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The Yamada-Watanabe-Engelbert theorem for general stochastic equations and inequalities*

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Abstract

A general version of the Yamada-Watanabe and Engelbert results relating existence and uniqueness of strong and weak solutions for stochastic equations is given. The results apply to a wide variety of stochastic equations including classical stochastic differential equations, stochastic partial differential equations, and equations involving multiple time transformations.

Key words: weak solution, strong solution, pathwise uniqueness, stochastic differential equations, stochastic partial differential equations, multidimensionalindex.

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1 Introduction

In the study of stochastic equations, it is common to distinguish between "strong" solutions and "weak" or distributional solutions. Roughly, a strong solution is one that exists for a given probability space and given stochastic inputs while existence of a weak solution simply ensures that a solution exists on some probability space for some stochastic inputs having the specified distributional properties. For example, given a Brownian motion W defined on a probability space $(\Omega, \mathcal{F}, P)$, for X to be a strong solution of the Itô equation

$$X(t) = X(0) + \int_0^t \sigma(X(s))dW(s) + \int_0^t b(X(s))ds, \tag{1.1}$$

X must be defined on $(\Omega, \mathcal{F}, P)$, be adapted to the filtration generated by W, and satisfy the identity (1.1) almost surely.

Similarly, strong uniqueness asserts that two solutions on the same probability space with the same stochastic inputs agree almost surely while weak uniqueness asserts that two solutions agree in distribution. (Precise definitions will be given later.)

For Itô equations, Yamada and Watanabe (20) proved that weak existence and strong uniqueness imply strong existence and weak uniqueness. Engelbert (7) extended this result to a somewhat more general class of equations and gave a converse in which the roles of existence and uniqueness are reversed, that is, weak uniqueness, in the sense that the joint distribution of X and W is uniquely determined, and strong existence imply strong uniqueness. Barlow (2) gave examples in which weak uniqueness holds, but there is no strong solution.

The issues addressed in these results arise naturally for any stochastic equation and extensions to other settings occur frequently in the literature (1; 12; 13; 18; 21). The goal of the present paper is to give general results that cover all these cases as well as other settings in which these questions have not yet been addressed.

The notion of weak uniqueness requiring that the *joint* distribution of the solution and the stochastic inputs be uniquely determined used by Engelbert and also by Jacod (12) is required in our extension of Engelbert's result as well, except in the simple setting of Section 2. For Itô equations, however, Cherny (6) has shown that it is sufficient to assume uniqueness in distribution for the solution X. In particular, uniqueness of the distribution of X implies uniqueness of the joint distribution of X and W. That result appears to depend heavily on the explicit construction of W from the solution X.

In Section 2, we consider simple equations of the form $\Gamma(X, Y) = 0$, where Y represents the stochastic inputs and X is the solution. In this setting, strong existence is essentially existence of a measurable selection and the results are straightforward; however, considering the problem in this simple setting helps clarify the definitions and leads to further insight into what is really "going on." In particular, we see that the result has little to do with equations, but is really a simple consequence of the convexity of the collection of joint distributions of solutions.

The Yamada-Watanabe and Engelbert results do not follow from the results in Section 2, because measurability issues are key to the notion of strong uniqueness that is useful for stochastic differential equations. In Section 3, we introduce compatibility restrictions that enforce the necessary measurability and give results that cover stochastic differential equations of the usual form, as well as stochastic partial differential equations and other equations involving infinite dimensional semimartingales. We discover again in this more structured setting that convexity of the collection of joint distributions of solutions is still the foundation of the result and hence the result extends to problems involving inequalities or any other conditions that determine convex subsets of the collection of joint distributions.

For a Polish space S, $M(S)$ will denote the collection of Borel measurable functions, $B(S)$ the bounded, Borel measurable functions, and $\mathcal{P}(S)$ the Borel probability measures on S. For an S-valued random variable Y, $\mu_Y \in \mathcal{P}(S)$ will denote its distribution.

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2 Simple stochastic equations

Let $S_1$and S_2 be Polish spaces, and let $\Gamma : S_1 \times S_2 \to \mathbb{R}$ be a Borel measurable function. Let Y be an S_2-valued random variable with distribution ν. We are interested in solutions of the equation

$$\Gamma(X,Y) = 0. \tag{2.2}$$

Stochastic equations, at least equivalent to an equation of the form (2.2), arise in many contexts. Ordinarily one is attempting to use the stochastic equation to specify a stochastic model and it is really the distribution of X or the joint distribution of (X,Y) that is of primary interest. Consequently, it is both natural and useful to think of the primary data of the problem to be Γ and ν rather than Γ and Y, and we define a solution of (2.2) to be any pair of random variables (X,Y) with values in $S_1 \times S_2$ such that

$$\Gamma(X,Y) = 0 \quad a.s. \text{ and } \mu_Y = \nu. \tag{2.3}$$

We will say that (X,Y) is a solution for (Γ,ν) if (2.3) holds.

Clearly, (X,Y) being a solution is a property of the joint distribution of (X,Y), and following the terminology of Engelbert (7) and Jacod (12), we refer to the joint distribution of (X,Y) as a *joint solution measure*. In particular, μ is a joint solution measure if $\mu(S_1 \times \cdot) = \nu$ and

$$\int_{S_1\times S_2} |\Gamma(x,y)|\mu(dx \times dy) = 0.$$

Without loss of generality, we can assume that Γ is bounded, so integrability is not an issue. Let $\mathcal{S}_{\Gamma,\nu} \subset \mathcal{P}(S_1 \times S_2)$ denote the collection of joint solution measures. Clearly, $\mathcal{S}_{\Gamma,\nu}$ is convex, and if Γ is continuous, then $\mathcal{S}_{\Gamma,\nu}$ is closed in the weak topology.

