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Hausdorff Dimension of the SLE Curve Intersected with the Real Line

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Abstract

We establish an upper bound on the asymptotic probability of an SLE(κ) curve hitting two small intervals on the real line as the interval width goes to zero, for the range $4 < \kappa < 8$. As a consequence we are able to prove that the random set of points in $\mathbb{R}$ hit by the curve has Hausdorff dimension $2 - 8/\kappa$, almost surely.

Key words: SLE, Hausdorff dimension, Two-point hitting probability.

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1 Introduction

In the seminal paper [RS05], Rohde and Schramm were able to prove that the Hausdorff dimension of an SLE(κ) curve is almost surely less than or equal to $\min(1 + \kappa/8, 2)$. The scaling properties of SLE immediately imply that the Hausdorff dimension of the curve must almost surely be a constant, and they conjectured that their bound was in fact sharp. In general though, proving a sharp lower bound on the dimension of a random set is a difficult task. In [Law99], Lawler describes a widely applicable and commonly used method for doing so. The required ingredient is a very precise estimate on the probability of two balls both intersecting the random set. Often this is referred to as a second moment method since it can be used to get bounds on the variance of the number of balls (of a certain radius) needed to cover the set. The second moment estimate is difficult as it has to precisely describe how the probability decays as the radius of the balls shrink to zero, and as the balls move closer and farther apart. In the case of the SLE curve, Beffara was able to establish the necessary second moment estimates in [Bef07]. Lawler [Law07] has recently announced a new proof of the lower bound by using a modified version of the second moment method that does not explicitly require an estimate on the two-ball hitting probability.

In this paper we prove a result on the almost sure Hausdorff dimension of another random set arising from the Schramm-Loewner Evolution, namely the set of points at which the curve intersects the real line. Let γ be a chordal SLE(κ) curve from zero to infinity in the upper half plane $\mathbb{H}$ of $\mathbb{C}$. The interaction of this curve with the real line depends very strongly on the well-known phase transitions of SLE. In the case $0 \leq \kappa \leq 4$ the curve is almost surely simple and intersects $\mathbb{R}$ only at zero. For $\kappa \geq 8$ the curve is space-filling and so $\gamma[0, \infty) \cap \mathbb{R} = \mathbb{R}$. For the purposes of this paper the most interesting range is $4 < \kappa < 8$, in which the curve intersects $\mathbb{R}$ on a random Cantor-like set of Hausdorff dimension less than 1. The fractal nature of $\gamma[0, \infty) \cap \mathbb{R}$ should not be surprising. When the curve does hit the real line it tends to linger for a while and hit other real points before wandering off into the upper half plane again, which gives the set of hit points enough irregularity to have a fractional dimension. The main result of this paper is the following:

Theorem 1.1. *For $4 < \kappa < 8$, the Hausdorff dimension of the set $\gamma[0, \infty) \cap \mathbb{R}$ is almost surely $2 - 8/\kappa$.*

It is worth noting that the dimension in Theorem 1.1 is the unique affine function of $1/\kappa$ that interpolates between the already known dimension values of 0 for $\kappa \leq 4$, and 1 for $\kappa \geq 8$. In contrast, the Hausdorff dimension of the SLE(κ) curve itself is an affine function of κ for $0 \leq \kappa \leq 8$.

We will prove Theorem 1.1 using the second moment method described in [Law99]. The asymptotics of certain hitting probabilities, already well established in a number of papers (see Section 2), give the upper bound on the dimension. New results of this paper, which establish the asymptotics of the SLE curve hitting two disjoint small intervals on the real line, give the lower bound.

An alternative (and independently obtained) proof of Theorem 1.1 was announced by Schramm and Zhou in [SZ07]. The main differences between our work and theirs are in the details and methods of proof, but there are two differences in the results. One one hand, Schramm and Zhou do not obtain explicit bounds on the probability that the SLE path hits two disjoint intervals (as we do here). Rather, instead of working with $\gamma[0,\infty)\cap\mathbb{R}$ directly, they use an explicit martingale to construct a measure (a so-called *Frostman measure*) on a particular subset of $\gamma[0,\infty)\cap\mathbb{R}$, which allows them to bound the Hausdorff dimension of both sets from below.

On the other hand, [SZ07] contains a variant of the Hausdorff dimension lower bound argument that applies in the range $\kappa \leq 4$ (which we do not consider). To describe the latter result, suppose that $\kappa \leq 4$ and let $B_\epsilon[a,b]$ denote the ϵ-neighbourhood of a fixed interval $[a,b]$ (with $0 < a < b$). If one covers this interval with ϵ^{-1} balls with radius of order ϵ, then a first moment estimate (similar to the one in this paper for $\kappa > 4$, or the one in Beffara's work [Bef07]) can be used to show that the expected number of these balls that an SLE(κ) curve hits decays like $\epsilon^{8/\kappa-2}$ as $\epsilon \downarrow 0$. One would then expect a second moment estimate to show that the probability that the SLE(κ) curve hits $B_\epsilon[a,b]$ at all decays like the same power of ϵ. Schramm and Zhou do not make this point explicitly, but they use a related analysis to determine when the intersection of the SLE(κ) curve with the graph of a certain kind of function is almost surely unbounded; in the language above, this amounts to showing how quickly a sequence ϵ_n has to decay in order for the probability that an SLE(κ) intersects only finitely many of the sets $B_{2^n\epsilon_n}[2^{n-1},2^n]$ to be one.

1.1 Preliminaries

In this paper we work exclusively with the chordal form of Loewner's equation in the upper half plane. Given a continuous, real-valued function $t \mapsto U_t, t \geq 0$, the map $g_t(z)$ is defined to be the unique solution to the initial value problem

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t},\ g_0(z) = z.$$

An important feature of the maps g_t is that they satisfy the hydrodynamic normalization at infinity, i.e. $g_t(z) = z + o(1)$ as $z \to \infty$. Schramm-Loewner Evolution, or more precisely chordal SLE(κ) from 0 to infinity in $\mathbb{H}$, corresponds to the choice $U_t = \sqrt{\kappa}B_t$, where B_t is a standard 1-dimensional Brownian motion (with filtration $\mathcal{F}_t = \sigma\{B_s : 0 \leq s \leq t\}$). The results of this paper hold exclusively for SLE(κ), but many of the lemmas we derive are deterministic in nature and hold for any continuous driving function. To emphasize this point and keep the deterministic results separate from the probabilistic ones we, for these lemmas, denote the driving function by U_t.

As most of the exponents in this paper usually involve terms in $1/\kappa$ rather than κ, we have chosen to use the slightly different SLE notation that has been championed by Lawler. Instead of κ he uses the parameter $a = 2/\kappa$, and the form of the Loewner equation defined by

$$\partial_t g_t(z) = \frac{a}{g_t(z) - B_t},\ g_0(z) = z. \tag{1}$$

For any $z \in \overline{\mathbb{H}}$ the function $g_t(z)$ is well-defined up to a random time T_z. It is clear from (1) that T_z is the first time t at which $g_t(z) - B_t = 0$. Let $K_t = \overline{\{z \in \mathbb{H} : T_z \leq t\}}$ which is a compact, connected subset of $\overline{\mathbb{H}}$ called the SLE hull. In [RS05] it was proven that for all values of κ the hull is generated by a curve $\gamma : [0,\infty) \to \overline{\mathbb{H}}$, i.e. for all t, $\mathbb{H}\backslash K_t$ is the unbounded connected component of $\mathbb{H}\backslash\gamma([0,t])$. If $1/4 < a < 1/2$ (corresponding to $4 < \kappa < 8$) then $K_\infty \cap \mathbb{R} = \mathbb{R}$ but $\gamma[0,\infty) \cap \mathbb{R}$ is a proper subset of $\mathbb{R}$. The latter fact is evident by observing that $\gamma[0,\infty) \cap \mathbb{R}$ is determined by the process T_x for $x \in \mathbb{R}$. If $x > y > 0$ then the curve intersects $\mathbb{R}$ between y and x iff $T_x > T_y$, and in the case $1/4 < a < 1/2$ there is always a positive probability of having $T_x = T_y$. In fact this last probability can be computed exactly (see [Law05, Propositions 6.8 & 6.34] for a detailed discussion), and it is from the asymptotics of this probability as $x \downarrow y$ that we obtain the upper bound on the Hausdorff dimension.

Two well known scaling properties of SLE (to be used throughout) are that T_x is identical in law to $x^2 T_1$, and that if γ is an SLE curve then $\gamma_r(t) := r^{-1}\gamma(r^2 t)$ is a curve identical in law to γ (see, e.g., [RS05]). The latter, combined with the symmetry of the SLE process about the imaginary axis, tells us that to compute the Hausdorff dimension of $\gamma[0,\infty) \cap \mathbb{R}$ it is enough to consider only $\gamma[0,\infty) \cap [0,1] = \gamma[0,T_1] \cap [0,1]$.

Scaling properties also immediately imply the following.

Lemma 1.2. *The Hausdorff dimension of $\gamma[0,T_1] \cap [0,1]$ is almost surely a constant.*

Proof. The following argument is by now standard (see [Bef04], for instance). Let $A_x = \gamma[0,T_x] \cap [0,x]$. The scaling relations tell us that A_x has the same law as xA_1 for all $x > 0$, and since Hausdorff dimension is unchanged under linear scaling we have $\dim_{\mathrm{H}} xA_1 = \dim_{\mathrm{H}} A_1$. Thus $\dim_{\mathrm{H}} A_x$ is equal in law to $\dim_{\mathrm{H}} A_1$ for all $x > 0$. But $\dim_{\mathrm{H}} A_x$ is a decreasing quantity as $x \downarrow 0$ so it converges almost surely, and its limit has the same distribution as $\dim_{\mathrm{H}} A_1$ and is $\mathcal{F}_{0+}$-measurable. By Blumenthal 0-1 Law the limit must be a constant. Hence $\dim_{\mathrm{H}} A_1$ is equal in law to a constant and therefore a constant itself. $\square$

1.2 Method of Calculating the Hausdorff Dimension

A standard procedure for calculating the Hausdorff dimension of random subsets of $[0,1]$ is described in [Law99]. The main idea is to finely partition the unit interval and compute statistics on the number of subintervals that intersect the random subset. For integer $n \geq 1$ and $1 \leq k \leq 2^n$, define $D_k^n = \{T(k2^{-n}) > T((k-1)2^{-n})\}$, which is the event that the SLE curve hits in the interval $[(k-1)2^{-n}, k2^{-n}]$. The next lemma shows how to prove the upper bound on the Hausdorff dimension.

