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Asymptotic Estimates Of The Green Functions And Transition Probabilities For Markov Additive Processes*

Kôhei UCHIYAMA
Department of Mathematics
Tokyo Institute of Technology
Oh-okayama, Meguro Tokyo 152-8551
e-mail: uchiyama@math.titech.ac.jp

Abstract

In this paper we shall derive asymptotic expansions of the Green function and the transition probabilities of Markov additive (MA) processes (ξ_n, S_n) whose first component satisfies Doeblin's condition and the second one takes valued in $\mathbf{Z}^d$. The derivation is based on a certain perturbation argument that has been used in previous works in the same context. In our asymptotic expansions, however, not only the principal term but also the second order term are expressed explicitly in terms of a few basic functions that are characteristics of the expansion. The second order term will be important for instance in computation of the harmonic measures of a half space for certain models. We introduce a certain aperiodicity condition, named Condition (AP), that seems a minimal one under which the Fourier analysis can be applied straightforwardly. In the case when Condition (AP) is violated the structure of MA processes will be clarified and it will be shown that in a simple manner the process, if not degenerate, are transformed to another one that satisfies Condition (AP) so that from it we derive either directly or indirectly (depending on purpose) the asymptotic expansions for the original process. It in particular is shown that if the MA processes is irreducible as a Markov process, then the Green function is expanded quite similarly to that of a classical random walk on $\mathbf{Z}^d$.

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Introduction

Let $(T, \mathcal{T})$ be a measurable space. Let (ξ_n, S_n) $(n = 0, 1, 2, \ldots)$ be a *Markov additive process* (*MA process* in abbreviation) taking values in the product space $T \times \mathbf{Z}^d$, namely it is a time homogeneous Markov process on the state space $T \times \mathbf{Z}^d$ whose one step transition law is such that the conditional distribution of $(\xi_n, S_n - S_{n-1})$ given (ξ_{n-1}, S_{n-1}) does not depend on the value of S_{n-1}.

Among various examples of MA processes those which motivated the present work are Markov processes moving in the Euclidian space whose transition laws are spatially periodic like the random walk on a periodic graph or the Brownian motion on a jungle-gym-like manifold. These processes are expected to be very similar to classical random walks or Brownian motions in various aspects but with different characteristic constants that must be determined. The results in this paper will serve as the base for verification of such similarity in fundamental objects of the processes like the hitting distribution to a set as well as for computation of the corresponding characteristic constants. (Another example is given in Remark 5.)

From the definition of MA process it follows that the process ξ_n is a Markov process on T and that with values of ξ_n being given for all n, the increments

$$Y_n := S_n - S_{n-1}$$

constitute a conditionally independent sequence. Let $p^n_T(\xi, d\eta)$ denote the n-step transition probability kernel of the ξ_n-process and P_ξ the probability law (on a measurable space $(\Omega, \mathcal{F})$ common to ξ) of the process (ξ_n, S_n) starting at $(\xi, 0)$. Set $p_T = p^1_T$. We suppose that p_T is ergodic, namely it admits a unique invariant probability measure, which we denote by μ, and that Doeblin's condition is satisfied. With the ergodic measure at hand the latter may be presented as follows:

(H.1) *there exist an integer* $\nu \geq 1$ *and a number* $\varepsilon > 0$ *such that*

$$\sup_{\xi \in T} p^\nu_T(\xi, A) \leq 1 - \varepsilon \quad \text{if} \quad \mu(A) \leq \varepsilon.$$

If there is no cyclically moving subset for the process ξ_n, this amounts to supposing that for some constant $\rho < 1$,

$$\text{(H.1}'\text{)} \qquad |p_T^n(\xi, A) - \mu(A)| \le \text{const}\, \rho^n \qquad (n = 1, 2, 3, \ldots)$$

uniformly in $A \in \mathcal{T}$ and $\xi \in T$. (See Remark 2 in the next section for the other case.) We also suppose that

$$\int \mu(d\xi) E_\xi[Y_1] = 0, \tag{1}$$

where E_ξ represents the expectation under P_ξ.

In this paper we shall derive asymptotic expansions of the Green function ($d \ge 2$) and the transition probabilities of the process (ξ_n, S_n) that satisfies (H.1) and (1). The derivation is based on a certain perturbation argument that has already been applied in previous works on (local) central limit theorems (cf. (6), (20), (17), (18) etc.) or on estimates of the Green functions (cf. (1), (9)) of MA processes. In our asymptotic expansions, however, not only the principal term but also the second order term are expressed explicitly in terms of a few basic functions that are characteristics of the expansion. The second order term will be important in computation of the harmonic measures for the walks restricted on a half space ((16)) (of which we shall give a brief outline after the statement of a main result) or other regions ((4),(14), (19), (22)).

We introduce a certain aperiodicity condition, named Condition (AP), that seems a minimal one under which the Fourier analysis can be applied straightforwardly. In the case when Condition (AP) is violated the structure of MA processes will be studied in detail and it will be shown that in a simple manner the process, if not degenerate, can be transformed to another one that satisfies Condition (AP) so that from it we derive either directly or indirectly (depending on purpose) the asymptotic expansions for the original process. It in particular is shown that if (ξ_n, S_n) is irreducible as a Markov process on $T \times \mathbf{Z}^d$, then the Green function is expanded quite similarly to that of a classical random walk on $\mathbf{Z}^d$ (Theorems 1 and 2). In the case when T is countable we also obtain a local limit theorem that is valid without assuming Condition (AP) nor irreducibility (Theorem 15). The results obtained here directly yield the corresponding ones for classical random walks on $\mathbf{Z}^d$, of which the estimates of transition probabilities valid uniformly for space-time variables outside parabolic regions seem not be in the existing literatures (Theorem 4, Corollary 6).

Other aspects of MA processes have been investigated under less restrictive settings than the present one by several authors: see (13) and (9) as well as references therein for renewal theorems or (21) for large deviations; a general theory of continuous time MA processes (construction, regularity property, Lévy-Ito-like decomposition etc.) is provided by (2). There are a lot of interesting examples of MA processes and several of them are found in (1). The term 'Markov additive process' is used by Cinlar (2). In some other works it is called by other names: a process defined on a Markov process, a random walk with internal states, a semi-Markov process or a Markov process with homogeneous second component etc.

The plan of the paper is as follows. In Section 1 we introduce basic notations and the aperiodicity condition (AP) and present the asymptotic expansions for the Green function and the transition probabilities. Several Remarks are made upon the results after that. The proofs of the asymptotic expansions are given for the Green function and for the transition probabilities in Sections 2 and 3, respectively, by assuming that the condition (AP) holds and ξ_n makes no cyclic transition, whereas the case of its making cyclic transitions and that when (AP) is violated are dealt with in Sections 4 and 5, respectively. In Section 6 we compute derivatives at 0 of $\kappa(\theta)$ the principal eigenvalue of a transfer operator. In Section 7 we show that the characteristic functions of S_n converge to zero geometrically fast away from the origin. The consequences of these two sections are taken for granted and applied in Sections 2 and 3. In the last section four Appendices are given: the first three provide certain standard proofs or statements of facts omitted in Sections 2 and 3; in the third we review a standard perturbation argument for the Fourier operator which is introduced in Section 2 and made use of in the proof of the main theorems.

1 Main Results

Let (ξ_n, S_n) be a MA process on $T \times \mathbf{Z}^d$ and p_T^n and P_ξ the transition probability of the first component and the probability law of the process, respectively, as in Introduction. We suppose the conditions (H.1) and (1) to be true unless otherwise stated explicitly. To state the main results we introduce some further notation. Let $\mathbf{p}$ stand for the integral operator whose kernel is p_T: $\mathbf{p}f = \int p_T(\cdot, d\eta)f(\eta)$. Define an $\mathbf{R}^d$-valued function h on T by

$$h(\xi) = E_\xi[Y_1]$$

and let c be a solution of $(1-\mathbf{p})c = h$ (μ-a.e.) that satisfies $\mu(c) := \int c d\mu = 0$, which, if it exists, is unique owing to the ergodicity of p_T. We shall impose some moment condition on the variable Y_1, which incidentally implies that h is bounded and that c exists and is bounded and given by

$$c(\xi) = (1-\mathbf{p})^{-1}h(\xi) := \lim_{z \uparrow 1} \sum_{n=0}^{\infty} z^n \mathbf{p}^n h(\xi). \tag{2}$$

(If (H.1′) is true, the last series is convergent without the convergence factors, hence $c(\xi) = \lim_{n\to\infty} E_\xi[S_n]$; see Remark 1 after Corollary 6 and Section 4 in the case when there are cyclically moving subsets; the convention that $(1-\mathbf{p})^{-1}$ is defined by the Abel summation method as above will be adopted throughout the paper.)

Put $\tilde{Y}_n = Y_n - c(\xi_{n-1}) + c(\xi_n)$. Then

$$E_\xi[\tilde{Y}_1] = (h - c + \mathbf{p}c)(\xi) = 0. \tag{3}$$

Let Q be the matrix whose components are the second moments of $\tilde{Y}_1$ under the equilibrium

$$P_\mu := \int \mu(d\xi)P_\xi$$

and denote its quadratic form by $Q(\theta)$:

$$Q(\theta) := \theta \cdot Q\theta := E_\mu(\tilde{Y}_1 \cdot \theta)^2, \tag{4}$$

where $\theta \in \mathbf{R}^d$, a d-dimensional column vector, $y \cdot \theta$ stands for the usual inner product in $\mathbf{R}^d$ and E_μ denotes the expectation by P_μ . Owing to (3) we have

$$Q(\theta) = \lim_{n \to \infty} \frac{1}{n} E_\mu |S_n \cdot \theta|^2$$

since $S_n - S_0 = \tilde{Y}_1 + \cdots + \tilde{Y}_n - c(\xi_n) + c(\xi_0)$.

We also define functions h^* and c^*, dual objects of h and c, by

$$h^*(\eta) = E_\mu[-Y_1 |\, \xi_1 = \eta],$$

and

$$c^* = (1 - \mathbf{p}^*)^{-1} h^*, \tag{5}$$

where $\mathbf{p}^*$ denotes the conjugate operator of $\mathbf{p}$ in $L^2(\mu)$: c^* is a unique solution of $(1-\mathbf{p}^*)c^* = h^*$ such that $\mu(c^*) = 0$ as before. Alternatively c^* may be defined as a μ-integrable function such that $\mu(c^* f) = -\sum_{n=0}^{\infty} E_\mu[Y_1 \mathbf{p}^n f(\xi_1)]$ for every bounded f.

The transition probability p of the process (ξ_n, S_n) may be expressed as

$$p((\xi, x), (d\eta, y)) := p_T(\xi, d\eta) q_{\xi,\eta}(y - x),$$

where $q_{\xi,\eta}(x) = P_\xi[Y_1 = x | \xi_1 = \eta]$. We denote by p^n the n-step transition probability which is defined by iteration for $n \geq 1$ and $p^0((\xi, x), \cdot\,) = \delta_{(\xi,x)}(\,\cdot\,)$ as usual.

We call the process (ξ_n, S_n) *symmetric* if $p((\xi, x), (d\eta, y))$ is symmetric relative to $\mu \times$the counting measure on $\mathbf{Z}^d$. From the expression of p given above we see that it is symmetric if and only if p_T is symmetric relative to μ and $q_{\xi,\eta}(x) = q_{\eta,\xi}(-x)$ for every x and almost every (ξ, η) with respect to $\mu(d\xi)p(\xi, d\eta)$. If this is the case, $h^* = h$ and $c^* = c$.

We introduce two fundamental conditions, one on irreducibility and the other on aperiodicity.

Irreducibility. A MA process (ξ_n, S_n) is called *irreducible* if there exists a set $T_\circ \in \mathcal{T}$ with $\mu(T_\circ) = 1$ such that if $\xi \in T_\circ$, $x \in \mathbf{Z}^d$ and $\mu(A) > 0$ $(A \in \mathcal{T})$, then for some n

$$P_\xi[\xi_n \in A, S_n = x] > 0. \tag{6}$$

Condition (AP). We say a MA process (ξ_n, S_n) or simply a walk S_n satisfies *Condition* (AP) or simply (AP) if there exists no proper subgroup H of the additive group $\mathbf{Z}^d$ such that for every positive integer n, the conditional law $P[S_n - S_0 = \cdot \,|\, \sigma\{\xi_0, \xi_n\}]$ on $\mathbf{Z}^d$ is supported by $H + a$ for some $a = a(\xi_0, \xi_n) \in \mathbf{Z}^d$ with P_μ-probability one; namely it satisfies (AP) if no proper subgroup H of $\mathbf{Z}^d$ fulfills the condition

$$\forall n \geq 1, \quad P_\mu\Big[\exists a \in \mathbf{Z}^d, P_\mu[S_n \in H + a \,|\, \sigma\{\xi_0, \xi_n\}] = 1\Big] = 1. \tag{7}$$

Here $\sigma\{X\}$ denotes the σ-fields generated by a random variable X. (See Corollary 22 in Section 7 for an alternative expression of (AP).)

Condition (AP) is stronger than the irreducibility and often so restrictive that many interesting MA processes do not satisfy it (some examples are given in **5.2** of Section 5). For the estimate of Green function we need only the irreducibility, while the local central limit theorem becomes somewhat complicated without (AP). In practice the results under the supposition of (AP) are enough, the problem in general cases of intrinsic interest often being reduced to them in a direct way as well as by means of a simple transformation of the processes (cf. Section 5).

We shall see in Section **5.4** that the matrix Q is **positive definite** if the process is irreducible. Here we see it to be true under (AP). Indeed, if $Q(\theta^\circ) = 0$ for some $\theta^\circ \neq 0$, then for every $n \geq 1$, $E_\mu|(S_n - c(\xi_0) + c(\xi_n)) \cdot \theta^\circ|^2 = nQ(\theta^\circ) = 0$, so that S_n is on the hyperplane perpendicular to θ° with P_μ-probability one, contradicting (AP).

We use the norm

$$\|x\| := \sigma\sqrt{x \cdot Q^{-1}x}, \quad \text{where} \quad \sigma := (\det Q)^{1/2d},$$

provided that Q is positive definite. Multiplication of Q by a constant does not change $\|x\|$; in particular, if $Q(\theta)$ is of the form $\sigma^2|\theta|^2$ (hence $\sigma^2 = \frac{1}{d}E_\mu|\tilde{Y}_1|^2$), then $\|x\|$ coincides with the usual Euclidian length $|x| := \sqrt{x_1^2 + \cdots + x_d^2}$ for $x \in \mathbf{R}$.

The Green functions. We define the Green function (actually a measure kernel) $G((\xi,x),(d\eta,y))$ for $d \geq 3$ by

$$G((\xi,x),(d\eta,y)) = \sum_{n=0}^{\infty} p^n((\xi,x),(d\eta,y)) \qquad (d \geq 3).$$

Put $\kappa_d = \frac{1}{2}\pi^{-d/2}\Gamma(\frac{1}{2}(d-2))$.

Theorem 1. *Let $d \geq 3$. Suppose that (ξ_n, S_n) is irreducible and that*

$$\sup_\xi E_\xi[|Y_1|^{2+m+\delta}] < \infty \quad \text{if} \quad d = 3$$

$$\sup_\xi E_\xi[|Y_1|^{d-2+m+\delta}] < \infty \quad \text{if} \quad d \geq 4$$

for some $\delta \in [0,1)$ such that $\delta \neq 0$ if $d = 4$, and for some nonnegative integer m. Then the Green function G admits the expansion (as $|x| \to \infty$)

$$G((\xi,0),(A,x)) = \frac{\kappa_d}{\sigma^2\|x\|^{d-2}}\mu(A) + T_1 + \cdots + T_m + \frac{\int_A R_m(x,\xi,\eta)\mu(d\eta)}{|x|^{d-2+m+\delta}} \tag{8}$$

($x \neq 0$, $A \in \mathcal{T}$) with $\lim_{|x|\to\infty}\sup_\xi \int |R_m(x,\xi,\eta)|\mu(d\eta) = 0$. Here $T_k = \int_A \Big[\{x^{3k}\}/\|x\|^{d-2+4k}\Big]\mu(d\eta) = O\Big(\mu(A)/\|x\|^{d-2+k}\Big)$ with $\{x^j\}$ representing a certain homogeneous polynomial of x of degree j whose coefficients, depending on η as well as ξ, are $L^1(\mu(d\eta))$-bounded uniformly in ξ; and $T_1 + \cdots + T_m$ is understood to be zero if $m = 0$. Moreover T_1 is of the form

$$T_1 = \frac{1}{\|x\|^{d+2}}\int_A \Big[U(x) + (d-2)\kappa_d\|x\|^2(Q^{-1}x)\cdot\Big(c(\xi) - c^*(\eta)\Big)\Big]\mu(d\eta), \tag{9}$$

where $U(x)$ is a homogeneous polynomial of degree 3 (given by (30) in Section 2) that does not depend on variables ξ, η. If (ξ_n, S_n) is symmetric (in the sense above), then $U = 0$ and $c^* = c$.

In the two dimensional case we define

$$G((\xi,x),(d\eta,y)) = \sum_{n=0}^{\infty}\Big[p^n((\xi,x),(d\eta,y)) - p^n((\xi,0),(d\eta,0))\Big] \qquad (d=2).$$

Theorem 2. *Let $d = 2$. Suppose that (ξ_n, S_n) is irreducible and that for some $\delta \in (0,1)$ and some integer $m \geq 0$, $\sup_\xi E_\xi[|Y_1|^{2+m+\delta}] < \infty$. Then G admits the following asymptotic expansion*

$$G((\xi,0),(A,x)) = -\frac{\mu(A)}{\pi\sigma^2}\log\|x\| + C_\xi(A) + T_1 + \cdots + T_m + \frac{\int_A R_m(x,\xi,\eta)\mu(d\eta)}{|x|^{d-2+m+\delta}}$$

$(x \neq 0,\ A \in \mathcal{T})$. Here T_k and R_m are as in the preceding theorem (but with $d = 2$) and $C_\xi(d\eta)$ is a bounded signed measure on T (given by (31) in the next section). Moreover

$$T_1 = \frac{1}{\|x\|^4}\int_A\left[U(x) + \frac{1}{\pi}\|x\|^2(Q^{-1}x)\cdot\Big(c(\xi) - c^*(\eta)\Big)\right]\mu(d\eta).$$

In Theorems 1 and 2, some explicit expressions are presented not only for the principal term but also for the second order term, i.e. T_1, in the expansion of G. The second order term sometimes plays an important role in applications. In (16) a random walk on a periodic graph, say (V,E) whose transition law is spatially periodic according to the periodicity of the graph is studied and the results above are applied to compute the hitting distribution of a hyper plane where the second order term mentioned above is involved. Typically the vertex set of the graph is of the form $V = \{u+\gamma : u \in F, \gamma \in \Gamma\}$ where F is a finite set of $\mathbf{R}^d$ and Γ is a d-dimensional lattice spanned by d linearly independent vectors $e_1,\ldots,e_d \in \mathbf{R}^d$. The random walk X_n moves on V whose transition law is invariant under the natural action of Γ so that $\pi(X_n)$ is a Markov chain on F, where $\pi_F : V \to F$ denotes the projection, and is viewed as a MA process on $T \times \mathbf{R}^d$. Suppose that there is a hyper plane M relative to which the reflection principle works so that if $\bar{v}$ denotes the mirror symmetric point of v relative to M and $u, v \in V \setminus M$, then the Green function G^+ of the walk X_n killed on M is given by $G^+(u,v) = G(u,v) - G(u,\bar{v})$ if u, v are on the same side of V separated by M and $G^+(u,v) = 0$ otherwise. Let V^+ be one of two parts of V separated by M and e a unit vector perpendicular to M. Then under suitable moment condition one can deduces from Theorems 1 and 2 that for $u, v \in V^+$,

$$G^+(u,v) = \frac{2\Gamma(d/2)|\det A|}{\pi^{d/2}\,Q(e)}\cdot\frac{\Big[(u+c(\xi))\cdot e\Big]\Big[(v+c^*(\eta))\cdot e\Big]}{\|v-u\|^d}\mu(\eta)\times\Big\{1+o(1)\Big\},$$

as $|v-u| \to \infty$ in such a manner that $(u\cdot e)(v\cdot e)/|u-v|^2 \to 0$, where $\xi = \pi_F(u)$, $\eta = \pi_F(v)$ and A is the matrix made of column vectors $e_1,\ldots,e_d$. The asymptotic formula for the hitting distribution of M is readily obtained form this.

In the next result the same assumption as in Theorem 2 is supposed.

Corollary 3. *Let $d = 2$. In the definition of G one may subtract $(2\pi\sigma^2 n)^{-1}\mu(d\eta)$ instead of $p^n((\xi,0),(d\eta,0))$; so define $\tilde{G}$ by*

$$\tilde{G}((\xi,x),(d\eta,y)) = \sum_{n=1}^{\infty}\Big[p^n((\xi,x),(d\eta,y)) - \frac{1}{2\pi\sigma^2 n}\mu(d\eta)\Big], \tag{10}$$

which only causes an alteration of the constant term $C_\xi(A)$ and an additional error term, $r(x)\mu(d\eta)$ say, of order $O(\|x\|^{-4})$ in the expansion of Theorem 2: the constant term is given by

$$(\pi\sigma^2)^{-1}\Big(-\gamma+\log(\sigma\sqrt{2}\,)\Big)\mu(A)$$

where γ is Euler's constant. ($r(x)$ appears only if ξ_n makes a cyclic transition.)

