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STOCHASTIC WEAK ATTRACTOR FOR A DISSIPATIVE EULER EQUATION

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Abstract In this paper a non autonomous dynamical system is considered, a stochastic one that is obtained from the dissipative Euler equation subject to a stochastic perturbation, an additive noise. Absorbing sets have been defined as sets that depend on time and attract from $-\infty$. A stochastic weak attractor is constructed in phase space with respect to two metrics and is compact in the lower one.

Keywords Dissipative Euler equation, random dynamical systems, attractors.

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1 Introduction

In the classical theory of attractors, we need to define a semigroup $S(t)$ on a metric space (X, d) which must be continuous on X (continuity ensures the invariance of the attracting set). We also need to have a concept of dissipativity and compactness for $S(t)$. Usually, the semigroup property follows from uniqueness.

Here, we are concerned with a stochastic dynamical system which is obtained from a partial differential equation perturbed by a random forcing term. It is dissipative but does not produce regularization; the dissipative Euler equation. An existence and uniqueness theorem is established when the initial data are in $\mathcal{W}$ (defined below). Dissipation occurs also in $\mathcal{W}$, but the continuity of the dynamic is not satisfied. Moreover the compactness of the flow does not occur with respect to the topology of $\mathcal{W}$, but in a lower sence. In a preceding paper see [3] and following the approach of [13], we have proved the existence of an attracting set using the dynamic of the shift operator in the space of paths. Hence, the non-uniqueness problem (we took only that $u_0 \in V$, defined below) was bypassed and a continuous semigroup was obtained. We have also introduced a notion of attractor relative to a pair (d, δ) of topologies where the attractor constructed was d-bounded and δ-compact (recall that Euler equation do not produce regularization), so that we obtain a weak attractor in path space. In this paper, we aim to construct a stochastic weak attractor in the phase space $\mathcal{W}$ for (4.1), where uniqueness holds. In this way, the dynamic is well defined and enjoys the evolution property. However, we are not able to prove its continuity. So, the results of [3] concerning the attractor in the weak sense can not be used here. Notice that in the classical theory of attractors, the continuity of $S(t)$ is necessary to prove the invariance property of the attractor. Instead of continuity we will give another property which ensures the invariance of the absorbing set and the existence of a weak attractor in phase space (for a deterministic dynamical system). Equation (4.1) is non autonomous, but the general theory for non autonomous deterministic systems can not be used here. In [4] and [5], attracting sets have been defined as sets that depend on time and attract any orbits from $-\infty$. Indeed, in this paper we will prove existence of weak random attractor at time 0, which attracts bounded sets from $-\infty$.

The paper is organised as follows, in section 2 we give the general theory of weak attractor for deterministic non-autonomous dynamical systems, that we extend to stochastic dynamical systems in section 3. In section 4 we apply it to the particular case of stochastic dissipative Euler equation (4.1).

2 Deterministic weak attractor for non autonomous systems

If d denotes the distance on a metric space X, for each pair (A, B) of subsets of X and each $x \in X$, we define

$$d(x, A) = \inf_{y \in A} d(x, y), \qquad d(B, A) = \inf_{x \in B} d(x, A).$$

Let $\mathcal{W}$ be a Banach metric space. We denote by $d_{\mathcal{W}}$ the metric on $\mathcal{W}$. Let us define a mapping $S(t, s)$ on $(\mathcal{W}, d_{\mathcal{W}})$, $-\infty < s \leq t < \infty$. Assume that another metric δ exists on $\mathcal{W}$. We introduce the following concept of continuity for the mapping $S(t,s)$;

$$\begin{aligned}&\text{if } x_n \text{ is } d_{\mathcal{W}}-\text{bounded and } \delta-\text{convergent to } x \text{ in } \mathcal{W}\\ &\text{then } S(t,s)x_n \text{ is } \delta-\text{convergent to } S(t,s)x, \text{ for all } s\le t.\end{aligned} \tag{2.1}$$

We assume that the family $S(t,s)$ satisfy the condition (2.1) and the following evolution property,

$$S(t,r)S(r,s)x = S(t,s)x \text{ for all } s\le r\le t \text{ and for all } x\in\mathcal{W}, \tag{2.2}$$

Definition 2.1 *Given $t\in R$, we say that $\mathcal{B}(t)\subset\mathcal{W}$ is a $d_{\mathcal{W}}$- absorbing set at time t if*

1. *$\mathcal{B}(t)$ is $d_{\mathcal{W}}$-bounded,*
2. *$\forall B_0$ $d_{\mathcal{W}}$-bounded, $\exists s_1(B_0)$ such that $S(t,s)B_0\subset\mathcal{B}(t)$, $\forall s\le s_1(B_0)$.*

Definition 2.2 *Given $t\in R$, we say that $\{S(t,s)\}_{t\ge s}$ is $d_{\mathcal{W}}/\delta$-uniformly compact at time t if for all $B\subset\mathcal{W}$ $d_{\mathcal{W}}$-bounded, there exists s_0 which may depend on B such that*

$$\bigcup_{s\le s_0} S(t,s)B$$

is relatively compact in $\mathcal{W}$ with the topology δ.

Definition 2.3 *The family $\{S(t,s)\}_{t\ge s}$ is asymptotically $d_{\mathcal{W}}/\delta$-compact if there exists a $d_{\mathcal{W}}$-absorbing set at time t, which is δ-compact*

Definition 2.4 *(Weak omega-limit set) For any set $B\subset\mathcal{W}$, we define the δ-weak omega-limit set of B at time t and write $\omega^\delta(B,t)$, as the set $\bigcap_{\tau\le t}\overline{\bigcup_{s\le\tau}S(t,s)B}^{\delta}$, where the closure is taken in the δ-topology of $\mathcal{W}$ (this set can be empty). It is characterized as follows: $x\in\omega^\delta(B,t)$ if and only if there exists a sequence $x_n\in B$ and a sequence $s_n\to-\infty$ such that*

$$\delta(S(t,s_n)x_n,x)\to 0 \quad \text{as} \quad n\to\infty.$$

Definition 2.5 *(Global weak attractor) We say that $\mathcal{A}(t)\subset\mathcal{W}$ is a $d_{\mathcal{W}}/\delta$ global weak attractor at time t if it verifies the following properties:*

(i) $\mathcal{A}(t)$ is not empty, it is $d_{\mathcal{W}}$-bounded and δ-compact,

(ii) $\mathcal{A}(\tau)$ is invariant by $S(\tau,s)$, i.e. $S(\tau,s)\mathcal{A}(s)=\mathcal{A}(\tau)$ for all $\tau\ge s\ge t$,

(iii) for every $d_{\mathcal{W}}$-bounded set $B\subset\mathcal{W}$, $\lim_{s\to-\infty}\delta(S(t,s)B,\mathcal{A}(t))=0$.