Lemma 1.3 ([Law99], Lemma 1)**.** *If $s \in (0,1)$ and there exists a $C < \infty$ such that for all sufficiently large n,*

$$\sum_{k=1}^{2^n} \boldsymbol{P}(D_k^n) \leq C 2^{sn}, \tag{2}$$

then almost surely $\dim_{\mathrm{H}} \gamma[0,T_1] \cap [0,1] \leq s$.

Showing that the same s is in fact a lower bound is usually a more difficult task, and it is accomplished by establishing the following estimates.

Lemma 1.4 ([Law99], Lemma 2)**.** *If $s \in (0,1)$, and there exists $C_1, C_2 \in (0,\infty)$ and $\delta \in (0,1/2)$ such that*

$$\boldsymbol{P}(D_k^n) \geq C_1 2^{-(1-s)n}, \text{ for } \delta \leq \frac{k}{2^n} \leq 1-\delta, \tag{3}$$

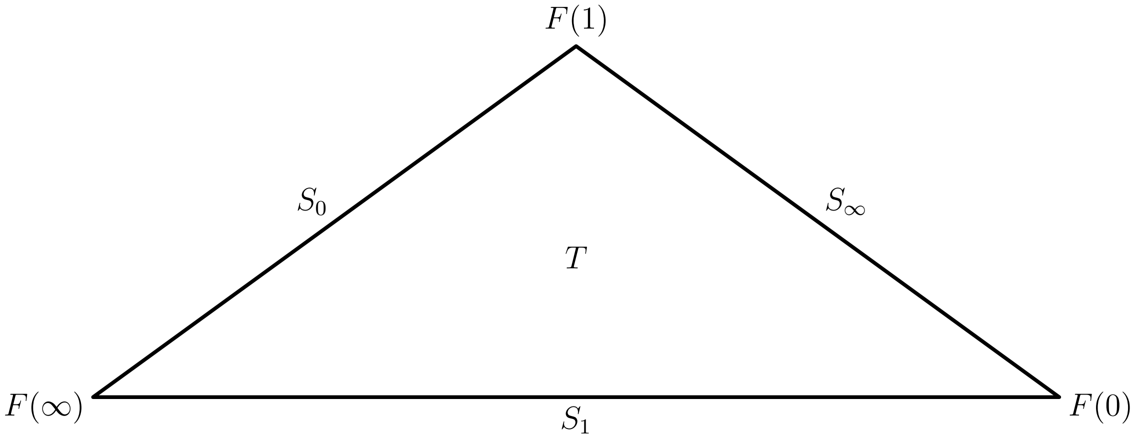

Figure 1: An example of the triangle T used in Proposition 2.1.

and

$$\boldsymbol{P}\left(D_j^n \cap D_k^n\right) \leq C_2 2^{-(1-s)n}(k-j)^{-(1-s)}, \text{ for } \delta \leq \frac{j}{2^n} < \frac{k}{2^n} \leq 1-\delta, \tag{4}$$

for all n sufficiently large, then there exists a $p = p(s, C_1, C_2, \delta) > 0$ such that

$$\boldsymbol{P}\left(\dim_{\mathrm{H}}\left(\gamma[0,T_1] \cap [\delta, 1-\delta]\right) \geq s\right) \geq p.$$

In the present paper we take $s = 2 - 8/\kappa = 2 - 4a$. Section 2 summarizes the results that give us (2). Establishing estimates (3) and (4) is the focus of Section 3. Combined with Lemma 1.2 these three estimates will prove Theorem 1.1.

2 The One-Interval Estimate

In this section we consider the probability of an SLE curve hitting a specified interval on the positive real axis. An exact formula exists and was first proven in [RS05]. Also see [Law05, Proposition 6.34] for another proof. We will make use of a more general version proven in [Dub03].

Proposition 2.1 ([Dub03, Proposition 1])**.** *For chordal SLE(κ) with $4 < \kappa < 8$, define $F : \mathbb{H} \to T$ to be a Schwarz-Christoffel map from $\mathbb{H}$ into an isosceles triangle T that sends $0, 1,$ and ∞ to the vertices, with interior angle $(4a-1)\pi$ at the vertex $F(1)$ and equal angles at the other two vertices (see Figure 1). Then*

$$F(z) = F(0)\boldsymbol{P}(T_z < T_1) + F(1)\boldsymbol{P}(T_z = T_1) + F(\infty)\boldsymbol{P}(T_z > T_1),$$

that is, the three swallowing probabilities are the weights that make $F(z)$ a convex combination of the three vertices $F(0), F(1),$ and $F(\infty)$.

The weights used in the above convex combination are commonly called the *barycentric coordinates* of the point $F(z)$ in the triangle T. Up to translation, scaling, and rotation of the triangle T, the map F is determined by the condition $F'(z) \propto z^{-2a}(1-z)^{4a-2}$ (here $f(z) \propto g(z)$ means $f(z) = \zeta g(z)$ for some $\zeta \in \mathbb{C}\backslash\{0\}$). In subsequent discussion, we will use the choice of F defined by

$$F(z) = \frac{\Gamma(2a)}{\Gamma(1-2a)\Gamma(4a-1)} \int_0^{1-z} \frac{d\xi}{\xi^{2-4a}(1-\xi)^{2a}}. \tag{5}$$

This is the choice of F for which no extra scaling or translation is required to express the hitting probability $\mathbf{P}(T_x < T_y)$, as in the next proposition. Note that the integral is single-valued in $\overline{\mathbb{H}}$ with $F(1) = 0$ and $F(0) = 1$ (the integral defining $F(0)$ is a standard beta integral).

We now use Proposition 2.1 to establish some further results that will be useful in later computations. Here and throughout this paper we will use the notation $f(s) \asymp g(s)$ to mean there exists constants $0 < C_1 < C_2$ such that $C_1 f(s) \le g(s) \le C_2 g(s)$, for all values of the parameter s.

Corollary 2.2. *If $x, y \in \mathbb{R}, x > y > 0$, then $\boldsymbol{P}(T_x > T_y) = F(y/x)$, and consequently*

$$\boldsymbol{P}(T_x > T_y) \asymp \left(\frac{x-y}{x}\right)^{4a-1}. \tag{6}$$

The constants implicit in $\asymp$ depend only on a. Moreover, if τ is any deterministic time or stopping time such that $\tau < T_y$, then

$$\boldsymbol{P}(T_x > T_y \mid \mathcal{F}_\tau) = F\left(\frac{g_\tau(y) - B_\tau}{g_\tau(x) - B_\tau}\right) \asymp \left(\frac{g_\tau(x) - g_\tau(y)}{g_\tau(x) - B_\tau}\right)^{4a-1}.$$

Proof. The exact expression for $\mathbf{P}(T_x > T_y) = \mathbf{P}(T_1 > T_{y/x})$ can be derived from Proposition 2.1 by using our choice of F to compute the barycentric coordinate of the $F(0)$ vertex. For (6), note that $v := y/x \in (0,1)$ and F is a decreasing function on $[0,1]$ with $F(0) = 1$ and $F(1) = 0$. Therefore it is enough to show that $F(v) \asymp (1-v)^{4a-1}$ for v slightly less than 1, which follows easily from (5). Combining the exact and approximate expressions with the Domain Markov Property (that is, mapping back to the upper half plane via g_τ) proves the last statement. □

We get (2) as an immediate result of Corollary 2.2, since

$$\begin{aligned}
\sum_{k=1}^{2^n} \mathbf{P}(D_k^n) &\asymp \sum_{k=1}^{2^n} \left(\frac{1}{k}\right)^{4a-1} \\
&= 2^{(2-4a)n} \sum_{k=1}^{2^n} \left(\frac{1}{k2^{-n}}\right)^{4a-1} 2^{-n}.
\end{aligned}$$

The summation term is a Riemann sum for $\int_0^1 u^{1-4a} du$, which is finite for $1/4 < a < 1/2$. This completes the proof of the upper bound estimate. The next two results will only be used in Section 3 but we mention them here as they are direct corollaries of Proposition 2.1.

Corollary 2.3. *There are fixed constants D_0, D_1, and D_∞, depending only on a, for which the three swallowing probabilities of Proposition 2.1 satisfy*

$$\begin{aligned}\boldsymbol{P}(T_z < T_1) &= D_0 \operatorname{dist}(F(z), S_0),\\ \boldsymbol{P}(T_z = T_1) &= D_1 \operatorname{dist}(F(z), S_1),\\ \boldsymbol{P}(T_z > T_1) &= D_\infty \operatorname{dist}(F(z), S_\infty),\end{aligned}$$

where S_0, S_1, and S_∞ are the lines that form the sides of T, opposite the vertices $F(0), F(1)$, and $F(\infty)$, respectively.

