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Gaussian fluctuations in complex sample covariance matrices

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Abstract

Let $X = (X_{i,j})_{m\times n}, m \geq n$, be a complex Gaussian random matrix with mean zero and variance $\frac{1}{n}$, let $S = X^*X$ be a sample covariance matrix. In this paper we are mainly interested in the limiting behavior of eigenvalues when $\frac{m}{n} \to \gamma \geq 1$ as $n \to \infty$. Under certain conditions on k, we prove the central limit theorem holds true for the k-th largest eigenvalues $\lambda_{(k)}$ as k tends to infinity as $n \to \infty$. The proof is largely based on the Costin-Lebowitz-Soshnikov argument and the asymptotic estimates for the expectation and variance of the number of eigenvalues in an interval. The standard technique for the RH problem is used to compute the exact formula and asymptotic properties for the mean density of eigenvalues. As a by-product, we obtain a convergence speed of the mean density of eigenvalues to the Marchenko-Pastur distribution density under the condition $|\frac{m}{n} - \gamma| = O(\frac{1}{n})$

Key words: Central limit theorem; the Costin-Lebowitz-Soshnikov theorem; Eigenvalues; RH problems; Sample covariance matrices

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1 Introduction and main results

Let $X = (X_{i,j})$ be a complex $m \times n$, $m \geq n$, random matrix, the entries of which are independent complex Gaussian random variables with mean zero and variance $\frac{1}{n}$, namely $Re(X_{i,j})$, $Im(X_{i,j})$ form a family of independent real Gaussian random variables each with mean value 0 and variance $\frac{1}{2n}$. Let $S = X^*X$, then S can be viewed as a sample covariance matrix of m samples of n dimensional random vectors and it is of fundamental importance in multivariate statistical analysis.

The complex sample covariance matrices was first studied by Goodman (5) and Khatri (9). The distribution $d\mu(S)$ of S is given by

$$d\mu(S) = \frac{1}{Z}(\det S)^{m-n}e^{-trS}dS, \quad \text{for } S \in M_n(\mathcal{C})_+,$$

where $Z = Z_{n,m} > 0$ is a normalization constant depending on m, n and

$$dS = (\prod_{j=1}^{n} dS_{jj}) \prod_{j<k} d(ReS_{jk})d(ImS_{jk}).$$

The measure $d\mu$ on the matrices produces naturally a measure on the corresponding n real eigenvalues λ_i. It turns out that this induced measure can be explicitly calculated and its joint density is given by

$$\rho_n(x_1, x_2, \cdots, x_n) = \frac{1}{Z}\prod_{j<k}(x_j - x_k)^2 \prod_{k=1}^{n} x_k^{m-n}e^{-nx_k}, \quad x_1, x_2, \cdots, x_n > 0, \tag{1.1}$$

where $Z = Z_{n,m} > 0$ is again a normalization constant depending on m, n.

It is the explicit form (1.1) (sometimes called the Laguerre unitary ensemble) that makes possible the deep and thorough asymptotic analysis, both inside the bulk and at the edge of the spectrum. There are actually many other well-known ensembles with such a determinantal point process representation in random matrices and random growth models. See (10) for recent works.

Let $V_{n,m}(x)$ be such that $e^{-nV_{n,m}(x)} = x^{m-n}e^{-nx}, x \in R_+$. Let $p_j(x), j \geq 0$, be a sequence of orthonormalized polynomials with γ_j the highest coefficient with respect to the weight function $e^{-nV_{n,m}(x)}$. That is,

$$\int_{R_+} p_j(x)p_k(x)e^{-nV_{n,m}(x)}dx = \delta_{jk}, \quad j, k \geq 0.$$

Define the kernel K_n by

$$K_n(x, y) = \sum_{k=0}^{n-1} p_k(x)p_k(y)e^{-\frac{nV_{n,m}(x)}{2}}e^{-\frac{nV_{n,m}(y)}{2}}, \quad x, y \in R_+, \tag{1.2}$$

then

$$\rho_n(x_1, x_2, \cdots, x_n) = \frac{1}{n!}\det(K_n(x_i, x_j))_{n\times n}.$$

Moreover, for $1 \le k \le n$, the k-marginal dimensional density is given by

$$\begin{aligned} \rho_{n,k}(x_1, x_2, \cdots, x_k) &= \int_{R_+^{n-k}} \rho_n(x_1, x_2, \cdots, x_n) dx_{k+1} \cdots dx_n \\ &= \frac{(n-k)!}{n!} \det(K_n(x_i, x_j))_{k \times k}. \end{aligned} \tag{1.3}$$

In particular, $\rho_{n,1}(x_1)$ describes the overall density of the eigenvalues.

The kernel $K_n(x, y)$ can also be represented by the so-called Christoffel-Darboux identity. For $x \ne y$ it holds

$$K_n(x, y) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x)}{x - y} e^{-\frac{nV_{n,m}(x)}{2}} e^{-\frac{nV_{n,m}(y)}{2}}, \tag{1.4}$$

and for $x = y$ one has

$$K_n(x, x) = \frac{\gamma_{n-1}}{\gamma_n} \left(p_n(x)p'_{n-1}(x) - p'_n(x)p_{n-1}(x)\right) e^{-nV_{n,m}(x)}. \tag{1.5}$$

The classic Marchenko-Pastur (11) theorem states that as $n \to \infty$ such that $\frac{m}{n} \to \gamma \ge 1$, almost surely

$$\frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i} \longrightarrow \mu_\gamma \tag{1.6}$$

in distribution, where μ_γ is the probability density function of the M-P distribution with parameter γ, namely

$$\mu_\gamma(x) = \begin{cases} \frac{1}{2\pi x} \sqrt{(x - \alpha)(\beta - x)}, & \text{for } \alpha \le x \le \beta, \\ 0, & \text{otherwise }, \end{cases} \tag{1.7}$$

and $\alpha = (\sqrt{\gamma} - 1)^2$ and $\beta = (\sqrt{\gamma} + 1)^2$.

A recent remarkable work is on a limiting distribution of the largest eigenvalue. Let $\lambda_{(1)} > \lambda_{(2)} > \cdots > \lambda_{(n)}$ be the ordered list of eigenvalues. While studying a random growth model of interest in probability, Johansson (8) derived the F_2 limit distribution for $\lambda_{(1)}$. Specifically speaking, define

$$\nu_{m,n} = (\sqrt{m} + \sqrt{n})^2, \quad \sigma_{m,n} = (\sqrt{m} + \sqrt{n}) \left(\frac{1}{\sqrt{m}} + \frac{1}{\sqrt{n}} \right)^{1/3}.$$

Then as $\frac{m}{n} \to \gamma \ge 1$,

$$\frac{n\lambda_{(1)} - \nu_{m,n}}{\sigma_{m,n}} \longrightarrow F_2 \tag{1.8}$$

in distribution, where F_2 is the Tracy-Widom distribution discovered in the Gaussian unitary ensemble (GUE). Analogs for $\lambda_{(k)}$ with k fixed has also been studied.

In this paper we deal with the distribution of $\lambda_{(k)}$ as n and k tend to infinity. Let

$$\alpha_{n,m} = \left(\sqrt{\frac{m}{n}} - 1 \right)^2, \quad \beta_{n,m} = \left(\sqrt{\frac{m}{n}} + 1 \right)^2.$$

Define

$$\mu_{n,m}(x) = \begin{cases} \frac{1}{2\pi x}\sqrt{(x-\alpha_{n,m})(\beta_{n,m}-x)}, & \text{for } \alpha_{n,m} \le x \le \beta_{n,m}, \\ 0 & \text{elsewhere.} \end{cases} \tag{1.9}$$

Our main results are as follows.

Theorem 1. *(the bulk case). Set*

$$G(t) = \int_{\alpha_{n,m}}^{t} \mu_{n,m}(x)dx, \quad \alpha_{n,m} \le t \le \beta_{n,m}$$

and $t = t_{n,k} = G^{-1}(\frac{k}{n})$ *where* $k = k(n)$ *is such that* $\frac{k}{n} \to a \in (0,1)$ *as* $n \to \infty$. *Then as* $\frac{m}{n} \to \gamma \ge 1$,

$$\frac{\sqrt{2}\pi\mu_{n,m}(t)(\lambda_{(n-k)} - t)}{\sqrt{\log n}} \longrightarrow N(0,1) \tag{1.10}$$

in distribution.

Theorem 2. *(the edge case). Let* $k = k(n)$ *be such that* $k \to \infty$ *but* $\frac{k}{n} \to 0$ *as* $n \to \infty$. *Then as* $\frac{m}{n} \to \gamma \ge 1$,

$$\frac{\lambda_{(k)} - \left(\beta_{n,m} - (\frac{3\pi\beta_{n,m}}{\sqrt{\beta_{n,m}-\alpha_{n,m}}}\frac{k}{n})^{2/3}\right)}{\frac{\sqrt{2}\beta_{n,m}^{2/3}}{(3\pi(\beta_{n,m}-\alpha_{n,m}))^{1/3}}\frac{\sqrt{\log k}}{n^{2/3}k^{1/3}}} \longrightarrow N(0,1) \tag{1.11}$$

in distribution.

Remarks: (1) Analogous results have recently been established for the eigenvalues of the GUE by (7). Our proofs are again based on the Costin-Lebowitz-Soshnikov theorem. One should be able to apply the same methodology to many other ensembles. A work on the discrete Krawtchouk ensemble and Hahn ensemble is in progress.

(2) It is remarkable that with regard to the Plancherel measure on the set of partitions λ of n, the rows $\lambda_1, \lambda_2, \lambda_3, \cdots$ of λ behaves, suitably scaled, like the 1st, 2nd, 3rd and so on eigenvalues of a random matrix from the GUE. On the other hand, the boundary of Young shape λ, a polygonal line, behaves like

$$\Omega(x) + \frac{U(x)}{n^{1/2}} + o\left(\frac{1}{n^{1/2}}\right), \tag{1.12}$$

where $\Omega(x)$ is closely connected to Wigner's semicircle law and $U(x)$ is a kind of generalized Gaussian random process. This is often referred to as Kerov's central limit theorem.

Borodin, Okounkov and Olshanski (1) established an exact determinantal formula for the correlation function of the Poissonized Plancherel measures. It would be expected that one could provide a proof of (1.12) using the Costin-Lebowitz-Soshnikov argument.

(3) It is well-known that $\mu_\gamma(x)$ is also the asymptotic distribution for zeros of orthogonal polynomial $p_n(x)$. Therefore it is natural to ask whether the k-th eigenvalue $\lambda_{(k)}$ fluctuates around the corresponding zero of $p_n(x)$. The above theorems partly confirm the conjecture although a rigorous argument is still open.

We say as usual that $\lambda_{(k)}$ obeys the central limit theorem if there exist $a_n > 0$ and b_n such that

$$\frac{\lambda_{(k)} - b_n}{a_n} \longrightarrow N(0,1) \quad \text{in distribution .}$$

Fix $x \in (-\infty, \infty)$. Let $I_n = [a_n x + b_n, \infty)$, $\#I_n$ stand for the number of eigenvalues in I_n. Then it holds true

$$\begin{aligned} P\left(\frac{\lambda_{(k)} - b_n}{a_n} \le x\right) &= P(\lambda_{(k)} \le a_n x + b_n) \\ &= P(\#I_n \le k). \end{aligned} \tag{1.13}$$

Hence it suffices to prove

$$P\left(\frac{\#I_n - E\#I_n}{\sqrt{Var\#I_n}} \le \frac{k - E\#I_n}{\sqrt{Var\#I_n}}\right) \longrightarrow \Phi(x), \quad x \in (-\infty, \infty), \tag{1.14}$$

where $\Phi(x)$ is the standard normal distribution function.

This in turn follows from the Costin-Lebowitz-Soshnikov theorem (2; 13) as long as

$$\frac{k - E\#I_n}{\sqrt{Var\#I_n}} \to x, \quad n \to \infty. \tag{1.15}$$

(1.15) will be used to determine the normalizing constants.

This paper is organized as follows. In Section 2 we shall give an exact formula and asymptotics for the mean density of eigenvalues $\lambda_1, \cdots, \lambda_n$ uniformly valid on the entire real axis based on the standard technique of asymptotic analysis of RH problems. Analogues for the GUE type ensemble have been studied and used to give a proof of a complete large N expansion for the partition function by Ercolani and McLaughlin (4). Just recently did Vanlessen (14) consider strong asymptotics of Laguerre type orthogonal polynomials on R_+ with respect to the weight function $x^\alpha e^{-Q(x)}$ where $\alpha > -1$ and Q denotes a polynomial with positive leading coefficient. As a direct consequence, we obtain the convergence speed of $\rho_{n,1}$ to μ_γ when $|\frac{m}{n} - \gamma| = O(\frac{1}{n})$. In the case when $m = [\gamma n]$, the speed is due to Götze and Tikhomirov (6) using recursive equations.

The proofs of Theorems 1 and 2 will be given in Section 3. Starting from (1.13)-(1.15), we need only compute the expectation and variance for the number of eigenvalues in an interval. The computation heavily depends on the exact formula and asymptotics for $\rho_{n,1}(x)$ and $K_n(x, y)$ in the entire real axis.

Throughout the paper, there are lots of positive numerical constants, whose values are of no importance. We shall use c, C for simplicity, which may take different values in different places.

2 The mean density of eigenvalues: exact formula and asymptotics

This section is devoted to the asymptotic analysis of orthogonal polynomials $p_n(x)$ and $p_{n-1}(x)$ with respect to the n dependent weight function $e^{-nV_{n,m}(x)}$, in which techniques are used for the asymptotic analysis of RH problems, first developed for singularity by Deift-Zhou. Deift (3) is a standard excellent reference in this area. For completeness of notations we summarize major steps of rigorous analysis, including a series of equivalent RH problems and explicit transformations below.

Let us start with the following RH problem:

(a) $U : \mathcal{C} \setminus [0, \infty) \to \mathcal{C}^{2\times 2}$ is analytic,

(b)

$$U_+(x) = U_-(x) \begin{pmatrix} 1 & e^{-nV_{n,m}(x)} \\ 0 & 1 \end{pmatrix}, \quad \text{for } x \in (0, \infty),$$

(c)

$$U(z) = \left(I + O(\frac{1}{z})\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \quad \text{as } z \to \infty.$$

We remark that a similar analysis was given for $x^\alpha e^{-nx}$ with $\alpha > -1$ by (14).

The unique solution of the RH problem for U is given by

$$\begin{pmatrix} \frac{1}{\gamma_n} p_n(z) & \frac{1}{\gamma_n} C(p_n e^{-nV_{n,m}})(z) \\ -2\pi i \gamma_{n-1} p_{n-1}(z) & -2\pi i \gamma_{n-1} C(p_n e^{-nV_{n,m}})(z) \end{pmatrix}, \quad z \in \mathcal{C} \setminus [0, \infty),$$

where $C(\cdot)$ denotes the Cauchy operator. In particular,

$$p_n(z) = \gamma_n U_{11}(z), \quad p_{n-1}(z) = \frac{i}{2\pi\gamma_{n-1}} U_{21}(z). \tag{2.1}$$

We have by (1.4) and (1.5)

$$K_n(x, y) = \frac{i}{2\pi} \frac{U_{11}(x)U_{21}(y) - U_{11}(y)U_{21}(x)}{x - y} e^{-\frac{nV_{n,m}(x)}{2}} e^{-\frac{nV_{n,m}(y)}{2}}, \quad x \neq y, \tag{2.2}$$

and

$$K_n(x, x) = -\frac{i}{2\pi} \left(U_{11}(x)U'_{21}(x) - U'_{11}(x)U_{21}(x)\right) e^{-nV_{n,m}(x)}, \quad x \in (0, \infty). \tag{2.3}$$

The equilibrium measure plays an important role in the asymptotic analysis of RH problems. It turns out that the measure with density function $\mu_{n,m}(x)$ is the equilibrium measure in the presence of the external field $V_{n,m}(x)$. It is actually easy to check that $\mu_{n,m}$ satisfies the Euler-Lagrange variational conditions: there exists a real number $l_{n,m}$ such that

$$2\int \log|x - y|\mu_{n,m}(y)dy - V_{n,m}(x) - l_{n,m} = 0, \quad \text{for } x \in [\alpha_{n,m}, \beta_{n,m}]$$

and

$$2\int \log|x - y|\mu_{n,m}(y)dy - V_{n,m}(x) - l_{n,m} < 0, \quad \text{for } x \in (-\infty, \infty) \setminus [\alpha_{n,m}, \beta_{n,m}].$$

In order to normalize the RH problem for U at infinity, we use the log-transform of the equilibrium measure. Define

$$g_n(z) = \int_{\alpha_{n,m}}^{\beta_{n,m}} \log(z-y)\mu_{n,m}(y)dy, \quad \text{for } z \in \mathcal{C} \setminus (-\infty, \beta_{n,m}],$$

where we take the principal branch of the logarithm, so that g_n is analytic in $\mathcal{C} \setminus (-\infty, \beta_{n,m}]$.