Theorem 2.6 *Let $\{S(t,s)\}_{t\ge s}$ be a family of mappings on a metric space $(\mathcal{W},d_{\mathcal{W}})$. Let δ be another metric on $\mathcal{W}$, such that $S(t,s):\mathcal{W}\to\mathcal{W}$ verifies (2.1) and the evolution property, for all $t\ge s$. Assume that at time t, there exists a $d_{\mathcal{W}}$-absorbing set $\mathcal{B}(t)$, that it is δ-compact. Then, there exists a $d_{\mathcal{W}}/\delta$ global weak attractor $\mathcal{A}(t)$, and $\mathcal{A}(t)=\overline{\bigcup_{B\in\mathcal{W}}\omega^\delta(B,t)}^{\delta}$. (the union is taken over all $d_{\mathcal{W}}$-bounded sets B)*

Proof. Here, we must prove (i), (ii) and (iii) of definition (2.5).

(i) Notice that if there exists a $d_{\mathcal{W}}$-absorbing set $\mathcal{B}(t)$ at time t, then the set $\omega^\delta(B,t) \subset \mathcal{B}(t)$ is non-empty, $d_{\mathcal{W}}$-bounded and δ-compact, as a consequence also $\mathcal{A}(t)$, being the union over $d_{\mathcal{W}}$-bounded sets.

(ii) Let us first prove that $S(\tau,s)\mathcal{A}(s) \subset \mathcal{A}(\tau)$. Let us take $y \in \mathcal{A}(s)$. By the caracterization of $\mathcal{A}(s)$, there exists a $d_{\mathcal{W}}$-bounded set $B_n \subset \mathcal{W}$, a sequence $y_n \in \omega^\delta(B_n,t)$ and a sequence $s_n \to -\infty$ such that $y_n \longrightarrow^\delta_{n\to\infty} y$. This yields that

$$\forall n \in N,\ \forall s_0 \le s,\ y_n \in \overline{\bigcup_{s' \le s_0} S(s,s')B_n}^\delta .$$

Hence, $\forall n \in N,\ \forall s_0 \le s,\ \exists s_k \le s_0,\ z_n^k \in B_n$, such that

$$S(s,s_k)z_n^k \to^\delta_{k\to\infty} y_n.$$

By (refevolution) and (2.1), we have that

$$S(\tau,s)S(s,s_k)z_n^k = S(\tau,s_k)z_n^k \to^\delta_{k\to\infty} S(\tau,s)y_n.$$

We deduce that

$$S(\tau,s)y_n \in \bigcap_{s_0 \le \tau} \overline{\bigcup_{r \le s_0} S(\tau,r)B_n}^\delta \subset \bigcup_{B \subset \mathcal{W}} \bigcap_{s_0 \le \tau} \overline{\bigcup_{r \le s_0} S(\tau,r)B}^\delta .$$

On the other hand

$$S(\tau,s)y_n \longrightarrow^\delta_{n\to\infty} S(\tau,s)y,$$

which implies that $S(\tau,s)y \in \mathcal{A}(\tau)$.

Now let us prove that $S(\tau,s)\mathcal{A}(s) \supset \mathcal{A}(\tau)$. Taken $x \in \mathcal{A}(\tau)$, there exist a $d_{\mathcal{W}}$-bounded set $B_n \subset \mathcal{W}$, a sequence $y_n \in \omega^\delta(B_n,t)$ such that $y_n \to^\delta x$ and there exist a sequence $y_n^k \in B_n$ and a sequence $s_k \to -\infty$ such that

$$S(\tau,s_k)y_n^k \to^\delta y_n.$$

On the other hand, there exists a $d_{\mathcal{W}}$-absorbing set at time t, that is $\mathcal{B}(t)$, so also at time $s \ge t$ (take $S(s,t)\mathcal{B}(t) = \mathcal{B}(s)$), which attracts all bounded sets of B_n of $\mathcal{W}$, i.e.

$$S(s,s_k)y_n^k \to^{d_{\mathcal{W}}} \phi_n \in \mathcal{B}(s).$$

The absorbing set being δ-compact, the mapping $S(\tau,s)$ is also δ-compact, hence there exists a subsequence $S(s,s_{k'})y_n^{k'}$, such that

$$S(s,s_{k'})y_n^{k'} \longrightarrow^\delta \phi_n.$$

Thus, $\phi_n \in \omega^\delta(B_n,s)$. By the evolution property (2.2) and the (2.1) we get

$$S(\tau,s_{k'})y_n^{k'} \to^\delta S(\tau,s)\phi_n.$$

Hence $S(\tau,s)\phi_n = y_n$. Since $\mathcal{B}(s)$ is δ-compact, there exists a subsequence $\phi_{n_k} \to^\delta \phi$ and $\phi \in \mathcal{A}(s)$ and finally $S(\tau,s)\phi = x$. Hence $x \in S(\tau,s)\mathcal{A}(s)$.

(iii) We argue by contradiction. Assume that there exist $\epsilon > 0$, a $d_{\mathcal{W}}$-bounded set B and sequences $s_n \to -\infty$ and $x_n \in B$ such that

$$\delta(S(t, s_n)x_n, \omega^{\delta}(B, t)) \geq \epsilon > 0, \text{ as } s_n \to -\infty.$$

Since there exists a $d_{\mathcal{W}}$-absorbing set at time t, $\mathcal{B}(t)$, there exists a sequence $y_n \in \mathcal{B}(t)$, such that

$$d_{\mathcal{W}}(S(t, s_n)x_n, y_n) \to 0, \text{ as } s_n \to -\infty.$$

$\mathcal{B}(t)$ being a δ-compact set, there exists a subsequence $y_{n_k} \to y$ and

$$\delta(S(t, s_{n_k})x_{n_k}, y) \to 0.$$

This yields that $y \in \omega^{\delta}(B, t)$, which contradicts the hypothesis. ∎

3 Stochastic weak attractor

Let $(\mathcal{W}, d_{\mathcal{W}})$ a complete metric space and $(\Omega, \mathcal{F}, \mathbf{P})$ a probability space. let us define another metric δ on $\mathcal{W}$. We consider a family of mapping $\{S(t, s.\omega)\}_{t\geq s, \omega\in\Omega} : \mathcal{W} \to \mathcal{W}$, satisfying for $\mathbf{P}$-a.e.$\omega \in \Omega$ the properties (2.1) and (2.2).