Proof. The statement is an example of the relationship between barycentric coordinates and *trilinear coordinates*, which describe the point $F(z)$ using the distances to the three sides of the triangle. The relationship is clear: the distance from $c_0F(0) + c_1F(1) + c_\infty F(\infty)$ to the line through $F(0)$ and $F(1)$ is a linear function of c_∞ (and similarly the distances to the other lines are linear functions of c_0 and c_1). □

Corollary 2.4. *For $0 < y < x$, $0 \le \theta \le \pi$, and $r \le (x-y)/4$,*

$$\boldsymbol{P}\left(T_{x+re^{i\theta}} < T_y\right) \asymp \frac{y^{1-2a}}{x^{2a}}(x-y)^{4a-2} r\sin\theta. \tag{7}$$

Proof. Let $z' = (x + re^{i\theta})/y$. By scaling and Corollary 2.3,

$$\mathbf{P}\left(T_{x+re^{i\theta}} < T_y\right) = \mathbf{P}\left(T_{z'} < T_1\right) = D_0 \operatorname{dist}\left(F(z'), S_0\right).$$

A useful tool for estimating a distance to the boundary of a domain is the Koebe 1/4 Theorem (see [Law05, Corollary 3.19]), which states that if $f : D \to D'$ is conformal and $z \in D$ then

$$\frac{\operatorname{dist}\left(f(z), D'\right)}{\operatorname{dist}\left(z, D\right)} \asymp |f'(z)|,$$

where the left and right hand constants implicit in $\asymp$ are 1/4 and 4, respectively. We claim that the conditions $0 < y < x$ is enough so that $F(z')$ is closest to side S_0 in T. Assuming this, it follows that

$$\operatorname{dist}\left(F(z'), S_0\right) \asymp |F'(z')| \operatorname{dist}\left(z', \partial\mathbb{H}\right) \propto |z'|^{-2a}|z'-1|^{4a-2}\operatorname{Im}(z').$$

Using that $r \le (x-y)/4$, we have $|z'| \asymp x/y$ and $|z'-1| \asymp (x/y-1)$. Clearly $\operatorname{Im}(z') = r\sin\theta/y$, from which the result follows.

Now we justify the claim that $F(z')$ is closest to the side S_0 in T. Let $\alpha \in [0, \pi/2)$. We will show that the curve $\phi(t) := F(1 + te^{i\alpha})$ lies inside the subtriangle T' bounded by S_0 and the two angle bisectors at the vertices $F(1)$ and $F(\infty)$, which proves that it is closest to S_0 in T. In the upper half plane the pre-image of the bisector at $F(1)$ is locally the vertical line from 1 to ∞, and the line $1 + te^{i\alpha}$ is to the right of this (and closer to the pre-image of S_0, see Figure 2). Therefore $\phi(t)$ is in the subtriangle T' for t small at least. But using $F'(z) \propto z^{-2a}(1-z)^{4a-2}$ it is easy to verify that

$$\partial_t \arg\phi'(t) = -2a\, \partial_t \arg\left(1 + te^{i\alpha}\right) \le 0,$$

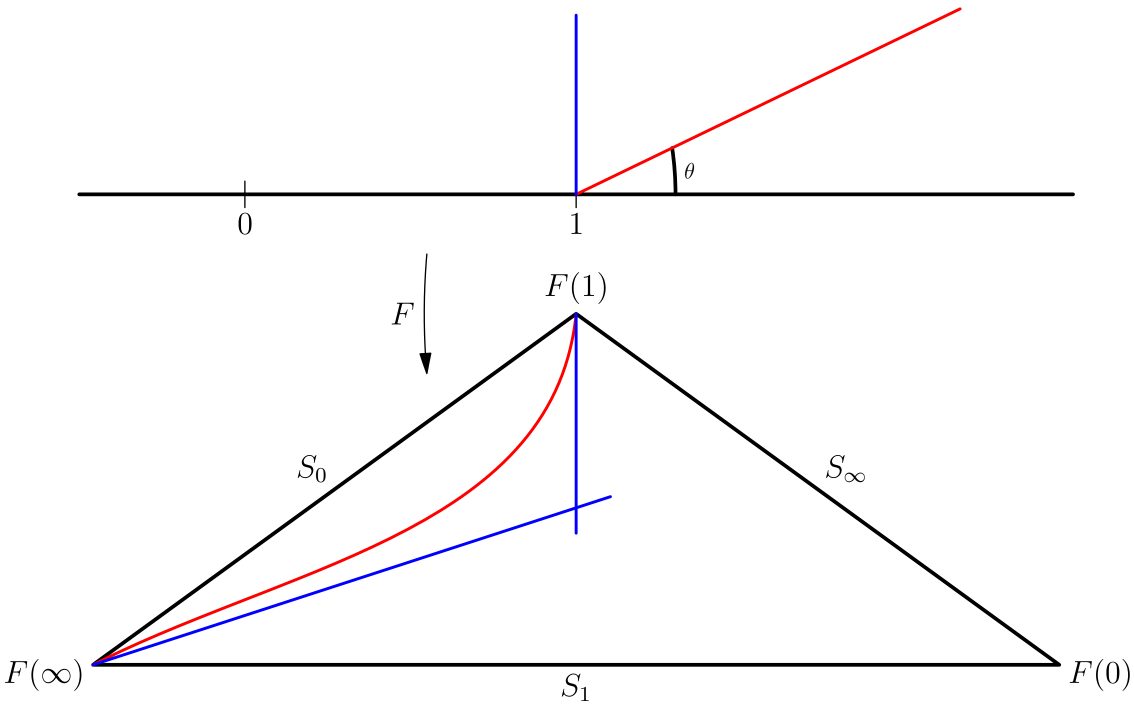

Figure 2: The image of the sector $0 \leq \arg(z-1) \leq \theta < \pi/2$ is, among the three sides of the triangle, always closest to side S_0. This is seen by noting that, in the upper-half plane, the sector begins on the side of the angle bisector at $F(1)$ that is closest to S_0, and then a curvature argument shows that the image of the sector must be curving *away* from the angle bisector. A similar argument shows the curve lies to the left of the image bisector at $F(\infty)$.

so that $\phi(t)$ must be curving *away* from the angle bisector at $F(1)$. Hence $\phi[0,\infty)$ lies on the side of the bisector closest to S_0. A similar argument shows that $\phi[0,\infty)$ also lies on the side of the angle bisector at $F(\infty)$ that is closest to S_0. Since $\mathrm{Re}(z')>1$, we have $z'=1+te^{i\alpha}$ for some $t>0$ and $\alpha\in[0,\pi/2)$, which proves the claim. $\square$

The constraint $r\le(x-y)/4$ was not crucial for the above estimates and certainly could have been improved, but it is all we will require for later use.

3 The Two-Interval Estimate

In this section we work towards establishing the estimates for Lemma 1.4. We already get (3) for free from Corollary 2.2 since

$$\mathbf{P}\left(D_k^n\right)\asymp k^{1-4a}\ge 2^{(1-4a)n},$$

by $k\le 2^n$. To prove the much more difficult bound (4) we require an estimate on the SLE curve hitting two small disjoint intervals. We use various tools from the theory of conformal mapping to accomplish this.

The case of adjacent intervals, corresponding to $k=j+1$ in (4), we will handle directly. In fact in this case the desired probability can be computed exactly, as the following lemma shows.

Lemma 3.1. *Let $0<x_1<x_2<x_3$ be real numbers. Then*

$$\boldsymbol{P}(T_{x_1}<T_{x_2}<T_{x_3})=\boldsymbol{P}(T_{x_1}<T_{x_2})+\boldsymbol{P}(T_{x_2}<T_{x_3})-\boldsymbol{P}(T_{x_1}<T_{x_3}).$$

Proof. The curve hitting in either interval $[x_1,x_2]$ or $[x_2,x_3]$ is equivalent to it hitting in $[x_1,x_3]$, from which the result follows. $\square$

From Lemma 3.1, the assumption $k2^{-n}>\delta$, and the approximation in (6), we have the existence of a constant C such that

$$\begin{aligned}\mathbf{P}\left(D_k^n\cap D_{k+1}^n\right)&\le C\left(\left(\frac{2^{-n}}{k2^{-n}}\right)^{4a-1}+\left(\frac{2^{-n}}{(k+1)2^{-n}}\right)^{4a-1}-\left(\frac{2\cdot 2^{-n}}{(k+1)2^{-n}}\right)^{4a-1}\right)\\&\le\left(\frac{1}{\delta}\right)^{4a-1}(2-2^{4a-1})2^{-(4a-1)n}\\&=C_\delta 2^{-(4a-1)n}.\end{aligned}$$

This is exactly (4) for $k-j=1$.

The rest of this section deals with $k-j\ge 2$. It is actually easier to discuss our proof of (4) if we use a notation involving continuous variables rather than discrete, so assume the two intervals are $(y,y+\epsilon)$ and $(x,x+\epsilon)$ with $0<\delta<y<x<1-\delta$ and $\epsilon>0$. Implicitly though we mean $x=k2^{-n},y=j2^{-n}$, and $\epsilon=2^{-n}$. In this notation, proving (4) is the same as showing that

$$\mathbf{P}\left(T_y<T_{y+\epsilon},T_x<T_{x+\epsilon}\right)\le C\frac{\epsilon^{2(4a-1)}}{(x-y)^{4a-1}}. \tag{8}$$

Since we are now assuming that $k - j \geq 2$, we have that $x - y = (k-j)2^{-n} \geq 2\epsilon$. The bound $\epsilon \leq (x-y)/2$ will be used later on.

We make a brief note about constants here. In moving from line to line we do not always explicitly indicate when the constants involved in a bound may change, usually preferring to fold the new constants into the generic value C. It is important to note that, in accordance with Lemma 1.4, any new constants depend only on a and δ and never x, y, or ϵ.

For the two-interval hitting probability we already know the probability of the curve hitting the first interval $(y, y+\epsilon)$, so we are clearly interested in the conditional probability of hitting the second interval $(x, x+\epsilon)$ at the time y is swallowed. Therefore we condition on $\mathcal{F}_{T_y}$ and arrive at

$$\begin{aligned} \mathbf{P}\left(T_y < T_{y+\epsilon}, T_x < T_{x+\epsilon}\right) &= \mathbf{E}\left[\mathbf{1}\left\{T_y < T_{y+\epsilon}\right\} \mathbf{E}\left[\mathbf{1}\left\{T_x < T_{x+\epsilon}\right\} \mid \mathcal{F}_{T_y}\right]\right] \\ &\asymp \mathbf{E}\left[\mathbf{1}\left\{T_y < T_{y+\epsilon}\right\} \left(\frac{g_{T_y}(x+\epsilon) - g_{T_y}(x)}{g_{T_y}(x+\epsilon) - B_{T_y}}\right)^{4a-1}\right], \end{aligned} \tag{9}$$

the last expression being a result of Corollary 2.2. This reduces the two-interval hitting probability to computing a certain moment, but only on the event $\{T_y < T_{y+\epsilon}\}$ rather than the full space. Needless to say this is a complicated calculation. Moreover, it is not a priori clear how the estimate (9) is related to the desired bound (8). The following two lemmas provide the link. We note here that these lemmas are deterministic in nature and apply to *any* continuous driving function U_t.