It is natural to hope that a solution for (Γ,ν) will have the property that $X = F(Y)$ for some measurable $F : S_2 \to S_1$, that is, Y completely characterizes the randomness in the problem. However, it is easy to see from simple examples that X will not be of this form in general.

Definition 2.1. *A solution (X,Y) for (Γ,ν) is a* strong solution *if there exists a Borel measurable function $F : S_2 \to S_1$ such that $X = F(Y)$ a.s.*

Existence of a strong solution is essentially the existence of a measurable selection from $\Gamma_y^0 = \{x : \Gamma(x,y) = 0\}$. In particular, if $\{(x,y) : \Gamma(x,y) = 0\}$ is closed and $\nu\{y : \Gamma_y^0 \neq \emptyset\} = 1$, there exists a strong solution. See Wagner (19).

Note that (X, Y) being a strong solution is a property of the distribution of (X, Y) but that the collection of joint solution measures corresponding to strong solutions need not be convex. In fact, it is convex only if there is at most one strong solution.

Lemma 2.2. *If $\mu \in \mathcal{P}(S_1 \times S_2)$ and $\mu(S_1 \times \cdot) = \nu$, then there exists a transition function η such that $\mu(dx \times dy) = \eta(y, dx)\nu(dy)$. If $\mu \in \mathcal{S}_{\Gamma,\nu}$, then μ corresponds to a strong solution if and only if there exists a Borel measurable $F : S_2 \to S_1$, such that $\eta(y, dx) = \delta_{F(y)}(dx)$ a.s. ν.*

Proof. The first statement is essentially just the existence of a regular conditional distribution. Since $P\{X \in B|Y\} = \eta(Y, B)$, if $\eta(y, dx) = \delta_{F(y)}(dx)$, $X = F(Y)$ a.s. Conversely, if $X = F(Y)$ a.s., then $P\{X \in B|Y\} = P\{F(Y) \in B|Y\} = \delta_{F(Y)}(B)$. □

There are several notions of uniqueness that are useful. The strongest notion is *pointwise uniqueness* (or *pathwise uniqueness* if S_1 and S_2 are function spaces).

Definition 2.3. Pointwise uniqueness *holds for (2.3) if X_1, X_2, and Y defined on the same probability space with $\mu_{X_1,Y}, \mu_{X_2,Y} \in \mathcal{S}_{\Gamma,\nu}$ implies $X_1 = X_2$ a.s.*

Engelbert (7) introduces a slightly weaker notion which in our present context is analogous to the following:

Definition 2.4. *For $\mu \in \mathcal{S}_{\Gamma,\nu}$, μ-*pointwise uniqueness *holds if X_1, X_2, and Y defined on the same probability space with $\mu_{X_1,Y} = \mu_{X_2,Y} = \mu$ implies $X_1 = X_2$ a.s.*

Remark 2.5. *If $\mu \in \mathcal{S}_{\Gamma,\nu}$ corresponds to a strong solution, then μ-pointwise uniqueness holds.*

Lemma 2.6. *If every solution is a strong solution, then pointwise uniqueness holds.*

Proof. Let G_1 and G_2 be functions corresponding to strong solutions and define

$$G_3(y, u) = \begin{cases} G_1(y) & u > 1/2 \\ G_2(y) & u \leq 1/2. \end{cases}$$

Then for Y and ξ independent, $\mu_Y = \nu$ and ξ uniform on $[0, 1]$,

$$\Gamma(G_3(Y, \xi), Y) = \Gamma(G_1(Y), Y)\mathbf{1}_{\{\xi > 1/2\}} + \Gamma(G_2(Y), Y)\mathbf{1}_{\{\xi \leq 1/2\}} = 0,$$

and hence $G_3(Y, \xi)$ is a solution. Alternatively, we could simply observe that μ defined by

$$\mu(D) = P\{(G_3(Y, \xi), Y) \in D\} = \frac{1}{2}P\{(G_1(Y), Y) \in D\} + \frac{1}{2}P\{(G_2(Y), Y) \in D\}$$

must be a solution by the convexity of $\mathcal{S}_{\Gamma,\nu}$. Since every solution is a strong solution, it follows that $G_1 = G_2$, ν-almost everywhere, and hence pointwise uniqueness holds. □

Lemma 2.7. *If $\mu \in \mathcal{S}_{\Gamma,\nu}$ and μ-pointwise uniqueness holds, then μ is the joint distribution for a strong solution, and hence, if there is a $\mu \in \mathcal{S}_{\Gamma,\nu}$ that does not correspond to a strong solution, then pointwise uniqueness does not hold.*

Proof. Let Y, ξ_1, and ξ_2 be independent with $\mu_Y = \nu$ and ξ_1 and ξ_2 uniformly distributed on $[0,1]$. By Lemma A.1, there exists $F : S_2 \times [0,1] \to S_1$, such that $(F(Y,\xi_1),Y)$ and $(F(Y,\xi_2),Y)$ have distribution μ. By μ-pointwise uniqueness, $F(Y,\xi_1) = F(Y,\xi_2)$ a.s. By Lemma A.2, there exists $G(y)$ such that $F(Y,\xi_1) = G(Y)$ a.s. implying that μ corresponds to a strong solution. □

Corollary 2.8. *If $\mathcal{S}_{\Gamma,\nu} \neq \emptyset$ and pointwise uniqueness holds, then on any probability space supporting a random variable Y with distribution ν, there exists a unique solution for (Γ,ν) given by a measurable function of Y.*

Two notions of *uniqueness in law* or *weak uniqueness* are useful.