We now list several important properties of the function $g_n(z)$:

• $g_n(z)$ is analytic for $z \in \mathcal{C}\setminus(-\infty, \beta_{n,m}]$, with continuous boundary values $g_{n,\pm}(z)$ on $(-\infty, \beta_{n,m}]$.

•

$$g_{n,+}(x) + g_{n,-}(x) - V_{n,m}(x) - l_{n,m} = 0, \quad \text{for } x \in [\alpha_{n,m}, \beta_{n,m}],$$

$$2g_n(x) - V_{n,m}(x) - l_{n,m} < 0, \quad \text{for } x \in (-\infty, \infty) \setminus [\alpha_{n,m}, \beta_{n,m}]. \tag{2.4}$$

•

$$g_{n,+}(x) - g_{n,-}(x) = 2\pi i, \quad \text{for } x \in (-\infty, \alpha_{n,m}),$$

$$g_{n,+}(x) - g_{n,-}(x) = 2\pi i \int_x^{\beta_{n,m}} \mu_{n,m}(y)dy, \quad \text{for } x \in [\alpha_{n,m}, \beta_{n,m}], \tag{2.5}$$

and

$$g_{n,+}(x) - g_{n,-}(x) = 0, \quad \text{for } x \in (\beta_{n,m}\infty).$$

•

$$e^{ng_n(z)} = z^n + O(z^{n-1}), \quad \text{as } z \to \infty.$$

Now we are ready to perform the transformation $U \to T$. Define the matrix valued function T as

$$T(z) = e^{-\frac{1}{2}nl_{n,m}\sigma_3} U(z) e^{-ng_n(z)\sigma_3} e^{\frac{1}{2}nl_{n,m}\sigma_3}, \quad \text{for } z \in \mathcal{C} \setminus (-\infty, \infty), \tag{2.6}$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Note that, the function $e^{ng_n(z)}$ has no jump across $(-\infty, \alpha_{n,m})$, so that T has an analytic continuation to $\mathcal{C} \setminus [\alpha_{n,m}, \infty)$. It is then straightforward to check, using the conditions of the RH problem for U, that T is the unique solution of the following equivalent RH problem:

(a) $T : \mathcal{C} \setminus [0, \infty) \to \mathcal{C}^{2\times 2}$ is analytic,

(b) $T_+(x) = T_-(x)\nu_T(x), \quad$ for $x \in (0, \infty)$, with

$$\nu_T(x) = \begin{cases} \begin{pmatrix} e^{-n(g_{n,+}(x) - g_{n,-}(x))} & 1 \\ 0 & e^{n(g_{n,+}(x) - g_{n,-}(x))} \end{pmatrix}, & \text{for } x \in (\alpha_{n,m}, \beta_{n,m}), \\ \begin{pmatrix} 1 & e^{n(2g_n(x) - V_{n,m}(x) - l_{n,m})} \\ 0 & 1 \end{pmatrix}, & \text{for } x \in (0, \infty) \setminus (\alpha_{n,m}, \beta_{n,m}), \end{cases}$$

(c) $T(z) = I + O(\frac{1}{z})$, as $z \to \infty$.

From (2.5) we see that the diagonal entries of $\nu_T(x)$ on $(\alpha_{n,m}, \beta_{n,m})$ are rapidly oscillating for large n. From (2.4) the jump matrix ν_T on $(0,\infty) \setminus (\alpha_{n,m}, \beta_{n,m})$ converges exponentially fast to the identity matrix as $n \to \infty$.

Now we will transform the oscillatory diagonal entries of the jump matrix $\nu_T(x)$ on $(\alpha_{n,m}, \beta_{n,m})$ into exponentially decaying off-diagonal entries. This lies the heart of the Deift-Zhou steepest descent method.

In order to perform the transformation $T \to S$ we will introduce scalar functions ψ_n and ξ_n. Define

$$\psi_n(z) = \frac{1}{2\pi i z}\sqrt{(z-\alpha_{n,m})(z-\beta_{n,m})} \quad \text{for } z \in \mathcal{C} \setminus [\alpha_{n,m}, \beta_{n,m}], \tag{2.7}$$

with principal branches of powers. So, the + boundary value of $\psi_n(z)$ on $[\alpha_{n,m}, \beta_{n,m}]$ is precisely $\mu_{n,m}(z)$. In particular, we have

$$\psi_{n,+}(x) = -\psi_{n,-}(x) = \mu_{n,m}(x), \quad \text{for } x \in [\alpha_{n,m}, \beta_{n,m}].$$

Now define

$$\xi_n(z) = -\pi i \int_{\beta_{n,m}}^{z} \psi_n(y)dy, \quad \text{for } z \in \mathcal{C} \setminus (-\infty, \beta_{n,m}), \tag{2.8}$$

where the path of integration does not cross the real axis.

The important feature of the function ξ_n is that $\xi_{n,+}$ and $\xi_{n,-}$ are purely imaginary on $(\alpha_{n,m}, \beta_{n,m})$ and satisfy for $x \in (\alpha_{n,m}, \beta_{n,m})$

$$\begin{aligned} 2\xi_{n,+}(x) &= -2\xi_{n,-}(x) \\ &= 2\pi i \int_x^{\beta_{n,m}} \mu_{n,m}(u)du \\ &= g_{n,+}(x) - g_{n,-}(x). \end{aligned}$$

So, $2\xi_{n,+}(z)$ and $2\xi_{n,-}(z)$ provide analytic extensions for $g_{n,+}(x) - g_{n,-}(x)$ into the upper half plane and lower half plane, respectively.

On $(-\infty,\infty) \setminus [\alpha_{n,m}, \beta_{n,m})$, ξ_n satisfies

$$\xi_{n,+}(x) - \xi_{n,-}(x) = 2\pi i, \quad \text{for } x \in (-\infty, \alpha_{n,m}),$$

$$2\xi_n(x) = 2g_n(x) - V_{n,m}(x) - l_{n,m}, \quad \text{for } x \in [\beta_{n,m}, \infty).$$

Further one can prove the existence of a $\delta_1 > 0$ such that

$$Re\xi_n(z) > 0, \quad \text{for } 0 < |Imz| < \delta_1, \quad \text{and } \alpha_{n,m} < Rez < \beta_{n,m}. \tag{2.9}$$

The jump matrix ν_T for T can be written in terms of the scalar function ξ_n as

$$\nu_T(x) = \begin{cases} \begin{pmatrix} e^{-2n\xi_{n,+}(x)} & 1 \\ 0 & e^{-2n\xi_{n,-}(x)} \end{pmatrix}, & \text{for } x \in (\alpha_{n,m}, \beta_{n,m}), \\ \\ \begin{pmatrix} 1 & e^{2n\xi_n(x)} \\ 0 & 1 \end{pmatrix}, & \text{for } x \in (0,\infty) \setminus (0, \beta_{n,m}). \end{cases} \tag{2.10}$$

A simple calculation, using the fact that $\xi_{n,+}(x)+\xi_{n,-}(x)=0$ for $x\in(\alpha_{n,m},\beta_{n,m})$, then shows that ν_T has on the interval $(\alpha_{n,m},\beta_{n,m})$ the following factorization,

$$\nu_T(x)=\begin{pmatrix}1 & 0\\ e^{-2n\xi_{n,-}(x)} & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ e^{-2n\xi_{n,+}(x)} & 1\end{pmatrix}. \tag{2.11}$$

Now we are ready to do the transformation $T\to S$. Let $\Sigma_S=\cup_{j=1}^4\Sigma_j$ be the oriented lens shaped contours as shown in Figure 2.1. The precise form of the lens (in fact of the lips Σ_1 and Σ_3) is not yet defined but for now we assume that it will contained in the region where (2.9) holds.

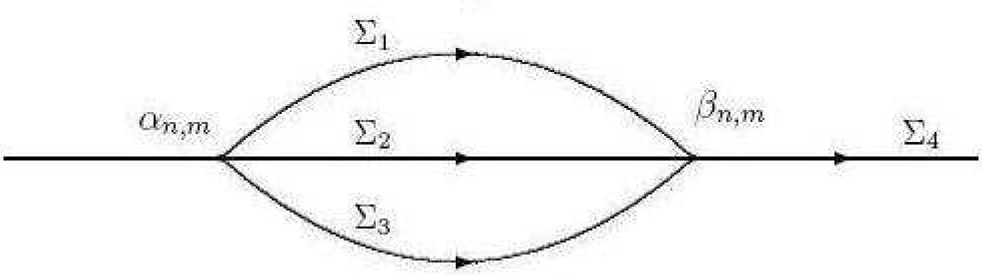

Figure 1: $\Sigma_S=\cup_{j=1}^4\Sigma_j$

Define an analytic matrix valued function S on $\mathcal{C}\setminus\Sigma_S$ as

$$S(z)=\begin{cases} T(z), & \text{for } z \text{ outside the lens,}\\[2ex] T(z)\begin{pmatrix}1 & 0\\ -e^{-2n\xi_n(z)} & 1\end{pmatrix}, & \text{for } z \text{ in the upper part of the lens,}\\[2ex] T(z)\begin{pmatrix}1 & 0\\ e^{-2n\xi_n(z)} & 1\end{pmatrix}, & \text{for } z \text{ in the lower part of the lens,}\end{cases} \tag{2.12}$$

with the upper part of the lens we mean the region between Σ_1 and Σ_2, and with the lower part of the lens the region between Σ_2 and Σ_3.

One can easily check, using (2.10), (2.11) and the conditions of the RH problem for T, that S satisfies the following RH problem:

(a) $S:\mathcal{C}\setminus\Sigma_S\to\mathcal{C}^{2\times2}$ is analytic,

(b) $S_+(x)=S_-(x)\nu_S(x),\quad$ for $x\in\Sigma_S$ with

$$\nu_S(x)=\begin{cases}\begin{pmatrix}1 & 0\\ e^{-2n\xi_n(x)} & 1\end{pmatrix}, & \text{for } x\in\Sigma_1\cup\Sigma_3,\\[2ex] \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, & \text{for } x\in\Sigma_2=(\alpha_{n,m},\beta_{n,m}),\\[2ex] \begin{pmatrix}1 & e^{2n\xi_n(x)}\\ 0 & 1\end{pmatrix}, & \text{for } x\in\Sigma_4=(0,\infty)\setminus(0,\beta_{n,m}),\end{cases} \tag{2.13}$$

(c) $S(z) = I + O(\frac{1}{z})$, as $z \to \infty$.

Note that the jump matrix ν_S on Σ_1, Σ_3, and Σ_4 converges exponentially fast to the identity matrix. We shall see that the leading order asymptotics of U will be determined by a solution P_∞, which is often referred to as the parametrix for the outside region, of the following RH problem:

(a) $P_\infty : \mathcal{C} \setminus (\alpha_{n,m}, \beta_{n,m}) \to \mathcal{C}^{2\times 2}$ is analytic,

(b)

$$P_{\infty,+}(x) = P_{\infty,-}(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{for } x \in (\alpha_{n,m}, \beta_{n,m}),$$

(c) $P_\infty(z) = I + O(\frac{1}{z})$, as $z \to \infty$.

It is well-known that P_∞ is given by

$$P_\infty(z) = \begin{pmatrix} \frac{a(z)+a^{-1}(z)}{2} & \frac{a(z)-a^{-1}(z)}{2i} \\ \\ \frac{a(z)-a^{-1}(z)}{-2i} & \frac{a(z)+a^{-1}(z)}{2} \end{pmatrix}, \quad \text{for } x \in \mathcal{C} \setminus (\alpha_{n,m}, \beta_{n,m}) \tag{2.14}$$

with

$$a(z) = \frac{(z-\beta_{n,m})^{1/4}}{(z-\alpha_{n,m})^{1/4}}, \quad \text{for } z \in \mathcal{C} \setminus (\alpha_{n,m}, \beta_{n,m}). \tag{2.15}$$

Before we can do the final transformation $S \to R$ we need to do a local analysis near $\alpha_{n,m}$ and $\beta_{n,m}$ since the jump matrices for S and P_∞ are not uniformly close to each other in the neighborhood of these points. We will only construct the parametrix P_n near the right endpoint $\beta_{n,m}$ below. As for the left endpoint, if $\frac{m}{n} \to \gamma > 1$, then $\alpha > 0$ and a similar construction is valid in the left endpoint $\alpha_{n,m}$. While in the case $\frac{m}{n} \to 1$, $\alpha = 0$ becomes a hard edge since all eigenvalues of covariance matrices are positive. The behavior of polynomials near the origin is described via Bessel function. We refer to (14) for modifications.

Let $U_{\beta_{n,m},\delta_2} = \{z \in \mathcal{C} : |z - \beta_{n,m}| < \delta_2\}$. Consider the following RH problem for P_n:

(a) $P_n : U_{\beta_{n,m},\delta_2} \setminus \Sigma_S \to \mathcal{C}^{2\times 2}$ is analytic,

(b) $P_{n,+}(x) = P_{n,-}(x)\nu_S(x)$, for $x \in \Sigma_S \cap U_{\beta_{n,m},\delta_2}$ with ν_S in (2.13),

(c) $P_n P_\infty(z)^{-1} = I + O(\frac{1}{n})$, as $n \to \infty$ uniformly for z on the boundary $\partial U_{\beta_{n,m},\delta}$ of the disk $U_{\beta_{n,m},\delta}$, and for δ in a compact subsets of $(0, \delta_2)$.

The construction of P_n is based on an auxiliary RH problem for Ψ in the ζ-plane with jumps on the following oriented contour γ_σ consisting of four straight rays (see Figure 2.2 below)

$$\gamma_{\sigma,1} : \arg\zeta = \sigma, \quad \arg\zeta = \pi, \quad \arg\zeta = -\sigma, \quad \arg\zeta = 0$$

with $\sigma \in (\frac{\pi}{3}, \pi)$. These four rays divide the complex plane into four regions.