Given $t \in R$ and $\omega \in \Omega$, we say that $\mathcal{B}(t, \omega)$ is $d_{\mathcal{W}}$-bounded absorbing set at time t if for all $d_{\mathcal{W}}$-bounded set $B \subset \mathcal{W}$, there exists $s_0(B)$, such that

$$S(t, s, \omega)B \subset \mathcal{B}(t, \omega)), \ \forall s \leq s_0.$$

We say that $\{S(t, s.\omega)\}_{t\geq s, \omega\in\Omega}$ is asymptotically $d_{\mathcal{W}}/\delta$-compact if there exists a measurable set $\Omega_0 \subset \Omega$ with measure one, such that for all $t \in R$, and all $\omega \in \Omega_0$, there exists a $d_{\mathcal{W}}$-absorbing set $\mathcal{B}(t, \omega)$, δ-compact.

Let us denote by $\mathcal{B}(\mathcal{W})$ the σ-algebra of the metric space $\mathcal{W}$. Let us give the following theorem (see [16])

Theorem 3.1 *Let X_1, X_2 two metric spaces such that $X_1 \subset X_2$ with continuous embedding. Then*

$$\mathcal{B}(X_2) \bigcap X_1 \subset \mathcal{B}(X_1).$$

Let us denote by $\mathcal{H}$ the separable metric space such that

$$\mathcal{W} \subset \mathcal{H},$$

with continuous embedding. The metric δ is the metric endowed on $\mathcal{H}$ and $\mathcal{W}$ is closed in $\mathcal{H}$. Assume the following condition

(**M**) for all $t \in R$, $x \in \mathcal{W}$, the mapping $(s, \omega) \to S(t, s, \omega)x$ is measurable from $(-\infty, t] \times \Omega, \mathcal{B}((-\infty, t]) \times \mathcal{F} \to (\mathcal{W}, \mathcal{B}(\mathcal{H}) \bigcap \mathcal{W})$.

As in section 2, we define the random weak omega-limit of a bounded set $B \subset \mathcal{W}$ at time t as:

$$\omega^\delta(B,t,\omega) = \bigcap_{\tau<t} \overline{\bigcup_{s<\tau} S(t,s,\omega)B}^\delta$$

and

$$\mathcal{A}(t,\omega) = \overline{\bigcup_{B\subset\mathcal{W}} \omega^\delta(B,t,\omega)}^\delta.$$

The set $\mathcal{A}(t,\omega)$ will be called the random attractor. As in section 2, we give the following theorem

Theorem 3.2 *Assume the condition* (**M**) *holds, assume that for each $t \in R$ there exists a measurable set $\Omega_t \subset \Omega$ with measure one, such that for all $\omega \in \Omega_t$ there exists an absorbing set, δ-compact. Then for* **P**.*a.e.* $\omega \in \Omega_t$, *the set $\mathcal{A}(t,\omega)$ is a measurable global weak attractor, i.e.*

(i) $\mathcal{A}(t,\omega)$ is not empty, it is $d_{\mathcal{W}}$-bounded and δ-compact,

(ii) $S(\tau,s,\omega)\mathcal{A}(s,\omega) = \mathcal{A}(\tau,\omega)$ for all $\tau \geq s \geq t$,

(iii) for every $d_{\mathcal{W}}$-bounded set $B \subset \mathcal{W}$, $\lim_{s\to-\infty} \delta(S(t,s,\omega)B, \mathcal{A}(t,\omega)) = 0$.

(iv)$\mathcal{A}(t,\omega)$ is measurable with respect to the **P**-*completion of $\mathcal{F}$.*

Proof. (i), (ii) and (iii) are consequences of theorem of section 2.
(iv)We say that a family $\mathcal{A}(t,\omega)$ of closed subsets of $\mathcal{W}$ is measurable if and only if for all $x \in \mathcal{W}$, the multifunction $\omega \to \delta(x,\mathcal{A}(\omega))$ is measurable (see [6]).
For all $x \in \mathcal{W}$, we have

$$\begin{aligned} \delta(x,\mathcal{A}(t,\omega)) &= \inf_{B\subset\mathcal{W}} \delta(x,\omega^\delta(B,t,\omega)) \\ &= \inf_{B\subset\mathcal{W}} \inf_{s\leq t} \delta(x,S(t,s,\omega)B). \end{aligned}$$

The norm $|\cdot|$ being the norm endowed by $\mathcal{H}$, $(\mathcal{W},|\cdot|)$ is a normed subspace of normed separable space $(\mathcal{H},|\cdot|)$, hence $(\mathcal{W},|\cdot|)$ is also separable. By the assumption (**M**), for all $t \in R$ and for all y_n in a dense subset of $B \subset \mathcal{W}$, the function

$$(s,\omega) \to \delta(x,S(t,s,\omega)y_n)$$

is measurable. By the property (2.1) and the separability of $(\mathcal{W},|\cdot|)$

$$(s,\omega) \to \delta(x,S(t,s)B)$$

is measurable. On the other hand, for each $\alpha \in R$

$$\left\{\omega \in \Omega, \inf_{s\leq t} \delta(x,S(t,s)B) < \alpha\right\} = \Pi_\Omega \left\{(s,\omega) \in]-\infty,t] \times \Omega, \delta(x,S(t,s)B) < \alpha\right\},$$

where Π_Ω is the canonical projection from $R \times \Omega$ in Ω. We deduce by the projection theorem (see [6]) that the set

$$\left\{\omega \in \Omega, \inf_{s\leq t} \delta(x,S(t,s)B) < \alpha\right\}$$

is measurable with respect to the $\mathbf{P}$ completion of $\mathcal{F}$. By the separability of $(\mathcal{W}, |\cdot|)$, we conclude the measurability of the set

$$\left\{\omega \in \Omega,\ \inf_{B \subset \mathcal{W}} \inf_{s \leq t} \delta(x, S(t,s)B) < \alpha\right\}.$$

This complete the proof. ∎

4 Application to stochastic dissipative Euler equation.

4.1 Mathematical setting and notations

We are concerned with a stochastic dissipative Euler equation for an incompressible fluid in an open bounded domain D of R^2, i.e.

