{b(t)\log t}{2\pi t} \int_{\mathbb{R}^2} f\left(\sqrt{\frac{b(t)}{t}} x\right) I_{\{x \in W_r(t)\}} dx \right\} \right) \\
&\ge \sup_{g \in \mathscr{H}_2} \left\{ \int_{\mathbb{R}^2} f(x) g^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^2} \langle \nabla g(x), \nabla g(x) \rangle dx \right\}.
\end{aligned} \tag{2.12}$$

(2). If $d = 3$ and $b(t)$ satisfies (1.7), then

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{b(t)} \log \mathbf{E} \left(\exp\left\{ \frac{b(t)}{2\pi r t} \int_{\mathbb{R}^3} f\left(\sqrt{\frac{b(t)}{t}} x\right) I_{\{x \in W_r(t)\}} dx \right\} \right) \\
&\ge \sup_{g \in \mathscr{H}_3} \left\{ \int_{\mathbb{R}^3} f(x) g^2(x) dx - \frac{1}{2} \int_{\mathbb{R}^3} \langle \nabla g(x), \nabla g(x) \rangle dx \right\}.
\end{aligned} \tag{2.13}$$

The proof of the lower bound

We now complete the proof of the lower bound. By (2.11) and Lemma 2.5, we have

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{b(t)}\log\sum_{m=0}^{\infty}\frac{\theta^m}{m!}\left(\frac{b(t)(\log t)^p}{t}\right)^{m/p}(\mathbf{E}V_r^m(t))^{1/p}\\ &\geq\sup_{g\in\mathscr{H}_2}\left\{2\pi\theta\int_{\mathbb{R}^2}f(x)g^2(x)dx-\frac{1}{2}\int_{\mathbb{R}^2}\langle\nabla g(x),\nabla g(x)\rangle dx\right\}.\end{aligned}$$

Taking supremum over all non-negative, bounded and uniformly continuous function f on $\mathbb{R}^2$ with $||f||_q=1$, the right hand side becomes

$$\begin{aligned}&\sup_{g\in\mathscr{H}_2}\left\{2\pi\theta\left(\int_{\mathbb{R}^2}|g(x)|^{2p}dx\right)^{1/p}-\frac{1}{2}\int_{\mathbb{R}^2}\langle\nabla g(x),\nabla g(x)\rangle dx\right\}\\ &=\frac{1}{p}\left(\frac{2(p-1)}{p}\right)^{p-1}(2\pi\theta)^p\kappa(2,p)^{2p}\end{aligned}\tag{2.14}$$

where the last step follows from Lemma A.2 in [11].

3 Laws of the iterated logarithm

We prove Theorem 1.2 in this section. The upper bound of the law of the iterated logarithm is a direct application of Theorem 1.1 and Borel-Cantelli lemma. So we only give proof of the lower bound. Because the proof of the case of $d=3$ is analogous, we only prove the case of $d=2$. That is

$$\limsup_{t\to\infty}\frac{(\log t)^p}{t(\log\log t)^{p-1}}V_r(t)\geq(2\pi)^p\left(\frac{2}{p}\right)^{p-1}\kappa(2,p)^{2p},\qquad a.s.\tag{3.1}$$

We first give some notations and a basic lemma which is the main tool to prove (3.1). For each $\bar{x}=(x_1,x_2,\cdots,x_p)\in(\mathbb{R}^2)^p$, write $\|\bar{x}\|=\max_{1\leq j\leq p}|x_j|$, and let $\mathbf{P}^{\bar{x}}$ denote the probability induced by the p independent Brownian motions $\beta_1(s),\beta_2(s),\cdots\beta_p(s)$ starting at $x_1,x_2,\cdots,x_p$, respectively. $\mathbf{E}^{\bar{x}}$ denotes the expectation correspondent to $\mathbf{P}^{\bar{x}}$. To be consistent with the notation we used before, we have $\mathbf{P}^{0,0,\cdots,0}=\mathbf{P},\mathbf{E}^{0,0,\cdots,0}=\mathbf{E}$.

Lemma 3.1. *(1). Let $d=2$ and $p\geq 2$ and let (1.4) hold. Then*

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{b(t)}\log\inf_{||\bar{x}||\leq\sqrt{\frac{t}{b(t)}}}\mathbf{P}^{\bar{x}}\left(V_r(t)\geq\lambda\frac{t}{(\log t)^p}b(t)^{p-1}\right)\\ &\geq-\frac{p}{2}(2\pi)^{-\frac{p}{p-1}}\kappa(2,p)^{-\frac{2p}{p-1}}\lambda^{\frac{p}{p-1}}.\end{aligned}\tag{3.2}$$

2). Let $d=3$ and $p=2$ and let (1.7) hold. Then

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{b(t)}\log\inf_{||\bar{x}||\leq\sqrt{\frac{t}{b(t)}}}\mathbf{P}^{\bar{x}}\left(V_r(t)\geq\lambda\sqrt{tb(t)^3}\right)\\ &\geq-(2\pi r)^{-\frac{4}{3}}\kappa(2,p)^{-\frac{8}{3}}\lambda^{\frac{2}{3}}.\end{aligned}\tag{3.3}$$

Proof. The proof is analogous to Lemma 7 in [12]. We only prove (3.2). For given $\bar{y} = (y_1, y_2, \cdots, y_p) \in (\mathbb{R}^2)^p$ and $m \geq 1$,

$$\begin{aligned}
\mathbf{E}^{\bar{y}} V_r^m(t) &= \mathbf{E}\left(\int_{\mathbb{R}^2} \prod_{j=1}^p I_{\{x+y_j \in W_r^j(t)\}} dx\right)^m \\
&= \int_{(\mathbb{R}^2)^m} \prod_{j=1}^p \mathbf{E}\left(\prod_{k=1}^m I_{\{x_k+y_j \in W_r^j(t)\}}\right) dx_1 \cdots dx_m \\
&\leq \prod_{j=1}^p \left(\int_{(\mathbb{R}^2)^m} \left(\mathbf{E}\left(\prod_{k=1}^m I_{\{x_k+y_j \in W_r^j(t)\}}\right)\right)^p dx_1 \cdots dx_m\right)^{1/p}. \\
&= \int_{(\mathbb{R}^2)^m} \left(\mathbf{E}\left(\prod_{k=1}^m I_{\{x_k \in W_r(t)\}}\right)\right)^p dx_1 \cdots dx_m \\
&= \mathbf{E}(V_r^m(t))
\end{aligned}$$

Therefore, by (2.1), we have

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{b(t)} \log \sum_{m=0}^\infty \frac{\theta^m}{m!} \left(\frac{b(t)(\log t)^p}{t}\right)^{m/p} (\sup_{\bar{y}} \mathbf{E}^{\bar{y}} V_r^m(t))^{1/p} \\
&\leq \frac{1}{p}\left(\frac{2(p-1)}{p}\right)^{p-1} (2\pi\theta)^p \kappa(2,p)^{2p}
\end{aligned} \tag{3.4}$$

It is easy to see from Theorem 4 in [12] that (3.2) is a consequence of (3.4) and the following

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{b(t)} \log \sum_{m=0}^\infty \frac{\theta^m}{m!} \left(\frac{b(t)(\log t)^p}{t}\right)^{m/p} \left(\inf_{||\bar{y}|| \leq \sqrt{l_t}} \mathbf{E}^{\bar{y}} V_r^m(t)\right)^{1/p} \\
&\geq \frac{1}{p}\left(\frac{2(p-1)}{p}\right)^{p-1} (2\pi\theta)^p \kappa(2,p)^{2p}.
\end{aligned} \tag{3.5}$$

For $\epsilon > 0$, set $B_t(x) = \left\{y; |x-y| \leq \epsilon\sqrt{l_t}\right\}$, $B_t = B_t(0)$ and

$$V_{r,\epsilon}(t) = \int_{\mathbb{R}^2} \prod_{j=1}^p \left(\int_{z\in B_t} \frac{1}{|B_t|} I_{\{x-z \in W_r^j(t)\}} dz\right) dx$$

where $|B_t|$ denotes the Lebesgue measure of the set B_t. For any function f on $\mathbb{R}^2$, define

$$f_\epsilon(x) = \frac{1}{\pi\epsilon^2} \int_{|y|\leq\epsilon} f(x+y) dy.$$

Let f be a non-negative, bounded and uniformly continuous function on $\mathbb{R}^2$ with $||f||_q = 1$. Similar to (2.10), for any integer $m \geq 1$ we have

$$\mathbf{E}\left(\int_{\mathbb{R}^2} f\left(\sqrt{l_t^{-1}}x\right)\left(\int_{z\in B_t} \frac{1}{|B_t|} I_{\{x-z\in W_r(t)\}} dz\right) dx\right)^m \leq (l_t)^{\frac{p-1}{p}m} (\mathbf{E}V_{r,\epsilon}^m(t))^{1/p} \tag{3.6}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^2} f\left(\sqrt{l_t^{-1}}x\right)\left(\int_{z\in B_t}\frac{1}{|B_t|}I_{\{x-z\in W_r(t)\}}dz\right)dx\\
=&\int_{\mathbb{R}^2} f\left(\sqrt{l_t^{-1}}(x+z)\right)\left(\int_{z\in B_t}\frac{1}{|B_t|}I_{\{x\in W_r(t)\}}dz\right)dx\\
=&l_t\int_{\mathbb{R}^2}\frac{1}{|B_t|}I_{\{x\in W_r(t)\}}\left(\int_{|z|\le\epsilon} f\left(\sqrt{l_t^{-1}}x+z\right)dz\right)dx\\
=&\int_{\mathbb{R}^2} I_{\{x\in W_r(t)\}}f_\epsilon\left(\sqrt{l_t^{-1}}x\right)dx.
\end{aligned}$$

Hence, we have

$$\mathbf{E}\left(\int_{\mathbb{R}^2} I_{\{x\in W_r(t)\}}f_\epsilon\left(\sqrt{l_t^{-1}}x\right)dx\right)^m \le (l_t)^{\frac{p-1}{p}m}\left(\mathbf{E}V_{r,\epsilon}^m(t)\right)^{1/p}$$

and

$$\begin{aligned}
&\sum_{m=0}^\infty \frac{\theta^m}{m!}\left(\frac{b(t)(\log t)^p}{t}\right)^{m/p}\left(\mathbf{E}V_{r,\epsilon}^m(t)\right)^{1/p}\\
\ge&\sum_{m=0}^\infty \frac{\theta^m}{m!}\left(\frac{b(t)\log t}{t}\right)^{m}\mathbf{E}\left(\int_{\mathbb{R}^2} I_{\{x\in W_r(t)\}}f_\epsilon\left(\sqrt{l_t^{-1}}x\right)dx\right)^m\\
=&\mathbf{E}\exp\left\{\theta\frac{b(t)\log t}{t}\int_{\mathbb{R}^2} I_{\{x\in W_r(t)\}}f_\epsilon\left(\sqrt{l_t^{-1}}x\right)dx\right\}.
\end{aligned}$$