Lemma 3.2. *Suppose that U_t is the driving function for the Loewner equation. Fix a point $x > 0$, and let $d_t(x) = dist(x, \partial K_t)$. Define $s_t = \sup K_t \cap \mathbb{R}$, and let $\eta_t := g_t(s_t+) := \lim_{x \downarrow s_t} g_t(x)$. Then for $t < T_x$,*

$$\frac{g_t(x) - \eta_t}{4 g_t'(x)} \leq d_t(x) \leq 4 \frac{g_t(x) - \eta_t}{g_t'(x)}.$$

In particular, if $T_y < T_x$, then

$$\frac{g_{T_y}(x) - U_{T_y}}{4 g_{T_y}'(x)} \leq d_{T_y}(x) \leq 4 \frac{g_{T_y}(x) - U_{T_y}}{g_{T_y}'(x)}.$$

Proof. Let $\tilde{K}_t$ be the reflection of the hull K_t across the real axis. Using the Schwarz reflection principle, the map g_t can be analytically extended as a map on $\mathbb{C} \backslash (K_t \cup \tilde{K}_t)$, which we then restrict to $\mathbb{C} \backslash (K_t \cup \tilde{K}_t \cup (-\infty, 0])$ so the domain is simply connected. The image of the extended g_t is $\mathbb{C} \backslash (-\infty, \eta_t]$. Noting that $d_t(x) = \operatorname{dist}(x, \partial(K_t \cup \tilde{K}_t))$ by symmetry, a direct application of the Koebe 1/4 Theorem gives that

$$\frac{D_t(x)}{4 d_t(x)} \leq g_t'(x) \leq \frac{4 D_t(x)}{d_t(x)}$$

where $D_t(x) = \operatorname{dist}(g_t(x), (-\infty, \eta_t]) = g_t(x) - \eta_t$. This gives the first statement, and for the special case one only has to note that $\eta_{T_y} = U_{T_y}$ since the tip of the SLE curve is on the positive real line at time T_y. ∎

Lemma 3.3. *Let U_t, x, and $d_t(x)$ be as in Lemma 3.2. Then*

$$\frac{g_{T_y}(x+\epsilon) - g_{T_y}(x)}{g_{T_y}(x+\epsilon) - U_{T_y}} \leq 4\frac{\epsilon}{d_{T_y}(x)}.$$

Moreover, if $d_{T_y}(x) > 4\epsilon$, then

$$\frac{g_{T_y}(x+\epsilon) - g_{T_y}(x)}{g_{T_y}(x+\epsilon) - U_{T_y}} \asymp \frac{\epsilon}{d_{T_y}(x)}.$$

Proof. Since $U_{T_y} \leq g_{T_y}(x) \leq g_{T_y}(x+\epsilon)$, we have

$$\frac{g_{T_y}(x+\epsilon) - g_{T_y}(x)}{g_{T_y}(x+\epsilon) - U_{T_y}} \leq 1,$$

and hence the claim is trivial if $d_{T_y}(x) \leq 4\epsilon$. In the case $d_{T_y}(x) > 4\epsilon$ note that

$$\frac{g_{T_y}(x+\epsilon) - U_{T_y}}{g_{T_y}(x+\epsilon) - g_{T_y}(x)} = 1 + \frac{g_{T_y}(x) - U_{T_y}}{g_{T_y}(x+\epsilon) - g_{T_y}(x)} \tag{10}$$

and by Lemma 3.2,

$$\frac{g_{T_y}(x) - U_{T_y}}{g_{T_y}(x+\epsilon) - g_{T_y}(x)} \asymp \frac{d_{T_y}(x) g'_{T_y}(x)}{g_{T_y}(x+\epsilon) - g_{T_y}(x)}, \tag{11}$$

where that the left and right constants implicit in $\asymp$ are $1/4$ and 4, respectively. The last term can be approximated using the Growth Theorem (see [Law05, Theorem 3.23]), which says that if $f : \{|z| < 1\} \to \mathbb{C}$ with $f(0) = 0$ and $f'(0) = 1$ then

$$\frac{|z|}{(1+|z|)^2} \leq |f(z)| \leq \frac{|z|}{(1-|z|)^2}.$$

The map

$$\tilde{g}_t(z) = \frac{g_t(z_0 + d_t(z_0)z) - g_t(z_0)}{d_t(z_0) g'_t(z_0)}$$

satisfies these conditions, where g_t is extended onto $\mathbb{C}\backslash(K_t \cup \tilde{K}_t \cup (-\infty, 0])$ as in Lemma 3.2. Setting $z_0 = x, t = T_y, z = \epsilon/d_{T_y}(x)$, and using the assumption that $4\epsilon < d_{T_y}(x)$ gives

$$\frac{(1-\epsilon/d_{T_y}(x))^2}{\epsilon/d_{T_y}(x)} \leq \frac{d_{T_y}(x) g'_{T_y}(x)}{g_{T_y}(x+\epsilon) - g_{T_y}(x)} \leq \frac{(1+\epsilon/d_{T_y}(x))^2}{\epsilon/d_{T_y}(x)}.$$

Combining this with (10) and (11) we have

$$1 + \frac{(1-\epsilon/d_{T_y}(x))^2}{4\epsilon/d_{T_y}(x)} \leq \frac{g_{T_y}(x+\epsilon) - U_{T_y}}{g_{T_y}(x+\epsilon) - g_{T_y}(x)} \leq 1 + 4\frac{(1+\epsilon/d_{T_y}(x))^2}{\epsilon/d_{T_y}(x)},$$

or, what is equivalent,

$$\frac{\epsilon/d_{T_y}(x)}{(1+\epsilon/d_{T_y}(x))^2 + 4\epsilon/d_{T_y}(x)} \leq \frac{g_{T_y}(x+\epsilon) - g_{T_y}(x)}{g_{T_y}(x+\epsilon) - U_{T_y}} \leq \frac{4\epsilon/d_{T_y}(x)}{(1+\epsilon/d_{T_y}(x))^2}.$$

Maximizing (minimizing) the denominator of the left (right) hand side produces

$$\frac{16}{41}\frac{\epsilon}{d_{T_y}(x)} \le \frac{g_{T_y}(x+\epsilon) - g_{T_y}(x)}{g_{T_y}(x+\epsilon) - U_{T_y}} \le 4\frac{\epsilon}{d_{T_y}(x)}.$$

$\square$

With Lemma 3.3 in hand the relation between (8) and (9) becomes more evident. By (9) and Lemma 3.3,

$$\mathbf{P}\left(T_y < T_{y+\epsilon}, T_x < T_{x+\epsilon}\right) \le C\epsilon^{4a-1}\mathbf{E}\left[\mathbf{1}\left\{T_y < T_{y+\epsilon}\right\} d_{T_y}(x)^{1-4a}\right]. \tag{12}$$

On the event $\{T_y < T_{y+\epsilon}\}$, it is important to note that $d_{T_y}(x)$ satisfies $0 \le d_{T_y}(x) \le x - y$. The upper bound comes from the simple observation that $\gamma(T_y)$ lies somewhere on the real line to the right of y. In fact, on $\{T_y < T_{y+\epsilon}\}$ it is even true that $\gamma(T_y) \in [y, y+\epsilon]$. The latter suggests that $d_{T_y}(x)$ should not be much less than $x - y$ either, since otherwise the SLE curve would have to touch somewhere on the real line before y, and then make an excursion in the upper half-plane that gets very close to x but then returns all the way back to the interval $[y, y+\epsilon]$. One expects such excursions to be rare. *If* it is true that $d_{T_y}(x)$ is roughly on the order of $x-y$, then (12) gives

$$\begin{aligned}
\mathbf{P}\left(T_y < T_{y+\epsilon}, T_x < T_{x+\epsilon}\right) &\le C\mathbf{P}\left(T_y < T_{y+\epsilon}\right)\epsilon^{4a-1}(x-y)^{1-4a} \\
&\le C\left(\frac{\epsilon}{y+\epsilon}\right)^{4a-1}\epsilon^{4a-1}(x-y)^{1-4a} \\
&\le C_\delta \epsilon^{2(4a-1)}(x-y)^{1-4a},
\end{aligned}$$

where the last inequality uses $y > \delta$. This is exactly (8). The rest of the paper proceeds with this line of attack in mind, and the crux of the remaining argument is showing that $d_{T_y}(x)$ is rarely small on the event $\{T_y < T_{y+\epsilon}\}$.

Consider the distribution function

$$G(r) = \mathbf{P}\left(T_y < T_{y+\epsilon}, d_{T_y}(x) \le r\right).$$

We use G to write the expectation in (12) as

$$\begin{aligned}
\mathbf{E}\left[\mathbf{1}\left\{T_y < T_{y+\epsilon}\right\} d_{T_y}(x)^{1-4a}\right] &= \int_0^{x-y} r^{1-4a}dG(r) \\
&= \int_0^{x-y}\int_r^\infty (4a-1)v^{-4a}dv\,dG(r) \\
&= \int_0^{x-y}(4a-1)v^{-4a}G(v)dv + \int_{x-y}^\infty (4a-1)v^{-4a}G(x-y)dv,
\end{aligned} \tag{13}$$

the last equality being an application of Fubini's Theorem. Consider the second integral first. For it we have

$$G(x-y) = \mathbf{P}\left(T_y < T_{y+\epsilon}\right) \asymp \left(\frac{\epsilon}{y+\epsilon}\right)^{4a-1} \le C_\delta \epsilon^{4a-1},$$

and again the last inequality uses $y > \delta$. Consequently

$$\int_{x-y}^{\infty} (4a-1)v^{-4a} G(x-y)dv \le C \frac{\epsilon^{4a-1}}{(x-y)^{4a-1}} \tag{14}$$

for some constant C depending only on a and δ.