$$du + ((u \cdot \nabla)u + \chi u)dt = (-\nabla p + f)dt + dW, \tag{4.1}$$

where u is the velocity of the fluid, p the pressure, f the external force, W is a Wiener process on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ (with expectation denoted by $\mathbf{E}$). The constant χ will be called the sticky viscosity. The term $-\chi u$ does not correspond to a constitutive equation and does not introduce the smoothing effects of the Navier-Stokes term Δu. For other comments see [10]. The velocity field u is in addition subject to the incompressibility condition

$$\nabla \cdot u(t,x) = 0, \quad t \in R, \quad x \in D, \tag{4.2}$$

the boundary condition

$$u \cdot n = 0 \quad \text{on } \partial D, \tag{4.3}$$

and it satisfies the initial condition

$$u(t_0) = u_{t_0}. \tag{4.4}$$

Let us introduce some functional spaces. Let $\mathcal{V}$ be the space of infinitely differentiable vector fields u on D with compact support strictly contained in D, satisfying $\nabla \cdot u = 0$. We introduce the space H of all measurable vector fields $u : D \longrightarrow R^2$ which are square integrable, divergence free, and tangent to the boundary

$$H = \left\{u \in \left[L^2(D)\right]^2;\ \nabla \cdot u = 0 \text{ in } D,\ u \cdot n = 0 \text{ on } \partial D\right\};$$

the meaning of the condition $u \cdot n = 0$ on ∂D for such vector fields is explained for instance in [14]. The space H is a separable Hilbert space with the inner product of $\left[L^2(D)\right]^2$, denoted in the sequel by $< .,. >$ (norm $|.|$). Let V be the following subspace of H:

$$V = \left\{u \in \left[H^1(D)\right]^2;\ \nabla \cdot u = 0 \text{ in } D,\ u \cdot n = 0 \text{ on } \partial D\right\};$$

The space V is a separable Hilbert space with the inner product of $\left[H^1(D)\right]^2$ (norm $\| \ . \ \|$). Identifying H with its dual space H', and H' with the corresponding natural subspace of the dual space V', we have the standard triple $V \subset H \subset V'$ with continuous dense injections. We

denote the dual pairing between V and V' by the inner product of H.
In what follows, we will denote by d the metric endowed by V and by δ the metric endowed by H. Let us denote by $|\cdot|_p$ the norm endowed by the Lebesgue space $L^p(D)$, when $p \neq 2$
From now on , we will assume that $W(t)$ is an infinite dimensional Brownian motion of the form $W(t,\omega) = \sum_{j=1}^{\infty} \sigma_j \beta_j(t,\omega) e_j$ where $\{\beta_j\}$ is a sequence of real independent towsided Brownian motions on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and we will assume the the following regularity space on W; the process $W = W(t,\omega)$ $\omega \in \Omega$, is an H- valued process such that for $\mathbf{P}$-a.e. $\omega \in \Omega$,

$$W(.,\omega) \in C\left(R, [H^4(D)]^2 \bigcap V\right) \tag{4.5}$$

with the mapping $(t,\omega) \to W(t,\omega)$ measurable in these topologies, and

$$\nabla \wedge W = 0 \text{ on } R \times \partial D, \tag{4.6}$$

where

$$\nabla \wedge W = D_1 W_2 - D_2 W_1.$$

The condition (4.5) can be written in this form

$$\sum_{j=1}^{\infty} \lambda_j^4 \sigma_j^2 < \infty, \tag{4.7}$$

where $\{\lambda_j\}_j$ and $\{e_j\}_j$ are respectively the eigenvalues and the eigenfunctions of the linear operator A defined below.
Let us introduce the unknown $v = u - z$ (in spite of $v = u - W$, the reason is that we want to use the ergodic properties of z), where z is solution of the following linear stochastic equation

$$dz + (\chi + \alpha) z dt = dW, \ \alpha \geq 0 \tag{4.8}$$

so that v is solution of the deterministic equation

$$\frac{dv}{dt} + (v+z) \cdot \nabla(v+z) + \chi v = -\nabla p + f + \alpha z. \tag{4.9}$$

4.2 Approximation scheme

Let us approximate (4.9) by the dissipative Navier-Stokes equations

$$\begin{cases} \frac{\partial v}{\partial t} + (v+z) \cdot \nabla(v+z) + \nabla p = \nu \Delta v - \chi v + f + \alpha z, & \text{in } (t_0, T) \times D \\ \nabla \cdot v = 0, & \text{in } (t_0, T) \times D \\ \nabla \wedge v = 0, & \text{on } (t_0, T) \times \partial D \\ v \cdot n = 0, & \text{on } (t_0, T) \times \partial D \\ v|_{t=t_0} = u_{t_0}, & \text{in } D \end{cases} \tag{4.10}$$

with $\nu > 0$. We have denoted by $\nabla \wedge v$ the vorticity, defined as

$$\nabla \wedge v = \partial_2 v_1 - \partial_1 v_2.$$

Let us define the continuous bilinear form on V

$$a(u,v) = \int_D \nabla u \cdot \nabla v - \int_{\partial D} k(\sigma) u(\sigma) \cdot v(\sigma) d\sigma,$$

where $k(\sigma)$ is the curvature. We set

$$D(A) = \left\{ u \in V \cap (H^2(D))^2, \nabla \wedge u|_{\partial D} = 0 \right\},$$

and define the linear operator $A : D(A) \longrightarrow H$, as

$$< Au, v >= a(u,v), \text{ for all } u, v \text{ in} V.$$

Let us also define the trilinear form on V

$$b(u,v,w) = \int_D (u \cdot \nabla) v \cdot w.$$

We define the bilinear operator $B(u,v) : V \times V \longrightarrow V'$, as $< B(u,v), z >= b(u,v.z)$ for all $z \in V$. Note that B verifies $< B(u,v), z >= - < B(u,z), v >$ and $< B(u,v), v >= 0$, when u, v and z are in suitable spaces (see for instance [14]).
We will write (4.10) in the following abstract form

$$\begin{cases} \frac{dv_\nu}{dt} + \nu A v_\nu + B(v_\nu + z, v_\nu + z) + \chi v_\nu = f + \alpha z, \\ v_\nu(t_0) = v_{t_0}, \end{cases} \tag{4.11}$$