The RH problem for Ψ:

(a) $\Psi : \mathcal{C} \setminus \gamma_\sigma \to \mathcal{C}^{2\times 2}$ is analytic,

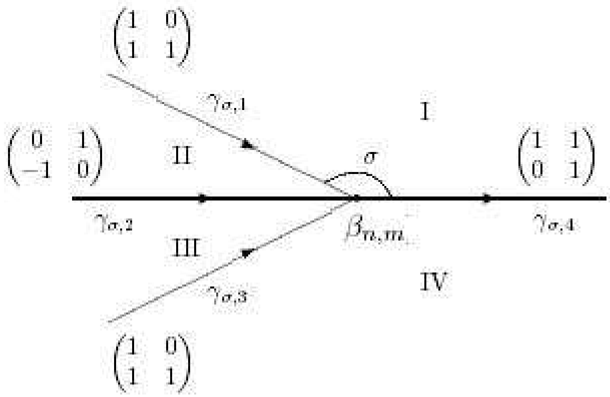

Figure 2: The oriented contour γ_σ

(b) $\Psi_+(\zeta) = \Psi_-(\zeta)\nu_1(\zeta)$, for $\zeta \in \gamma_\sigma$ with ν_1 as shown in Figure 2.2, i.e.,

$$\nu_1(\zeta) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \zeta \in \gamma_{\sigma,1}$$

(c) Ψ has the following asymptotic behavior at infinity,

$$\begin{aligned}\Psi(\zeta) \quad &\sim \quad \zeta^{-\frac{\sigma_3}{4}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\ \times \quad &\left[I + \sum_{k=1}^{\infty} \frac{1}{2} \left(\frac{2}{3} \zeta^{3/2} \right)^{-k} \begin{pmatrix} (-1)^k (s_k + t_k) & s_k - t_k \\ (-1)^k (s_k - t_k) & s_k + t_k \end{pmatrix} \right] e^{-\frac{\pi i \sigma_3}{4}} e^{-\frac{2}{3}\zeta^{3/2}\sigma_3}\end{aligned}$$

as $\zeta \to \infty$, uniformly for $\zeta \in \mathcal{C} \setminus \gamma_\sigma$ in a compact subset of $(\frac{\pi}{3}, \pi)$. Here

$$s_k = \frac{\Gamma(3k + \frac{1}{2})}{54^k k! \Gamma(k + \frac{1}{2})}, \quad t_k = -\frac{6k+1}{6k-1} s_k, \qquad k \geq 1.$$

It is well-known that Ψ is defined by

$$\Psi(\zeta) = \sqrt{2\pi} e^{-\frac{\pi i}{12}} \begin{cases} \begin{pmatrix} A(\zeta) & A(\omega^2\zeta) \\ A'(\zeta) & \omega^2 A'(\omega^2\zeta) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3}, & \text{for } \zeta \in I, \\ \\ \begin{pmatrix} A(\zeta) & A(\omega^2\zeta) \\ A'(\zeta) & \omega^2 A'(\omega^2\zeta) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & \text{for } \zeta \in II, \\ \\ \begin{pmatrix} A(\zeta) & -\omega^2 A(\omega\zeta) \\ A'(\zeta) & -A'(\omega\zeta) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } \zeta \in III, \\ \\ \begin{pmatrix} A(\zeta) & -\omega^2 A(\omega\zeta) \\ A'(\zeta) & -A'(\omega\zeta) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3}, & \text{for } \zeta \in IV. \end{cases} \tag{2.16}$$

The idea is now to construct $\Psi(f_n(z))$ for appropriate biholomorphic maps $f_n : U_{\beta_{n,m},\delta_2} \to f_n(U_{\beta_{n,m},\delta_2})$ with $f_n(\beta_{n,m}) = 0$. We will choose these biholomorphic maps to compensate for the factor $e^{-n\xi_n(z)\sigma_3}$.

Define

$$f_n(z) = n^{2/3}\phi_n(z), \quad \text{for } z \in U_{\beta_{n,m},\delta_2}, \tag{2.17}$$

with $\phi_n(z)$ defined in the following proposition.

Proposition. There exists a $\delta_2 > 0$ such that there are biholomorphic maps $\phi_n : U_{\beta_{n,m},\delta_2} \to \phi_n(U_{\beta_{n,m},\delta_2})$ satisfying

(1) There is a constant $c_0 > 0$ such that for all $z \in U_{\beta_{n,m},\delta_2}$ and all $n \geq 1$ the derivative of ϕ_n can be estimated by

$$c_0 < |\phi_n'(z)| < \frac{1}{c_0}, \quad |\arg \phi_n'(z)| < \frac{\pi}{15}.$$

(2)

$$\phi_n(U_{\beta_{n,m},\delta_2} \cap (-\infty,\infty)) = \phi_n(U_{\beta_{n,m},\delta_2}) \cap (-\infty,\infty)$$

and

$$\phi_n(U_{\beta_{n,m},\delta_2} \cap \mathcal{C}_\pm) = \phi_n(U_{\beta_{n,m},\delta_2}) \cap \mathcal{C}_\pm.$$

(3)

$$-\frac{2}{3}\phi_n(z)^{3/2} = \xi_n(z), \quad \text{for } z \in U_{\beta_{n,m},\delta_2} \setminus (-\infty, \beta_{n,m}).$$

Proof. Define

$$\hat{\phi}_n(z) = \frac{2\beta_{n,m}}{(\beta_{n,m}-\alpha_{n,m})^{1/2}} \cdot \frac{-\frac{3}{2}\xi_n(z)}{(z-\beta_{n,m})^{3/2}}, \quad \text{for } z \in U_{\beta_{n,m},\delta_2} \setminus (-\infty, \beta_{n,m}). \tag{2.18}$$

Note that the function $\hat{\phi}_n(z)$ has no jump across $(\alpha_{n,m}, \beta_{n,m})$ so that $\hat{\phi}_n(z)$ has analytic continuation to $\mathcal{C} \setminus ((-\infty, \alpha_{n,m}) \cup \{\beta_{n,m}\})$. From (2.7) and (2.8) it follows that

$$\begin{aligned} \hat{\phi}_n(z) &= 1 + \frac{3}{2(z-\beta_{n,m})^{3/2}} \cdot \frac{\beta_{n,m}}{(\beta_{n,m}-\alpha_{n,m})^{1/2}} \\ &\quad \times \int_{\beta_{n,m}}^{z} \left(\frac{(\omega-\alpha_{n,m})^{1/2}}{\omega} - \frac{(\beta_{n,m}-\alpha_{n,m})^{1/2}}{\beta_{n,m}} \right) (\omega-\beta_{n,m})^{1/2} d\omega. \end{aligned} \tag{2.19}$$

Using Cauchy's theorem there exists an n_2 such that for all $n \geq n_2$ and $|s-\beta_{n,m}| \leq 1/4$,

$$\begin{aligned} &\left| \frac{(s-\alpha_{n,m})^{1/2}}{s} - \frac{(\beta_{n,m}-\alpha_{n,m})^{1/2}}{\beta_{n,m}} \right| \\ &= |(s-\beta_{n,m})| \frac{1}{2\pi i} \oint_{|\omega-\beta_{n,m}|=\frac{1}{2}} \frac{\frac{(\omega-\alpha_{n,m})^{1/2}}{\omega} - \frac{(\beta_{n,m}-\alpha_{n,m})^{1/2}}{\beta_{n,m}}}{\omega-\beta_{n,m}} \cdot \frac{d\omega}{\omega - s} \\ &\leq |s-\beta_{n,m}| \sup_{|\omega-\beta_{n,m}|=\frac{1}{2}} \left| \frac{(\omega-\alpha_{n,m})^{1/2}}{\omega} - \frac{(\beta_{n,m}-\alpha_{n,m})^{1/2}}{\beta_{n,m}} \right| \\ &\leq C|s-\beta_{n,m}|. \end{aligned}$$

Inserting this into (2.19) we obtain

$$|\hat{\phi}_n(z) - 1| \le C|z - \beta_{n,m}|, \quad \text{for } |z - \beta_{n,m}| \le 1/4,\ n \ge n_2. \tag{2.20}$$

Therefore the isolated singularity of $\hat{\phi}_n(z)$ at $\beta_{n,m}$ is removable so that $\hat{\phi}_n(z)$ is analytic in $\mathcal{C} \setminus (-\infty, \alpha_{n,m}]$, and there exists $\delta > 0$ such that $Re\hat{\phi}_n(z) > 0$ for all $|z - \beta_{n,m}| \le 1/4$ and $n \ge n_2$. This yields

$$\phi_n(z) = \left(\frac{(\beta_{n,m} - \alpha_{n,m})^{1/2}}{2\beta_{n,m}} \right)^{2/3} (z - \beta_{n,m}) \hat{\phi}_n(z)^{2/3} \tag{2.21}$$

is analytic for $z \in U_{\beta_{n,m},\delta}$.

Observe that $\phi_n(z)$ is uniformly (in n and z) bounded in $U_{\beta_{n,m},\delta}$, i.e.,

$$\sup_{n \ge n_2} \sup_{z \in U_{\beta_{n,m},\delta}} |\phi_n(z)| < \infty. \tag{2.22}$$

This implies, by using Cauchy's theorem for derivatives, that $\phi_n''(z)$ is also uniformly (in n and z) bounded in $U_{\beta_{n,m},\delta}$ for a smaller δ, i.e.,

$$\sup_{n \ge n_2} \sup_{z \in U_{\beta_{n,m},\delta}} |\phi_n''(z)| < \infty. \tag{2.23}$$

Since $\hat{\phi}_n(\beta_{n,m}) = 1$, we have $\phi_n'(\beta) = \left(\frac{(\beta_{n,m} - \alpha_{n,m})^{1/2}}{2\beta_{n,m}} \right)^{2/3}$, so that

$$\left| \phi_n'(z) - \left(\frac{(\beta_{n,m} - \alpha_{n,m})^{1/2}}{2\beta_{n,m}} \right)^{2/3} \right| = \left| \int_{\beta_{n,m}}^{z} \phi_n''(s) ds \right| \le C|z - \beta_{n,m}|.$$

Therefore, there exists $0 < \delta_2 < \delta$ such that for all $n \ge n_2$ the $\phi_n(z)$ are injective and hence biholomorphic in $U_{\beta_{n,m},\delta}$. We conclude (1). Now (2) and (3) easily follows from (1), so the proof is complete.

For later reference, observe that the biholomorphic maps is given by

$$f_n(z) = \left(\frac{(\beta_{n,m} - \alpha_{n,m})^{1/2}}{2\beta_{n,m}} \right)^{2/3} n^{2/3} (z - \beta_{n,m}) \hat{f}_n(z), \quad \text{for } z \in U_{\beta_{n,m},\delta}, \tag{2.24}$$

where $\hat{f}_n(z) = \hat{\phi}_n(z)^{2/3}$ with $\hat{\phi}_n$ given by (2.18).

Furthermore it follows that $\hat{f}_n$ is analytic and uniformly (in n and z) bounded in $U_{\beta_{n,m},\delta}$ for some $\delta > \delta_2$ and that $\hat{f}_n(\beta_{n,m}) = 1$. Therefore

$$\begin{aligned} |\hat{f}_n(z) - 1| &= \left| \frac{1}{2\pi i} \oint_{|s - \beta_{n,m}| = \frac{\delta + \delta_2}{2}} \frac{\hat{f}_n(s) - \hat{f}_n(\beta_{n,m})}{s - \beta_{n,m}} \cdot \frac{ds}{s - z} \right| \cdot |z - \beta_{n,m}| \\ &\le C|z - \beta_{n,m}| \end{aligned}$$

for $z \in U_{\beta_{n,m},\delta}$ and $n \ge n_2$.

Now suppose that Σ_S is defined in $U_{\beta_{n,m},\delta_2}$ as the inverse f_n-image of $\gamma_\sigma \cap f_n(U_{\beta_{n,m},\delta_2})$. Then $\Psi(f_n(z))$ is analytic in $U_{\beta_{n,m},\delta_2} \setminus \Sigma_S$ and satisfies jump relations,

$$\Psi_+(f_n(z)) = \begin{cases} \Psi_-(f_n(z)) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } z \in (\Sigma_1 \cup \Sigma_3) \cap U_{\beta_{n,m},\delta_2}, \\ \Psi_-(f_n(z)) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{for } z \in \Sigma_2 \cap U_{\beta_{n,m},\delta_2} = (\beta_{n,m} - \delta_2, \beta_{n,m}), \\ \Psi_-(f_n(z)) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & \text{for } z \in \Sigma_4 \cap U_{\beta_{n,m},\delta_2} = (\beta_{n,m}, \beta_{n,m} + \delta_2). \end{cases}$$

Define

$$E_n(z) = \frac{1}{\sqrt{2}} P^\infty(z) e^{\frac{\pi i}{4}\sigma_3} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} f_n(z)^{\frac{\sigma_3}{4}}, \quad \text{for } z \in U_{\beta_{n,m},\delta_2} \tag{2.25}$$

and

$$P_n(z) = E_n(z)\Psi(f_n(z))e^{-n\xi_n(z)}, \qquad \text{for } z \in U_{\beta_{n,m},\delta_2} \setminus f_n^{-1}(\gamma_\sigma). \tag{2.26}$$

Then P_n solves the desired RH problem, and

$$P_n(z)P^\infty(z)^{-1} \sim I + \sum_{k=1}^{\infty} \Delta_k(z) \frac{1}{n^k} \quad \text{ as } n \to \infty \tag{2.27}$$

uniformly for z in compact subsets of $0 < |z - \beta_{n,m}| < \delta_2$ and σ in a compact subsets of $(\frac{\pi}{3}, \pi)$, where $\Delta_k(z)$ is a meromorphic 2×2 matrix valued function given by

$$\Delta_k(z) = \frac{1}{2(-\xi_n(z))^k} P^\infty(z) \begin{pmatrix} (-1)^k(s_k + t_k) & i(s_k - t_k) \\ -i(-1)^k(s_k - t_k) & s_k + t_k \end{pmatrix} P^\infty(z)^{-1}.$$

We will perform the final transformation of our RH problems. Let $\delta_0 = \min\{\delta_1, \delta_2\}$. Fix $\delta \in (0, \delta_0)$ and $\nu \in (\frac{2\pi}{3}, \frac{5\pi}{6})$. There exists a σ such that $f_n^{-1}(\gamma_{\sigma,1}) \cap \partial U_{\beta_{n,m},\delta} = \{1 + \delta e^{iv}\}$. By the symmetry $\bar{f}_n(z) = f_n(\bar{z})$ we have then $f_n^{-1}(\gamma_{\sigma,3}) \cap \partial U_{\beta_{n,m},\delta} = \{1 + \delta e^{-iv}\}$. We then define Σ_S in $U_{\beta_{n,m},\delta}$ as the inverse f_n image of γ_σ and define Σ_S in $U_{\alpha_{n,m},\delta}$ as the inverse $\tilde{f}_n$ image of $\tilde{\gamma}_{\tilde{\sigma}}$ such that $\tilde{f}_n^{-1}(\tilde{\gamma}_{\tilde{\sigma},1}) \cap \partial\tilde{U}_{\alpha_{n,m},\delta} = \{-\delta e^{-iv}\}$ and $\tilde{f}_n^{-1}(\tilde{\gamma}_{\tilde{\sigma},3}) \cap \partial\tilde{U}_{\alpha_{n,m},\delta} = \{-\delta e^{iv}\}$.