Corollary 3 is actually a corollary of Theorem 2 and Theorem 4 below. For a proof, see the end of the proof of Theorem 2 given in the next section.

We shall prove these results first under the supposition of (AP), and then reduce the general case to them . In the case when the condition (AP) does not hold, it is natural to consider the minimal subgroup H that satisfies (7) and the process, denoted by a_n, that is the projection of S_n on the quotient group $\mathbf{Z}^d/H$. If the process (ξ_n, a_n), which is a Markov process on $T\times(\mathbf{Z}^d/H)$, is ergodic (namely it has a unique invariant probability measure), then the Green function is shown to well behave. This ergodicity of course follows from the *irreducibility* of (ξ_n, S_n) (see Lemma 10). On the other hand, if it is not ergodic, the formulae in the theorems above must be suitably modified for obvious reason.

Local Central Limit Theorem. The method that is used in the proof of Theorems 1 and 2 also applies to the derivation of local central limit theorems just as in the case of classical random walks on $\mathbf{Z}^d$. We give an explicit form of the second order term as in the estimate of the Green function given above. The next order term is computable in principle though quite complicated (cf. Corollary 5).

In the expansion of the Green function there is no trace of cyclic transitions of ξ_n (if any), which is reflected in the transition probability $p^n((\xi,x),(A,y))$ for obvious reason. In general there may be cyclically moving subsets of T so that the set T can be partitioned into a finite number of mutually disjoint subsets $T_0,\dots,T_{\tau-1}$ ($\tau\ge 1$) such that for $j,k,\ell\in\{0,\dots,\tau-1\}$,

$$p_T^{n\tau+\ell}(\xi,A)=p_T^{n\tau+\ell}(\xi,A\cap T_k)\quad\text{if}\quad \xi\in T_j,\ k=j+\ell\ (\mathrm{mod}\ \tau). \tag{11}$$

To state the next result it is convenient to introduce the probability measures μ_j ($j=0,\dots,\tau-1$) on T which are defined by

$$\mu_j(A)=\tau\mu(A\cap T_j)\qquad(A\in\mathcal{T}). \tag{12}$$

If $\tau=1$, we set $\mu_0=\mu$.

Theorem 4. *Let τ and μ_j be as above. Suppose that Condition (AP) holds and $\sup_\xi E_\xi[|Y_1|^{k+\delta}]<\infty$ for some $k\ge 2$ and $\delta\in[0,1)$. Then, if $\xi\in T_j$, $A\in\mathcal{T}$, $0\le\ell\le\tau-1$, and $n=m\tau+\ell>0$,*

$$\begin{aligned}&(2\pi\sigma^2 n)^{d/2}P^n((\xi,0),(A,x)) \qquad (13)\\ &=\exp\left(-\frac{\|x\|^2}{2\sigma^2 n}\right)\Big[\mu_{j+\ell}(A)+P^{n,k}(x)\Big]+o\left(\frac{1}{\sqrt{n}^{\,k-2+\delta}}\wedge\frac{n}{|x|^{k+\delta}+1}\right),\end{aligned}$$

as $n+|x|\to\infty$. Here $a\wedge b$ stands for the minimum of a and b; $P^{n,k}(x)$ is a polynomial of x such that $P^{n,2}\equiv 0$ and if $k\ge 3$

$$P^{n,k}(x)=\frac{1}{\sqrt{n}}P_1^A\left(\frac{x}{\sqrt{n}}\right)+\cdots+\frac{1}{\sqrt{n}^{\,k-2}}P_{k-2}^A\left(\frac{x}{\sqrt{n}}\right)$$

where $P_j^A(y)$ is a polynomial (depending on ξ, ℓ as well as A but determined independently of m) of degree at most $3j$ and being odd or even according as j is odd or even. The first polynomial P_1^A is given by

$$P_1^A(y) = \int_A \Big[H(\sqrt{Q^{-1}}\, y) + Q^{-1}y \cdot \Big(c(\xi) - c^*(\eta)\Big)\Big]\mu_{j+\ell}(d\eta) \quad (y \in \mathbf{R}^d)$$

where H (defined via (40) (valid also for the case $\tau \geq 2$)) is a linear combination of Hermite polynomials (in d-variables) of degree three with coefficient independent of ξ, A; in particular H is identically zero if the process is symmetric.

The error estimate in Theorem 4 is fine: the expansions of the Green functions as in Theorems 1 and 2 can be obtained from Theorem 4 (except for $d = 4$) in view of the inequality

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}^d}\left(\frac{1}{\sqrt{n}^{k-2+\delta}} \wedge \frac{n}{|x|^{k+\delta}+1}\right) \leq \frac{C}{|x|^{d+k-4+\delta}} \tag{14}$$

if $d = 2, \delta > 0$ or $d = 3, \delta \geq 0$; and a similar one with $C/|x|^{k+\delta}$ on the right side for $d \geq 5$ (we need to multiply the right side by $\log|x|$ for $d = 4$).

Corollary 5. *Suppose that (ξ_n, S_n) satisfies (AP) and $\sup_\xi E_\xi[|Y_1|^6] < \infty$. Then, if $\xi \in T_j$, $A \in \mathcal{T}$, $\ell = 0, \dots, \tau - 1$, and $n = m\tau + \ell$,*

$$\begin{aligned} &(2\pi\sigma^2 n)^{d/2} p^n((\xi, 0), (A, x)) \\ &= \exp\left(-\frac{\|x\|^2}{2\sigma^2 n}\right)\left[\mu_{j+\ell}(A) + \frac{1}{\sqrt{n}}P_1^A\left(\frac{x}{\sqrt{n}}\right) - \frac{a^A(\xi)}{2n}\right] + O\left(\frac{1+|x|^2}{n^2}\right)\mu_{j+\ell}(A) \end{aligned} \tag{15}$$

as $n \to \infty$ uniformly for $|x| \leq C\sqrt{n}$ and $\xi \in T$ (with C being any positive constant). Here $a^A(\xi)$ is a certain function on T.

In the special case when the ξ_n process degenerates into a constant we have the result for a classical random walk on $\mathbf{Z}^d$ which is an extension of that of Spitzer (22) . Its n-step transition probability $p^n(x, y)$ is given in the form $p^n(y - x)$.

Corollary 6. *Suppose that the random walk is strongly aperiodic in the sense of (22), $\sum p^1(x)x = 0$ and $\sum p^1(x)|x|^{k+\delta}] < \infty$ for some $k \geq 2$ and $\delta \in [0, 1)$. Then*

$$(2\pi\sigma^2 n)^{d/2} p^n(x) = \exp\left(-\frac{\|x\|^2}{2\sigma^2 n}\right)\left[1 + P^{n,k}(x)\right] + o\left(\frac{1}{\sqrt{n}^{k-2+\delta}} \wedge \frac{n}{|x|^{k+\delta}+1}\right),$$

as $n + |x| \to \infty$. Here $P^{n,k}(x)$ is a polynomial of x as described in Theorem 4 except that P_j^A therein is independent of A ($\kappa(\theta)$ in (40) is nothing but the characteristic function of p^1).

<u>*Remarks for Theorems 1 to 4.*</u>

Remark **1** If there are cyclically moving subsets of T for the process (ξ_n) as in (12), the relation (H.1′) does not hold any more. However, under Doeblin's condition and ergodicity, it holds that for $j, k, \ell \in \{0, \ldots, \tau - 1\}$,

$$\text{(H.1}''\text{)} \quad \sup_{\xi \in T_j} |p_T^{n\tau+\ell}(\xi, A) - \mu_k(T_k)| \le \text{const}\, \rho^n \quad \text{if} \quad k = j + \ell \pmod{\tau}$$

uniformly in $\xi \in T_j, A \in \mathcal{T}$. (For proof see Doob (3), Section V.5 (especially pages 205-206, 207). The function c, defined as a solution of $(1 - \mathbf{p})c = h$, is obtained by taking the Abel sum as in (2) of possibly divergent series $\sum \mathbf{p}^n h$; c^* may be similarly given. (In the special case when every T_i consists of one point, the process is a deterministic cyclic motion on a finite set and $c = h + (1 - \tau^{-1})\mathbf{p}h + (1 - 2\tau^{-1})\mathbf{p}^2 h + \cdots + \tau^{-1}\mathbf{p}^{\tau-1}h$. For the general case see Section 4.)

Remark **2** One might consider a class of MA processes which satisfy Condition (AP) and wish to prove that the estimates as given above hold uniformly for the processes in the class (cf. (10), (26)). For expecting such uniform estimates to be true it is reasonable to suppose that the following bounds hold uniformly for the class:
(i) $\sup_\xi E_\xi |Y_1|^{k+\delta} \le C$; $\exists\lambda > 0$, $\inf_{|\theta|=1} Q(\theta)/|\theta|^2 > \lambda$
(ii) $\|(\mathbf{p} - \Pi)^n\|_{L^\infty(\mu)} \le C\rho^n$ for all n $(\rho < 1)$;
(iii) $\exists n^\circ, \forall\varepsilon > 0, \exists\alpha < 1$, $\sup_{|\theta|>\varepsilon, \theta\in[-\pi,\pi]^d} E_\mu \Big| E_\mu \Big[e^{iS_{n^\circ}\cdot\theta} \,\Big|\, \sigma\{\xi_0, \xi_{n^\circ}\} \Big] \Big| \le \alpha$
(provided that τ is bounded), where Π is the principal part of $\mathbf{p}$ made of eigenprojections corresponding to eigenvalues of modulus unity (see Section 6). (For (iii) see (83) of Section 7.) Unfortunately it is not fully clear whether the uniformity of these bounds is sufficient since it does not seem to provide appropriate bounds for derivatives of $\kappa(\theta), M_\theta(f)$ (cf. Section 2 for the notation).

Remark **3** For the local central limit theorem the zero mean condition (1) is not essential. With the mean vector $b = E_\mu[Y_1]$ we have only to replace x by $x - nb$ on the right sides of (13) and (15) to have the corresponding formulas. The same proofs as in the case of mean zero go through if h and h^* are defined by $h(\xi) = E_\xi[Y_1] - b$ and $h^*(\eta) = b - E_\mu[Y_1|\xi_1 = \eta]$, respectively.

Remark **4** We have supposed the Doeblin's condition (H.1) to hold, which amounts to supposing the uniform bound of the exponentially fast convergence (H.1$''$). We may replace it by the $L^p(\mu)$ $(p \ge 1)$ bound under some auxiliary condition on p_T (which is satisfied eg. if T is a countable set), although the estimates stated in Theorems above must be generally not uniform relative to $\xi_0 = \xi$ any more. (Cf. (1) for a such extension.)

Remark **5** The case when the distributions of S_n is not supported by any lattice can be dealt with in a similar way under some reasonable conditions. If the asymptotic formulae are for measure kernels and understood in a weak sense (cf. the last section of (25)), it suffices to suppose, in place of Condition (AP), that for some positive integer n°,

$$P_\mu\Big[\limsup_{|\theta|\to\infty} \Big| E_\mu[\exp(iS_{n^\circ}\cdot\theta) \,|\, \sigma\{\xi_0, \xi_{n^\circ}\}] \Big| < 1\Big] > 0. \tag{16}$$

If it is for the density, we need suppose a more restrictive one, eg, the condition that for some integers $n^\circ, k \ge 1$

$$\sup_{\xi,\eta} \int_{\mathbf{R}^d} \Big| E_\xi[\exp(iS_{n^\circ}\cdot\theta) \,|\, \xi_{n^\circ} = \eta] \Big|^k d\theta < \infty, \tag{17}$$

which in particular implies that if $2j \geq k$, the conditional distribution of S_{jn° with ξ_0, ξ_{n° being fixed has a square integrable density. It follows from each of (16) and (17) that for every $\varepsilon > 0$

$$\sup_{|\theta|>\varepsilon} \Big| E_\mu[\exp(iS_{jn^\circ} \cdot \theta) \,|\, \sigma\{\xi_0, \xi_{n^\circ}\}] \Big| < 1$$

with positive P_μ-probability, so that the estimates as given in Section 7 can be verified in a similar way as therein. It may be worth pointing out that to have an expansion analogous to the right side of (8) the first several terms $P^1, \ldots, P^\ell$ must be discarded from the Green function G since they may possibly behave very badly even when they possess densities.

Let (M_n, u_n) $(n \geq 1)$ be an i.i.d. sequence of pais of random $d \times d$-matrices M_n and random d-vectors u_n and define $\xi_0 = I$ (unit matrix), $\xi_n = M_1 \cdots M_n$, $S_0 = 0$ and recursively $S_{n+1} = S_n + \xi_n u_{n+1}$. The pair (ξ_n, S_n) is then a Markov additive process starting from $(I, 0)$. If M_n are taken from the special orthogonal group $SO(d)$, the conditions (16) as well as (H.1) is satisfied under mild restrictions on the distribution of (M_1, u_1) (cf. (1)). This model is s closely related to the random difference equation $Y_n = M_n Y_{n-1} + u_n$: in fact, given Y_0, Y_n has the same distribution as $S_n + \xi_n Y_0$. (Cf. (15), (1) for more general cases of M_n.)

Other expressions of Q. In some of previous works the covariance matrix Q is expressed in apparently different forms, which we here exhibit. Define

$$Q^\circ(\theta) := \theta \cdot Q^\circ \theta = E_\mu\Big[(Y_1 \cdot \theta)^2\Big].$$

Let R be another symmetric matrix defined through the quadratic form:

$$R(\theta) = 2E_\mu\Big[(-Y_1 \cdot \theta)(c(\xi_1) \cdot \theta)\Big].$$

Then $R(\theta) = 2\mu((c^* \cdot \theta)(h \cdot \theta)) = 2\mu((h^* \cdot \theta)(c \cdot \theta))$ and

$$Q = Q^\circ - R. \tag{18}$$

In fact from the identity $E_\cdot[Y_1] - c + \mathbf{p}c = 0$, we deduce the equality

$$E_\cdot|\tilde{Y}_1 \cdot \theta|^2 = E_\cdot[\,|Y_1 \cdot \theta|^2 + 2\{Y_1 \cdot \theta\}\{c(\xi_1) \cdot \theta\}] - |c \cdot \theta|^2 + \mathbf{p}|c \cdot \theta|^2,$$

which, on integrating by μ, yields (18). In particular, if Q is isotropic, namely $Q(\theta) = \sigma^2|\theta|^2$, then $\sigma^2 d = E_\mu|Y_1|^2 + 2E_\mu[Y_1 \cdot c(\xi_1)]$. If the process (ξ_n, S_n) is symmetric, then $R(\theta)$, being equal to $2\mu(h \cdot \theta(1-\mathbf{p})^{-1} h \cdot \theta)$, is the central limit theorem variance for the sequence $h(\xi_n) \cdot \theta$ under the stationary process measure P_μ, in particular $\sigma^2 \leq (\det Q^\circ)^{1/2d}$. The last inequality is not necessarily true in the asymmetric case (see Example 6.1 of (16)).

Let $m(\xi, \eta)$ be the first moment of the conditional law of Y_1 given $(\xi_0, \xi_1) = (\xi, \eta)$, namely $m(\xi, \eta) = E_\xi[Y_1 \,|\, \xi_1 = \eta]$. Then $h(\xi) = \int p_T(\xi, d\eta) m(\xi, \eta)$, and we infer that

$$R(\theta) = -2\sum_{n=1}^{\infty} E_\mu\Big[(m(\xi_0, \xi_1) \cdot \theta)(m(\xi_n, \xi_{n+1}) \cdot \theta)\Big].$$

Let Q_m denote the central limit theorem variance for the sequence $m(\xi_n, \xi_{n+1})$. Then $Q_m(\theta) = E_\mu\Big[\Big(m(\xi_0, \xi_1)\cdot\theta\Big)^2\Big] - R(\theta)$. Hence Q may be expressed as the sum of two variances:

$$Q(\theta) = E_\mu\Big[\Big|\Big(Y_1 - m(\xi_0, \xi_1)\Big)\cdot\theta\Big|^2\Big] + Q_m(\theta). \tag{19}$$

The expression (4) appears in (6) and (1), whereas (18) and (19) appear in (18) and (24), respectively.

2 Proofs of Theorems 1 and 2

Let f be a bounded measurable function on T. We are to compute

$$E_\xi[e^{iS_n\cdot\theta} f(\xi_n)] = \sum_{x\in\mathbf{Z}^d} e^{ix\cdot\theta} E_\xi[f(\xi_n); S_n = x], \qquad \theta \in \mathbf{R}^d.$$

Define the following complex measure kernel:

$$p_\theta(\xi, d\eta) = E_\xi[e^{iY_1\cdot\theta}; \xi_1 \in d\eta] \tag{20}$$

and let $\mathbf{p}_\theta$ be the integral operator given by it. Then

$$E_\xi[e^{iS_n\cdot\theta} f(\xi_n)] = \mathbf{p}_\theta^n f(\xi).$$

We regard the operator $\mathbf{p}_\theta$ as a perturbation of $\mathbf{p}$. In doing so, we suppose for simplicity that there are no cyclically moving subsets of T for the process (ξ_n) so that we can use (H.1$'$). (See Section 4 in general cases.) Under this supposition it follows that if $\delta_\circ > 0$ is small enough and $|\theta| \le \delta_\circ$, then there are an eigenfunction e_θ and an eigen-complex-measure μ_θ such that $\mathbf{p}_\theta e_\theta = \kappa(\theta)e_\theta$, $\mu_\theta\mathbf{p}_\theta = \kappa(\theta)\mu_\theta$, $\mu(e_\theta) = \mu_\theta(e_\theta) = 1$ and

$$\mathbf{p}_\theta = \kappa(\theta)e_\theta \otimes \mu_\theta + \mathbf{r}_\theta, \tag{21}$$

with $\kappa(\theta) \to 1$ as $|\theta| \to 0$ and $\limsup_{n\to\infty} \|\mathbf{r}_\theta^n\|^{1/n} < (\rho+1)/2$ uniformly for $|\theta| \le \delta_\circ$ (see Appendix D). Here $e_\theta \otimes \mu_\theta$ is a projection operator defined by $(e_\theta \otimes \mu_\theta)f = \mu_\theta(f)e_\theta$. Since it commutes with $\mathbf{p}_\theta$, we have $\mathbf{p}_\theta^n = [\kappa(\theta)]^n e_\theta \otimes \mu_\theta + \mathbf{r}_\theta^n$. Accordingly for $n = 1, 2, \dots$

$$E_\xi[e^{iS_n\cdot\theta} f(\xi_n)] = [\kappa(\theta)]^n e_\theta(\xi)\mu_\theta(f) + \mathbf{r}_\theta^n f(\xi), \tag{22}$$

and, on performing summation over n,

$$\sum_{n=0}^\infty E_\xi[e^{iS_n\cdot\theta} f(\xi_n)] = \frac{\kappa(\theta)M_\xi(\theta)}{1-\kappa(\theta)} + R_\xi(\theta), \tag{23}$$

where we put $R_\xi(\theta) = \sum_{n=0}^\infty \mathbf{r}_\theta^n f(\xi)$ (with $\mathbf{r}_\theta^0 f = f$) and

$$M_\xi(\theta) = e_\theta(\xi)\mu_\theta(f). \tag{24}$$

These are valid only for $|\theta| \le \delta_\circ$. We may extend M_ξ and κ to arbitrary functions that are sufficiently smooth and $|\kappa| < 1$ for $|\theta| > \delta_\circ$ and define $R_\xi(\theta)$ as the remainder. Set $\Delta = [-\pi, \pi)^d$

and choose these extensions of M_ξ and κ so that they vanish in a neighborhood of the boundary $\partial\Delta$. It holds that

$$\int_T G((\xi,0),(d\eta,x))f(\eta) = \frac{1}{(2\pi)^d}\int_\Delta \frac{\kappa(\theta)M_\xi(\theta)}{1-\kappa(\theta)}e^{-i\theta\cdot x}d\theta + \frac{1}{(2\pi)^d}\int_\Delta R_\xi(\theta)e^{-i\theta\cdot x}d\theta. \tag{25}$$

In the rest of this section we suppose that the process (ξ_n, S_n) satisfies (AP). (The general case will be treated in Section 5.) In this proof we further suppose that $\sup_\xi E_\xi[|Y_1|^m] < \infty$ for every $m = 0, 1, 2, \ldots$. (See Appendix A for the case when it is only supposed that $\sup_\xi E_\xi[|Y_1|^{k+\delta}] < \infty$.) Then for each m both $\kappa(\theta)$ and $M_\xi(\theta)$ are m-times differentiable functions of θ and there exists a positive constant $r = r_m < 1$ such that

$$\sup_{\theta\in\Delta,\,|\theta|>\delta_\circ} |\nabla^m E_\mu[e^{iS_n\cdot\theta}f(\xi_n)]| = O(r^n) \qquad (n\to\infty).$$

Here (and later on without exception) the gradient operator ∇ acts on a function of θ. The proof of this estimate is postponed to Section 7 (Lemma 20). From it together with (21) it follows that the second term on the right side of (25) approaches zero as $|x|\to\infty$ faster than $|x|^{-m}$ for every m.