$t \in [t_0, T)$. Now, we consider the classical Faedo-Galerkin approximation scheme, we obtain the following problem

$$\begin{cases} \frac{dv_{n\nu}}{dt} + \nu A v_{n\nu} + P_n B(v_{n\nu} + z_n, v_{n\nu} + z_n) + \chi v_{n\nu} = P_n f + \alpha z_n, \\ v_{n\nu}(t_0) = P_n v_{t_0}. \end{cases} \tag{4.12}$$

where P_n is the orthogonal projector in H over the subspace spanned by $e_1, e_2, ..., e_n$, the first n eigenfunctions of A, i.e.

$$P_n x = \sum_{i=1}^{n} < x, e_i > e_i, \ x \in H.$$

We can look for energy estimates satisfied by $v_{n\nu}$. We multiply (4.12) by $v_{n\nu}$ and integrate over D; by using the incompressibility condition,

$$\begin{aligned} |b(v_{n\nu} + z_n, v_{n\nu} + z_n, v_{n\nu})| &= |b(v_{n\nu} + z_n, z_n, v_{n\nu})| \\ &\leq |\int_D v_{n\nu} \cdot \nabla z_n \cdot v_{n\nu}| + |\int_D z_n \cdot \nabla z_n \cdot v_{n\nu}| \\ &\leq \frac{\chi}{4} |v_{n\nu}|^2 + C(\chi)(|\nabla z_n|^2 |z_n|^2_\infty + |v_{n\nu}|^2 |\nabla z_n|^2_\infty). \end{aligned}$$

On the other hand there exists an arbitrary $\epsilon > 0$, such that

$$\int_{\partial D} k(\sigma) u(\sigma) \cdot v(\sigma) d\sigma \leq \epsilon \parallel u \parallel^2 + C_\epsilon |v|^2. \tag{4.13}$$

Therefore,

$$|a(v_{n\nu}, v_{n\nu})| \leq |\nabla v_{n\nu}|^2 + \epsilon |\nabla v_{n\nu}|^2 + C_\epsilon |v_{n\nu}|^2.$$

Using the above estimates, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |v_{n\nu}(t)|^2 &+ \nu(1-\epsilon)|\nabla v_{n\nu}(t)|^2 \leq \frac{-\chi}{2} |v_{n\nu}(t)|^2 + C_\epsilon |v_{n\nu}(t)|^2 \\ &+ C(\chi)(|f(t)|^2 + |z_n(t)|^2_\infty |\nabla z_n(t)|^2 + |v_{n\nu}(t)|^2 |\nabla z_n(t)|^2_\infty), \end{aligned}$$

for an arbitrary $\epsilon > 0$.
We integrate over (t_0, t). For all $\nu \le \nu_0$, and in particular for $\epsilon = 1/2$, we obtain

$$\begin{aligned}
|v_{n\nu}(t)|^2 &+ \nu\int_{t_0}^t |\nabla v_{n\nu}(s)|^2 ds \le |v_{n\nu}(t_0)|^2 + (-\chi + 2\nu C_{1/2})\int_{t_0}^t |v_{n\nu}(s)|^2 ds \\
&+ C(\chi)\int_{t_0}^t (|f(s)|^2 + |\nabla z_n|^2 |z_n(s)|_\infty^2) + \alpha^2 |z_n(s)|^2 ds \\
&+ \int_{t_0}^t |v_{n\nu}(s)|^2 |\nabla z_n(s)|_\infty^2. \qquad (4.14)
\end{aligned}$$

Applying Gronwall lemma, we get

$$\begin{aligned}
|v_{n\nu}(t)|^2 &\le |v_{n\nu}(t_0)|^2 e^{\int_{t_0}^t (C_1(\chi,\nu) + |\nabla z_n(s)|_\infty^2) ds} + C(\chi)\int_{t_0}^t (|f(s)|^2 + \alpha^2 |z_n(s)|^2 \\
&+ |\nabla z_n(s)|^2 |z_n(s)|_\infty^2) e^{\int_s^t (C_1(\chi,\nu) + |\nabla z_n(\sigma)|_\infty^2) d\sigma} ds. \qquad (4.15)
\end{aligned}$$

where $C_1(\chi, \nu) = -\chi + 2\nu C_{1/2}$.
Inequality (4.15) implies that $v_{n\nu}$ remains in a bounded set of $L^\infty(t_0, T; H)$, for every $T \ge 0$. Then we go back to (4.14) which shows that $v_{n\nu}$ remains in a bounded set of $L^2(t_0, T; V)$. On the other hand, we get from the equation and the estimates below proved that $\frac{\partial v_{n\nu}}{\partial t}$ is bounded in $L^2(t_0, T; V')$. Hence by a compactness argument, we obtain that

$$v_{n\nu} \to v_\nu, \quad \text{in } L^2((t_0, T) \times D) \text{ strongly},$$

$$v_{n\nu} \rightharpoonup v_\nu, \quad \text{in } L^2((t_0, T; V) \times D) \text{ weakly},$$

so that we can pass to the limit on n (see [14]), and we obtain that $v_{n\nu}$ verifies the equation (4.11) in the sence of distributions. Moreover $(v_{n\nu})$ belongs in $C([t_0, T]; H) \cap L^2(t_0, T; V)$.
Let us denote by $\xi = \nabla \wedge u$, $F = \nabla \wedge f$, $z_r = \nabla \wedge z$, $g = \nabla \wedge W$ and set $\beta = \xi - z_r$, we have

$$\begin{cases} d\beta_\nu + \nu A\beta_\nu + B(u_{nu}, \beta_\nu) + \chi\beta_\nu = F dt + \alpha z_r - (u_\nu \cdot \nabla) z_r, \\ \beta_\nu(t_0) = \nabla \wedge u_\nu(t_0) - z_r(t_0). \end{cases} \qquad (4.16)$$

From the incompressibilty condition, we have that

$$< B(u_\nu, \beta_\nu), \beta_\nu >= 0.$$

We multiply equation (4.16) by β_ν and integrate over (t_0, t), and we obtain

$$\begin{aligned}
|\beta_\nu(t)|^2 &+ \chi\int_{t_0}^t |\beta_\nu(s)|^2 ds + 2\nu\int_{t_0}^t |\nabla\beta_\nu(s)|^2 ds \le |\beta(t_0)|^2 \\
&+ C(\chi)\int_{t_0}^t (|F(s)|^2 + \alpha|z_r(s)|^2 + |u_\nu(s)|^2 |\nabla z_r(s)|_\infty^2).
\end{aligned}$$