Let $\Sigma_R = \Sigma_S \cup \partial U_{\beta_{n,m},\delta} \cup \partial\tilde{U}_{\alpha_{n,m},\delta}$. Note that the contour Σ_R depends on n (and also on δ, ν). However, we easily see that Σ_i''', $i = 1, \cdots, 4$ are independent of n. Define a matrix valued function $R : \mathcal{C} \to \Sigma_R($ depending on the parameters $n, \delta, v)$ as

$$R(z) = \begin{cases} S(z)P_n(z)^{-1}, & \text{for } z \in U_{\beta_{n,m},\delta} \setminus \Sigma_S, \\ S(z)\tilde{P}_n(z)^{-1}, & \text{for } z \in \tilde{U}_{\alpha_{n,m},\delta} \setminus \Sigma_S, \\ S(z)P^\infty(z)^{-1}, & \text{for } z \text{ elsewhere.} \end{cases} \tag{2.28}$$

By the definition, R has jumps on the contour Σ_R. However, one can show that R has only jumps on the reduced contour $\hat{\Sigma}_R = \Sigma_1''' \cup \Sigma_3''' \cup \Sigma_4''' \cup \partial U_{\beta_{n,m},\delta} \cup \partial\tilde{U}_{\alpha_{n,m},\delta}$, and R is a solution of the following RH problem on the contour $\hat{\Sigma}_R$:

(a) $R : \mathcal{C} \setminus \hat{\Sigma}_R \to \mathcal{C}^{2\times 2}$ is analytic,
(b) $R_+(x) = R_-(x)\nu_R(x)$, for $x \in \hat{\Sigma}_R$ with

$$\nu_R(z) = \begin{cases} P_n(z)P^\infty(z)^{-1}, & \text{for } z \in \partial U_{\beta_{n,m},\delta}, \\ \tilde{P}_n(z)P^\infty(z)^{-1}, & \text{for } z \in \partial \tilde{U}_{\alpha_{n,m},\delta}, \\ P^\infty(z)\nu_S(z)P^\infty(z)^{-1}, & \text{for } z \in \Sigma_1''' \cup \Sigma_3''' \cup \Sigma_4''', \end{cases}$$

(c) $R(z) = I + O(\frac{1}{z})$, as $z \to \infty$.
The point of the matter is that R has the following asymptotic expansion in powers of n^{-1},

$$R(z) \sim I + \frac{1}{n}\sum_{k=0}^{\infty} r_k(z)n^{-k} \quad \text{as } n \to \infty \tag{2.29}$$

uniformly for δ in compact subsets of $(0, \delta_0)$ and for $v \in (\frac{2\pi}{3}, \frac{5\pi}{6})$ and $z \in \mathcal{C} \setminus \Sigma_R$, i.e., there are $C_l > 0$ for each $l \geq 1$ such that

$$\sup_{z \in \mathcal{C}\setminus\Sigma_R, \delta\in(0,\delta_0), v\in(\frac{2\pi}{3},\frac{5\pi}{6})} ||R(z) - I - \frac{1}{n}\sum_{k=0}^{l} r_k(z)n^{-k/m}|| \leq \frac{C_l}{n^{(l+1)/m}}. \tag{2.30}$$

By going back in the series of transformations $U \overset{(2.6)}{\longrightarrow} T \overset{(2.12)}{\longrightarrow} S \overset{(2.28)}{\longrightarrow} R$, we can find the asymptotics of U. This is summarized as follows.

Theorem 3. *(1) For $x \in (\alpha_{m,n} + \delta,\ \beta_{m,n} - \delta)$, $\delta > 0$,*

$$\rho_{n,1}(x) = \mu_{n,m}(x) - \frac{\beta_{m,n} - \alpha_{m,n}}{4n\pi(\beta_{m,n} - x)(x - \alpha_{m,n})}\cos\left(n\pi\int_x^{\beta_{m,n}} \mu_{n,m}(u)du\right) + O\left(\frac{1}{n^2}\right). \tag{2.31}$$

(2) For $x \in (0, \infty) \setminus (\alpha_{m,n} - \delta,\ \beta_{m,n} + \delta)$, $\delta > 0$,

$$\rho_{n,1}(x) = \frac{2n(\beta_{m,n} - \alpha_{m,n})}{(x - \alpha_{m,n})(x - \beta_{m,n})}e^{2n\xi_n(x)} + O\left(\frac{1}{n^2}\right)e^{2n\xi_n(x)}. \tag{2.32}$$

(3) For $x \in (\beta_{m,n} - \delta,\ \beta_{m,n} + \delta)$, $\delta > 0$,

$$\begin{aligned} &n\rho_{n,1}(x) \\ &= \left(\frac{f_n'(x)}{4f_n(x)} - \frac{a'(x)}{a(x)}\right)[2A(f_n(x))A'(f_n(x))] + f_n'(x)[A'(f_n(x))^2 - f_n(x)A(f_n(x))^2] \\ &\quad + O\left(\frac{1}{n^2}\right)\left[a^2(x)f_n^{-\frac{1}{2}}(x)A'(f_n(x))^2 + a^{-2}(x)f_n^{\frac{1}{2}}(x)A(f_n(x))^2\right]. \end{aligned} \tag{2.33}$$

For $x \in (\alpha_{m,n} - \delta,\ \alpha_{m,n} + \delta)$, $\delta > 0$

$$\begin{aligned} &n\rho_{n,1}(x) \\ &= -\left(\frac{\tilde{f}_n'(x)}{4\tilde{f}_n(x)} + \frac{a'(x)}{a(x)}\right)[2A(\tilde{f}_n(x))A'(\tilde{f}_n(x))] - \tilde{f}_n'(x)[A'(\tilde{f}_n(x))^2 - \tilde{f}_n(x)A(\tilde{f}_n(x))^2] \\ &\quad + O\left(\frac{1}{n^2}\right)\left[a^2(x)\tilde{f}_n^{-\frac{1}{2}}(x)A'(\tilde{f}_n(x))^2 + a^{-2}(x)\tilde{f}_n^{\frac{1}{2}}(x)A(\tilde{f}_n(x))^2\right], \end{aligned} \tag{2.34}$$

where

$$\tilde{f}_n(x) = \begin{cases} -\left(\frac{3\pi}{2} n \int_{\alpha_{n,m}}^{x} \mu_{n,m}(u)du\right)^{2/3}, & \text{if } x > \alpha_{n,m}, \\ \left(\frac{3\pi}{2} n \int_{x}^{\alpha_{n,m}} \mu_{n,m}(u)du\right)^{2/3}, & \text{if } x \le \alpha_{n,m}. \end{cases}$$

Proof (1). We have for $x \in (\alpha_{m,n} + \delta, \beta_{m,n} - \delta)$, $\delta > 0$,

$$\begin{pmatrix} U_{11}(x) \\ U_{21}(x) \end{pmatrix} = e^{\frac{1}{2} n l_{n,m} \sigma_3} R(x) P_\infty(x) \begin{pmatrix} 1 & 0 \\ e^{-2\xi_n(x)} & 1 \end{pmatrix} \begin{pmatrix} e^{n(g_n(x) - \frac{1}{2} l_{n,m})} \\ 0 \end{pmatrix}. \tag{2.35}$$

Taking derivatives at both sides of (2.35) yields

$$\begin{aligned} \begin{pmatrix} U'_{11}(x) \\ U'_{21}(x) \end{pmatrix} &= e^{\frac{1}{2} n l_{n,m} \sigma_3} R'(x) P_\infty(x) \begin{pmatrix} 1 & 0 \\ e^{-2\xi_n(x)} & 1 \end{pmatrix} \begin{pmatrix} e^{n(g_n(x) - \frac{1}{2} l_{n,m})} \\ 0 \end{pmatrix} \\ &\quad + e^{\frac{1}{2} n l_{n,m} \sigma_3} R(x) P'_\infty(x) \begin{pmatrix} 1 & 0 \\ e^{-2\xi_n(x)} & 1 \end{pmatrix} \begin{pmatrix} e^{n(g_n(x) - \frac{1}{2} l_{n,m})} \\ 0 \end{pmatrix} \\ &\quad + e^{\frac{1}{2} n l_{n,m} \sigma_3} R(x) P_\infty(x) \begin{pmatrix} 1 & 0 \\ e^{-2\xi_n(x)} & 1 \end{pmatrix}' \begin{pmatrix} e^{n(g_n(x) - \frac{1}{2} l_{n,m})} \\ 0 \end{pmatrix} \\ &\quad + e^{\frac{1}{2} n l_{n,m}} R(x) P_\infty(x) \begin{pmatrix} 1 & 0 \\ e^{-2\xi_n(x)} & 1 \end{pmatrix} \begin{pmatrix} e^{n(g_n(x) - \frac{1}{2} l_{n,m})} \\ 0 \end{pmatrix}' \\ &=: \begin{pmatrix} a_{11}(x) \\ a_{21}(x) \end{pmatrix} + \begin{pmatrix} b_{11}(x) \\ b_{21}(x) \end{pmatrix} + \begin{pmatrix} c_{11}(x) \\ c_{21}(x) \end{pmatrix} + \begin{pmatrix} d_{11}(x) \\ d_{21}(x) \end{pmatrix}. \end{aligned} \tag{2.36}$$

First, it is easy to see

$$\det \begin{pmatrix} U_{11}(x) & d_{11}(x) \\ U_{21}(x) & d_{21}(x) \end{pmatrix} = 0. \tag{2.37}$$

Also, $P_\infty(x)$ can be written as

$$P_\infty(x) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} a(x) & 0 \\ 0 & a^{-1}(x) \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}, \tag{2.38}$$

so we have

$$\begin{aligned} P'_\infty(x) &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} a(x) & 0 \\ 0 & a^{-1}(x) \end{pmatrix}' \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \\ &= \frac{a'(x)}{2a(x)} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} a(x) & 0 \\ 0 & a^{-1}(x) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}. \end{aligned}$$

A simple algebraic computation gives

$$\det \begin{pmatrix} U_{11}(x) & b_{11}(x) \\ U_{21}(x) & b_{21}(x) \end{pmatrix} = i \frac{a'(x)}{a(x)} \cos \left(n\pi \int_x^{\beta_{m,n}} \mu_{n,m}(u)du \right) e^{nV_{n,m}(x)}. \tag{2.39}$$

Since

$$\begin{pmatrix} 1 & 0 \\ e^{-2\xi_n(x)} & 1 \end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ -2\xi_n'(x)e^{-2\xi_n(x)} & 1 \end{pmatrix}$$

and

$$\begin{aligned} \xi_n'(x) &= -\pi i \psi_n(x) \\ &= -\frac{1}{2x}\sqrt{(x-\alpha_{n,m})(x-\beta_{n,m})}, \end{aligned}$$

then

$$\begin{aligned} \det \begin{pmatrix} U_{11}(x) & c_{11}(x) \\ U_{21}(x) & c_{21}(x) \end{pmatrix} &= -2n\xi_n'(x)e^{nV_{n,m}(x)} \\ &= \frac{n}{x}\sqrt{(x-\alpha_{n,m})(x-\beta_{n,m})}\, e^{nV_{n,m}(x)}. \end{aligned} \tag{2.40}$$

Let

$$\begin{pmatrix} x_1(x) \\ x_2(x) \end{pmatrix} = P_\infty(x) \begin{pmatrix} 1 & 0 \\ e^{-2\xi_n(x)} & 1 \end{pmatrix} \begin{pmatrix} e^{n(g_n(x)-\frac{1}{2}l_{n,m})} \\ 0 \end{pmatrix} \tag{2.41}$$

and

$$R(x) = \begin{pmatrix} R_{11}(x) & R_{12}(x) \\ R_{21}(x) & R_{22}(x) \end{pmatrix}. \tag{2.42}$$

Then inserting (2.41) and (2.42) into (2.35) and (2.36), we obtain

$$\det \begin{pmatrix} U_{11}(x) & a_{11}(x) \\ U_{21}(x) & a_{21}(x) \end{pmatrix} = \begin{pmatrix} x_1(x) & x_2(x) \end{pmatrix} \begin{pmatrix} r_{11}(x) & r_{12}(x) \\ r_{21}(x) & r_{22}(x) \end{pmatrix} \begin{pmatrix} x_1(x) \\ x_2(x) \end{pmatrix}, \tag{2.43}$$

where and in the sequel

$$r_{11}(x) = \begin{pmatrix} R_{11}(x) & R_{11}'(x) \\ R_{21}(x) & R_{21}'(x) \end{pmatrix}, \quad r_{12}(x) = \begin{pmatrix} R_{11}(x) & R_{11}'(x) \\ R_{22}(x) & R_{22}'(x) \end{pmatrix},$$

$$r_{21}(x) = \begin{pmatrix} R_{12}(x) & R_{12}'(x) \\ R_{21}(x) & R_{21}'(x) \end{pmatrix}, \quad r_{22}(x) = \begin{pmatrix} R_{12}(x) & R_{12}'(x) \\ R_{22}(x) & R_{22}'(x) \end{pmatrix}.$$

Note that

$$\begin{pmatrix} x_1(x) \\ x_2(x) \end{pmatrix} = P_\infty(x) \begin{pmatrix} e^{n\xi_n(x)} \\ e^{-n\xi_n(x)} \end{pmatrix} e^{\frac{nV_{n,m}(x)}{2}}$$

and the vector $P_\infty(x) \begin{pmatrix} e^{n\xi_n(x)} \\ e^{-n\xi_n(x)} \end{pmatrix}$ is bounded uniformly in n and $x \in (\alpha_{m,n}+\delta, \beta_{m,n}-\delta)$.
Also, $R(\cdot)$ is piecewise analytic, it follows from (2.29)

$$|r_{11}|, |r_{22}| = O\left(\frac{1}{n}\right), \quad |r_{12}|, |r_{21}| = O\left(\frac{1}{n^2}\right).$$

Hence, we have by (2.43)

$$\left|\det\begin{pmatrix} U_{11}(x) & a_{11}(x) \\ U_{21}(x) & a_{21}(x) \end{pmatrix}\right| = O\left(\frac{1}{n^2}\right) e^{nV_{n,m}(x)}. \tag{2.44}$$

Combining (2.37),(2.39),(2.40) and (2.44) yields the desired result.
(2). For all $x \in (0,\infty) \setminus (\alpha_{n,m} - \delta, \beta_{n,m} + \delta)$, we have

$$\begin{pmatrix} U_{11}(x) \\ U_{21}(x) \end{pmatrix} = e^{\frac{1}{2}nl_{n,m}\sigma_3} R(x) P_\infty(x) \begin{pmatrix} e^{n(g_n(x) - \frac{1}{2}l_{n,m})} \\ 0 \end{pmatrix}. \tag{2.45}$$

Taking derivatives at both sides yields

$$\begin{aligned} \begin{pmatrix} U'_{11}(x) \\ U'_{21}(x) \end{pmatrix} &= e^{\frac{1}{2}nl_{n,m}\sigma_3} R'(x) P_\infty(x) \begin{pmatrix} e^{n(g_n(x) - \frac{1}{2}l_{n,m})} \\ 0 \end{pmatrix} \\ &\quad + e^{\frac{1}{2}nl_{n,m}\sigma_3} R(x) P'_\infty(x) \begin{pmatrix} e^{n(g_n(x) - \frac{1}{2}l_{n,m})} \\ 0 \end{pmatrix} \\ &\quad + e^{\frac{1}{2}nl_{n,m}\sigma_3} R(x) P_\infty(x) \begin{pmatrix} e^{n(g_n(x) - \frac{1}{2}l_{n,m})} \\ 0 \end{pmatrix}' \\ &=: \begin{pmatrix} a_{11}(x) \\ a_{21}(x) \end{pmatrix} + \begin{pmatrix} b_{11}(x) \\ b_{21}(x) \end{pmatrix} + \begin{pmatrix} c_{11}(x) \\ c_{21}(x) \end{pmatrix}. \end{aligned} \tag{2.46}$$

Now it is easy to see

$$\det\begin{pmatrix} U_{11}(x) & c_{11}(x) \\ U_{21}(x) & c_{21}(x) \end{pmatrix} = 0. \tag{2.47}$$

Also, a similar algebra to (2.39) shows

$$\det\begin{pmatrix} U_{11}(x) & b_{11}(x) \\ U_{21}(x) & b_{21}(x) \end{pmatrix} = -i\frac{a'(x)}{a(x)} e^{2n\xi_n(x)} e^{nV_{n,m}(x)}. \tag{2.48}$$

Let

$$\begin{pmatrix} x_1(x) \\ x_2(x) \end{pmatrix} = P_\infty(x) \begin{pmatrix} e^{n(g_n(x) - \frac{1}{2}l_{n,m})} \\ 0 \end{pmatrix},$$

then it obviously holds

$$\begin{pmatrix} x_1(x) \\ x_2(x) \end{pmatrix} = P_\infty(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\xi_n(x)} e^{\frac{nV_{n,m}(x)}{2}}.$$

Since $P_\infty(x)$ is bounded uniformly in n and $x \in (-\infty,\infty) \setminus (\alpha - \delta, \beta + \delta)$, then one can similarly prove

$$\left|\det\begin{pmatrix} U_{11}(x) & a_{11}(x) \\ U_{21}(x) & a_{21}(x) \end{pmatrix}\right| = O\left(\frac{1}{n^2}\right) e^{2n\xi_n(x)} e^{nV_{n,m}(x)}. \tag{2.49}$$

Combining (2.47), (2.48) and (2.49) yields the desired result.