Definition 2.9. Joint uniqueness in law *(or* weak joint uniqueness*) holds for (2.3) if $\mathcal{S}_{\Gamma,\nu}$ contains at most one measure.* Uniqueness in law *(or* weak uniqueness*) holds if all $\mu \in \mathcal{S}_{\Gamma,\nu}$ have the same marginal distribution on S_1.*

Proposition 2.10. *The following are equivalent:*

a) Pointwise uniqueness.

b) μ-pointwise uniqueness for every $\mu \in \mathcal{S}_{\Gamma,\nu}$.

c) Joint uniqueness in law.

d) Uniqueness in law.

Proof. Clearly, pointwise uniqueness implies μ-pointwise uniqueness for every $\mu \in \mathcal{S}_{\Gamma,\nu}$. If μ-pointwise uniqueness holds for every $\mu \in \mathcal{S}_{\Gamma,\nu}$, then every solution is strong by Lemma 2.7 and pointwise uniqueness follows by Lemma 2.6.

To see that pointwise uniqueness implies joint uniqueness in law, let $\mu, \gamma \in \mathcal{S}_{\Gamma,\nu}$ and let Y, ξ_1, and ξ_2 be independent, $\mu_Y = \nu$, and ξ_1 and ξ_2 uniform on $[0,1]$. Then there exist $F_\mu : S_2 \times [0,1] \to S_1$ and $F_\gamma : S_2 \times [0,1] \to S_1$ such that $(F_\mu(Y,\xi_1),Y)$ has distribution μ and $(F_\gamma(Y,\xi_2),Y)$ has distribution γ. By pointwise uniqueness $F_\mu(Y,\xi_1) = F_\gamma(Y,\xi_2)$ a.s., so $\mu = \gamma$.

The fact that (c) implies (d) is immediate.

Finally, we show that (d) implies (a). Suppose pointwise uniqueness does not hold. Then there exist X_1, X_2, Y defined on the same probability space such that $\mu_{X_1,Y}, \mu_{X_2,Y} \in \mathcal{S}_{\Gamma,\nu}$, that is $\Gamma(X_1,Y) = \Gamma(X_2,Y) = 0$ a.s., and $P\{X_1 \neq X_2\} > 0$. There must exist $f : S_1 \to \mathbb{R}$ such that $P\{f(X_1) \neq f(X_2)\} > 0$. Define

$$X = \begin{cases} X_1 & f(X_1) > f(X_2) \\ X_2 & f(X_2) \geq f(X_1). \end{cases}$$

Then

$$\Gamma(X,Y) = \Gamma(X_1,Y)\mathbf{1}_{\{f(X_1)>f(X_2)\}} + \Gamma(X_2,Y)\mathbf{1}_{\{f(X_2)\geq f(X_1)\}} = 0 \quad a.s.,$$

so (X,Y) is a solution, but since $f(X) = f(X_1) \vee f(X_2)$, X cannot have the same distribution as X_1 or X_2 and hence uniqueness in law does not hold. □

We close this section with the observation that if we drop any mention of the equation (2.3) and simply require that $\mathcal{S}_{\Gamma,\nu}$ be a convex subset of $\mathcal{P}(S_1 \times S_2)$ such that $\mu \in \mathcal{S}_{\Gamma,\nu}$ implies $\mu(S_1 \times \cdot) = \nu$ and say that (X, Y) is a solution for (Γ, ν) if $\mu_{XY} \in \mathcal{S}_{\Gamma,\nu}$, then all of the definitions continue to make sense and all of the results continue to hold, except for Proposition 2.10. In Proposition 2.10, the equivalence of (a) and (b) continues to hold, (a) implies (c), and (c) implies (d).

3 Stochastic equations with compatibility restrictions

Let E_1 and E_2 be Polish spaces and let $D_{E_i}[0,\infty)$, be the Skorohod space of cadlag E_i-valued functions. Let Y be a process in $D_{E_2}[0,\infty)$. By $\mathcal{F}_t^Y$, we mean $\sigma(Y(s), s \le t)$.

Definition 3.1. *A process X in $D_{E_1}[0,\infty)$ is* compatible *with Y if for each $t \ge 0$ and $h \in B(D_{E_2}[0,\infty))$,*

$$E[h(Y)|\mathcal{F}_t^{X,Y}] = E[h(Y)|\mathcal{F}_t^Y] \quad a.s. \tag{3.4}$$

The notion of compatibility is essentially (4.5) of Jacod (12). It is central to the extension of the results in Section 2 to stochastic differential equations and other, more general, stochastic equations. Buckdahn, Engelbert, and Răşcanu (5) state an equivalent condition in terms of martingales. If Y has independent increments, then compatibility can be restated as an independence condition. Proof of the following lemma is straightforward.

Lemma 3.2. *X is compatible with Y if and only if every $\{\mathcal{F}_t^Y\}$-martingale, is an $\{\mathcal{F}_t^{X,Y}\}$-martingale.*

If Y has independent increments, then X is compatible with Y if and only if for each $t \ge 0$, $(Y(t+\cdot) - Y(t))$ is independent of $\mathcal{F}_t^{X,Y}$.

We generalize the notion of compatibility in order to allow for stochastic equations involving processes with index sets other than $[0,\infty)$. Note that if $\mathcal{B}_\alpha^{S_1}$ is a sub-σ-algebra of $\mathcal{B}(S_1)$ and X is an S_1-valued random variable on $(\Omega, \mathcal{F}, P)$, then $\mathcal{F}_\alpha^X \equiv \{\{X \in D\} : D \in \mathcal{B}_\alpha^{S_1}\}$ is the sub-σ-algebra of $\mathcal{F}$ generated by $\{h(X) : h \in B(\mathcal{B}_\alpha^{S_1})\}$, where $B(\mathcal{B}_\alpha^{S_1})$ is the collection of $h \in B(S_1)$ that are $\mathcal{B}_\alpha^{S_1}$-measurable.