Hence, by Gronwall inequality, we have

$$\begin{aligned}
&|\beta_\nu(t)|^2 \le |\beta_\nu(t_0)|^2 e^{-\chi(t-t_0)} \\
&+ C(\chi)\int_{t_0}^t (|F(s)|^2 + \alpha|z_r(s)|^2 + |u_{n\nu}(s)|^2 |\nabla z_r(s)|_\infty^2) e^{-\chi(t-s)} ds. \qquad (4.17)
\end{aligned}$$

Now let us introduce the elliptic problem

$$\begin{cases} \triangle u_\nu = -\nabla^\perp \xi_\nu, \\ u_\nu|_{\partial D} = 0, \\ \xi_\nu|_{\partial D} = 0. \end{cases} \tag{4.18}$$

Where $\nabla^\perp = (D_2, -D_1)$. We multiply the first equation by u_ν and integrate over D. By an integration by parts and using the estimate (4.13), we have for an arbitrary $\epsilon > 0$ (in particular for $\epsilon = 1/2$),

$$|\nabla u_\nu(t)|^2 \le C(|u_\nu(t)|^2 + |\xi_{n\nu}(t)|^2).\ t \in (t_0, T).$$

Collecting the estimates (4.15) and (4.17), we have that

$$\| v_\nu(t) \|^2 \le \| v_\nu(t_0) \|^2 \mathrm{e}^{-C_1(\chi,\nu)t} + \chi^{-1} \int_{t_0}^t \| f(s) \|^2 \mathrm{e}^{-C_1(\chi,\nu)(t-s)} ds. \tag{4.19}$$

The estimates (4.19) and (4.14) show that v_ν remains uniformly bounded in $L^\infty(t_0, T; V) \cap L^2(t_0, T; V)$. By the same argument used before, we prove that $\frac{\partial v_\nu}{\partial t}$ is bounded in $L^2(t_0, T; V')$. Hence by a compactness argument, we can extract a subsequence (also denoted v_ν) such that

$$v_\nu \rightharpoonup v \text{ in } L^2((t_0, T; V),$$

and

$$v_\nu \longrightarrow v \text{ strongly in } L^2((t_0, T) \times D).$$

Therefore, we can pass to the limit on n in the equation (4.9) and v satisfies (4.9) in the distribution sence.
Let us give the definition of global weak solution.

Definition 4.1 *We shall say that a stochastic process $u(t, \omega)$ is a global weak solution of the equation (4.1) over the time interval (t_0, T) if for $\mathbf{P}$ a.e. ω*

$$u(., \omega) \in C[t_0, T; H) \cap L^\infty_{loc}(t_0, T; V) \cap L^2_{loc}(t_0, T; V),$$

$$\begin{aligned} &< u(t) - u(t_0), \phi > + \chi \int_{t_0}^t < u(s), \phi > + \int_{t_0}^t < B(u(s), u(s)), \phi > \\ &= \int_{t_0}^t < f(s), \phi > + < W(t) - W(t_0), \phi >, \end{aligned}$$

for all $\phi \in \mathcal{V}$ and for all $T \ge t \ge t_0$.

As a consequence, we give the following theorem

Theorem 4.2 *(a) Assume that (4.5) and (4.6) hold. Assume that $u_{t_0} \in V$ and $f \in L^2(t_0, T, V)$. Then, on each interval (t_0, T) there exists at least a weak global solution for (4.1) with the initial condition $u(t_0) = u_{t_0}$ satisfying for $\mathbf{P}$- a.e.$\omega \in \Omega$*

$$u(., \omega) \in C[t_0, T], ; H) \cap L^2(t_0, T; V).$$

Moreover, u is measurable in these topologies and satisfies for $\mathbf{P}$*-a.e.* $\omega \in \Omega$ *and for all* $t \in (t_0, T)$

$$u(.,\omega) \in L^\infty(t_0, T; V),$$

$$\begin{aligned} &|u(t) - z(t)|^2 \le |u(t_0) - z(t_0)|^2 \mathrm{e}^{\int_{t_0}^t (-\chi + |\nabla z|_\infty^2)} \\ + \quad & C(\chi) \int_{t_0}^t (|f(\sigma)|^2 + \alpha^2 |z|^2 + |\nabla z|^2 |z|_\infty^2) \mathrm{e}^{\int_\sigma^t (-\chi + |\nabla z|_\infty^2)}, \end{aligned} \tag{4.20}$$

$$\begin{aligned} &|\xi(t) - z_r(t)|^2 \le |\xi(t_0) - z_r(t_0)|^2 \mathrm{e}^{-\chi(t-t_0)} \\ + \quad & C(\chi) \int_{t_0}^t (|F|^2 + \alpha^2 |z_r|^2 + |u|^2 |\nabla z_r|_\infty^2) \mathrm{e}^{-\chi(t-\sigma)} d\sigma. \end{aligned} \tag{4.21}$$

and

$$|\nabla u(t)|^2 \le C|\xi(t)|^2 + |u(t)|^2. \tag{4.22}$$

(b) If in addition for given $\gamma \in (0,1)$, $\sum_{j=1}^\infty \lambda_j^{4+\gamma} \sigma_j^2 < \infty$, *and* $\xi_{t_0} \in L^\infty(D)$ *and* $f \in L^\infty(D)$ *then,* $\mathbf{P}$*-a.e.*$\omega \in \Omega$

$$\xi(.,\omega) \in L^\infty(D \times (t_0, T)),$$

and the solution is unique. Moreover it is progressively measurable in these topologies and it satisfies for all $\mathbf{P}$*-a.s.*$\omega \in \Omega$ *and for all* $t \in (t_0, T)$

$$\begin{aligned} |\xi(t) - z_r(t)|_\infty \quad &\le \quad |\xi(t_0) - z_r(t_0)|_\infty \mathrm{e}^{-\chi(t-t_0)} \\ &+ \quad \int_{t_0}^t (|F|_\infty + \alpha |z_r|_\infty + |u|_\infty |\nabla z_r|_\infty) \mathrm{e}^{-\chi(t-\sigma)} d\sigma, \end{aligned} \tag{4.23}$$

and

$$|\nabla u(t)|_\infty \le |u(t)|_{W^{1,4}} \le C \left(|\xi(t)|_4^4 + |u(t)|_4^4 \right)^{1/4}, \tag{4.24}$$

*where z is solution of problem (*4.8*).*

Proof. (a) Proved above.