We shall derive in Section 6 the following identities

$$\nabla\kappa(0) = 0;\ \ (\theta\cdot\nabla)^2\kappa(0) = -Q(\theta); \tag{26}$$

$$\nabla e_\theta\Big|_{\theta=0} = ic;\ \ \nabla\mu_\theta(f)\Big|_{\theta=0} = -i\mu(c^*f) \tag{27}$$

(Proposition 18 and Lemma 20). Here the somewhat abusing notation $(\theta\cdot\nabla)^2\kappa(0)$ stands for $\sum_{k,j}\theta_k\theta_j\nabla_k\nabla_j\kappa(0)$, which may be expressed by another one $\mathrm{Tr}(\theta^2\nabla^2\kappa(0))$, where θ^2 is understood to be a $d\times d$-matrix whose (k,j) entry is $\theta_k\theta_j$ and similarly for ∇^2. We infer from (26) that

$$\frac{\kappa(\theta)}{1-\kappa(\theta)} = \frac{2}{Q(\theta)}\left(1 + \frac{(\theta\cdot\nabla)^3\kappa(0)}{3Q(\theta)} + \frac{\{\theta^6\}}{Q^2(\theta)} + \cdots\right),$$

where $\{\theta^k\}$ denotes a homogeneous polynomial of $\theta\in\mathbf{R}^d$ of degree k. On the other hand by (27) we find that $\nabla M_\xi(0) = i[c(\xi)\mu(f) - \mu(c^*f)]$. It therefore follows that

$$\frac{\kappa(\theta)M_\xi(\theta)}{1-\kappa(\theta)} = \frac{2\mu(f)}{Q(\theta)} + \frac{iB_f(\theta)}{Q(\theta)} + \frac{\{\theta^6\}}{Q^3(\theta)} + \cdots, \tag{28}$$

where $B_f(\theta)$ is a real function given by

$$B_f(\theta) = \frac{2(\theta\cdot\nabla)^3\kappa(0)}{i3Q(\theta)}\mu(f) + 2[c(\xi)\mu(f) - \mu(c^*f)]\cdot\theta. \tag{29}$$

Let $d\geq 3$. Then, following the usual manner of evaluation of Fourier integrals (cf. Appendix B), we deduce from (25) together with what is remarked right after it that for $\xi,\eta\in T$,

$$\begin{aligned} &G((\xi,0),(d\eta,x)) - \frac{\kappa_d}{\sigma^2\|x\|^{d-2}}\mu(d\eta) \\ &= \frac{U(x) + (d-2)\kappa_d\|x\|^2(Q^{-1}x)\cdot\Big(c(\xi) - c^*(\eta)\Big)}{\|x\|^{d+2}}\mu(d\eta) + \frac{\{x^6\}\mu(d\eta)}{\|x\|^{d+6}} + \cdots \end{aligned}$$

where $U(x)$ is a homogeneous polynomial of degree 3 and given by

$$U(x) = \frac{\|x\|^{d+2}}{(2\pi)^d} \int_{\mathbf{R}^d} \frac{2(\theta \cdot \nabla)^3 \kappa(0)}{3Q^2(\theta)} e^{-ix\cdot\theta} d\theta. \tag{30}$$

In the case when the process (ξ_n, Y_n) is symmetric, it is clear that $c^* = c$; the equality $U = 0$ follows from Proposition 18 (iii) of Section 6.

Let $d = 2$. The proof may proceed as in (25) or (5). We outline it in below to identify the second order term. The proof is based on the Fourier inversion formula

$$G_\xi^f(x) := \int_T G((\xi, 0), (d\eta, x)) f(\eta) = \int_\Delta \sum_{n=0}^{\infty} E_\xi[e^{iS_n\cdot\theta} f(\xi_n)](e^{-ix\cdot\theta} - 1)\frac{d\theta}{(2\pi)^2}.$$

Set

$$\psi_\xi^f(\theta) = \sum_{n=0}^{\infty} E_\xi[e^{iS_n\cdot\theta} f(\xi_n)] - \frac{2\mu(f)}{Q(\theta)}$$

and make the decomposition

$$G_\xi^f(x) = \mu(f) \int_\Delta \frac{2}{Q(\theta)}(\cos x\cdot\theta - 1)\frac{d\theta}{4\pi^2} + \int_\Delta \psi_\xi^f(\theta)(e^{-ix\cdot\theta} - 1)\frac{d\theta}{4\pi^2}.$$

Then one deduces that the first integral on the right side equals

$$-\frac{1}{\sigma^2\pi}\log\|x\| + \frac{1}{\sigma^2\pi}(\log 2 - \gamma) - \frac{1}{4\pi^2}\int_{\Delta\setminus B}\frac{2d\theta}{Q(\theta)} - \int_{\mathbf{R}^2\setminus\Delta}\frac{2e^{-ix\cdot\theta}}{Q(\theta)}\frac{d\theta}{4\pi^2},$$

where $B = \{\theta : Q(\theta) \le \sigma^2\}$ $(\subset \Delta)$ and γ is Euler's constant (cf. (22),(5)). (Here the last integral is not absolutely convergent and needs to be defined as a principal value in a suitable sense (cf. Lemma 2 of (5)).) If we define

$$C_\xi(f) = \frac{\mu(f)}{\sigma^2\pi}(\log 2 - \gamma) - \frac{\mu(f)}{4\pi^2}\int_{\Delta\setminus B}\frac{2d\theta}{Q(\theta)} - \int_\Delta \psi_\xi^f(\theta)\frac{d\theta}{4\pi^2}, \tag{31}$$

then

$$\begin{aligned} G_\xi^f(x) &= -\frac{\mu(f)}{\sigma^2\pi}\log\|x\| + C_\xi(f) \\ &\quad + \int_\Delta \psi_\xi^f(\theta)e^{-ix\cdot\theta}\frac{d\theta}{4\pi^2} - \int_{\mathbf{R}^2\setminus\Delta}\frac{2e^{-ix\cdot\theta}d\theta}{Q(\theta)}\cdot\frac{\mu(f)}{4\pi^2}. \end{aligned} \tag{32}$$

By (28) and (23) we have

$$\psi_\xi^f(\theta) = R_\xi(\theta) + \frac{iB_f(\theta)}{Q(\theta)} + \frac{\{\theta^6\}}{Q^3(\theta)} + \cdots$$

in a neighborhood of the origin. With this we estimate the sum of the last two integrals in (32) as in the case $d \ge 3$.

Proof of Corollary 3. This proof amounts to computing the sum $S_\xi^A := \sum_{n=1}^\infty [p^n((\xi, 0), (A, 0)) - (2\pi\sigma^2 n)^{-1}\mu(A)]$ in a roundabout way, which is to result in $C_\xi^A + S_\xi^A + E_\xi[\xi \in A] = [(-\gamma +$

$\log\sigma\sqrt{2}\,)/\pi\sigma^2]\mu(A)$. In view of Theorem 4 (see (14)) and Theorem 2 it suffices to show that as $|x| \to \infty$,

$$\sum_{n=1}^{\infty} \frac{1}{2\pi\sigma^2 n}\Big(e^{-\|x\|^2/2\sigma^2 n} - 1\Big) = \frac{-\log\|x\| - \gamma + \log\sigma\sqrt{2}}{\pi\sigma^2} + O(\|x\|^{-4}) \tag{33}$$

as well as, for dealing with the case when ξ_n is cyclic of period $\tau\ (\geq 2)$, that

$$\sum_{k=1}^{\infty}\sum_{j=0}^{\tau-1}\left[\frac{e^{-\|x\|^2/2\sigma^2(k\tau+\ell)}}{2\pi\sigma^2(k\tau+\ell)} - \frac{e^{-\|x\|^2/2\sigma^2(k\tau+j)}}{2\pi\sigma^2(k\tau+j)}\right] = O(\|x\|^{-4}). \tag{34}$$

These estimates will be shown in Section 8 (Appendix C).

REMARK **6** Consider the case when ξ_n makes a cyclic transition and/or S_n does not satisfy (AP). As in (22) (p.310) we modify the process (ξ_n, S_n) by adding $\delta \in (0,1)$ of the probability that it does not move at each step and multiplying by $1-\delta$ the original probabilities: for the new law P'_ξ

$$P'_\xi[\xi_1 = \xi, S_1 = 0] = \delta + (1-\delta)P_\xi[\xi_1 = \xi, S_1 = 0].$$

This transforms the cyclically moving ξ_n into noncyclic one and many processes that do not satisfy (AP) into those satisfying (AP), but not all as it always does in the case of random walks on $\mathbf{Z}^d$ (see Section 5 for a counter example as well as relevant matters). Denoting by $\mathbf{p}'$, κ' etc. the corresponding objects for the modified process, we have $\mathbf{p}' = \delta\mathbf{1} + (1-\delta)\mathbf{p}$, $h' = (1-\delta)h$ and $c' = c$. Thus

$$\mathbf{p}'_\theta = \delta\mathbf{1} + (1-\delta)\mathbf{p}_\theta = \kappa'(\theta)e_\theta \otimes \mu_\theta + \mathbf{r}'_\theta$$

with $\kappa'(\theta) = \delta + (1-\delta)\kappa(\theta)$ and $\mathbf{r}'_\theta = \delta(\mathbf{1} - e_\theta \otimes \mu_\theta) + (1-\delta)\mathbf{r}_\theta$. In particular, $\nabla\kappa'(\theta) = (1-\delta)\nabla\kappa(\theta)$ (implying $Q' = (1-\delta)Q$ etc.) and both μ_θ and e_θ remain the same. If $d \geq 3$, we have

$$G' = (1-\delta)^{-1}G;$$

hence the required estimate of G follows from that of G' straightforwardly, provided that the modified process satisfies (AP) for all sufficiently small $\delta > 0$.

3 Proof of Local Central Limit Theorem.

By means of the expression (23) together with (28) one can derive an asymptotic expansion of the transition probability $p^n(\alpha, d\beta)$ in a usual manner. Here we first review the derivation of the expansion in the case when $|x|/\sqrt{n}$ is bounded above mainly for identification of the second order term and then discuss it in the case when $|x|/\sqrt{n}$ is bounded off zero. It is supposed that $\tau = 1$, namely there are no cyclically moving sets for ξ_n process (see section 4 in the case $\tau \geq 2$).

The case $|x|/\sqrt{n} < C$. We consider mostly the case $k = 5$, namely $E_\xi|Y_1|^{5+\delta} < \infty$. Owing to (26) and (27) we see

$$M_\xi(\theta) = \mu(f) + i\Big(c(\xi)\mu(f) - \mu(c^*f)\Big)\cdot\theta + \{\theta^2\} + \cdots; \tag{35}$$

$$\kappa(\theta) = 1 - \frac{1}{2}Q(\theta) + \frac{1}{6}(\theta \cdot \nabla)^3\kappa(0) + \{\theta^4\} + \cdots . \tag{36}$$

(Remember that $M_\xi(\theta) = e_\theta(\xi)\mu_\theta(f)$.) By the second relation we have

$$n \log[\kappa(\theta/\sqrt{n})] = -\frac{1}{2}Q(\theta) + \frac{1}{6\sqrt{n}}(\theta \cdot \nabla)^3\kappa(0) + \frac{1}{n}\{\theta^4\} + \frac{1}{n^{3/2}}\{\theta^5\} + \cdots . \tag{37}$$

Therefore on using (22)

$$\begin{aligned}
&E_\xi[e^{iS_n\cdot\theta/\sqrt{n}}f(\xi_n)] \\
&= e^{-\frac{1}{2}Q(\theta)}\left(1 + \frac{(\theta\cdot\nabla)^3\kappa(0)}{6\sqrt{n}} + \frac{\{\theta^4\}+\{\theta^6\}}{n} + \frac{\{\theta^5\}+\{\theta^7\}+\{\theta^9\}}{n^{3/2}} + \cdots\right) \times \\
&\quad \times \left(\mu(f) + \frac{i}{\sqrt{n}}\Big(c(\xi)\mu(f) - \mu(c^*f)\Big)\cdot\theta + \frac{\{\theta^2\}}{n} + \frac{\{\theta^3\}}{n^{3/2}} + \cdots\right) \\
&= e^{-\frac{1}{2}Q(\theta)}\left\{\mu(f) + \frac{1}{\sqrt{n}}\left[i\Big(c(\xi)\mu(f) - \mu(c^*f)\Big)\cdot\theta + \frac{1}{6}(\theta\cdot\nabla)^3\kappa(0)\mu(f)\right]\right\} \\
&\quad + e^{-\frac{1}{2}Q(\theta)}[P(\theta,n) + R(\theta,n)]
\end{aligned} \tag{38}$$

with $P(\theta,n) = n^{-1}(\{\theta^2\}+\{\theta^4\}+\{\theta^6\}) + n^{-3/2}(\{\theta^3\}+\{\theta^5\}+\{\theta^7\}+\{\theta^9\})$ and

$$|R(\theta,n)| = o\left(\frac{\mu(|f|)(1+|\theta|^{5+\delta})}{\sqrt{n}^{3+\delta}}\right) \qquad \text{for} \quad |\theta| < n^{1/6},$$

provided that $E_\mu|Y_1|^{5+\delta} < \infty$. Now

$$\begin{aligned}
&(2\pi\sigma^2 n)^{d/2}\int_T p^n((\xi,0),(d\eta,x))f(\eta) \\
&= \frac{\sigma^d}{(2\pi)^{d/2}}\int_{\sqrt{n}\Delta} E_\xi[e^{iS_n\cdot\theta/\sqrt{n}}f(\xi_n)]e^{-ix\cdot\theta/\sqrt{n}}\,d\theta \\
&= \exp\left(-\frac{\|x\|^2}{2\sigma^2 n}\right)\left[\mu(f) + H_\circ\left(\frac{x}{\sqrt{n}}\right)\frac{\mu(f)}{\sqrt{n}} + \frac{1}{n}Q^{-1}x\cdot\Big(c(\xi)\mu(f) - \mu(c^*f)\Big)\right] \\
&\quad + \frac{\sigma^d}{(2\pi)^{d/2}}\int_{\mathbf{R}^d} P(\theta,n)e^{-\frac{1}{2}Q(\theta)}e^{-ix\cdot\theta/\sqrt{n}}\,d\theta + o\left(\frac{\mu(|f|)}{\sqrt{n}^{3+\delta}}\right),
\end{aligned} \tag{39}$$

where $H_\circ(y)$ is an odd polynomial of degree three defined by

$$H_\circ(y) = \frac{\sigma^d e^{\|y\|^2/2\sigma^2}}{6(2\pi)^{d/2}}\int_{\mathbf{R}^d} e^{-\frac{1}{2}Q(\theta)}(\theta\cdot\nabla)^3\kappa(0)e^{-iy\cdot\theta}\,d\theta, \tag{40}$$

so that $H(x) := H_\circ(\sqrt{Q}x)$ is a linear combination of Hermite polynomials of degree three. For the verification one divides the range of integration $\sqrt{n}\Delta$ into three parts according as $|\theta| \le n^{1/6}$; $n^{1/6} < |\theta| \le \lambda_\circ\sqrt{n}$; $|\theta| > \lambda_\circ\sqrt{n}$ where $\lambda_\circ$ is a positive constant small enough that $\kappa(\theta)| < 1$ if $0 < |\theta| \le \lambda_\circ$; use Condition (AP) to estimate the last part with the help of Proposition 20 of Section 7.

Since the Fourier transform of a function of the form $\{\theta^j\}e^{-\frac{1}{2}Q(\theta)}$ is a Gaussian density times a polynomial of degree j (hence odd or even according as j is odd or even), the formula (39) is nothing but the formula stated in the theorem in the case when $|x|/\sqrt{n}$ is bounded above. The constant term of the transform being equal to the integral over $\mathbf{R}^d$ of the function, the constant $a^A(\xi)$ in the formula (15) is given by

$$\begin{aligned} a^f(\xi) &= \frac{-\sigma^d}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} e^{-\frac{1}{2}Q(\theta)} \Bigg[(\theta\cdot\nabla)^2 M_\xi(0) + \tfrac{1}{3}(\theta\cdot\nabla) M_\xi(0)(\theta\cdot\nabla)^3\kappa(0) \\ &\quad - \left(\left[\tfrac{1}{2}Q(\theta)\right]^2 - \tfrac{1}{12}(\theta\cdot\nabla)^4\kappa(0) - \left[\tfrac{1}{6}(\theta\cdot\nabla)^3\kappa(0)\right]^2 \right) \mu(f) \Bigg] \, d\theta. \end{aligned} \tag{41}$$

The contribution of $P(\theta, n)$ with the term $-a^f(\xi)/2n$ subtracted is bounded by

$$n^{-1}[(|x|/\sqrt{n})^2 + \cdots] + n^{-3/2}[(|x|/\sqrt{n}) + \cdots] \le C_1|x|^2/n^2$$

for $|x| < C\sqrt{n}$; and if $\sup_\xi E_\xi[|Y_1|^6] < \infty$, then that of $R(\theta, n)$ is bounded by C_2/n^2. This verifies Corollary 5.

The case $|x|/\sqrt{n} > C$. Put

$$\Psi_n(\theta) = E_\xi[e^{iS_n\cdot\theta} f(\xi_n)];$$

and $\omega = x/|x|$, $\nabla_\omega = \omega \cdot \nabla$. The proof is based on the following identities:

$$\frac{i^k|x|^k}{n^{k/2}} (2\pi\sigma^2 n)^{d/2} \int_T p^n((\xi,0),(d\eta,x)) f(\eta) = \int_{\sqrt{n}\,\Delta} e^{-ix\cdot\theta/\sqrt{n}} \; (\nabla_\omega^k \Psi_n)\left(\frac{\theta}{\sqrt{n}}\right) \frac{d\theta}{n^{k/2}}; \tag{42}$$

$$\frac{i^k|x|^k}{n^{k/2}} \exp\left(-\frac{\|x\|^2}{2n\sigma^2}\right) = \int_{R^d} e^{-ix\cdot\theta/\sqrt{n}} \, \nabla_\omega^k (e^{-\frac{1}{2}Q(\theta)}) d\theta \tag{43}$$

(where $\nabla_\omega^k = (\nabla_\omega)^k$) as well as the relation (22): $\Psi_n(\theta) = [\kappa(\theta)]^n M_\xi(\theta) + \mathbf{r}_\theta^n f(\xi)$ (where $M_\xi(\theta) = e^{ic(\xi)\cdot\theta} e_\theta(\xi)\mu_\theta(f)$). The method of using these identities is an extension of that found in Spitzer (22) in which $k = 2$. The arguments given below are mostly the same as for the classical random walk on $\mathbf{Z}^d$, but for the case $k \ge 3$ they seem not be in existing literatures.

We are to analyze the difference

$$D_n(\theta) := ((\nabla_\omega)^k \Psi_n)\left(\frac{\theta}{\sqrt{n}}\right) \frac{1}{n^{k/2}} - (\nabla_\omega)^k (e^{-\frac{1}{2}Q(\theta)}).$$

Lemma 7. *If $\sup_\xi E_\xi[|Y_1|^{k+\delta}] < \infty$, then*

$$D_n(\theta) = e^{-\frac{1}{2}Q(\theta)} \left[P_{n,k}(\theta) + nR_{n,k}(\theta) \right] \quad (|\theta| \le n^{1/6}) \tag{44}$$

with $R_{n,k}(\theta) = o\Big(\left[(1+|\theta|)/\sqrt{n}\right]^{k+\delta}\Big)$. Here $P_{n,2}(\theta) = 0$ and for $k \ge 3$,

$$P_{n,k}(\theta) = \sum_{j=1}^{k-2} P_j(\theta)/\sqrt{n}^{\,j} \tag{45}$$

where $P_j(\theta)$ is a polynomial of θ of degree at most $k+3j$. Moreover the function $R_{n,k}(y)$ satisfies

$$R_{n,k}(\theta) - R_{n,k}(\theta+\eta) = o\left(\left[\frac{1+|\theta|}{\sqrt{n}}\right]^k \left(\frac{|\eta|}{\sqrt{n}}\right)^\delta\right) \tag{46}$$

uniformly in ω, θ and η, provided that $|\eta|, |\theta| \le n^{1/6}$.