Definition 3.3. *Let $\mathcal{A}$ be an index set and for each $\alpha \in \mathcal{A}$, let $\mathcal{B}_\alpha^{S_1}$ be a sub-σ-algebra of $\mathcal{B}(S_1)$ and $\mathcal{B}_\alpha^{S_2}$ be a sub-σ-algebra of $\mathcal{B}(S_2)$. Let Y be an S_2-valued random variable. An S_1-valued random variable X is* compatible *with Y if for each $\alpha \in \mathcal{A}$ and each $h \in B(S_2)$,*

$$E[h(Y)|\mathcal{F}_\alpha^X \vee \mathcal{F}_\alpha^Y] = E[h(Y)|\mathcal{F}_\alpha^Y] \quad a.s., \tag{3.5}$$

where $\mathcal{F}_\alpha^X \equiv \{\{X \in D\} : D \in \mathcal{B}_\alpha^{S_1}\}$ and $\mathcal{F}_\alpha^Y \equiv \{\{Y \in D\} : D \in \mathcal{B}_\alpha^{S_2}\}$. The collection $\mathcal{C} \equiv \{(\mathcal{B}_\alpha^{S_1}, \mathcal{B}_\alpha^{S_2}) : \alpha \in \mathcal{A}\}$ will be referred to as a compatibility structure *and we will say X is $\mathcal{C}$-compatible with Y when we want to emphasize the particular choice of $\mathcal{C}$.*

Definition 3.4. *X is* adapted *to Y if for each $\alpha \in \mathcal{A}$, $\mathcal{F}_\alpha^X \subset \mathcal{F}_\alpha^Y$.*

Remark 3.5. *Clearly, if X is adapted to Y it is compatible with Y. If $S_1 = D_{E_1}[0,\infty)$, $S_2 = D_{E_2}[0,\infty)$, $\mathcal{A} = [0,\infty)$ and $\mathcal{B}_t^{S_i}$ is the σ-algebra generated by the coordinate maps $\pi_s^i : z \in S_i \to z(s) \in E_i$ for $s \le t$, then Definitions 3.1 and 3.3 are the same.*

Note that (3.5) is equivalent to requiring that for each $h \in B(S_2)$,

$$\inf_{f \in B(\mathcal{B}_\alpha^{S_1} \times \mathcal{B}_\alpha^{S_2})} E[(h(Y) - f(X,Y))^2] = \inf_{f \in B(\mathcal{B}_\alpha^{S_2})} E[(h(Y) - f(Y))^2], \tag{3.6}$$

so compatibility is a property of the joint distribution of (X,Y).

Definition 3.6. *Let $\mathcal{C}$ be a compatibility structure for S_1, S_2 and $\mu \in \mathcal{P}(S_1 \times S_2)$. μ is $\mathcal{C}$-*compatible with Y*, if (X,Y) having distribution μ implies X is $\mathcal{C}$-compatible with Y.*

Lemma 3.7. *X is compatible with Y if and only if for each $\alpha \in \mathcal{A}$ and each $g \in B(\mathcal{B}_\alpha^{S_1})$,*

$$E[g(X)|Y] = E[g(X)|\mathcal{F}_\alpha^Y] \quad a.s. \tag{3.7}$$

Proof. Suppose that X is compatible with Y. Then for $f \in B(S_2)$ and $g \in B(\mathcal{B}_\alpha^{S_1})$,

$$\begin{aligned} E[f(Y)g(X)] &= E[E[f(Y)|\mathcal{F}_\alpha^X \vee \mathcal{F}_\alpha^Y]g(X)] \\ &= E[E[f(Y)|\mathcal{F}_\alpha^Y]g(X)] \\ &= E[E[f(Y)|\mathcal{F}_\alpha^Y]E[g(X)|\mathcal{F}_\alpha^Y]] \\ &= E[f(Y)E[g(X)|\mathcal{F}_\alpha^Y]], \end{aligned}$$

and (3.7) follows.

Conversely, suppose (3.7) holds. For $f \in B(S_2)$, $g \in B(\mathcal{B}_\alpha^{S_1})$, and $h \in B(\mathcal{B}_\alpha^{S_2})$ we have

$$\begin{aligned} E[E[f(Y)|\mathcal{F}_\alpha^Y]g(X)h(Y)] &= E[E[f(Y)|\mathcal{F}_\alpha^Y]E[g(X)|\mathcal{F}_\alpha^Y]h(Y)] \\ &= E[f(Y)E[g(X)|Y]h(Y)] \\ &= E[f(Y)g(X)h(Y)], \end{aligned}$$

and compatibility follows. $\square$

Lemma 3.8. *Let $\mathcal{C}$ be a compatibility structure and $\nu \in \mathcal{P}(S_2)$. Let $\mathcal{S}_{\mathcal{C},\nu}$ be the collection of $\mu \in \mathcal{P}(S_1 \times S_2)$ with the following properties:*

a) $\mu(S_1 \times \cdot) = \nu$

b) If (X,Y) has distribution μ, then X is $\mathcal{C}$-compatible with Y.

Then $\mathcal{S}_{\mathcal{C},\nu}$ is convex.

Proof. Note that the right side of (3.6) is determined by ν, so $\mu \in \mathcal{S}_{\mathcal{C},\nu}$ if $\mu(S_1 \times \cdot) = \nu$ and

$$\int_{S_1 \times S_2} (h(y) - f(x,y)^2 \mu(dx \times dy) \ge \inf_{g \in B(\mathcal{B}_\alpha^{S_2})} \int_{S_2} (h(y) - g(y))^2 \nu(dy),$$

for each $h \in B(S_2)$, each $\alpha \in \mathcal{A}$, and each $f \in B(\mathcal{B}_\alpha^{S_1} \times \mathcal{B}_\alpha^{S_2})$. Each of these inequalities is preserved under convex combinations. $\square$

In what follows, Γ will denote a collection of constraints that determine convex subsets of $\mathcal{P}(S_1 \times S_2)$, and $\mathcal{S}_{\Gamma,\mathcal{C},\nu}$ will denote the convex subset of $\mu \in \mathcal{P}(S_1 \times S_2)$ such that μ fulfills the constraints in Γ, μ is $\mathcal{C}$-compatible, and $\mu(S_1 \times \cdot) = \nu$.