(b) To prove (4.23), we use the maximum principle on the scalar equation (4.16), where the Galerkin approximation is used. We obtain (4.23) for v_ν which is also true for v (when ν goes to 0).
The first part of the estimate (4.24) is given by the continuous embedding $W^{1,4}(D) \subset L^\infty(D)$. We have to check an estimate for $|\nabla u|_4$. Write the equation (4.18) as

$$-D_i^2 u_j = D_j^\perp \xi.$$

Here, we use the summation on repetitive indices. Multiply the above equation by $|\nabla u|^2 u_j$, and integrate over D. By integration by parts, we have that

$$- < D_i^2 u_j |\nabla u|^2, u_j >= -1/3 \int_D |\nabla u|^4 + +1/3 \int_{\partial D} k(\sigma) |u|^2 |\nabla u|^2.$$

The boundary integral is estimated as

$$\begin{aligned}
\int_{\partial D} k(\sigma)|u|^2|\nabla u|^2 &\leq C|\,|\nabla u|^2|_{H^{-1/2}(\partial D)}|\,|u|^2|_{H^{1/2}(\partial D)} \\
&\leq |\,|\nabla u|^2||\,|u|^2|_{H^1} \\
&= \left(\int_D |\nabla u|^4\right)^{1/2}\left(\int_D |u|^4 + 4\int_D |u|^2|\nabla u|^2\right)^{1/2},
\end{aligned}$$

Using two times the Holder inequality and Young inequality, we obtain that for an arbitrary $\epsilon > 0$, it is less than $|u|_4^4 + \epsilon|\nabla u|_4^4$. now we turn to the second term of the equation. By the boundary condition $\xi|_{\partial D} = 0$, and for an arbitrary real number $\epsilon > 0$, we get

$$\begin{aligned}
< D_j^{\perp}\xi|\nabla u|^2, u_j > &= \int_D |\xi|^2|\nabla u|^2 \\
&\leq |\xi|_4^4 + \epsilon|\nabla u|_4^4.
\end{aligned}$$

Collecting all the estimates, we obtain (4.24).
Uniqueness. The proof of uniqueness is given in [2] for $\chi = 0$, but it readily extends to $\chi \neq 0$.
Measurability. When $u_{t_0} \in V$, the processes $v_{n\nu}(t,\omega)$ are progressively measurable in H, by construction. Analyzing the limiting procedures in the previous steps we deduce that also $v(t,\omega)$ is progressively measurable in H and so $u(t,\omega)$ being the difference of two measurable processes $v(t,\omega)$ and $z(t,\omega)$. The embedding of V in H being continuous, we can deduce the progressive measurability of the solution in V. Moreover when $\xi_{t_0} \in L^\infty(D)$, the mollification of the Navier-Stokes solution is progressively measurable in the space of mollifers (the operation being continuous), yielding the progressive measurability in the required topology for its pointwise limit. This completes the proof. ∎

4.3 Existence of stochastic weak attractor

For sake of simplicity, we assume that $f \in V$. We recall that in this section, we are dealing with the metrics $d_{\mathcal{W}}$, d and δ introduced in section 2, i.e. d is the metric endowed by V, δ and $d_{\mathcal{W}}$ are the ones endowed respectively by H and $\mathcal{W}$. The metric $d_{\mathcal{W}}$ is defined by

$$d_{\mathcal{W}}(f,g) = d(f,g) + d_\infty(\nabla \wedge f, \nabla \wedge g),$$

where d_∞ is the metric of $L^\infty(D)$. We denote by $|.|_{\mathcal{W}}$ the norm induced by $\mathcal{W}$. We denote by

$$\mathcal{W} = \{f \in V, \text{ such that } \nabla \wedge f \in L^\infty(D)\}.$$

We define the family $\{S(t,s,\omega)\}_{t\geq s,\omega\in\Omega}$ by

$$\begin{aligned}
S(t,s,\omega):\quad \mathcal{W} &\longrightarrow \mathcal{W} \\
u_s(\omega) &\longrightarrow u(t,s,\omega)
\end{aligned}$$

Lemma 4.3 *The family of mappings $\{S(t,s,\omega)\}_{t\geq s,\omega\in\Omega}$ associated to the Euler equation (4.1) verify for $\mathbf{P}$.a.e. $\omega \in \Omega$ the evolution property (2.2) on $\mathcal{W}$ and the condition (2.1).*

Proof. The evolution property is obvious, by the uniqueness of the solution in $\mathcal{W}$. Now let us fix $\omega \in \Omega$ and let us take $t \geq s$. Take a sequence u_s^n such that it is $\mathbf{P}$.a.e. $\omega \in \Omega$ $d_{\mathcal{W}}$-bounded and δ-convergent to u_s. From the estimates of section 3, we deduce that $u^n(t)$ is $\mathbf{P}$.a.e. $\omega \in \Omega$ $d_{\mathcal{W}}$-equibounded and that $u^n \in L^\infty(0,T;V) \cap W^{1,2}(0,T;H)$. On the other hand, we have from the equation (4.1) that $\mathbf{P}$.a.e. $\omega \in \Omega$

$$
\begin{aligned}
&|(u^n(t) - u^n(s)) - (W(t) - W(s))| \\
&\leq \int_s^t |(u^n(r)\cdot\nabla)u^n(r)|dr + \int_s^t |-\chi u^n(r) + f(r)|dr + \int_s^t |W(r)|dr \\
&\leq (|u^n|_{L^\infty(0,T;V)}|u^n|_{L^\infty(0,T;H)} + |u^n|_{L^\infty(0,T;H)} + |f| + |W|_{L^\infty(0,T;H)})|t-s| \\
&\leq C|t-s|.
\end{aligned}
$$

for all $T \geq t \geq s \geq t_0$. From Ascoli-Arzelà theorem, there exists a subsequence $u_{n_k}(t)$ δ-convergent to u(t) uniformly on $[t_0, T]$. ∎

We will prove the following lemma

Lemma 4.4 *Let $\gamma \in (0,1)$ be fixed. For any $\epsilon > 0$, there exists $\alpha_0 > 0$ such that for all $\alpha \geq \alpha_0$*

$$\mathbf{E}|z(t)|^2_{(H^{2+\gamma}(D))^2} < \epsilon,$$

for all $t \in R$.