By lemma 2.5, we have

$$\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{b(t)}\log\sum_{m=0}^\infty \frac{\theta^m}{m!}\left(\frac{b(t)(\log t)^p}{t}\right)^{m/p}(\mathbf{E}V_{r,\epsilon}^m(t))^{1/p}\\
\ge&\sup_{g\in\mathscr{H}_2}\{2\pi\theta\int_{\mathbb{R}^2} f_\epsilon(x)g^2(x)-\frac{1}{2}\int_{\mathbb{R}^2}\langle\nabla g(x),\nabla g(x)\rangle dx\}\\
=&\sup_{g\in\mathscr{H}_2}\left\{2\pi\theta\int_{\mathbb{R}^2} f(x)(g^2)_\epsilon(x)-\frac{1}{2}\int_{\mathbb{R}^2}\langle\nabla g(x),\nabla g(x)\rangle dx\right\}.
\end{aligned}$$

Taking supremum over all non-negative, bounded and uniformly continuous function f on $\mathbb{R}^2$ with $||f||_q=1$ in the last inequality, we get

$$\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{b(t)}\log\sum_{m=0}^\infty \frac{\theta^m}{m!}\left(\frac{b(t)(\log t)^p}{t}\right)^{m/p}\left(\mathbf{E}V_{r,\epsilon}^m(t)\right)^{1/p}\\
\ge&\sup_{g\in\mathscr{H}_2}\left\{2\pi\theta\left(\int_{\mathbb{R}^2}|(g^2)_\epsilon(x)|^p dx\right)^{1/p}-\frac{1}{2}\int_{\mathbb{R}^2}\langle\nabla g(x),\nabla g(x)\rangle dx\right\}.
\end{aligned}\tag{3.7}$$

Take $l_t = \frac{t}{b(t)}$. Then

$$\begin{aligned}
\mathbf{E}^{\bar{y}} V_r^m(t) &= \mathbf{E}\left(\int_{\mathbb{R}^2} \prod_{j=1}^p I_{\{x+y_j \in W_r^j(t)\}} dx\right)^m \\
&= \int_{(\mathbb{R}^2)^m} \prod_{j=1}^p \mathbf{E}\left(\prod_{k=1}^m I_{\{x_k+y_j \in W_r^j(t)\}}\right) dx_1 \cdots dx_m \\
&\geq \int_{(\mathbb{R}^2)^m} \prod_{j=1}^p \mathbf{E}\left(\prod_{k=1}^m I_{\{x_k+y_j \in W_r^j([l_t,t])\}}\right) dx_1 \cdots dx_m \\
&\geq \int_{(\mathbb{R}^2)^m} \prod_{j=1}^p \mathbf{E}\left(\prod_{k=1}^m I_{\{\beta_j(l_t) \in B_t(y_j)\}} I_{\{x_k+y_j-\beta_j(l_t) \in \tilde{W}_r^j(t-l_t)\}}\right) dx_1 \cdots dx_m
\end{aligned}$$

where $\tilde{\beta}_j(t) = \beta_j(t+l_t) - \beta_j(l_t)$. Notice that

$$\begin{aligned}
&\mathbf{E}\left(\prod_{k=1}^m I_{\{\beta_j(l_t) \in B_t(y_j)\}} I_{\{x_k+y_j-\beta_j(l_t) \in \tilde{W}_r^j(t-l_t)\}}\right) \\
=&\mathbf{E}\left(I_{\{\beta_j(l_t) \in B_t(y_j)\}} \prod_{k=1}^m I_{\{x_k+y_j-\beta_j(l_t) \in \tilde{W}_r^j(t-l_t)\}}\right) \\
=&\int_{z \in B_t(y_j)} p_{l_t}(z) \mathbf{E}\left(\prod_{k=1}^m I_{\{x_k+y_j-z \in \tilde{W}_r^j([0,t-l_t])\}}\right) dz \\
\geq& \min_{1\leq j\leq p} \inf_{z \in B_t(y_j)} p_{l_t}(z) \int_{z \in B_t} \mathbf{E}\left(\prod_{k=1}^m I_{\{x_k-z \in W_r^j(t-l_t)\}}\right) dz \\
=&\gamma(t) \int_{z \in B_t} \mathbf{E}\left(\prod_{k=1}^m I_{\{x_k-z \in W_r^j(t-l_t)\}}\right) dz
\end{aligned}$$

where $\gamma(t) = \min_{1\leq j\leq p} \inf_{z\in B_t(y_j)} p_{l_t}(z) = \min_{1\leq j\leq p} \inf_{z\in B_t(y_j)} \frac{1}{2\pi l_t} e^{-z^2/2l_t}$ and $p_{l_t}(z)$ is the probability density of $\beta(l_t)$). We have

$$\begin{aligned}
\mathbf{E}^{\bar{y}} V_r^m(t) &\geq (\gamma(t))^p \int_{(\mathbb{R}^2)^m} \prod_{j=1}^p \left(\int_{z \in B_t} \mathbf{E}\left(\prod_{k=1}^m I_{\{x_k-z \in W_r^j(t-l_t)\}}\right) dz\right) dx_1 \cdots dx_m \\
=&(\gamma(t))^p \mathbf{E}\left(\int_{(\mathbb{R}^2)^m} \prod_{j=1}^p \left(\int_{z \in B_t} \prod_{k=1}^m I_{\{x_k-z \in W_r^j(t-l_t)\}} dz\right) dx_1 \cdots dx_m\right) \\
=&(\gamma(t))^p \mathbf{E}\left(\int_{(\mathbb{R}^2)^m} \int_{z_1 \in B_t} \cdots \int_{z_p \in B_t} \prod_{j=1}^p \prod_{k=1}^m I_{\{x_k-z_j \in W_r^j(t-l_t)\}} dz_1 \cdots dz_p dx_1 \cdots dx_m\right)
\end{aligned}$$

$$
\begin{aligned}
=&(\gamma(t))^p \mathbf{E}\left(\int_{z_1\in B_t}\cdots\int_{z_p\in B_t}\left(\int_{\mathbb{R}^2}\prod_{j=1}^p I_{\{x-z_j\in W_r^j(t-l_t)\}}dx\right)^m dz_1\cdots dz_p\right)\\
\geq&(\gamma(t))^p|B_t|^p \mathbf{E}\left(\int_{z_1\in B_t}\cdots\int_{z_p\in B_t}\frac{1}{|B_t|^p}\left(\int_{\mathbb{R}^2}\prod_{j=1}^p I_{\{x-z_j\in W_r^j(t-l_t)\}}dx\right)dz_1\cdots dz_p\right)^m\\
=&(\gamma(t))^p|B_t|^p \mathbf{E}\left(\int_{\mathbb{R}^2}\frac{1}{|B_t|}\prod_{j=1}^p\left(\int_{z\in B_t} I_{\{x-z\in W_r^j(t-l_t)\}}dz\right)dx\right)^m
\end{aligned}
$$

where the fifth step follows from Jensen's inequality. It follows from $\inf_{t>0}\gamma(t)|B_t|>0$ that for some constant $\delta>0$ and for any $t>0$,

$$
\inf_{||\bar{y}||\leq\sqrt{l_t}}\mathbf{E}^{\bar{y}}V_r^m(t)\geq\delta\mathbf{E}\left(\int_{\mathbb{R}^2}\frac{1}{|B_t|}\prod_{j=1}^p(\int_{z\in B_t}I_{\{x-z\in W_r^j(t-l_t)\}}dz)dx\right)^m.
$$

By (3.7) with t replaced by $t-l_t$, we obtain

$$
\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{b(t)}\log\sum_{m=0}^\infty\frac{\theta^m}{m!}\left(\frac{b(t)(\log t)^p}{t}\right)^{m/p}\left(\inf_{||\bar{y}||\leq\sqrt{l_t}}\mathbf{E}^{\bar{y}}V_r^m(t)\right)^{1/p}\\
&\geq\sup_{g\in\mathscr{H}_2}\left\{2\pi\theta(\int_{\mathbb{R}^2}|(g^2)_\epsilon(x)|^pdx)^{1/p}-\frac{1}{2}\int_{\mathbb{R}^2}\langle\nabla g(x),\nabla g(x)\rangle dx\right\}.
\end{aligned}
$$

Finally, we let $\epsilon\to 0^+$ on the right hand. Then (3.5) follows from (2.14). □

We now complete the proof of (3.1). Let $n_k=k^k$, $k\geq 1$. Since $V_r(n_{k+1})\geq V_r([n_k,n_{k+1}])$, we only need to prove that for any

$$
\lambda<(2\pi)^p\,(2/p)^{p-1}\,\kappa(2,p)^{2p},
$$

$$
\limsup_{k\to\infty}\frac{(\log n_{k+1})^p}{n_{k+1}(\log\log n_{k+1})^{p-1}}V_r([n_k,n_{k+1}])\geq\lambda\qquad a.s. \tag{3.8}
$$

First, applying (3.2) with $b(t)=\log\log\max\{t,27\}$, we get

$$
\sum_k\inf_{||\bar{x}||\leq\sqrt{n_{k+1}/\log\log n_{k+1}}}\mathbf{P}^{\bar{x}}\left\{V_r(n_{k+1}-n_k)\geq\frac{n_{k+1}(\log\log n_{k+1})^{p-1}}{(\log n_{k+1})^p}\lambda\right\}=\infty. \tag{3.9}
$$

Let us consider the $2p$-dimensional Brownian motion $\bar{\beta}(s)=(\beta_1(s),\cdots,\beta_p(s))$. Then by the law of the iterated logarithm of the Brownian motion and $\sqrt{n_k\log\log n_k}=o(\sqrt{n_{k+1}/\log\log n_{k+1}})$ as $k\to\infty$, we have that with probability 1, the events

$$
\left\{||\bar{\beta}(n_k)||\leq\sqrt{n_{k+1}/\log\log n_{k+1}}\right\},\quad k=1,2,\cdots
$$

eventually hold. Therefore, (3.9) implies

$$
\sum_k\mathbf{P}^{\bar{\beta}(n_k)}\left\{V_r(n_{k+1}-n_k)\geq\frac{n_{k+1}(\log\log n_{k+1})^{p-1}}{(\log n_{k+1})^p}\lambda\right\}=\infty,\quad a.s \tag{3.10}
$$

and so by the Markov property and the conditional Borel-Cantelli lemma, (3.8) holds.

4 High Moment estimates

In this section we prove Lemmas 2.2 and 2.3. Our proofs are analogous to the case of the range of random walk ([24][12]). There also exist some technical difficulties. For examples, the range R_n of the random walk has a natural upper bound n, but $|W_r(t)| \le t$ is not true. In the case of Wiener sausage, the behavior of the Wiener sausage within a short time should be concerned.

We first prove Lemma 2.2, which is a conclusion of the following lemma.