We remark that when $x \geq \beta_{n,m} + \delta$,

$$\xi_n(x) = -\int_{\beta_{n,m}}^{x} \frac{1}{2u}\sqrt{(u-\alpha_{n,m})(u-\beta_{n,m})}\, du < 0.$$

This shows $\rho_{n,1}(x)$ is exponentially small in n and $x \geq \beta_{n,m} + \delta$. Same to $x \leq \alpha_{n,m} - \delta$.

(3). We need only to give a proof in the case of $x \in (\beta_{n,m} - \delta, \beta_{n,m} + \delta)$ since the other case is similar.

For $x \in (\beta_{n,m} - \delta, \beta_{n,m} + \delta)$, we have

$$\begin{aligned}\begin{pmatrix} U_{11}(x) \\ U_{21}(x) \end{pmatrix} &= \frac{1}{\sqrt{2}} e^{\frac{1}{2} n l_{n,m} \sigma_3} R(x) P_\infty(x) e^{\frac{\pi i}{4}\sigma_3} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} f_n(z)^{\frac{\sigma_3}{4}} \Psi(f_n(x)) \\ &\quad \times e^{-n\xi_n(x)\sigma_3} \begin{pmatrix} 1 & 0 \\ e^{-2n\xi_n(x)} & 1 \end{pmatrix} \begin{pmatrix} e^{n(g_n(x) - \frac{1}{2} l_{n,m})} \\ 0 \end{pmatrix}. \end{aligned} \tag{2.50}$$

Taking derivatives at both sides (2.50) yields

$$\begin{aligned} &\begin{pmatrix} U'_{11}(x) \\ U'_{21}(x) \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} e^{\frac{1}{2} n l_{n,m} \sigma_3} R'(x) P_\infty(x) e^{\frac{\pi i}{4}\sigma_3} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} f_n(x)^{\frac{\sigma_3}{4}} \Psi(f_n(x)) \\ &\quad \times e^{-n\xi_n(x)\sigma_3} \begin{pmatrix} 1 & 0 \\ e^{-2n\xi_n(x)} & 1 \end{pmatrix} \begin{pmatrix} e^{n(g_n(x) - \frac{1}{2} l_{n,m})} \\ 0 \end{pmatrix} \\ &\quad + \frac{1}{\sqrt{2}} e^{\frac{1}{2} n l_{n,m} \sigma_3} R(z) P_\infty(x)' e^{\frac{\pi i}{4}\sigma_3} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} f_n(x)^{\frac{\sigma_3}{4}} \Psi(f_n(x)) \\ &\quad \times e^{-n\xi_n(x)\sigma_3} \begin{pmatrix} 1 & 0 \\ e^{-2n\xi_n(x)} & 1 \end{pmatrix} \begin{pmatrix} e^{n(g_n(x) - \frac{1}{2} l_{n,m})} \\ 0 \end{pmatrix} \\ &\quad + \frac{1}{\sqrt{2}} e^{\frac{1}{2} n l_{n,m} \sigma_3} R(x) P_\infty(x) e^{\frac{\pi i}{4}\sigma_3} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} f'_n(x)^{\frac{\sigma_3}{4}} \Psi(f_n(x)) \\ &\quad \times e^{-n\xi_n(x)\sigma_3} \begin{pmatrix} 1 & 0 \\ e^{-2n\xi_n(x)} & 1 \end{pmatrix} \begin{pmatrix} e^{n(g_n(z) - \frac{1}{2} l_{n,m})} \\ 0 \end{pmatrix} \\ &\quad + \frac{1}{\sqrt{2}} e^{\frac{1}{2} n l_{n,m} \sigma_3} R(x) P_\infty(x) e^{\frac{\pi i}{4}\sigma_3} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} f_n(x)^{\frac{\sigma_3}{4}} \Psi'(f_n(x)) \\ &\quad \times e^{-n\xi_n(x)\sigma_3} \begin{pmatrix} 1 & 0 \\ e^{-2n\xi_n(x)} & 1 \end{pmatrix} \begin{pmatrix} e^{n(g_n(x) - \frac{1}{2} l_{n,m})} \\ 0 \end{pmatrix} \\ &\quad + \frac{1}{\sqrt{2}} e^{\frac{1}{2} n l_{n,m} \sigma_3} R(x) P_\infty(x) e^{\frac{\pi i}{4}\sigma_3} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} f_n(x)^{\frac{\sigma_3}{4}} \Psi(f_n(x)) \\ &\quad \times \left(e^{-n\xi_n(x)\sigma_3} \begin{pmatrix} 1 & 0 \\ e^{-2n\xi_n(x)} & 1 \end{pmatrix} \begin{pmatrix} e^{n(g_n(x) - \frac{1}{2} l_{n,m})} \\ 0 \end{pmatrix} \right)' \\ &=: \begin{pmatrix} a_{11}(x) \\ a_{21}(x) \end{pmatrix} + \begin{pmatrix} b_{11}(x) \\ b_{21}(x) \end{pmatrix} + \begin{pmatrix} c_{11}(x) \\ c_{21}(x) \end{pmatrix} + \begin{pmatrix} d_{11}(x) \\ d_{21}(x) \end{pmatrix} + \begin{pmatrix} e_{11}(x) \\ e_{21}(x) \end{pmatrix}. \end{aligned} \tag{2.51}$$

Again, it is easy to see

$$\det\begin{pmatrix} U_{11}(x) & e_{11}(x) \\ U_{21}(x) & e_{21}(x) \end{pmatrix} = 0. \tag{2.52}$$

Also, note that the following equations hold true

$$\begin{pmatrix} a(x) & 0) \\ 0 & a^{-1}(x) \end{pmatrix}' = \frac{a'(x)}{a(x)}\begin{pmatrix} a(x) & 0) \\ 0 & a^{-1}(x) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} e^{-n\xi_n(x)\sigma_3}\begin{pmatrix} 1 & 0 \\ e^{-2n\xi_n(x)} & 1 \end{pmatrix}\begin{pmatrix} e^{n(g_n(x)-\frac{1}{2}l_{n,m})} \\ 0 \end{pmatrix} = \begin{pmatrix} e^{nV_{n,m}(x)} \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} e^{\frac{\pi i}{4}\sigma_3}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} e^{\frac{\pi i}{4}\sigma_3}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\left(f_n(x)^{\frac{\sigma_3}{4}}\right)' = \frac{f_n(x)'}{4f_n(x)} f_n(x)^{\frac{\sigma_3}{4}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} A(x) & A(\omega^2 x) \\ A'(x) & \omega^2 A'(\omega^2 x) \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix}\begin{pmatrix} A(x) & A(\omega^2 x) \\ A'(x) & \omega^2 A'(\omega^2 x) \end{pmatrix}.$$

Now some calculations show

$$\begin{aligned} &\det\begin{pmatrix} U_{11}(x) & b_{11}(x)+c_{11}(x)+d_{11}(x) \\ U_{21}(x) & b_{11}(x)+c_{11}(x)+d_{11}(x) \end{pmatrix} \\ &= \left(\frac{f_n'(x)}{4f_n(x)} - \frac{a'(x)}{a(x)}\right) 2A(f_n(x))A'(f_n(x)) \\ &\quad + f_n'(x)[A'(f_n(x))^2 - f_n(x)A(f_n(x))^2]. \end{aligned} \tag{2.53}$$

Let

$$\begin{aligned} \begin{pmatrix} x_1(x) \\ x_2(x) \end{pmatrix} &= \frac{1}{\sqrt{2}} P_\infty(x) e^{\frac{\pi i}{4}\sigma_3}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} f_n(x)^{\frac{\sigma_3}{4}} \Psi(f_n(x)) e^{-n\xi_n(x)\sigma_3} \\ &\quad \times \begin{pmatrix} 1 & 0 \\ e^{-2n\xi_n(x)} & 1 \end{pmatrix}\begin{pmatrix} e^{n(g_n(x)-\frac{1}{2}l_{n,m})} \\ 0 \end{pmatrix}, \end{aligned}$$

then it holds true

$$\begin{pmatrix} x_1(x) \\ x_2(x) \end{pmatrix} = \sqrt{\pi}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} -a(x) f_n^{-\frac{1}{4}}(x) A'(f_n(x)) \\ a^{-1}(x) f_n^{\frac{1}{4}}(x) A(f_n(x)) \end{pmatrix} e^{\frac{nV_{n,m}(x)}{2}}. \tag{2.54}$$

Noting that

$$\left|\det\begin{pmatrix} r_{11}(x) & r_{12}(x) \\ r_{21}(x) & r_{22}(x) \end{pmatrix}\right| = O\left(\frac{1}{n^2}\right),$$

we have

$$\left|\det\begin{pmatrix} U_{11}(x) & a_{11}(x) \\ U_{21}(x) & a_{21}(x) \end{pmatrix}\right|$$
$$= O\left(\frac{1}{n^2}\right)\left[a^2(x)f_n^{-\frac{1}{2}}(x)A'(f_n(x))^2 + a^{-2}(x)f_n^{\frac{1}{2}}(x)A(f_n(x))^2\right]. \tag{2.55}$$

Combining (2.52) (2.53) and (2.55) yields the desired result

$$n\rho_{n,1}(x)$$
$$= \left(\frac{f_n'(x)}{4f_n(x)} - \frac{a'(x)}{a(x)}\right)2A(f_n(x))A'(f_n(x)) + f_n'(x)[A'(f_n(x))^2 - f_n(x)A(f_n(x))^2]$$
$$+O\left(\frac{1}{n^2}\right)\left[a^2(x)f_n^{-\frac{1}{2}}(x)A'(f_n(x))^2 + a^{-2}(x)f_n^{\frac{1}{2}}(x)A(f_n(x))^2\right].$$

The proof of Theorem 3 is now complete.

Corollary. *We have for $x \in (\alpha_{n,m} + \frac{C}{n^{2/3}}, \beta_{n,m} - \frac{C}{n^{2/3}})$,*

$$|\rho_{n,1}(x) - \mu_{n,m}(x)| \leq \frac{C}{n(x-\alpha_{n,m})(\beta_{n,m}-x)}. \tag{2.56}$$

In addition, assume $|\frac{m}{n} - \gamma| = O(\frac{1}{n})$, then we have for $x \in (\alpha + \frac{C}{n^{2/3}}, \beta - \frac{C}{n^{2/3}})$,

$$|\rho_{n,1}(x) - \mu_\gamma(x)| \leq \frac{C}{n(x-\alpha)(\beta-x)}. \tag{2.57}$$

Proof. Let us first show (2.56). By (2.31), it suffices to prove for $\delta > 0$

$$|\rho_{n,1}(x) - \mu_{n,m}(x)| \leq \frac{C}{n(x-\alpha_{n,m})(\beta_{n,m}-x)}, \quad x \in (\beta_{n,m} - \delta, \beta_{n,m} - \frac{C}{n^{2/3}}) \tag{2.58}$$

and

$$|\rho_{n,1}(x) - \mu_{n,m}(x)| \leq \frac{C}{n(x-\alpha_{n,m})(\beta_{n,m}-x)}, x \in (\alpha_{n,m} + \frac{C}{n^{2/3}}, \alpha_{n,m} + \delta). \tag{2.59}$$

We shall prove only (2.58) using (2.33) since the other is similar.
Recall that the Airy function is bounded on the real line, and for $r > 0$ (see (7),(13))

$$A(-r) = \frac{1}{\sqrt{\pi}r^{1/4}}\left\{\cos\left(\frac{2}{3}r^{3/2} - \frac{\pi}{4}\right) + O\left(\frac{1}{r^{3/2}}\right)\right\} \tag{2.60}$$

and

$$A'(-r) = \frac{r^{1/4}}{\sqrt{\pi}}\left\{\sin\left(\frac{2}{3}r^{3/2} - \frac{\pi}{4}\right) + O\left(\frac{1}{r^{3/2}}\right)\right\}. \tag{2.61}$$

Then it is easy to see that $|A(x)A'(x)| = O(1)$. Also, noting that

$$f_n(x) = -\left(\frac{3\pi}{2}n\int_x^{\beta_{n,m}}\mu_{n,m}(u)du\right)^{2/3}, \qquad \text{for } x \leq \beta_{n,m}, \tag{2.62}$$

we have

$$\frac{f_n'(x)}{4f_n(x)} - \frac{a'(x)}{a(x)} = O(1). \tag{2.63}$$

When $x \in (\beta_{n,m} - \delta, \beta_{n,m} - \frac{C}{n^{2/3}})$,

$$|f_n(x)| \geq cn^{2/3}(\beta_{n,m} - x) \geq C. \tag{2.64}$$

This together with (2.60) and (2.61) implies

$$A'(f_n(x))^2 = \frac{|f_n(x)|^{1/2}}{\pi}\left\{\sin\left(\frac{2}{3}|f_n(x)|^{3/2} - \frac{\pi}{4}\right) + O\left(\frac{1}{|f_n(x)|^{3/2}}\right)\right\}^2, \tag{2.65}$$

and

$$f_n(x)A(f_n(x))^2 = \frac{|f_n(x)|^{1/2}}{\pi}\left\{\cos\left(\frac{2}{3}|f_n(x)|^{3/2} - \frac{\pi}{4}\right) + O\left(\frac{1}{|f_n(x)|^{3/2}}\right)\right\}^2. \tag{2.66}$$

Thus we have by (2.65) and (2.66)

$$\begin{aligned}
&f_n'(x)[A'(f_n(x))^2 - f_n(x)A(f_n(x))^2] \\
&= \frac{1}{\pi}f_n'(x)|f_n(x)|^{1/2}[\left\{\cos\left(\frac{2}{3}|f_n(x)|^{3/2} - \frac{\pi}{4}\right) + O\left(\frac{1}{|f_n(x)|^{3/2}}\right)\right\}^2 \\
&\quad + \left\{\sin\left(\frac{2}{3}|f_n(x)|^{3/2} - \frac{\pi}{4}\right) + O\left(\frac{1}{|f_n(x)|^{3/2}}\right)\right\}^2] \\
&= \frac{1}{\pi}f_n'(x)|f_n(x)|^{1/2}\left(1 + O\left(\frac{1}{|f_n(x)|^{3/2}}\right)\right).
\end{aligned} \tag{2.67}$$

Now (2.58) easily follows from (2.33) and the fact that $\frac{1}{\pi}f_n'(x)|f_n(x)|^{1/2} = \mu_{n,m}(x)$ for $x \leq \beta_{n,m}$. Next we prove (2.57). Note that $\alpha_{n,m} \to \alpha, \beta_{n,m} \to \beta$, so if $x \in (\alpha + \frac{C}{n^{2/3}}, \beta - \frac{C}{n^{2/3}})$ and $|\frac{m}{n} - \gamma| = O(\frac{1}{n})$, then $x \in (\alpha_{n,m} + \frac{C}{n^{2/3}}, \beta_{n,m} - \frac{C}{n^{2/3}})$ for some $C > 0$ (possibly different) and sufficiently large n.

A simple calculation shows

$$\begin{aligned}
|\mu_{n,m}(x) - \mu_\gamma(x)| &\leq \frac{1}{2\pi x}\sqrt{\beta - x}\left|\sqrt{x - \alpha_{n,m}} - \sqrt{x - \alpha}\right| \\
&\quad + \frac{1}{2\pi x}\sqrt{x - \alpha_{n,m}}\left|\sqrt{\beta_{n,m} - x} - \sqrt{\beta - x}\right| \\
&\leq C\left|\frac{m}{n} - \gamma\right|\frac{1}{\sqrt{(x - \alpha)(\beta - x)}}.
\end{aligned} \tag{2.68}$$

Thus we easily derive from (2.56) and (2.68) that (2.57) is true under the hypothesis $|\frac{m}{n} - \gamma| = O(\frac{1}{n})$, as desired.