Examples of convex constraints include finiteness conditions

$$h(X,Y) < \infty \quad a.s.,$$

moment conditions

$$E[|h(X,Y)|] < \infty,$$

inequalities

$$h(X,Y) \leq g(X,Y) \text{ or } E[h(X,Y)] \leq E[g(X,Y)],$$

equations

$$h(X,Y) = 0,$$

and limit requirements

$$\lim_{n\to\infty} E[|h_n(X,Y)|] = 0. \tag{3.8}$$

Example 3.9. (Stochastic differential equations.) Suppose U is a process in $D_{\mathbb{R}^d}[0,\infty)$, V is an $\mathbb{R}^m$-valued semimartingale with respect to the filtration $\{\mathcal{F}_t^{U,V}\}$, and $H : D_{\mathbb{R}^d}[0,\infty) \to D_{\mathbb{M}^{d\times m}}[0,\infty)$, $\mathbb{M}^{d\times m}$ the $d\times m$ dimensional matrices, is Borel measurable satisfying $H(x,t) = H(x(\cdot \wedge t),t)$. Then X is a solution of

$$X(t) = U(t) + \int_0^t H(X,s-)dV(s)$$

if X is compatible with $Y = (U,V)$ and

$$\lim_{n\to\infty} E[1 \wedge |X(t) - U(t) - \sum_k H(X,\frac{k}{n})(V(\frac{k+1}{n} \wedge t) - V(\frac{k}{n} \wedge t))|] = 0, \quad t \geq 0,$$

giving a collection of conditions of the form of (3.8).

Similarly, equations involving the quadratic variation $[X]$ of X can be handled by including the constraint

$$\lim_{n\to\infty} E[1 \wedge |[X]_t - \sum_k (X(\frac{k+1}{n}) - X(\frac{k}{n}))^2|] = 0.$$

Equations involving local times $L^X(t,y)$ as, for example, in Engelbert and Schmidt (8), Barlow and Perkins (3), and Le Gall (16) can be handled by including constraints of the form

$$\lim_{n\to\infty} E[1 \wedge |L^X(t,y) - \epsilon^{-1} \int_0^t \mathbf{1}_{[y,y+\epsilon]}(X(s))d[X_s]|] = 0$$

or

$$\int_{-\infty}^{\infty} f(y)L^X(t,y)dy = \int_0^t f(X(s))d[X]_s.$$

Stochastic differential equations driven by Poisson random measures, infinite systems of stochastic differential equations, and stochastic partial differential equations can be formulated in a similar manner.

Remark 3.10. *For Example 3.9, we could avoid the limit in (3.8) by applying results of Karandikar (14) that give a Borel measurable mapping $\Lambda : D_{\mathbb{M}^{d\times m}}[0,\infty) \times D_{\mathbb{R}^m}[0,\infty) \to D_{\mathbb{R}^d}[0,\infty)$ such that*

$$\int_0^t H(X,s-)dV(s) = \Lambda(H(X,\cdot),V,t).$$

Definition 2.1 continues to apply in the current setting. In the context of stochastic differential equations, a strong solution is sometimes defined to be a solution X that is $\{\mathcal{F}_t^Y\}$-adapted. The following lemma shows that under the compatibility restriction, the two definitions are equivalent.

Lemma 3.11. *Suppose that X is compatible with Y. If $X = F(Y)$ for some measurable $F : S_2 \to S_1$, then X is adapted to Y.*

Proof. For $f \in B(\mathcal{B}_\alpha^{S_1})$, Lemma 3.7 gives

$$f(X) = E[f(F(Y))|Y] = E[f(F(Y))|\mathcal{F}_\alpha^Y],$$

and the adaptedness follows. $\square$

In order to take into account the compatibility requirement, we must change the definition of pointwise uniqueness.

Definition 3.12. *Let X_1, X_2, and Y be defined on the same probability space. Let X_1 and X_2 be S_1-valued and Y be S_2-valued. (X_1,X_2) are* jointly compatible *with Y if*

$$E[f(Y)|\mathcal{F}_\alpha^{X_1} \vee \mathcal{F}_\alpha^{X_2} \vee \mathcal{F}_\alpha^Y] = E[f(Y)|\mathcal{F}_\alpha^Y], \quad \alpha \in \mathcal{A}, f \in B(S_2).$$

Pointwise uniqueness *holds for compatible solutions of (Γ,ν), if for every triple of processes (X_1,X_2,Y) defined on the same sample space such that $\mu_{X_1,Y},\mu_{X_2,Y} \in \mathcal{S}_{\Gamma,\mathcal{C},\nu}$ and (X_1,X_2) is jointly compatible with Y, $X_1 = X_2$ a.s.*

The modification of the definition for μ-pointwise uniqueness is similar.

Lemma 3.13. *If every $\mu \in \mathcal{S}_{\Gamma,\mathcal{C},\nu}$ corresponds to a strong solution, then pointwise uniqueness holds.*

Proof. Using the convexity of $\mathcal{S}_{\Gamma,\mathcal{C},\nu}$, the proof is the same as for Lemma 2.6. $\square$

The following theorem extends Proposition 1 and Corollary 1 of Yamada and Watanabe (20), Theorem 3 of Engelbert (7), and Theorem 8.3 of Jacod (12).