We need the same upper bound for the first integral in (13), which requires an upper bound on $G(r)$. By definition, $G(r)$ is the probability of an SLE curve coming within a specified distance r of the point x *before* continuing on to hit the interval $(y, y+\epsilon)$. To estimate $G(r)$ our strategy will be to decompose any such curve into the path from zero to where it first hits the semi-circle of radius r centered at x, and then from the semi-circle to the interval $(y, y+\epsilon)$ (see Figure 3). The probability of the curve hitting the semi-circle (before swallowing y) will be estimated directly, and the probability of the curve going from the semi-circle to $(y, y+\epsilon)$ will be estimated using the conformal invariance property and some considerations of harmonic measure.

We split the first integral in (13) into two parts:

$$\int_0^{x-y} (4a-1)v^{-4a} G(v)dv = \int_0^{\frac{x-y}{4}} (4a-1)v^{-4a} G(v)dv + \int_{\frac{x-y}{4}}^{x-y} (4a-1)v^{-4a} G(v)dv. \tag{15}$$

Using that $G(r)$ is an increasing function of r,

$$\begin{aligned} \int_{\frac{x-y}{4}}^{x-y} (4a-1)v^{-4a} G(v)dv &\le \int_{\frac{x-y}{4}}^{x-y} (4a-1) \left(\frac{x-y}{4}\right)^{-4a} G(x-y)dv \\ &\le C \frac{\epsilon^{4a-1}}{(x-y)^{4a-1}}, \end{aligned} \tag{16}$$

which is the same upper bound in (14). For the integral from zero to $(x-y)/4$ we therefore only need an upper bound on $G(r)$ for r small, namely $r \le (x-y)/4$. Again the condition $r \le (x-y)/4$ is arbitrary, but it is all we will require later on.

Now we show how to estimate the probability of the SLE curve going from the semi-circle to the interval $(y, y+\epsilon)$. Define the stopping time $\tau_r = \inf\{t \ge 0 : |\gamma(t) - x| = r\}$. The event $\{d_{T_y}(x) \le r\}$ is the same as the event $\{\tau_r < T_y\}$, and both are clearly $\mathcal{F}_{\tau_r}$-measurable. We condition on $\mathcal{F}_{\tau_r}$ to compute the probability of the curve going from the semi-circle to $(y, y+\epsilon)$, so that

$$G(r) = \mathbf{P}\left(T_y < T_{y+\epsilon}, d_{T_y}(x) \le r\right) \asymp \mathbf{E}\left[\mathbf{1}\left\{d_{T_y}(x) \le r\right\} \left(\frac{g_{\tau_r}(y+\epsilon) - g_{\tau_r}(y)}{g_{\tau_r}(y+\epsilon) - B_{\tau_r}}\right)^{4a-1}\right]. \tag{17}$$

The following lemma gives an upper bound on (17). Again we should note that the lemma is essentially deterministic in nature and holds for any continuous driving function U_t.

Lemma 3.4. *Suppose $\tau_r < T_y$. Then there exists a constant $C > 0$, depending only on a and δ, such that*

$$\frac{g_{\tau_r}(y+\epsilon) - g_{\tau_r}(y)}{g_{\tau_r}(y+\epsilon) - U_{\tau_r}} \le C \frac{\epsilon r}{(x-y)^2}. \tag{18}$$

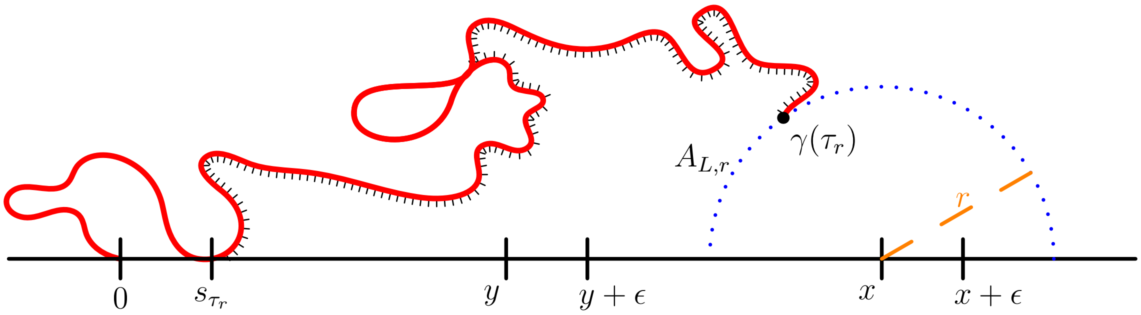

Figure 3: The SLE hull at time τ_r. The right hand side of the hull is highlighted with tick marks.

The proof first gives a way of exactly computing the left hand side of (18) using the harmonic measure of certain boundary segments of the hull $\mathbb{H}\backslash K_{\tau_r}$, and then the upper bound is arrived at by estimating the harmonic measure terms. Throughout the rest of the paper we let β denote a standard complex Brownian motion (independent of the driving function for the Loewner equation), and for $z \in \mathbb{C}$ let $\mathbf{P}_z$ and $\mathbf{E}_z$ denote probabilities and expectations for Brownian motion assuming $\beta_0 = z$. Moreover, given a domain $D \subset \mathbb{C}$ we define $\tau_D = \inf\{t \geq 0 : \beta_t \notin D\}$.

Proof of Lemma 3.4. Let $x_1 < x_2$ be real numbers. If $L > 0$, then in the upper half-plane

$$\begin{aligned}\mathbf{P}_{iL}\left(\beta(\tau_{\mathbb{H}}) \in [x_1, x_2]\right) &= \int_{x_1}^{x_2} \frac{L}{\pi(x^2 + L^2)} dx \\ &= \frac{x_2 - x_1}{\pi L} + O(L^{-2}),\end{aligned}$$

which implies

$$x_2 - x_1 = \lim_{L \uparrow \infty} \pi L \cdot \mathbf{P}_{iL}\left(\beta(\tau_{\mathbb{H}}) \in [x_1, x_2]\right).$$

Consequently,

$$\frac{g_{\tau_r}(y+\epsilon) - g_{\tau_r}(y)}{g_{\tau_r}(y+\epsilon) - U_{\tau_r}} = \lim_{L \uparrow \infty} \frac{\mathbf{P}_{iL}\left(\beta(\tau_{\mathbb{H}}) \in [g_{\tau_r}(y), g_{\tau_r}(y+\epsilon)]\right)}{\mathbf{P}_{iL}\left(\beta(\tau_{\mathbb{H}}) \in [U_{\tau_r}, g_{\tau_r}(y+\epsilon)]\right)} \tag{19}$$

Using the conformal invariance of Brownian motion, we can compute the above harmonic measures in the domain $\mathbb{H}\backslash K_{\tau_r}$ rather than $\mathbb{H}$. Define

$$\mathcal{A}_1 = \{\beta(\tau_{\mathbb{H}\backslash K_{\tau_r}}) \in [y, y+\epsilon]\}, \;\; \mathcal{A}_2 = \{\beta(\tau_{\mathbb{H}\backslash K_{\tau_r}}) \in [s_{\tau_r}, y+\epsilon] \cup \{\text{right side of } K_{\tau_r}\}\},$$

where s_t is as in Lemma 3.2. Note $s_{\tau_r} < y$ since $\tau_r < T_y$. By conformal invariance,

$$\begin{aligned}\mathbf{P}_{iL}\left(\beta(\tau_{\mathbb{H}}) \in [g_{\tau_r}(y), g_{\tau_r}(y+\epsilon)]\right) &= \mathbf{P}_{g_{\tau_r}^{-1}(iL)}(\mathcal{A}_1), \\ \mathbf{P}_{iL}\left(\beta(\tau_{\mathbb{H}}) \in [U_{\tau_r}, g_{\tau_r}(y+\epsilon)]\right) &= \mathbf{P}_{g_{\tau_r}^{-1}(iL)}(\mathcal{A}_2).\end{aligned}$$

Since g_t is normalized so that $g_t(z) = z + o(1)$ as $z \to \infty$, it follows from (19) that

$$\frac{g_{\tau_r}(y+\epsilon) - g_{\tau_r}(y)}{g_{\tau_r}(y+\epsilon) - U_{\tau_r}} = \lim_{L \uparrow \infty} \frac{\mathbf{P}_{iL}(\mathcal{A}_1)}{\mathbf{P}_{iL}(\mathcal{A}_2)}. \tag{20}$$

At time τ_r it is clear that the semi-circle $|z-x|=r$ is naturally divided into a left arc and a right arc by the point $\gamma(\tau_r)$ (see Figure 3). The left arc we will refer to as $A_{L,r}$ and the right one as $A_{R,r}$. In the domain $\mathbb{H}\backslash K_{\tau_r}$ it is clear that the left arc $A_{L,r}$ naturally "shields" the right side of K_{τ_r} and the segment $[s_{\tau_r}, y+\epsilon]$, since any Brownian motion started near infinity that hits these boundaries must have passed through $A_{L,r}$ first. Hence define the stopping time

$$\sigma_r = \tau_{\mathbb{H}\backslash K_{\tau_r}} \wedge \inf\{t \geq 0 : \beta_t \in A_{L,r}\}.$$

Using the Strong Markov Property, the Brownian path from iL to $[y, y+\epsilon]$ can be decomposed into the path from iL to $\beta(\sigma_r) \in A_{L,r}$ plus an independent Brownian path from $\beta(\sigma_r)$ to $[y, y+\epsilon]$. Hence

$$\mathbf{P}_{iL}(\mathcal{A}_1) = \mathbf{E}_{iL}\left[\mathbf{P}_{\beta(\sigma_r)}(\mathcal{A}_1)\right].$$

Likewise a similar expression can be derived for the denominator of (20), and upon taking the ratio of the two we have

$$\frac{g_{\tau_r}(y+\epsilon) - g_{\tau_r}(y)}{g_{\tau_r}(y+\epsilon) - U_{\tau_r}} = \lim_{L\uparrow\infty} \frac{\mathbf{E}_{iL}\left[\mathbf{P}_{\beta(\sigma_r)}(\mathcal{A}_1)\right]}{\mathbf{E}_{iL}\left[\mathbf{P}_{\beta(\sigma_r)}(\mathcal{A}_2)\right]}.$$

Note $\mathbf{P}_{\beta(\sigma_r)}(\mathcal{A}_1) = \mathbf{P}_{\beta(\sigma_r)}(\mathcal{A}_2) = 0$ if $\beta(\sigma_r) \notin A_{L,r}$.