3 Proofs of Theorems 1 and 2

In this section we shall give the proof of Theorems 1 and 2. By virtue of (1.13), (1.14) and (1.15), we shall basically focus on the computation of expectation and variance, which are given in the following lemmas.

Lemma 1. *Let t_n be such that $\beta_{n,m} - t_n \to 0$ and $n(\beta_{n,m} - t_n)^{3/2} \geq C$ for some $C > 0$. Let $I_n = [t_n, \infty)$, then*

$$E\#I_n = \frac{\sqrt{\beta_{n,m} - \alpha_{n,m}}}{3\pi\beta_{n,m}} n(\beta_{n,m} - t_n)^{3/2}(1 + o(1)). \tag{3.1}$$

Proof. For some $\delta > 0$, we have

$$\begin{aligned} E\#I_n &= n\int_{t_n}^{\infty} \rho_{n,1}(x)dx \\ &= n\int_{t_n}^{\beta_{n,m}+\delta} \rho_{n,1}(x)dx + n\int_{\beta_{n,m}+\delta}^{\infty} \rho_{n,1}(x)dx. \end{aligned} \tag{3.2}$$

By virtue of (2.32), for $x \geq \beta_{n,m} + \delta$, $\rho_{n,1}(x)$ is exponentially small in n and exponentially decaying in x. So it holds

$$\int_{\beta_{n,m}+\delta}^{\infty} n\rho_{n,1}(x)dx = O(1). \tag{3.3}$$

For $\beta_{n,m} - \delta \leq x \leq \beta_{n,m} + \delta$, recall

$$\begin{aligned} &n\rho_{n,1}(x) \\ &= \left(\frac{f_n'(x)}{4f_n(x)} - \frac{a'(x)}{a(x)}\right)[2A(f_n(x))A'(f_n(x))] + f_n'(x)[A'(f_n(x))^2 - f_n(x)A(f_n(x))^2] \\ &\quad +O\left(\frac{1}{n^2}\right)\left[a^2(x)f_n^{-\frac{1}{2}}(x)A'(f_n(x))^2 + a^{-2}(x)f_n^{\frac{1}{2}}(x)A(f_n(x))^2\right]. \end{aligned} \tag{3.4}$$

Here

$$a(x) = \frac{(x - \beta_{n,m})^{1/4}}{(x - \alpha_{n,m})^{1/4}}$$

and

$$f_n(x) = \begin{cases} -\left(\frac{3\pi}{2}n\int_x^{\beta_{n,m}} \mu_{n,m}(u)du\right)^{2/3}, & \text{if } x \leq \beta_{n,m}, \\ \left(\frac{3\pi}{2}n\int_{\beta_{n,m}}^{x} \mu_{n,m}(u)du\right)^{2/3}, & \text{if } x > \beta_{n,m}. \end{cases} \tag{3.5}$$

We now look at the different terms in the asymptotic expression for $n\rho_{n,1}(x)$ above. Observe that (2.60) and (2.61) give the asymptotic expansion of Airy function for large negative values.

We need also the following asymptotic expansion of Airy function for large positive values (see (12), (13)): for $r > 0$

$$A(r) = \frac{1}{2\pi^{1/2} r^{1/4}} e^{-\frac{2}{3}r^{3/2}} \left(1 + O\left(\frac{1}{r^{3/2}}\right)\right) \tag{3.6}$$

and

$$A'(r) = \frac{r^{1/4}}{2\pi^{1/2}} e^{-\frac{2}{3}r^{3/2}} \left(1 + O\left(\frac{1}{r^{3/2}}\right)\right). \tag{3.7}$$

Hence it follows for any $x \in (-\infty, \infty)$

$$|A(x)A'(x)| = O(1) \tag{3.8}$$

and

$$\left(\frac{f_n'(x)}{4f_n(x)} - \frac{a'(x)}{a(x)}\right) = O(1). \tag{3.9}$$

Thus we have

$$\int_{t_n}^{\beta_{n,m}+\delta} \left(\frac{f_n'(x)}{4f_n(x)} - \frac{a'(x)}{a(x)}\right) 2A(f_n(x))A'(f_n(x))dx = O(1). \tag{3.10}$$

Also, integrating the third term of (3.4) only gives a contribution of order n^{-1}. The main contribution comes from the second term. In fact, a primitive function can be found for this expression:

$$\begin{aligned}
&\int_{t_n}^{\beta_{n,m}+\delta} f_n'(x)[A'(f_n(x))^2 - f_n(x)A(f_n(x))^2]dx \\
&= \int_{f_n(t_n)}^{f_n(\beta_{n,m}+\delta)} \left(A'(y)^2 - yA(y)^2\right) dy \\
&= -\left[\frac{2}{3}\left(y^2A'(y)^2 - yA(y)^2\right) - \frac{1}{3}A(y)A'(y)\right]_{f_n(t_n)}^{f_n(\beta_{n,m}+\delta)} \\
&= \frac{2}{3}\left(f_n(t_n)^2A(f_n(t_n))^2 - f_n(t_n)A'(f_n(t_n))^2\right) \\
&\quad -\frac{1}{3}A(f_n(t_n))A'(f_n(t_n)) + O(e^{-cn}),
\end{aligned} \tag{3.11}$$

where $c > 0$.

One can now use (2.60) and (2.61) to get the desired result. Indeed, $|f_n(t_n)| \geq C$ under the hypothesis, so we have

$$f_n(t_n)^2A(f_n(t_n))^2 = \frac{1}{\pi}|f_n(t_n)|^{3/2}\left\{\cos\left(\frac{2}{3}|f_n(t_n)|^{3/2} - \frac{\pi}{4}\right) + O\left(\frac{1}{|f_n(t_n)|^{3/2}}\right)\right\} \tag{3.12}$$

and

$$f_n(t_n)A'(f_n(t_n))^2 = \frac{1}{\pi}|f_n(t_n)|^{3/2}\left\{\sin\left(\frac{2}{3}|f_n(t_n)|^{3/2} - \frac{\pi}{4}\right) + O\left(\frac{1}{|f_n(t_n)|^{3/2}}\right)\right\}. \tag{3.13}$$

Thus it follows

$$
\begin{aligned}
&\frac{2}{3}\left(f_n(t_n)^2 A(f_n(t_n))^2 - f_n(t_n)A'(f_n(t_n))^2\right)\\
&= \frac{1}{\pi}|f_n(t_n)|^{3/2} + O(1)\\
&= n\int_{t_n}^{\beta_{n,m}} \mu_{n,m}(u)du + O(1)\\
&= \frac{\sqrt{\beta_{n,m}-\alpha_{n,m}}}{3\pi\beta_{n,m}} n(\beta_{n,m}-t_n)^{3/2}(1+o(1)),
\end{aligned}
\tag{3.14}
$$

where in the last equation we use the condition $\beta_{n,m} - t_n \to 0$. We now conclude the proof.

Lemma 2. *Assume t is defined as in Theorem 1. Let $t_n = t + x\frac{\sqrt{\log n}}{n}$ for $x \in (-\infty, \infty)$, $I_n = [t_n, \infty)$. Then*

$$
E\#I_n = n - k - \mu_{n,m}(t)x\sqrt{\log n} + O(1). \tag{3.15}
$$

Proof. We use the fact again

$$
E\#I_n = n\int_{t_n}^{\infty} \rho_{n,1}(x)dx. \tag{3.16}
$$

The above integral (3.16) can be written as

$$
\begin{aligned}
\int_{t_n}^{\infty} \rho_{n,1}(x)dx &= \int_{t_n}^{\beta_{n,m}-\delta} (\rho_{n,1}(x) - \mu_{n,m}(x))dx + \int_{t_n}^{\beta_{n,m}} \mu_{n,m}(x)dx\\
&\quad + \int_{\beta_{n,m}-\delta}^{\infty} \rho_{n,1}(x)dx - \int_{\beta_{n,m}-\delta}^{\beta_{n,m}} \mu_{n,m}(x)dx,
\end{aligned}
\tag{3.17}
$$

where $\delta > 0$ is a constant.

We now look at the integrals at the right hand side of (3.17). First, one easily sees from (2.56) that

$$
\int_{t_n}^{\beta_{n,m}-\delta} (\rho_{n,1}(x) - \mu_{n,m}(x))dx = O\left(\frac{1}{n}\right). \tag{3.18}
$$

Also, using the Taylor expansion for $\int_{t_n}^{\cdot} \mu_{n,m}(x)dx$ we have

$$
\begin{aligned}
\int_{t_n}^{\beta_{n,m}} \mu_{n,m}(x)dx &= 1 - \int_{\alpha_{n,m}}^{t_n} \mu_{n,m}(x)dx\\
&= 1 - \frac{k}{n} - \mu_{n,m}(t)(t_n - t) + O((t_n - t)^2).
\end{aligned}
\tag{3.19}
$$

Next we shall prove

$$
\int_{\beta_{n,m}-\delta}^{\infty} \rho_{n,1}(x)dx - \int_{\beta_{n,m}-\delta}^{\beta_{n,m}} \mu_{n,m}(x)dx = O\left(\frac{1}{n}\right). \tag{3.20}
$$

By the argument in Lemma 1 above, it suffices to show

$$\int_{\beta_{n,m}-\delta}^{\beta_{n,m}+\delta} f_n'(x)[A'(f_n(x))^2 - f_n(x)A(f_n(x))^2]dx - n\int_{\beta_{n,m}-\delta}^{\beta_{n,m}} \mu_{n,m}(x)dx = O(1). \tag{3.21}$$

In turn, by a primitive function argument used in (3.11), we need to prove

$$\begin{aligned}&\frac{2}{3}[f_n(\beta_{n,m}-\delta)^2 A(f_n(\beta_{n,m}-\delta))^2 - f_n(\beta_{n,m}-\delta)A'(f_n(\beta_{n,m}-\delta))^2] \\ &= n\int_{\beta_{n,m}-\delta}^{\beta_{n,m}} \mu_{n,m}(x)dx + O(1). \end{aligned}\tag{3.22}$$

Indeed, since $|f_n(\beta_{n,m}-\delta)| \geq cn^{2/3}(\beta_{n,m}-\delta) \to \infty$, then by (2.60) and (2.61) again we have

$$\begin{aligned}&\frac{2}{3}\left[f_n(\beta_{n,m}-\delta)^2 A(f_n(\beta_{n,m}-\delta))^2 - f_n(\beta_{n,m}-\delta)A'(f_n(\beta_{n,m}-\delta))^2\right] \\ &= \frac{2}{3\pi}\left|f_n(\beta_{n,m}-\delta)\right|^{3/2}\left[1 + O\left(\frac{1}{|f_n(\beta_{n,m}-\delta)|^{3/2}}\right)\right] \\ &= n\int_{\beta_{n,m}-\delta}^{\beta_{n,m}} \mu_{n,m}(x)dx + O(1). \end{aligned}\tag{3.23}$$

Now combining (3.17)-(3.20) yields the desired result (3.15), so the proof of Lemma 2 is complete.

We now turn to the variance of the number of eigenvalues in I_n. The computation is based on a fact due to (7) that

$$Var(\#I_n) = \int_{I_n}\int_{I_n^c} K_n(x,y)^2 dxdy. \tag{3.24}$$

Observe that by (2.1)

$$K_n(x,y) = \frac{i}{2\pi}\frac{1}{x-y}\det\begin{pmatrix} U_{11}(x) & U_{11}(y) \\ U_{21}(x) & U_{21}(y)\end{pmatrix} e^{-\frac{nV_{n,m}(x)}{2}}e^{-\frac{nV_{n,m}(y)}{2}}. \tag{3.25}$$

We will compute the above determinant in two basic cases separately. The first case is when $t \leq \beta_{n,m} - \delta$ for some $\delta > 0$, i.e., in the bulk. The second case is when $\beta_{n,m} - t \to 0$ as $n \to \infty$, i.e., near the right spectrum edge.

Lemma 3. *Assume $t \leq \beta_{n,m} - \delta$ for some $\delta > 0$. Let $I_n = [t, \infty)$, then*

$$Var(\#I_n) = \frac{1}{2\pi^2}\log n(1 + o(1)). \tag{3.26}$$

Proof. First consider the sub-domain where both variables are in the bulk:

$$\Gamma = \{(x,y) : t \leq x \leq \beta_{n,m} - \delta, \alpha_{n,m} + \delta \leq y \leq t\}.$$

For $z \in (\alpha_{n,m}+\delta, \beta_{n,m}-\delta)$, define $\theta(z)$ to be such that

$$\cos\theta(z) = \frac{z-(\beta_{n,m}+\alpha_{n,m})/2}{(\beta_{n,m}-\alpha_{n,m})/2}. \tag{3.27}$$

Then a simple computation shows

$$a(z)+a^{-1}(z) = \frac{(\beta_{n,m}-\alpha_{n,m})^{1/2}}{(z-\beta_{n,m})^{1/4}(z-\alpha_{n,m})^{1/4}} e^{\frac{i}{2}\theta(z)} \tag{3.28}$$

and

$$a(z)-a^{-1}(z) = -\frac{(\beta_{n,m}-\alpha_{n,m})^{1/2}}{(z-\beta_{n,m})^{1/4}(z-\alpha_{n,m})^{1/4}} e^{-\frac{i}{2}\theta(z)}. \tag{3.29}$$

So it follows from (2.14) that

$$P_\infty(z) = \frac{(\beta_{n,m}-\alpha_{n,m})^{1/2}}{2(z-\beta_{n,m})^{1/4}(z-\alpha_{n,m})^{1/4}} \begin{pmatrix} e^{\frac{i}{2}\theta(z)} & ie^{-\frac{i}{2}\theta(z)} \\ -ie^{-\frac{i}{2}\theta(z)} & e^{\frac{i}{2}\theta(z)} \end{pmatrix}. \tag{3.30}$$

Define

$$\alpha(z) = \frac{1}{2}\theta(z) - in\xi_n(z), \quad \beta(z) = -\frac{1}{2}\theta(z) - in\xi_n(z). \tag{3.31}$$

Then inserting (3.30) and (3.31) into (2.35) yields

$$\begin{aligned} & \begin{pmatrix} U_{11}(z) \\ U_{21}(z) \end{pmatrix} \\ = \; & e^{\frac{n}{2}l_{n,m}\sigma_3} R(z) P_\infty(z) \begin{pmatrix} e^{n\xi_n(z)} \\ e^{-n\xi_n(z)} \end{pmatrix} e^{\frac{nV_{n,m}(z)}{2}} \\ = \; & \frac{(\beta_{n,m}-\alpha_{n,m})^{1/2}}{2(z-\beta_{n,m})^{1/4}(z-\alpha_{n,m})^{1/4}} e^{\frac{n}{2}l_{n,m}\sigma_3} R(z) \begin{pmatrix} e^{i\alpha(z)} + ie^{-i\alpha(z)} \\ -ie^{\beta(z)} + e^{-i\beta(z)} \end{pmatrix} e^{\frac{nV_{n,m}(z)}{2}} \\ = \; & \frac{(\beta_{n,m}-\alpha_{n,m})^{1/2}}{2(z-\beta_{n,m})^{1/4}(z-\alpha_{n,m})^{1/4}} e^{\frac{n}{2}l_{n,m}\sigma_3} R(z) \begin{pmatrix} (1-i)(\cos\alpha(z)+\sin\alpha(z)) \\ (1+i)(\cos\beta(z)+\sin\beta(z)) \end{pmatrix} e^{\frac{nV_{n,m}(z)}{2}}. \end{aligned} \tag{3.32}$$