Proof of Lemma 7. We can expand $\nabla_\omega^m \kappa(\theta)$ for $m < k$ into a Taylor series up to the order $k-m$ with the error estimate of $o(|\theta|^{k-m+\delta})$ to see that

$$(\nabla_\omega^m \kappa)\left(\frac{\theta}{\sqrt{n}}\right) = (\nabla_\omega^m)\kappa(0) + \cdots + \frac{(\theta\cdot\nabla)^{k-m}(\nabla_\omega^m \kappa)(0)}{(k-m)!\sqrt{n}^{k-m}} + o\left(\frac{|\theta|^{k-m+\delta}}{\sqrt{n}^{k-m+\delta}}\right)$$

and for each $\nu = 0, 1, 2, \ldots$,

$$\begin{aligned}\kappa^{n-\nu}\left(\frac{\theta}{\sqrt{n}}\right) \quad &= \quad e^{-\frac{1}{2}Q(\theta)}\left[1 + \sum_{j=1}^{k-2}\frac{\{\theta^{j+2}\} + \cdots + \{\theta^{3j}\}}{\sqrt{n}} + o\left(\frac{1+|\theta|^{k+\delta}}{\sqrt{n}^{k-2+\delta}}\right)\right] \times \\ &\times \left(1 + \frac{\nu}{2n}Q(\theta) + \cdots + \frac{\{\theta^k\}}{n^{k/2}} + o\left(\left(\frac{|\theta|}{\sqrt{n}}\right)^{k+\delta}\right)\right),\end{aligned} \tag{47}$$

as $n \to \infty$ uniformly for $|\theta| \le n^{1/6}$.
The function $n^{-k/2}(\nabla_\omega^k \kappa^n)\,(\theta/\sqrt{n})$ is expanded in the form

$$I_n(\theta) := \frac{1}{n^{k/2}}\sum_\alpha C_\alpha\,(n)_\nu\,\kappa^{n-\nu}\left(\frac{\theta}{\sqrt{n}}\right)\prod_{j=1}^{\nu}((\nabla_\omega)^{\alpha_j}\kappa)\left(\frac{\theta}{\sqrt{n}}\right) \tag{48}$$

where $(n)_\nu = n(n-1)\cdots(n-\nu+1)$; the summation extends over all the multi-indices $\alpha = (\alpha_1, ..., \alpha_\nu)$ such that

$$1 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_\nu; \quad \alpha_1 + \cdots + \alpha_\nu = k;$$

$\nu = \nu_a \in \{1, 2, ..., k\}$; and C_α is a certain constant associated with α. Let $\ell = \ell(\alpha)$ denote the number of 1 in the sequence $\alpha_1, ..., \alpha_\nu$. Then we have the dichotomy

(1) $\quad \ell + k = 2\nu \quad$ if $\alpha_i \le 2$ for every i,

(2) $\quad \ell + k > 2\nu \quad$ if $\alpha_i \ge 3$ for some i.

We accordingly decompose $I_n = I_n^{(1)} + I_n^{(2)}$.
The corresponding expansion of $(\nabla_\omega)^k(e^{-\frac{1}{2}Q(\theta)})$ is given by

$$II_n(\theta) := \sum_\alpha C_\alpha(-\omega\cdot Q\theta)^\ell[-Q(\omega)]^{\nu-\ell}e^{-\frac{1}{2}Q(\theta)}$$

where the sum is restricted to those α for which $\alpha_i \le 2$ for every i.

For the rest of the proof we suppose that $k \geq 3$ (the argument is easily adapted to the case $k = 2$). Noticing that $n^{-k/2}(n)_\nu = \sqrt{n}^{\ell}(1 + O(1/n))$ if $\ell + k = \nu$, and

$$\sqrt{n}\nabla_\omega \kappa(\theta/\sqrt{n}) = -\omega \cdot Q\theta + \left(\{\theta^2\}/\sqrt{n}\right)(1 + o(1)),$$

$$(\nabla_\omega)^2 \kappa(\theta/\sqrt{n}) = -Q(\omega) + \left(\{\theta\}/\sqrt{n}\right)(1 + o(1)),$$

we observe that $I_n^{(1)} - II_n$ is of the form

$$e^{-\frac{1}{2}Q(\theta)}\left[\sum_{j=1}^{k-2} P_j^1(\theta)/\sqrt{n}^j + o\left(n\left(\frac{|\theta|}{\sqrt{n}}\right)^{k+\delta}\right)\right]$$

like that required for D_n. As for the terms of $I_n^{(2)}$, since $|\nabla^j \kappa| \leq C_j$ $(j = 1, 2, ..., k)$, from each factor $(\nabla_\omega^m)\kappa(\theta/\sqrt{n})$ with $m > 2$ there arises a factor $1/\sqrt{n}^{m-2}$, so that $I_n^{(2)}$ is also of the same form as above (but without the term of order $1/\sqrt{n}$). These together with the smoothness of M concludes that D_n is given as in (44) with $P_{n,k}(\theta)$ of the form $\sum_{j=1}^{k-2} P_j(\theta)/\sqrt{n}^j$. (A little reflection shows that the highest degree term of $P_j(\theta)$ equals $(-\omega \cdot Q\theta)^k \times \{\theta^{3j}\}$ (notice that $(\nabla_\omega \kappa)(\theta/\sqrt{n}) = -\omega \cdot Q\theta/\sqrt{n} + \cdots$), so that its degree is $k + 3j$.) It remains to show (46). But this follows from the the facts that among the factors in (48) κ and $(\nabla_\omega)^{\alpha_j}\kappa$ with $\alpha_j < k$ are differentiable and

$$\nabla_\omega^k \kappa(y) - \nabla_\omega^k \kappa(y') = o(|y - y'|^\delta) \quad (y - y' \to 0).$$

This completes the proof of the lemma. □

We resume the proof of Theorem 4. First we prove it in the case $\delta = 0$. Recalling the basic relations (42), (43), and (22) it is routine (as indicated in the case $|x|/\sqrt{n} < C$) to deduce from Lemma 7 that uniformly for $|x|/\sqrt{n} > C$,

$$\begin{aligned}
&(2\pi\sigma^2 n)^{d/2} \int_T p^n((\xi, 0), (d\eta, x)) f(\eta) \\
&= \exp\left(-\frac{\|x\|^2}{2\sigma^2 n}\right)\left[\mu(f) + \frac{n^{k/2}}{i^k |x|^k}\tilde{P}_{k,n}(x/\sqrt{n})\right] + \left(\frac{\sqrt{n}}{|x|}\right)^k \left[o\left(\frac{1}{\sqrt{n}^{k-2}}\right) + O\left(e^{-\varepsilon n^{1/3}}\right)\right],
\end{aligned}$$

where $\tilde{P}_{k,n}(y)$ is the polynomial appearing in the Fourier transform of $e^{-\frac{1}{2}Q(\theta)}P_{k,n}(\theta)$. Since both the formula (15) of Theorem 4 and the above one are valid uniformly for $C < |x|/\sqrt{n} < C^{-1}$ with arbitrary $0 < C < 1$, the polynomial following $e^{-\|x\|^2/2n\sigma^2}$ (as its multiple) on the right side of the latter must agree with that of the former within the indicated error estimate. This yields the required formula in the case $\delta = 0$.

In the case $\delta > 0$. Let $R_{n,k}$ be the function introduced in Lemma 7. It suffices to prove that for $r := |x| > C\sqrt{n}$,

$$J_n^{(1)} = \sqrt{n}^d \int_{n^{1/6}\Delta} e^{-\frac{1}{2}Q(\theta)} R_{n,k}(\theta) e^{-ix\cdot\theta/\sqrt{n}} d\theta = o\left(\frac{1}{\sqrt{n}^k r^\delta}\right),$$

and

$$J_n^{(2)} = \sqrt{n}^d \int_{\sqrt{n}\Delta \setminus n^{1/6}\Delta} (\nabla_\omega^k \Psi_n)\left(\frac{\theta}{\sqrt{n}}\right) e^{-ix\cdot\theta/\sqrt{n}} d\theta = o\left(\frac{1}{\sqrt{n}^k r^\delta}\right). \tag{49}$$

For the first estimate we are to apply (46). To this end we set $\eta_{n,r} = (\pi\sqrt{n}/r)\omega$, where $\omega = x/r$, Then, by virtue of the factor $e^{-\frac{1}{2}Q(\theta)}$ in the integrand, the relation $\eta_{n,r}\cdot x/\sqrt{n} = \pi$, and the fact that the volume of the symmetric difference of $n^{1/6}\Delta$ and $n^{1/6}\Delta - \eta_{n,r}$ is $O(r^{-1}\sqrt{n}\cdot n^{(d-1)/6})$, we see that

$$\begin{aligned} J_n^{(1)} &= -\sqrt{n}^{\,d}\int_{n^{1/6}\Delta} e^{-\frac{1}{2}Q(\theta+\eta_{n,r})}R_{n,k}(\theta+\eta_{n,r})e^{-ix\cdot\theta/\sqrt{n}}d\theta + O(r^{-1}e^{-\varepsilon n^{1/3}}) \\ &= \frac{\sqrt{n}^{\,d}}{2}\int_{n^{1/6}\Delta}\Big[e^{-\frac{1}{2}Q(\theta)} - e^{-\frac{1}{2}Q(\theta+\eta_{n,r})}\Big]R_{n,k}(\theta+\eta_{n,r})e^{-ix\cdot\theta/\sqrt{n}}d\theta \\ &\quad + \frac{\sqrt{n}^{\,d}}{2}\int_{n^{1/6}\Delta} e^{-\frac{1}{2}Q(\theta}\Big[R_{n,k}(\theta) - R_{n,k}(\theta+\eta_{n,r})\Big]e^{-ix\cdot\theta/\sqrt{n}}d\theta \\ &\quad + O(r^{-1}e^{-\varepsilon n^{1/3}}). \end{aligned}$$

Since $R_{n,k}(\theta+\eta_{n,r}) = o\Big((1+|\theta|)^{k+\delta}/\sqrt{n}^{\,k+\delta}\Big)$ provided that $r > C\sqrt{n}$, the first term on the right side is $o(|\eta_{n,r}|/\sqrt{n}^{\,k+\delta}) = o(1/\sqrt{n}^{\,k}r^{\delta})$. On using (46) the second term also is $o(1/\sqrt{n}^{\,k}r^{\delta})$. Thus the required estimate for $J_n^{(1)}$ is established.

For the verification of (49) we set $\varphi_n(\theta) = (\nabla_\omega^k\Psi_n)(\theta/\sqrt{n})$. Since Ψ_n is periodic, we have in the same way as above

$$\begin{aligned} &\sqrt{n}^{\,d}\int_{\sqrt{n}\,\Delta\,\setminus\, n^{1/6}\Delta}\varphi_n(\theta)e^{-ix\cdot\theta/\sqrt{n}}d\theta \\ &= \frac{1}{2}(\sqrt{n}\,)^d\int_{\sqrt{n}\,\Delta\,\setminus\, n^{1/6}\Delta}[\varphi_n(\theta) - \varphi_n(\theta+\eta_{n,r})]e^{-ix\cdot\theta/\sqrt{n}}d\theta + O(r^{-1}e^{-\varepsilon n^{1/3}}). \end{aligned}$$

For every $\varepsilon > 0$, if $|\theta| < \varepsilon\sqrt{n}$, then $|\varphi_n(\theta) - \varphi_n(\theta+\eta_{n,r})| \le C_3 e^{-\varepsilon_1|\theta|^2}r^{-\delta}$, which shows that the last integral restricted to $\varepsilon\sqrt{n}\,\Delta\setminus n^{1/6}\Delta$ is dominated by $C_3r^{-\delta}e^{-\varepsilon' n^{1/3}}$. By Lemma 23 the same integral but restricted to $\sqrt{n}\,\Delta\setminus\varepsilon\sqrt{n}\,\Delta$ is estimated as $O(r^{-\delta}e^{-\varepsilon n})$. These verify (49). The proof of Theorem 4 is now complete. $\square$

4 Cyclic Transitions of ξ_n

We here advance formal analytical procedures for dealing with the case when the process ξ_n cyclically moves, although what modification is to be done is intuitively clear.

In a such case as describes in Remark 1 (namely $p_T(\xi, T_j) = \mathbf{1}(\xi\in T_{j-1})$ for $j = 1,\dots,\tau$ (mod τ) with $\tau\ge 2$), we have

$$\int \chi_{j-1}(\xi)\mu(d\xi)p_T(\xi, d\eta) = \chi_j(\eta)\mu(d\eta) \quad \text{where} \quad \chi_j(\xi) = \mathbf{1}(\xi\in T_j). \tag{50}$$

Any powers of $\omega = e^{i2\pi/\tau}$ are eigenvalues, and

$$e_j(\xi) = \sum_{k=0}^{\tau-1}\omega^{kj}\chi_k(\xi)$$

is the eigenfunction of eigenvalue ω^j; these are orthonormal in $L^2(\mu)$, so that $\Pi_j = e_j \otimes (\bar{e}_j\mu)$ (namely $\Pi_j f = (\int f\bar{e}_j d\mu)e_j$), $j = 0, \dots, \tau - 1$, are projection operators that are orthogonal to one another. By (50) these also commute with $\mathbf{p}$. The remainder $\mathbf{r}$ which is defined by

$$\mathbf{p} = \Pi_0 + \omega\Pi_1 + \cdots + \omega^{\tau-1}\Pi_{\tau-1} + \mathbf{r}$$

commutes with $\mathbf{p}$ and its spectral radius (in $L^\infty(\mu)$) is less than 1 by virtue of (H.1$''$) (see (88)). Since $\Pi_0 h = \mu(h) = 0$ and the divergent series $1 + \omega^j + \omega^{2j} + \cdots$ is summable to $(1 - \omega^j)^{-1}$ ($j \neq 0(\mod \tau)$) in the Abel summation method, the function c is given by

$$c = (1 - \omega)^{-1}\Pi_1 h + \cdots + (1 - \omega^{\tau-1})^{-1}\Pi_{\tau-1} h + \sum_{n=1}^{\infty} \mathbf{r}^n h.$$

On estimation of the Green function. In the proof of Theorems 1 and 2 given in Section 2 we supposed that $\tau = 1$. In the general case we have only to slightly modify it (also cf. Remark 6). Indeed, instead of (21) we have

$$\mathbf{p}_\theta = \kappa(\theta)e_\theta \otimes \mu_\theta + \kappa_1(\theta)\Pi_1^\theta + \cdots + \kappa_{\tau-1}(\theta)\Pi_{\tau-1}^\theta + \mathbf{r}_\theta, \tag{51}$$

where Π_j^θ are the projection operators that commutes with one another and also with $\mathbf{p}_\theta$ as well as $e_\theta \otimes \mu_\theta$, and converges to Π_j as $|\theta| \to 0$ and also $\kappa_j(\theta) \to \omega^j$ as $|\theta| \to 0$. For simplicity consider the case $\tau = 2$. Then this merely gives rise to the additional term $[\kappa_1(\theta)]^n \Pi_1^\theta f(\xi)$ on the right side of (22). Owing to (AP), if $\theta \in \Delta \setminus \{0\}$, then $\limsup \|\mathbf{p}_\theta^n\|_{L^\infty(\mu)}^{1/n} < 1$ (cf. Proposition 20) and in particular $|\kappa_1(\theta)| < 1$. Hence, making summation over n, we have on the right side of (23) the additional term $\kappa_1(\theta)[1 - \kappa_1(\theta)]^{-1}\Pi_1^\theta f(\xi)$, which is sufficiently smooth in a neighborhood of the origin of θ space so that its contribution is negligible.

On proof of local central limit theorem. The local central limit theorem is sensitive to the cyclic motion of ξ_n. For its proof we look at the process in intervals of τ. To be precise consider the MA process $(\xi_{n\tau}, S_{n\tau})$, $n = 0, 1, \dots$ as well as the operator $\mathbf{p}^\tau$, and denote by $h^{(\tau)}, c^{(\tau)}, \mathbf{p}_\theta^{(\tau)}$ the corresponding functions and operator. Then $h^{(\tau)} = h + \mathbf{p}h + \cdots \mathbf{p}^{\tau-1}h$, and $c^{(\tau)} = (1 - \mathbf{p}^\tau)^{-1}h^{(\tau)} = h + \mathbf{p}h + \mathbf{p}^2 h + \cdots$. Hence

$$c^{(\tau)} = c, \quad \mathbf{p}_\theta^{(\tau)} = (\mathbf{p}_\theta)^\tau. \tag{52}$$

By the second identity we have the decomposition

$$\mathbf{p}_\theta^\tau = \lambda_1(\theta)e_\theta^1 \otimes \mu_\theta^1 + \cdots + \lambda_{\tau-1}(\theta)e_\theta^{\tau-1} \otimes \mu_\theta^{\tau-1} + \mathbf{r}_\theta^{(\tau)}. \tag{53}$$

where e_θ^j and μ_θ^j are the eigenfunction and eigen-complex-measure of $\mathbf{p}_\theta^{(\tau)}$, respectively, such that $\chi_k e_\theta^j = \delta_{k,j} e_\theta^j$ and $\chi_k \mu_\theta^j = \delta_{k,j}\mu_\theta^j$ ($j, k = 0, \dots, \tau - 1$; $\chi_k(\xi) = \mathbf{1}(\xi \in T_k)$). On the other hand from (51) we also have

$$\mathbf{p}_\theta^\tau = [\kappa(\theta)]^\tau e_\theta \otimes \mu_\theta + [\kappa_1(\theta)]^\tau \Pi_1^\theta + \cdots + [\kappa_{\tau-1}(\theta)]^\tau \Pi_{\tau-1}^\theta + \mathbf{r}_\theta^\tau.$$

Comparing these two decompositions we first infer that $\mathbf{r}_\theta^{(\tau)} = \mathbf{r}_\theta^\tau$ and then, recalling that $\Pi_j^\theta \to \Pi_j$ as $|\theta| \to 0$ and using the uniqueness theorem for spectral representation of finite dimensional linear operator, that all the coefficients $\lambda_j(\theta)$ and $[\kappa(\theta)]^\tau$ must coincide with one another:

$$\kappa^{(\tau)}(\theta) := \lambda_1(\theta) = \cdots = \lambda_{\tau-1}(\theta) = [\kappa(\theta)]^\tau.$$

Since $\nabla\kappa(0) = 0$, it follows that $\nabla^2\kappa^{(\tau)}(0) = \tau\nabla^2\kappa(0)$, $\nabla^3\kappa^{(\tau)}(0) = \tau\nabla^3\kappa(0)$; in particular (or directly from $E_\xi[\tilde{Y}] = 0$)

$$Q^{(\tau)}(\theta) := E_\mu\Big((S_\tau - c(\xi_0) + c(\xi_\tau))\cdot\theta\Big)^2 = \tau Q(\theta). \tag{54}$$

We apply (38) to $(\xi_{m\tau}, S_{m\tau})$ with θ and f replaced by $\theta/\sqrt{\tau}$ and

$$f_\ell^\theta(\eta) = E_\eta[e^{iS_\ell\cdot\theta/\sqrt{m\tau+\ell}} f(\xi_\ell)],$$

respectively. Let $\mu_j = \tau\chi_j\mu$. Then for $\xi \in T_j$ and $n = m\tau + \ell$,

$$\begin{aligned} E_\xi[e^{iS_n\cdot\theta/\sqrt{n}} f(\xi_n)] &= E_\xi[e^{iS_{m\tau}\cdot(\theta/\sqrt{\tau})/\sqrt{n/\tau}} f_\ell^\theta(\xi_{m\tau})] \\ &= e^{-\frac{1}{2}Q(\theta)}\Big\{\mu_j(f_\ell^\theta) + \frac{1}{\sqrt{m}}\Big[i\Big(c(\xi)\mu_j(f_\ell^\theta) - \mu_j(c^* f_\ell^\theta)\Big)\cdot(\theta/\sqrt{\tau}) \\ &\qquad + \tfrac{1}{6}((\theta/\sqrt{\tau})\cdot\nabla)^3\kappa^{(\tau)}(0)\mu_j(f_\ell^\theta)\Big] + \cdots\Big\}. \end{aligned}$$

Observe that $c^*\mathbf{p}^\ell = c^* - h^*(\mathbf{1} + \mathbf{p} + \cdots + \mathbf{p}^{\ell-1})$ and

$$\mu_j(f_\ell^\theta) = \mu_{j+\ell}(f) + \frac{i}{\sqrt{m\tau}}\tau\mu\Big((h^*\cdot\theta)(\mathbf{1} + \mathbf{p} + \cdots + \mathbf{p}^{\ell-1})(f\chi_{j+\ell})\Big) + O(1/m)$$

(notice the identity $\tau\mu(g\mathbf{p}^k(f\chi_{j+\ell})) = \mu_{j+\ell-k}(g\mathbf{p}^k f)$ (for any g)). We then deduce from these equalities that for $\xi \in T_j$,

$$\begin{aligned} &E_\xi[e^{iS_n\cdot\theta/\sqrt{n}} f(\xi_n)] \\ &= e^{-\frac{1}{2}Q(\theta)}\Big\{\mu_{j+\ell}(f) + \frac{1}{\sqrt{m\tau}}\Big[i\Big(c(\xi)\mu_{j+\ell}(f) - \mu_{j+\ell}(c^* f)\Big)\cdot\theta + \tfrac{1}{6}(\theta\cdot\nabla)^3\kappa(0)\mu_{j+\ell}(f)\Big] + \cdots\Big\}. \end{aligned} \tag{55}$$

By inverting the Fourier transform as carried out in Section 3 we find the formula of Theorem 4 under the condition (AP). It is noticed that the expansion (55) itself is true whether (AP) holds or not.

5 The Case When (AP) Is Violated.

In this section we consider the case when Condition (AP) is violated, in other words, there exists a proper subgroup H for which the condition (7) holds. Throughout this section we denote by H the minimum of such subgroups. (The minimum exists since the class of H satisfying (7) is closed under intersection.) The arguments in this section are mostly algebraic and apply without the condition (H.1) except for the matters that obviously require (H.1) in this paper.