Now we take an arbitrary point $z \in A_{L,r}$ and find an upper bound on $\mathbf{P}_z(\mathcal{A}_1)$ and a lower bound on $\mathbf{P}_z(\mathcal{A}_2)$. The upper bound on $\mathbf{P}_z(\mathcal{A}_1)$ is easy, since any Brownian path going from z to $[y, y+\epsilon]$ in $\mathbb{H}\backslash K_{\tau_r}$ is also a Brownian path going from z to $[y, y+\epsilon]$ in $\mathbb{H}$. Hence

$$\begin{aligned}
\pi\mathbf{P}_z(\mathcal{A}_1) &\leq \pi\mathbf{P}_z(\beta(\tau_{\mathbb{H}}) \in [y, y+\epsilon]) \\
&= \arg(z-y-\epsilon) - \arg(z-y) \\
&= \arg\left(1 - \frac{\epsilon}{z-y}\right)
\end{aligned}$$

Figure 4 provides a geometric proof, using only $\epsilon \leq (x-y)/2$ and $r \leq (x-y)/4$, that for some constant $C > 0$

$$\arg\left(1 - \frac{\epsilon}{z-y}\right) \leq C\frac{\epsilon \mathrm{Im} z}{(x-y)^2}.$$

Hence for all $z \in A_{L,r}$

$$\mathbf{P}_z(\mathcal{A}_1) \leq C\frac{\epsilon \mathrm{Im} z}{(x-y)^2}. \tag{21}$$

For $z \in A_{L,r}$ we need a lower bound on $\mathbf{P}_z(\mathcal{A}_2)$. Let

$$\mathcal{A}_3 = \mathcal{A}_2 \cap \{\beta[0, \tau(\mathbb{H}\backslash K_{\tau_r})] \cap A_{R,r} = \emptyset\}.$$

Then $\mathcal{A}_3$ consists of paths in $\mathbb{H}\backslash K_{\tau_r}$ that exit the domain in $[s_{\tau_r}, y+\epsilon]$ or the right side of K_{τ_r} but don't pass through the right arc $A_{R,r}$ of the semi-circle. Let $V_1 = (-\infty, y+\epsilon) \cup (x+r, \infty) \cup \{\text{right side of } A_{R,r}\}$, and

$$\mathcal{A}_4 = \{\beta(\tau(\mathbb{H}\backslash A_{R,r})) \in V_1\}.$$

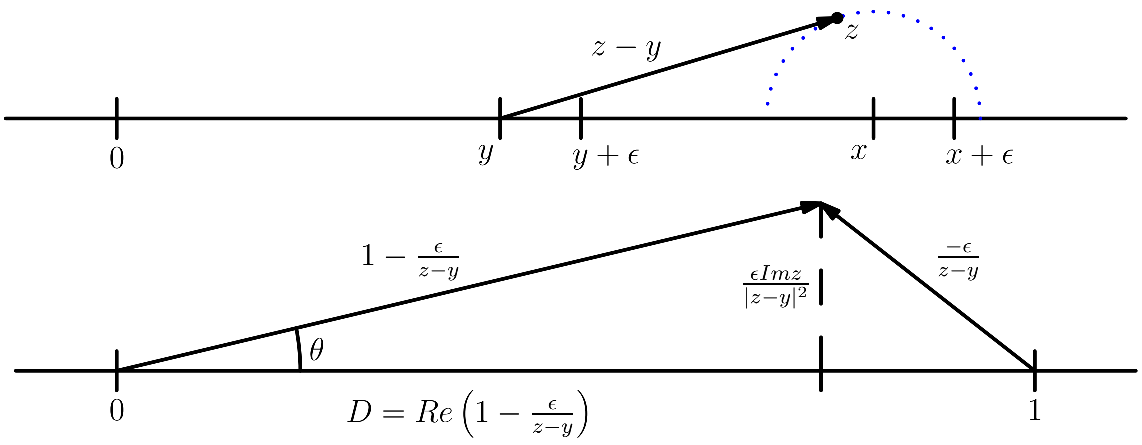

Figure 4: Using $r \le (x-y)/4$ it follows that $|z-y| \ge \frac{3}{4}(x-y)$. Then by $\epsilon \le (x-y)/2$ we have $\frac{\epsilon}{|z-y|} \le \frac{2}{3}$. Thus $D \ge 1/3$. But then $\arg\left(1-\frac{\epsilon}{z-y}\right) = \theta \le \tan\theta = \frac{1}{D}\frac{\epsilon \mathrm{Im} z}{|z-y|^2} \le \frac{16}{3}\frac{\epsilon \mathrm{Im} z}{(x-y)^2}$.

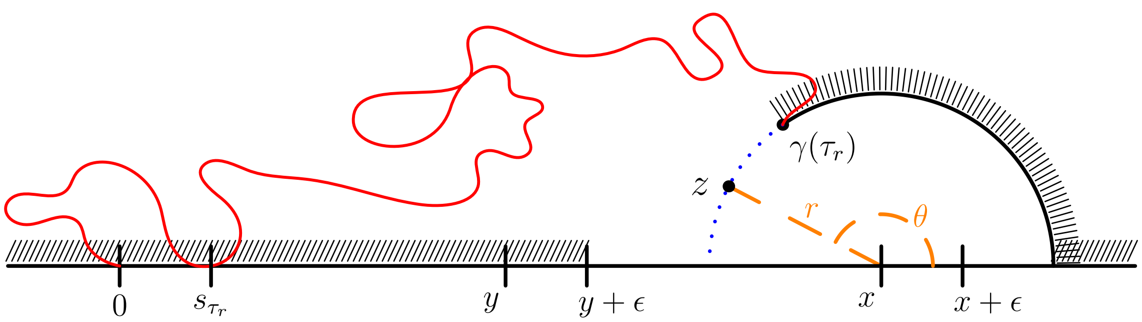

Figure 5: The domain $\mathbb{H}\backslash A_{R,r}$ indicated by solid black boundaries, with the curve $\gamma([0,\tau_r])$ sitting inside it. The boundary segment V_1 is highlighted by tick marks. Any Brownian path started at z that exits $\mathbb{H}\backslash A_{R,r}$ on V_1 is also a Brownian path in $\mathbb{H}\backslash K_{\tau_r}$ that exits $\mathbb{H}\backslash K_{\tau_r}$ on $[s_{\tau_r}, y+\epsilon]$ or the right side of K_{τ_r}.

Topological considerations show that any path in $\mathcal{A}_4$, started at $z \in A_{L,r}$, must have exited the domain $\mathbb{H}\backslash K_{\tau_r}$ on $[s_{\tau_r}, y+\epsilon]$ or the right side of K_{τ_r} (see Figures 3 and 5), so that $\mathcal{A}_4 \subset \mathcal{A}_3$. Therefore $\mathbf{P}_z(\mathcal{A}_2) \geq \mathbf{P}_z(\mathcal{A}_3) \geq \mathbf{P}_z(\mathcal{A}_4)$. Using basic conformal mappings the probability $\mathbf{P}_z(\mathcal{A}_4)$ can be computed explicitly, but for our purposes a lower bound is sufficient. Map the domain $\mathbb{H}\backslash A_{R,r}$ into a strip with a slit via $z \mapsto \log((z-x)/r)$, as shown in Figure 6(a). Call the image domain D and let V_2 be the image of V_1. Let $\theta = \arg(z-x)$, $\phi = \arg(\gamma(\tau_r) - x)$, so that

$$\begin{aligned}\mathbf{P}_z(\mathcal{A}_4) = \mathbf{P}_{i\theta}\left(\beta(\tau_D) \in V_2\right) &\geq \mathbf{P}_{i\theta}\left(\beta(\tau_D) \in [0,\infty) \cup \{\text{right side of } [0, i\phi]\}\right) \\ &= \frac{1}{2}\mathbf{P}_{i\theta}\left(\beta(\tau_D) \in \mathbb{R} \cup [0, i\phi]\right).\end{aligned}$$

The last equality is by symmetry. Any Brownian path in the strip $S = \mathbb{R} \times [0, \pi i]$ that exits S on $\mathbb{R}$ is also a Brownian path in D that exits D on $\mathbb{R} \cup [0, i\phi]$, so that

$$\begin{aligned}\mathbf{P}_{i\theta}(\beta(\tau_D) \in \mathbb{R} \cup [0, i\phi]) &\geq \mathbf{P}_{i\theta}(\beta(\tau_S) \in \mathbb{R}) \\ &= \frac{\pi - \theta}{\pi} \\ &\geq \frac{\sin(\pi-\theta)}{\pi} \\ &= \frac{\sin\theta}{\pi} \\ &\geq C\frac{\mathrm{Im} z}{r}.\end{aligned}$$

Therefore there is a constant $C > 0$ such that

$$\mathbf{P}_z(\mathcal{A}_2) \geq C\frac{\mathrm{Im} z}{r}. \tag{22}$$

Finally by (21) and (22),

$$\mathbf{P}_{\beta(\sigma_r)}(\mathcal{A}_1) \leq C\frac{\epsilon \mathrm{Im}\beta(\sigma_r)}{(x-y)^2}, \quad \mathbf{P}_{\beta(\sigma_r)}(\mathcal{A}_2) \geq C\frac{\mathrm{Im}\beta(\sigma_r)}{r},$$

so that

$$\frac{\mathbf{E}_{iL}\left[\mathbf{P}_{\beta(\sigma_r)}(\mathcal{A}_1)\right]}{\mathbf{E}_{iL}\left[\mathbf{P}_{\beta(\sigma_r)}(\mathcal{A}_2)\right]} \leq C\frac{\epsilon r}{(x-y)^2}.$$