Thus using (2.29) and (3.32) we have for $(x,y) \in \Gamma$,

$$\begin{aligned} & \det \begin{pmatrix} U_{11}(x) & U_{11}(y) \\ U_{21}(x) & U_{21}(y) \end{pmatrix} \\ & = \frac{(\beta_{n,m}-\alpha_{n,m})}{4(x-\beta_{n,m})^{1/4}(x-\alpha_{n,m})^{1/4}(y-\beta_{n,m})^{1/4}(y-\alpha_{n,m})^{1/4}} e^{\frac{nV_{n,m}(x)}{2}} e^{\frac{nV_{n,m}(y)}{2}} \\ & \quad \times \det \begin{pmatrix} (1-i)(\cos\alpha(x)+\sin\alpha(x)) & (1-i)(\cos\alpha(y)+\sin\alpha(y)) \\ (1+i)(\cos\beta(x)+\sin\beta(x)) & (1+i)(\cos\beta(y)+\sin\beta(y)) \end{pmatrix} + O\left(\frac{1}{n}\right) \\ & = \frac{\beta_{n,m}-\alpha_{n,m}}{2(x-\beta_{n,m})^{1/4}(x-\alpha_{n,m})^{1/4}(y-\beta_{n,m})^{1/4}(y-\alpha_{n,m})^{1/4}} e^{\frac{nV_{n,m}(x)}{2}} e^{\frac{nV_{n,m}(y)}{2}} \\ & \quad \times [(\cos\alpha(x)+\sin\alpha(x))(\cos\beta(y)+\sin\beta(y)) \\ & \qquad -(\cos\alpha(y)+\sin\alpha(y))(\cos\beta(x)+\sin\beta(x))] + O\left(\frac{1}{n}\right). \end{aligned} \tag{3.33}$$

Noting that

$$\cos\alpha(z)+\sin\alpha(z)=2\cos\left(\alpha(z)-\frac{\pi}{4}\right)\cos\frac{\pi}{4}$$

and

$$\cos\beta(z)+\sin\beta(z)=2\cos\left(\beta(z)-\frac{\pi}{4}\right)\cos\frac{\pi}{4},$$

we derive from (3.25) and (3.33) that

$$\begin{aligned}&K_n(x,y)\\&=\frac{\beta_{n,m}-\alpha_{n,m}}{2\pi(\beta_{n,m}-x)^{1/4}(x-\alpha_{n,m})^{1/4}(\beta_{n,m}-y)^{1/4}(y-\alpha_{n,m})^{1/4}}\\&\times\frac{[\cos(\alpha(x)-\frac{\pi}{4})\cos(\beta(y)-\frac{\pi}{4})-\cos(\alpha(y)-\frac{\pi}{4})\cos(\beta(x)-\frac{\pi}{4})]+O(\frac{1}{n})}{(x-y)}.\end{aligned}\tag{3.34}$$

To prepare for integration we now divide Γ into four disjoint sub-domains. Let

$$\Gamma_0=\left\{(x,y):t\le x\le t+\frac{1}{n},t-\frac{1}{n}\le y\le t\right\},$$

$$\Gamma_1^1=\left\{(x,y):t\le x\le t+\frac{\beta_{n,m}-t}{r(n)},t-\frac{t+\beta_{n,m}}{r(n)}\le y\le t-\frac{1}{n}\right\},$$

$$\Gamma_1^2=\left\{(x,y):t+\frac{1}{n}\le x\le t+\frac{\beta_{n,m}-t}{r(n)},t-\frac{1}{n}\le y\le t\right\},$$

where $r(n)=\log n$, and let

$$\Gamma_2=\Gamma-\Gamma_0\cup\Gamma_1^1\cup\Gamma_1^2.$$

Let us first calculate the integral over Γ_0. When $(x,y)\in\Gamma_0$, one can use the Taylor expansion for the function $\theta(\cdot)$ at t,

$$\theta(x)=\theta(t)+O\left(\frac{1}{n}\right),\qquad\theta(y)=\theta(t)+O\left(\frac{1}{n}\right).\tag{3.35}$$

From the obvious fact $\beta(z)=\alpha(z)-\theta(z)$ it follows

$$\begin{aligned}&\cos\left(\alpha(x)-\frac{\pi}{4}\right)\cos\left(\alpha(y)-\frac{\pi}{4}-\theta(y)\right)-\cos\left(\alpha(y)-\frac{\pi}{4}\right)\cos\left(\alpha(x)-\frac{\pi}{4}-\theta(x)\right)\\&=\cos\left(\alpha(x)-\frac{\pi}{4}\right)\cos\left(\alpha(y)-\frac{\pi}{4}-\theta(t)\right)-\cos\left(\alpha(y)-\frac{\pi}{4}\right)\cos\left(\alpha(x)-\frac{\pi}{4}-\theta(t)\right)\\&\quad+O\left(\frac{1}{n}\right)\\&=\sin\theta(t)\sin(\alpha(x)-\alpha(y))+O\left(\frac{1}{n}\right).\end{aligned}\tag{3.36}$$

Since both x and y are close to t, and

$$\sin\theta(t)=\frac{(\beta_{n,m}-t)^{1/2}(t-\alpha_{n,m})^{1/2}}{(\beta_{n,m}-\alpha_{n,m})/2},\tag{3.37}$$

then

$$\frac{\beta_{n,m}-\alpha_{n,m}}{2(\beta_{n,m}-x)^{1/4}(x-\alpha_{n,m})^{1/4}(\beta_{n,m}-y)^{1/4}(y-\alpha_{n,m})^{1/4}(x-y)}=1+O\left(\frac{1}{n}\right). \tag{3.38}$$

Also, let for $z\in(\alpha_{n,m},\beta_{n,m})$

$$F(z)=\int_z^{\beta_{n,m}}\frac{\sqrt{(u-\alpha_{n,m})(\beta_{n,m}-u)}}{u}du, \tag{3.39}$$

then it follows from (2.7), (2.8) and (3.31)

$$\begin{aligned}\alpha(x)-\alpha(y) &= -in(\xi_n(x)-\xi_n(y))+O\left(\frac{1}{n}\right)\\ &= \frac{n}{2}(F(x)-F(y))+O\left(\frac{1}{n}\right).\end{aligned} \tag{3.40}$$

Thus we have by combining (3.34)-(3.40) together,

$$K_n(x,y)=\frac{\frac{1}{\pi}\sin[\frac{n}{2}(F(x)-F(y))]+O(\frac{1}{n})}{x-y},\quad (x,y)\in\Gamma_0, \tag{3.41}$$

which in turn implies

$$K_n(x,y)\le Cn,\qquad (x,y)\in\Gamma_0. \tag{3.42}$$

Hence

$$\int_{\Gamma_0}K_n(x,y)^2dxdy=O(1). \tag{3.43}$$

A similar argument to (3.41) gives

$$K_n(x,y)=\frac{\frac{1}{\pi}\sin[\frac{n}{2}(F(x)-F(y))]+O(\frac{1}{r(n)})}{x-y},\quad (x,y)\in\Gamma_1^1. \tag{3.44}$$

In order to calculate the integral of $K_n(x,y)^2$ over Γ_1^1, we shall use an elementary fact

$$\int_a^b dx\int_c^d\frac{1}{(x-y)^2}dy=\ln\frac{(b-d)(a-c)}{(a-d)(b-c)}. \tag{3.45}$$

It immediately follows from (3.45) that

$$\begin{aligned}\int_{\Gamma_1^1}\frac{1}{(x-y)^2}dxdy &= \log\frac{n(\beta_{n,m}^2-t^2)}{2r(n)}\\ &= \log n+O(\ln r(n)).\end{aligned} \tag{3.46}$$

Also, one easily sees

$$
\begin{aligned}
&\int_{\Gamma_1^1} \frac{\cos[n(F(y)-F(x))]}{(x-y)^2} dxdy \\
&= \int_t^{t+\frac{\beta_{n,m}-t}{r(n)}} dx \left[\frac{\sin[n(F(y)-F(x))]}{nF'(y)(x-y)^2} \right]_{t-(\beta_{n,m}+1)/r(n)}^{t-1/n} \\
&\quad - \int_t^{t+\frac{\beta_{n,m}-t}{r(n)}} dx \int_{t-\frac{t+\beta_{n,m}}{r(n)}}^{t-\frac{1}{n}} \sin[n(F(y)-F(x))] \left(\frac{1}{nF'(y)(x-y)^2} \right)_y' dy \\
&=: I_1 - I_2. \qquad (3.47)
\end{aligned}
$$

Both the integrals are easy to estimate:

$$
\begin{aligned}
|I_1| &\le \frac{C}{n} \int_t^{t+\frac{\beta_{n,m}-t}{r(n)}} \left[\frac{1}{(x-(t-\frac{1}{n}))^2} + \frac{1}{(x-(t-\frac{t+\beta_{n,m}}{r(n)}))^2} \right] dx \\
&= O(1). \qquad (3.48)
\end{aligned}
$$

Note that

$$
\left(\frac{1}{F'(y)(x-y)^2} \right)_y' = -\frac{F''(y)}{F'(y)^2} \cdot \frac{1}{(x-y)^2} + \frac{2}{F'(y)(x-y)^3},
$$

and both $F'(y)$ and $F''(y)$ are bounded in $(\alpha_{n,m}+\delta, \beta_{n,m}-\delta)$. We easily have

$$
|I_2| \le \frac{C}{n} \int_{\Gamma_1^1} \frac{1}{(x-y)^3} dxdy = O(1). \qquad (3.49)
$$

Combining (3.48) and (3.49) yields

$$
\begin{aligned}
\int_{\Gamma_1^1} K_n(x,y)^2 dxdy &= \int_{\Gamma_1^1} \frac{\sin^2[n(F(y)-F(x))] + O(\frac{1}{r(n)})}{\pi^2 (x-y)^2} dxdy \\
&= \frac{1}{2\pi^2} \int_{\Gamma_1^1} \frac{1-\cos[n(F(y)-F(x))]}{(x-y)^2} dxdy + O(1) \\
&= \frac{1}{2\pi^2} \log n - \frac{1}{2\pi^2} \int_{\Gamma_1^1} \frac{\cos[n(F(y)-F(x))]}{(x-y)^2} dxdy + O(1) \\
&= \frac{1}{2\pi^2} \log n + O(1). \qquad (3.50)
\end{aligned}
$$

Also, it is easy to see

$$
K_n(x,y) = O\left(\frac{1}{|x-y|} \right), \quad (x,y) \in \Gamma,
$$

then using (3.45) again, we have

$$
\begin{aligned}
\int_{\Gamma_1^2} K_n(x,y)^2 dxdy &\le C \int_{\Gamma_1^2} \frac{1}{(x-y)^2} dxdy \\
&= \ln \frac{2n(\beta_{n,m}-t)}{n(\beta_{n,m}-t)+r(n)} = O(1). \qquad (3.51)
\end{aligned}
$$

Similarly, it follows

$$\begin{aligned}\int_{\Gamma_2} K_n(x,y)^2 dxdy &\leq C\int_{\Gamma_2} \frac{1}{(x-y)^2}dxdy \\ &= O(\ln(r(n))). \end{aligned} \tag{3.52}$$

Finally, we need to look at the sub-domain $\Gamma_3 =: I_n \times I_n^c - \Gamma$. We divide Γ_3 into the following four sub-domians:

$$\Gamma_3^1 = \{(x,y) : \alpha_{n,m} - \delta \leq x \leq \alpha_{n,m} + \delta,\ \beta_{n,m} - \delta \leq x \leq \beta_{n,m} + \delta\}$$

$$\Gamma_3^2 = \{(x,y) : \alpha_{n,m} - \delta \leq x \leq \alpha_{n,m} + \delta,\ x \geq \beta_{n,m} + \delta\}$$

$$\Gamma_3^3 = \{(x,y) : x \geq \alpha_{n,m} + \delta,\ \beta_{n,m} - \delta \leq x \leq \beta_{n,m} + \delta\}$$

$$\Gamma_3^4 = \{(x,y) : x \geq \alpha_{n,m} + \delta,\ x \geq \beta_{n,m} + \delta\}$$

The asymptotic expression for $K_n(x,y)$ is different in Γ_3^i but there are no difficulties. One can just take the absolute values in the integral since $|x-y| \geq \beta_{n,m} - \alpha_{n,m}$ and the result

$$\int_{\Gamma_3} K_n(x,y)^2 dxdy = O(1). \tag{3.53}$$

This together with (3.43), (3.50), (3.51) and (3.52) now concludes the proof.

Lemma 4. *Assume t is such that $\beta_{n,m} - t \to 0$ and $n(\beta_{n,m} - t)^{3/2} \geq C$ for some $C > 0$. Let $I_n = [t, \infty)$, then*

$$Var(\#I_n) = \frac{1}{2\pi^2}\log\left[n(\beta_{n,m} - t)^{3/2}\right](1 + o(1)). \tag{3.54}$$

Proof. The argument is similar to that of Lemma 3 but requires a finer partition. Now we divide $I_n \times I_n^c$ into the following nine sub-domains:

$$\Omega_0 = \{(x,y) : t \leq x \leq t + \varepsilon,\ t - \varepsilon \leq y \leq t\},$$

$$\Omega_1 = \left\{(x,y) : t \leq x \leq t + \frac{\beta_{n,m} - t}{r(n)},\ t - \frac{\beta_{n,m} - t}{r(n)} \leq y \leq t - \varepsilon\right\},$$

$$\Omega_2 = \left\{(x,y) : t + \epsilon \leq x \leq t + \frac{\beta_{n,m} - t}{r(n)},\ t - \epsilon \leq y \leq t\right\},$$

$$\Omega_3 = \left\{(x,y) : t \leq x \leq \beta_{n,m} - \frac{C}{n},\ \beta_{n,m} - \delta \leq y \leq t - \frac{\beta_{n,m} - t}{r(n)}\right\},$$

$$\Omega_4 = \left\{(x,y) : t + \frac{\beta_{n,m} - t}{r(n)} \leq x \leq \beta_{n,m} - \frac{C}{n},\ t - \frac{\beta_{n,m} - t}{r(n)} \leq y \leq t\right\},$$

$$\Omega_5 = \left\{(x,y) : \beta_{n,m} - \frac{C}{n} \leq x \leq \beta_{n,m} + \frac{C}{n},\ \beta_{n,m} - \delta \leq y \leq t\right\}$$

$$\Omega_6 = \left\{(x,y) : \beta_{n,m} + \frac{C}{n} \le x \le \beta_{n,m} + \delta,\ \beta_{n,m} - \delta \le y \le t\right\},$$
$$\Omega_7 = \{(x,y) : t \le x \le \beta_{n,m} + \delta,\ \alpha_{n,m} - \delta \le y \le \beta_{n,m} - \delta\},$$
$$\Omega_8 = \{(x,y) : x \ge \beta_{n,m} + \delta,\ \text{ or } y \le \alpha_{n,m} - \delta\},$$

where $\delta > 0$, $C > 0$, $r(n)$ and ε are defined by

$$\frac{1}{r(n)} = \max\left(\sqrt{\beta_{n,m} - t}\,,\ \frac{1}{\log\left[n(\beta_{n,m} - t)^{3/2}\right]}\right)$$

and

$$\varepsilon = \frac{1}{n(\beta_{n,m} - t)^{1/2}}.$$

In the corner Ω_0 we shall use the representation (1.2). By use of the Cauchy-Schwarz inequality we have

$$K_n^2(x,y) \le K_n(x,x)K_n(y,y).$$

Having separated the variables one can now use the calculations of the expected value giving

$$\int_{t-\varepsilon}^{t} \int_{t}^{t+\varepsilon} K_n^2(x,y)dxdy = O(1). \tag{3.55}$$