We divide this section into four parts. In the first one we introduce a new MA process, denoted by $(\hat{\xi}_n, \hat{S}_n)$, which is obtained from (ξ_n, S_n) by a simple transformation and prove that Condition

(AP) is satisfied for it. In the next part we present several examples, which exhibit certain possibilities about ergodicity of the process $(\hat{\xi}_n)$. In the third we see that the degenerate case where the dimension of H is less than d may be reduced to a non-degenerate case. In the last part the non-degenerate case is considered. It is shown that if $(\hat{\xi}_n)$ is ergodic then the expansions of the Green function in Section 1 are valid without any modification (this will complete the proof of the results of Section 1); if it is non-ergodic, the expansions are still valid except for a constant factor and for a suitable restriction on the combination of initial and terminal points (depending on an ergodic component). Also as an asymptotic form of transition probability we present a fairly clear picture in the case when T is countable.

5.1. Pick up a representative, $a \in \mathbf{Z}^d$ say, of each coset in the quotient group $\mathbf{Z}^d/H$ and let K be the set of such a's, so that each $x \in \mathbf{Z}^d$ is uniquely represented as $x = y + a$ with $y \in H$, $a \in K$. According to this representation of x we define π_K by $\pi_K(x) = a$. If $\hat{T} = T \times K$, this gives rise to the mapping $T \times \mathbf{Z}^d \mapsto \hat{T} \times H$ which maps $(\xi, x) \in T \times \mathbf{Z}^d$ to $(\hat{\xi}, \hat{y})$ where $\hat{\xi} = (\xi, \pi_K(x))$ and $\hat{x} = x - \pi_K(x)$; and accordingly the new process, $(\hat{\xi}_n, \hat{S}_n)$, taking on values in $\hat{T} \times H$, is induced from (ξ_n, S_n):

$$\hat{S}_n = S_n - a_n \quad \text{and} \quad \hat{\xi}_n = (\xi_n, a_n) \qquad \text{where} \quad a_n = \pi_K(S_n). \tag{56}$$

Clearly $(\hat{\xi}_n, \hat{S}_n)$ is a MA process on $\hat{T} \times H$.

We denote by $\hat{p}_{\hat{T}}$ the transition probability for $\hat{\xi}_n$, by $\hat{P}_{\xi,a}$ the law of $(\hat{\xi}_n, \hat{S}_n)$ starting at $((\xi, a), 0))$ $\in \hat{T} \times H$ and by $\hat{Q}$ the covariance matrix of $\hat{S}_1 - \hat{S}_0 + \hat{c}(\hat{\xi}_1) - \hat{c}(\hat{\xi}_0)$ with obvious notation of $\hat{c}$. Clearly

$$\hat{p}_{\hat{T}}((\xi, a), d\eta \times K) = p_T(\xi, d\eta);$$

from this it follows that if $\hat{\mu}$ is an invariant measure for the $\hat{\cdot}$ process, then

$$\hat{\mu}(d\xi \times K) = \mu(d\xi), \tag{57}$$

which in turn shows $\hat{E}_{\hat{\mu}}[\hat{S}_1 - \hat{S}_0] = 0$.

Proposition 8. *For every invariant measure $\hat{\mu}$, Condition (AP) holds for the process $(\hat{\xi}_n, \hat{S}_n)$ that is regarded as a MA process on $\hat{T} \times H$.*

Proof. We first notice that owing to (7), there exists a measurable function $a : T \times T \mapsto K$ such that

$$P_\mu[\pi_K(Y_1) = a(\xi_0, \xi_1)] = 1. \tag{58}$$

Since $\hat{S}_1 - \hat{S}_0 = Y_1 - a_1 + a_0$, we have $a_1 = \pi_K(a_0 + a(\xi_0, \xi_1))$ a.s. Let H' be a subgroup of H for which $\exists a' \in H, \hat{P}_{\xi,a}[\, \hat{S}_1 - \hat{S}_0 \in H' + a' \,|\, \sigma\{\xi_1, a_1\}] = 1$ $(d\hat{\mu}(d\xi, a)\hat{P}_{\xi,a}$-a.s.). Then, owing to (57) as well as independence of Y_1 from a_0,

$$\exists a' \in H, \; P_\mu[\, Y_1 \in H' + a' + a(\xi_0, \xi_1) \,|\, \sigma\{\xi_0, \xi_1\}] = 1 \quad (P_\mu - \text{a.s.}),$$

so that $H = H'$ by the minimality of H. Hence (AP) holds for $(\hat{\xi}_n, \hat{S}_n)$. $\square$

5.2. Even when S_n is aperiodic in the sense that for every proper subgroup H' of $\mathbf{Z}^d$, $\mu(\{\xi \in T : \exists a \in \mathbf{Z}^d,\, P_\xi[Y_1 \in H' + a] = 1\}) < 1$, there are various cases of the process $\hat{\xi}_n$: it can be cyclic, non-ergodic, or non-cyclic and ergodic as exhibited in the examples given below. If $\hat{\xi}_n$ is not ergodic or $\hat{\tau} > \tau$, the formulas of Theorems 1, 2 and 4 must be suitably modified.

Examples. In these examples T is a quotient group $\mathbf{Z}/k\mathbf{Z} \cong \{0, \dots, k-1\}$ with $k = 2$ or 3; (ξ_n) is noncyclic and S_n is aperiodic in the sense stated above except for the example (5); but Condition (AP) is not satisfied.

(1) Let $T = \{0, 1\}$, $d = 1$; $p_T > 0$ for every entry;

$$q_{\xi,\xi}(\pm 1) > 0, \quad q_{\xi,\xi+1}(x) > 0 \ \text{ for } \ x = 0, \pm 2 \quad \text{and} \quad q_{\xi,\eta}(x) = 0 \ \text{ otherwise.}$$

Then $S_n = \xi_n - \xi_0 + n \pmod 2$ a.s.(P_μ); $H = 2\mathbf{Z}$, $K = \{0, 1\}$; $\hat{\xi}_n$ is ergodic with two cyclically moving subsets

$$\hat{T}_0 = \{(0, 0), (1, 1)\} \quad \text{and} \quad \hat{T}_1 = \{(0, 1), (1, 0)\}.$$

(2) Let $T = \{0, 1, 2\}$, $d = 1$; $p_T(\xi, \eta) > 0$ if and only if $\xi \neq \eta$;

$$q_{\xi,\xi+1}(0) > 0, \quad q_{\xi,\xi+2}(\pm 1) > 0 \quad \text{and} \quad q_{\xi,\eta}(x) = 0 \ \text{ otherwise.}$$

Then $S_n = \xi_n - \xi_0 - n \pmod 2$ a.s.(P_μ); $H = 2\mathbf{Z}$; $\hat{\xi}_n$ is ergodic and noncyclic.

(3) (This example is a skeleton of the one treated in (26) where T is a continuum.)
Let $T = \{0, 1\}$, $d = 2$; $p_T > 0$ for every entry;

$$q_{\xi,\eta}(x) > 0 \ \text{ if } \xi \neq \eta \text{ and } x \in \{\pm 1, \pm i\},$$

$$q_{0,0}(\pm(1+i)) > 0, \quad q_{1,1}(\pm(1+i)) > 0 \ \text{ and } \quad q_{\xi,\eta}(x) = 0 \ \text{ otherwise.}$$

Here complex notation is used so that the lattice points are denoted by $x_1 + ix_2$. Then $H = \{x_1 + ix_2 : x_1 + x_2 = 0 \pmod 2\}$, $K = \{0, 1\}$ (or alternatively $K = \{0, i\}$); $\hat{\xi}_n$ is non-ergodic with two ergodic components:

$$\hat{T}^{(1)} = \{(0, 0), (1, 1)\} \quad \text{and} \quad \hat{T}^{(2)} = \{(0, 1), (1, 0)\}.$$

Condition (AP) remains violated even after we make the δ modification described in Remark 6.

(4) Let $T = \{0, 1, 2\}$, $d = 2$; $p_T > 0$ for every entry;

$$q_{\xi,\xi}((k, 0)) = 0, \ q_{\xi,\xi+1}((k, 1)) > 0, \ q_{\xi,\xi-1}((k, -1)) > 0 \ \text{ for } k = 0, \pm 1;$$

and $q_{\xi,\eta}(x) = 0$ otherwise. Then $H = \mathbf{Z} \times 3\mathbf{Z}$. Observe that the second component of $S_n - S_0$ is congruent to $\xi_n - \xi_0 \pmod 3$ (in other words $a(\xi, \eta) = (0, \eta - \xi)$ in the proof of Proposition 8), so that $(\hat{\xi}_n)$ has three ergodic components. We need to take $n^\circ = 2$ for finding H.

(5) Let $T = \{0, 1\}$, $d = 1$; $p_T(0, 1) = p_T(1, 0) = 1$; $q_{0,1}(\pm 1) > 0, q_{1,0}(\pm 2) > 0$ and $q_{\xi,\eta}(x) = 0$ otherwise. Then ξ_n is cyclic with $\tau = 2$; $H = 2\mathbf{Z}$; and

$$S_n = \lfloor n/2 \rfloor + \mathbf{1}(\xi_0 = 0, \xi_n = 1) \pmod 2 \quad \text{a.s.}(P_\mu),$$

where $\lfloor a \rfloor$ denotes the largest integer that does not exceeds a; $\hat{\xi}_n$ is ergodic and cyclic. (See Lemma 14 of **5.4.3** for general setting to this example.)

It may well be pointed out that c is related to $\hat{c}$ in a simple formula (see (70)) and the matrix Q is often easier to compute by means of the original process (ξ_n, S_n) than the $\hat{}$ process.

5.3. In all the examples above we have $\sharp K < \infty$, which, however, is not generally true. Given a Markov process ξ_n on T satisfying (H.1), we take a measurable function $\varphi : T \mapsto \mathbf{Z}^d$ and an initial random variable S_0 and define S_n by

$$S_n = \varphi(\xi_n) - \varphi(\xi_0) + S_0, \tag{59}$$

which is clearly MA and satisfies (7) with $H = \{0\}$ (moreover for a suitable φ the walk S_n may be irreducible in usual sense if T is large enough). Clearly $\sharp K = \infty$ and $Q = 0$. The converse is also true.

Proposition 9. *Suppose that $\{(\xi, \xi) : \xi \in T\}$ is $\mathcal{T} \times \mathcal{T}$-measurable. If $H = \{0\}$ (namely (7) is satisfied with $H = \{0\}$), then S_n is given in the form (59) with P_μ-probability one.*

Proof. Let $H = \{0\}$. Then there exist $\mathbf{Z}^d$-valued measurable functions $\varphi_n(\xi, \eta)$ (for $n \geq 1$) such that with P_μ-probability one,

$$Y_n = \varphi_1(\xi_{n-1}, \xi_n) \quad \text{and} \quad \varphi_1(\xi_0, \xi_1) + \cdots + \varphi_1(\xi_{n-1}, \xi_n) = \varphi_n(\xi_0, \xi_n). \tag{60}$$

We divide the rest of proof into two steps.

Step 1. By the ergodicity of ξ_n we have $n^{-1}\varphi_n(\xi_0, \xi_n) \to 0$ (P_μ-a.s.) since $E_\mu[\varphi_1(\xi_{n-1}, \xi_n)] = 0$. In this step we deduce from it that

$$\varphi_1(\xi_0, \xi_1)\mathbf{1}(\xi_1 = \xi_0) = 0 \quad (P_\mu\text{-a.s.}), \tag{61}$$

where $\mathbf{1}(\mathcal{S})$ denotes the indicator function of the statement $\mathcal{S}$. Set $A = \{\xi \in T : p_T(\xi, \{\xi\}) > 0\}$. Then by virtue of (60) $\varphi_n(\xi, \xi) = n\varphi_1(\xi, \xi)$ for $\xi \in A$ (μ-a.s.). On the other hand for all $k \geq 1$ and for almost all two points ξ and η from A relative to $\mu(d\xi)p_T^k(\xi, d\eta)$, we have the equality $\varphi_{k+1}(\xi, \eta) = \varphi_1(\xi, \xi) + \varphi_k(\xi, \eta) = \varphi_k(\xi, \eta) + \varphi_1(\eta, \eta)$, showing that $\varphi_1(\xi, \xi)$ for $\xi \in A$ equals a constant element, x° say, except a μ-null set of A. It therefor follows that $\varphi_{n+k}(\xi, \eta) = nx^\circ + \varphi_k(\xi, \eta)$ on $A \times A$ ($\mu(d\xi)p_T^k(\xi, d\eta)$-a.s.). This is consistent to what is stated at the beginning of this step only if $x^\circ = 0$. Thus we have (61).

Step 2. We may suppose $p_T(\xi, \{\xi\}) > 0$ for every $\xi \in T$ since otherwise we have only to consider the δ-transformation in Remark 6, which owing to (61) does not change the function φ_1 with the understanding that $\varphi_1(\xi, \xi) = 0$ for all ξ. It follows that $\varphi_k(\xi, \xi) = 0$ for every ξ and every integer $k \geq 1$. Let $\xi^\circ \in T$ be such that (60) holds for all n with P_{ξ°-probability one. Then for any n and k and for $p_T^n(\xi^\circ, \cdot)$-almost every ξ,

$$\int p_T^k(\xi, d\eta)\mathbf{1}\Big(\varphi_n(\xi^\circ, \xi) + \varphi_k(\xi, \eta) = \varphi_{n+k}(\xi^\circ, \eta)\Big) = 1.$$

By applying the relations $\varphi_k(\xi, \xi) = 0$ and $p_T^k(\xi, \{\xi\}) > 0$, we see that for all n and k,

$$\varphi_n(\xi^\circ, \xi) = \varphi_{n+k}(\xi^\circ, \xi) \quad \text{for } p_T^n(\xi^\circ, \cdot)\text{-almost every } \xi,$$

and infer from this that there exists a function φ such that for all n,

$$\varphi(\xi) = \varphi_n(\xi^\circ, \xi) \quad \text{for } p_T^n(\xi^\circ, \cdot)\text{-almost every } \xi\ .$$

Applying Fubini's theorem we deduce from the equation $\varphi_n(\xi^\circ,\xi)+\varphi_1(\xi,\eta)=\varphi_{n+1}(\xi^\circ,\eta)$ that $\varphi_1(\xi,\eta)=\varphi(\eta)-\varphi(\xi)$ for $p_T^n(\xi^\circ,d\xi)p_T(\xi,d\eta)$-almost all (ξ,η) for all n, hence for $\mu(d\xi)p_T(\xi,d\eta)$-almost all (ξ,η). The proof of the proposition is complete. $\square$

In general H may be isomorphic to $\mathbf{Z}^m$:

$$H \cong \mathbf{Z}^m \quad \text{with} \quad 0<m<d,$$

so that $\sharp K=\infty$ and $Q\neq 0$ whereas Q is not positive definite. In such a case, letting $\bar{H}$ be the largest subgroup of $\mathbf{Z}^d$ such that $H\subset \bar{H}\cong \mathbf{Z}^m$, we can find another subgroup $H^\circ\cong \mathbf{Z}^{d-m}$ so that $\mathbf{Z}^d=\bar{H}+H^\circ$ and $K=K'+H^\circ$ (direct sum) where $K'=\bar{H}/H$ (the quotient group); this induces the decomposition

$$S_n := \hat{S}_n + a'_n + \varphi(\xi_n)-\varphi(\xi_0)+(S_0-\hat{S}_0-a'_0) \tag{62}$$

with φ a function on T taking on values in the lattice H° and a'_n a K'-valued process, such that if $\hat{\xi}_n=(\xi_n,a'_n)$, then $(\hat{\xi}_n,\hat{S}_n)$ is a MA process on $(T\times K')\times H$ which satisfies (AP); clearly $\sharp K'<\infty$. The proof is immediate from Proposition 9.

Example. Let $d=2$ and $H=\{(5k,3k):k\in\mathbf{Z}\}$. Then $K=H^\circ=\{(2k,k):k\in\mathbf{Z}\}\cong\mathbf{Z}$ (hence $K'=\{0\}$).

Lemma 10. *If (ξ_n,S_n) is irreducible (cf. (6)), then $\sharp K<\infty$ and $\hat{\xi}_n$ is ergodic, and vice versa.*

Proof. If $\sharp K=\infty$, (ξ_n,S_n) cannot be irreducible owing to the decomposition (62). If $\hat{\xi}_n$ is not ergodic, (ξ_n,S_n) cannot be irreducible; thus the first half of the lemma. The converse follows from Proposition 8. $\square$

5.4. In what follows we suppose that

$$\sharp K<\infty,$$

which is satisfied under the irreducibility. By virtue of (56) $\lim n^{-1}E_\mu|S_n-\hat{S}_n|^2=0$; hence $Q=\hat{Q}$, in particular Q is <u>*positive definite*</u> according to Proposition 8. Applications of Theorems proven under (AP) to the process $(\hat{S}_n,\hat{\xi}_n)$ yield the expansions of Green functions and transition probabilities of it, from which we can derive those for (S_n,ξ_n). In this subsection we obtain such results in a rather direct way.

Without essential loss of generality we also suppose that S_n is irreducible in the sense that for every proper sub-group H' of $\mathbf{Z}^d$, $P_\mu[Y_1\in H']<1$.

5.4.1. *The Green function in the case when $\hat{\xi}_n$ is ergodic.*

In view of Lemma 10 the following lemma completes the proof of the results of Section 1.

Lemma 11. *Suppose that $\hat{\xi}_n$ is ergodic. Then the expansions of the Green functions in Theorems 1 and 2 and Corollary 3 hold true.*

The expansions in Theorems 1 and 2 and Corollary 3 are derived from the estimates of $E_\xi[e^{iS_n\cdot\theta}f(\xi_n)]$ in a neighborhood $|\theta|<\varepsilon$. In the proof of Theorem 11 we shall see that even in the case when (AP) is violated for the walk S_n the computation based only on such estimates leads to correct results, provided that $\hat{\xi}_n$ is ergodic.

Proof of Lemma 11. We first notice that by (58)

$$\hat{p}_{\hat{T}}((\xi,a),d\eta\times\{b\}) = \mathbf{1}\Big(b=\pi_K(a+a(\xi,\eta))\Big)p_T(\xi,d\eta),$$

and then that if $\hat{\xi}_n$ is ergodic, the unique invariant measure is given by

$$\hat{\mu}(d\xi\times\{a\}) = (\sharp K)^{-1}\mu(d\xi) \tag{63}$$

since the right side is always an invariant measure for $\hat{\xi}_n$.

In view of Proposition 8 we can apply the results of Sections 2 through 4 to the $\hat{\cdot}$ process with a Fourier domain $\hat{\Delta}$ in place of Δ. If $a\in K$, $b=\pi_K(x)$ and $x=\hat{x}+b$,

$$\begin{aligned} P_n((\xi,a),(A,x)) &= \hat{P}_n((\xi,a),0),(A\times\{b\},\hat{x})) \\ &= \frac{\sharp K}{(2\pi)^d}\int_{\hat{\Delta}}\hat{E}_{\xi,a}[e^{i\hat{S}_n\cdot\theta};\,\xi_n\in A, a_n=b]e^{-i\theta\cdot\hat{x}}d\theta \end{aligned} \tag{64}$$

since $|\hat{\Delta}|=(2\pi)^d/\sharp K$. (For the present purpose we may put $a=0$ but this proof will apply to the nonergodic case.) Owing to the relation $\hat{S}_n+b=S_n=S_n-S_0+a$ (P_ξ-a.s.) and the additivity property of the walk S_n, we can rewrite the right side above as

$$\frac{\sharp K}{(2\pi)^d}\int_{\hat{\Delta}}E_\xi[e^{iS_n\cdot\theta};\,\xi_n\in A, a_n=b-a\ (\mathrm{mod}\ K)]e^{-i\theta\cdot(x-a)}d\theta. \tag{65}$$

Clearly $\hat{p}_{\hat{T}}$ satisfies Doeblin's condition, so that the distribution of (ξ_n,a_n) converges to $\hat{\mu}$ geometrically fast. Suppose $\hat{\tau}=1$. Then, we can discard the event $a_n=b-a$ (mod K) and the factor $\sharp K$ simultaneously up to an error of order $o(\rho_1^n)$ (with $0<\rho_1<1$), which results in

$$P_n((\xi,a),(A,x)) = \frac{1}{(2\pi)^d}\int_{\hat{\Delta}}E_\xi[e^{iS_n\cdot\theta};\,\xi_n\in A]e^{-i\theta\cdot(x-a)}d\theta + o(\rho_1^n). \tag{66}$$

In carrying out the Fourier integration we use this expression on the ε-neighborhood of $\theta=0$ and (64) on the rest to follow the computation of Section 2. The case $\hat{\tau}>1$ can be dealt with as before (see Section 4). This proves Lemma 11. □

The next lemma, though not used in this paper, is sometimes useful to translate results for (ξ_n,S_n) to those for $(\hat{\xi}_n,\hat{S}_n)$ and vice versa.

Lemma 12. *If $\hat{\xi}_n$ is ergodic, then $\hat{U}=U$ and for $\xi\in T$ and $a\in K$,*

$$\hat{c}(\xi,a)=c(\xi)+a-g \quad \textit{and} \quad \hat{c}^*(\xi,a)=c^*(\xi)+a-g. \tag{67}$$

where $g:=[\sharp K]^{-1}\sum_{a\in K}a$ (namely $g=\int a\,d\hat{\mu}(\xi,a)$ owing to (63)).