This proves the lemma. □

Lemma 3.4 gives us half of the bound on $G(r)$. Indeed, combining Lemma 3.4 with (17) gives

$$G(r) \leq C\left(\frac{\epsilon r}{(x-y)^2}\right)^{4a-1} \mathbf{P}\left(d_{T_y}(x) \leq r\right). \tag{23}$$

Now we are only left to estimate the term $\mathbf{P}(d_{T_y}(x) \leq r) = \mathbf{P}(\tau_r < T_y)$. A lower bound is easy, since if the curve swallows any point on the semi-circle $|z - x| = r$ before y is swallowed then $\tau_r < T_y$. The probability of z being swallowed before y is known exactly by Proposition 2.1, and

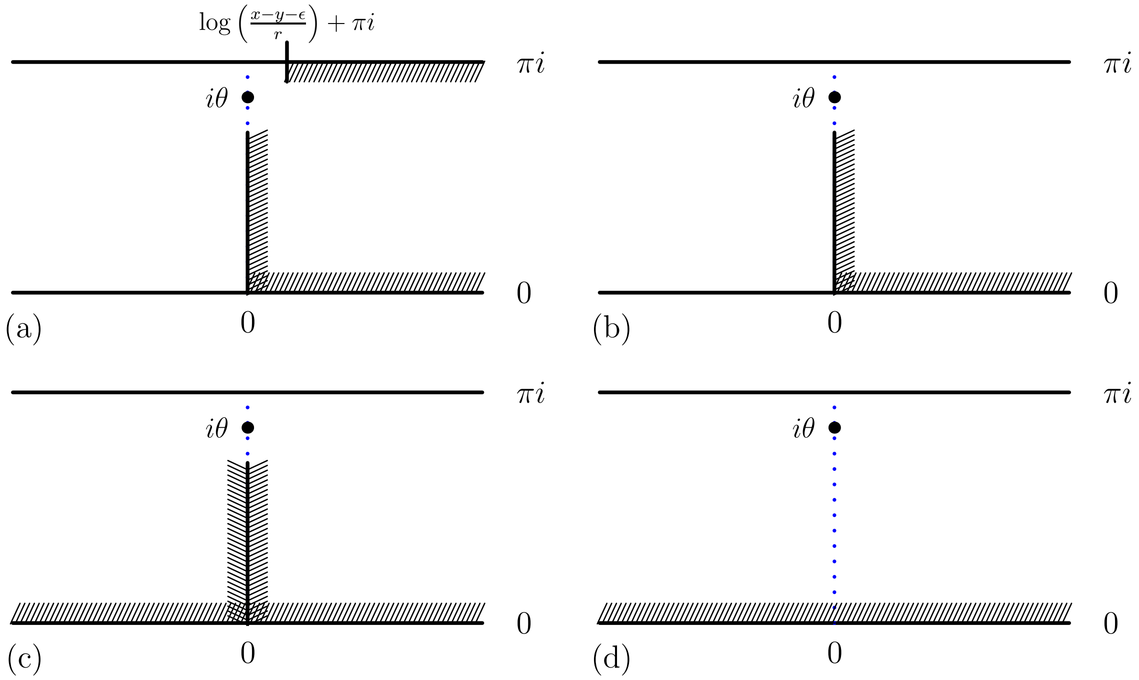

Figure 6: (a) The image of the domain $\mathbb{H}\backslash A_{R,r}$ and the point z under the map $w \mapsto \log\left(\frac{w-x}{r}\right)$. The point z goes to $i\theta$, from which we measure all the harmonic measure terms. The tick marks highlight the boundary segment referred to as V_2. (b) Removing some of the tick marks from (a) only makes the harmonic measure smaller. (c) By symmetry, the harmonic measure in (c) is twice the harmonic measure in (b). (d) Removing the slit from (c) only decreases the harmonic measure.

is well approximated by Corollary 2.4. In fact, choosing $\theta = \pi/2$ in Corollary 2.4 gives a lower bound

$$c' \frac{y^{1-2a}}{x^{2a}}(x-y)^{4a-2} r \le \mathbf{P}\left(\tau_r < T_y\right)$$

for some constant $c' > 0$. We claim that there is a $C > 0$, independent of x, y, and r, such that

$$\mathbf{P}\left(\tau_r < T_y\right) \le C \frac{y^{1-2a}}{x^{2a}}(x-y)^{4a-2} r, \tag{24}$$

at least for $r \le (x-y)/4$. First we suppose that this is true and show how to get the upper bound estimate (8). From (24) and (23)

$$G(r) \le C \frac{y^{1-2a}}{x^{2a}} \frac{\epsilon^{4a-1} r^{4a}}{(x-y)^{4a}} \le C_\delta \frac{\epsilon^{4a-1} r^{4a}}{(x-y)^{4a}},$$

the last inequality coming from $0 < \delta < y < x < 1-\delta$. Substituting this into the first integral of (15) gives

$$\int_0^{\frac{x-y}{4}} v^{-4a} G(v) dv \le C \frac{\epsilon^{4a-1}}{(x-y)^{4a-1}}. \tag{25}$$

As discussed in (13) and (15), the term $\mathbf{E}\left[\mathbf{1}\left\{T_y < T_{y+\epsilon}\right\} d_{T_y}(x)^{1-4a}\right]$ can be broken into three parts, and then, by (14), (16), and (25), each part is bounded above by $C\epsilon^{4a-1}(x-y)^{1-4a}$. Hence $\mathbf{E}\left[\mathbf{1}\left\{T_y < T_{y+\epsilon}\right\} d_{T_y}(x)^{1-4a}\right] \le C\epsilon^{4a-1}(x-y)^{1-4a}$, and substituting this into (12) we get that

$$\mathbf{P}\left(T_y < T_{y+\epsilon}, T_x < T_{x+\epsilon}\right) \le C \frac{\epsilon^{2(4a-1)}}{(x-y)^{4a-1}}.$$

This last bound is exactly (8).

The rest of this section is dedicated to proving (24).

Lemma 3.5. *Let $w_k = -2^{-k-1} + (1 - 3 \cdot 2^{-k-1})\frac{\pi}{2} i$ for $k = 1, 2, \ldots,$ and for $k = -1, -2, \ldots$ let $w_k = \overline{w_{-k}}$. Let $z_k = x + r\exp\{w_k + \frac{\pi}{2} i\}$. Then*

$$\boldsymbol{P}\left(\bigcup_{|k|\ge 1} T_{z_k} < T_y\right) \le \sum_{|k|\ge 1} \boldsymbol{P}(T_{z_k} < T_y) \asymp \frac{y^{1-2a}}{x^{2a}}(x-y)^{4a-2} r$$

Proof. The first inequality is trivial, and using Corollary 2.4

$$\begin{aligned}
\sum_{|k|\ge 1} \mathbf{P}\left(T_{z_k} < T_y\right) &\asymp \frac{y^{1-2a}}{x^{2a}}(x-y)^{4a-2} \sum_{|k|\ge 1} r\exp\{-2^{-|k|-1}\}\sin(\pi - 3\cdot 2^{-|k|-2}\pi) \\
&\asymp \frac{y^{1-2a}}{x^{2a}}(x-y)^{4a-2} \sum_{|k|\ge 1} r\sin(3\cdot 2^{-|k|-2}\pi) \\
&\asymp \frac{y^{1-2a}}{x^{2a}}(x-y)^{4a-2} r.
\end{aligned}$$

□

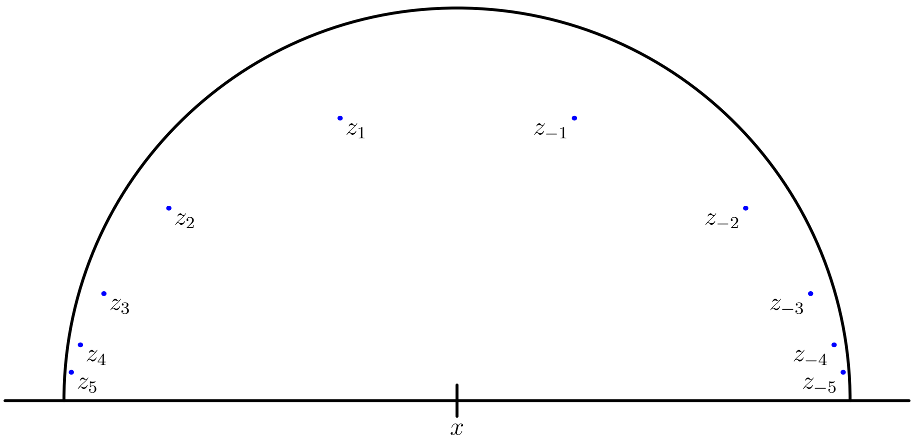

Figure 7: The semi-circle of radius r centered at x with the points z_k inside.

Notice that the points z_k sit inside the semi-circle $|z-x|=r$ (see Figure 7), and so if $T_{z_k} < T_y$ for some k then $\tau_r < T_y$. Conversely, the z_k have been chosen in such a way that if $\tau_r < T_y$ then it's likely that $T_{z_k} < T_y$ for some k. We prove this last statement shortly, but to do so we first require a small lemma on harmonic measure.