Theorem 3.14. *Suppose $\mathcal{S}_{\Gamma,\mathcal{C},\nu} \neq \emptyset$. The following are equivalent:*

a) Pointwise uniqueness holds for compatible solutions.

b) Joint uniqueness in law holds for compatible solutions and there exists a strong, compatible solution.

Proof. Assume (a). As in the proof of Proposition 2.10, let $\mu, \gamma \in \mathcal{S}_{\Gamma,\mathcal{C},\nu}$ and let Y, ξ_1, and ξ_2 be independent, $\mu_Y = \nu$, and ξ_1 and ξ_2 uniform on $[0,1]$. Then there exist $F_\mu : S_2 \times [0,1] \to S_1$ and $F_\gamma : S_2 \times [0,1] \to S_1$ such that $(F_\mu(Y,\xi_1), Y)$ has distribution μ and $(F_\gamma(Y,\xi_2), Y)$ has distribution γ. We claim that $X_1 = F_\mu(Y,\xi_1)$ and $X_2 = F_\gamma(Y,\xi_2)$ are jointly compatible. Recalling that compatibility is a distributional assumption, by Lemma 3.7, for $f \in B(\mathcal{B}_\alpha^{S_1})$

$$E[f(F_\mu(Y,\xi_1))|Y,\xi_2] = E[f(F_\mu(Y,\xi_1))|Y] = E[f(F_\mu(Y,\xi_1))|\mathcal{F}_\alpha^Y].$$

Consequently, for $f \in B(S_2)$, $g_1, g_2 \in B(\mathcal{B}_\alpha^{S_1})$, and $h \in B(\mathcal{B}_\alpha^{S_2})$,

$$\begin{aligned}
&E[f(Y)g_1(X_1)g_2(X_2)h(Y)] \\
&\quad = E[f(Y)E[g_1(X_1)|Y,\xi_2]g_2(X_2)h(Y)] \\
&\quad = E[f(Y)E[g_1(X_1)|\mathcal{F}_\alpha^Y]g_2(X_2)h(Y)] \\
&\quad = E[E[f(Y)|\mathcal{F}_\alpha^{X_2} \vee \mathcal{F}_\alpha^Y]E[g_1(X_1)|\mathcal{F}_\alpha^Y]g_2(X_2)h(Y)] \\
&\quad = E[E[f(Y)|\mathcal{F}_\alpha^Y]E[g_1(X_1)|Y,\xi_2]g_2(X_2)h(Y)] \\
&\quad = E[E[f(Y)|\mathcal{F}_\alpha^Y]g_1(X_1)g_2(X_2)h(Y)],
\end{aligned}$$

giving the joint compatibility. By pointwise uniqueness, $F_\mu(Y,\xi_1) = F_\gamma(Y,\xi_2)$ a.s., so $\mu = \gamma$, and the solution is strong by Lemma A.2.

Conversely, the strong solution must give the unique $\mu \in \mathcal{S}_{\Gamma,\mathcal{C},\nu}$. Consequently, by Lemma 2.2, there exists $F : S_2 \to S_1$ such that $\mu_{XY} = \mu$ implies $X = F(Y)$ almost surely, and pointwise uniqueness follows. □

The proof of the following generalization of Theorem 2 of Engelbert (7) is similar.

Theorem 3.15. *Let $\mu \in \mathcal{S}_{\Gamma,\mathcal{C},\nu}$. Then μ-pointwise uniqueness holds if and only if the solution corresponding to μ is strong.*

If μ-pointwise uniqueness holds for every $\mu \in \mathcal{S}_{\Gamma,\mathcal{C},\nu}$, then every solution is strong and pointwise uniqueness holds.

Example 3.16. (Spatial birth and death processes.) Equations for birth and death processes of the following form are studied in (10). For $t \geq 0$, η_t is a counting measure on $\mathbb{R}^d$ giving the locations of the particles that are alive at time t. $\lambda(x,\eta_t)$ denotes the birth rate of a new particle at location x at time t and $\delta(x,\eta_t)$ denotes the death rate of an existing particle located at x, that is, the probability of a new particle being born in the set K in the time interval $(t, t+\Delta t]$ is approximately $\int_K \lambda(x,\eta_t)dx\Delta t$ for small Δt and if there is a particle at x at time t, then the probability that it dies in $(t, t+\Delta t]$ is approximately $\delta(x,\eta_t)\Delta t$. For simplicity, assume δ and λ are bounded.

To formulate the corresponding stochastic equation, let ξ be a Poisson random measure on $\mathbb{R}^d \times [0,\infty)^3$ with mean measure $dx \times ds \times e^{-r}dr \times du$, where dx denotes Lebesgue measure on $\mathbb{R}^d$, ds, Lebesgue measure on $[0,\infty)$, etc. Let $\eta_0 = \sum_{i=1}^\infty \delta_{x_i}$ be a counting measure on $\mathbb{R}^d$, where the x_i denote the locations of the initial population. Define

$$\widehat{\eta}_0 = \sum_{i=1}^\infty \delta_{(x_i,\tau_i)},$$

where the $\{\tau_i\}$ are independent unit exponentials, independent of η_0 and ξ. The birth and death process $\eta = \{\eta_t, t \geq 0\}$ should satisfy

$$\begin{aligned}\eta_t(D) &= \int_{D\times[0,t]\times[0,\infty)^2} \mathbf{1}_{[0,\lambda(x,\eta_{s-})]}(u)\mathbf{1}_{(\int_s^t \delta(x,\eta_v)\,dv,\infty)}(r)\xi(dx\times ds\times dr\times du)\\ &\qquad + \int_{D\times[0,\infty)} \mathbf{1}_{(\int_0^t \delta(x,\eta_s)\,ds,\infty)}(r)\widehat{\eta}_0(dx,dr),\end{aligned} \tag{3.9}$$

for every Borel set D with finite Lebesgue measure. η should be compatible with $Y = (\xi, \widehat{\eta}_0)$, where we take $\mathcal{A} = [0,\infty)$ and

$$\mathcal{F}_t^Y \equiv \mathcal{F}_t^{\xi,\widehat{\eta}_0} = \sigma(\xi(D\times[0,s]\times[0,r]\times[0,u]) : |D| < \infty, 0 \leq s \leq t, r \geq 0, u \geq 0) \vee \sigma(\widehat{\eta}_0).$$