For $\triangle \subset R$, denote by

$$W_r(\triangle) = \bigcup_{s\in\triangle} B_r(\beta(s)), \quad V_r(\triangle) = |W_r^1(\triangle) \cap W_r^2(\triangle) \cap \cdots \cap W_r^p(\triangle)|.$$

Let us make a decomposition of the domain $[0, t]$ which is come from Chen ([12]). Let $a \ge 2$ be an integer and let $t_0 = 0$ and $t_1, t_2, \cdots, t_a$ be positive real numbers satisfying $t_0 + t_1 + \cdots + t_a = t$. Write

$$\triangle_i = [t_0 + t_1 + \cdots + t_{i-1}, t_0 + t_1 + \cdots + t_i], \qquad i = 1, \cdots, a$$

and

$$A = \int_{\mathbb{R}^d} \prod_{j=1}^{p} \sum_{i=1}^{a} I_{\{y \in W_r^j(\triangle_i)\}} dy.$$

Then it is obvious from $I_{\{y\in W_r^j(t)\}} \le \sum_{i=1}^a I_{\{y\in W_r^j(\triangle_i)\}}$ that

$$V_r(t) = \int_{\mathbb{R}^d} \prod_{j=1}^{p} I_{\{y\in W_r^j(t)\}} dy \le A. \tag{4.1}$$

Lemma 4.1. *For any integer $m \ge 1$,*

$$(\mathbf{E}A^m)^{\frac{1}{p}} \le \sum_{\substack{k_1+k_2+\cdots+k_a=m \\ k_1,k_2,\cdots,k_a\ge 0}} \frac{m!}{k_1!k_2!\cdots k_a!} \prod_{i=1}^{a} \left(\mathbf{E}(V_r^{k_i}(t_i))\right)^{\frac{1}{p}}. \tag{4.2}$$

Consequently, for any $\theta > 0$,

$$\sum_{m=0}^{\infty} \frac{\theta^m}{m!} (\mathbf{E}A^m)^{\frac{1}{p}} \le \prod_{i=1}^{a} \sum_{m=0}^{\infty} \frac{\theta^m}{m!} \left(\mathbf{E}(V_r^m(t_i))\right)^{\frac{1}{p}}. \tag{4.3}$$

Proof.

$$
\begin{aligned}
(\mathbf{E}A^m)^{\frac{1}{p}} &= \left(\int_{(\mathbb{R}^d)^m} \mathbf{E}\left(\prod_{k=1}^{m}\prod_{j=1}^{p}\sum_{i=1}^{a} I_{\{y_k\in W_r^j(\triangle_i)\}}\right) dy_1 dy_2\cdots dy_m\right)^{\frac{1}{p}}\\
&= \left(\int_{(\mathbb{R}^d)^m} \left(\mathbf{E}\left(\prod_{k=1}^{m}\sum_{i=1}^{a} I_{\{y_k\in W_r(\triangle_i)\}}\right)\right)^p dy_1 dy_2\cdots dy_m\right)^{\frac{1}{p}}\\
&= \left(\int_{(\mathbb{R}^d)^m} \left(\sum_{i_1,\cdots,i_m=1}^{a} \mathbf{E}(I_{\{y_1\in W_r(\triangle_{i_1})\}}\cdots I_{\{y_m\in W_r(\triangle_{i_m})\}})\right)^p dy_1\cdots dy_m\right)^{\frac{1}{p}}\\
&\le \sum_{i_1,\cdots,i_m=1}^{a}\left(\int_{(\mathbb{R}^d)^m}\Big(\mathbf{E}(I_{\{y_1\in W_r(\triangle_{i_1})\}}\cdots I_{\{y_m\in W_r(\triangle_{i_m})\}})\Big)^p dy_1\cdots dy_m\right)^{\frac{1}{p}}
\end{aligned}
$$

Given integers $i_1, i_2, \cdots, i_m$ between 1 and a, let $k_1, k_2\cdots, k_a$ be the number of occurrences of $i=1, i=2, \cdots i=a$, respectively. Then $k_1+k_2+\cdots+k_a=m$. In order to get (4.2), we need only to prove

$$
\int_{(\mathbb{R}^d)^m}(\mathbf{E}(I_{\{y_1\in W_r(\triangle_{i_1})\}}\cdots I_{\{y_m\in W_r(\triangle_{i_m})\}}))^p dy_1\cdots dy_m \le \prod_{i=1}^{a}\mathbf{E}(V_r^{k_i}(t_i)). \tag{4.4}
$$

Let $\{\mathscr{F}_s, s\ge 0\}$ denote the σ-algebra filter generated by $\{\beta(s), s\ge 0\}$. Then

$$
\begin{aligned}
&\int_{(\mathbb{R}^d)^m}\Big(\mathbf{E}(I_{\{y_1\in W_r(\triangle_{i_1})\}}\cdots I_{\{y_m\in W_r(\triangle_{i_m})\}})\Big)^p dy_1\cdots dy_m\\
&=\int_{(\mathbb{R}^d)^{m-k_a}}\left(\int_{(\mathbb{R}^d)^{k_a}}\left(\mathbf{E}\left(\prod_{i=1}^{a} I_{\{y_1^i\in W_r(\triangle_i)\}}\cdots I_{\{y_{k_i}^i\in W_r(\triangle_i)\}}\right)\right)^p dy_1^a\cdots dy_{k_a}^a\right)\\
&\qquad\qquad dy_1^1\cdots dy_{k_1}^1\cdots dy_1^{a-1}\cdots dy_{k_{a-1}}^{a-1}
\end{aligned}
$$

and

$$\begin{aligned}
&\int_{(\mathbb{R}^d)^{k_a}} \left\{ \mathbf{E}\left(\prod_{i=1}^{a} I_{\{y_1^i \in W_r(\triangle_i)\}} \cdots I_{\{y_{k_i}^i \in W_r(\triangle_i)\}} \right) \right\}^p dy_1^a \cdots dy_{k_a}^a \\
&= \int_{(\mathbb{R}^d)^{k_a}} \left\{ \mathbf{E}\left\{ \mathbf{E}\left(\prod_{i=1}^{a} I_{\{y_1^i \in W_r(\triangle_i)\}} \cdots I_{\{y_{k_i}^i \in W_r(\triangle_i)\}} | \mathscr{F}_{t-t_a} \right) \right\} \right\}^p dy_1^a \cdots dy_{k_a}^a \\
&= \int_{(\mathbb{R}^d)^{k_a}} \left\{ \mathbf{E}\left(\prod_{i=1}^{a-1} I_{\{y_1^i \in W_r(\triangle_i)\}} \cdots I_{\{y_{k_i}^i \in W_r(\triangle_i)\}} \mathbf{E}_{\beta(t-t_a)} (I_{\{y_1^a \in W_r(t_a)\}} \cdots I_{\{y_{k_a}^a \in W_r(t_a)\}}) \right) \right\}^p \\
&\qquad dy_1^a \cdots dy_{k_a}^a \\
&= \mathbf{E}\left\{ \prod_{j=1}^{p} \prod_{i=1}^{a-1} I_{\{y_1^i \in W_r^j(\triangle_i)\}} \cdots I_{\{y_{k_i}^i \in W_r^j(\triangle_i)\}} \times \right. \\
&\qquad \left. \int_{(\mathbb{R}^d)^{k_a}} \prod_{j=1}^{p} \mathbf{E}_{\beta_j(t-t_a)} (I_{\{y_1^a \in W_r^j(t_a)\}} \cdots I_{\{y_{k_a}^a \in W_r^j(t_a)\}}) dy_1^a \cdots dy_{k_a}^a \right\} \\
&\le \mathbf{E}\left\{ \prod_{j=1}^{p} \prod_{i=1}^{a-1} I_{\{y_1^i \in W_r^j(\triangle_i)\}} \cdots I_{\{y_{k_i}^i \in W_r^j(\triangle_i)\}} \times \right. \\
&\qquad \left. \prod_{j=1}^{p} \left(\int_{(\mathbb{R}^d)^{k_a}} \left(\mathbf{E}_{\beta_j(t-t_a)} (I_{\{y_1^a \in W_r^j(t_a)\}} \cdots I_{\{y_{k_a}^a \in W_r^j(t_a)\}}) \right)^p dy_1^a \cdots dy_{k_a}^a \right)^{\frac{1}{p}} \right\} \\
&\le \mathbf{E}\left\{ \prod_{j=1}^{p} \prod_{i=1}^{a-1} I_{\{y_1^i \in W_r^j(\triangle_i)\}} \cdots I_{\{y_{k_i}^i \in W_r^j(\triangle_i)\}} \times \right. \\
&\qquad \left. \left(\int_{(\mathbb{R}^d)^{k_a}} \left(\mathbf{E}(I_{\{y_1^a \in W_r^1(t_a)\}} \cdots I_{\{y_{k_a}^a \in W_r^1(t_a)\}}) \right)^p dy_1^a \cdots dy_{k_a}^a \right) \right\} \\
&= \mathbf{E}\left\{ \prod_{j=1}^{p} \prod_{i=1}^{a-1} I_{\{y_1^i \in W_r^j(\triangle_i)\}} \cdots I_{\{y_{k_i}^i \in W_r^j(\triangle_i)\}} \right\} \mathbf{E}(V_r^{k_a}(t_a)).
\end{aligned}$$

Repeating this process gives (4.4). (4.3) is easy to get from (4.2). □

We now prove Lemma 2.3. We first give the following useful lemma.

Lemma 4.2. *For any integer $m \ge 1$, $\mathbf{E}(V_r^m(t)) \le (m!)^p (\mathbf{E}(V_r(t)))^m$.*

Proof. Set $T(y) = \inf\{s \ge 0; |\beta(s) - y| \le r\}$ *and* $T_j(y) = \inf\{s \ge 0; |\beta_j(s) - y| \le r\}$. $\sum_m$ denotes the set of all permutations on $\{1,2,\cdots,m\}$. Then

$$\begin{aligned}\mathbf{E}(V_r^m(t)) = & \ \mathbf{E}\left(\int_{(\mathbb{R}^d)^m}\prod_{i=1}^m\prod_{j=1}^p I_{\{y_i\in W_r^j(t)\}}dy_1\cdots dy_m\right) \\ = & \ \int_{(\mathbb{R}^d)^m}\mathbf{E}\left(\prod_{i=1}^m I_{\{y_i\in W_r(t)\}}\right)^p dy_1\cdots dy_m.\end{aligned} \tag{4.5}$$

It is easy to see that $\mathbf{P}\{y\in W_r(t)\}=\mathbf{P}\{T(y)\le t\}$, and $\mathbf{P}\{y\in W_r^j(t)\}=\mathbf{P}\{T_j(y)\le t\}$. Hence, by the strong Markov property, we have

$$\begin{aligned}&\mathbf{E}(V_r^m(t))\\ &\le \int_{(\mathbb{R}^d)^m}(\sum_{\sigma\in\sum_m}\mathbf{P}\left(T(y_{\sigma(1)})\le T(y_{\sigma(2)})\le\cdots\le T(y_{\sigma(m)})\le t\right)^p dy_1\cdots dy_m\\ &\le \int_{(\mathbb{R}^d)^m}(m!)^p\left(\mathbf{P}\left(T(y_1)\le T(y_2)\le\cdots\le T(y_m)\le t\right)\right)^p dy_1\cdots dy_m\\ &= \int_{(\mathbb{R}^d)^m}(m!)^p\left\{\mathbf{E}\left\{\mathbf{E}(I_{T(y_1)\le T(y_2)\le\cdots\le T(y_m)\le t}|\mathscr{F}_{T(y_{m-1})})\right\}\right\}^p dy_1\cdots dy_m\\ &\le (m!)^p\int_{(\mathbb{R}^d)^m}\left\{\mathbf{E}\left\{I_{\{T(y_1)\le T(y_2)\le\cdots\le T(y_{m-1})\le t\}}\mathbf{E}_{\beta(T(y_{m-1}))}(I_{\{T(y_m)\le t\}})\right\}\right\}^p dy_1\cdots dy_m\\ &= (m!)^p\int_{(\mathbb{R}^d)^m}\mathbf{E}\left\{\prod_{j=1}^p I_{\{T_j(y_1)\le T_j(y_2)\le\cdots\le T_j(y_{m-1})\le t\}}\mathbf{E}_{\beta_j(T(y_{m-1}))}(I_{\{T_j(y_m)\le t\}})\right\}dy_1\cdots dy_m\end{aligned}$$