Proof. The process $z(t) = \int_{-\infty}^t \mathrm{e}^{(t-s)(-\chi-\alpha)}dW(s)$ solution of (4.8) is an ergodic and stationary process with continous trajectory in $(H^4(D))^2$.

$$
\begin{aligned}
\mathbf{E}|z(t)|^2_{(H^{2+\gamma}(D))^2} &= \mathbf{E}|A^{\frac{2+\gamma}{2}}z(t)|^2 \\
&= \mathbf{E}\left|\sum_{j=1}^\infty \int_{-\infty}^t A^{\frac{2+\gamma}{2}}\mathrm{e}^{(t-s)(-\chi-\alpha}\sigma_j e_j d\beta_j(s)\right|^2 \\
&\leq \mathbf{E}\sum_{j=1}^\infty \left|\int_{-\infty}^t \lambda_j^{\frac{2+\gamma}{2}}\mathrm{e}^{(t-s)(-\chi-\alpha)}\sigma_j d\beta_j(s)\right|^2 \\
&\leq \sum_{j=1}^\infty \int_{-\infty}^t \lambda_j^{2+\gamma}\mathrm{e}^{2(t-s)(-\chi-\alpha)}\sigma_j^2 ds \\
&\leq \frac{1}{2(\chi+\alpha)}\sum_{j=1}^\infty \lambda_j^{2+\gamma}\sigma_j^2.
\end{aligned}
$$

Write

$$\frac{1}{2(\chi+\alpha)}\sum_{j=1}^\infty \lambda_j^{2+\gamma}\sigma_j^2 = \frac{1}{2(\chi+\alpha)}\sum_{j=1}^N \lambda_j^{2+\gamma}\sigma_j^2 + \frac{1}{2(\chi+\alpha)}\sum_{N+1}^\infty \lambda_j^{2+\gamma}\sigma_j^2,$$

Using the condition (4.7), we can choose α such that the first term of the right hand of the above equality is less than $\epsilon/2$ and N such that the second term is less than $\epsilon/2$. This completes the proof. ∎

Lemma 4.5 *There exists an absorbing set at time 0 wich is δ-compact.*

Proof. By previous lemma, in particular for $t=0$, we can choose α such that

$$-\chi + \mathbf{E}|z(t)|^2_{H^{2+\gamma}} \leq -\frac{\chi}{2}.$$

By the ergodicity of z we have that

$$\frac{1}{-t_0}\int_{t_0}^0 |z|^2_{H^{2+\gamma}} \to \mathbf{E}|z(0)|^2_{H^{2+\gamma}} \text{ as } t_0 \to -\infty.$$

So for $\omega \in \Omega$ there exists $\tau(\omega) < 0$ such that

$$\int_{t_0}^0 (-\chi + |z(r)|^2_{H^{2+\gamma}})dr \leq -\frac{\chi}{4}(-s),\ \forall t_0 < \tau(\omega).$$

Moreover, by the continuity of the trajectories of z

$$\sup_{\tau(\omega)\leq s\leq 0}\int_{t_0}^0 (-\chi + |z(r)|^2_{H^{2+\gamma}})dr \leq C(\omega) < \infty$$

for some constant $C(\omega)$.
Therefore for $t=0$, the estimate (4.20) and (4.21) yield

$$\begin{aligned} & |u(0)-z(0)|^2 \leq |u(t_0)-z(t_0)|^2 \mathrm{e}^{\chi(t_0,\omega)+C(\omega)} \\ + \quad & C(\chi)\int_{t_0}^0 (|f|^2 + \alpha^2|z(r)|^2 + |\nabla z(r)|^2|z(r)|^2_\infty)\mathrm{e}^{\chi(t_0,\omega)+C(\omega)}dr, \end{aligned} \tag{4.25}$$

Since $|z(s)|^2_{H^{3+\gamma}}$ has at most a polynomial growth, when $t_0 \to -\infty$, the right hand side of (4.25), (4.21), and (4.23) are almost surely bounded. Collecting all the estimates of theorem 4.2, there exists a random constant $r(\omega)$ such that $\mathbf{P}$.a.e. $\omega \in \Omega$

$$|u(0)|^2_{\mathcal{W}} \leq r(\omega).$$

We deduce the existence of a $d_{\mathcal{W}}$-bounded and δ-compact random ball which absorbs $d_{\mathcal{W}}$-sets of $\mathcal{W}$ at time 0 from $-\infty$. ∎

As a consequence we give the following theorem

Theorem 4.6 *Under conditions of theorem 4.2, there exists a random weak attractor for (4.1) at time 0, which attracts bounded sets from $-\infty$.*

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References

[1] C. Bardos, *Existence et unicité de la solution de l'équation d'Euler en dimensions deux*, Jour. Math. Anal. Appl. **40**, 1972, 769-780.

[2] H. Bessaih, F. Flandoli, *2-D Euler equations perturbed by noise*, Nonlinear. Diff. Eq. and Appl, Vol **6**, Issue 1, 1999, pp 35-54.

[3] H. Bessaih, F. Flandoli, *Weak attractor for a dissipative Euler equation.*, submitted.

[4] H.Crauel, F.Flandoli, *Attractors for random dynamical systems*, Probab.Theory Relat. Fields, **100**, 1994, 365-393.

[5] H.Crauel, A. Debussche, F.Flandoli, *Random attractors*, J.Dyn.Diff.Eq, 1995.

[6] C. Castaing, M. Valadier,*Convex analysis and measurable multifunctions*, Lecture Notes in Mathematics 580, Springer-Verlag, Berlin 1977.

[7] V. Chepyzhov, M. I. Vishik, *Nonautonomous evolution equations and their attractors*, Russian J. Math. Phys. **1**, 1993, 165-190.

[8] G. Da Prato, J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, 1992.

[9] F. Flandoli, B.Schmalfuss, *Weak solutions and attractors for the 3-dimensional Navier-Stokes equation with nonregular force*, To appear in J. Dyn. Diff. Eq.

[10] G. Gallavotti, *Ipotesi per una introduzione alla Meccanica Dei Fluidi*, Quaderni del Consiglio Nazionale delle Ricerche, Gruppo Nazionale di Fisica Matematica, no 52, 1996.

[11] J.L. Lions, *Équations Différentielles Opérationelles et Problèmes aux limites*, Springer-Verlag, Berlin, 1961.

[12] P.L. Lions, *Mathematical Topics in Fluid Mechanics*, Vol 1, *Incompressible Models*, Oxford Sci. Publ, Oxford, 1996.

[13] G. Sell, *Global attractors for the 3D Navier-Stokes Equations*, J.Dyn.Diff.Eq, **8**, 1, 1996.

[14] R. Temam, *Navier-Stokes Equations*, North-Holland 1984.

[15] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Applied Mathematical Sciences, Vol **68**, Springer Verlag, New York, 1988.

[16] M.J. Vishik, A.V. Fursikov, *Mathematical Problems of Statistical Hydromechanics*, Kluwer, Dordrecht, 1980.