Compatibility then becomes

$$E[h(\xi,\widehat{\eta}_0)|\mathcal{F}_t^{\eta,\xi,\widehat{\eta}_0}] = E[h(\xi,\widehat{\eta}_0)|\mathcal{F}_t^{\xi,\widehat{\eta}_0}].$$

This identity is simply the requirement that $\xi(D\times(t,t+s]\times[0,r]\times[0,u])$ is independent of $\mathcal{F}_t^{\eta,\xi,\widehat{\eta}_0}$ for all D satisfying $|D| < \infty$ and $t, s, r, u \geq 0$.

Example 3.17. (Backward stochastic differential equations.) Let Y be a stochastic process with sample paths in $D_{E_2}[0,T]$. Let $H : D_{E_2}[0,T] \to \mathbb{R}^d$ and $f : [0,T] \times D_{\mathbb{R}^d}[0,T] \times D_{E_2}[0,T] \to \mathbb{R}^d$. Require X to be compatible with Y, $E[|H(Y)|] < \infty$,

$$E[\int_0^T |f(s,X(s),Y(s))|ds] < \infty,$$

and for each $0 \leq t \leq T$,

$$X(t) = E[H(Y) + \int_t^T f(s,X(s),Y(s))ds|\mathcal{F}_t^{X,Y}]. \tag{3.10}$$

Note that (3.10) is equivalent to

$$E[(X(t) - H(Y) - \int_t^T f(s,X(s),Y(s))ds)h(X(\cdot\wedge t),Y(\cdot\wedge t))] = 0$$

for all $h \in B(D_{\mathbb{R}^d}[0,T]\times D_{E_2}[0,T])$, so (3.10) is a convex constraint. Theorem 3.15 then gives Proposition 5.3 of (5).

Example 3.18. (Forward-backward stochastic differential equations.) Consider

$$\begin{aligned} U(t) &= U(0) + \int_0^t \sigma(s,U(s),V(s))dW(s) + \int_0^t b(s,U(s),V(s))ds,\\ V(t) &= E[g(U(T)) + \int_t^T h(s,U(s),V(s))ds|\mathcal{F}_t],\end{aligned}$$

where U and V are required to be $\{\mathcal{F}_t\}$-adapted and W is required to be an $\{\mathcal{F}_t\}$-Brownian motion. Assume that U takes values in $\mathbb{R}^k$, V in $\mathbb{R}^l$, and W in $\mathbb{R}^m$. Any solution of the system will also satisfy the system with $\mathcal{F}_t$ replaced by $\mathcal{F}_t^{U,V,W}$ and the requirement then becomes that $X = (U, V)$ be compatible with $Y = (W, U(0))$.

Translating the problems into our setting, ν is the joint distribution of $(W, U(0))$, and the requirements that give the convex constraints are

$$\int_0^t |\sigma(s, U(s), V(s))|^2 ds + \int_0^t |b(s, U(s), V(s))| ds < \infty \quad a.s.,$$

$$U(t) = \lim_{h \to 0+} (U(0) + \int_0^t \frac{1}{h} \int_{s-h}^s \sigma(r, U(r), V(r)) dr dW(s) + \int_0^t b(s, U(s), V(s)) ds) \tag{3.11}$$

in probability, for each $0 < t \le T$, and

$$E[(V(t) - g(U(T)) - \int_t^T h(s, U(s), V(s)) ds) h(U(\cdot \wedge t), V(\cdot \wedge t), W(\cdot \wedge t))] = 0$$

for each $0 \le t \le T$ and each $h \in B(D_{\mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R}^m}[0, \infty))$. Note that the expression on the right of (3.11) is a Borel measurable function on $D_{\mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R}^m}[0, \infty)$ and the limit will exist (that is, the stochastic integral will exist) provided (U, V) is compatible with W.

Proposition 4.2 of (1) follows from Theorem 3.15. See also (17).

Example 3.19. (Multiparameter stochastic differential equations.) Yeh (21) considers stochastic differential equations that can be written in the form

$$X(t, s) = U(t, s) + \int_{[0,t] \times [0,s]} \alpha(u, v, X) W(du \times dv) + \int_{[0,t] \times [0,s]} \beta(u, v, X) du dv,$$

where W is Gaussian white noise on $[0, \infty)^2$ with $E[W(A)W(B)] = m(A \cap B)$, for Lebesgue measure m, and α and β are nonanticipating in the sense that

$$\alpha(u, v, X) = \alpha(u, v, X(\cdot \wedge u, \cdot \wedge v)).$$

$\mathcal{A} = [0, \infty)^2$ and $\mathcal{F}_\alpha^X = \sigma(X(t, s), t \le \alpha_1, s \le \alpha_2)$ and similarly for $\mathcal{F}_\alpha^Y$, where $Y(t, s) = W([0, t] \times [0, s])$. The theorem of (21) then follows from Theorem 3.15.