Proof. Define a function φ on $\hat{T}$ by $\varphi(\xi,a)=a$. Then, since $\hat{S}_1-\hat{S}_0=Y_1-a_1+a_0$,

$$\hat{h}(\xi,a)=h(\xi)-\hat{\mathbf{p}}\varphi+a=(1-\hat{\mathbf{p}})(c+\varphi),$$

showing the first relation of (67). (Recall $\hat{\mu}(\hat{c})=0$.) For any bounded function $f(\xi,a)$,

$$\hat{\mu}(\varphi\hat{\mathbf{p}}f)=\hat{E}_{\hat{\mu}}[a_0 f(\xi_1,a_1)].$$

Noticing $\hat{\mathbf{p}}f(\xi,a) = \int p_T(\xi,d\eta)f(\eta,\pi_K(a(\xi,\eta)+a))$ and employing (63), we also have that if $\bar f(\xi) = (\sharp K)^{-1}\sum_a f(\xi,a)$,

$$\hat\mu(c^*(1-\hat{\mathbf{p}})f) = \mu(c^*(1-\mathbf{p})\bar f) = -\hat E_{\hat\mu}[Y_1 f(\xi_1,a_1)].$$

Therefore

$$\hat\mu((c^*+\varphi)(1-\hat{\mathbf{p}})f) = -\hat E_{\hat\mu}[(Y_1 - a_1 + a_0)f(\xi_1,a_1)],$$

which shows the second relation of (67). Taking $f=1$ in (28), making use of (67) and looking at (30) we see $\hat U = U$. □

5.4.2. *The Green function in the case when $\hat\xi_n$ is non-ergodic.*

Let $\hat\xi_n$ be not ergodic. Then $\hat T$ is decomposed into more than one ergodic components. We regard K as the quotient group $\mathbf{Z}^d/H$.

Lemma 13. *Let m be the number of ergodic components of $\hat\xi_n$. Then $m \le \sharp K$ and there exist a subgroup K' of K and a decomposition $T = \sum_{\langle a\rangle\in K/K'} T(\langle a\rangle)$ such that $m = \sharp(K/K')$ and the class of sets*

$$\hat T^{(\langle a\rangle)} := \sum_{b\in K} T(\langle b-a\rangle)\times\{b\} = \sum_{\langle b\rangle\in K/K'} T(\langle b-a\rangle)\times\langle b\rangle, \qquad \langle a\rangle \in K/K'$$

makes the ergodic decomposition of $\hat T = T\times K$, where $\langle a\rangle \in K/K'$ ($a\in K$) is identified with a coset $a+K'$ ($\subset K$); the corresponding invariant measures are given by

$$\hat\mu^{(\langle a\rangle)}(A\times\{b\}) = \frac{1}{\sharp K'}\mu(A\cap T(\langle b-a\rangle)), \qquad A\in\mathcal{T}, b\in K, \tag{68}$$

respectively.

Proof. Pick up an ergodic component $E\subset\hat T$ and set

$$T(a) = \{\xi\in T : (\xi,a)\in E\} \qquad (a\in K),$$

so that $E = \sum_a T(a)\times\{a\}$. Since (ξ_n,a_n) is a MA process on $\hat T$, namely the distribution of the increment $a_n - a_{n-1}$ is determined by the value of ξ_{n-1} independently of a_{n-1}, for every $a\in T$

$$\hat T^{(a)} := \sum_b T(b)\times\{a+b\}$$

must be an ergodic component. If $\hat T^{(a)}$ are distinct from one another, then $m = \sharp K$ and the sets $\hat T^{(a)}$ ($a\in K$) constitute the ergodic decomposition of $\hat T$; if this is the case $T(a)$ must be disjoint since $\hat T^{(a)}$ are disjoint. (68) is a consequence of the fact that the measure given by (63) and hence its restriction to any ergodic component is an invariant measure of $\hat\xi_n$. In general, if $\hat T^{(c)} = \hat T^{(a)}$, then $T(a+b-c) = T(b)$ for every b, so that $\hat T^{(a-c)} = \hat T^{(0)}$. Hence $K' := \{b\in K : \hat T^{(b)} = \hat T^{(0)}\}$ is a subgroup of K and we have only to regard $\hat T^{(a)}$ as being labeled by a representative of the class $a+K'\in K/K'$. □

According to Lemma 13 the formulas in Section 1 is modified as follows.

Let $G^{(\langle a^\circ\rangle)}$ denote the Green function for the process restricted on an irreducible component $\{(\xi,x):(\xi,\pi_K(x))\in\hat T^{(\langle a^\circ\rangle)}\}$, where $\hat T^{(\langle a^\circ\rangle)}$ is an ergodic component for $(\hat\xi_n)$ described in Lemma 13. Then a version of the formula corresponding to (8) is stated as follows: if

$$\pi_K(x)=a,\ \xi\in T(\langle a-a^\circ\rangle),\ \pi_K(y)=b,\ A\subset T(\langle b-a^\circ\rangle) \tag{69}$$

and $w=y-x$, then

$$\begin{aligned}&G^{(\langle a^\circ\rangle)}((\xi,x),(A,y))-\frac{\sharp(K/K')\kappa_d}{\sigma^2\|w\|^{d-2}}\mu(A)\\ &=\frac{\sharp(K/K')}{\|w\|^{d+2}}\int_A\Big[U(w)+(d-2)\kappa_d\|w\|^2(Q^{-1}w)\cdot\Big(c(\xi)-c^*(\eta)\Big)\Big]\,\mu(d\eta)+\cdots\end{aligned}$$

(no change except for the factor $\sharp(K/K')$ and for the restriction on the combinations of the initial point (ξ,a) and the terminal set $A\times\{b\}$) .

The modification for two dimensional case is similar.

For the proof we may proceed as for Theorem 11 except that we divide by $\sharp K'$ instead of $\sharp K$ when we discard the event $a_n=b-a \pmod H$ in the formula (65), which gives rise to the factor $\sharp(K/K')$ to the right side of (66).

We have an analogue of Lemma 12 also in the case when $\hat\xi_n$ is nonergodic: it may read that $\hat U=U$ and

$$\hat c(\xi)=c(\xi)+a-g,\quad \hat c(\xi)=c^*(\xi)+a-g\quad\text{if}\ \ \xi\in T(\langle a-a^\circ\rangle). \tag{70}$$

5.4.3. *Local central limit theorems.*

Under our assumption that $\sharp K<\infty$ we have the following result in place of the decompositions (59) and (62).

Lemma 14. *Suppose that T is countable. There then exist an element e° of K and a mapping φ of T into K such that if $\tau=1$ (namely ξ_n is not cyclic), then*

$$a_n=\varphi(\xi_n)-\varphi(\xi_0)+ne^\circ\pmod H\quad P_\mu\text{-a.s.}; \tag{71}$$

if $\tau>1$, then either (71) or the following relation holds

$$\begin{aligned}a_n\ &=\ \varphi(\xi_n)-\varphi(\xi_0)+(m+\mathbf{1}(\ell+j\ge\tau))e^\circ\pmod H\quad P_\xi\text{-a.s.}\\ &\quad \textit{if}\ \ n=m\tau+\ell,\ \xi\in T_j\ \ (0\le j,\ell<\tau,m\ge 0).\end{aligned} \tag{72}$$

Proof. Let t be the smallest positive integer such that $p_T^t(\xi,\{\xi\})>0$ for some $\xi\in T$ and choose ξ° so that $p_T^t(\xi^\circ,\{\xi^\circ\})>0$. Foe each n there exists a K-valued function, say $\varphi_n(\xi,\eta)$, such that $a_n-a_0=\varphi_n(\xi_0,\xi_n)$ a.s. (φ_1 is the same as $a(\xi,\eta)$ in (58)). We set $\bar e=\varphi_t(\xi^\circ,\xi^\circ)$. Then as in the step 1 of Proposition 9 we find that $\varphi_t(\xi,\xi)=\bar e$ if $p_T^t(\xi,\{\xi\})>0$. The rest of the proof is done in several steps.

Step 1. Let $t=1$. Then necessarily $\tau=1$ and (71) holds with $e^\circ=\bar e$. In fact by the irreducibility of ξ_n we have $\varphi_{n+k}(\xi^\circ,\xi)=k\bar e+\varphi_n(\xi^\circ,\xi)\pmod H$ for all positive k and all sufficiently large n, showing that $\varphi(\xi):=\varphi_n(\xi^\circ,\xi)-n\bar e\pmod H$ is independent of n for all sufficiently large n.

The required relation now follows from this together with the equation $\varphi_{n+1}(\xi^\circ, \eta) = \varphi_n(\xi^\circ, \xi) + \varphi_1(\xi, \eta) \pmod H$.

Step 2. Let $t > 1$ (thus $\xi_1 \neq \xi_0$ a.s.) and set $e' = t^{-1}\bar{e}$. The set $K' := \{ke' + a : a \in K, k = 1, \dots, t-1\}$ $(\subset t^{-1}\mathbf{Z}^d)$ may be naturally considered as a finite additive group in which $ke' + a$ and $\ell e' + b$ are identified if $(k - \ell)e' + a - b = 0 \pmod H$. We show that there exists a mapping φ' of T into K' such that

$$\varphi_1(\xi_0, \xi_1) = \varphi'(\xi_1) - \varphi'(\xi_0) + e' \quad \text{a.s.} \tag{73}$$

For the proof we consider a δ-transform of ξ_n introduced in Remark 6. Let U be a random variable taking on values 0 or 1 with probabilities δ and $1 - \delta$, respectively. Suppose U is independent of (ξ_n, S_n). Let $\xi'_0 = \xi_0$ and define ξ'_1 by $\xi'_1 = \xi_1$ if $U = 1$ and $\xi'_1 = \xi_0$ if $U = 0$. Given $a'_0 \in K'$ arbitrarily, define a'_1 by $a'_1 - a'_0 = a_1 - a_0$ if $U = 1$ and $a'_1 - a'_0 = e'$ if $U = 0$. Since the distribution of $a'_1 - a'_0$ does not depend on a'_0, this determines a MA process on $T \times K'$ such that $\xi'_1 = \xi'_0$ with a positive probability. Therefore, by Step 1 there exists a mapping φ' of T into K' such that $a'_1 - a'_0 = \varphi'(\xi'_1) - \varphi'(\xi'_0) + e'$ *a.s.* But we have $a'_1 - a'_0 = \varphi_1(\xi_0, \xi_1)$ a.s. on the event $\xi'_1 = \xi_1 \neq \xi_0$, whence (73) must hold.

Step 3. Let $t > 1$ and $\tau = 1$. Taking $n = kt + 1$ with k large enough we infer from (73) that $\varphi_n(\xi^\circ, \xi^\circ) = k\bar{e} + e'$. Since both $\varphi_n(\xi^\circ, \xi^\circ)$ and $\bar{e}$ are in K, we have $e' \in K$, hence (71) with $e^\circ = \bar{e}$ and $\varphi = \varphi'$.

Step 4. Let $\tau > 1$. Then as in Step 3 we see $\tau e' \in K$. Owing to the identity $\varphi'(\xi_\tau) = \varphi'(\xi_0) + \tau e^\circ$ together with the irreducibility of $\xi_{n\tau}$ this allows us to choose φ' so that $\varphi'(\xi) \in K$ for $\xi \in T_0$. Define $e^\circ = \tau e'$. Noticing $\varphi'(\eta) - \varphi'(\xi) + ke' \in K$ if $\eta \in T_k$ and $\xi \in T_0$, we define $\varphi(\eta) := \varphi'(\eta) + ke'$ if $\eta \in T_k$ $(0 \leq k < \tau)$, so that φ is K-valued. It is now immediate to see (72). □

Remark **7**. (i) In the case $\tau > 1$ the two conditions (71) and (72) are not exclusive of each other: if $\tau^{-1}e^\circ \in \mathbf{Z}^d$, then on suitably modifying φ and regarding $\tau^{-1}e^\circ$ as an element of K, which we rewrite as e°, the latter is reduced to the former. On recalling that $P_\mu[a_0 = 0] = 1$ these formulas actually give expressions for the increments $a_n - a_0$ when S_0 is not necessarily 0.

(ii) The condition (72) is equivalently expressed as

$$a_k - a_{k-1} = \varphi(\xi_k) - \varphi(\xi_{k-1}) + \mathbf{1}(\xi_k \in T_0)e^\circ. \tag{74}$$

A recipe for finding e° may be found from (71) and (74) as well as in the proof of Lemma 14. In the case when $p_T^\tau(\xi, \xi) > 0$ for some $\xi = \xi^\circ$ in particular, it is given by $e^\circ = a_\tau$ a.s.(P_{ξ°) where a_τ is necessarily nonrandom under the premise.

(iii) A simplest example of (72) is provided by (5) of **5.2** for which $\tau = 2$. Therein we have taken $T_0 = \{1\}, T_1 = \{0\}$ and $\varphi(0) = \varphi(1) = 0$, whereas for the alternative choice $T_0 = \{0\}, T_1 = \{1\}$ we may set $\varphi(j) = \xi_j$, so that $\varphi(1) - \varphi(0) = 1 \pmod 2$, and (72) is written as $a_n = 1 + \lfloor n/2 \rfloor + \mathbf{1}(\xi_0 = 1, \xi_n = 0) \pmod 2$ a.s.(P_μ).

In the rest of this section we suppose that T is **countable**. This supposition is used only through Lemma 14. It is recalled that our only hypothesis here is $\sharp K < \infty$; the irreducibility may fail to hold.

Theorem 15. *Let φ and e° be as in Lemma 14. Suppose that T is countable. Then, without assuming Condition (AP) the local central limit theorems in Section 1 remain true if the right* *sides of the formulas (13) and (15) are multiplied by the function*

$$(\sharp K)\mathbf{1}(x = \varphi(\eta) - \varphi(\xi) + ne^\circ \ (mod\ H))$$

or

$$(\sharp K)\mathbf{1}\Big(x = \varphi(\eta) - \varphi(\xi) + (m + \mathbf{1}(\ell + j \geq \tau))e^\circ \ (mod\ H)\Big)$$

according as (71) or (72) holds.

Proof. Suppose that (71) holds for simplicity. Owing to it the transition probability and the indicator function above vanish simultaneously. To identify the asymptotic form of the former we set $K^\flat = \{u \in [-\pi,\pi)^d : u \cdot x \in (2\pi)\mathbf{Z} \text{ for all } x \in H\}$. From the relation $S_n - a_n \in H$ and $a_n - (\varphi(\xi_n) - \varphi(\xi_0) + ne^\circ) \in H$ (P_μ-a.s.) it then follows that if $u \in K^\flat$,

$$e^{iS_n\cdot(\theta+u)} = e^{i(\varphi(\xi_n)-\varphi(\xi_0)+ne^\circ)\cdot u} e^{iS_n\cdot\theta} \quad P_\mu\text{-a.s.}$$

or on multiplying $f(\xi_n)$ and taking expectation,

$$(\mathbf{p}^n_{\theta+u} f)(\xi) = \mathbf{p}^n_\theta\Big(e^{i(\varphi(\cdot)-\varphi(\xi)+ne^\circ)\cdot u} f\Big)(\xi) \quad \mu\text{-a.s.}. \tag{75}$$

Recalling what is noticed at the beginning of this proof we apply this identity with f such that $f(\eta) = 0$ whenever $x \neq \varphi(\eta) - \varphi(\xi) + ne^\circ$ (mod H), so that it reduces to $\mathbf{p}^n_{\theta+u} f = e^{ix\cdot u}\mathbf{p}^n_\theta f$. Hence we have only to sum up the contributions of the integrals on neighborhoods of $u \in K^\flat$ to see that the same computation as in Section 3 leads to the desired result owing to the identity $\sharp K^\flat = \sharp K$. □

From the relation (75) and Proposition 20 we obtain the following corollary.

Corollary 16. *For Condition (AP) to be true it is necessary and sufficient that the operator norm of $\mathbf{p}_\theta$ is less than one for each $\theta \in \Delta \setminus \{0\}$.*

Proof. If the condition (AP) is violated, then we have (75) with $u \in \Delta \setminus \{0\}$ and, taking $f(\eta) = e^{-i\varphi(\eta)\cdot u}$ and $\theta = 0$ therein, $\mathbf{p}^n_u f(\xi) = e^{i(-\varphi(\xi)+ne^\circ)\cdot u}$, showing that the operator norm of $\mathbf{p}_u$ equals 1. The converse assertion follows from Proposition 20 (with $m = 0$).

It is interesting to describe the ergodic decomposition given in Lemma 13 by means of φ and e° of Lemma 14. Let s be the order of e° and K° the cyclic subgroup of K generated by e°: $K^\circ = \{0, e^\circ, \ldots, (s-1)e^\circ\}$. The following proposition shows that this K° agrees with the subgroup K' of K appearing in Lemma 13 if $(s,\tau) = 1$ (namely s and τ are relatively prime) and (71) holds. We denote by $\langle a\rangle$ the coset $a + K^\circ$.

Proposition 17. *Let s, $\langle a\rangle$ and K° be as above. If the process $(\hat{\xi}_n)$ is ergodic, then $K = K^\circ$. Conversely if $K^\circ = K$, $(s,\tau) = 1$ and (71) holds, then the process $(\hat{\xi}_n)$ is ergodic. Generally, if $(s,\tau) = 1$ and (71) holds, then the set $T(\langle a\rangle)$ introduced in Lemma 13 is given by $T(\langle a\rangle) = \varphi^{-1}(\langle a - a^\circ\rangle)$ for some $a^\circ \in K$. In particular, if $s = 1$, namely $e^\circ = 0$, then $\sum_{a\in K}\varphi^{-1}(\{a - a^\circ\}) \times \{a\}$ is an ergodic component for $(\hat{\xi}_n)$ for each $a^\circ \in K$.*

Proof. We consider only the case $a^\circ = 0$. It is ready to see that $\sum_{\langle a\rangle} \varphi^{-1}(\langle a\rangle) \times \langle a\rangle$ is an invariant set for $(\hat{\xi}_n)$; hence the first assertion of the theorem. Suppose that (71) holds. Then for $(\hat{\xi}_n)$ to be ergodic on this set it is sufficient that for each $a \in K$ and $a' \in \langle 0\rangle$,

$$P_\xi[\,\exists n \geq 0, \varphi(\xi_n) - \varphi(\xi) + ne^\circ + a' = a, \xi_n \in A] > 0 \quad \text{for } \mu\text{-almost all } \; \xi \in \varphi^{-1}(\langle 0\rangle)$$

whenever $A \subset T(\langle a\rangle)$ and $\mu(A) > 0$. For $n = ms + k$ the equation $\varphi(\xi_n) - \varphi(\xi) + ne^\circ + a' = a$ is reduced to $\varphi(\xi_{ms+k}) = a_1 - ke^\circ$ with $a_1 := \varphi(\xi) - a' + a \in \langle a\rangle$; hence by choosing k so that $\mu(A\cap\varphi^{-1}(\{a_1 - ke^\circ\})) > 0$, we infer from the assumption $(s,\tau) = 1$ that $P_\xi[\,\exists m \geq 1, \varphi(\xi_{ms+k}) = a_1 - ke^\circ, \xi_{ms+k} \in A] > 0$ for μ-almost all $\xi \in \varphi^{-1}(\langle 0\rangle)$. Thus the sufficient condition mentioned above is fulfilled. □

If $(s,\tau) \neq 1$, the ergodic decomposition may be finer than that given in Proposition 17 depending on how the cyclically moving sets T_j is related to φ. For instance, if K'' is a subgroup of K such that $K = K' + K''$ (direct sum) and the cyclically moving subsets T_j are of the form $T_j = \varphi^{-1}(-je^\circ + K'')$, then $\sum_{a\in K''} \varphi^{-1}(\langle a - b\rangle) \times \{je^\circ + a\}$ is an ergodic component for each pair (j,b) ($j = 0,\ldots,s$, $b \in K''$). In particular, there can be s distinct ergodic components even in the case $K' = K$.

Remark **8**. In the case $\sharp T < \infty$ a local central limit theorem for MA processes as given in Theorem 15 (but up to the principal order term) is obtained by Krámli and D. Szász (18) under the condition that the covariance matrix Q is positive definite and (ξ_n) makes no cyclic transition. Their approach is somewhat different from ours. Keilson and Wishart (17), studying a central limit theorem for MA processes on $T \times \mathbf{R}$ with $\sharp T < \infty$, show among others that $Q = 0$ if and only if it is degenerate in the sense that the walk is represented as in (59). Our proof is applicable to MA processes on $T \times \mathbf{R}^d$.

6 Derivatives of $\kappa(\theta)$ and $M_\xi(\theta)$ at 0

In this section we compute the derivatives of the principal eigenvalue $\kappa(\theta)$ based on the perturbation method of which we shall review in Appendix D. Let p_T and $\mathbf{p}$ be a probability kernel on T and the bounded operator on $L^\infty(\mu)$ associated with it as defined in Section 1. We suppose that the basic assumptions mentioned in Introduction (i.e., (H.1) and (1)) hold.

Let $\mathbf{p}_\theta$ be the operator with the kernel defined by (20): $\mathbf{p}_\theta f(\xi) = E_\xi[e^{iY_1\cdot\theta} f(\xi_1)]$. Denote its principal eigenvalue by $\kappa(\theta)$ (for $|\theta|$ small enough); let e_θ and μ_θ be the corresponding eigenfunction and its dual object which are normalized so that $\mu(e_\theta) = \mu_\theta(e_\theta) = 1$ as in Section 2.