and by Hölder's inequality,

$$\int_{\mathbb{R}^d} \mathbf{E}\left\{\prod_{j=1}^{p} I_{\{T_j(y_1)\le T_j(y_2)\le\cdots\le T_j(y_{m-1})\le t\}} \mathbf{E}_{\beta_j(T(y_{m-1}))}(I_{\{T_j(y_m)\le t\}})\right\} dy_m$$

$$= \mathbf{E}\left\{\prod_{j=1}^{p} I_{\{T_j(y_1)\le T_j(y_2)\le\cdots\le T_j(y_{m-1})\le t\}} \int_{\mathbb{R}^d}\prod_{j=1}^{p} \mathbf{E}_{\beta_j(T(y_{m-1}))}(I_{\{T_j(y_m)\le t\}}) dy_m\right\}$$

$$\le \mathbf{E}\left\{\prod_{j=1}^{p} I_{\{T_j(y_1)\le T_j(y_2)\le\cdots\le T_j(y_{m-1})\le t\}} \prod_{j=1}^{p}\left(\int_{\mathbb{R}^d} (\mathbf{E}_{\beta_j(T(y_{m-1}))}(I_{\{T_j(y_m)\le t\}}))^p dy_m\right)^{\frac{1}{p}}\right\}$$

$$= \mathbf{E}\left\{\prod_{j=1}^{p} I_{\{T_j(y_1)\le T_j(y_2)\le\cdots\le T_j(y_{m-1})\le t\}} \int_{\mathbb{R}^d} \left(\mathbf{E}_{\beta(T(y_{m-1}))}(I_{\{T(y_m)\le t\}})\right)^p dy_m\right\}$$

$$= \mathbf{E}\left(\prod_{j=1}^{p} I_{\{T_j(y_1)\le T_j(y_2)\le\cdots\le T_j(y_{m-1})\le t\}}\right) \times \mathbf{E}V_r(t).$$

Repeating this procedure yields the conclusion of Lemma 4.2. $\square$

Remark 4.1. *We borrow the method from ([24]) in which the analogous result for intersections of the ranges of random walks is given by LeGall and Rosen. But we need a different analysis when applying the Markov property at time $T(y_k)$ due to $V_r(t)$ is a continuous version of the intersections of the range of random walks.*

Using Lemma 4.2, we get the following high moment estimates.

Lemma 4.3. *There is a constant C depending only on d and p such that:*
(1). When $d = 2$ and $p \ge 2$,

$$\mathbf{E}(V_r^m(t)) \le (m!)^{p-1} C^m \left(\frac{t}{(\log t)^p}\right)^m, \quad m = 1, 2, \cdots, \quad t \ge 3. \tag{4.6}$$

(2). When $d = 3$ and $p = 2$,

$$\mathbf{E}(V_r^m(t)) \le (m!)^{\frac{3}{2}} C^m t^{\frac{m}{2}}, \quad m = 1, 2, \cdots, t \ge 3. \tag{4.7}$$

Proof. We only prove (4.6) because the proof of (4.7) is analogous. We first consider $m \le (\log t)^{\frac{p-1}{p-2}}$. Let $l(m,t) = \frac{t}{m} + 1$, write l for $l(m,t)$ for simplicity. Then, by Corollary 2.2 and Lemma 4.2 and

Lemma 2.1,

$$
\begin{aligned}
&(\mathbf{E}V_r^m(t))^{\frac{1}{p}} \\
\le & \sum_{\substack{k_1+k_2+\cdots+k_m=m\\ k_1,k_2,\cdots,k_m\ge 0}} \frac{m!}{k_1!k_2!\cdots k_m!}(\mathbf{E}(V_r^{k_1}(l)))^{\frac{1}{p}}(\mathbf{E}(V_r^{k_2}(l)))^{\frac{1}{p}}\cdots(\mathbf{E}(V_r^{k_m}(l)))^{\frac{1}{p}} \\
\le & \sum_{\substack{k_1+k_2+\cdots+k_m=m\\ k_1,k_2,\cdots,k_m\ge 0}} \frac{m!}{k_1!k_2!\cdots k_m!}k_1!k_2!\cdots k_m!(\mathbf{E}(V_r(l)))^{\frac{k_1}{p}}(\mathbf{E}V_r(l))^{\frac{k_2}{p}}\cdots(\mathbf{E}V_r(l))^{\frac{k_m}{p}} \\
= & \begin{pmatrix} 2m-1 \\ m \end{pmatrix} m!(\mathbf{E}V_r(l))^{m/p} \\
\le & \begin{pmatrix} 2m-1 \\ m \end{pmatrix} m!C^m\left(\frac{t/m}{(\log t)^p}\right)^{m/p} \\
\le & (m!)^{\frac{p-1}{p}}C^m\left(\frac{t}{(\log t)^p}\right)^{m/p}.
\end{aligned}
$$

By the help of Lemma 4.2 with $p=1$, we have $\mathbf{E}W_r^m(t)\le m!(\mathbf{E}W_r(t))^m$. For the case $m\ge(\log t)^{\frac{p-1}{p-2}}$,

$$
\begin{aligned}
&\mathbf{E}V_r^m(t)\le \mathbf{E}W_r^m(t)\le m!(\mathbf{E}W_r(t))^m \\
&\le m!C^m\left(\frac{t}{\log t}\right)^m \le (m!)^{p-1}C^m\left(\frac{t}{(\log t)^p}\right)^m\left(\frac{(\log t)^{p-1}}{m^{p-2}}\right)^m \\
&\le m!^{p-1}C^m\left(\frac{t}{(\log t)^p}\right)^m.
\end{aligned}
$$

Therefore, (4.6) is valid. □

Remark 4.2. *The proof is based on a decomposition of the time interval $[0,t]$ which comes from Lemma 1 in [12]. When m is large enough, the behavior of the Wiener sausage within a short time must be concerned.*

The proof of Lemma 2.3. (1). For given $(y_1,y_2,\cdots,y_p)\in(\mathbb{R}^2)^p$ and $m\ge 1$,

$$
\begin{aligned}
\mathbf{E}^{(y_1,y_2,\cdots,y_p)}V_r^m(t) &= \mathbf{E}\left(\int_{\mathbb{R}^2}\prod_{j=1}^p I_{\{x+y_j\in W_r^j(t)\}}dx\right)^m \\
&= \int_{(\mathbb{R}^2)^m}\prod_{j=1}^p\mathbf{E}\left(\prod_{k=1}^m I_{\{x_k+y_j\in W_r^j(t)\}}\right)dx_1\cdots dx_m \\
&\le \prod_{j=1}^p\left(\int_{(\mathbb{R}^2)^m}\left(\mathbf{E}\left(\prod_{k=1}^m I_{\{x_k+y_j\in W_r^j(t)\}}\right)\right)^p dx_1\cdots dx_m\right)^{1/p}. \\
&= \int_{(\mathbb{R}^2)^m}\left(\mathbf{E}\left(\prod_{k=1}^m I_{\{x_k\in W_r(t)\}}\right)\right)^p dx_1\cdots dx_m \\
&= \mathbf{E}(V_r^m(t)).
\end{aligned}
$$

Therefore, by Hölder's inequality, we have

$$\begin{aligned}
&\left(\frac{(\log t)^p}{t}\right)^{m/(p-1)} \sup_{y_1,y_2,\cdots,y_p} \mathbf{E}^{(y_1,y_2,\cdots,y_p)}(V_r(t))^{m/(p-1)} \\
&\le \left(\frac{(\log t)^p}{t}\right)^{m/(p-1)} \sup_{y_1,y_2,\cdots,y_p} \left\{\mathbf{E}^{(y_1,y_2,\cdots,y_p)} V_r^m(t)\right\}^{1/(p-1)} \\
&\le \left(\frac{(\log t)^p}{t}\right)^{m/(p-1)} \left\{\mathbf{E} V_r^m(t)\right\}^{1/(p-1)} \\
&\le \tilde{C}^m m!
\end{aligned}$$

where the last step follows from (4.6) and $\tilde{C}$ is a positive constant. Lemma 2.3 follows from a Taylor expansion.

(2). By Lemma 4.3 and Jensen's inequality, we have

$$\mathbf{E}V_r^{2m/3}(t) \le (\mathbf{E}V_r^m(t))^{\frac{2}{3}} \le m! C^m t^{m/3}, \quad m \ge 1.$$

For given $(y_1, y_2) \in (\mathbb{R}^3)^2$ and $m \ge 1$,

$$\begin{aligned}
\mathbf{E}^{(y_1,y_2)} V_r^m(t) &= \mathbf{E}\left(\int_{\mathbf{R}^2} \prod_{j=1}^{2} I_{\{x+y_j \in W_r^j(t)\}} dx\right)^m \\
&= \int_{(\mathbf{R}^2)^m} \prod_{j=1}^{2} \mathbf{E}\left(\prod_{k=1}^{m} I_{\{x_k+y_j \in W_r^j(t)\}}\right) dx_1 \cdots dx_m \\
&\le \prod_{j=1}^{2} \left(\left(\int_{(\mathbf{R}^2)^m} \mathbf{E}^2 \left(\prod_{k=1}^{m} I_{\{x_k+y_j \in W_r^j(t)\}}\right) dx_1 \cdots dx_m\right)\right)^{1/2} \\
&= \int_{(\mathbf{R}^2)^m} \mathbf{E}^2 \left(\prod_{k=1}^{m} I_{\{x_k \in W_r(t)\}}\right) dx_1 \cdots dx_m \\
&= \mathbf{E}(V_r^m(t)).
\end{aligned}$$

Therefore, one can get (2.8) by Taylor expansion.

□

5 Exponential moment estimates of the Wiener sausage

In this section, we give some exponential moment estimates of single Wiener sausage which will be used in the proof of Lemma 2.5.

From ([27]), we have the following asymptotic behavior of the expectation of the Wiener sausage:

$$\mathbf{E}|W_r(t)| = \begin{cases} \frac{2\pi t}{\log t} + \frac{2\pi t}{(\log t)^2}(1+\gamma-\log 2+2\log r) + o(\frac{t}{(\log t)^2}) & d=2 \\ 2\pi r t + 4\sqrt{2\pi} r^2 t^{1/2} + o(t^{1/2}) & d=3 \end{cases}. \tag{5.1}$$

Lemma 5.1. *(1). As $d = 2$, for any $\theta > 0$,*

$$\sup_{t\geq 3} \mathbf{E} \exp\left\{\frac{\theta \log t}{t}|W_r(t)|\right\} < \infty. \tag{5.2}$$

(2). As $d \geq 3$, for any $\theta > 0$,

$$\sup_{t\geq 3} \mathbf{E} \exp\left\{\frac{\theta}{t}|W_r(t)|\right\} < \infty. \tag{5.3}$$

Proof. We only prove (5.2). By Lemma 4.2 with $p = 1$, we have

$$\mathbb{E}|W_r(t)|^m \leq m!(\mathbb{E}|W_r(t)|)^m.$$

By (5.1) and the Taylor's expansion, we can easily get (5.2) holds for some $\theta_0 > 0$. For any $\theta > 0$, we choose $\delta > 0$ such that $\theta < \theta_0[\delta^{-1}]$ and denote $k_t = [\delta t]$, Then

$$\begin{aligned}
\mathbf{E} \exp\left\{\frac{\theta \log t}{t}|W_r(t)|\right\} &\leq \mathbf{E} \exp\left\{\frac{\theta \log t}{t}|W_r(k_t)|\right\}^{[\delta^{-1}]+1} \\
&\leq \mathbf{E} \exp\left\{\frac{\theta_0 \log k_t}{k_t}|W_r(k_t)|\right\}^{[\delta^{-1}]+1}
\end{aligned}$$

where the first inequality is due to the subadditivity of the Wiener sausage and the Markov property of Brownian motion. $\square$

Next, we will give an exponential estimate for $|W_r(t)| - \mathbb{E}|W_r(t)|$. The proof is analogous to Theorem 5.4 in [2]. The following lemma plays an important role in the proof of Lemma 5.3, which was given by Bass, Chen and Rosen ([1]).