Now let us look at the sub-domain Ω_1. We shall prove the following estimate for $K_n(x,y)$:

$$\begin{aligned} K_n(x,y) &= \frac{1}{\pi(x-y)}\{\sin\left[\frac{n}{2}(F(x) - F(y))\right] \\ &\qquad + O\left(\frac{1}{n(\beta_{n,m} - x)^{3/2}}\right) + O\left(\frac{1}{n(\beta_{n,m} - y)^{3/2}}\right)\}. \end{aligned} \tag{3.56}$$

To see this, we use for $z \in (\beta_{n,m} - \delta, \beta_{n,m} + \delta)$

$$\begin{aligned} \begin{pmatrix} U_{11}(z) \\ U_{21}(z) \end{pmatrix} &= \frac{1}{\sqrt{2}} e^{\frac{1}{2}nl_{n,m}\sigma_3} R(z) P_\infty(z) e^{\frac{\pi i}{4}\sigma_3} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} f_n(z)^{\frac{\sigma_3}{4}} \Psi(f_n(z)) \\ &\quad \times e^{-n\xi_n(z)\sigma_3} \begin{pmatrix} 1 & 0 \\ e^{-2n\xi_n(z)} & 1 \end{pmatrix} \begin{pmatrix} e^{n(g_n(z) - \frac{1}{2}l_{n,m})} \\ 0 \end{pmatrix}. \end{aligned} \tag{3.57}$$

Some algebraic calculations show

$$\begin{aligned} &\begin{pmatrix} U_{11}(z) \\ U_{21}(z) \end{pmatrix} \\ &= \sqrt{\pi} e^{\frac{1}{2}nl_{n,m}\sigma_3} R(z) \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} -a(x) f_n(x)^{-\frac{1}{4}} A'(f_n(x)) \\ a^{-1}(x) f_n(x)^{\frac{1}{4}} A(f_n(x)) \end{pmatrix} e^{\frac{nV_{n,m}(z)}{2}}. \end{aligned} \tag{3.58}$$

By virtue of (2.29), one easily obtains for $x, y \in (\beta_{n,m} - \delta, \beta_{n,m} + \delta)$

$$
\begin{aligned}
&K_n(x,y) \\
&= \frac{i}{2\pi(x-y)} \det \begin{pmatrix} U_{11}(x) & U_{11}(y) \\ U_{21}(x) & U_{21}(y) \end{pmatrix} e^{-\frac{nV_{n,m}(x)}{2}} e^{-\frac{nV_{n,m}(y)}{2}} \\
&= \frac{1}{x-y} \left\{ \det \begin{pmatrix} -a(x) f_n(x)^{-\frac{1}{4}} A'(f_n(x)) & -a(y) f_n(y)^{-\frac{1}{4}} A'(f_n(y)) \\ a^{-1}(x) f_n(x)^{\frac{1}{4}} A(f_n(x)) & a^{-1}(y) f_n(y)^{\frac{1}{4}} A(f_n(y)) \end{pmatrix} + O(\frac{1}{n}) \right\} \\
&= \frac{1}{x-y} \{ (\frac{a(y)}{a(x)} f_n(x)^{\frac{1}{4}} A(f_n(x)) f_n(y)^{-\frac{1}{4}} A'(f_n(y)) \\
&\qquad - \frac{a(x)}{a(y)} f_n(y)^{\frac{1}{4}} A(f_n(y)) f_n(x)^{-\frac{1}{4}} A'(f_n(x))) + O(\frac{1}{n}) \}. \qquad (3.59)
\end{aligned}
$$

Now assume $(x, y) \in \Omega_1$. Using (2.60) and (2.61) for large positive r, we have

$$
\begin{aligned}
&f_n(x)^{\frac{1}{4}} A(f_n(x)) f_n(y)^{-\frac{1}{4}} A'(f_n(y)) - f_n(y)^{\frac{1}{4}} A(f_n(y)) f_n(x)^{-\frac{1}{4}} A'(f_n(x)) \\
&= \frac{1}{\pi} \sin\left[\frac{n}{2}(F(x) - F(y))\right] + O\left(\frac{1}{n(\beta_{n,m} - x)^{3/2}}\right) + O\left(\frac{1}{n(\beta_{n,m} - y)^{3/2}}\right) \qquad (3.60)
\end{aligned}
$$

Moreover, $\frac{a(x)}{a(y)}$ is asymptotically equal to 1. Indeed,

$$
\begin{aligned}
&\left[\frac{a(x)}{a(y)}\right]^2 - 1 \\
&= \frac{(x-\beta_{n,m})^{\frac{1}{2}}(y-\alpha_{n,m})^{\frac{1}{2}} - (y-\beta_{n,m})^{\frac{1}{2}}(x-\alpha_{n,m})^{\frac{1}{2}}}{(y-\beta_{n,m})^{\frac{1}{2}}(x-\alpha_{n,m})^{\frac{1}{2}}} \\
&= \frac{(x-\alpha_{n,m})^{\frac{1}{2}}[(x-\alpha_{n,m})^{\frac{1}{2}} - (y-\beta_{n,m})^{\frac{1}{2}}]}{(y-\beta_{n,m})^{\frac{1}{2}}(x-\alpha_{n,m})^{\frac{1}{2}}} + \frac{(x-\beta_{n,m})^{\frac{1}{2}}[(y-\alpha_{n,m})^{\frac{1}{2}} - (x-\alpha_{n,m})^{\frac{1}{2}}]}{(y-\beta_{n,m})^{\frac{1}{2}}(x-\alpha_{n,m})^{\frac{1}{2}}} \\
&= \frac{(x-\beta_{n,m})^{\frac{1}{2}} - (y-\beta_{n,m})^{\frac{1}{2}}}{(y-\beta_{n,m})^{\frac{1}{2}}} + \frac{(x-\beta_{n,m})^{\frac{1}{2}}[(y-\alpha_{n,m})^{\frac{1}{2}} - (x-\alpha_{n,m})^{\frac{1}{2}}]}{(y-\beta_{n,m})^{\frac{1}{2}}(x-\alpha_{n,m})^{\frac{1}{2}}} \\
&= \frac{x-y}{(y-\beta_{n,m})^{\frac{1}{2}}((x-\beta_{n,m})^{\frac{1}{2}} + (y-\beta_{n,m})^{\frac{1}{2}})} \\
&\quad + \frac{(x-\beta_{n,m})^{\frac{1}{2}}(y-x)}{(y-\beta_{n,m})^{\frac{1}{2}}(x-\alpha_{n,m})^{\frac{1}{2}}((y-\alpha_{n,m})^{\frac{1}{2}} + (x-\alpha_{n,m})^{\frac{1}{2}})} \\
&= O\left(\frac{1}{r(n)}\right), \qquad (3.61)
\end{aligned}
$$

which implies

$$\frac{a(x)}{a(y)} = 1 + O\left(\frac{1}{r(n)}\right). \qquad (3.62)$$

Similarly, we have

$$\frac{a(y)}{a(x)} = 1 + O\left(\frac{1}{r(n)}\right). \qquad (3.63)$$

Inserting (3.60), (3.62) and (3.63) into (3.59) immediately yield (3.56).

The following computation is completely similar to that of (3.46) and (3.47) except that $F'(y)$ and $F''(y)$ are no longer bounded.

By (3.45) again,

$$\begin{aligned}\int_{\Omega_1} \frac{1}{(x-y)^2} dxdy &= \log \frac{\frac{\beta_{n,m}-t}{r(n)} + \varepsilon}{\varepsilon} \\ &= \log \left[n(\beta_{n,m} - t)^{3/2} \right] + O(\ln r(n)). \end{aligned} \tag{3.64}$$

Also, it holds

$$\begin{aligned}&\int_{\Omega_1} \frac{\cos[n(F(y) - F(x))]}{(x-y)^2} dxdy \\ &= \int_t^{t+\frac{\beta_{n,m}-t}{r(n)}} \left[\frac{\sin[n(F(y) - F(x))]}{nF'(y)(x-y)^2} \right]_{t-(t+\beta_{n,m})/r(n)}^{t-1/n} dx \\ &\quad - \int_t^{t+\frac{\beta_{n,m}-t}{r(n)}} dx \int_{t-\frac{t+\beta_{n,m}}{r(n)}}^{t-\frac{1}{n}} \sin[n(F(y) - F(x))] \left(\frac{1}{nF'(y)(x-y)^2} \right)'_y dy \\ &=: I_1 - I_2. \end{aligned} \tag{3.65}$$

Both the integrals are easy to estimate: we take derivatives twice at both sides of (3.39) to get

$$F'(y) = -\frac{\sqrt{(y - \alpha_{n,m})(\beta_{n,m} - y)}}{y}$$

and

$$F''(y) = -\left(\frac{\sqrt{y - \alpha_{n,m}}}{y} \right)'_y \sqrt{\beta_{n,m} - y} + \frac{\sqrt{y - \alpha_{n,m}}}{2y\sqrt{\beta_{n,m} - y}}.$$

Thus we have

$$\begin{aligned}|I_1| &\le \frac{C}{n\sqrt{\beta_{n,m} - t}} \int_t^{t+\frac{\beta_{n,m}-t}{r(n)}} \left[\frac{1}{(x - (t - \frac{1}{n}))^2} + \frac{1}{(x - (t - \frac{t+\beta_{n,m}}{r(n)}))^2} \right] dx \\ &= O(1). \end{aligned} \tag{3.66}$$

Also,

$$\left(\frac{1}{F'(y)(x-y)^2} \right)'_y = -\frac{F''(y)}{F'(y)^2} \cdot \frac{1}{(x-y)^2} + \frac{2}{F'(y)(x-y)^3}$$

which gives

$$\begin{aligned}|I_2| &\le \frac{C}{n} \int_{\Omega_1} \left(\frac{1}{(\beta_{n,m} - y)^{3/2}(x-y)^2} + \frac{1}{(\beta_{n,m} - y)^{1/2}(x-y)^3} \right) dxdy \\ &= O(1). \end{aligned} \tag{3.67}$$

Combining (3.64)-(3.67), we have

$$\begin{aligned}\int_{\Omega_1} K_n(x,y)^2 dxdy &= \int_{\Omega_1} \frac{\sin^2[\frac{2}{n}(F(y)-F(x))]+O(\frac{1}{r(n)})}{\pi^2(x-y)^2}dxdy + \int_{\Omega_1}\frac{O(1)}{(x-y)^2}dxdy \\ &= \frac{1}{2\pi^2}\int_{\Omega_1}\frac{1-\cos[n(F(y)-F(x))]}{(x-y)^2}dxdy + O(\log(r(n))) \\ &= \frac{1}{2\pi^2}\log\left[n(\beta_{n,m}-t)^{3/2}\right] + O(\log(r(n))). \end{aligned} \tag{3.68}$$

The calculations made above can also be applied to the small slice Ω_2:

$$\int_{\Omega_2} K_n(x,y)^2 dxdy = O(\log(r(n))). \tag{3.69}$$

Next let us look at the integral in $\Omega_3 \cup \Omega_4 \cup \Omega_5$. We shall prove in these sub-domains

$$\left|\frac{a(y)}{a(x)} f_n(x)^{\frac{1}{4}} A(f_n(x)) f_n(y)^{-\frac{1}{4}} A'(f_n(y))\right| \le C\frac{|\beta_{n,m}-y|^{1/4}}{|\beta_{n,m}-x|^{1/4}} \tag{3.70}$$

and

$$\left|\frac{a(x)}{a(y)} f_n(y)^{\frac{1}{4}} A(f_n(y)) f_n(x)^{-\frac{1}{4}} A'(f_n(x))\right| \le C\frac{|\beta_{n,m}-y|^{1/4}}{|\beta_{n,m}-x|^{1/4}}. \tag{3.71}$$

For $\beta_{n,m}-\delta \le y < t$, a simple calculation shows

$$cn^{2/3}(\beta_{n,m}-y) \le |f_n(y)| \le Cn^{2/3}(\beta_{n,m}-y). \tag{3.72}$$

So, by virtue of (2.60) and (2.61) and $n^{2/3}(\beta_{n,m}-y) \ge n^{2/3}(\beta_{n,m}-t) \to \infty$, we have

$$\left|f_n(y)^{\frac{1}{4}} A(f_n(y))\right| \le C, \quad \left|f_n(y)^{-\frac{1}{4}} A'(f_n(y))\right| \le C. \tag{3.73}$$

For $t \le x < \beta_{n,m}$, in the exactly same way, we have

$$cn^{2/3}(\beta_{n,m}-x) \le |f_n(x)| \le Cn^{2/3}(\beta_{n,m}-x), \tag{3.74}$$

which together with (3.73) implies

$$\begin{aligned}\left|\frac{a(y)}{a(x)} f_n(x)^{\frac{1}{4}} A(f_n(x)) f_n(y)^{-\frac{1}{4}} A'(f_n(y))\right| &\le C\left|\frac{a(y)}{a(x)}\right| \\ &\le C\frac{|\beta_{n,m}-y|^{1/4}}{|\beta_{n,m}-x|^{1/4}}\end{aligned} \tag{3.75}$$

i.e., (3.70) is true for $t \le x < \beta_{n,m}$ and $\beta_{n,m}-\delta \le y < t$.

To prove (3.71), we need only to consider the case $f_n(x) \to 0$ since the other case is simpler. When $f_n(x) \to 0$, $|A'(f_n(x))| \le C$ which easily follows from $A'(0) = -\frac{1}{3^{1/3}\Gamma(1/3)}$. Thus we have by (3.73) and (3.74)

$$\begin{aligned}\left|\frac{a(x)}{a(y)} f_n(y)^{\frac{1}{4}} A(f_n(y)) f_n(x)^{-\frac{1}{4}} A'(f_n(x))\right| &\le C\left|\frac{a(x)}{a(y)} f_n(x)^{-\frac{1}{4}}\right| \\ &= \frac{C}{n^{1/6}(\beta_{n,m}-y)^{1/4}}.\end{aligned} \tag{3.76}$$

Since $y \le t \le x$ and $n^{2/3}(\beta_{n,m} - t) \to \infty$, then

$$\begin{aligned} |\beta_{n,m} - y|^{1/2} n^{1/6} &\ge |\beta_{n,m} - t|^{1/2} n^{1/6} \\ &\ge |\beta_{n,m} - t|^{1/4} \\ &\ge |\beta_{n,m} - x|^{1/4}, \end{aligned} \tag{3.77}$$

which yields (3.71).

So far, we have proved (3.70) and (3.71) for $t \le x \le \beta_{n,m}$ and $\beta_{n,m} - \delta \le y \le t$. The case $\beta_{n,m} \le x \le \beta_{n,m} + \frac{C}{n}$ can be similarly proved using (3.6) and (3.7) and noting that

$$f_n(x) = \left(\frac{3}{2} \int_{\beta_{n,m}}^{x} \frac{\sqrt{(u - \alpha_{n,m})(u - \beta_{n,m})}}{u} du \right)^{2/3}, \quad x > \beta_{n,m}.$$

From (3.70) and (3.71) it follows

$$K_n(x,y) = O\left(\left(\frac{\beta_{n,m} - y}{\beta_{n,m} - x} \right)^{1/2} \frac{1}{(x-y)^2} \right) \quad \text{in } \Omega_3 \cup \Omega_4 \cup \Omega_5. \tag{3.78}$$

It is now sufficient to calculate the integrals of $K_n(x,y)^2$ in $\Omega_i, i = 3,4,5$ are $O(\log(r(n)))$. The calculation is straightforward and almost same as in the case of GUE (see (7)), so some details will be skipped.

In the sub-domain Ω_6 one can perform the same calculations as in

$$\left\{ (x,y) : t \le x \le \beta_{n,m} + \frac{C}{n}, \beta_{n,m} - \delta \le y \le t \right\}$$

and the contribution is $O(1)$.

In Ω_7 one can use the fact $x - y \ge \delta$ to show that the contribution form this domain is $O(1)$. In Ω_8 one can easily get $K_n(x,y)$ is exponentially small in n and exponentially decaying in x (or y). Thus the contribution from this domain is $o(1)$. The proof of Lemma 4 is complete.

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