If $E_\mu|Y_1|^{k+\delta} < \infty$, then the function $\kappa(\theta) = \mu(\mathbf{p}_\theta e_\theta) = \mu_\theta(\mathbf{p}_\theta 1)$ is k-times continuously differentiable and the k-th derivative satisfies that

$$\nabla^k\kappa(\theta) - \nabla^k\kappa(\theta') = o(|\theta - \theta'|^\delta) \tag{76}$$

as $|\theta - \theta'| \to 0$ and similarly for e_θ and μ_θ. (See (89) through (91) in Appendix D.)

From $\mu(e_\theta) = 1$ it follows that $\mu(\nabla^k e_\theta) = 0$ ($k = 1,2,\ldots$). By differentiating both sides of $\kappa(\theta) = \mu(\mathbf{p}_\theta e_\theta)$, we have

$$\nabla\kappa(\theta) = \mu((\nabla\mathbf{p}_\theta)e_\theta) + \mu(\mathbf{p}_\theta\nabla e_\theta), \tag{77}$$

so that $\nabla\kappa(0) = \mu\Big(\nabla\mathbf{p}_\theta 1\Big|_{\theta=0}\Big) = iE_\mu[Y_1]$. Thus $\nabla\kappa(0) = 0$.

Differentiating the relation $\mathbf{p}_\theta e_\theta = \kappa(\theta)e_\theta$ at 0 with the help of $\nabla\mathbf{p}_\theta\Big|_{\theta=0} 1 = ih$ we obtain the identity $ih + \mathbf{p}\nabla e_\theta\Big|_{\theta=0} = \nabla e_\theta\Big|_{\theta=0}$ and rewrite it as $ih = (\mathbf{1}-\mathbf{p})\nabla e_\theta\Big|_{\theta=0}$ to see that $\nabla e_\theta\Big|_{\theta=0} = ic$ (owing to $\mu(\nabla e_\theta) = 0$). Taking this into account, we once more perform differentiation to find that

$$\nabla^2\mathbf{p}_\theta\Big|_{\theta=0} 1 + i2\nabla\mathbf{p}_\theta\Big|_{\theta=0} c = \nabla^2\kappa(0) + (1-\mathbf{p})\nabla^2 e_\theta\Big|_{\theta=0}. \tag{78}$$

(Recall the convention that θ^2 stands for the matrix $(\theta_k\theta_j)_{1\le j,k\le d}$.) Integrating both sides by μ, we conclude that $\nabla^2\kappa(0) = -E_\mu[Y_1^2] - 2E_\mu[Y_1 c(\xi_1)] = -Q$ (see (18)).

Proposition 18. *Suppose that $E_\mu|Y_1|^j < \infty$ for $j = 1, 2$ or 3 in* (i), (ii) *or* (iii) *below , respectively. Then*

(i) $\nabla\kappa(0) = 0$; $\nabla e_\theta\Big|_{\theta=0} = ic$.

(ii) $(\theta\cdot\nabla)^2\kappa(0) = -Q(\theta)$.

(iii) $(\theta\cdot\nabla)^3\kappa(0) = -iE_\mu(\theta\cdot\tilde{Y}_1)^3 + 3i\mu\Big((c^* - c)\cdot\theta\, m_\theta^{(2)}\Big)$

where $m_\theta^{(2)} = E_\xi(\theta\cdot\tilde{Y}_1)^2$. (Recall $\tilde{Y}_1 = Y_1 - c(\xi_0) + c(\xi_1)$.)

If the process (ξ_n, S_n) is symmetric, then $(\theta\cdot\nabla)^3\kappa(0) = 0$.

Proof. The assertions (i) and (ii) have already been proved. For the computation of $(\theta\cdot\nabla)^3\kappa(0)$ it is more transparent to do it by means of the operator $\tilde{\mathbf{p}}_\theta$ that is defined by the kernel $e^{-ic(\xi)\cdot\theta} p_\theta(\xi, d\eta) e^{ic(\eta)\cdot\theta}$, in other words

$$\tilde{\mathbf{p}}_\theta f(\xi) = E_\mu[e^{i\tilde{Y}_1\cdot\theta} f(\xi_1)\,|\,\xi_0 = \xi] \tag{79}$$

for bounded f. The corresponding eigenvectors, which are normalized in the same way as e_θ and μ_θ, are given by $\tilde{e}_\theta = e^{-ic\cdot\theta}e_\theta/\mu(e^{-ic\cdot\theta}e_\theta)$ and $\tilde{\mu}_\theta(f) = \mu(e^{-ic\cdot\theta}e_\theta)\mu_\theta(e^{ic\cdot\theta}f)$ with the eigenvalue $\kappa(\theta)$ being the same. We use analogues of the equalities (77) and (78) for the $\tilde{}$ system. By the identity $\nabla e_\theta|_{\theta=0} = ic$, we see that $\nabla\tilde{e}_\theta|_{\theta=0} = 0$. Twice differentiating the analogue of (77), integrating by μ and making use of the identity $\mu(\nabla^k\tilde{e}_\theta) = 0$ we obtain

$$\nabla^3\kappa(0) = \mu\nabla^3\tilde{\mathbf{p}}_\theta\Big|_{\theta=0} 1 + 3\mu\nabla\tilde{\mathbf{p}}_\theta\Big|_{\theta=0}\Big(\nabla^2\tilde{e}_\theta\Big|_{\theta=0}\Big);$$

also, by the analogue of (78), $(\theta\cdot\nabla)^2\tilde{e}_0(\xi) = (1-\mathbf{p})^{-1}(Q(\theta) - m_\theta)(\xi)$. These combined yield

$$(\theta\cdot\nabla)^3\kappa(0) = -iE_\mu(\theta\cdot\tilde{Y}_1)^3 + 3iE_\mu\Big[(\tilde{Y}_1\cdot\theta)(1-\mathbf{p})^{-1}\left(Q(\theta) - m_\theta^{(2)}\right)\Big].$$

Now (iii) follows from the formula

$$\mu\nabla\tilde{\mathbf{p}}_\theta\Big|_{\theta=0}(1-\mathbf{p})^{-1}f = iE_\mu\Big[\tilde{Y}_1(1-\mathbf{p})^{-1}f(\xi_1)\Big] = i\mu((c-c^*)f),$$

which is easily verified on recalling the definitions of c and c^*. If (ξ_n, S_n) is symmetric, then for $w := E_\mu(\theta\cdot\tilde{Y}_1)^3$ we have $w = -w$, showing $w = 0$, thus $(\theta\cdot\nabla)^3\kappa(0) = 0$. $\square$

Lemma 19. *Let c^* be the function defined by (5). Then*

$$\nabla\mu_\theta(f)\Big|_{\theta=0} = -i\mu(c^* f). \tag{80}$$

Proof. On differentiating $\mu_\theta \mathbf{p}_\theta = \kappa(\theta)\mu_\theta$ it follows that $\Big[\nabla\mu_\theta\mathbf{p} + \mu\nabla\mathbf{p}_\theta\Big]_{\theta=0} = \nabla\mu_\theta\Big|_{\theta=0}$. Noticing $\mu\nabla\mathbf{p}_\theta\Big|_{\theta=0} 1 = 0$, we may rewrite it in the form

$$\nabla\mu_\theta\Big|_{\theta=0} = \mu\nabla\mathbf{p}_\theta\Big|_{\theta=0}(1-\mathbf{p})^{-1}.$$

Substituting the identity $\mu(\nabla\mathbf{p}_\theta\Big|_{\theta=0} f) = iE_\mu[Y_1 f(\xi_1)]$, we see that $\nabla\mu_\theta\Big|_{\theta=0} f = iE_\mu\Big[Y_1(1-\mathbf{p})^{-1}f(\xi_1)\Big] = -i\mu(c^* f)$. Consequently (80) follows. □

Remark 9 Introducing the matrix function $Q_\xi, \xi \in T$ defined by

$$Q_\xi := -\nabla^2\mathbf{p}_\theta 1\Big|_{\theta=0}(\xi) - 2i\nabla\mathbf{p}_\theta c\Big|_{\theta=0} = E_\xi[Y_1^2 + 2Y_1 c(\xi_1)],$$

we obtain from (78) the identity $\nabla^2 e_\theta\Big|_{\theta=0}(\xi) = (1-\mathbf{p})^{-1}(Q - Q_\cdot)(\xi)$. In a similar way we deduce from $\mu_\theta\mathbf{p}_\theta = \kappa(\theta)\mu_\theta$

$$\nabla^2\mu_\theta\Big((1-\mathbf{p})f\Big)\Big|_{\theta=0} = E_\mu\Big[(-Y_1^2 + 2c^*(\xi_0)Y_1 + Q)f(\xi_1)\Big],$$

(f is a $\mathbf{R}$-valued function on T) or; rewriting it,

$$\nabla^2\mu_\theta(f)\Big|_{\theta=0} = \mu\Big(f\cdot(1-\mathbf{p}^*)^{-1}(Q - Q^*_\cdot)\Big), \text{ where } Q^*_\xi = E_\mu\Big[Y_1^2 - 2c^*(\xi_0)Y_1 \Big| \xi_1 = \xi\Big].$$

These together with (80) and the identity $\nabla e_\theta\Big|_{\theta=0} = ic$ yield

$$(\theta\cdot\nabla)^2 M_\xi(0) = \theta\cdot\Big[\mu(f)(1-\mathbf{p})^{-1}(Q - Q_\cdot)(\xi) + \mu\Big(f\cdot(1-\mathbf{p}^*)^{-1}(Q - Q^*_\cdot)\Big) + 2c(\xi)\mu(c^* f)\Big]\theta.$$

(See (24) for M_ξ.) It follows (see (28)) that the third order term in the expansion of G is of the form

$$\frac{\{x^4\}\mu(f)}{\|x\|^{d+4}} + \int_\Delta\left[\frac{\frac{2i}{3}\theta\cdot[c(\xi)\mu(f) - \mu(c^* f)](\theta\cdot\nabla)^3\kappa(0)}{(2\pi)^d Q^2(\theta)} + \frac{(\theta\cdot\nabla)^2 M_\xi(0)}{(2\pi)^d Q(\theta)}\right]e^{-ix\cdot\theta}d\theta,$$

where $\{x^4\}$ is independent of (ξ,η). There is an occasion where this expression is useful (cf. (16)).

7 Estimation of $\int_{|\theta|>\varepsilon}|E[e^{iS_n\cdot\theta}f(\xi_n)]|d\theta$

In this section we prove the following proposition. Recall that $\Delta = [-\pi,\pi)^d$.

Proposition 20. *Let m be a non-negative integer. Suppose that $\sup_\xi E_\xi[|Y_1|^m] < \infty$ and Condition (AP) holds for S_n. Then, for each $\varepsilon > 0$ there exists a positive constant $r < 1$ such that if $f(\xi)$ is bounded,*

$$\sup_{\theta\in\Delta,\,|\theta|>\varepsilon}\sup_{\xi\in T}\left|\nabla^m E_\xi[e^{iS_n\cdot\theta}f(\xi_n)]\right| = O(r^n) \qquad (n\to\infty).$$

In the proof of Proposition 20 given below we need to express Condition (AP) of the walk S_n in terms of the characteristic functions

$$\psi_{\xi,\eta}(\theta) = E_\xi\left[e^{iY_1\cdot\theta}\,\middle|\,\xi_1 = \eta\right]. \tag{81}$$

To this end we first prove a preliminary result that we formulate in a general setting.

Lemma 21. *Let $X_\lambda, \lambda \in \Lambda$ be a family of random variables taking on values in $\mathbf{Z}^d$ and $\nu(d\lambda)$ a probability measure on Λ and suppose that $P[X_\lambda = x]$ is λ-measurable. Denote by F_λ the support of the law of X_λ: $F_\lambda = \{x \in \mathbf{Z}^d : P[X_\lambda = x] > 0\}$. The following two conditions are equivalent.*
(i) $\int_\Lambda \left|E\left[e^{iX_\lambda\cdot\theta}\right]\right|\nu(d\lambda) = 1$ *for some* $\theta \in \Delta\setminus\{0\}$.
(ii) *There is a proper subgroup H of $\mathbf{Z}^d$ such that $\nu(\{\lambda : \exists a \in \mathbf{Z}^d,\ F_\lambda - a \subset H\}) = 1$.*

Proof. For a nonempty set $F \subset \mathbf{Z}^d$, taking any $x \in F$, we denote by $[F]$ the smallest subgroup including $F - x$. Clearly $[F]$ does not depend on x. Now suppose that the equality in (i) of the lemma holds true for a $\theta \in \Delta\setminus\{0\}$ and let H be the set of all x such that $x\cdot\theta \in 2\pi\mathbf{Z}$. Then for ν-almost all λ, $\left|E\left[e^{iX_\lambda\cdot\theta}\right]\right| = 1$, or equivalently, $x\cdot\theta \in 2\pi\mathbf{Z}$ for all $x \in [F_\lambda]$, so that for ν-almost all λ, $[F_\lambda] \subset H$. Since H is a proper subgroup we have (ii). The converse is obvious. □

Corollary 22. *Condition (AP) holds if and only if there exists a positive integer n° such that for each proper subgroup H of $\mathbf{Z}^d$,*

$$P_\mu\left[\exists a \in \mathbf{Z}^d, P_\mu[S_{n^\circ} \in H + a \,|\, \sigma\{\xi_0, \xi_{n^\circ}\}] = 1\right] < 1. \tag{82}$$

Proof. It suffices to show that Condition (AP) is violated if for every $n^\circ \ge 1$ there exists a proper subgroup H for which the probability in (82) equals unity. In view of the preceding lemma this follows from the inequality

$$|E_\xi[e^{iS_m\cdot\theta}\,|\,\xi_m = \eta]\,| \le \int_T \left|E_\xi[e^{iS_k\cdot\theta}\,|\,\xi_k = \xi']\right|\left|E_{\xi'}[e^{iS_{m-k}\cdot\theta}\,|\,\xi_{m-k} = \eta]\right| P_\xi[\xi_k \in d\xi'\,|\,\xi_m = \eta]$$

($1 \le k < m$) since it shows that if the probability in (82) equals unity for $n^\circ = m$, then it does for every $n^\circ \le m$ and since if this condition (with the same m) is satisfied by two subgroups, so is by the intersection of them. □

Studying a Markov chain on $\mathbf{Z}^d$ with a transition law having a certain periodicity structure Takenami (24) introduces a condition analogous to that in the corollary 22 and proves it to be satisfied by the Markov chain under a certain circumstance. Babillot (1) and Givarc'h (6) call a MA process aperiodic if the condition (82) holds with $n^\circ = 1$ for each proper subgroup H.

Proof of Proposition 20. We suppose that the condition of Corollary 22 holds with $n^\circ = 1$ for simplicity. Let $\psi_{\xi,\eta}(\theta)$ be defined by (81). From the preceding lemma it then follows that

$P_\mu[\,|\psi_{\xi_0,\xi_1}(\theta)| < 1] > 0$ for every $\theta \in \Delta \setminus \{0\}$. Since $(-1) \vee \log|\psi_{\xi_0,\xi_1}(\theta)|$ is uniformly bounded and continuous with respect to θ, this yields the inequality

$$\sup_{\theta\in\Delta,\,|\theta|>\varepsilon} E_\mu\Big[(-1) \vee \log|\psi_{\xi_0,\xi_1}(\theta)|\Big] < 0 \tag{83}$$

for every $\varepsilon > 0$.

Define $\mathbf{p}_{\lambda,\theta}$ for $\lambda \geq 0$ by

$$\mathbf{p}_{\lambda,\theta} f(\xi) = E_\xi\Big[\exp\Big\{\lambda\Big((-1) \vee \log|\psi_{\xi_0,\xi_1}(\theta)|\Big)\Big\} f(\xi_1)\Big].$$

From the relation (89) (in Appendix D) applied with $\mathbf{p}_{\lambda,\theta}$ in place of $\mathbf{p}_\theta$ we see that there exists a constant $\lambda_0 \in (0,1]$ such that the circle $C_\rho = \{|1-z| = (1-\rho)/2\}$ encloses an eigenvalue of $\mathbf{p}_{\lambda,\theta}$, $\kappa(\lambda,\theta)$ say, and the rest of the spectrum is inside the circle $|z| = (1+\rho)/2$ for all $\lambda \leq \lambda_0$ and $\theta \in \mathbf{R}^d$ and the projection operator $\Pi_{\lambda,\theta}$ onto the eigenspace of eigenvalue $\kappa(\lambda,\theta)$ is given by the integral of the resolvent along the circle C_ρ as in (90). Hence as before we see that $\kappa(\lambda,\theta)$ is jointly continuous in $(\lambda,\theta) \in [0,\lambda_0] \times \mathbf{R}^d$ and

$$\frac{\partial\kappa}{\partial\lambda}(0,\theta) = E_\mu\Big[(-1) \vee \log|\psi_{\xi_0,\xi_1}(\theta)|\Big].$$

From the decomposition $(\mathbf{p}_{\lambda,\theta})^n = [\kappa(\lambda,\theta)]^n \Pi_{\lambda,\theta} + (\mathbf{r}_{\lambda,\theta})^n$ where the spectral radius of $\mathbf{r}_{\lambda,\theta}$ is less than $(1+\rho)/2$, we see that (uniformly in ξ)

$$\lim_{n\to\infty} \frac{1}{n} \log(\mathbf{p}_{\lambda,\theta})^n 1(\xi) = \log\kappa(\lambda,\theta),$$

which in particular implies that $\kappa(\lambda,\theta)$ is real and non-increasing in λ. These combined with (83) show that

$$\sup_{\theta\in\Delta,\,|\theta|>\varepsilon} \kappa(\lambda_0,\theta) < 1.$$

Now, taking a function $f(\xi)$ such that $\|f\|_\infty \leq 1$ we have

$$|E_\xi[e^{iS_n\cdot\theta} f(\xi_n)]| \leq E_\xi\Bigg[\prod_{j=0}^{n-1} |\psi_{\xi_j,\xi_{j+1}}(\theta)|\Bigg]$$

and also

$$\begin{aligned} E_\xi\Bigg[\prod_{j=0}^{n-1} |\psi_{\xi_j,\xi_{j+1}}(\theta)|\Bigg] &\leq E_\xi\Bigg[\exp\Bigg\{\sum_{j=0}^{n-1} \lambda_0[(-1) \vee \log|\psi_{\xi_j,\xi_{j+1}}(\theta)|]\Bigg\}\Bigg] \\ &= (\mathbf{p}_{\lambda_0,\theta})^n 1(\xi) \\ &\leq [(1+o(1))\kappa(\lambda_0,\theta)]^n, \end{aligned}$$

where $o(1)$ is uniform both in θ and in ξ. Consequently

$$\limsup_{n\to\infty} \frac{1}{n} \sup_{\theta\in\Delta,\,|\theta|>\varepsilon} \sup_{\xi\in T} \log|E_\xi[e^{iS_n\cdot\theta} f(\xi_n)]| \leq \sup_{\theta\in\Delta,\,|\theta|>\varepsilon} \kappa(\lambda_0,\theta) < 1.$$

Thus the assertion of the proposition has been proved in the case $m = 0$. For $m = 1$ we see

$$|\nabla E_\xi[e^{iS_n\cdot\theta}f(\xi_n)]| \le \sum_k E_\xi\Big[\prod_{j=0}^{k-1}|\psi_{\xi_j,\xi_{j+1}}| \times |\nabla\psi_{\xi_k,\xi_{k+1}}| \times \prod_{j=k+1}^{n-1}|\psi_{\xi_j,\xi_{j+1}}|\Big]$$

and we may proceed as above. The cases $m > 1$ are similarly dealt with. □

By the same argument we verify the following

Lemma 23. *Let m be a non-negative integer and $\delta \in [0,1)$. Suppose that $\sup_\xi E_\xi[|Y_1|^{m+\delta}] < \infty$ and S_n satisfies (AP). Then, for each $\varepsilon > 0$ there exists a positive constant $r < 1$ such that if $f(\xi)$ is bounded,*

$$\sup_{\theta\in\Delta,\,|\theta|>\varepsilon}\sup_{\xi\in T}\Big|\nabla^m E_\xi[e^{iS_n\cdot\theta}f(\xi_n)] - \nabla^m E_\xi[e^{iS_n\cdot(\theta+\eta)}f(\xi_n)]\Big| = o(|\eta|^\delta r^n)$$

as $n\to\infty$. Here $o(|\eta|^\delta r^n)$ is uniform for $|\eta| < 1$, so that the infinite sum over $n \ge 0$ of the supremums on the left side is $o(|\eta|^\delta)$.