Lemma 5.2. *Let $0 < p \leq 1$ and $\{Y_k(\zeta)\}_{k\geq 1}$ be a family (indexed by ζ) of sequences of i.i.d real valued random functions such that $\mathbf{E}Y_k(\zeta) = 0$ and*

$$\lim_{\theta\to 0} \sup_{\zeta} \mathbf{E} e^{\theta |Y_1(\zeta)|^p} = 1.$$

Then for some $\lambda > 0$,

$$\sup_{n,\zeta} \mathbf{E} \exp\left\{\lambda \Big|\sum_{k=1}^{n} Y_k(\zeta)/\sqrt{n}\Big|^p\right\} < \infty.$$

Lemma 5.3. *Set $\bar{W}_r(t) = |W_r(t)| - \mathbf{E}|W_r(t)|$. Then there is a constant C such that:*

(1). When $d = 2$,

$$\sup_{t\geq 27} \mathbf{E} \exp\left\{\frac{C(\log t)^2}{t}|\bar{W}_r(t)|\right\} < \infty. \tag{5.4}$$

(2). When $d = 3$,

$$\sup_{t\geq 27} \mathbf{E} \exp\left\{\frac{C}{t^{1/3}(\log t)^2}|\bar{W}_r(t)|^{\frac{2}{3}}\right\} < \infty. \tag{5.5}$$

Proof. (1). For $t \geq 27$, set $N = [2(\log 2)^{-1} \log\log t]$, so that $2^N \sim (\log t)^2$. Write

$$
\begin{aligned}
|W_r(t)| = & \sum_{k=1}^{2^N} |W_r([(k-1)t2^{-N}, kt2^{-N}])| \\
& - \sum_{j=1}^{N} \sum_{k=1}^{2^{j-1}} |W_r([(2k-2)t2^{-j}, (2k-1)t2^{-j}]) \cap W_r([(2k-1)t2^{-j}, (2k)t2^{-j}])|
\end{aligned}
$$

and set

$$\beta_k = |W_r([(k-1)t2^{-N}, kt2^{-N}])|$$

and

$$\alpha_{j,k} = |W_r([(2k-2)t2^{-j}, (2k-1)t2^{-j}]) \cap W_r([(2k-1)t2^{-j}, (2k)t2^{-j}])|.$$

Then

$$\bar{W}_r(t) = \sum_{k=1}^{2^N} \bar{\beta}_k - \sum_{j=1}^{N} \sum_{k=1}^{2^{j-1}} \bar{\alpha}_{j,k}$$

where $\bar{\beta}_k = \beta_k - \mathbf{E}\beta_k$ and $\bar{\alpha}_{j,k} = \alpha_{j,k} - \mathbf{E}\alpha_{j,k}$. By Lemma 5.1, we have that

$$\sup_{t \geq 27} \mathbf{E} \exp\left\{ \lambda \left(\frac{\log t2^{-N}}{t2^{-N}} \right) |\bar{\beta}_1| \right\} < \infty.$$

Therefore, by Lemma 5.2, there is a $\theta > 0$ such that

$$\sup_{t \geq 27} \mathbf{E} \exp\left\{ \theta 2^{-N/2} \left(\frac{\log t2^{-N}}{t2^{-N}} \right) \left| \sum_{k=1}^{2^N} \bar{\beta}_k \right| \right\} < \infty.$$

By choice of N it is easy to see that there is a $C > 0$ independent of t such that

$$2^{-N/2} \frac{\log t2^{-N}}{t2^{-N}} \geq \frac{C\left(\log t\right)^2}{t}.$$

So there is some $C > 0$ such that

$$\sup_{t \geq 27} \mathbf{E} \exp\left\{ C \left(\frac{(\log t)^2}{t} \right) \left| \sum_{k=1}^{2^N} \bar{\beta}_k \right| \right\} < \infty.$$

Next, We need only to show that for some $C > 0$,

$$\sup_{t \geq 27} \mathbf{E} \exp\left\{ C \left(\frac{(\log t)^2}{t} \right) \left| \sum_{j=1}^{N} \sum_{k=1}^{2^{j-1}} \bar{\alpha}_{j,k} \right| \right\} < \infty. \tag{5.6}$$

Set $\widetilde{V}_r(t) = |W_r(t) \cap W_r'(t)|$, where $W_r'(t)$ is an independent copy of $W_r(t)$. Then, for each $1 \le j \le N$, $\{\bar{\alpha}_{j,1}, \bar{\alpha}_{j,2}, \cdots, \bar{\alpha}_{j,2^{j-1}}\}$ is a sequence of i.i.d random variables with the same distribution as $\widetilde{V}_r(t2^{-j})$. By Lemma 2.3 (with $p = 2$), there is a $\delta > 0$ such that

$$\sup_{t \ge 27} \sup_{j \le N} \mathbf{E} \exp\left\{ \frac{\delta(\log t2^{-j})^2}{t2^{-j}} |\bar{\alpha}_{j,1}| \right\} < +\infty.$$

By Lemma 5.2 again, there is a $\bar{\theta} > 0$ such that

$$\sup_{t \ge 27} \sup_{j \le N} \mathbf{E} \exp\left\{ \frac{\bar{\theta} 2^{-j/2}(\log t)^2}{t2^{-j}} |\sum_{k=1}^{2^{j-1}} \bar{\alpha}_{j,k}| \right\} < +\infty.$$

Hence for some $c > 0$,

$$M(c) \equiv \sup_{t \ge 27} \sup_{j \le N} \mathbf{E} \exp\left\{ \frac{c 2^{j/2}(\log t)^2}{t} |\sum_{k=1}^{2^{j-1}} \bar{\alpha}_{j,k}| \right\} < +\infty.$$

Set $\lambda_N = \prod_{j=1}^{N}(1 - 2^{-j/2})$ and $\lambda = \prod_{j=1}^{\infty}(1 - 2^{-j/2})$. Using Hölder's inequality with $1/p = 1 - 2^{-N/2}$ and $1/q = 2^{-N/2}$ we have

$$\begin{aligned}
&\mathbf{E} \exp\left\{ \frac{\lambda_N c(\log t)^2}{t} \left| \sum_{j=1}^{N} \sum_{k=1}^{2^{j-1}} \bar{\alpha}_{j,k} \right| \right\} \\
&\le \left(\mathbf{E} \exp\left\{ \frac{\lambda_{N-1} c(\log t)^2}{t} \left| \sum_{j=1}^{N-1} \sum_{k=1}^{2^{j-1}} \bar{\alpha}_{j,k} \right| \right\} \right)^{1-2^{-N/2}} \\
&\times \left(\mathbf{E} \exp\left\{ \frac{\lambda_N c 2^{N/2}(\log t)^2}{t} \left| \sum_{k=1}^{2^{N-1}} \bar{\alpha}_{N,k} \right| \right\} \right)^{2^{-N/2}} \\
&\le \mathbf{E} \exp\left\{ \frac{\lambda_{N-1} c(\log t)^2}{t} \left| \sum_{j=1}^{N-1} \sum_{k=1}^{2^{j-1}} \bar{\alpha}_{j,k} \right| \right\} \times M(c)^{2^{-N/2}}
\end{aligned}$$

where the second inequality follows from the fact that $\lambda_N < 1$. Repeating this procedure,

$$\mathbf{E} \exp\left\{ \frac{\lambda_N c(\log t)^2}{t} \left| \sum_{j=1}^{N} \sum_{k=1}^{2^{j-1}} \bar{\alpha}_{j,k} \right| \right\} \le M(c)^{2^{-1/2} + \cdots + 2^{-N/2}} \le M(c).$$

So we have

$$\mathbf{E}\exp\left\{\frac{\lambda c(\log t)^2}{t}|\sum_{j=1}^{N}\sum_{k=1}^{2^{j-1}}\bar{\alpha}_{j,k}|\right\}\le M(c)<+\infty.$$

So (5.6) holds.

(2). For $t\ge 27$, set $N=[(\log 2)^{-1}\log t]$, so that $2^N\sim t$. From the proof of (5.4), we have

$$\bar{W}_r(t)=\sum_{k=1}^{2^N}\bar{\beta}_k-\sum_{j=1}^{N}\sum_{k=1}^{2^{j-1}}\bar{\alpha}_{j,k},$$

where

$$\beta_k=|W_r([(k-1)\frac{t}{2^N},k\frac{t}{2^N}])|$$

and

$$\alpha_{j,k}=|W_r([(2k-2)\frac{t}{2^j},(2k-1)\frac{t}{2^j}])\cap W_r([(2k-1)\frac{t}{2^j},(2k)\frac{t}{2^j}])|.$$

Then

$$|\bar{W}_r(t)|^{2/3}\le|\sum_{k=1}^{2^N}\bar{\beta}_k|^{2/3}+\sum_{j=1}^{N}|\sum_{k=1}^{2^{j-1}}\bar{\alpha}_{j,k}|^{2/3}.$$

By Lemma 5.1 and $2^N\sim t$, for some $\lambda>0$

$$\sup_{t\ge 27}\mathbf{E}\exp\left\{\lambda|\bar{\beta}_1|\right\}<\infty.$$

So there exists $\lambda>0$ such that

$$\sup_{t\ge 27}\mathbf{E}\exp\left\{\lambda|\bar{\beta}_1|^{2/3}\right\}<\infty.$$

Notice that $\beta_1,\cdots,\beta_{2^N}$ is an i.i.d sequence with $\mathbf{E}\bar{\beta}_1=0$. By lemma 5.2, there exists $\theta>0$ such that

$$\sup_{t\ge 27}\mathbf{E}\exp\left\{\theta 2^{-N/3}|\sum_{k=1}^{2^N}\bar{\beta}_k|^{2/3}\right\}<\infty.$$

So there exists $C>0$ such that

$$\sup_{t\ge 27}\mathbf{E}\exp\left\{\frac{C}{t^{1/3}}|\sum_{k=1}^{2^N}\bar{\beta}_k|^{2/3}\right\}<\infty.$$

Next, we need only to show that for some $C>0$,

$$\sup_{t\ge 27}\mathbf{E}\exp\left\{\frac{C}{t^{1/3}(\log t)^2}\sum_{j=1}^{N}|\sum_{k=1}^{2^{j-1}}\bar{\alpha}_{j,k}|^{2/3}\right\}<\infty. \tag{5.7}$$