Example 3.20. (Time-change equations.) Kurtz (15) and Holly and Stroock (11) (see Chapters 6 and 11 of (9)) characterize processes as solutions of systems of the form

$$X_k(t) = Y_k(\int_0^t \beta_k(s, X) ds), \tag{3.12}$$

where $X = (X_1, X_2, \ldots)$, $\beta_k(t, X) = \beta_k(t, X(\cdot \wedge t)) \ge 0$, and the Y_k are independent Markov processes. Set $\tau_k(t) = \int_0^t \beta_k(s, X) ds$, and for $\alpha \in [0, \infty)^\infty$, define

$$\mathcal{F}_\alpha^Y = \sigma(Y_k(s_k) : s_k \le \alpha_k, k = 1, 2, \ldots)$$

and

$$\mathcal{F}_\alpha^X = \sigma(\{\tau_1(t) \le s_1, \tau_2(t) \le s_2, \ldots\} : s_i \le \alpha_i, i = 1, 2, \ldots, t \ge 0).$$

Let $A_k \subset B(E_k) \times B(E_k)$ be a linear operator with $(1, 0) \in A_k$, let ν_k^0 denote the distribution of $Y_k(0)$, and assume that the distribution of Y_k is uniquely determined by the requirement that Y_k

be a solution of the martingale problem for (A_k, ν_k^0). Define $\mathcal{D}(H_k) = \{f_k \in \mathcal{D}(A_k) : \inf_y f_k(y) > 0\}$ and $H_k f_k = A_k f_k / f_k$. Then for $f_i \in \mathcal{D}(H_i)$, $i = 1, \ldots, k$,

$$M_{f_1,\ldots,f_k}(\alpha) = \prod_{i=1}^{k} f_i(Y_i(\alpha)) \exp\{-\int_0^{\alpha_i} H_i f_i(Y_i(s))ds\}$$

is a martingale with respect to the filtration $\{\mathcal{F}_\alpha^Y\}$ and compatibility is equivalent to the requirement that $M_{f_1,\ldots,f_k}$ be a martingale with respect to $\{\mathcal{F}_\alpha^X \vee \mathcal{F}_\alpha^Y\}$ for all k and all $f_i \in \mathcal{D}(H_i)$. $\tau(t) = (\tau_1(t), \tau_2(t), \ldots)$ is a stopping time with respect to $\{\mathcal{F}_\alpha^X \vee \mathcal{F}_\alpha^Y\}$ in the sense that

$$\{\tau(t) \le \alpha\} = \{\tau_1(t) \le \alpha_1, \tau_2(t) \le \alpha_2, \ldots\} \in \mathcal{F}_\alpha^X \vee \mathcal{F}_\alpha^Y, \quad \alpha \in [0,\infty)^\infty,$$

and it follows that a compatible solution of (3.12) is a solution of the multiple random time change problem as defined in Section 3 of (15).

A Appendix

Lemma A.1. *If $\mu \in \mathcal{P}(S_1 \times S_2)$ has S_2-marginal ν, then there exists $F : S_2 \times [0,1] \to S_1$ such that if Y and ξ are independent, $\mu_Y = \nu$ and ξ uniform on $[0,1]$, then $(F(Y,\xi), Y)$ has distribution μ.*

Proof. There exists Borel measurable $H : \mathcal{P}(S_1) \times [0,1] \to S_1$ such that $H(\zeta, \xi)$ has distribution ζ for each $\zeta \in \mathcal{P}(S_1)$. (See Blackwell and Dubins (4) for a construction with particularly nice topological properties.) Let η be the transition function satisfying $\mu(dx \times dy) = \eta(y, dx)\nu(dy)$, and define $F(y,u) = H(\eta(y,\cdot), u)$. Then $(F(Y,\xi), Y)$ has distribution μ. □

Lemma A.2. *Let Y, ξ_1, and ξ_2 be independent with values in S_2, S_3, and S_4. Suppose $F_1 : S_2 \times S_3 \to S_1$ and $F_2 : S_2 \times S_4 \to S_1$ are Borel measurable. If*

$$F_1(Y,\xi_1) = F_2(Y,\xi_2) \quad a.s.,$$

then there exists a Borel measurable $F : S_2 \to S_1$ such that $F_1(Y,\xi_1) = F_2(Y,\xi_2) = F(Y)$ almost surely.

Proof. For $f \in B(S_1)$

$$f(F_1(Y,\xi_1)) = E[f(F_2(Y,\xi_2))|Y,\xi_1] = \int_{S_4} f(F_2(Y,u))\mu_{\xi_2}(du) \quad a.s.$$

Taking $f = \mathbf{1}_D$, we see that $P\{\mu_{\xi_2}\{u : F_2(Y,u) \in D\} \in \{0,1\}\} = 1$, and if $\{A_i\}$ is a partition of S_1, then

$$P\{\max_i \mu_{\xi_2}\{u : F_2(Y,u) \in A_i\} = 1\} = 1. \tag{A.1}$$

Let d_1 be a metric for S_1, and for $x \in S_1$, let $B_\epsilon(x) = \{y : d_1(x,y) \le \epsilon\}$. Let $\{x_i\}$ be dense in S_1, and define $\tau_n(y) = \min\{i : \mu_{\xi_2}\{u : F_2(y,u) \in B_{1/n}(x_i)\} = 1\}$, with $\min \emptyset = \infty$. Define

$$F_n(y) = \begin{cases} x_{\tau_n(y)} & \tau_n(y) < \infty \\ x_0 & \tau_n(y) = \infty. \end{cases}$$

If $n < m$ and $\tau_m(y) < \infty$, then $\tau_n(y) < \infty$ and $d_1(F_n(y), F_m(y)) \leq n^{-1} + m^{-1}$. If $\tau_n(y) = \infty$, then $\tau_m(y) = \infty$ and $F_n(y) = F_m(y) = x_0$. It follows that $\{F_n(y)\}$ is Cauchy for every y, and hence, $F(y) = \lim_{n\to\infty} F_n(y)$ exists. By (A.1), $\tau_n(Y) < \infty$ and $d_1(F_2(Y, \xi_2), F_n(Y)) \leq n^{-1}$ almost surely. It follows that $F_2(Y, \xi_2) = F(Y)$ almost surely. $\square$

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