8 Appendices

A. In this appendix we suppose that $E_\mu|Y_1|^{k+\delta} < \infty$ for an integer $k \ge 2$ and $\delta \in (0,1)$ and indicate a method for computing the Fourier transform $\Big[\Big((1-\kappa)^{-1} - 2/Q\Big)M_\xi\Big]^\wedge(y)$ $(y \in \mathbf{R}^d)$ under this moment condition (cf. (25) in the case $\delta = 0$, where it is somewhat involved if $d = 3$). Under this moment condition M_ξ and κ can be expanded in Taylor polynomial of degree k with the error term $o(|\theta|^{k+\delta})$. We then observe that

$$\Big[\frac{1}{1-\kappa(\theta)} - \frac{2}{Q(\theta)}\Big]M_\xi(\theta) = \frac{\{\theta^3\}}{Q^2(\theta)} + \cdots + \frac{\{\theta^{3(k-1)}\}}{Q^k(\theta)} + \frac{\varepsilon_k(\theta)}{Q^2(\theta)}. \tag{84}$$

Here ε_k is a C^k -class function such that $\varepsilon_k(\theta) = o(|\theta|^{k+\delta})$ and every k-th derivative of it, $\nabla^\alpha\varepsilon_k$ say, satisfies

$$|\nabla^\alpha\varepsilon_k(\theta) - \nabla^\alpha\varepsilon_k(\theta')| \le o(|\theta-\theta'|^\delta) \qquad (\alpha = (\alpha_1,\ldots,\alpha_d), \textstyle\sum\alpha_j = k).$$

For estimation of Fourier integral of the error term $\varepsilon_k(\theta)/Q^2(\theta)$ we repeat integration by parts $k-2$, $k-1$ or k times according as $d = 2$, $d = 3$ or $d \ge 4$. To complete the proof of Theorems 1 and 2 it now is sufficient to prove the next lemma.

Lemma 24. *Let $a(\theta)$, $|\theta| < 3/2$, be a function such that $a(0) = 0$ and for some positive constants $\delta < 1$ and K_a*

$$|a(\theta) - a(\theta')| \le K_a|\theta-\theta'|^\delta,$$

and $h(\theta)$ a bounded, Borel measurable function such that it is differentiable for $\theta \ne 0$ and $|\nabla h(\theta)| \le K_h/|\theta|$. Let $0 \le \nu \le d$. Then

$$\int_{|\theta|\le 1} a(\theta)\frac{h(\theta)}{|\theta|^\nu}e^{-ix\cdot\theta}d\theta \le C|x|^{-\delta}\log|x| \qquad (|x| > 2\pi),$$

where the constant C may be taken as $\lambda_{\nu,d}(K_h + \|h\|_\infty)K_a[(1-\delta)\delta]^{-1}$ with a constant $\lambda_{\nu,d}$ depending only on ν and d. If either $\nu < d$ or a is differentiable for $\theta \ne 0$ with $|\nabla a(\theta)| = O(|\theta|^{-1+\delta})$, then the right hand side may be replaced by $C'|x|^{-\delta}$.

Proof. First we compute the integral restricted on $\{\theta : |x\cdot\theta| < \pi\}$. Put

$$f(\theta) = a(\theta)h(\theta)/|\theta|^{\nu}.$$

Then, on using $|a(\theta)| \le K_a|\theta|^{\delta}$ and $\nu \le d$,

$$\left|\int_{|x\cdot\theta|<\pi,|\theta|\le 1} f(\theta)e^{-ix\cdot\theta}d\theta\right| \le K_a\|h\|_\infty \int_{|x\cdot\theta|\le\pi,\ |\theta|\le 1} \frac{d\theta}{|\theta|^{d-\delta}} \le \frac{C_1K_a}{\delta}\left(\frac{\pi}{|x|}\right)^{\delta}, \tag{85}$$

where $C_1 = 2\|h\|_\infty \int_{\mathbf{R}^{d-1}} d\theta'/\sqrt{1+|\theta'|^2}^{\,d-\delta} \le \lambda_d\|h\|_\infty/(1-\delta)$. For the remainder of the integral we may restrict it to $\{\theta : x\cdot\theta > \pi\}$ since the other half is similar. Thus we consider the integral

$$J = \int_{x\cdot\theta>\pi,\ |\theta|\le 1} f(\theta)e^{-ix\cdot\theta}d\theta.$$

Let $r = |x|$ and $\omega = x/r$ and shift the variable θ by $\omega\pi/r$ so that $x\cdot\theta$ is transformed to $x\cdot\theta+\pi$ and the integral is to

$$J = -\int_{x\cdot\theta>0,\ |\theta+\omega\pi/r|\le 1} f(\theta+\omega\pi/r)e^{-ix\cdot\theta}d\theta,$$

which differs from $-\int_{x\cdot\theta>\pi,\ |\theta|\le 1} f(\theta+\omega\pi/r)e^{-ix\cdot\theta}d\theta$, at most by

$$\int_{|\theta|\le 1}[I(\pi < x\cdot\theta \le 2\pi) + I(1<|\theta|<1+\pi/r)]|f(\theta)|d\theta \le \frac{C_1K_a(2\pi)^{\delta}}{\delta r^{\delta}} + \frac{\lambda_d\|ah\|_\infty}{r}.$$

Accordingly

$$J = \frac{1}{2}\int_{x\cdot\theta>\pi,\ |\theta|\le 1}[f(\theta) - f(\theta+\omega\pi/r)]e^{-ix\cdot\theta}d\theta + O(r^{-\delta}).$$

Now we apply the assumptions on h and a to have

$$|f(\theta) - f(\theta+\omega\pi/r)| \le \frac{(\nu\|h\|_\infty + K_h)K_a}{|\theta|^{\nu+1-\delta}r} + \frac{\|h\|_\infty K_a}{|\theta|^{\nu}r^{\delta}}$$

valid for θ satisfying $x\cdot\theta > \pi$. The estimate of the lemma is immediately inferred from the following ones: $\int_{|\theta|\le 1}|\theta|^{-\nu}d\theta < \infty$ if $\nu < d$;

$$\int_{x\cdot\theta>\pi,\ |\theta|\le 1}|\theta|^{-(d+1-\delta)}d\theta = O(r^{1-\delta}); \quad \int_{x\cdot\theta>\pi,\ |\theta|\le 1}|\theta|^{-d}d\theta = O(\log(1/r)).$$

If a is differentiable for $\theta \ne 0$ with $|\nabla a(\theta)| = O(|\theta|^{-1+\delta})$, then the integration by parts directly gives the required estimate of J. This completes the proof of the lemma.

B. Our evaluation of Fourier integrals on the torus Δ made in Section 3 is based on the following two formulae (i) and (ii) as well as the results of the preceding section: the former two are used to dispose of the integral on a neighborhood of origin and the latter ones are on the rest.

(i) Let D be a d-dimensional bounded domain containing the origin and having piece-wise smooth boundary ∂D. Let g be a function on $\mathbf{R}^d$ of the form $\{\theta^k\}/|\theta|^{s+k}$ with k a non-negative integer and s a real number such that $s < d$. Then for every integer n satisfying $n \geq d - s$,

$$\int_D g(\theta)e^{ix\cdot\theta}d\theta = g^\wedge(x) + \sum_{l=1}^{n} \frac{B_{D,l}(x)}{(i|x|)^l} - \frac{1}{(i|x|)^n}\int_{D^c}(-\omega\cdot\nabla)^n g(\theta)e^{ix\cdot\theta}d\theta \qquad (x \in \mathbf{R}^d).$$

Here $\omega = x/|x|$ and $B_{D,l}$ denotes the boundary integral $\int_{\partial D}(-\omega\cdot\nabla)^n g(\theta)e^{ix\cdot\theta}\omega\cdot d\mathbf{S}$; if $n = d-s$ the last integral is not absolutely convergent and must be understood to be the principal value; $g^\wedge$ denotes the Fourier transform $\int_{\mathbf{R}^d} g(\theta)e^{ix\cdot\theta}d\theta$ in the sense of Schwartz distribution on the punctured space $\mathbf{R}\setminus\{0\}$, namely $g^\wedge(x)$ $(x \neq 0)$ is identified by the relation $\int g\varphi^\wedge\, dx = \int g^\wedge\, \varphi dx$ to hold for every smooth function φ that vanishes outside a compact set of $\mathbf{R}^d \setminus \{0\}$. Proof is standard (see eg. Lemma 2.1 of (25)).

(ii) If φ_k is a homogeneous harmonic polynomial of degree k, then for $s \in \mathbf{R}$,

$$\frac{1}{(2\pi)^d}\int_{\mathbf{R}^d}\frac{\varphi_k(\theta)e^{-ix\cdot\theta}}{|\theta|^{2s}}d\theta = \frac{i^k\lambda_k(s)\varphi_k(-x)}{|x|^{d+2(k-s)}} \qquad (k = 0,1,2,\ldots; s < \tfrac{1}{2}(k+d)),$$

where

$$\lambda_k(s) = \frac{\Gamma(\frac{1}{2}d + k - s)}{\pi^{d/2}2^{2s-k}\Gamma(s)} \quad \text{if} \quad s \notin \{0,-1,-2,\ldots\}$$

and $\lambda_k(s) = 0$ otherwise; the Fourier transform is in the sense of distribution on $\mathbf{R}^d \setminus \{0\}$ as in (i). (Cf. (23) in the case $k < 2s$; for the other case make analytic continuation as a function of s.) It is recalled that any homogeneous polynomial $\{\theta^n\}$ is expressed in a finite sum of the form $\varphi_n(\theta) + |\theta|^2\varphi_{n-2} + \cdots$.

C. Here is given proofs of (33) and (34). For the first one it suffices to prove that

$$\sum_{n=1}^{\infty}\left(\frac{e^{-r^2/n}}{n} - \frac{1}{n}\right) = -2\log r - 2\gamma + O(r^{-4}) \tag{86}$$

as $r \to \infty$. For the proof we make the decomposition

$$\sum_{n=1}^{\infty}\left(\frac{e^{-r^2/n}}{n} - \frac{1}{n}\right) = \sum_{1\leq n\leq r^2}\frac{e^{-r^2/n}}{n} - \sum_{1\leq n\leq r^2}\frac{1}{n} + \sum_{n>r^2}^{\infty}\left(\frac{e^{-r^2/n}}{n} - \frac{1}{n}\right).$$

When r is large, the first term on the right side may be written as

$$\sum_{1\leq n\leq r^2}\frac{e^{-r^2/n}}{n} = \sum_{1\leq n\leq k}\frac{e^{-r^2/n}}{n/r^2}\frac{1}{r^2} = \int_{\frac{1}{2}/r^2}^{(k+\frac{1}{2})/r^2}\frac{1}{u}e^{-1/u}du + O(r^{-4}), \tag{87}$$

where $k = \lfloor r^2 \rfloor$ (the largest integer that does not exceed r^2) and similarly for the last term. By elementary computation we see that for $a > 0$

$$\int_0^a \frac{1}{u}e^{-1/u}du + \int_a^\infty \frac{1}{u}\left[e^{-1/u} - 1\right]du = \int_0^\infty (\log t)e^{-t}dt + \log a,$$

where the first term on the right side equals $-\gamma$. Taking $a = (k+\frac{1}{2})/r^2$ in the last formula and combining it with observations preceding it we conclude that

$$\sum_{n=1}^{\infty}\left(\frac{e^{-r^2/n}}{n} - \frac{1}{n}\right) = -\gamma + \log\frac{k+\frac{1}{2}}{r^2} - \sum_{n=1}^{k}\frac{1}{n} + O(r^{-4}) \qquad (k = \lfloor r^2 \rfloor).$$

The required relation (86) now follows from this by substituting $\sum_{n=1}^{k}\frac{1}{n} = \gamma + \log(k+1/2) + \frac{1}{24k^2} + \cdots$.

For the proof of (34) set $f(u) = \frac{1}{u}e^{-r^2/\tau u}$ and $F(t) = \sum_{m=1}^{\infty}[f(m+t) - f(m)]$. Then by the same argument as made in (87), $F'(t) = \sum_{m=1}^{\infty} f'(m+t) = \int_{\frac{1}{2}/r^2}^{\infty} f'(u+t)du + O(r^{-4}) = O(r^{-4})$ as $r \to \infty$ uniformly for $0 \le t \le 1$. Hence $F(\ell/\tau) - F(j/\tau) = O(r^{-4})$, showing (34) .

D. In this appendix we briefly review a standard perturbation method. Let p_T and $\mathbf{p}$ be a probability kernel on T and the bounded operator on $L^\infty(\mu)$ associated with it as defined in Section 1. From the supposition that p_T has a unique invariant probability measure, it follows that the eigenfunction of $\mathbf{p}$ with eigenvalue 1 is constant. Suppose (H.1$''$) to be true (cf. Remark 1) and define the operator Π by $\Pi = \Pi_0 + \omega\Pi_1 + \cdots + \omega^{\tau-1}\Pi_{\tau-1}$ if there is τ (≥ 2) cyclically moving classes for the process ξ_n (see Section 4 for Π_j and ω) and $\Pi = 1 \otimes \mu$ otherwise. Since $\mathbf{p}\Pi_j = \Pi_j\mathbf{p} = \omega^j\Pi_j$, we have $\Pi\mathbf{p} = \mathbf{p}\Pi = \Pi^2$. By a simple computation we see

$$\Pi^\ell = \sum_{k=0}^{\tau-1}\omega^{\ell k}\Pi_k = \sum_{k=0}^{\tau-1}\chi_{k-\ell}\otimes\mu_k \qquad (\ell = 0, 1, \ldots, \tau-1);$$

in particular $\Pi^\tau = \Pi_0 + \Pi_1 + \cdots + \Pi_{\tau-1} = \sum_{k=0}^{\tau-1}\chi_k \otimes \mu_k$ is the projection to the finite dimensional space spanned by eigenfunctions of eigenvalues $1, \omega, \ldots, \omega^{\tau-1}$. Since $(\mathbf{p}-\Pi)^n = \mathbf{p}^n - \Pi^n = \mathbf{p}^{m\tau+\ell} - \Pi^\ell$ for $n = m\tau + \ell$ ($\ell = 0, \ldots, \tau-1$), it follows from (H.1$''$) that

$$\limsup \|(\mathbf{p}-\Pi)^n\|_{L^\infty(\mu)}^{1/n} \le \rho\,(<1) \tag{88}$$

($\|\cdot\|_{L^\infty(\mu)}$ denotes the operator norm in $L^\infty(\mu)$). Thus the spectrum other than $1, \omega, \ldots, \omega^{\tau-1}$ is enclosed by the circle $|z| = \rho$. If $z \notin \{1, \omega, \ldots, \omega^{\tau-1}\}$ with $|z| > \rho$, the resolvent $R(z) = (\mathbf{p}-z)^{-1}$ can be represented by a complex measure kernel. Indeed, this is certainly true if $|z| > 1$, and for all the other values of z to be considered the resolvents are analytic continuations of one another by means of finite applications of the formula $R(z_2) = \sum_{n=0}^{\infty}(z_2 - z_1)^n R^{n+1}(z_1)$ (valid at least for z_2 such that $(|z_1 - z_2|\|R(z_1)\|_{L^\infty(\mu)} < 1)$.

Let $\mathbf{p}_\theta$ be the operator with the kernel defined by (20): $\mathbf{p}_\theta f(\xi) = E_\xi[e^{iY_1\cdot\theta}f(\xi_1)]$. Let Σ_θ denote the spectrum of $\mathbf{p}_\theta$ and $R_\theta(z)$ the resolvent operator of $\mathbf{p}_\theta$ at $z \notin \Sigma_\theta$, namely $R_\theta(z) = (\mathbf{p}_\theta - z)^{-1}$. Then by the identity $\mathbf{p}_\theta - z = (\mathbf{p}-z)[I - R(z)(\mathbf{p}-\mathbf{p}_\theta)]$, we see

$$R_\theta(z) = R(z) + \sum_{n=1}^{\infty}[R(z)(\mathbf{p}-\mathbf{p}_\theta)]^n R(z), \tag{89}$$

provided $|\theta|$ is small enough and z is in a compact set of $\mathbf{C}\setminus\Sigma_0$. Since $\mathbf{p}_\theta$ is continuous (or smooth under existence of moments) with respect to θ, $R_\theta(z)$ is also continuous (resp. smooth) in a neighborhood of the origin. Since $\rho < 1$, we can find a positive number $\delta_\circ$ so that if $|\theta| < \delta_\circ$

and $r = \frac{1}{2}\min\{1-\rho, |\omega - 1|\}$, then the spectrum Σ_θ is divided by the circle $C_r = \{|1-z| = r\}$ into two parts in such a manner that the part outside C_r is contained in one of the open disks $|z| < (1+\rho)/2$, $|z - \omega^j| < r$, $j = 1, \dots, \tau - 1$, and the part inside continuously moves to 1 as θ approaches the origin; and that the latter consists of a single eigenvalue, $\kappa(\theta)$ say, which is simple (cf. (12): either p.34 or p.212). Let Π_0^θ denote the projection operator corresponding to this eigenspace. Then $\|(\mathbf{p}_\theta - \Pi_0^\theta)^n\|_{L^\infty(\mu)} \le C[(1+\rho)/2]^n$ and

$$\Pi_0^\theta = e_\theta \otimes \mu_\theta = \frac{1}{2\pi i}\int_{|z-1|=r} R_\theta(z)dz. \tag{90}$$

Here e_θ is an eigenfunction for the eigenvalue $\kappa(\theta)$ and μ_θ is a dual object: these may be defined (if $\delta_\circ$ is small enough) by

$$e_\theta = \frac{1}{\Xi(\theta)}\Pi_0^\theta 1, \quad \mu_\theta = \mu\Pi_0^\theta, \tag{91}$$

with $\Xi(\theta) = \mu\Pi_0^\theta 1$, and the product $e_\theta \otimes \mu_\theta$ stands for the operator given by the complex measure kernel $e_\theta(\xi)\mu_\theta(d\eta)$. They are normalized so that $\mu(e_\theta) = \mu_\theta(e_\theta) = 1$.

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References

[1] M. Babillot, Théorie du renouvellement pour des chaînes semi-markoviennes transientes, Ann. Inst. H. Poincaré (4) **24** (1988), 507-569. MR0978023

[2] E. Cinlar, Markov additive processes I, II, Z.Wahr.verw. Beb. **24** (1972), 85-93, 95-121. MR0329047

[3] J. L. Doob, Stochastic Processes, John Wiley & Sons, New York, 1953. MR0058896

[4] Y. Fukai, Hitting distribution to a quadrant of two-dimensional random walk, Kodai Math. J. **23** (2000), 35-80. MR1749384

[5] Y. Fukai and K. Uchiyama, Potential kernel for two-dimensional random walk, Ann. Probab. **24** (1996), 1979-1992. MR1415236

[6] Y. Givarc'h, Application d'un théoreme limite local à la transience et à la récurrence des marches de Markov, Lecture Notes, **1096**, 301-332, Springer-Verlag, 1984.

[7] K. Ito and H. P. McKean, Potentials and random walks, Illinois J. Math. **4**, (1960), 119-132 MR0121581

[8] B.V. Gnedenko and A.N. Kolmogorov, Limit distributions for sums of independent random variables (translated from Russian), Addison-Wesley, Reading 1954 MR0062975

[9] T. Höglund, A multi-dimensional renewal theorem for finite Markov chains, Ark. Math. (1990), 273-287 MR1084016

[10] T. Hara, R. Hofstad, and G. Slade. Critical two-point functions and the lace expansion for spread-out high dimensional percolation and related models, Ann. Prob., **31** (2003), 349-408. MR1959796

[11] I. A. Ibragimov and Yu. V. Linnik, Independent and stationary sequences of random variables, Wolters-Noordhoff Publishing Groningen, The Netherlands, 1971 MR0322926

[12] T. Kato, Perturbation Theory for Linear Operators, Springer, Berlin-Heidelberg-New York, 1980.

[13] H. Kesten, Renewal theory for functionals of a Markov chain with general state space, Ann. Probab. **2** (1974), 355-386. MR0365740

[14] H. Kesten, Hitting probabilities of random walks on $\mathbf{Z}^d$, Stochastic Process. Appl. **25** (1987), 165-184. MR0915132

[15] H. Kesten, Random difference equations and renewal theory for products of random matrices, Acta Math. **131** (1973), 207-248. MR0440724

[16] T. Kazami and K. Uchiyama, Random walks on periodic graphs. (to appear in TAMS)

[17] J. Keilson and D. M. G. Wishart, A central limit theorem for processes defined on a finite Markov chain, Proc. Cambridge Philos. Soc. **60** (1964), 547-567. MR0169271

[18] A. Krámli and D. Szász, Random walks with internal degrees of freedom, Z. Wahr. verw. Gebiete, **63** (1983), 85-95. MR0699788

[19] G. F. Lawler and V. Limic, The Beurling estimate for a class of random walks, Elec. Jour. Probab. **9** (2004), 846-861. MR2110020

[20] Nagaev, Some limit theorems for stationary Markov chains, Th. Proba. Appl. **2** (1957), 378-406. MR0094846

[21] P. Ney and E. Nummelin, Markov additive processes I. Eigenvalue properties and limit theorems, Ann. Probab. (1987) 561-592; Markov additive processes II. Large deviations, Ann. Probab. (1987), 593-609. MR0885131

[22] F. Spitzer, Principles of Random Walks, Van Nostrand, Princeton, 1964. MR0171290

[23] E.M. Stein, Singular integrals and differentiable properties of functions, Princeton University Press, 1970. MR0290095

[24] T. Takenami, Local limit theorem for random walk in periodic environment, to appear in Osaka Jour. Math. MR1951520

[25] K. Uchiyama, Green's functions for random walks on $\mathbf{Z}^d$, Proc. London Math. Soc. **77** (1998), 215-240. MR1625467

[26] K. Uchiyama, Newtonian potential and heat kernel on jungle gym, preprint