Set $\widetilde{V}_r(t) = |W_r(t) \cap W_r'(t)|$, where $W_r'(t)$ is an independent copy of $W_r(t)$. Then, for each $1 \le j \le N$, $\{\bar{\alpha}_{j,1}, \bar{\alpha}_{j,2}, \cdots, \bar{\alpha}_{j,2^{j-1}}\}$ is an i.i.d sequence with the same distribution as $\widetilde{V}_r(\frac{t}{2^j})$. By Lemma 5.1, for some $\theta > 0$,

$$\sup_{t\ge 27}\sup_{j\le N}\mathbf{E}\exp\left\{\frac{\theta}{(\frac{t}{2^j})^{1/3}}|\bar{\alpha}_{j,1}|^{2/3}\right\} < +\infty.$$

By Lemma 5.2 again, for some $\bar{\theta} > 0$,

$$\bar{M}(\bar{\theta}) := \sup_{t\ge 27}\sup_{j\le N}\mathbf{E}\exp\left\{\frac{\bar{\theta}}{t^{1/3}}|\sum_{k=1}^{2^{j-1}}\bar{\alpha}_{j,k}|^{2/3}\right\} < +\infty.$$

Write $\lambda_N = \prod_{j=1}^N (1 - j^{-2})$ and $\lambda_\infty = \prod_{j=1}^\infty (1 - j^{-2})$. Then using Hölder's inequality with $1/p = 1 - N^{-2}$ and $1/q = N^{-2}$ we have

$$\begin{aligned}
&\mathbf{E}\exp\left\{\frac{\lambda_N\bar{\theta}(\log 2)^2}{t^{1/3}(\log t)^2}\sum_{j=1}^N|\sum_{k=1}^{2^{j-1}}\bar{\alpha}_{j,k}|^{2/3}\right\}\\
&\le \left(\mathbf{E}\exp\left\{\frac{\lambda_{N-1}\bar{\theta}(\log 2)^2}{t^{1/3}(\log t)^2}|\sum_{j=1}^{N-1}|\sum_{k=1}^{2^{j-1}}\bar{\alpha}_{j,k}|^{2/3}\right\}\right)^{1-N^{-2}}\\
&\times\left(\mathbf{E}\exp\left\{\frac{\lambda_N\bar{\theta}(\log 2)^2N^2}{t^{1/3}(\log t)^2}|\sum_{k=1}^{2^{N-1}}\bar{\alpha}_{N,k}|^{2/3}\right\}\right)^{N^{-2}}\\
&\le \mathbf{E}\exp\left\{\frac{\lambda_{N-1}c(\log 2)^2}{t^{1/3}(\log t)^2}|\sum_{j=1}^{N-1}\sum_{k=1}^{2^{j-1}}\bar{\alpha}_{j,k}|^{2/3}\right\}\times\bar{M}(\bar{\theta})^{N^{-2}}
\end{aligned}$$

where the second inequality follows from the fact that $\lambda_N < 1$. Repeating this procedure,

$$\mathbf{E}\exp\left\{\frac{\lambda_N\bar{\theta}(\log 2)^2}{t^{1/3}(\log t)^2}\sum_{j=1}^N|\sum_{k=1}^{2^{j-1}}\bar{\alpha}_{j,k}|^{2/3}\right\} \le \bar{M}(\bar{\theta})^{1+2^{-2}+\cdots+N^{-2}}.$$

So (5.7) holds.

$\square$

Remark 5.1. *The analogous results for the range of the random walk R_n are proved in [2] and [13]. When $d = 3$, the author of [13] shows that there is a $\theta > 0$ such that*

$$\sup_{n\ge 1}\mathbb{E}\exp\left\{\theta\frac{|R_n - \mathbb{E}R_n|}{n^{2/3}}\right\} < \infty,$$

where the fact that $R_n \le n$ is used. In the case of Wiener sausage, we can not find a proper upper bound for $|W_r(t)|$. That is why we can not improve the restriction on $b(t)$ for $d=3$.

6 Some estimates of the Feynman-Kac semigroup

In this section we show Lemma 2.5. The proof is also based on a decomposition of the domain $[0,t]$ and the exponential moment estimates of the Wiener sausage similar to [12].

We first give a weak convergence theorem of the Wiener sausage due to LeGall ([20]).

Lemma 6.1. *Let $f(x)$ be a bounded, continuous function on $\mathbb{R}^d$.*
(1). When $d=2$,

$$\frac{\log t}{2\pi t}\int_{\mathbb{R}^2} f\left(\frac{x}{\sqrt{t}}\right) I_{\{x\in W_r(t)\}}dx \xrightarrow{d} \int_0^1 f(\beta(s))ds, \qquad t\to+\infty. \tag{6.1}$$

(2). When $d\ge 3$,

$$\frac{1}{2\pi\, tr}\int_{\mathbb{R}^d} f\left(\frac{x}{\sqrt{t}}\right) I_{\{x\in W_r(t)\}}dx \xrightarrow{d} \int_0^1 f(\beta(s))ds, \qquad t\to+\infty. \tag{6.2}$$

Proof. We only prove (6.1). By scaling, $W_r(t)\overset{d}{=}\sqrt{t}W_{\frac{r}{\sqrt{t}}}(1)$. Set $\epsilon=\frac{r}{\sqrt{t}}$. Then

$$\begin{aligned}
&\frac{\log t}{2\pi\, t}\int_{\mathbb{R}^2} f\left(\frac{x}{\sqrt{t}}\right) I_{\{x\in W_r(t)\}}dx\\
&=\frac{\log t}{2\pi\, t}\int_{\mathbb{R}^2} f\left(\frac{x}{\sqrt{t}}\right) I_{\{x\in \sqrt{t}W_{r/\sqrt{t}}(1)\}}dx\\
&=\frac{1}{\pi}\log\frac{r}{\varepsilon}\int_{\mathbb{R}^2} f(x)I_{\{x\in W_\epsilon(1)\}}dx\\
&\xrightarrow{d}\int_0^1 f(\beta(s))ds \quad as \quad t\to\infty,
\end{aligned}$$

where the last step is a direct consequence of Theorem 3.1 in [20]. ☐

Denote by

$$\mathscr{H}_d=\{g\in L^2(\mathbb{R}^d); ||g||_2=1, ||\nabla g||_2<\infty\}$$

and

$$\triangle_i=[(i-1)l_t, il_t], i=1,2,\cdots. \tag{6.3}$$

We now use the exponential moment estimates of the Wiener sausage and the Feynman-Kac semigroup approach to prove the following lemma.

Lemma 6.2. *Let f be bounded and continuous on $\mathbb{R}^d$.*
(1). When $d=2$,

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{b(t)}\log\mathbf{E}\left\{\exp\left\{\frac{b(t)\log(l_t)}{2\pi t}\sum_{i=1}^{[b(t)]}\int_{\mathbb{R}^2}f\left(\sqrt{\frac{b(t)}{t}}x\right)I_{\{x\in W_r(\triangle_i)\}}dx\right\}\right\}\\&\geq\sup_{g\in\mathscr{H}_2}\left\{\int_{\mathbb{R}^2}f(x)g^2(x)dx-\frac{1}{2}\int_{\mathbb{R}^2}\langle\nabla g(x),\nabla g(x)\rangle dx\right\}.\end{aligned}\tag{6.4}$$

(2). When $d=3$,

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{b(t)}\log\mathbf{E}\left\{\exp\left\{\frac{b(t)}{2\pi rt}\sum_{i=1}^{[b(t)]}\int_{\mathbb{R}^3}f\left(\sqrt{\frac{b(t)}{t}}x\right)I_{\{x\in W_r(\triangle_i)\}}dx\right\}\right\}\\&\geq\sup_{g\in\mathscr{H}_3}\left\{\int_{\mathbb{R}^3}f(x)g^2(x)dx-\frac{1}{2}\int_{\mathbb{R}^3}\langle\nabla g(x),\nabla g(x)\rangle dx\right\}.\end{aligned}\tag{6.5}$$

Proof. The proof is analogous to Lemma 5 in [12]. We only prove (6.4). Define the linear operator T_t on $L^2(\mathbb{R}^2)$ as follows: for any $\xi\in L^2(\mathbb{R}^2)$,

$$(T_t\xi)(x)=\mathbf{E}_x\left(\exp\left\{\frac{\log l_t}{2\pi l_t}\int_{\mathbb{R}^2}f\left(\sqrt{l_t^{-1}}y\right)I_{\{y\in W_r(l_t)\}}dy\right\}\xi(\beta(l_t))\right).$$

By Lemma 2.4, T_t is self-adjoint. Let g be an infinitely differentiable function on $\mathbb{R}^2$ satisfying $g(x)=0$ for all $x\notin[-M,M]^2$ and $\int_{\mathbb{R}^2}|g(x)|^2dx=1$. Set

$$\xi_t(x)=\frac{g\left(\sqrt{l_t^{-1}}x\right)}{\sqrt{\int_{\mathbb{R}^2}g^2\left(\sqrt{l_t^{-1}}y\right)dy}}=l_t^{-1/2}g\left(\sqrt{l_t^{-1}}x\right),\quad x\in\mathbb{R}^2.$$

Let $p_t(x)$ be the probability density of $\beta(t)$. Let $\{\mathscr{F}_s,s\geq 0\}$ denote the σ-algebra filter generated by $\{\beta(s),s\geq 0\}$. Then by the Markov property, we have

$$\begin{aligned}&\mathbf{E}\exp\left\{\frac{\log l_t}{2\pi l_t}\sum_{i=2}^{[b(t)]}\int_{\mathbb{R}^2}f\left(\sqrt{l_t^{-1}}x\right)I_{\{x\in W_r(\triangle_i)\}}dx\right\}\\&=\int_{\mathbb{R}^2}p_{l_t}(y)\mathbf{E}_y\exp\left\{\frac{\log l_t}{2\pi l_t}\sum_{i=1}^{[b(t)]-1}\int_{\mathbb{R}^2}f\left(\sqrt{l_t^{-1}}x\right)I_{\{x\in W_r(\triangle_i)\}}dx\right\}dy\\&\geq\frac{1}{\sup_y|g(y)|^2}\left\{\int_{\mathbb{R}^2}g^2\left(\sqrt{l_t^{-1}}y\right)dy\right\}\int_{\mathbb{R}^2}p_{l_t}(y)\xi_t(y)\times\\&\quad\mathbf{E}_y\left(\exp\left\{\frac{\log l_t}{2\pi l_t}\sum_{i=1}^{[b(t)]-1}\int_{\mathbb{R}^2}f\left(\sqrt{l_t^{-1}}x\right)I_{\{x\in W_r(\triangle_i)\}}dx\right\}\xi_t(\beta(([b(t)]-1)l_t))\right)dy\end{aligned}$$

and

$$\mathbf{E}_y\left(\exp\left\{\frac{\log l_t}{2\pi l_t}\sum_{i=1}^{[b(t)]-1}\int_{\mathbb{R}^2} f\left(\sqrt{l_t^{-1}}x\right) I_{\{x\in W_r(\triangle_i)\}}dx\right\}\xi_t(\beta(([b(t)]-1)l_t))\right)dy$$

$$\begin{aligned}
=&\mathbf{E}_y\left(\mathbf{E}_y\left(\exp\left\{\frac{\log l_t}{2\pi l_t}\sum_{i=1}^{[b(t)]-1}\int_{\mathbb{R}^2} f\left(\sqrt{l_t^{-1}}x\right)I_{\{x\in W_r(\triangle_i)\}}dx\right\}\right.\right.\\
&\qquad\left.\left.\xi_t(\beta(([b(t)]-1)l_t))\middle|\mathscr{F}_{([b(t)]-2)l_t}\right)\right)\\
=&\mathbf{E}_y\left(\exp\left\{\frac{\log l_t}{2\pi l_t}\sum_{i=1}^{[b(t)]-2}\int_{\mathbb{R}^2} f\left(\sqrt{l_t^{-1}}x\right)I_{\{x\in W_r(\triangle_i)\}}dx\right\}\times\right.\\
&\qquad\left.\mathbf{E}_{\beta(([b(t)]-2)l_t)}\left(\exp\left\{\frac{\log l_t}{2\pi l_t}\int_{\mathbb{R}^2} f\left(\sqrt{l_t^{-1}}x\right)I_{\{x\in W_r(l_t)\}}dx\right\}\xi_t(\beta(l_t))\right)\right)\\
=&\mathbf{E}_y\left(\exp\left\{\frac{\log l_t}{2\pi l_t}\sum_{i=1}^{[b(t)]-2}\int_{\mathbb{R}^2} f\left(\sqrt{l_t^{-1}}x\right) I_{\{x\in W_r(\triangle_i)\}}dx\right\}T_t\xi_t(\beta(([b(t)]-2)l_t))\right)\\
=&T_t^{[b(t)]-1}\xi_t(y).
\end{aligned}$$