Lemma 3.6. *Let S denote the strip $\mathbb{R} \times [0, \pi i]$ and let the w_k be as in Lemma 3.5. There exists a universal constant $l > 0$ such that if $\phi : [0,1] \to S$ is a non-self-crossing curve (possibly having multiple points) with Re $\phi(t) > 0$ for $t \in [0,1)$, Im $\phi(0) = \pi$, and Re $\phi(1) = 0$ (see Figure 9), and H is the hull that ϕ generates (i.e. the complement of the unbounded connected component of $S \backslash \phi[0,\infty)$), then $\boldsymbol{P}_{w_k}\left(\beta(\tau_{S\backslash H}) \in \{\text{right side of } \phi\}\right) \geq l$ and $\boldsymbol{P}_{w_k}\left(\beta(\tau_{S\backslash H}) \in \{\text{left side of } \phi\}\right) \geq l$, for some k.*

Proof. First consider the sets

$$\mathcal{R}_1 = \left\{ x+iy : |x| \leq \frac{1}{5} + \frac{1}{10}, |y| \leq \frac{\pi}{8} + \frac{1}{10} \right\},$$
$$\mathcal{R}_2 = \left\{ x+iy : |x| \leq \frac{1}{5}, |y| \leq \frac{\pi}{8} \right\},$$

and $\mathcal{R} = \mathcal{R}_1 \backslash \mathcal{R}_2$. A sketch of $\mathcal{R}$ is given in Figure 8. Note that $w_0 := -1/4 \in \mathcal{R}$. Let $\mathcal{L}$ be the line segment from $-\pi i/8$ to $-\pi i/8 - i/10$, and $\mathcal{L}'$ be the complex conjugate of the set of points in $\mathcal{L}$. Consider a Brownian particle started at w_0 and killed when it hits the boundary of $\mathcal{R}$. There is a positive probability that the particle arrives at $\mathcal{L}$ in the clockwise direction before it arrives there in the counterclockwise direction, call this probability l. By symmetry this is also the probability that the particle first reaches $\mathcal{L}'$ in the counterclockwise direction. An important feature of this probability l is that it is invariant under scalings and translations of the rectangle $\mathcal{R}$. We now cover the imaginary axis from 0 to πi with scaled and translated versions of $\mathcal{R}$ that send w_0 to the various w_k, as in Figure 9. The idea is that the tip of the curve $\phi(1)$ lies inside one of the rectangles in Figure 9, and then for this rectangle if the Brownian particle travels from w_k to $\mathcal{L}$ in the clockwise direction before reaching it in the counterclockwise direction then

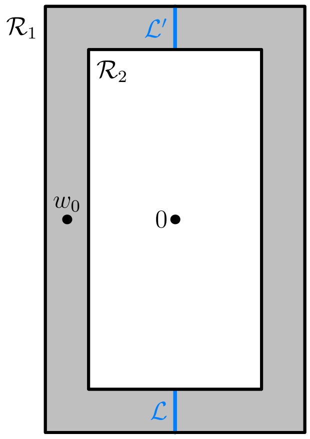

Figure 8: The set $\mathcal{R}$ (the shaded region). We let l be the probability that a Brownian particle started at w_0 hits $\mathcal{L}$ in the clockwise direction before hitting it in the counterclockwise direction.

it must have hit the right hand side of the curve ϕ. The next paragraph provides the details of this argument.

Let $\theta = \operatorname{Im}\phi(1) \in [0, \pi]$. Choose the integer k as follows: if $\theta \geq \pi/2$ then let $k \geq 1$ be such that $(1 - 2^{-k+1})\pi/2 \leq \theta - \pi/2 \leq (1 - 2^{-k})\pi/2$, otherwise let $k \leq -1$ be such that $(1 - 2^{k+1})\pi/2 \leq \pi/2 - \theta \leq (1 - 2^{k})\pi/2$. Then take the rectangle $\mathcal{R}$ and the point w_0, scale them by a factor of $2^{-|k|+1}$, and translate both so that the point w_0 coincides with point w_k. By construction the point $\phi(1)$ lies somewhere on the vertical line subdividing the inner rectangle $\mathcal{R}_2$, and the curve $\phi(t)$ divides the set $\mathcal{R}$. An example with $\theta \in [\pi/2, 3\pi/4]$ and $k = 1$ is shown in Figure 9. For topological reasons, a Brownian particle started at w_k that hits the line segment $\mathcal{L}$ in the clockwise direction must have intersected the right side of ϕ along the way. This shows that $\mathbf{P}_{w_k}\left(\beta(\tau_{S\setminus H} \in \{\text{right side of } \phi\}\right) \geq l$. A completely symmetrical argument proves the lemma for the left hand side of ϕ. □

Lemma 3.7. *Let z_k be as in Lemma 3.5. There exists a $c > 0$ such that*

$$\boldsymbol{P}\left(\bigcup_{|k|\geq 1} T_{z_k} < T_y \,\middle|\, \tau_r < T_y\right) \geq c,$$

for all $r \leq \frac{x-y}{4}$. The constant c is independent of x, y, and r.

Proof. We will actually prove the stronger statement

$$\mathbf{P}\left(T_{z_k} < T_y \text{ for some k } \mid \mathcal{F}_{\tau_r}\right) \geq c\mathbf{1}\left\{\tau_r < T_y\right\}.$$

Let

$$\hat{g}_t(z) = \frac{g_t(z) - U_t}{g_t(y) - U_t}, \tag{26}$$

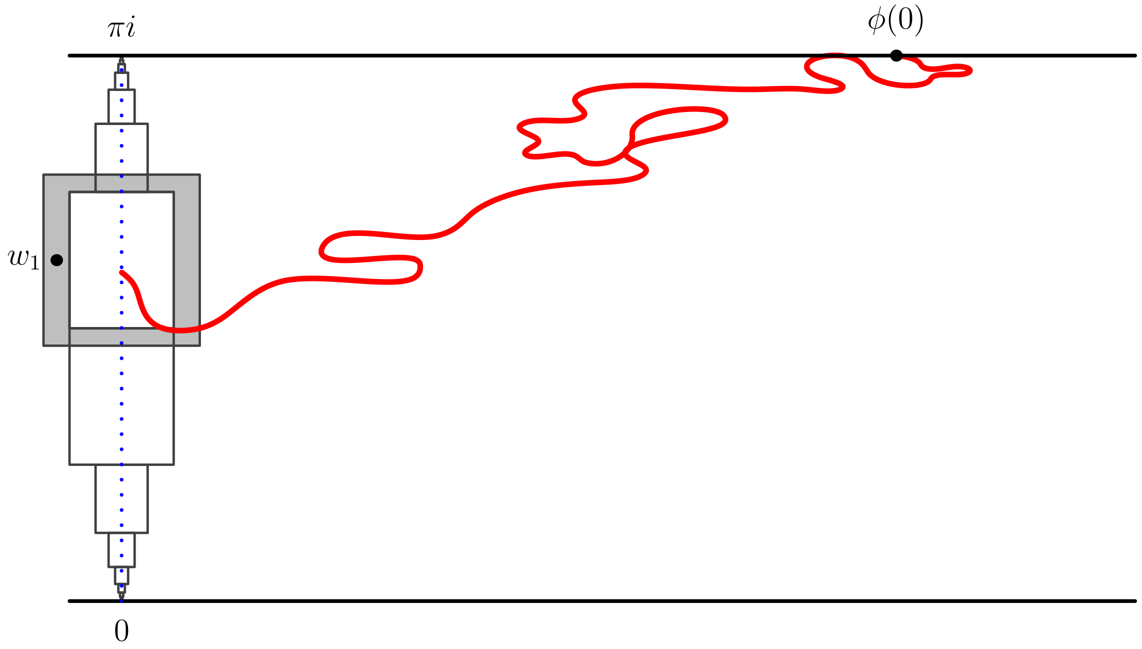

Figure 9: The imaginary axis is covered by scaled and shifted versions of the rectangle $\mathcal{R}_2$. The point $\phi(1)$ must lie inside one of them, in this case it's the rectangle corresponding to $k = 1$. From the point w_1 the harmonic measure of each side of the curve must be at least l.

which is well-defined for $t < T_y$, maps from $\mathbb{H}\backslash K_t \to \mathbb{H}$ and sends $\gamma(t) \to 0, y \to 1$, and $\infty \to \infty$. Also let $H_t = F \circ \hat{g}_t : \mathbb{H}\backslash K_t \to T$, where F is the Schwarz-Christoffel map from Lemma 2.1 and T is the triangle that F maps into. By the Domain Markov Property and Corollary 2.3,

$$P\left(T_z < T_y \mid \mathcal{F}_t\right) = D_0 \operatorname{dist}(H_t(z), S_0), \text{ for } t < T_y \wedge T_z.$$

Since $|z_k - x| \leq r$ we know $T_{z_k} \geq \tau_r$, so that

$$P\left(T_{z_k} < T_y \mid \mathcal{F}_{\tau_r}\right) = D_0 \operatorname{dist}(H_{\tau_r}(z_k), S_0), \text{ for } \tau_r < T_y.$$

Clearly then it is enough to find a $c > 0$ such that $\operatorname{dist}(H_{\tau_r}(z_k), S_0) \geq c$ for some k. Again we turn to harmonic measure estimates. Let l be the universal constant from Lemma 3.6 and consider a point $w \in T$ such that a Brownian particle in T, started at w, has at least probability l of hitting the side S_1 before any other, and also probability l of hitting S_∞ before any other side of T. Then w cannot be arbitrarily close to S_0, otherwise the probability of hitting one of the sides S_1 or S_∞ would have to be small, so there exists a constant $c = c(l, a)$ such that $\operatorname{dist}(w, S_0) \geq c$. Hence it is enough to show that for some k, a Brownian particle in T, started at $H_{\tau_r}(z_k)$, has at least probability l of hitting side S_1 first, and also probabilty l of hitting side S_∞ first. Using the conformal invariance of Brownian motion, and noting that the map $H_{\tau_r}^{-1}$ identifies the sides S_1, S_∞ of T with the boundaries $U_1 = (-\infty, 0) \cup \{\text{left side of } K_{\tau_r}\}, U_\infty = [0, y] \cup \{\text{right side of } K_{\tau_r}\}$ of $\mathbb{H}\backslash K_{\tau_r}$ (respectively), this is equivalent to showing a Brownian particle in $\mathbb{H}\backslash K_{\tau_r}$, started at z_k, has probability at least l of hitting the boundary segment U_1 first, and probability at least l of hitting the boundary segment U_∞ first. But Lemma 3.6 already proves this last statement; all that is left to do is to map $\mathbb{H}$ to the strip U via $z \mapsto \log((z-x)/r)$ and note that the points z_k go to the points w_k. □

Lemmas 3.7 and 3.5 now combine to show

$$\mathbf{P}(\tau_r < T_y) \leq \frac{1}{c}\mathbf{P}\left(\bigcup_{|k|\geq 1} T_{z_k} < T_y\right) \leq C'\frac{y^{1-2a}}{x^{2a}}(x-y)^{4a-2}r.$$

This completes the proof of (24), and also of the two-interval estimate (8).

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