Therefore

$$\mathbf{E}\left(\exp\left\{\frac{\log l_t}{2\pi l_t}\sum_{i=2}^{[b(t)]}\int_{\mathbb{R}^2} f\left(\sqrt{l_t^{-1}}x\right) I_{\{x\in W_r(\triangle_i)\}}dx\right\}\right)\geq\delta\int_{\mathbb{R}^2}\xi_t(y)T_t^{[b(t)]-1}\xi_t(y)dy$$

where

$$\delta=\frac{\inf_{t>0} l_t \inf_{y\in[-M,M]^2}\frac{1}{2\pi l_t}\exp\left\{-\frac{|y|^2}{2l_t}\right\}}{\sup_y|g(y)|^2}.$$

In order to estimate $\int_{\mathbb{R}^2}\xi_t(y)T_t^{[b(t)]-1}\xi_t(y)dy$, let us consider the spectral representation of T_t:

$$T_t=\int_0^\infty \lambda dP_\lambda^t$$

where $\{P_\lambda^t,\lambda\geq 0\}$ is the spectral measure of T_t. Then by $\int|g|^2dx=1$, we see that $\langle P_\lambda^t\xi_t,\xi_t\rangle$ is a probability measure on $\mathbb{R}^2$, and by Jensen's inequality we have that

$$\begin{aligned}
\langle\xi_t(x),T_t^{[b(t)]-1}\xi_t(x)\rangle=&\int_0^\infty\lambda^{[b(t)]-1}d\langle P_\lambda^t\xi_t,\xi_t\rangle\\
&\geq\left(\int_0^\infty\lambda d\langle P_\lambda^t\xi_t,\xi_t\rangle\right)^{[b(t)]-1}=\langle\xi_t,T_t\xi_t\rangle^{[b(t)]-1}.
\end{aligned}$$

Therefore,

$$\liminf_{t\to\infty}\frac{1}{b(t)}\log\mathbf{E}\left(\exp\left\{\frac{\log l_t}{2\pi l_t}\sum_{i=2}^{[b(t)]}\int_{\mathbb{R}^2} f\left(\sqrt{l_t^{-1}}x\right) I_{\{x\in W_r(\triangle_i)\}}dx\right\}\right)$$
$$\geq \liminf_{t\to\infty}\log\langle\xi_t, T_t\xi_t\rangle.$$

Next we calculate $\langle\xi_t, T_t\xi_t\rangle$. Since

$$\begin{aligned}
&\langle\xi_t, T_t\xi_t\rangle\\
=&\left\{\int_{\mathbb{R}^2} g^2\left(\sqrt{l_t^{-1}}y\right)dy\right\}^{-1}\int_{\mathbb{R}^2} g\left(\sqrt{l_t^{-1}}x\right)\\
&\times\mathbf{E}_x\left(\exp\left\{\frac{\log l_t}{2\pi l_t}\int_{\mathbb{R}^2} f\left(\sqrt{l_t^{-1}}y\right)I_{\{y\in W_r(l_t)\}}dy\right\}g\left(\sqrt{l_t^{-1}}\beta(l_t)\right)\right)dx\\
=&l_t^{-1}\int_{\mathbb{R}^2} g\left(\sqrt{l_t^{-1}}x\right)\\
&\times\mathbf{E}\left(\exp\left\{\frac{\log l_t}{2\pi l_t}\int_{\mathbb{R}^2} f\left(\sqrt{l_t^{-1}}(x+y)\right)I_{\{y\in W_r(l_t)\}}dy\right\}g\left(\sqrt{l_t^{-1}}(x+\beta(l_t))\right)\right)dx\\
=&\int_{\mathbb{R}^2} g(x)\mathbf{E}\left(\exp\left\{\frac{\log l_t}{2\pi l_t}\int_{\mathbb{R}^2} f\left(x+\sqrt{l_t^{-1}}y\right)I_{\{y\in W_r(l_t)\}}dy\right\}g\left(x+\sqrt{l_t^{-1}}\beta(l_t)\right)\right)dx,
\end{aligned}$$

it follows from (6.1), Lemma 6.1 and dominated convergence theorem that

$$\lim_{t\to\infty}\log\langle\xi_t, T_t\xi_t\rangle = \log\int_{\mathbb{R}^2} g(x)\mathbf{E}_x\left(\exp\left\{\int_0^1 f(\beta(s))ds\right\}g(\beta(1))\right)dx.$$

Let us consider the following the semigroup of linear operators on $L^2(\mathbb{R}^2)$ defined by

$$T_t^f g(x) = \mathbf{E}_x\left(\exp\left\{\int_0^t f(\beta(s))ds\right\}g(\beta(t))\right).$$

Then by Feymman-Kac formula, we see that the generator $\mathscr{A}^f$ of T_t^f is self-adjoint and satisfies

$$\mathscr{A}^f g = \frac{1}{2}\triangle g + fg, \quad g\in C^\infty(\mathbb{R}^2).$$

Let $\{P_\lambda^f, \lambda\geq 0\}$ is the spectral measure of $\mathscr{A}^f$. Then by $\int|g|^2dx = 1$, we see that $\langle P_\lambda^f g, g\rangle$ is a probability on $\mathbb{R}^2$, and by Jensen's inequality we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} g(x)\mathbf{E}_x\left(\exp\left\{\int_0^1 f(\beta(s))ds\right\}g(\beta(1))\right)dx\\
=&\langle g, T_1^f g\rangle = \int_0^\infty e^\lambda d\langle P_\lambda^f g, g\rangle\\
\geq &\exp\left\{\int_0^\infty \lambda d\langle P_\lambda^f g, g\rangle\right\} = \exp\{\langle g, \mathscr{A}^f g\rangle\}\\
=&\exp\left\{\int_{\mathbb{R}^2} f(x)g^2(x)dx - \frac{1}{2}\int_{\mathbb{R}^2}\langle\nabla g(x), \nabla g(x)\rangle dx\right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\liminf_{t\to\infty}\frac{1}{b(t)}\log\mathbf{E}\left\{\exp\left\{\frac{\log l_t}{2\pi l_t}\sum_{i=1}^{[b(t)]}\int_{\mathbb{R}^2} f(\sqrt{l_t^{-1}}x)I_{\{x\in W_r(\triangle_i)\}}dx\right\}\right\}\\
\geq &\sup_{g\in\mathscr{H}_2}\left\{\int_{\mathbb{R}^2} f(x)g^2(x)dx - \frac{1}{2}\int_{\mathbb{R}^2}\langle\nabla g(x), \nabla g(x)\rangle dx\right\}.
\end{aligned}$$

□

Next, we will use Lemma 5.3 to prove that $|W_r(t)|$ is exponentially equivalent to $\sum_{i=1}^{[b(t)]}|W_r(\triangle_i)|$. So we can transfer Lemma 6.2 to Lemma 2.5.

Lemma 6.3. *Let $\epsilon > 0$ be fixed but arbitrary.*

(1). If $d = 2$ and $b(t)$ satisfies (1.4), then

$$\limsup_{t\to\infty}\frac{1}{b(t)}\log\mathbf{P}\left(\left|\sum_{i=1}^{[b(t)]}|W_r(\triangle_i)| - |W_r(t)|\right| \geq \epsilon\frac{t}{\log t}\right) = -\infty. \tag{6.6}$$

(2). If $d = 3$ and $b(t)$ satisfies (1.7), then

$$\limsup_{t\to\infty}\frac{1}{b(t)}\log\mathbf{P}\left(\left|\sum_{i=1}^{[b(t)]}|W_r(\triangle_i)| - |W_r(t)|\right| \geq \epsilon t\right) = -\infty. \tag{6.7}$$

Proof. (1). By Chebyshev's inequality, we need only to prove there exits a constant $\theta > 0$ such that

$$\lim_{t\to\infty}\frac{1}{b(t)}\log\mathbf{E}\exp\left\{\frac{\theta(\log t)^2}{t}\left|\sum_{i=1}^{[b(t)]}|W_r(\triangle_i)| - |W_r(t)|\right|\right\} < \infty. \tag{6.8}$$

By triangle inequality, we need only to prove there exits a constant $\theta > 0$ such that

$$\lim_{t\to\infty}\frac{1}{b(t)}\log\mathbf{E}\exp\left\{\frac{\theta(\log t)^2}{t}\left|\sum_{i=1}^{[b(t)]}\mathbf{E}|W_r(\triangle_i)| - \mathbf{E}|W_r(t)|\right|\right\} < \infty \tag{6.9}$$

$$\lim_{t\to\infty}\frac{1}{b(t)}\log\mathbf{E}\exp\left\{\frac{\theta(\log t)^2}{t}\left|\bar{W}_r(t)\right|\right\}<\infty \tag{6.10}$$

and

$$\lim_{t\to\infty}\frac{1}{b(t)}\log\mathbf{E}\exp\left\{\frac{(\theta\log t)^2}{t}\left|\sum_{i=1}^{[b(t)]}(|W_r(\triangle_i)|-\mathbf{E}|W_r(\triangle_i)|)\right|\right\}<\infty. \tag{6.11}$$

By (5.1), one can easily get

$$\sum_{i=1}^{[b(t)]}E|W_r(\triangle_i)|-\mathbf{E}|W_r(t)|=O\left(\frac{t\log b(t)}{(\log t)^2}\right)$$

and so (6.9) holds.

By Lemma 5.3, (6.10) holds.

It remains to show (6.11). In fact, by the Markov property, we have

$$\begin{aligned}&\mathbf{E}\exp\left\{\frac{\theta(\log t)^2}{t}\left|\sum_{i=1}^{[b(t)]}(|W_r(\triangle_i)|-\mathbf{E}|W_r(\triangle_i)|)\right|\right\}\\ \le&\left(\mathbf{E}\exp\left\{\frac{\theta(\log t)^2}{t}\left||W_r(\triangle_1)|-\mathbf{E}|W_r(\triangle_1)|)\right|\right\}\right)^{[b(t)]}\end{aligned}$$

which implies (6.11).

(2). Set $\alpha>0$. Notice that

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{b(t)}\log\mathbf{P}\left(\left|\sum_{i=1}^{[b(t)]}|W_r(\triangle_i)|-|W_r(t)|\right|\ge\epsilon t\right)\\ =&\limsup_{t\to\infty}\frac{1}{b(t)}\log\mathbf{P}\left(\left|\sum_{i=1}^{[b(t)]}|W_r(\triangle_i)|-|W_r(t)|\right|^{2/3}\ge\epsilon t^{2/3}\right)\\ \le&\limsup_{t\to\infty}\frac{1}{b(t)}\log\left(e^{-\epsilon\alpha t^{1/3}/(\log t)^2}\times\mathbf{E}\exp\left\{\frac{\alpha}{t^{1/3}(\log t)^2}\left|\sum_{i=1}^{[b(t)]}|W_r(\triangle_i)|-|W_r(t)|\right|^{2/3}\right\}\right)\\ =&-\infty+\limsup_{t\to\infty}\frac{1}{b(t)}\log\mathbf{E}\exp\left\{\frac{\alpha}{t^{1/3}(\log t)^2}\left|\sum_{i=1}^{[b(t)]}|W_r(\triangle_i)|-|W_r(t)|\right|^{2/3}\right\}.\end{aligned}$$

By triangle inequality, in order to get (6.7), we need only to prove there exists a constant $\alpha>0$ such that

$$\limsup_{t\to\infty}\frac{1}{b(t)}\log\mathbf{E}\exp\left\{\frac{\alpha}{t^{1/3}(\log t)^2}\left|\sum_{i=1}^{[b(t)]}\mathbf{E}|W_r(\triangle_i)|-\mathbf{E}|W_r(t)|\right|^{2/3}\right\}<\infty \tag{6.12}$$

$$\limsup_{t\to\infty}\frac{1}{b(t)}\log\mathbf{E}\exp\left\{\frac{\alpha}{t^{1/3}(\log t)^2}\left||W_r(t)|-\mathbf{E}|W_r(t)|\right|^{2/3}\right\}<\infty \tag{6.13}$$

and

$$\limsup_{t\to\infty}\frac{1}{b(t)}\log\mathbf{E}\exp\left\{\frac{\alpha}{t^{1/3}(\log t)^2}\left|\sum_{i=1}^{[b(t)]}|W_r(\triangle_i)|-\mathbf{E}|W_r(\triangle_i)|\right|^{2/3}\right\}<\infty. \tag{6.14}$$

By (5.1), it is easy to get

$$\sum_{i=1}^{[b(t)]}\mathbf{E}|W_r(\triangle_i)|-\mathbf{E}|W_r(t)|=O(\sqrt{tb(t)})$$

and so (6.12) holds.

By Lemma 5.3, there exists a constant $c>0$ such that

$$K:=\sup_{t\geq 3}\mathbf{E}\exp\left\{\frac{c}{t^{1/3}(\log t)^2}\left|\bar{W}_r(t)\right|^{2/3}\right\}<\infty.$$

Therefore, (6.13) holds.

It remains to show (6.14). In fact, by the Markov property,

$$\begin{aligned}&\mathbf{E}\exp\left\{\frac{\alpha}{t^{1/3}(\log t)^2}\left|\sum_{i=1}^{[b(t)]}(|W_r(\triangle_i)|-\mathbf{E}|W_r(\triangle_i)|)\right|^{2/3}\right\}\\ &\leq\left(\mathbf{E}\exp\left\{\frac{\alpha}{t^{1/3}(\log t)^2}\left||W_r(\triangle_1)|-\mathbf{E}|W_r(\triangle_1)|\right|^{2/3}\right\}\right)^{[b(t)]}\end{aligned}$$

which implies (6.14).

□

The proof of Lemma 2.5. We only prove (2.12). Noticed that

$$\begin{aligned}\int_{\mathbf{R}^2}f\left(\sqrt{\frac{b(t)}{t}}x\right)I_{\{x\in W_r(t)\}}dx\geq&\sum_{i=1}^{[b(t)]}\int_{\mathbf{R}^2}f(\sqrt{\frac{b(t)}{t}}x)I_{\{x\in W_r(\triangle_i)\}}dx\\ &-\left|\int_{\mathbf{R}^2}f\left(\sqrt{\frac{b(t)}{t}}x\right)\left(\sum_{i=1}^{[b(t)]}I_{\{x\in W_r(\triangle_i)\}}-I_{\{x\in W_r(t)\}}\right)dx\right|.\end{aligned}$$

For any $\epsilon>0$, set

$$A_\epsilon:=\left\{\left|\int_{\mathbf{R}^2}f\left(\sqrt{\frac{b(t)}{t}}x\right)\left(\sum_{i=1}^{[b(t)]}I_{\{x\in W_r(\triangle_i)\}}-I_{\{x\in W_r(t)\}}\right)dx\right|\geq\epsilon\frac{t}{\log t}\right\}.$$

So,

$$
\begin{aligned}
J := &\mathbf{E}\exp\left\{\frac{b(t)\log t}{t}\int_{\mathbf{R}^2} f\left(\sqrt{\frac{b(t)}{t}}x\right) I_{\{x\in W_r(t)\}}dx\right\}\\
&\geq \mathbf{E}\left[\exp\left\{\frac{b(t)\log t}{t}\sum_{i=1}^{[b(t)]}\int_{\mathbf{R}^2} f\left(\sqrt{\frac{b(t)}{t}}x\right) I_{\{x\in W_r(\triangle_i)\}}dx\right.\right.\\
&\qquad \left.\left. -\left|\int_{\mathbf{R}^2} f\left(\sqrt{\frac{b(t)}{t}}x\right)\left(\sum_{i=1}^{[b(t)]} I_{\{x\in W_r(\triangle_i)\}} - I_{\{x\in W_r(t)\}}\right)dx\right|\right\} I_{A_\epsilon^c}\right]\\
&\geq e^{-\epsilon b(t)}\mathbf{E}\exp\left\{\frac{b(t)\log t}{t}\sum_{i=1}^{[b(t)]}\int_{\mathbf{R}^2} f\left(\sqrt{\frac{b(t)}{t}}x\right) I_{\{x\in W_r(\triangle_i)\}}dx\right\}\\
&\qquad -\mathbf{E}\left[\exp\left\{\frac{b(t)\log t}{t}\sum_{i=1}^{[b(t)]}\int_{\mathbf{R}^2} f\left(\sqrt{\frac{b(t)}{t}}x\right) I_{\{x\in W_r(\triangle_i)\}}dx\right\} I_{A_\epsilon}\right]\\
&=: J_1 - J_2.
\end{aligned}
$$

Therefore,

$$
\max\left\{\liminf_{t\to\infty}\frac{1}{b(t)}\log J,\quad \limsup_{t\to\infty}\frac{1}{b(t)}\log J_2\right\} \geq \liminf_{t\to\infty}\frac{1}{b(t)}\log J_1.
$$

By lemma 6.2, we get

$$
\liminf_{t\to\infty}\frac{1}{b(t)}\log J_1 \geq -\epsilon + \sup_{g\in\mathscr{H}_2}\left\{\int_{\mathbf{R}^2} f(x)g^2(x)dx - \frac{1}{2}\int_{\mathbf{R}^2}\langle\nabla g(x),\nabla g(x)\rangle dx\right\}. \tag{6.15}
$$

By Cauchy-Schwartz inequality,

$$
\begin{aligned}
J_2 \leq &\left[\mathbf{E}\exp\left\{\frac{2b(t)\log t}{t}\sum_{i=1}^{[b(t)]}\int_{\mathbf{R}^2} f\left(\sqrt{\frac{b(t)}{t}}x\right) I_{\{x\in W_r(\triangle_i)\}}dx\right\}\right]^{1/2}\times\\
&\left[\mathbf{P}\left(\left|\int_{\mathbf{R}^2} f\left(\sqrt{\frac{b(t)}{t}}x\right)\left(\sum_{i=1}^{[b(t)]} I_{\{x\in W_r(\triangle_i)\}} - I_{\{x\in W_r(t)\}}\right)dx\right| \geq \epsilon\frac{t}{\log t}\right)\right]^{1/2}.
\end{aligned}
$$

Because f is bounded, we can assume $\inf_x f(x) = m$. Write $||f||_\infty = \sup_x f(x)$. Noticed that

$$\begin{aligned}
&\left|\int_{\mathbf{R}^2} f\left(\sqrt{\frac{b(t)}{t}}x\right)\left(\sum_{i=1}^{[b(t)]} I_{\{x\in W_r(\triangle_i)\}} - I_{\{x\in W_r(t)\}}\right)dx\right| \\
&\le \left(\int_{\mathbf{R}^2}\left(f\left(\sqrt{\frac{b(t)}{t}}x\right) - m\right)\left(\sum_{i=1}^{[b(t)]} I_{\{x\in W_r(\triangle_i)\}} - I_{\{x\in W_r(t)\}}\right)dx\right. \\
&+ |m|\int_{R^2}\left(\sum_{i=1}^{[b(t)]} I_{\{x\in W_r(\triangle_i)\}} - I_{\{x\in W_r(t)\}}\right)dx \\
&\le (||f-m||_\infty + |m|)\left(\sum_{i=1}^{[b(t)]} |W_r(\triangle_i)| - |W_r(t)|\right).
\end{aligned}$$

Applying Lemma 6.3, we obtain

$$\limsup_{t\to\infty}\frac{1}{b(t)}\log\mathbf{P}\left(\left|\int_{\mathbf{R}^2} f\left(\sqrt{\frac{b(t)}{t}}x\right)\left(\sum_{i=1}^{[b(t)]} I_{\{x\in W_r(\triangle_i)\}} - I_{\{x\in W_r(t)\}}\right)dx\right| \ge \epsilon\frac{t}{\log t}\right) = -\infty. \tag{6.16}$$

By (6.15) and (6.16), we need only to prove

$$\limsup_{t\to\infty}\frac{1}{b(t)}\log\mathbf{E}\exp\left\{\frac{2b(t)\log t}{t}\sum_{i=1}^{[b(t)]}\int_{\mathbf{R}^2} f\left(\sqrt{\frac{b(t)}{t}}x\right) I_{\{x\in W_r(\triangle_i)\}}dx\right\} < \infty. \tag{6.17}$$

Using the Markov property,

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{b(t)}\log\mathbf{E}\exp\left\{\frac{2b(t)\log t}{t}\sum_{i=1}^{[b(t)]}\int_{\mathbf{R}^2} f\left(\sqrt{\frac{b(t)}{t}}x\right) I_{\{x\in W_r(\triangle_i)\}}dx\right\} \\
&\le \limsup_{t\to\infty}\frac{1}{b(t)}\log\mathbf{E}\exp\left\{||f||_\infty\frac{2b(t)\log t}{t}\sum_{i=1}^{[b(t)]}|W_r(\triangle_i)|\right\} \\
&= \limsup_{t\to\infty}\frac{1}{b(t)}\log\mathbf{E}\exp\left\{||f||_\infty\frac{2b(t)\log t}{t}|W_r(\triangle_1)|\right\}^{[b(t)]} \\
&\le \log\sup_{t\ge 27}\mathbf{E}\exp\left\{||f||_\infty\frac{2b(t)\log t}{t}|W_r(\triangle_1)|\right\}.
\end{aligned}$$

By Lemma 5.1, (6.17) holds.

□

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